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UNIQUE CONTINUATION PRINCIPLE FOR SPECTRAL PROJECTIONS OF SCHRÖDINGER OPERATORS AND OPTIMAL WEGNER ESTIMATES FOR NON-ERGODIC RANDOM SCHRÖDINGER OPERATORS

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ABSTRACT. We prove a unique continuation principle for spectral projections of Schrödinger operators. We consider a Schrödinger operator $H = -\Delta + V$ on $\mathrm{L}^2(\mathbb{R}^d)$, and let H_Λ denote its restriction to a finite box Λ with either Dirichlet or periodic boundary condition. We prove unique continuation estimates of the type $\chi_I(H_\Lambda) W \chi_I(H_\Lambda) \geq \kappa\, \chi_I(H_\Lambda)$ with $\kappa > 0$ for appropriate potentials $W \geq 0$ and intervals I. As an application, we obtain optimal Wegner estimates at all energies for a class of non-ergodic random Schrödinger operators with alloy-type random potentials ('crooked' Anderson Hamiltonians). We also prove optimal Wegner estimates at the bottom of the spectrum with the expected dependence on the disorder (the Wegner estimate improves as the disorder increases), a new result even for the usual (ergodic) Anderson Hamiltonian. These estimates are applied to prove localization at high disorder for Anderson Hamiltonians in a fixed interval at the bottom of the spectrum.

1. INTRODUCTION

Let $H = -\Delta + V$ be a Schrödinger operator on $\mathrm{L}^2(\mathbb{R}^d)$. Given a box (or cube) $\Lambda = \Lambda_L(x_0) \subset \mathbb{R}^d$ with side of length L and center $x_0 \in \mathbb{R}^d$, let $H_\Lambda = -\Delta_\Lambda + V_\Lambda$ denote the restriction of H to the box Λ with either Dirichlet or periodic boundary condition: Δ_Λ is the Laplacian with either Dirichlet or periodic boundary condition and V_Λ is the restriction of V to Λ. (We will abuse the notation and simply write V for V_Λ, i.e., $H_\Lambda = -\Delta_\Lambda + V$ on $\mathrm{L}^2(\Lambda)$.) By a unique continuation principle for spectral projections (UCPSP) we will mean an estimate of the form

$$\chi_I(H_\Lambda) W \chi_I(H_\Lambda) \geq \kappa\, \chi_I(H_\Lambda), \tag{1.1}$$

where χ_I is the characteristic function of an interval $I \subset \mathbb{R}$, $W \geq 0$ is a potential, and $\kappa > 0$ is a constant.

If V and W are bounded $\mathbb{Z}^d$-periodic potentials, $W \geq 0$ with $W > 0$ on some open set, Combes, Hislop and Klopp [CHK1, Section 4], [CHK2, Theorem 2.1] proved a UCPSP for H_Λ with periodic boundary condition, for boxes $\Lambda = \Lambda_L(x_0) \subset \mathbb{R}^d$ with $L \in \mathbb{N}$ and $x_0 \in \mathbb{Z}^d$ and arbitrary bounded intervals I, with a constant $\kappa > 0$ depending on d, I, V, W but not on the box Λ. Their proof uses the unique continuation principle and Floquet theory. Germinet and Klein [GK4, Theorem A.6] proved a modified version of this result, using Bourgain and Kenig's quantitative unique continuation principle [BK, Lemma 3.10] and Floquet theory, obtaining control of the constant κ in terms of the relevant parameters.

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Rojas-Molina and Veselić recently proved "scale-free unique continuation estimates" for Schrödinger operators [RV, Theorem 2.1] (see also [R2, Theorem A.1.1]). They consider a Schrödinger operator $H = -\Delta + V$, where V is only required to be bounded, and its restrictions H_Λ to boxes Λ with side $L \in \mathbb{N}$ with either Dirichlet or periodic boundary condition. They decompose the box Λ into unit boxes, and for each unit box pick a ball of (a fixed) radius δ contained in the unit box, and let W be the potential given by the sum of the characteristic functions of those balls. Using a version of the quantitative unique continuation principle [RV, Theorem 3.1], they prove that if ψ is an eigenfunction of H_Λ with eigenvalue E (more generally, if $|\Delta\psi| \le |(V-E)\psi|$), then

$$\|W\psi\|_2^2 \ge \kappa \|\psi\|_2^2, \tag{1.2}$$

where the constant $\kappa > 0$ depends only on d, V, δ, E, and is locally bounded on E. Since (1.2) is just the UCPSP (1.1) when $I = \{E\}$, this raises the question of the validity of a UCPSP in this setting, posed as an open question by Rojas-Molina and Veselić [RV].

In this article we prove a UCPSP for Schrödinger operators (Theorem 1.1), giving an affirmative answer to the open question in [RV]. The proof is based on the quantitative unique continuation principle derived by Bourgain and Klein [BKl, Theorem 3.2], restated here as Theorem 2.1. This version of the quantitative unique continuation principle, as the original result of Bourgain and Kenig [BK, Lemma 3.10] and the version of Germinet and Klein [GK4, Theorem A.1], allows for approximate solutions of the stationary Schrödinger equation. ([RV, Theorem 3.1] requires $|\Delta\psi| \le |V\psi|$.) Theorem 2.1 can be applied not only to eigenfunctions of a Schrödinger operator H, but also to approximate eigenfunctions, i.e., arbitrary $\psi \in \operatorname{Ran}\chi_{[E-\gamma,E+\gamma]}(H)$, with the error controlled by $\|(H-E)\psi\|_2 \le \gamma\|\psi\|_2$. (See the derivation of [GK4, Theorem A.6] from [GK4, Theorem A.1].) The notion of "dominant boxes", introduced by Rojas-Molina and Veselić [RV, Subsection 5.2] (see also [R2, Appendix A]), plays an important role in the derivation of Theorem 1.1 from Theorem 2.1.

Using Theorem 1.1, we obtain (Theorems 1.4 and 1.5) optimal Wegner estimates (i.e., with the correct dependence on the volume and interval length) at all energies for a class of non-ergodic random Schrödinger operators with alloy-type random potentials (called crooked Anderson Hamiltonians in Definition 1.2). As a consequence, we get optimal Wegner estimates for Delone-Anderson models at all energies (Remark 1.6). We also prove (Theorem 1.7) optimal Wegner estimates at the bottom of the spectrum for crooked Anderson Hamiltonians that have the expected dependence on the disorder (in particular, the Wegner estimate improves as the disorder increases), a new result even for the usual (ergodic) Anderson Hamiltonian. Using Theorem 1.7, we prove localization at high disorder for Anderson Hamiltonians in a fixed interval at the bottom of the spectrum (Theorem 1.8); such a result was previously known only with a covering condition [GK2, Theorem 3.1].

We use two norms on $\mathbb{R}^d$:

$$|x| = |x|_2 := \left(\sum_{j=1}^{d} |x_j|^2\right)^{\frac{1}{2}} \quad \text{and} \quad |x|_\infty := \max_{j=1,2,\ldots,d} |x_j|, \tag{1.3}$$

where $x = (x_1, x_2, \dots, x_d) \in \mathbb{R}^d$. Distances between sets in $\mathbb{R}^d$ will be measured with respect to norm $|x|$. The ball centered at $x \in \mathbb{R}^d$ with radius $\delta > 0$ is given by

$$B(x, \delta) := \left\{ y \in \mathbb{R}^d;\ |y - x| < \delta \right\}. \tag{1.4}$$

The box (or cube) centered at $x \in \mathbb{R}^d$ with side of length L is

$$\Lambda_L(x) = x +]-\tfrac{L}{2}, \tfrac{L}{2}[^d = \left\{ y \in \mathbb{R}^d;\ |y - x|_\infty < \tfrac{L}{2} \right\}; \tag{1.5}$$

we set

$$\widehat{\Lambda}_L(x) = \Lambda_L(x) \cap \mathbb{Z}^d. \tag{1.6}$$

Given subsets A and B of $\mathbb{R}^d$, and a function φ on the set B, we set $\varphi_A := \varphi \chi_{A \cap B}$. In particular, given $x \in \mathbb{R}^d$ and $\delta > 0$ we write $\varphi_{x,\delta} := \varphi_{B(x,\delta)}$. We let $\mathbb{N}_{\text{odd}}$ denote the set of odd natural numbers. If K is an operator on a Hilbert space, $\mathcal{D}(K)$ will denote its domain. By a constant we will always mean a finite constant. We will use $C_{a,b,\dots}$, $C'_{a,b,\dots}$, $C(a, b, \dots)$, etc., to denote a constant depending only on the parameters $a, b, \dots$.

Theorem 1.1. *Let $H = -\Delta + V$ be a Schrödinger operator on $\mathrm{L}^2(\mathbb{R}^d)$, where V is a bounded potential. Fix $\delta \in]0, \frac{1}{2}]$, let $\{y_k\}_{k \in \mathbb{Z}^d}$ be sites in $\mathbb{R}^d$ with $B(y_k, \delta) \subset \Lambda_1(k)$ for all $k \in \mathbb{Z}^d$, and set*

$$W = \sum_{k \in \mathbb{Z}^d} \chi_{B(y_k, \delta)}. \tag{1.7}$$

Given $E_0 > 0$, set $K = K(V, E_0) = 2\|V\|_\infty + E_0$. Consider a box $\Lambda = \Lambda_L(x_0)$, where $x_0 \in \mathbb{Z}^d$ and $L \in \mathbb{N}_{\text{odd}}$, $L \geq 72\sqrt{d}$. There exists a constant $M_d > 0$, such that, defining $\gamma = \gamma(d, K, \delta) > 0$ by

$$\gamma^2 = \tfrac{1}{2} \delta^{M_d \left(1 + K^{\frac{2}{3}}\right)}, \tag{1.8}$$

then for any closed interval $I \subset]-\infty, E_0]$ with $|I| \leq 2\gamma$ we have

$$\chi_I(H_\Lambda) W \chi_I(H_\Lambda) \geq \gamma^2 \chi_I(H_\Lambda). \tag{1.9}$$

Theorem 1.1 is proved in Section 2. It is derived from the quantitative unique continuation principle given in [BKl, Theorem 3.2] using the "dominant boxes" introduced by Rojas-Molina and Veselić [RV, Subsection 5.2], [R2, Appendix A].

Combes, Hislop and Klopp used the UCPSP to prove Wegner estimates for Anderson Hamiltonians, random Schrödinger operators on $\mathrm{L}^2(\mathbb{R}^d)$ with $q\mathbb{Z}^d$-periodic background potential ($q \in \mathbb{N}$) and alloy-type random potentials located in the lattice $\mathbb{Z}^d$; the estimate (1.1) replaces the covering condition required by Combes and Hislop [CH]. They obtained optimal Wegner estimates at all energies for these ergodic random Schrödinger operators [CHK2, Theorem 1.3].

Rojas-Molina and Veselić used (1.2) to prove Wegner estimates at all energies, optimal up to an additional factor of $|\log |I||^d$ ($|I|$ denotes the length of the interval I), for a class of non-ergodic random Schrödinger operators on $\mathrm{L}^2(\mathbb{R}^d)$ with alloy-type random potentials, including Delone-Anderson models [RV, Theorem 4.4]. They also proved optimal Wegner estimates at the bottom of the spectrum [RV, Theorem 4.11]

These non-ergodic random Schrödinger operators are 'crooked' versions of the usual (ergodic) Anderson Hamiltonian. Theorem 1.1 leads to optimal Wegner estimates at all energies for crooked Anderson Hamiltonians. (In particular, we obtain optimal Wegner estimates for Delone-Anderson models at all energies; see Remark 1.6.)

Definition 1.2. *A* crooked Anderson Hamiltonian *is a random Schrödinger operator on* $\mathrm{L}^2(\mathbb{R}^d)$ *of the form*

$$H_{\boldsymbol{\omega}} := H_0 + V_{\boldsymbol{\omega}}, \tag{1.10}$$

where:

(i) $H_0 = -\Delta + V^{(0)}$*, where the the background potential* $V^{(0)}$ *is bounded and* $\inf \sigma(H_0) = 0$.

(ii) $V_{\boldsymbol{\omega}}$ *is a* crooked alloy-type random potential*:*

$$V_{\boldsymbol{\omega}}(x) := \sum_{j \in \mathbb{Z}^d} \omega_j u_j(x), \quad \text{with} \quad u_j(x) = v_j(x - y_j), \tag{1.11}$$

where, for some $\delta_- \in]0, \frac{1}{2}]$ *and* $u_-, \delta_+, M \in]0, \infty[$*:*

(a) $\{y_j\}_{j \in \mathbb{Z}^d}$ *are sites in* $\mathbb{R}^d$ *with* $B(y_j, \delta_-) \subset \Lambda_1(j)$ *for all* $j \in \mathbb{Z}^d$*;*

(b) the single site potentials $\{v_j\}_{j \in \mathbb{Z}^d}$ *are measurable functions on* $\mathbb{R}^d$ *with*

$$u_- \chi_{B(0,\delta_-)} \le v_j \le \chi_{\Lambda_{\delta_+}(0)} \quad \text{for all} \quad j \in \mathbb{Z}^d; \tag{1.12}$$

(c) $\boldsymbol{\omega} = \{\omega_j\}_{j \in \mathbb{Z}^d}$ *is a family of independent random variables whose probability distributions* $\{\mu_j\}_{j \in \mathbb{Z}^d}$ *are non-degenerate with*

$$\operatorname{supp} \mu_j \subset [0, M] \quad \text{for all} \quad j \in \mathbb{Z}^d. \tag{1.13}$$

If the background potential $V^{(0)}$ is $q\mathbb{Z}^d$-periodic with $q \in \mathbb{N}$, and $y_j = j$ and $v_j = v_0$ for all $j \in \mathbb{Z}^d$, then $H_{\boldsymbol{\omega}}$ is the usual (ergodic) Anderson Hamiltonian.

Given a crooked Anderson Hamiltonian $H_{\boldsymbol{\omega}}$, we will use the following notation, definitions, and observations:

- We let $V_\infty^{(0)} := \left\| V^{(0)} \right\|_\infty$, and set

$$U(x) := \sum\nolimits_{j \in \mathbb{Z}^d} u_j(x), \quad \text{so} \quad U_\infty := \|U\|_\infty \le (2 + \delta_+)^d. \tag{1.14}$$

- We have

$$\|V_{\boldsymbol{\omega}}\|_\infty \le M U_\infty, \quad \text{and hence} \quad \left\| V^{(0)} + V_{\boldsymbol{\omega}} \right\|_\infty \le V_\infty^{(0)} + M U_\infty. \tag{1.15}$$

- We set

$$W := \sum_{j \in \mathbb{Z}^d} \chi_{B(y_j, \delta_-)} = \chi_{\cup_{j \in \mathbb{Z}^d} B(y_j, \delta_-)}, \tag{1.16}$$

  and note that

$$0 \le W \le u_-^{-1} U, \quad W^2 = W, \quad \text{and} \quad \|W\|_\infty = 1. \tag{1.17}$$

- We will consider only boxes $\Lambda = \Lambda_L(x_0)$, where $x_0 \in \mathbb{Z}^d$ and $L \in \mathbb{N}_{\text{odd}}$. For such a box Λ we define finite volume crooked Anderson Hamiltonians, with either Dirichlet or periodic boundary condition, by

$$H_{\boldsymbol{\omega}, \Lambda} = H_{0,\Lambda} + V_{\boldsymbol{\omega}}^{(\Lambda)} \quad \text{on} \quad \mathrm{L}^2(\Lambda), \tag{1.18}$$

  where $H_{0,\Lambda}$ is the restriction of H_0 to Λ with the specified boundary condition, and

$$V_{\boldsymbol{\omega}}^{(\Lambda)}(x) := \sum_{j \in \widehat{\Lambda}} \omega_j u_j(x) \quad \text{for} \quad x \in \mathbb{R}^d. \tag{1.19}$$

We also set

$$U^{(\Lambda)}(x) := \sum_{j\in\widehat{\Lambda}} u_j(x) \le U(x), \tag{1.20}$$

$$W^{(\Lambda)}(x) := \sum_{j\in\widehat{\Lambda}} \chi_{B(y_j,\delta_-)}(x)) \le u_-^{-1} U^{(\Lambda)}(x), \tag{1.21}$$

and note that $W^{(\Lambda)}(x) = W(x)$ for $x \in \Lambda$.

- We write $P_{\boldsymbol{\omega},\Lambda}(B) := \chi_B(H_{\boldsymbol{\omega},\Lambda})$ for a Borel set $B \subset \mathbb{R}$.
- Given a box Λ, we set $S_\Lambda(t) := \max_{j\in\widehat{\Lambda}} S_{\mu_j}(t)$ for $t \ge 0$, where $S_\mu(t) := \sup_{a\in\mathbb{R}} \mu([a, a+t])$ denotes the concentration function of the probability measure μ. We also set $S(t) := \sup_{j\in\mathbb{Z}^d} S_{\mu_j}(t)$ for $t \ge 0$.

Remark 1.3. *We defined a* normalized *crooked Anderson Hamiltonian. Requiring* $\inf \sigma(H_0) = 0$ *is just a convenience. It suffices to have* $v_j \le u_+$ *for all* $j \in \mathbb{Z}^d$ *for some* $u_+ \in]0, \infty[$ *in* (1.12) *(we took* $u_+ = 1$*), and we need only* $\operatorname{supp} \mu_j \subset [M_-, M_+]$ *for all* $j \in \mathbb{Z}^d$ *with* $M_\pm \in \mathbb{R}$ *in* (1.13)*. Since an unrenormalized crooked Anderson Hamiltonian is always equal to a renormalized crooked Anderson Hamiltonian plus a constant (see the argument in* [GK4, Subsection 2.1]*), there is no loss of generality in taking* $H_{\boldsymbol{\omega}}$ *as in Definition 1.2.*

Let $H_{\boldsymbol{\omega}}$ be a crooked Anderson Hamiltonian $H_{\boldsymbol{\omega}}$. Using the UCPSP of Theorem 1.1 with $H = H_0$ and W as in (1.16), we can simply follow the proof in [CHK2] obtaining the following extension of their results for crooked Anderson Hamiltonians.

Theorem 1.4. *Let* $H_{\boldsymbol{\omega}}$ *be a crooked Anderson Hamiltonian. Given* $E_0 > 0$*, set* $K_0 = E_0 + 2V_\infty^{(0)}$*, and define* $\gamma_0 = \gamma_0(d, K_0, \delta_-) > 0$ *by*

$$\gamma_0^2 = \tfrac{1}{2}\delta_-^{M_d\left(1+K_0^{\frac{2}{3}}\right)}, \tag{1.22}$$

where $M_d > 0$ *is the constant of Theorem 1.1. Then for any closed interval* $I \subset]-\infty, E_0]$ *with* $|I| \le 2\gamma_0$ *and any box* $\Lambda = \Lambda_L(x_0)$*, where* $x_0 \in \mathbb{Z}^d$ *and* $L \in \mathbb{N}_{\mathrm{odd}}$*,* $L \ge 72\sqrt{d} + \delta_+$*, we have*

$$\mathbb{E}\left\{\operatorname{tr} P_{\boldsymbol{\omega},\Lambda}(I)\right\} \le C_{d,\delta_\pm,u_-,V_\infty^{(0)},E_0}\left(1 + M^{2^{2+\frac{\log d}{\log 2}}}\right) S_\Lambda(|I|)\,|\Lambda|\,. \tag{1.23}$$

We may also use Theorem 1.1 with $H = H_0 + V_{\boldsymbol{\omega}}^{(\Lambda)}$ and W as in (1.16), obtaining the UCPSP (1.9) with a constant γ independent of $\boldsymbol{\omega}$. In Lemma 3.1 we show how this implies a Wegner estimate. Combining Theorem 1.1 and Lemma 3.1 yields the following optimal Wegner estimate.

Theorem 1.5. *Let* $H_{\boldsymbol{\omega}}$ *be a crooked Anderson Hamiltonian. Given* $E_0 > 0$*, set* $K = E_0 + 2\left(V_\infty^{(0)} + MU_\infty\right)$*, and define* $\gamma = \gamma(d, K, \delta_-) > 0$ *by*

$$\gamma^2 = \tfrac{1}{2}\delta_-^{M_d\left(1+K^{\frac{2}{3}}\right)}, \tag{1.24}$$

where $M_d > 0$ *is the constant of Theorem 1.1. Then for any closed interval* $I \subset]-\infty, E_0]$ *with* $|I| \le 2\gamma$ *and any box* $\Lambda = \Lambda_L(x_0)$*, where* $x_0 \in \mathbb{Z}^d$ *and* $L \in \mathbb{N}_{\mathrm{odd}}$*,*

$L \geq 72\sqrt{d} + \delta_+$, *we have*

$$\mathbb{E}\left\{\operatorname{tr} P_{\boldsymbol{\omega},\Lambda}(I)\right\} \leq C_{d,\delta_+,V_\infty^{(0)}} \left(u_-^{-2}\gamma^{-4}(1+E_0)\right)^{2^{1+\frac{\log d}{\log 2}}} S_\Lambda(|I|)\,|\Lambda|\,. \tag{1.25}$$

Theorems 1.4 and 1.5 are proved in Section 3. They both give optimal Wegner estimates valid at all energies, but the constants in (1.23) and (1.25) differ on their dependence on the relevant parameters.

Remark 1.6 (The Delone-Anderson model)**.** *Theorems 1.4 and 1.5 can be applied to the Delone-Anderson model, improving the Wegner estimate of* [RV, Theorem 4.4]*. The Delone-Anderson Hamiltonian is defined almost exactly as in Definition 1.2, the difference being that the crooked alloy-type random potential of* (1.11) *is replaced by the Delone-Anderson random potential*

$$V_{\boldsymbol{\omega}}(x) := \sum_{j\in\mathbb{D}} \omega_j u_j(x), \quad \textit{with} \quad u_j(x) = v_j(x-j), \tag{1.26}$$

where:

- *(i)* $\mathbb{D} \subset \mathbb{Z}^d$ *is a Delone set, i.e., there exist scales* $0 < K_1 < K_2$ *such that* $\#\left(\mathbb{D} \cap \Lambda_{K_1}(x)\right) \leq 1$ *and* $\#\left(\mathbb{D} \cap \Lambda_{K_2}(x)\right) \geq 1$ *for all* $x \in \mathbb{R}^d$*, where* $\#A$ *denotes the cardinality of the set* A*;*
- *(ii)* $\boldsymbol{\omega} = \{\omega_j\}_{j\in\mathbb{D}}$ *and* $\{v_j\}_{j\in\mathbb{D}}$ *are as in Definition 1.2 with* $\mathbb{D}$ *substituted for* $\mathbb{Z}^d$*.*

We set $R = 2\min\left\{r \in \mathbb{N};\ r \geq \frac{K_2}{2} + \delta_-\right\}$*, and fix* $y_k \in \mathbb{D} \cap \Lambda_{K_2}(k)$ *for each* $k \in RZ^d$*; note that* $B(y_k, \delta_-) \subset \Lambda_R(k)$*. We set* $\mathbb{D}_1 = \{y_k\}_{k\in RZ^d}$ *and* $\mathbb{D}_2 = \mathbb{D} \setminus \mathbb{D}_1$*, and decompose the Delone-Anderson random potential similarly to* [RV, Eq. (21)]*:*

$$V_{\boldsymbol{\omega}}(x) = V_{\boldsymbol{\omega}^{(1)}}(x) + V_{\boldsymbol{\omega}^{(2)}}(x), \tag{1.27}$$

$$\textit{where} \quad \boldsymbol{\omega}^{(i)} = \{\omega_j\}_{j\in\mathbb{D}_i} \quad \textit{and} \quad V_{\boldsymbol{\omega}^{(i)}}(x) := \sum_{j\in\mathbb{D}_i} \omega_j u_j(x) \quad \textit{for} \quad i = 1, 2.$$

Note that $V_{\boldsymbol{\omega}^{(2)}} \geq 0$*, and, since* $\mathbb{D}$ *is a Delone set, there exists a constant* $V_\infty^{(2)}$ *such that* $\|V_{\boldsymbol{\omega}^{(2)}}\|_\infty \leq V_\infty^{(2)}$ *for* $\mathbb{P}$*-a.e.* $\boldsymbol{\omega}^{(2)}$*. We set*

$$H_{\boldsymbol{\omega}^{(1)}}^{(\boldsymbol{\omega}^{(2)})} := -\Delta + V^{(0,\boldsymbol{\omega}^{(2)})} + V_{\boldsymbol{\omega}^{(1)}}, \quad \textit{where} \quad V^{(0,\boldsymbol{\omega}^{(2)})} = V^{(0)} + V_{\boldsymbol{\omega}^{(2)}}, \tag{1.28}$$

and note that

$$\left\| V^{(0,\boldsymbol{\omega}^{(2)})} \right\|_\infty \leq V_\infty^{(0)} + V_\infty^{(2)} \quad \textit{for} \quad \mathbb{P}\textit{-a.e. } \boldsymbol{\omega}^{(2)}. \tag{1.29}$$

If we had $R = 1$*,* $H_{\boldsymbol{\omega}^{(1)}}^{(\boldsymbol{\omega}^{(2)})}$ *would be a crooked Anderson Hamiltonian with background potential* $V^{(0,\boldsymbol{\omega}^{(2)})}$ *and alloy-type potential* $V_{\boldsymbol{\omega}^{(1)}}^{(1)}$*, but would not be not normalized as in Definition 1.2 since we we only have* $\inf\sigma\left(-\Delta + V^{(0,\boldsymbol{\omega}^{(2)})}\right) \geq 0$*. But Theorems 1.4 and 1.5 hold as stated with the same constants if we only required* $\inf\sigma(H_0) \geq 0$ *in Definition 1.2. Moreover, Theorems 1.1, 1.4 and 1.5 are valid with boxes of side* R *instead of boxes of side* 1*, except that all the constants would depend on* R*. We can thus apply Theorems 1.4 and 1.5, averaging only with respect to* $\boldsymbol{\omega}^{(1)}$*, to obtain Wegner estimates for* $H_{\boldsymbol{\omega}^{(1)}}^{(\boldsymbol{\omega}^{(2)})}$ *with* $S_\Lambda(t) := \max_{j\in\mathbb{D}_1\cap\Lambda} S_{\mu_j}(t)$*, with constants independent of* $\boldsymbol{\omega}^{(2)}$ *for* $\mathbb{P}$*-a.e.* $\boldsymbol{\omega}^{(2)}$ *in view of* (1.29)*. We thus conclude that the Wegner estimates of Theorems 1.4 and 1.5 are valid for the Delone-Anderson* *model, with $V_\infty^{(0)} + V_\infty^{(2)}$ substituted for $V_\infty^{(0)}$ and the constants also depending on the scale R.*

The constants in the Wegner estimates (1.23) and (1.25) grow fast with the disorder. To see that, consider $H_{\boldsymbol{\omega},\lambda} = H_0 + \lambda V_{\boldsymbol{\omega}}$, where H_0 and $V_{\boldsymbol{\omega}}$ are as in Definition 1.2 and $\lambda > 0$ is the disorder parameter. $H_{\boldsymbol{\omega},\lambda}$ can be rewritten as a crooked Anderson Hamiltonian $H_{\boldsymbol{\omega}}^{(\lambda)} = H_0 + V_{\boldsymbol{\omega}}$ in the form of Definition 1.2 by replacing the probability distributions $\{\mu_j\}_{j\in\mathbb{Z}^d}$ by the probability distributions $\left\{\mu_j^{(\lambda)}\right\}_{j\in\mathbb{Z}^d}$, where $\mu_j^{(\lambda)}$ is the probability distribution of the random variable $\lambda\omega_j$, that is,

$$\mu_j^{(\lambda)}(B) = \mu_j(\lambda^{-1}B) \quad \text{for all Borels sets} \quad B \subset \mathbb{R}. \tag{1.30}$$

We clearly have $S_{\mu_j^{(\lambda)}}(t) = S_{\mu_j}(\frac{t}{\lambda})$, and it follows from (1.13) that

$$\operatorname{supp} \mu_j^{(\lambda)} \subset [0, M_\lambda], \quad \text{where} \quad M_\lambda = \lambda M. \tag{1.31}$$

Applying the Wegner estimates (1.23) and (1.25) to $H_{\boldsymbol{\omega},\lambda}$ we get (we omit the dependence on the constants from Definition 1.2)

$$\mathbb{E}\left\{\operatorname{tr} P_{\boldsymbol{\omega},\lambda,\Lambda}(I)\right\} \le C_{E_0}\left(1 + \lambda^{2^{2+\frac{\log d}{\log 2}}}\right) S_\Lambda(\lambda^{-1}|I|)\,|\Lambda| \qquad \text{from (1.23)}, \tag{1.32}$$

$$\mathbb{E}\left\{\operatorname{tr} P_{\boldsymbol{\omega},\lambda,\Lambda}(I)\right\} \le C_{E_0}\mathrm{e}^{c_{E_0}\left(1+\lambda^{\frac{2}{3}}\right)} S_\Lambda(\lambda^{-1}|I|)\,|\Lambda| \qquad \text{from (1.25)}. \tag{1.33}$$

The constants in these Wegner estimates grow as the disorder increases.

The Wegner estimate (1.32) is what one gets for the usual Anderson Hamiltonian from [CHK2] without further assumptions. But if the crooked Anderson Hamiltonian satisfies the covering condition $U^{(\Lambda)} \ge \alpha\chi_\Lambda$ for some $\alpha > 0$, the UCPSP (1.1) holds trivially on $\mathrm{L}^2(\Lambda)$ for all intervals I with $H = H_{0,\Lambda}$ or $H = H_{\boldsymbol{\omega},\Lambda}$, $W = U^{(\Lambda)}$, and $\kappa = \alpha$, so, either proceeding as in [CH] if we use (1.1) with $H = H_0$, or using Lemma 3.1 if we take $H = H_{\boldsymbol{\omega}}$ in (1.1), we get an optimal Wegner estimates of the form

$$\mathbb{E}\left\{\operatorname{tr} P_{\boldsymbol{\omega},\Lambda}(I)\right\} \le C_{d,\delta_+,\alpha,V_\infty^{(0)},E_0} S_\Lambda(|I|)\,|\Lambda|\,. \tag{1.34}$$

Note that the constant does not depend on M, so introducing the disorder parameter λ we get

$$\mathbb{E}\left\{\operatorname{tr} P_{\boldsymbol{\omega},\lambda,\Lambda}(I)\right\} \le C_{d,\delta_+,\alpha,V_\infty^{(0)},E_0} S_\Lambda(\lambda^{-1}|I|)\,|\Lambda|\,. \tag{1.35}$$

In other words, the constant in the Wegner estimate improves as the disorder increases.

Up to now an estimate like (1.35) had not been proven for Anderson Hamiltonians without the covering condition. While we are not able to prove this estimate at all energies without the covering condition, we can prove them at the bottom of the the spectrum, a new result even for the usual (ergodic) Anderson Hamiltonian.

We write $H_\Lambda^{(D)}$ to denote the restriction of a Schrödinger operator H to the box Λ with Dirichlet boundary condition, and set $P_\Lambda^{(D)}(B) := \chi_B(H_\Lambda^{(D)})$. We recall that Dirichlet boundary condition implies $\inf\sigma(H_\Lambda^{(D)}) \ge \inf\sigma(H)$.

Given a crooked Anderson Hamiltonian $H_{\boldsymbol{\omega}}$, we define finite volume operators $H_{\boldsymbol{\omega},\Lambda}^{(D)} = H_{0,\Lambda}^{(D)} + V_{\boldsymbol{\omega}}^{(\Lambda)}$, and let $P_{\boldsymbol{\omega},\Lambda}^{(D)}(B) := \chi_B(H_{\boldsymbol{\omega},\Lambda}^{(D)})$. We set $H(t) = H_0 + tu_-W$

for $t \geq 0$, and note

$$0 \leq E(t) := \inf \sigma(H(t)) \leq E_\Lambda^{(D)}(t) := \inf \sigma(H_\Lambda^{(D)} t)). \tag{1.36}$$

By our normalization $E(0) = 0$, and it follows from the min-max principle that $0 \leq E(t_2) - E(t_1) \leq (t_2 - t_1)u_-$ for $0 \leq t_1 \leq t_2$. We may thus define

$$E(\infty) := \lim_{t\to\infty} E(t) = \sup_{t\geq 0} E(t) \in [0, \infty]. \tag{1.37}$$

If $W = I$ we have $E(\infty) = \infty$. But if not, that is, if $\Upsilon = \mathbb{R}^d \setminus \overline{\cup_{j\in\mathbb{Z}^d} B(y_j, \delta_-)} \neq \emptyset$, letting $H_{0,\Upsilon}^{(D)}$ denote the restriction of H_0 to Υ with Dirichlet boundary condition, we get

$$E(t) \leq E(\Upsilon) := \inf \sigma(H_{0,\Upsilon}^{(D)}) < \infty \text{ for } t \geq 0 \quad \Longrightarrow \quad E(\infty) \leq E(\Upsilon) < \infty. \tag{1.38}$$

More importantly, Rojas-Molina and Veselić proved that $E(\infty) > 0$ [RV, Theorem 4.9], [R2, Theorem A.3.1]. By a similar argument, we establish strictly positive lower bounds for $E(t)$ and $E(\infty)$ in Lemma 4.2.

Theorem 1.7. *Let $H_{\boldsymbol{\omega}}$ be a crooked Anderson Hamiltonian. Then $E(\infty) > 0$. Let $E_1 \in]0, E(\infty)[$, so we have*

$$\kappa = \kappa(H_0, u_- W, E_1) = \sup_{s>0;\ E(s)>E_1} \frac{E(s) - E_1}{s} > 0, \tag{1.39}$$

and consider a box $\Lambda = \Lambda_L(x_0)$ with $x_0 \in \mathbb{Z}^d$ and $L \in \mathbb{N}_{\text{odd}}$, $L \geq 2 + \delta_+$. Then

$$P_{\boldsymbol{\omega},\Lambda}^{(D)}(] - \infty, E_1])U^{(\Lambda)} P_{\boldsymbol{\omega},\Lambda}^{(D)}(] - \infty, E_1]) \geq \kappa P_{\boldsymbol{\omega},\Lambda}^{(D)}(] - \infty, E_1]), \tag{1.40}$$

and for any closed interval $I \subset] - \infty, E_1]$ we have

$$\mathbb{E}\left\{\operatorname{tr} P_{\boldsymbol{\omega},\Lambda}^{(D)}(I)\right\} \leq C_{d,\delta_+,V_\infty^{(0)}} \left(\kappa^{-2}(1 + E_1)\right)^{2^{1+\frac{\log d}{\log 2}}} S_\Lambda(|I|)\, |\Lambda|\,. \tag{1.41}$$

In particular, for all disorder $\lambda > 0$ we have

$$\mathbb{E}\left\{\operatorname{tr} P_{\boldsymbol{\omega},\lambda,\Lambda}^{(D)}(I)\right\} \leq C_{d,\delta_+,V_\infty^{(0)}} \left(\kappa^{-2}(1 + E_1)\right)^{2^{1+\frac{\log d}{\log 2}}} S_\Lambda(\lambda^{-1} |I|)\, |\Lambda|\,. \tag{1.42}$$

for any closed interval $I \subset] - \infty, E_1]$

Theorem 1.7 is proven in Section 4. We use Lemma 4.1, a slight extension of an abstract UCPSP due to Boutet de Monvel, Lenz, and Stollmann [BoLS, Theorem 1.1], to prove (1.40). The estimate (1.41) then follows from Lemma 3.1. Since κ in (1.39) does not depend on M, Lemma 3.1 gives a constant in the Wegner estimate (1.41) independent of M, so (1.42) follows.

Theorem 1.7 is the missing link for proving localization at high disorder for Anderson Hamiltonians in a fixed interval at the bottom of the spectrum. This was previously known only with a covering condition $U^{(\Lambda)} \geq \alpha\chi_\Lambda$, where $\alpha > 0$ [GK2, Theorem 3.1].

We state the theorem in the generality of crooked Anderson Hamiltonians. (The bootstrap multiscale analysis can be adapted for crooked Anderson Hamiltonians [R1, R2].) By complete localization on an interval I we mean that for all $E \in I$ there exists $\delta(E) > 0$ such that we can perform the bootstrap multiscale analysis on the interval $(E - \delta(E), E + \delta(E))$, obtaining Anderson and dynamical localization; see [GK1, GK2, GK3].

Theorem 1.8. *Let $H_{\boldsymbol{\omega},\lambda}$ be a crooked Anderson Hamiltonian with disorder $\lambda > 0$, and suppose the single-site probability distributions $\{\mu_j\}_{j\in\mathbb{Z}^d}$ satisfy $S(t) := \sup_{j\in\mathbb{Z}^d} S_{\mu_j}(t) \le Ct^\theta$ for all $t \ge 0$, where $\theta \in]0,1]$ and C is a constant. Given $E_1 \in]0, E(\infty)[$, there exists $\lambda(E_1) < \infty$ (depending also on $d, V_\infty^{(0)}, u_-, \delta_\pm, U, \theta, C$), such that $H_{\boldsymbol{\omega},\lambda}$ exhibits complete localization on the interval $[0, E_1[$ for all $\lambda \ge \lambda(E_1)$.*

Theorem 1.8 is proven in Section 4.

2. Unique continuation principle for spectral projections

In this section we prove Theorem 1.1. We start by recalling the quantitative unique continuation principle as given in [BKl, Theorem 3.2].

Theorem 2.1. *Let Ω be an open subset of $\mathbb{R}^d$ and consider a real measurable function V on Ω with $\|V\|_\infty \le K < \infty$. Let $\psi \in \mathrm{H}^2(\Omega)$ be real valued and let $\zeta \in \mathrm{L}^2(\Omega)$ be defined by*

$$-\Delta\psi + V\psi = \zeta \quad \text{a.e. on} \quad \Omega. \tag{2.1}$$

Let $\Theta \subset \Omega$ be a bounded measurable set where $\|\psi_\Theta\|_2 > 0$. Set

$$Q(x,\Theta) := \sup_{y\in\Theta} |y-x| \quad \text{for} \quad x \in \Omega. \tag{2.2}$$

Consider $x_0 \in \Omega \setminus \overline{\Theta}$ such that

$$Q = Q(x_0,\Theta) \ge 1 \quad \text{and} \quad B(x_0, 6Q+2) \subset \Omega. \tag{2.3}$$

Then, given

$$0 < \delta \le \min\left\{\operatorname{dist}(x_0,\Theta), \tfrac{1}{2}\right\}, \tag{2.4}$$

we have

$$\left(\frac{\delta}{Q}\right)^{m_d\left(1+K^{\frac{2}{3}}\right)\left(Q^{\frac{4}{3}}+\log\frac{\|\psi_\Omega\|_2}{\|\psi_\Theta\|_2}\right)} \|\psi_\Theta\|_2^2 \le \|\psi_{x_0,\delta}\|_2^2 + \delta^2\|\zeta_\Omega\|_2^2, \tag{2.5}$$

where $m_d > 0$ is a constant depending only on d.

Note the condition $\delta \le \frac{1}{2}$ in (2.4) instead of $\delta \le \frac{1}{24}$ as in [BKl, Eq. (3.2)]. All that is needed in (2.4) is an upper bound $\delta \le \delta_0$; the constant m_d in (2.5) then depending on δ_0.

Note that for $\psi \in \mathrm{L}^2(\Lambda)$ we have $\psi = \psi_\Lambda$ in our notation, and hence $\|\psi\|_2 = \|\psi_\Lambda\|_2$.

Theorem 2.2. *Let $H = -\Delta + V$ be a Schrödinger operator on $\mathrm{L}^2(\mathbb{R}^d)$, where V is a bounded potential with $\|V\|_\infty \le K$. Fix $\delta \in]0, \frac{1}{2}]$, let $\{y_k\}_{k\in\mathbb{Z}^d}$ be sites in $\mathbb{R}^d$ with $B(y_k,\delta) \subset \Lambda_1(k)$ for all $k \in \mathbb{Z}^d$. Consider a box $\Lambda = \Lambda_L(x_0)$, where $x_0 \in \mathbb{Z}^d$ and $L \in \mathbb{N}_{\mathrm{odd}}$, $L \ge 72\sqrt{d}$. Then for all real-valued $\psi \in \mathcal{D}(\Delta_\Lambda)$ we have*

$$\delta^{M_d\left(1+K^{\frac{2}{3}}\right)}\|\psi_\Lambda\|_2^2 \le \sum_{k\in\widehat{\Lambda}} \|\psi_{y_k,\delta}\|_2^2 + \delta^2\|((-\Delta+V)\psi)_\Lambda\|_2^2, \tag{2.6}$$

where $M_d > 0$ is a constant depending only on d.

Proof. Without loss of generality we take $x_0 = 0$, so $\Lambda = \Lambda_L(0)$ with $L \in \mathbb{N}_{\text{odd}}$, $L \geq 72\sqrt{d}$. As in [GK4, Proof of Corollary A.2], we extend V and functions $\varphi \in \mathrm{L}^2(\Lambda)$ to $\mathbb{R}^d$ as follows.

Dirichlet boundary condition: Given $\varphi \in \mathrm{L}^2(\Lambda)$, we extend it to a function $\widetilde{\varphi} \in \mathrm{L}^2_{\text{loc}}(\mathbb{R}^d)$ by setting $\widetilde{\varphi} = \varphi$ on Λ and $\widetilde{\varphi} = 0$ on $\partial\Lambda$, and requiring

$$\widetilde{\varphi}(x) = -\widetilde{\varphi}(x + (L - 2\widehat{x_j})\mathrm{e}_j) \quad \text{for all} \quad x \in \mathbb{R}^d \quad \text{and} \quad j \in \{1, 2\ldots, d\}, \tag{2.7}$$

where $\{\mathrm{e}_j\}_{j=1,2\ldots,d}$ is the canonical orthonormal basis in $\mathbb{R}^d$, and for each $t \in \mathbb{R}$ we define $\hat{t} \in]-\frac{L}{2}, \frac{L}{2}]$ by $t = kL + \hat{t}$ with $k \in \mathbb{Z}$. We also extend the potential V to a potential $\widehat{V}$ on $\mathbb{R}^d$ by by setting $\widehat{V} = V$ on Λ and $V = 0$ on $\partial\Lambda$, and requiring that for all $x \in \mathbb{R}^d$ and $j \in \{1, 2\ldots, d\}$ we have

$$\widehat{V}(x) = \widehat{V}(x + (L - 2\widehat{x_j})\mathrm{e}_j). \tag{2.8}$$

Note that $\|\widehat{V}\|_\infty = \|V\|_\infty \leq K$. Moreover, $\psi \in \mathcal{D}(\Delta_\Lambda)$ implies $\widetilde{\psi} \in \mathrm{H}^2_{\text{loc}}(\mathbb{R}^d)$ and

$$\widetilde{(-\Delta + V)\psi} = (-\Delta + \widehat{V})\widetilde{\psi}. \tag{2.9}$$

Periodic boundary condition: We extend $\varphi \in \mathrm{L}^2(\Lambda)$ and V to periodic functions $\widetilde{\varphi}$ and $\widehat{V}$ on $\mathbb{R}^d$ of period L; note $\|\widehat{V}\|_\infty = \|V\|_\infty \leq K$. Moreover, $\psi \in \mathcal{D}(\Delta_\Lambda)$ implies $\widetilde{\psi} \in \mathrm{H}^2_{\text{loc}}(\mathbb{R}^d)$ and we have (2.9).

We now take $Y \in \mathbb{N}_{\text{odd}}$, $Y < \frac{L}{2}$ (to be specified later), and note that since L is odd, we have

$$\overline{\Lambda} = \bigcup_{k \in \widehat{\Lambda}} \overline{\Lambda_1(k)}. \tag{2.10}$$

It follows that for all $\varphi \in \mathrm{L}^2(\Lambda)$ we have (see [RV, Subsection 5.2])

$$\sum_{k \in \widehat{\Lambda}} \left\|\widetilde{\varphi}_{\Lambda_Y(k)}\right\|_2^2 \begin{cases} \leq (2Y)^d \left\|\varphi_\Lambda\right\|_2^2 & \text{for Dirichlet boundary condition} \\ = Y^d \left\|\varphi_\Lambda\right\|_2^2 & \text{for periodic boundary condition} \end{cases}. \tag{2.11}$$

We now fix $\psi \in \mathcal{D}(\Delta_\Lambda)$. Following Rojas-Molina and Veselić, we call a site $k \in \widehat{\Lambda}$ *dominating* (for ψ) if

$$\left\|\psi_{\Lambda_1(k)}\right\|_2^2 \geq \tfrac{1}{2(2Y)^d} \left\|\widetilde{\psi}_{\Lambda_Y(k)}\right\|_2^2. \tag{2.12}$$

Letting $\widehat{D} \subset \widehat{\Lambda}$ denote the collection of dominating sites, Rojas-Molina and Veselić [RV, Subsection 5.2] observed that it follows from (2.11), (2.12), and (2.10), that

$$\sum_{k \in \widehat{D}} \left\|\psi_{\Lambda_1(k)}\right\|_2^2 \geq \tfrac{1}{2} \left\|\psi_\Lambda\right\|_2^2. \tag{2.13}$$

We define a map $J \colon \widehat{D} \to \widehat{\Lambda}$ by

$$J(k) = \begin{cases} k + 2\mathrm{e}_1 & \text{if} \quad k + 2\mathrm{e}_1 \in \widehat{\Lambda} \\ k - 2\mathrm{e}_1 & \text{if} \quad k + 2\mathrm{e}_1 \notin \widehat{\Lambda} \end{cases}. \tag{2.14}$$

Note that J is well defined,

$$\# J^{-1}(\{j\}) \leq 2 \quad \text{for all} \quad j \in \widehat{\Lambda}, \tag{2.15}$$

and recalling (2.2),

$$Q(y_{J(k)}, \Lambda_1(k)) = \tfrac{1}{2}\sqrt{24 + d} \leq \tfrac{5}{2}\sqrt{d} \quad \text{for all} \quad k \in \widehat{D}. \tag{2.16}$$

Choosing

$$Y=\min\left\{n\in\mathbb{N}_{\mathrm{odd}};n>2\left(\left(2+\tfrac{\sqrt{d}}{2}\right)+\left(3\sqrt{24+d}+2\right)\right)\right\}\leq 40\sqrt{d}, \tag{2.17}$$

we have $Y<\frac{L}{2}$ and

$$B(y_{J(k)},6Q(y_{J(k)},\Lambda_1(k))+2)\subset\Lambda_Y(k)\quad\text{for all}\quad k\in\widehat{D}. \tag{2.18}$$

For each $k\in\widehat{D}$ we may thus apply Theorem 2.1 with $\Omega=\Lambda_Y(k)$ and $\Theta=\Lambda_1(k)$, using (2.16) and (2.12), obtaining

$$\delta^{m'_d\left(1+K^{\frac{2}{3}}\right)}\left\|\psi_{\Lambda_1(k)}\right\|_2^2\leq\left\|\psi_{y_{J(k)},\delta}\right\|_2^2+\delta^2\left\|\widetilde{\zeta}_{\Lambda_Y(k)}\right\|_2^2, \tag{2.19}$$

where $\zeta=(-\Delta+V)\psi$ and $m'_d>0$ is a constant depending only on d. Summing over $k\in\widehat{D}$ and using (2.13), (2.15), (2.11), and (2.17), and we get

$$\begin{aligned}\tfrac{1}{2}\delta^{m'_d\left(1+K^{\frac{2}{3}}\right)}\left\|\psi_\Lambda\right\|_2^2&\leq 2\sum_{k\in\widehat{\Lambda}}\left\|\psi_{y_k,\delta}\right\|_2^2+(2Y)^d\delta^2\left\|\zeta_\Lambda\right\|_2^2 \\ &\leq 2\sum_{k\in\widehat{\Lambda}}\left\|\psi_{y_k,\delta}\right\|_2^2+(80\sqrt{d})^d\delta^2\left\|\zeta_\Lambda\right\|_2^2,\end{aligned} \tag{2.20}$$

so (2.6) follows. □

Comment. *The final version of* [RV] *uses a map similar to* (2.14)*, see* [RV, Subsection 5.3].

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. Given $E_0>0$, set $K=K(V,E_0)=2\left\|V\right\|_\infty+E_0$, and let γ be given by (1.8), where $M_d>0$ is the constant in Theorem 2.2. Let $I\subset]-\infty,E_0]$ be a closed interval with $|I|\leq 2\gamma$. Since $\sigma(H_\Lambda)\subset[-\left\|V\right\|_\infty,\infty[$ for any box Λ, without loss of generality we assume $I=[E-\gamma,E+\gamma]$ with $E\in[-\left\|V\right\|_\infty,E_0]$, so

$$\left\|V-E\right\|_\infty\leq\left\|V\right\|_\infty+\max\{E_0,\left\|V\right\|_\infty\}\leq K. \tag{2.21}$$

Moreover, for any box Λ we have

$$\left\|(H_\Lambda-E)\,\psi\right\|_2\leq\gamma\left\|\psi\right\|_2\quad\text{for all}\quad\psi\in\operatorname{Ran}\chi_I(H_\Lambda). \tag{2.22}$$

Let Λ be a box as in Theorem 2.2 and $\psi\in\operatorname{Ran}\chi_I(H_\Lambda)$. If ψ is real-valued, it follows from Theorem 2.2, (1.8), and (2.22) that

$$2\gamma^2\left\|\psi\right\|_2^2\leq\sum_{k\in\widehat{\Lambda}}\left\|\psi_{y_k,\delta}\right\|_2^2+\gamma^2\left\|\psi\right\|_2^2, \tag{2.23}$$

yielding

$$\gamma^2\left\|\psi\right\|_2^2\leq\sum_{k\in\widehat{\Lambda}}\left\|\psi_{y_k,\delta}\right\|_2^2=\left\|W\psi\right\|_2^2, \tag{2.24}$$

where the equality follows from (1.7). For arbitrary $\psi\in\operatorname{Ran}\chi_I(H_\Lambda)$, we write $\psi=\operatorname{Re}\psi+i\operatorname{Im}\psi$, and note that $\operatorname{Re}\psi,\operatorname{Im}\psi\in\operatorname{Ran}\chi_I(H_\Lambda)$, $\left\|\psi\right\|_2^2=\left\|\operatorname{Re}\psi\right\|_2^2+\left\|\operatorname{Im}\psi\right\|_2^2$, and, since W is real-valued, $\left\|W\psi\right\|_2^2=\left\|W\operatorname{Re}\psi\right\|_2^2+\left\|W\operatorname{Im}\psi\right\|_2^2$. Recalling $W^2=W$, we conclude that

$$\gamma^2\left\langle\psi,\psi\right\rangle=\gamma^2\left\|\psi\right\|_2^2\leq\left\|W\psi\right\|_2^2=\left\langle\psi,W\psi\right\rangle \tag{2.25}$$

for all $\psi\in\operatorname{Ran}\chi_I(H_\Lambda)$, proving (1.9). □

3. Wegner estimates

In this section we prove Theorems 1.4 and 1.5.

Note that for a crooked Anderson Hamiltonian $H_{\boldsymbol{\omega}}$ and a box Λ, we always have

$$\sigma(H_{0,\Lambda}) \subset [-\alpha, \infty[\quad \text{and} \quad \sigma(H_{\boldsymbol{\omega},\Lambda}) \subset [-\alpha, \infty[, \tag{3.1}$$

where $\alpha = 0$ for Dirichlet boundary condition and $\alpha = V_\infty^{(0)}$ for periodic boundary condition.

Proof of Theorem 1.4. Let $H_{\boldsymbol{\omega}}$ be a be a crooked Anderson Hamiltonian. Given $E_0 > 0$, set $K_0 = E_0 + 2V_\infty^{(0)}$, and define γ_0 by (1.22). We apply Theorem 1.1 with $H = H_0$ and W as in (1.16), concluding that for any closed interval $I \subset]-\infty, E_0]$ with $|I| \leq 2\gamma_0$ and any box Λ as in the hypotheses of the theorem, we have, using also (1.17),

$$\chi_I(H_{0,\Lambda}) \leq \gamma_0^{-2} \chi_I(H_{0,\Lambda}) W^{(\Lambda)} \chi_I(H_{0,\Lambda}) \leq u_-^{-1} \gamma_0^{-2} \chi_I(H_{0,\Lambda}) U^{(\Lambda)} \chi_I(H_{0,\Lambda}). \tag{3.2}$$

In view of (3.1), it suffices to take $I \subset [-\alpha, E_0]$. We can now follow the proof in [CHK2], using (3.2) instead of [CHK2, Theorem 2.1], and keeping careful track of the dependence of the constants on the relevant parameters, obtaining (1.23). □

We now turn to the proof of Theorem 1.5. We start by showing that, given a crooked Anderson Hamiltonian $H_{\boldsymbol{\omega}}$, the UCPSP (1.1), with $H = H_{\boldsymbol{\omega}}$, $W = U$, and a constant κ independent of $\boldsymbol{\omega}$ implies a Wegner estimate.

Lemma 3.1. *Let $H_{\boldsymbol{\omega}}$ be a crooked Anderson Hamiltonian. Let $I \subset]-\infty, E_0]$ be a closed interval and $\Lambda = \Lambda_L(x_0)$ a box centered at $x_0 \in \mathbb{Z}^d$ with $L \in \mathbb{N}_{\text{odd}}$, $L \geq 2 + \delta_+$. Suppose there exists a constant $\kappa > 0$ such that*

$$P_{\boldsymbol{\omega},\Lambda}(I) U^{(\Lambda)} P_{\boldsymbol{\omega},\Lambda}(I) \geq \kappa P_{\boldsymbol{\omega},\Lambda}(I) \quad \textit{with probability one.} \tag{3.3}$$

Then

$$\mathbb{E}\left\{\operatorname{tr} P_{\boldsymbol{\omega},\Lambda}(I)\right\} \leq C_{d,\delta_+,V_\infty^{(0)}} \left(\kappa^{-2}(1+E_0)\right)^{2^{1+\frac{\log d}{\log 2}}} S_\Lambda(|I|)\,|\Lambda|\,. \tag{3.4}$$

Proof. We fix the box Λ, let $P = P_{\boldsymbol{\omega},\Lambda}(I)$ for a closed interval $I \subset]-\infty, E_0]$, and simply write U for $U^{(\Lambda)}$. Then it follows from (3.3), using (3.1), that

$$\begin{aligned}
\operatorname{tr} P &\leq \kappa^{-1} \operatorname{tr} PUP = \kappa^{-1} \operatorname{tr} \sqrt{U} P \sqrt{U} \leq \kappa^{-2} \operatorname{tr} \sqrt{U} PUP \sqrt{U} = \kappa^{-2} \operatorname{tr} PUPU \\
&= \kappa^{-2} \operatorname{tr} PUPUP \leq \kappa^{-2}(1+\alpha+E_0) \operatorname{tr} PU(H_{\boldsymbol{\omega},\Lambda}+1+\alpha)^{-1}UP \\
&\leq \kappa^{-2}(1+\alpha+E_0) \operatorname{tr} PU(H_{0,\Lambda}+1+\alpha)^{-1}UP \\
&= \kappa^{-2}(1+\alpha+E_0) \operatorname{tr} UPU(H_{0,\Lambda}+1+\alpha)^{-1} \\
&= \kappa^{-2}(1+\alpha+E_0) \sum_{i,j\in\widehat{\Lambda}} \operatorname{tr} \sqrt{u_j} P \sqrt{u_i} T_{ij},
\end{aligned} \tag{3.5}$$

where

$$T_{ij} = \sqrt{u_i}(H_{0,\Lambda}+1+\alpha)^{-1}\sqrt{u_j} \quad \text{for} \quad i,j \in \widehat{\Lambda}. \tag{3.6}$$

We now proceed as in [CHK2, Eqs. (2.10)-(2.16)], adapting [CHK2, Lemma A.1]. Using $\operatorname{supp} u_j \subset \Lambda_{1+\delta_+}(j)$, the resolvent identity (several times), trace estimates, and the Combes-Thomas estimate we obtain

$$\|T_{ij}\|_1 \leq C_1 \mathrm{e}^{c_1|i-j|} \quad \text{for all} \quad i,j \in \widehat{\Lambda} \quad \text{with} \quad |i-j|_\infty \geq 2+\delta_+, \tag{3.7}$$

where the constants C_1 and c_1 depend only on $d, \delta_+, V_\infty^{(0)}$. Given $i \in \widehat{\Lambda}$, we set

$$\mathcal{J}_i = \left\{ j \in \widehat{\Lambda};\ ; |i-j|_\infty < 2 + \delta_+ \right\}; \quad \text{note that} \quad \# \mathcal{J}_i \leq (2+\delta_+)^d . \tag{3.8}$$

We have

$$\sum_{i,j \in \widehat{\Lambda}} \operatorname{tr} \sqrt{u_j} P \sqrt{u_i} T_{ij} = \sum_{i \in \widehat{\Lambda}} \left\{ \sum_{j \in \mathcal{J}_i^c} \operatorname{tr} \sqrt{u_j} P \sqrt{u_i} T_{ij} + \sum_{j \in \mathcal{J}_i} \operatorname{tr} \sqrt{u_j} P \sqrt{u_i} T_{ij} \right\}. \tag{3.9}$$

Using spectral averaging [CHK2, Lemma 2.1] and (3.7) we get

$$\mathbb{E} \left| \sum_{i \in \widehat{\Lambda}} \sum_{j \in \mathcal{J}_i^c} \operatorname{tr} \sqrt{u_j} P \sqrt{u_i} T_{ij} \right| \leq C_2 S_\Lambda(|I|) \, |\Lambda| , \tag{3.10}$$

where C_2 depends only on $d, \delta_+, V_\infty^{(0)}$.

Now let

$$T_\Lambda = \sum_{i \in \widehat{\Lambda}} \sum_{j \in \mathcal{J}_i} \sqrt{u_i} T_{ij} \sqrt{u_j} = \sum_{i \in \Lambda} \sum_{j \in \mathcal{J}_i} u_i (H_{0,\Lambda} + 1 + \alpha)^{-1} u_j , \tag{3.11}$$

so

$$\sum_{i \in \widehat{\Lambda}} \sum_{j \in \mathcal{J}_i} \operatorname{tr} \sqrt{u_j} P \sqrt{u_i} T_{ij} = \operatorname{tr} P T_\Lambda . \tag{3.12}$$

Proceeding as in in [CHK2, Eqs. (A.4)-(A.5)], we get

$$|\operatorname{tr} P T_\Lambda| \leq \left(\sum_{j=1}^m \tfrac{\sigma_j}{2^j \sigma_1 \ldots \sigma_{j-1}} \right) \operatorname{tr} P + \tfrac{1}{2^m \sigma_1 \ldots \sigma_m} \operatorname{tr} P \left(T_\Lambda T_\Lambda^* \right)^{2^{m-1}} , \tag{3.13}$$

for all $m \in \mathbb{N}$, $\sigma_j > 0$ for $j = 1, 2, \ldots, m$, and $\sigma_0 = 1$. We take $\beta = \left(\kappa^{-2} (1 + E_0) \right)^{-1}$ and choose $\sigma_j = \beta^{2^{j-1}}$, so

$$|\operatorname{tr} P T_\Lambda| \leq \beta \left(1 - 2^{-m} \right) \operatorname{tr} P + 2^{-m} \beta^{1 - 2^m} \operatorname{tr} P \left(T_\Lambda T_\Lambda^* \right)^{2^{m-1}} . \tag{3.14}$$

It follows from (3.5), (3.9), (3.10), (3.12), (3.14) that

$$\begin{aligned} \mathbb{E} \operatorname{tr} P \leq{} & C_2 \kappa^{-2} (1 + E_0 + \alpha) S_\Lambda(|I|) \, |\Lambda| + \left(1 - 2^{-m} \right) \mathbb{E} \operatorname{tr} P \\ & + 2^{-m} \left(\kappa^{-2} (1 + \alpha + E_0) \right)^{2^m} \mathbb{E} \left\{ \operatorname{tr} P \left(T_\Lambda T_\Lambda^* \right)^{2^{m-1}} \right\} , \end{aligned} \tag{3.15}$$

so

$$\begin{aligned} \mathbb{E} \operatorname{tr} P \leq{} & C_2 2^m \kappa^{-2} (1 + \alpha + E_0) S_\Lambda(|I|) \, |\Lambda| \\ & + \left(\kappa^{-2} (1 + \alpha + E_0) \right)^{2^m} \mathbb{E} \left\{ \operatorname{tr} P \left(T_\Lambda T_\Lambda^* \right)^{2^{m-1}} \right\} . \end{aligned} \tag{3.16}$$

We now estimate $\mathbb{E} \left\{ \operatorname{tr} P \left(T_\Lambda T_\Lambda^* \right)^{2^{m-1}} \right\}$ as in [CHK2, Lemma A.1]. Since we have $u_i (H_{0,\Lambda} + 1 + \alpha)^{-1} u_j \in \mathcal{T}_q$ (i.e., $\operatorname{tr} \left| u_i (H_{0,\Lambda} + 1 + \alpha)^{-1} u_j \right|^q < \infty$) for $q > \frac{d}{2}$, letting

$$m_d = \min \left\{ m \in \mathbb{N}; \quad 2^{m-1} > \tfrac{d}{4} \right\} = \min \left\{ m \in \mathbb{N}; \quad m > \tfrac{\log d}{\log 2} - 1 \right\} , \tag{3.17}$$

we obtain, similarly to [CHK2, Eq. (A.8)]

$$\left\| \left(T_\Lambda T_\Lambda^* \right)^{2^{m_d - 1}} \right\|_1 \leq C_{d, \delta_+, V_\infty^{(0)}} \, |\Lambda| , \tag{3.18}$$

and conclude, using spectral averaging as in [CHK2, Eqs. (2.17)-(2.19)], that

$$\left|\mathbb{E}\left\{\operatorname{tr} P\left(T_\Lambda T_\Lambda^*\right)^{2^{m_d-1}}\right\}\right| \le C'_{d,\delta_+,V_\infty^{(0)}} S_\Lambda(|I|)\,|\Lambda| \tag{3.19}$$

Putting together (3.16) and (3.19) we get

$$\mathbb{E}\operatorname{tr} P \le C_{d,\delta_+,V_\infty^{(0)}}\left(\kappa^{-2}(1+\alpha+E_0)\right)^{2^{m_d}} S_\Lambda(|I|)\,|\Lambda|\,, \tag{3.20}$$

and (3.4) follows, changing the constant to absorb α in case of periodic boundary condition. □

We are ready to prove Theorem 1.5.

Proof of Theorem 1.5. Let $H_{\boldsymbol{\omega}}$ be a crooked Anderson Hamiltonian. Given $E_0 > 0$, set $K = E_0 + 2\left(V_\infty^{(0)} + MU_\infty\right)$, and define γ by (1.24). Given a box Λ as in the theorem, we apply Theorem 1.1 with $H = H_0 + V_{\boldsymbol{\omega}}^{(\Lambda)}$ and W as in (1.16), concluding that for any closed interval $I \subset]-\infty, E_0]$ with $|I| \le 2\gamma$ we have, using also (1.21),

$$\chi_I(H_{\boldsymbol{\omega},\Lambda}) \le \gamma^{-2}\chi_I(H_{\boldsymbol{\omega},\Lambda})W^{(\Lambda)}\chi_I(H_{\boldsymbol{\omega},\Lambda}) \le u_-^{-1}\gamma^{-2}\chi_I(H_{\boldsymbol{\omega},\Lambda})U^{(\Lambda)}\chi_I(H_{\boldsymbol{\omega},\Lambda}). \tag{3.21}$$

We now apply Lemma 3.1, getting (1.25). □

4. At the bottom of the spectrum

The following lemma is a slight extension of [BoLS, Theorem 1.1].

Lemma 4.1. *Let H_0 be a self-adjoint operator on a Hilbert space $\mathcal{H}$, bounded from below, and let $Y \ge 0$ be a bounded operator on $\mathcal{H}$. Let $H(t) = H_0 + tY$ for $t \ge 0$, and set $E(t) = \inf\sigma(H(t))$, a non-decreasing function of t. Let $E(\infty) = \lim_{t\to\infty} E(t) = \sup_{t\ge 0} E(t)$. Suppose $E(\infty) > E(0)$. Given $E_1 \in]E(0), E(\infty)[$, let*

$$\kappa = \kappa(H_0, Y, E_1) = \sup_{s>0;\ E(s)>E_1} \frac{E(s) - E_1}{s} > 0. \tag{4.1}$$

Then for all bounded operators $V \ge 0$ on $\mathcal{H}$ and Borel sets $B \subset]-\infty, E_1]$ we have

$$\chi_B(H_0+V)Y\chi_B(H_0+V) \ge \kappa\chi_B(H_0+V). \tag{4.2}$$

Proof. Fix $E_1 \in]E(0), E(\infty)[$. For all Borel sets $B \subset]-\infty, E_1]$ we have, writing $P_V(B) = \chi_B(H_0+V)$,

$$P_V(B)(H_0+V)P_V(B) \le E_1 P_V(B). \tag{4.3}$$

Since $E_1 \in]E(0), E(\infty)[$, there is $s > 0$ such that $E(s) > E_1$. Then,

$$P_V(B)(H(s)+V-sY-E_1)P_V(B) = P_V(B)(H_0+V-E_1)P_V(B) \le 0, \tag{4.4}$$

and hence, using $V \ge 0$,

$$\begin{aligned} sP_V(B)YP_V(B) &\ge P_V(B)(H(s)+V-E_1)P_V(B) \\ &\ge P_V(B)(H(s)-E_1)P_V(B) \ge (E(s)-E_1)P_V(B). \end{aligned} \tag{4.5}$$

The estimate (4.2) follows □

To use Lemma 4.1 we must show that $E(\infty) > E(0)$. This will follow from the following lemma.

Lemma 4.2. *Let H_0, u_-, W be as in Definition 1.2 and (1.16), set $H(t) = H_0 + tu_-W$ for $t \geq 0$, and let $E(t) = \inf \sigma(H(t))$, $E(\infty) = \lim_{t\to\infty} E(t) = \sup_{t\geq 0} E(t)$. Then*

$$E(t) \geq tu_-\delta_-^{M_d\left(1+\left(V_\infty^{(0)}+2tu_-\right)^{\frac{2}{3}}\right)} \quad \textit{for all} \quad t \geq 0, \tag{4.6}$$

so we conclude that

$$E(\infty) \geq \sup_{t\in[0,\infty[} t\delta_-^{M_d\left(1+\left(V_\infty^{(0)}+2t\right)^{\frac{2}{3}}\right)} > 0. \tag{4.7}$$

Proof. By our normalization $E(0) = 0$, and it follows from the min-max principle that $0 \leq E(t_2) - E(t_1) \leq (t_2 - t_1)u_-$ for $0 \leq t_1 \leq t_2$. Thus $E(\infty) \in [0, \infty]$ is well defined.

Given a box $\Lambda = \Lambda_L(x_0)$, where $x_0 \in \mathbb{Z}^d$ and $L \in \mathbb{N}_{\text{odd}}$, $L \geq 72\sqrt{d}$, set $E_\Lambda^{(D)}(t) = \inf \sigma(H_\Lambda^{(D)}(t))$. Note that $E_\Lambda^{(D)}(t) \geq E(t) \geq 0$ for all $t \geq 0$ since we have Dirichlet boundary condition, and we also have

$$E_\Lambda^{(D)}(t) \leq \inf \sigma(-\Delta_\Lambda^{(D)}) + tu_- = d\left(\tfrac{\pi}{L}\right)^2 + tu_-. \tag{4.8}$$

Since $H_\Lambda^{(D)}(t)$ has compact resolvent, there exists $\psi(t) \in \mathcal{D}(\Delta_\Lambda^{(D)})$, $\|\psi(t)\| = 1$, such that $H_\Lambda^{(D)}(t)\psi(t) = E_\Lambda^{(D)}(t)\psi(t)$. Applying Theorem 2.2 with $H = H_\Lambda^{(D)}(t) - E_\Lambda^{(D)}(t)$ and $\psi = \psi(t)$, and using (1.16) and (1.17), we get (see [RV, Proof of Theorem 4.9] for a similar argument)

$$\delta_-^{M_d\left(1+\left\|V^{(0)}+tu_-W-E_\Lambda^{(D)}(t)\right\|_\infty^{\frac{2}{3}}\right)} \leq \langle \psi(t), W\psi(t)\rangle. \tag{4.9}$$

Using (4.8), we get

$$\langle \psi(t), W\psi(t)\rangle \geq \delta_-^{M_d\left(1+\left(V_\infty^{(0)}+2tu_-+d\left(\frac{\pi}{L}\right)^2\right)^{\frac{2}{3}}\right)} \quad \text{for all} \quad t \geq 0. \tag{4.10}$$

It follows that

$$\begin{aligned} E_\Lambda^{(D)}(t) &\geq E_\Lambda^{(D)}(0) + tu_-\delta_-^{M_d\left(1+\left(V_\infty^{(0)}+2tu_-+d\left(\frac{\pi}{L}\right)^2\right)^{\frac{2}{3}}\right)} \\ &\geq tu_-\delta_-^{M_d\left(1+\left(V_\infty^{(0)}+2tu_-+d\left(\frac{\pi}{L}\right)^2\right)^{\frac{2}{3}}\right)}. \end{aligned} \tag{4.11}$$

Taking $\Lambda = \Lambda_L(0)$ and noting that $\lim_{L\to\infty} E_\Lambda^{(D)}(t) = E(t)$, we get

$$E(t) \geq tu_-\delta_-^{M_d\left(1+\left(V_\infty^{(0)}+2tu_-\right)^{\frac{2}{3}}\right)} \quad \text{for all} \quad t \geq 0, \tag{4.12}$$

so we have (4.6), and hence (4.7), since

$$E(\infty) \geq \sup_{t\in[0,\infty[} tu_-\delta_-^{M_d\left(1+\left(V_\infty^{(0)}+2tu_-\right)^{\frac{2}{3}}\right)} = \sup_{t\in[0,\infty[} t\delta_-^{M_d\left(1+\left(V_\infty^{(0)}+2t\right)^{\frac{2}{3}}\right)}. \tag{4.13}$$

□

We can now prove Theorem 1.7.

Proof of Theorem 1.7. Let $H_{\boldsymbol{\omega}}$ be a be a crooked Anderson Hamiltonian. By Lemma 4.2 we have $E(\infty) > 0$, so we can pick $E_1 \in]0, E(\infty)[$, and we have (1.39).

Consider a box $\Lambda = \Lambda_L(x_0)$, where $x_0 \in \mathbb{Z}^d$ and $L \in \mathbb{N}_{\text{odd}}$, $L \geq 2 + \delta_+$. Using (1.36), we get

$$\kappa(H_{0,\Lambda}^{(D)}, u_- W^{(\Lambda)}, E_1) \geq \kappa = \kappa(H_0, u_- W, E_1) > 0, \tag{4.14}$$

and Lemma 4.1 then gives (1.40). Applying Lemma 3.1 we get (1.42). □

We now turn to Theorem 1.8.

Proof of Theorem 1.8. Let $H_{\boldsymbol{\omega},\lambda}$ be a crooked Anderson Hamiltonian with disorder $\lambda > 0$, and assume $S(t) \leq Ct^\theta$, $\theta \in]0, 1]$. By Theorem 1.7, $E(\infty) > 0$, so we fix $E_1 \in]0, E(\infty)[$. Let us pick $E_2 \in]E_1, E(\infty)[$ and $t^* > 0$ such that $E(t^*) \geq E_2$.

Now let Λ be a box as in Theorem 1.7. Then

$$\mathbb{P}\left\{H_{\boldsymbol{\omega},\lambda,\Lambda}^{(D)} \geq E_2\right\} \geq 1 - |\Lambda|\, S\left(\lambda^{-1}[0, t^*]\right) \geq 1 - C\left(\lambda^{-1} t^*\right)^{-\theta} |\Lambda|\,. \tag{4.15}$$

Moreover, we have the Wegner estimate (1.42) (we omit the dependence on parameters):

$$\mathbb{E}\left\{\operatorname{tr} P_{\boldsymbol{\omega},\lambda,\Lambda}^{(D)}(I)\right\} \leq C_{E_1}\left(\lambda^{-1}|I|\right)^{-\theta} |\Lambda|\,. \tag{4.16}$$

for any closed interval $I \subset]-\infty, E_1]$ and boxes Λ as in Theorem 1.7.

Using (4.15) and (4.16), we can prove Theorem 1.8 by following the proof of [GK2, Theorem 3.1]. □

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