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Hereditary Subclasses and Closures of Invariant-Defined Graph Classes

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Hereditary Subclasses and Closures of Invariant-Defined Graph Classes

by

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Abstract

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A graph $G = (V(G), E(G))$ is an ordered pair consisting of a finite vertex set $V(G)$ and an edge set $E(G)$ of unordered pairs of distinct vertices. Given a graph G, a graph H is an *induced subgraph* of G if $V(H) \subseteq V(G)$ and $E(H)$ is exactly the set of edges among $V(H)$ inherited from G. A class $\mathcal{A}$ of graphs is *hereditary* if $\mathcal{A}$ contains all induced subgraphs of graphs in $\mathcal{A}$. Hereditary classes are particularly interesting because each can be characterized by a set of forbidden induced subgraphs. Where a class $\mathcal{A}$ is not hereditary, we define the *hereditary subclass* of $\mathcal{A}$ to be the largest hereditary class contained within $\mathcal{A}$, and the *hereditary closure* to be the smallest hereditary class containing $\mathcal{A}$. In this dissertation, I study the hereditary subclasses and closures of important non-hereditary graph classes, defined by relationships among invariants.

Given a graph, we can color its vertices such that no two adjacent vertices are the same color. If such a coloring exists with k colors, we call G *k-colorable.* The *chromatic number* $\chi(G)$ is the minimum number k such that G is k-colorable. Computing $\chi(G)$ generally is NP-complete, as is evaluating whether G is k-colorable for $k \geq 3$. The sum and product of the chromatic number of a graph G and its complement $\overline{G}$ are both bounded in terms of $|V(G)|$. The families of graphs satisfying each of these inequalities with equality are known, but little is known about how far a given graph is from meeting either the lower or upper bounds with equality.

Nordhaus and Gaddum proved in 1956 that the sum of the chromatic number χ of a graph G and its complement is at most $|G| + 1$. The Nordhaus-Gaddum graphs are the class of graphs satisfying this inequality with equality, and are well-understood. In this dissertation I consider a hereditary generalization: graphs G for which all induced subgraphs H of G satisfy $\chi(H) + \chi(\overline{H}) \geq |H|$. I characterize the forbidden induced subgraphs of this class and find its intersection with a number of common classes, including line graphs. I also discuss χ-boundedness and algorithmic results.

I then generalize further: for a fixed constant a, let the a-hereditary-Nordhaus-Gaddum graphs be those satisfying $\chi(G) + \omega(G) \geq |G| + 1 - a$ for all induced subgraphs. I bring to light a substantive connection between polyominoes and graph coloring, furthering the early work of Finck. Using polyominoes, for both the a-hereditary-Nordhaus-Gaddum graphs and a generalization of sum-perfect graphs, I show that there are finitely many forbidden induced subgraphs, resolving a question of Litjens, Polak, and Sivaraman, and provide a minimum and upper bound on their order. I provide a partial forbidden induced subgraph characterization of these graph classes. Lastly, I show an equivalence between a-hereditary-Nordhaus-Gaddum graphs, bounded-apex-split graphs, and generalized sum-perfect graphs based on their shared bipartite forbidden induced subgraphs.

Another set of problems relate to degree sequences and realizations. In most cases, multiple graphs have the same degree sequence, but a select number of graphs have a unique degree sequence. We say a graph with degree sequence π is a *unigraph* if it is isomorphic to every graph that has degree sequence π. The class of unigraphs is not hereditary and in this dissertation I study the related hereditary class HCU, the hereditary closure of unigraphs, consisting of all graphs induced in a unigraph. I characterize the class HCU in multiple ways making use of the tools of a decomposition due to Tyshkevich and a partial order on degree sequences due to Rao. I show that all unigraphs are apex-perfect graphs. I also provide a new characterization of the class that consists of unigraphs for which all induced subgraphs are also unigraphs.

Given a class $\mathcal{A}$ of graphs, I define the class of $\mathcal{A}$-unigraphs to be graphs identifiable from degree sequence and membership in $\mathcal{A}$. While these classes are often not hereditary, I provide characterizations of the largest hereditary subclass contained in the bipartite-unigraphs, the perfect-unigraphs, the forest-unigraphs, the chordal-unigraphs, and the weakly-chordal-unigraphs. I also characterize the largest hereditary subclass contained in the bipartite-unigraphs in terms of structure, degree sequence, and a partial order on degree sequences due to Rao.

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Chapter 1

Introduction

The introduction proceeds as follows: in Section 1.1 we give an overview of basic graph theory definitions and notation used in this dissertation. We refer the reader to Diestel's *Graph Theory* as a resource for any additional background [23]. We introduce hereditary graph classes in Section 1.2, discussing bipartite, perfect, threshold, split, and chordal graphs, and give a summary of important questions in the field. In Section 1.3 we outline the principal topics of the following chapters.

1.1 Definitions and Notation

A *graph* $G = (V(G), E(G))$ is an ordered pair consisting of a finite vertex set $V(G)$ and an edge set $E(G)$ of unordered pairs of distinct vertices. Consequently, all graphs in this dissertation are finite and simple (i.e., they have neither loops nor duplicate edges). Given two vertices x and y such that $\{x, y\} \in E(G)$, we say x and y are *adjacent* and denote their shared edge by xy. Then, the vertices x and y are *neighbors*; more generally, let $N(x) \subseteq V(G)$ be the set of vertices adjacent to x, which we call the *neighbor set* of x. An edge xy is called *incident* to its endpoints x and y.

Given two graphs G and H, we say that a function $f : V(G) \to V(H)$ is a *morphism* if for $vw \in E(G)$ we have $f(v)f(w) \in E(H)$. If the reverse implication is true, then f is an *embedding*. An *isomorphism* is a bijective morphism whose inverse is also a morphism. Where there exists an isomorphism between G and H, we say the graphs are *isomorphic*. Given a graph G, a graph H is a *subgraph* of G with $V(H) \subseteq V(G)$ for which the inclusion map $f : V(H) \to V(G)$ is a morphism (i.e., $E(H) \subseteq E(G)$). Where the inclusion map is an embedding, we call H the subgraph of G *induced* on vertex set $V(H)$, and say that H is an *induced subgraph* of G. We use $G - \{v\}$ to denote the subgraph of G induced on $V(G) - \{v\}$; we use $G - H$ to denote the subgraph of G induced on $V(G) - V(H)$. If no induced subgraph of G is isomorphic to H, then G is called *H-free.*

We introduce a number of simple graph classes and operations referenced throughout this dissertation. The *complement* of a graph G is the graph $\overline{G}$ with vertex set $V(G)$ and "opposite" edge set $E = \{xy : x, y \in V(G) \text{ and } xy \notin E(G)\}$. A graph G is a *clique* (also called a *complete graph*) if it contains an edge between all pairs of vertices, and a *stable set* (also called an *independent set*) if $E(G) = \emptyset$. We denote the clique on n vertices by K_n and its complement the stable set on n vertices by $\overline{K_n}$. We also frequently discuss whether an induced subgraph is (isomorphic to) a clique or a stable set.

A *path* is a sequence of distinct vertices $v_1 v_2 \ldots v_m$ such that $v_i v_{i+1} \in E(G)$ for all $1 \leq i \leq m-1$. We call this a path between vertices v_1 and v_m, or with endpoints v_1 and v_m. The *path graph* P_n on $n \geq 2$ vertices is the graph with vertex set $\{v_1, \ldots, v_n\}$ and edge set $\{v_1v_2, v_2v_3, \ldots, v_{n-1}v_n\}$. A cycle is a sequence of distinct vertices $v_1, \ldots, v_m$ such that $v_i v_{i+1} \in E(G)$ and $v_1 v_m \in E(G)$. Such a cycle is called an *odd cycle* or an *even cycle* depending on the parity of m. The *cycle graph* C_n on $n \geq 3$ vertices is the graph with vertex set $\{v_1, \ldots, v_n\}$ and edge set $\{v_1v_2, v_2v_3, \ldots, v_{n-1}v_n, v_1v_n\}$. The graphs $K_5, \overline{K_5}, P_5$, and C_5 are shown in Figure 1.1. We frequently seek out induced path graphs and cycle graphs as subgraphs, and will abuse notation to call these induced paths and cycles.

A graph is *connected* if for all pairs of vertices v, w, there exists a path with endpoints v and w. If not, a graph G is *disconnected*, and its *connected components* are the maximal connected induced subgraphs of G. Given two graphs G and H with disjoint vertex sets, the *disjoint sum* of G and H, written $G + H$, is the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$. The disconnected graph $K_3 + K_2$ in Figure 1.1 uses this notation. We use kG to denote the disjoint sum of k distinct copies of G.

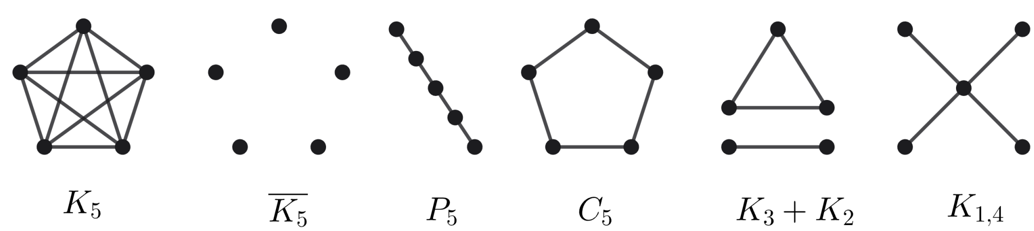

Figure 1.1: Six named graphs on five vertices.

Given a graph G with vertex set $V(G) = \{v_1, \ldots v_n\}$, the *degree* of a vertex v_i is the number of edges incident to the vertex. We denote this by $\deg_G(v_i)$ and omit the subscript where the choice of graph is obvious. The *degree sequence* $\pi = (d_1, \ldots, d_n)$ of a graph G is the multiset of degrees of the vertices of G, listed in nonincreasing order. Alternatively, we write $\pi = (d_1^{a_1}, d_2^{a_2}, \ldots, d_k^{a_k})$, where each d_i is distinct and a_i indicates its multiplicity in π. For example, the degree sequence of P_5 is $(2, 2, 2, 1, 1)$ and the degree sequence of C_5 is $(2, 2, 2, 2, 2)$. Given a vertex v, we call v *isolated* (or an isolated vertex) if $\deg_G(v) = 0$, and *dominating* (or a dominating vertex) if $\deg_G(v) = |V(G)| - 1$. A subset A of vertices is *complete to* a subset B of vertices if $ab \in E(G)$ for all $a \in A$ and $b \in B$. The subsets A and B are *anti-complete to* one another if no such edges ab exist in $E(G)$.

Given a graph G and a multiset π of nonnegative integers, we say that G *realizes* π or is a *realization* of π if π is the degree sequence of G. Importantly, this correspondence is not bijective: a multiset might have no realizations (for instance, if its sum is odd) or multiple realizations. For instance, in Figure 1.1, both P_5 and $K_3 + K_2$ are realizations of the degree sequence $(2, 2, 2, 1, 1)$, but they are not isomorphic. We call π *graphic* if it has at least one realization; such multisets can be characterized as follows:

Theorem 1.1. *[25] A non-increasing sequence $\pi = (d_1, \dots, d_n)$ is graphic if and only if it has even sum and satisfies the kth Erdős-Gallai inequality*

$$\sum_{i=1}^{k} d_i \leq k(k-1) + \sum_{i=k+1}^{n} \min\{k, d_i\}$$

with equality for all k, $1 \leq k \leq n$.

When G is the unique realization of a degree sequence π, up to isomorphism, we call G a *unigraph* and π *unigraphic*.

1.2 Hereditary Graph Classes

A class $\mathcal{A}$ of graphs is *hereditary* if $\mathcal{A}$ contains all induced subgraphs of graphs in $\mathcal{A}$. As an example, consider the class of cliques. Given a clique, any subset of its vertices induces a smaller clique, so this class is hereditary. By contrast, consider the class of cycle graphs. Given a cycle, any two-vertex subset of its vertices induces K_2 or $\overline{K_2}$, neither of which is a cycle. Hence the class of cycle graphs is not hereditary. The class of *trees* – connected, acyclic graphs – are also non-hereditary, since most trees have disconnected induced subgraphs. However, the class of all acyclic graphs, called *forests*, is hereditary, since any induced subgraph of an acyclic graph is still acyclic.

A slightly deeper example of a hereditary class are the bipartite graphs. A graph G is *bipartite* if there exists a partition of its vertex set into two independent sets; we call this partition a *bipartition*. Since any induced subgraph of a bipartite graph can inherit the vertex set partition, it too is a bipartite graph. We take the opportunity to mention two subclasses: a graph is a *complete bipartite graph* if, in some bipartition, the two independent sets of vertices are complete to one another. We denote the complete bipartite graph with independent sets of size m and n by $K_{m,n}$. Where $m = 1$, we call the graph a *star graph*; see Figure 1.1 for an example.

Hereditary classes are particularly interesting because each can be characterized by a set of forbidden induced subgraphs: for any hereditary class $\mathcal{A}$, there exists a unique minimal set $\mathcal{F}$ of forbidden graphs such that $G \in \mathcal{A}$ if and only if G contains no element of $\mathcal{F}$ as an induced subgraph. The set $\mathcal{F}$ is unique and minimal: no graph in $\mathcal{F}$ is induced in another graph of $\mathcal{F}$. While any graphs containing elements of $\mathcal{F}$ as induced subgraphs are forbidden from $\mathcal{A}$ as well, we refer to the (minimal) graphs in $\mathcal{F}$ as *forbidden induced subgraphs* of $\mathcal{A}$. One main objective in structural graph theory research is to characterize hereditary classes in terms of their forbidden induced subgraphs, and to use these characterizations to solve other properties of these classes. We use the remainder of this section to introduce several important hereditary graph classes and their properties, including perfect graphs, threshold graphs and split graphs.

The hereditary class of cliques is characterized by exactly one forbidden induced subgraph, $\overline{K_2}$. Most hereditary classes, however, are characterized by larger finite or infinite families of forbidden induced subgraphs. For instance, a graph is bipartite if and only if it contains no odd cycle $(C_3, C_5, C_7, \ldots)$ as an induced subgraph. Such cycles cannot be partitioned into two independent sets, and therefore any graph containing an odd cycle as an induced subgraph cannot be partitioned into two independent sets. Furthermore, this constitutes a minimum set of forbidden induced subgraphs for bipartite graphs, since every proper subgraph of an odd cycle graph is bipartite.

These characterizations are often quite difficult to solve. Consider the related classes of k-partite graphs, that is, graphs where the vertex set can be partitioned into $k \geq 3$ independent sets. Determining whether a graph is k-partite is NP-complete, and no known forbidden induced subgraph characterization exists.

Many of the hereditary classes that I study are partially or fully defined by the following three invariants. The *clique number* $\omega(G)$ is the size of the largest clique in a graph G, and the *independence number* $\alpha(G)$ is the size of the largest stable set in a graph G. We can also color the vertices of a graph such that no two adjacent vertices are the same color. If such a coloring exists with k colors, we call G *k-colorable.* The *chromatic number* $\chi(G)$ is the minimum number k such that G is k-colorable. Computing $\chi(G)$ generally is NP-complete, as is evaluating whether G is k-colorable for $k \geq 3$. Note this is equivalent to determining whether a graph is k-partite, described above.

Of course, the chromatic number of any graph must be at least as large as the clique number. Graphs in the hereditary subclass of graphs for which $\chi(G) \geq \omega(G)$ holds with equality – that is, graphs G for which all induced subgraphs H satisfy $\chi(H) = \omega(H)$ – have the best possible coloring properties, and are accordingly called *perfect graphs.* Perfect graphs are known to be closed under complementation, and are exactly characterized by Chudnovsky et al. [16] in the renowned 2006 Strong Perfect Graph Theorem.

Theorem 1.2. *[16] A graph is perfect if and only if it contains none of* $\{C_5, C_7, \overline{C_7}, C_9, \overline{C_9}, \ldots\}$

as an induced subgraph.

Perfect graphs are important due to how they link local, structural information (the absence of odd induced cycles and their complements) to global optimization results on the clique and chromatic numbers. Numerous papers have been published on them in the last 50 years, with over 500 citing [16] alone.

We can measure how well-colorable (i.e., close to perfect) a hereditary class is by giving a function that relates the chromatic number to the clique number of graphs in that class. For a hereditary class $\mathcal{A}$, $f : \mathbb{N} \to \mathbb{N}$ is a χ-bounding function on $\mathcal{A}$ if for all $G \in \mathcal{A}$, $\chi(G) \leq f(\omega(G))$. For perfect graphs, the function $f(x) = x$ is χ-bounding, and this is best possible. Determining a χ-bound for a hereditary class is an important problem, particularly to ask if a given class is linearly or polynomially χ-bounded. Much ink has been spilt, for instance, on partial solutions to the Gyárfás-Sumner conjecture that for all trees T, the family of T-free graphs is polynomially χ-bounded [35] [64]. See [61] for a survey of known results in χ-boundedness.

Where a hereditary class $\mathcal{A}$ is χ-bounded by $f(x) = x + 1$, we call the class *apex-perfect*. From every graph in $\mathcal{A}$ it is possible to remove just one vertex and produce a perfect graph. Apex-perfect graphs form a hereditary class, but the forbidden induced subgraph characterization problem remains open. Throughout the dissertation, we will prove several new results about apex-perfect classes.

There are a number of well-studied hereditary subclasses of perfect graphs. We discuss threshold graphs, split graphs, and chordal graphs throughout this dissertation, and give an overview here. Some of this material is repeated in subsequent chapters when the theorems are used.

A graph G is said to be *threshold* if there exist a function $w : V(G) \to \mathbb{R}$ and a real number t such that there is an edge between two distinct vertices u and v if and only if $w(u) + w(v) > t$ [18]. The class of threshold graphs has been studied in great detail. See Mahadev and Peled's book for more information [52]. Threshold graphs are identifiable by forbidden induced subgraphs and with a vertex ordering ([19]), among myriad other characterizations.

Theorem 1.3. *Given a graph G, the following are equivalent:*

(i) *G is threshold.*

(ii) *G contains no induced $2K_2, P_4, C_4$.*

(iii) *$V(G)$ can be given an ordering $v_1, \ldots, v_n$ such that for every i, $1 \leq i \leq n$, v_i is adjacent to all or none of $v_1, \ldots, v_{i-1}$.*

Split graphs form a slightly broader class and contain the threshold graphs. A *split graph* is a graph in which the vertex set can be partitioned into a clique and a stable set. For a split graph G, the partition $V(G) = K \cup S$ is called a *KS-partition* if K is a clique and S is a stable set. The class of split graphs was first defined by Gyarfas and Lehel in [36], and independently by Földes and Hammer in [28]. See Collins and Trenk's chapter on split graphs in *Topics in Algorithmic Graph Theory* for more information [22]. As a hereditary class, split graphs can be characterized by forbidden induced subgraphs.

A class $\mathcal{A}$ of graphs is *closed under degree sequences* if for all degree sequences π, either all realizations of π are in $\mathcal{A}$ or no realization of π is in $\mathcal{A}$. The class of split graphs is closed under degree sequences, essentially because we can identify the KS-partition of a graph, should one exist, from the degree sequence alone.

Theorem 1.4. *[28] [39] Given a graph G with degree sequence $\pi = (d_1, \dots, d_n)$, the following are equivalent:*

1. *G is a split graph.*
2. *G contains no induced $2K_2, C_4, C_5$.*
3. *Given $m = \max\{i | d_i \geq i - 1\}$, π satisfies*

$$\sum_{i=1}^{m} d_i = m(m-1) + \sum_{i=m+1}^{n} d_i.$$

The class of *chordal graphs* is another subclass of perfect graphs. A graph G is *chordal* if every induced cycle in it is a triangle [24]. Equivalently, chordal graphs contain no induced cycles on four or more vertices. Since cycles C_n for $n \geq 6$ contain $2K_2$ as an induced subgraph, it follows from Theorem 1.4 that split graphs are chordal. See [33] for more information on chordal graphs.

Where a class $\mathcal{A}$ is not hereditary, we define the *hereditary subclass* of $\mathcal{A}$ to be the largest hereditary class contained within $\mathcal{A}$, and the *hereditary closure* to be the smallest hereditary class containing $\mathcal{A}$. Equivalently, the hereditary subclass of $\mathcal{A}$ is the set of graphs whose induced subgraphs are contained in $\mathcal{A}$, and we often call graphs in this class *hereditary-$\mathcal{A}$-graphs* (hereditary unigraphs, hereditary Nordhaus-Gaddum graphs, etc.). For example, the class of perfect graphs is the hereditary subclass of graphs G satisfying $\chi(G) = \omega(G)$. The hereditary closure of $\mathcal{A}$ is the set of induced subgraphs of elements of $\mathcal{A}$. The class of forests is the hereditary closure of the (non-hereditary) class of trees.

In this research, we ask a number of questions of given classes: how can we define and characterize the class? Is the class hereditary, and if so, what is its forbidden induced subgraph characterization? Is the class closed under degree sequences? If the class is not hereditary, how can we describe the hereditary subclass and hereditary closure?

Beyond characterization, we also consider the properties of given classes, particularly regarding algorithmic optimization. In general, it is NP-hard to determine the clique number, independence number, or chromatic number of a graph, but this can be improved for specific classes. For instance, perfect graphs can be identified in polynomial time, and the clique (and thus chromatic) number can be determined in polynomial time due to algorithms of Chudnovsky [15] and Grötschel, Lovász, and Schrijver [34], respectively. For a given class, can either of these invariants be determined in polynomial time or better? What about membership in the class? The classes we study are often defined by relationships among invariants, so these questions are especially relevant to any invariants used in the definition.

1.3 Overview of Dissertation

Chapters 2 and 3 relate to graph classes defined by invariants generally, and the Nordhaus-Gaddum inequality specifically. The relationship between the clique number $\omega(G)$ and the chromatic number $\chi(G)$ is well-studied, and their sum and product are both bounded in terms of $|V(G)|$.

Theorem 1.5 (Nordhaus, Gaddum [54])**.** *For a graph G with n vertices, $2\sqrt{n} \leq \chi(G) + \chi(\overline{G}) \leq n+1$ and $n \leq \chi(G) \cdot \chi(\overline{G}) \leq \frac{(n+1)^2}{4}$.*

Nordhaus and Gaddum proved in 1956 that the sum of the chromatic number χ of a graph G and its complement is at most $|G|+1$. The families of graphs satisfying each of these inequalities with equality are known (see [1] for a survey of these results), but little is known about how far a given graph is from meeting either the lower or upper bounds with equality. In Chapter 2, we define a-NG to be the class of graphs within distance a of satisfying the upper bound of the first inequality with equality, i.e., $G \in a$-NG if and only if $\chi(G) + \chi(\overline{G}) \geq n + 1 - a$. We characterize the hereditary subclass of 1-NG with an explicit family $\mathcal{F}$ of 52 forbidden induced subgraphs. We also show that the resulting hereditary class has very nice algorithmic and coloring properties, including a linear χ-bound. Furthermore, the clique and chromatic numbers of graphs in 1-HNG can be found in polynomial time. The forbidden induced subgraphs for 1-HNG extend the characterization of sum-perfect graphs in [50], including all six-vertex bipartite graphs with a perfect matching and their complements, and numerous seven- and eight-vertex graphs containing an induced C_5.

In Chapter 3, we consider a-HNG in general, and obtain partial lists of forbidden induced subgraphs for the classes. Our proofs rely on a substantive connection between these graphs and polyominoes, and we study the latter in some depth. In addition to showing that each bipartite graph with a perfect matching is a forbidden induced subgraph of a-HNG

for some a, we also establish that a-HNG is linearly χ-bounded and has finitely many forbidden induced subgraphs, resolving an open question of [50]. Many of these results apply to generalizations of sum-perfect graphs, which we additionally show have finitely many forbidden induced subgraphs. We conclude the chapter by establishing a correspondence between a-HNG, generalized sum-perfect graphs, and a generalization of split graphs.

Chapters 4 and 5 address hereditary classes related to unigraphs and generalizations of unigraphs. Recall that a graph with degree sequence π is a *unigraph* if it is isomorphic to every graph that has degree sequence π. The class of unigraphs is by definition closed under degree sequence realization, but as we show in Chapter 4, it is not hereditary. Barrus characterized the hereditary subclass HU [2], [4], and in this dissertation we resolve the characterization of the class HCU. We give an explicit characterization of the structure of graphs in HCU as the composition of smaller graphs from any of seven infinite families and their complements (composition here is defined by Tyshkevich [67] as a means of merging the cliques of graphs, and will be introduced fully in Chapter 4.) Another characterization is entirely different: we give rules to determine if the realizations of a degree sequence are contained in HCU. We characterize HCU in terms of a set of Rao-minimal forbidden degree sequences and a set of minimal forbidden induced subgraphs. We say that a degree sequence d *Rao-contains* a degree sequence e if there exists a realization of e induced in a realization of d [58]. Moreover, we present a characterization of HU in terms of Rao-minimal forbidden degree sequences. Lastly, we show the χ-bounding function on unigraphs is close to perfect, pointing to a strong relationship between graph realization and graph coloring.

Most graphs are not unigraphs, however, and we seek to understand what information (additional to the degree sequence) would suffice to identify the graph up to isomorphism. In other words, what graphs – within the restricted universe of a particular graph class – are the only realizations of their degree sequences? Given a class $\mathcal{A}$ of graphs, we define the class of $\mathcal{A}$-unigraphs to be graphs identifiable from degree sequence and membership in $\mathcal{A}$. While these classes are often not hereditary, we provide characterizations in Chapter 5 of the largest hereditary subclass contained in the bipartite-unigraphs, the perfect-unigraphs, the forest-unigraphs, the chordal-unigraphs, and the weakly-chordal-unigraphs (defined in that chapter). We also characterize the hereditary bipartite-unigraphs in terms of structure and in terms of a partial order on degree sequences due to Rao.

Chapter 2 is published as "Hereditary Nordhaus-Gaddum graphs" with Vaidyanathan Sivaraman, in *Discrete Applied Mathematics*. The vast majority of material in Chapter 4 is from "The hereditary closure of the unigraphs," co-authored with Michael Barrus and Ann Trenk and published in *Discrete Mathematics* [6]. While the published version of the paper included the statement of Theorems 4.34 and 4.32 without proof, we include the proofs in this dissertation. These proofs are also given in the arXiv version of the paper [7]. Section 4.3 is also new to this dissertation and included within Chapter 4. I am grateful to my co-authors for their permission to utilize our shared work in this dissertation.

Chapter 2

Hereditary Nordhaus-Gaddum Graphs

This chapter is published with Vaidyanathan Sivaraman in [62].

2.1 Introduction

In this chapter we define a new family of graph classes generalizing the Nordhaus-Gaddum graphs, and introduce the largest hereditary subclasses of each of these classes. Throughout, all graphs are finite and simple. We begin by introducing four graph invariants used throughout the chapter. Let $G = (V(G), E(G))$ be a graph. Let n denote $|V(G)|$ and $\overline{G}$ denote the complement of G. The *clique number* $\omega(G)$ is the largest positive integer k such that the complete graph on k vertices, denoted K_k, is a subgraph of G, and the *independence number* $\alpha(G)$ is the largest positive integer l such that $\overline{K_l}$ is an induced subgraph of G. The *chromatic number* $\chi(G)$ is the smallest positive integer k such that the vertices of G can be properly colored with k colors (a proper coloring is one where adjacent vertices receive different colors). The *clique cover number* $\theta(G)$ is the smallest positive integer l such that $V(G)$ can be partitioned into l sets, each inducing a clique. It follows from the definitions that $\omega(\overline{G}) = \alpha(G)$, $\chi(\overline{G}) = \theta(G)$, $\omega(G) \leq \chi(G) \leq n$, and $\alpha(G) \leq \theta(G) \leq n$.

The relationship between the chromatic number of a graph and that of its complement was first studied by Nordhaus and Gaddum in 1956 [54]. They proved that the sum of the two invariants is bounded above by $n + 1$:

$$\chi(G) + \theta(G) \leq n + 1 \tag{2.1}$$

Since then several researchers have studied the same problem for various graph invariants. See [1] for a survey. The class of graphs satisfying Inequality 2.1 with equality are called the Nordhaus-Gaddum graphs (we denote this class 0-NG). The class 0-NG has been widely studied, with three increasingly elegant structural characterizations [27] [63] [20]. Cheng, Collins, and Trenk gave a degree sequence characterization of 0-NG and an enumeration of the class [14].

Notably, 0-NG is not a hereditary class, closed under taking induced subgraphs. As an example, the cycle graph on 5 vertices, C_5, has $\chi = \theta = 3$ and $n = 5$, so satisfies Inequality 2.1 with equality. Its induced subgraph the path graph P_4 has $\chi = \theta = 2$ and $n = 4$, so is not in 0-NG. A natural question is to understand the largest hereditary subclass of 0-NG, i.e. the set of graphs for which all induced subgraphs are Nordhaus-Gaddum. It turns out that this class coincides with a class well-studied in the context of induced subgraphs, namely, the threshold graphs. We provide a proof in Proposition 2.3, later in this section. Threshold graphs have been relaxed and generalized in several different ways (see for instance [52] [74] [9]), and this chapter will give another.

The class 0-NG is small and contains few important subclasses, aside from the threshold graphs. Nevertheless, a number of classes have interesting intersections. One is the $(C_4, 2K_2)$-free graphs (the pseudo-split graphs), defined in [12]. Pseudo-split graphs containing an induced C_5 satisfy Inequality 2.1 with equality, and form one of three subclasses of 0-NG in Collins and Trenk's characterization [20]. Blászik [12] showed all pseudo-split graphs satisfy the relaxed bound $\chi(G) + \theta(G) \geq n$.

We build on this generalization to introduce a parameter a to Inequality 2.1 and consider graphs satisfying the following inequality for each choice of $a \geq 0$:

$$\chi(G) + \theta(G) \geq n + 1 - a. \tag{2.2}$$

Given $a \geq 0$, a graph G is *a-Nordhaus-Gaddum* if G satisfies Inequality 2.2 with equality. The generalized classes of a-Nordhaus-Gaddum graphs are again not hereditary; as an example, C_{2a+5} has $\chi = 3$ and $\theta = a+3$, so is an element of a-NG, but its induced subgraph P_{2a+4} has $\chi = 2$ and $\theta = a + 2$, so is not an element of a-NG. We hence define G to be *a-hereditary-Nordhaus-Gaddum* if every induced subgraph H of G satisfies 2.2 with equality. We denote the class of a-Nordhaus-Gaddum graphs by a-NG and the class of a-hereditary-Nordhaus-Gaddum graphs by a-HNG. For all $a \geq 0$, a-NG and a-HNG are both closed under complementation, and a-HNG is hereditary.

We have the following chain of inclusions:

Proposition 2.1. 0-$HNG \subset$ 0-$NG \subset$ 1-$HNG \subset$ 1-$NG \subset$ 2-$HNG \subset \ldots$

Proof. Fix $a \geq 0$. By definition a-$HNG \subset a$-NG. Suppose for a contradiction that a-NG $\not\subset (a+1)$-HNG, so there exists $G \in a$-NG with an induced subgraph $H \notin (a+1)$-NG. Let K be the induced subgraph with vertex set $V(G) - V(H)$. Then $\chi(G) + \theta(G) \geq |G| + 1 - a$

and $\chi(H) + \theta(H) < |H| + 1 - (a+1)$. By Inequality 2.1, $\chi(K) + \theta(K) \leq |K| + 1$. As a result:

$$\chi(G) + \theta(G) \leq \chi(H) + \chi(K) + \theta(H) + \theta(K) < |H| + |K| + 1 - a = |G| + 1 - a,$$

contradicting G's inclusion in a-NG. Thus instead a-$NG \subset (a+1)$-HNG.

Since C_{2a+5} is in a-NG but not a-HNG, and P_{2a+4} is in $(a+1)$-HNG but not a-NG, all inclusions are strict. □

We introduce more notation used throughout the chapter, then discuss a number of important graph classes and their relation to a-NG and a-HNG. Given positive integers k, l, we denote the complete bipartite graph with k vertices in one half of the bipartition and l in the other by $K_{k,l}$. We use $+$ to denote the disjoint union of two graphs. Given a vertex v (resp. induced subgraph H), let $G - \{v\}$ (resp., $G - H$) denote the induced subgraph on vertex set $V(G) - \{v\}$ (resp., $V(G) - V(H)$). A vertex v is *complete to* $H \subseteq V(G)$ if v is adjacent to all vertices in H. A vertex subset A is complete to H if all $v \in A$ are complete to H. The neighborhood of a vertex v in G (resp., in induced subgraph H) is denoted $N(v)$ (resp., $N_H(v)$). A vertex v is *isolated* in G (resp., in H) if $N(v)$ (resp. $N_H(v)$) is empty, and *dominating* if $N(v) = V(G) - \{v\}$. We refer to two induced subgraphs as isolated from one another where there are no edges with one endpoint in each subgraph.

Recall that a graph G is said to be *threshold* if there exist a function $w : V(G) \to \mathbb{R}$ and a real number t such that there is an edge between two distinct vertices u and v if and only if $w(u) + w(v) > t$ [18]. The class of threshold graphs has been studied in great detail. See Mahadev and Peled's book for more information [52]. Threshold graphs are identifiable by forbidden induced subgraphs and with a vertex ordering ([19]), among myriad other characterizations.

Theorem 2.2. *Given a graph G, the following are equivalent:*

(i) G is threshold.

(ii) G contains no induced $2K_2, P_4, C_4$.

(iii) $V(G)$ can be given an ordering $v_1, \ldots, v_n$ such that for every i, $1 \leq i \leq n$, v_i is adjacent to all or none of $v_1, \ldots, v_{i-1}$.

Together, these two characterizations show that threshold graphs are exactly 0-HNG.

Proposition 2.3. *The class* 0-HNG *is exactly the class of threshold graphs.*

Proof. Given $a \geq 0$ and a graph G with an isolated or dominating vertex v, G satisfies Inequality 2.2 with equality if and only if $G - \{v\}$ satisfies Inequality 2.2 with equality. Hence Theorem 2.2 (iii) implies that threshold graphs are contained in 0-HNG. For the converse it suffices to see from Theorem 2.2(ii), that none of $P_4, 2K_2, C_4$, the three forbidden induced subgraphs for the class of threshold graphs, is in 0-HNG. □

A graph G is *split* if its vertices can be partitioned into a clique and a stable set [28]. A graph G is *chordal* if every induced cycle in is isomorphic to C_3 [24]. It is *weakly chordal* if every induced cycle in it or its complement is isomorphic to C_3 or C_4 [40]. A graph G is *perfect* if $\chi(H) = \omega(H)$ for all induced subgraphs H of G [11]. The Strong Perfect Graph Theorem [16] shows a graph is perfect if and only if it contains none of $\{C_5, C_7, \overline{C_7}, C_9, \overline{C_9}, \ldots\}$ as an induced subgraph.

We also define a new class generalizing perfect graphs: a graph is *apex-perfect* if it contains a vertex whose deletion results in a perfect graph. Although hereditary, the forbidden induced subgraph characterization of this class remains unknown.

By the forbidden induced subgraph characterizations, we have the following chain of inclusions on graph classes:

$$\text{Threshold} \subset \text{Split} \subset \text{Chordal} \subset \text{Weakly Chordal} \subset \text{Perfect} \subset \text{Apex-Perfect}.$$

We compare the hereditary classes *a*-HNG to the above hereditary classes. First, 0-HNG is exactly the class of threshold graphs. In a split graph, $\omega(G) + \alpha(G) \geq |G|$ [39], so since split graphs are perfect, $\chi(G) + \theta(G) = \omega(G) + \alpha(G) \geq |G|$, it follows that split graphs are a subclass of 1-HNG. For $a \geq 1$, *a*-HNG is not a subclass of chordal, weakly chordal, or perfect graphs, since it contains C_5. The converse also holds: chordal, weakly chordal, and perfect graphs do not constitute subclasses of *a*-HNG, since P_{2a+4} is in each of these classes but not in *a*-HNG. We prove later in Theorem 2.12 that all graphs in 1-HNG are apex-perfect.

Another generalization of perfect graphs is to χ-bounding functions. For a hereditary class $\mathcal{G}$, $f : \mathbb{N} \to \mathbb{N}$ is a χ-bounding function on $\mathcal{G}$ if for all $G \in \mathcal{G}$, $\chi(G) \leq f(\omega(G))$. For perfect graphs, the function $f(x) = x$ is χ-bounding, and this is of course best possible. See [61] for a survey of hereditary classes with known χ-bounding functions. Since all graphs in 1-HNG are apex-perfect, it follows as Theorem 2.13 that 1-HNG is χ-bounded by the function $f(x) = x + 1$, which is best possible.

Given a graph G, it holds that $\omega(G)+\alpha(G) \leq n+1$. Where this bound holds with equality for all induced subgraphs H of G, it follows that G is threshold [50]. More interesting is the generalization of threshold graphs to the class of sum-perfect graphs. A graph is *sum-perfect* if for all induced subgraphs H, $\omega(H) + \alpha(H) \geq n$. One of the authors, together with Litjens and Polak, provides a forbidden induced subgraph characterization of the class [50]. It follows from the characterization that the class of sum-perfect graphs is a subclass of the weakly-chordal graphs, and hence perfect. Since $\omega(G) \leq \chi(G)$ and $\alpha(G) \leq \theta(G)$, sum-perfect graphs are also 1-HNG. We make use of this inclusion in Theorem 2.9, our forbidden induced subgraph characterization of 1-HNG.

The line graph of G, denoted $L(G)$, has the edges of G as its vertices. Two vertices in $L(G)$ are adjacent if as edges in G they are incident to a shared vertex. Line graphs translate questions about edges into questions about vertices, and so are an important tool for simplifying otherwise-intractable problems. The class of line graphs is hereditary and there is a beautiful characterization of this class in terms of its forbidden induced subgraphs [10]. We characterize the intersection of 1-HNG and line graphs in Theorem 2.14.

A *claw* is an induced subgraph that is isomorphic to $K_{1,3}$; its trivalent vertex is called its *center*. A graph is *claw-free* if it contains no claw. Since the claw is one of the forbidden induced subgraphs of line graphs, claw-free graphs are an important generalization of line graphs. They also have strong algorithmic properties (see the survey [26]). We characterize the intersection of 1-HNG and claw-free graphs in Theorem 2.16. Section 5 ends with characterizations of the intersection of 1-HNG with two simple, essential classes: bipartite graphs and triangle-free graphs.

The chapter is structured as follows. In Section 2 we study the effect of vertex deletion on χ and θ, which is connected to forbidden induced subgraphs of a-HNG. From here, the remaining sections of the chapter pertain only to the smallest class 1-HNG. In Section 3 we extend the forbidden induced subgraph characterization of sum-prefect graphs [50] to characterize the forbidden induced subgraphs of 1-HNG. In Section 4 we show graphs in 1-HNG are apex-perfect, thereby giving a best possible χ-bounding function on the class. Section 5 centers on the intersection of 1-HNG with, respectively, the classes of line graphs, claw-free graphs, and triangle-free graphs. In Section 6 we provide a number of optimization results about 1-HNG, showing that inclusion in 1-HNG, $\omega(G), \alpha(G), \chi(G)$, and $\theta(G)$ can all be computed in polynomial time.

2.2 Vertex Deletion in Minimum Colorings and Clique Coverings

In this section we present several results about vertex deletion and forbidden induced subgraphs of a-HNG. First, since a-HNG is closed under complementation, so too is its set of forbidden induced subgraphs, which we record as a remark.

Remark 2.4. *For all $a \geq 0$, the set of forbidden induced subgraphs of a-HNG is closed under complementation.*

Second, each vertex in a graph can contribute at most one color to a coloring and add at most one clique to a clique covering. Since a proper coloring of a graph G is formally a partition of $V(G)$ into stable sets, we call each stable set in a proper coloring a *color class*, that is, the set of vertices with the same color. A vertex v is *χ-distinct* in G if there exists a minimum proper coloring of G where v is the only vertex in its color class. Similarly, v is *θ-distinct* in G if there exists a minimum proper clique covering where v is the only vertex in its clique. Such a minimum proper coloring or clique covering is called *v-distinct*. Deleting a vertex v from G will decrement χ or θ if and only if v is χ-distinct or θ-distinct, respectively, which we record in the following proposition.

Proposition 2.5. *Given $v \in G$, $\chi(G - \{v\}) = \chi(G) - 1$ if and only if v is χ-distinct. Otherwise $\chi(G - \{v\}) = \chi(G)$. Analogously, $\theta(G - \{v\}) = \theta(G) - 1$ if and only if v is θ-distinct, and otherwise $\theta(G - \{v\}) = \theta(G)$.*

Proof. Let $v \in G$. Clearly $\chi(G - \{v\}) = \chi(G)$ or $\chi(G) - 1$. If v is χ-distinct, then choose a v-distinct coloring. The restriction of this coloring to $G - \{v\}$ gives a proper coloring of $G - \{v\}$ with $\chi(G) - 1$ colors, so $\chi(G - \{v\}) \leq \chi(G) - 1$. Hence they are equal. For the reverse direction, if $\chi(G - \{v\}) = \chi(G) - 1$, then fix a proper coloring of $G - \{v\}$ using $\chi(G) - 1$ colors. Add a new color class containing only v to produce a proper coloring of G using $\chi(G)$ colors. Hence the coloring is minimal in G, and we conclude that v is χ-distinct. An analogous proof holds for $\theta(G)$. $\square$

The following result is used to identify θ-distinct vertices; an analogous result holds for χ-distinct vertices.

Proposition 2.6. *For any $v \in V(G)$ such that $N(v)$ induces a stable set, v is θ-distinct in G if its neighbors are not θ-distinct in $G - \{v\}$.*

Proof. Let $v \in V(G)$ with $N(v)$ inducing a stable set. We prove the contrapositive. Suppose v is not θ-distinct in G. Then for all minimum proper clique coverings of G, v shares a clique with one of its neighbors. Choose a minimum proper clique covering. This clique covering restricted to $G - \{v\}$ is then a proper clique covering with some $w \in N(v)$ in a distinct clique. Proposition 2.5 implies that $\theta(G - \{v\}) = \theta(G)$, so the clique covering restricted to $G - \{v\}$ is minimum. Thus w is θ-distinct in $G - \{v\}$. $\square$

Identifying distinct vertices is critical because forbidden induced subgraphs cannot contain distinct vertices of either type.

Proposition 2.7. *If G is a forbidden induced subgraph of a-HNG, then $G \in (a+1)$-HNG, and no vertex of G is χ-distinct or θ-distinct.*

Proof. Let $v \in V(G)$. Since G is a forbidden induced subgraph of a-HNG, it follows that $G - \{v\} \in a$-HNG. Thus $\chi(G) + \theta(G) \leq n - a$ and $\chi(G - \{v\}) + \theta(G - \{v\}) \geq n - a$. Since $\chi(G) \geq \chi(G - \{v\})$ and $\theta(G) \geq \theta(G - \{v\})$, it follows that $\chi(G) = \chi(G - \{v\})$, $\theta(G) = \theta(G - \{v\}$, and $\chi(G) + \theta(G) = n - a$. Thus $G \in (a + 1)$-HNG, and by Proposition 2.5, v is not χ-distinct or θ-distinct. □

The above two propositions combine to give the following corollary, which we use extensively in our characterization of 1-HNG in Section 3.

Corollary 2.8. *If there exists $v \in V(G)$ such that $N(v)$ induces a stable set and no vertex in $N(v)$ is θ-distinct in $G - \{v\}$, then G is not a forbidden induced subgraph of a-HNG for any $a \geq 0$.*

2.3 A Forbidden Induced Subgraph Characterization of 1-HNG

Let $\mathcal{F}$ be the set of 52 forbidden induced subgraphs of 1-HNG found computationally on 6 to 8 vertices, shown in Figure 2.1. Let $\mathcal{F_S}$ denote those graphs in $\mathcal{F}$ containing no induced C_5, and $\mathcal{F_C}$ denote graphs in $\mathcal{F}$ containing an induced C_5. The subscript S will denote that these are forbidden induced subgraphs of the sum-perfect graphs; see Theorem 2.10. The set $\mathcal{F_S}$ contains 24 graphs on 6 vertices and 2 graphs on 7 vertices. Twelve of the 24 graphs on 6 vertices are bipartite graphs with matching number $\nu = 3$. Note that a bipartite graph has $\nu = 3$ if and only if it has an induced perfect matching on 6-vertices, if and only if it contains a $3K_2$ subgraph. The other 12 are complements of these graphs. The two graphs on 7 vertices are the sun graph with a pendant (degree one) vertex attached to a degree two vertex, and its complement.

The set $\mathcal{F_C}$ contains 22 graphs on 7 vertices and 4 graphs on 8 vertices, and is also by necessity closed under complementation.

From here, we prove that $\mathcal{F}$ is exactly the set of forbidden induced subgraphs for 1-HNG.

Theorem 2.9. *A graph G is in 1-HNG if and only if it contains no element of $\mathcal{F}$ as an induced subgraph.*

This work is made much simpler for graphs in 1-HNG without C_5 by the fact that these graphs are exactly the sum-perfect graphs, and their forbidden induced subgraph characterization is known. We present the characterization by Litjens, Polak, and Sivaraman [50].

Theorem 2.10. *A graph G is sum-perfect if and only if it is $\mathcal{F_S} \cup \{C_5\}$-free.*

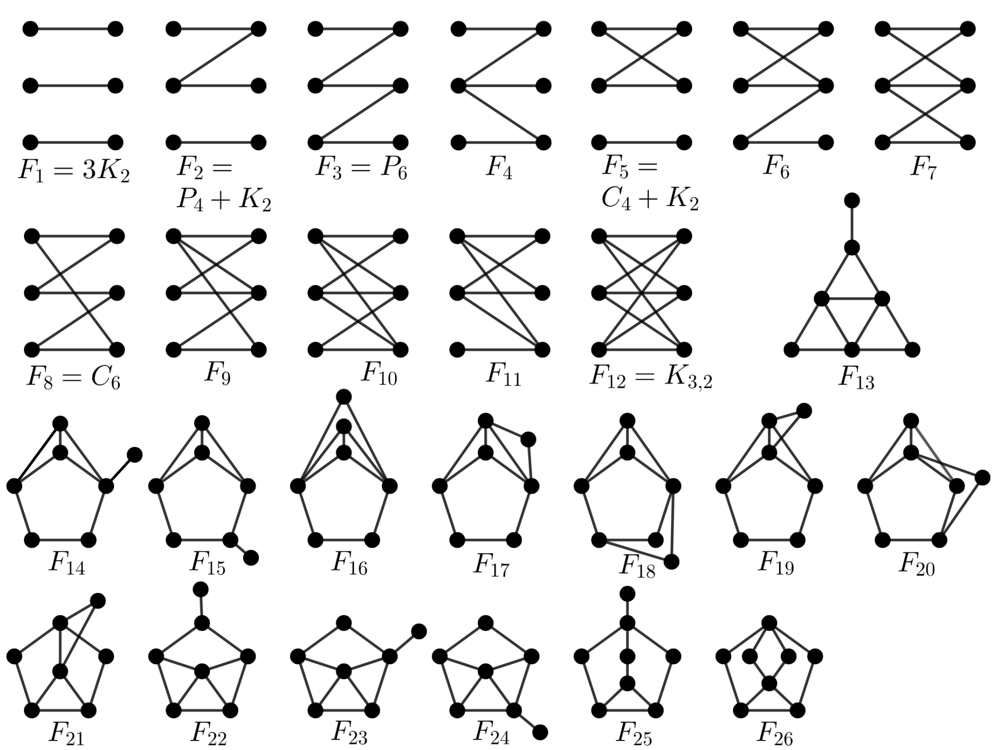

Figure 2.1: These graphs, together with their complements, comprise the set $\mathcal{F}$ of forbidden induced subgraphs of 1-HNG.

Our proof of Theorem 2.9 will separately address graphs that do and do not contain an induced C_5. In anticipation of the former, we introduce notation and a proposition summarizing the contents of $\mathcal{F}_\mathcal{C}$. If a graph G contains no element of $\mathcal{F}$ as an induced subgraph, then the possible coexisting vertices outside C_5 and their adjacencies are limited.

Let G be a graph containing vertices $C = \{c_1, c_2, c_3, c_4, c_5\} \in V(G)$ that induce a copy of C_5 along edges $c_1c_2, c_2c_3, c_3c_4, c_4c_5$, and c_1c_5. (We will choose such a subgraph C frequently throughout this chapter, and always with the given edge set.) Let $D = V(G) - C$; we say that a vertex $v \in D$ is of type $N(v) \cap C$, according to its adjacencies in C_5, such as $\{c_1, c_2, c_4\}, \{c_3, c_5\}$, or even $\emptyset$. Multiple vertices in D may be of the same type. The automorphism group of C_5 is the dihedral group on 5 elements. Throughout, we will work up to symmetry by this automorphism group, and relabel C, up to symmetry, so that any chosen vertex or vertices are of lowest possible type. For instance, if we assume the existence of a vertex with one neighbor in C, we would call it of type $\{c_1\}$, up to symmetry. With a vertex of type $\{c_2, c_4\}$ or $\{c_2, c_5\}$, we would apply the requisite relabeling of C to have a vertex of type $\{c_1, c_3\}$.

The following remark summarizes the pairs of vertices, up to symmetry, not in C_5 that can occur within D without containing an induced graph from $\mathcal{F}$.

Proposition 2.11. *Let G be a graph containing no element of $\mathcal{F}$ as an induced subgraph, and let $C = \{c_1, c_2, c_3, c_4, c_5\} \in V(G)$ induce a copy of C_5 with edges c_1c_2, c_2c_3, c_3c_4, c_4c_5, and c_1c_5. Let $D = V(G) - C$ and let $v \in D$.*

If v is of type $\emptyset$, then for all $w \in D$, either w is adjacent to v and of type $\{c_1, c_2, c_3\}$, $\{c_1, c_2, c_3, c_4\}$ or $\{c_1, c_2, c_3, c_4, c_5\}$, up to symmetry, or w is non-adjacent to v and of any type.

If v is of type $\{c_1\}$, up to symmetry, then for all $w \in D$, either w is adjacent to v and of type $\{c_3, c_4\}$ or $\{c_1, c_2, c_3, c_4, c_5\}$, or w is non-adjacent to v and of type $\emptyset, \{c_1\}$, $\{c_1, c_2\}$, $\{c_1, c_3\}$, $\{c_1, c_4\}$, $\{c_1, c_5\}$, $\{c_3, c_4\}$, $\{c_1, c_2, c_4\}$, $\{c_1, c_2, c_5\}$, $\{c_1, c_3, c_4\}$, $\{c_1, c_3, c_5\}$, or $\{c_1, c_2, c_3, c_4, c_5\}$.

If v is of type $\{c_1, c_2\}$, up to symmetry, then for all $w \in D$, either w is adjacent to v and of type $\{c_4\}$, $\{c_1, c_2\}$, $\{c_1, c_2, c_3\}$, $\{c_1, c_2, c_4\}$, $\{c_1, c_2, c_5\}$, $\{c_1, c_2, c_3, c_4\}$, $\{c_1, c_2, c_3, c_5\}$, $\{c_1, c_2, c_4, c_5\}$ or $\{c_1, c_2, c_3, c_4, c_5\}$, or w is non-adjacent to v and of type $\emptyset$, $\{c_1\}$, $\{c_4\}$, $\{c_1, c_2\}$,$\{c_1, c_4\}$, $\{c_2, c_4\}$, or $\{c_1, c_2, c_4\}$.

If v is of type $\{c_1, c_3\}$, up to symmetry, then for all $w \in D$, either w is adjacent to v and of type $\{c_1, c_3, c_4, c_5\}$ or $\{c_1, c_2, c_3, c_4, c_5\}$, or w is non-adjacent to v and of type $\emptyset$, $\{c_1\}$, $\{c_3\}$, $\{c_1, c_3\}$, $\{c_1, c_4\}$, $\{c_1, c_5\}$, $\{c_3, c_4\}$, $\{c_3, c_5\}$, $\{c_1, c_3, c_4\}$, $\{c_1, c_3, c_5\}$, or $\{c_1, c_2, c_3, c_4, c_5\}$.

If v is of type $\{c_1, c_2, c_3\}$, up to symmetry, then for all $w \in D$, either w is adjacent to v and of type $\emptyset$, $\{c_1, c_2\}$, $\{c_2, c_3\}$, $\{c_1, c_2, c_3\}$, $\{c_1, c_2, c_4\}$, $\{c_1, c_2, c_5\}$, $\{c_2, c_3, c_4\}$, $\{c_2, c_3, c_5\}$, $\{c_1, c_2, c_3, c_4\}$, $\{c_1, c_2, c_3, c_5\}$ or $\{c_1, c_2, c_3, c_4, c_5\}$, or w is non-adjacent to v and of type $\emptyset$ or $\{c_2\}$.

If v is of type $\{c_1, c_2, c_4\}$, up to symmetry, then for all $w \in D$, either w is adjacent to v and of type $\{c_1, c_2\}$, $\{c_1, c_2, c_3\}$, $\{c_1, c_2, c_4\}$, $\{c_1, c_2, c_5\}$, $\{c_1, c_2, c_3, c_4\}$, $\{c_1, c_2, c_3, c_5\}$, $\{c_1, c_2, c_4, c_5\}$ or $\{c_1, c_2, c_3, c_4, c_5\}$, or w is non-adjacent to v and of type $\emptyset$, $\{c_1\}$, $\{c_2\}$, $\{c_4\}$, $\{c_1, c_2\}$, $\{c_1, c_4\}$, $\{c_2, c_4\}$, $\{c_1, c_2, c_4\}$, or $\{c_1, c_2, c_3, c_5\}$.

If v is of type $\{c_1, c_2, c_3, c_4\}$, up to symmetry, then for all $w \in D$, either w is adjacent to v and of type $\{c_1, c_2\}$, $\{c_1, c_4\}$, $\{c_2, c_3\}$, $\{c_3, c_4\}$, $\{c_1, c_2, c_3\}$, $\{c_2, c_3, c_4\}$, $\{c_1, c_2, c_4\}$, $\{c_1, c_3, c_4\}$, $\{c_2, c_3, c_5\}$, $\{c_1, c_2, c_3, c_4\}$ or $\{c_1, c_2, c_3, c_4, c_5\}$, or w is non-adjacent to v and of type $\emptyset$ or $\{c_2, c_3, c_5\}$.

If v is of type $\{c_1, c_2, c_3, c_4, c_5\}$, then for all $w \in D$, either w is adjacent to v and of any type, or w is non-adjacent to v and of type $\emptyset$, $\{c_1\}$, $\{c_2\}$, $\{c_3\}$, $\{c_4\}$, $\{c_5\}$, $\{c_1, c_3\}$, $\{c_1, c_4\}$, $\{c_2, c_4\}$, $\{c_2, c_5\}$, or $\{c_3, c_5\}$.

Proof. It is straightforward to verify the result: we provide pseudocode below for an algorithm to do so. For v of any type out of $\emptyset$, $\{c_1\}$, $\{c_1, c_2\}$, $\{c_1, c_3\}$, $\{c_1, c_2, c_3\}$, $\{c_1, c_2, c_4\}$, $\{c_1, c_2, c_3, c_4\}$, or $\{c_1, c_2, c_3, c_4, c_5\}$, if w is of a listed type and adjacency to v, then the subgraph induced on vertices
$\{v, w, c_1, c_2, c_3, c_4, c_5\}$ is neither isomorphic to nor contains an element of $\mathcal{F}$. If w is not of a listed type and adjacency to v, then the subgraph induced on vertices $\{v, w, c_1, c_2, c_3, c_4, c_5\}$ is isomorphic to or contains an element of $\mathcal{F}$. □

Let $N_G(v)$ denote the neighbor set of v in graph G. We present an algorithm for verification of Theorem 2.11. The algorithm produces all possible pairs of vertex types; the user must delete those equivalent up to symmetry.

The algorithm requires as input the elements of $\mathcal{F}$. We first initialize a graph S with vertex set $\{c_1, c_2, c_3, c_4, c_5, c_6, c_7\}$ and edge set $\{c_6c_1, c_6c_2, c_6c_3, c_6c_4, c_6c_5, c_6c_7, c_7c_1, c_7c_2, c_7c_3, c_7c_4, c_7c_5\}$, such that c_6 and c_7 are dominating vertices, and no edges are present among the other five vertices (line 1). From here we compute all possible subsets of the edge set $E(S)$ (line 2), then combine each with the edges of a fixed five-cycle on vertex set $\{c_1, \ldots, c_5\}$ to form all possible graphs on seven vertices with a fixed copy of C_5 (line 3). Within those graphs, we consider equivalence classes by isomorphism (with c_6 and c_7 fixed or mapping to each other), and retain the canonical representative of each class, with lexicographically smallest neighbor sets for c_6 and c_7 (line 4). We call this set D.

We initialize the return set R (line 5), then for each graph in D, we check if it contains an element of $\mathcal{F}$ as an induced subgraph (lines 6-7). If not, we append it to R along with a codification of whether c_6 and c_7 are adjacent in the graph (lines 8 to 11). The algorithm concludes by producing the list of permissible pairs of vertex types (written as neighbor sets), together with whether c_6 and c_7 are adjacent (line 12). The algorithm can be found on the following page.

We will use Theorem 2.11 profusely throughout our proof of Theorem 2.9, which we now take up.

Proof. The reverse direction is straightforward to check: for each of the 52 graphs in $\mathcal{F}$, each has $\chi(G) + \theta(G) = |G| - 1$, but for all proper induced subgraphs H, $\chi(H) + \theta(H) \geq |H|$.

For the forward direction, let G be a graph containing none of the graphs in $\mathcal{F}$ as an induced subgraph. We consider separately graphs that do and do not contain an induced subgraph isomorphic to C_5. Suppose, first, that G does not contain C_5 as an induced subgraph. Hence by Theorem 2.10, G is sum-perfect, so $\omega(G) + \alpha(G) \geq n$. Since $\omega(G) \leq \chi(G)$ and $\alpha(G) \leq \theta(G)$, it follows that $\chi(G) + \theta(G) \geq n$ and therefore $G \in$ 1-HNG.

Algorithm Permissible Vertex Type Pairs

Require: The elements of $\mathcal{F}$.

Ensure: The possible types of the two vertices not contained in C_5 and whether that pair of vertices may be adjacent to each other (indicated by a 0 if they may be non-adjacent and a 1 if they may be adjacent).

$S := \{\{c_1, c_2, c_3, c_4, c_5, c_6, c_7\}, \{c_6c_1, c_6c_2, c_6c_3, c_6c_4, c_6c_5, c_6c_7, c_7c_1, c_7c_2, c_7c_3, c_7c_4, c_7c_5\}$

$A := \{E : E \subseteq E(S)\}$

$B := \{(V(G), E \cup \{c_1c_2, c_2c_3, c_3c_4, c_4c_5, c_1c_5\}) : E \in A\}$

$D := \{G \in B :$ for all $H \in B$ such that there exists an isomorphism $f : V(G) \to V(H)$ mapping $\{v_6, v_7\}$ to itself, $(N_G(v_6), N_G(v_7))$ is lexicographically less than or equal to $(N_H(v_6), N_H(v_7))$.

$R := \emptyset$

for all $G \in D$ **do**

    if G contains no element of $\mathcal{F}$ as an induced subgraph **then**

        (if $c_6c_7 \notin E(G)$**)** $a := 0$ **or (if** $c_6c_7 \in E(H)$**)** $a := 1$

        $R := R \cup \{(N_G(c_6) - \{c_7\}, N_G(c_7) - \{c_6\}, a)\}$

    end if

end for

return R

Otherwise, let G contain an induced copy C of C_5. Suppose that vertices $\{c_1, c_2, c_3, c_4, c_5\} \in V(G)$ induce a copy of C_5 with edges v_1v_2, v_2v_3, v_3v_4, v_4v_5, and v_1v_5. Let $C = \{c_1, c_2, c_3, c_4, c_5\}$ and $D = V(G) - C$. In any graph, isolated vertices are θ-distinct and dominating vertices are χ-distinct. We may assume by Proposition 2.7 that G has neither isolated nor dominating vertices. It is also straightforward to check that if $|D| \leq 2$, then $G \in 1\text{-}HNG$, so we may assume $|D| \geq 3$.

We will exploit Proposition 2.11 to identify possible vertex types in G. Then, we show either (i), there exists a θ-distinct vertex in G, or (ii), there exists some $v \in V(G)$ whose neighborhood is isolated and no vertex in it is θ-distinct in $G - \{v\}$. If (i) holds, G cannot be a forbidden induced subgraph by Proposition 2.7, and if (ii) holds, Corollary 2.8 implies the same. The proof consists of four cases: first, where there exists a vertex of type $\emptyset$. Since the forbidden induced subgraphs of $1\text{-}HNG$ are closed under complementation (see Remark 2.4), we can assume for subsequent cases that no vertex is of type $\emptyset$ or $\{c_1, c_2, c_3, c_4, c_5\}$. The second case is where there exists a vertex of type $\{c_1\}$, up to symmetry. Again, for subsequent cases, we may assume there are no vertices of types $\{c_1\}$, $\{c_2\}$, $\{c_3\}$, $\{c_4\}$, $\{c_5\}$, $\{c_1, c_2, c_3, c_4\}$, $\{c_1, c_2, c_3, c_5\}$, $\{c_1, c_2, c_4, c_5\}$, $\{c_1, c_3, c_4, c_5\}$, or $\{c_2, c_3, c_4, c_5\}$. The third case is where there exists a vertex of type $\{c_1, c_3\}$, up to symmetry. For the fourth case, we may assume there are only vertices of types $\{c_1, c_2\}$, $\{c_1, c_5\}$, $\{c_2, c_3\}$, $\{c_3, c_4\}$, $\{c_4, c_5\}$, $\{c_1, c_2, c_4\}$, $\{c_1, c_3, c_4\}$, $\{c_1, c_3, c_5\}$, $\{c_2, c_3, c_5\}$, or $\{c_2, c_4, c_5\}$, and suppose up to symmetry that there exists a vertex of type $\{c_1, c_2\}$.

Case 1: D contains a vertex of type $\emptyset$.

Let Z be the set of vertices of type $\emptyset$ and let $N(Z)$ be the set of vertices with a neighbor in Z. By Proposition 2.11, Z induces a stable set, and by assumption, both Z and $N(Z)$ are nonempty (else a vertex in Z would be isolated). By Proposition 2.11, $N(Z)$ can contain vertices of types $\{c_1, c_2, c_3\}, \{c_1, c_2, c_3, c_4\}$, and $\{c_1, c_2, c_3, c_4, c_5\}$, up to symmetry. Any vertices of these types must be adjacent by Proposition 2.11, so $N(Z)$ induces a clique.

We now show $N(Z)$ is complete to $B = D - Z - N(Z)$. First, let $w \in N(Z)$ be a vertex of type $\{c_1, c_2, c_3\}$, up to symmetry. Proposition 2.11 implies w is complete to B except for vertices of type $\{c_2\}$. However, if there exists a vertex of type $\{c_2\}$, G contains F_2 as an induced subgraph. By Proposition 2.11, D contains no vertices of types $\{c_1\}, \{c_3\}, \{c_4\}, \{c_5\}, \{c_1, c_3\}, \{c_1, c_4\}, \{c_2, c_4\}, \{c_2, c_5\}$, or $\{c_3, c_5\}$, so any vertex of type $\{c_1, c_2, c_3, c_4, c_5\}$ in $N(Z)$ is also complete to B.

Second, let $w \in N(Z)$ be of type $\{c_1, c_2, c_3, c_4\}$. Proposition 2.11 implies w is complete to B except for possibly vertices of type $\{c_2, c_3, c_5\}$. However, given a vertex x of type $\{c_2, c_3, c_5\}$ not adjacent to w, G contains F_{24} as an induced subgraph. Furthermore, by Proposition 2.11, D contains no vertices of types $\{c_1\}$, $\{c_2\}$ $\{c_3\}$, $\{c_4\}$, $\{c_5\}$, $\{c_1, c_3\}$, $\{c_2, c_4\}$, $\{c_2, c_5\}$, or $\{c_3, c_5\}$. More subtly, D also does not contain a vertex of type $\{c_1, c_4\}$ that is not adjacent to some vertex of type $\{c_1, c_2, c_3, c_4, c_5\}$, else G contains an induced $\overline{F_2}$. Thus any vertex of type $\{c_1, c_2, c_3, c_4, c_5\}$ in $N(Z)$ is complete to B.

The third and final option is that $N(Z)$ contains only vertices of type $\{c_1, c_2, c_3, c_4, c_5\}$. By Proposition 2.11, any vertex of type $\{c_1, c_2, c_3, c_4, c_5\}$ is adjacent to all other vertices in B except possibly those of type $\{c_1\}$ and $\{c_1, c_3\}$, up to symmetry. If there exists a vertex of type $\{c_1, c_2, c_3, c_4, c_5\}$ that is non-adjacent to a vertex of type $\{c_1\}$ in D, up to symmetry, then $\overline{G}$ contains a vertex of type $\emptyset$ with a neighbor of type $\{c_1, c_2, c_3\}$. If there exists a vertex of type $\{c_1, c_2, c_3, c_4, c_5\}$ that is non-adjacent to a vertex of type $\{c_1, c_3\}$ in D, up to symmetry, then $\overline{G}$ contains a vertex of type $\emptyset$ with a neighbor of type $\{c_1, c_2, c_3, c_4\}$. Since 1-HNG is closed under complementation, we can assume $N(Z)$ contains a vertex of type $\{c_1, c_2, c_3\}$ or $\{c_1, c_2, c_3, c_4\}$, both resolved above. Hence we can proceed with the assumption that $N(Z)$ is complete to B.

Fix a minimum proper clique covering of $C \cup B$. We can extend this to a minimum proper clique covering of $C \cup B \cup N(Z)$ by adding $N(Z)$ to the clique containing c_2. Since $N(Z)$ is complete to B, the clique covering is proper. This extends to a proper clique covering of G by giving each vertex in Z a distinct clique, so $\theta(G) \leq \theta(C \cup B) + |Z|$. Since Z induces a stable set and is isolated to $C \cup B$, $\theta(G) \geq \theta(C \cup B) + |Z|$. We conclude the specified clique covering is minimum. Since it is v-distinct for any $v \in Z$, Proposition 2.7 implies that G is not a forbidden induced subgraph of 1-HNG.

Case 2: D contains a vertex of type $\{c_1\}$.

Suppose that D contains no vertices of type $\emptyset$ or $\{c_1, c_2, c_3, c_4, c_5\}$, but does contain a vertex of type $\{c_1\}$, up to symmetry. By Proposition 2.11, D may also contain vertices of type $\{c_1\}$, $\{c_1, c_2\}$, $\{c_1, c_3\}$, $\{c_1, c_4\}$, $\{c_1, c_5\}$, $\{c_3, c_4\}$, $\{c_1, c_2, c_4\}$, $\{c_1, c_2, c_5\}$, $\{c_1, c_3, c_4\}$, and $\{c_1, c_3, c_5\}$. Of these, vertices of type $\{c_1\}$ may only be adjacent to vertices of type $\{c_3, c_4\}$.

Suppose there exists v of type $\{c_1\}$ not adjacent to any vertex of type $\{c_3, c_4\}$, so $N(v) = \{c_1\}$ is a stable set. Suppose for a contradiction there exists a c_1-distinct clique covering of $H = G - \{v\}$. Since $N(c_2) \subseteq N(c_1) - \{c_3\}$ with the exception of c_3, it follows that c_2 and c_3 share a clique. By an analogous argument, c_4 and c_5 share a clique, and so no vertex in $H - C_5$ shares a clique with any vertex in C_5. By assumption, $|D| \geq 3$, so $|H - C_5|$ is nonempty. Choose a clique K in $H - C_5$; there are three possibilities:

(i) K has no vertex of type $\{c_3, c_4\}$. If so, c_1 is complete to K and can be added to it, contradicting the clique covering's minimality.

(ii) K has a vertex of type $\{c_3, c_4\}$ but no vertex of type $\{c_1\}$. If so, K contains only vertices of types $\{c_3, c_4\}$ and $\{c_1, c_3, c_4\}$, so we add c_3 to K and combine c_1 and c_2 into one clique, contradicting the assumption that the clique covering is minimum.

(iii) K has a vertex of type $\{c_3, c_4\}$ and a vertex of type $\{c_1\}$. These two vertices, together with C_5 and v, induce F_{25} or F_{26} in G, for a contradiction.

Instead, we conclude H has no c_1-distinct clique covering, and Corollary 2.8 implies G is not a forbidden induced subgraph of 1-HNG.

Otherwise, every vertex of type $\{c_1\}$ is adjacent to a vertex of type $\{c_3, c_4\}$. If there exist two vertices of type $\{c_1\}$, they cannot be adjacent by Proposition 2.11. If they are adjacent to a common vertex of type $\{c_3, c_4\}$, G contains an induced F_{26}, and if they are adjacent to different vertices of type $\{c_3, c_4\}$, G contains an induced F_{25}. Thus G has exactly one vertex v of type $\{c_1\}$. If there exist two vertices of type $\{c_3, c_4\}$ adjacent to v, then G contains either F_3 or F_4 as an induced subgraph. Hence v is adjacent to exactly one vertex w of type $\{c_3, c_4\}$. The vertex w is adjacent to all other vertices of type $\{c_3, c_4\}$, else G contains an induced F_3, and to all vertices of type $\{c_1, c_3, c_4\}$, else G contains an induced F_{17}. Hence w is complete to D.

Here, $N(v) = \{c_1, w\}$, which induces a stable set. We claim both c_1 and w are not θ-distinct in $H = G - \{v\}$. Suppose for a contradiction there exists a w-distinct clique covering of H. Thus $D - \{v\}$ shares a clique with c_1, c_2 and c_3 share a clique, and c_4 and c_5 share a clique. However, the clique covering is not minimum: put c_1 with c_2 and add c_3 and w to $D - \{v\}$ to produce a strictly smaller clique covering of H. The covering is proper since w is complete to D. Suppose for a contradiction there exists a c_1-distinct clique covering of H. Thus c_2 and c_3 share a clique in that covering, and c_4 and c_5 share a clique. Put c_3 in w's clique and add c_1 to c_2's clique to produce a proper, strictly smaller clique covering. Instead, we conclude H has no c_1-distinct or w-distinct clique covering, and Corollary 2.8 implies G is not a forbidden induced subgraph of 1-HNG.

Case 3: D **contains a vertex of type** $\{c_1, c_3\}$**.**

By ruling out the previous cases, D contains no vertices with zero, one, four, or five neighbors in C, but does contain a vertex v of type $\{c_1, c_3\}$, up to symmetry and complementation. By Proposition 2.11, D may also contain vertices of type $\{c_1, c_3\}, \{c_1, c_4\}, \{c_1, c_5\}, \{c_3, c_4\}$, $\{c_3, c_5\}, \{c_1, c_3, c_4\}$, and $\{c_1, c_3, c_5\}$. Any vertices of type $\{c_1, c_3\}$, $\{c_1, c_4\}$, or $\{c_3, c_5\}$ are isolated in D. However, D contains vertices of type $\{c_1, c_4\}, \{c_3, c_4\}$, or $\{c_1, c_3, c_4\}$ if and only if it does not contain vertices of type $\{c_1, c_5\}, \{c_3, c_5\}$, or $\{c_1, c_3, c_5\}$. Up to symmetry, we may assume the former.

Here, $N(v) = \{c_1, c_3\}$, which induces a stable set. We claim both c_1 and c_3 are not θ-distinct in $H = G - \{v\}$. Suppose for a contradiction there exists a c_1-distinct clique covering of H. Thus c_2 and c_3 share a clique, c_4 and c_5 share a clique, D contains no vertex of type $\{c_1, c_4\}$. and no vertex in $H - C_5$ shares a clique with any vertex in C_5. Since $H - C_5$ is nonempty and complete to c_3, join c_3 to any clique in $H - C_5$ and combine c_1 and c_2 to produce a proper clique covering of H with fewer cliques. If there exists a c_3-distinct clique covering of H, then c_1 and c_2 share a clique. Since c_3 cannot be added to any other clique, $H - C_5$ contains only vertices of type $\{c_1, c_4\}$ and by assumption must have at least 2 such vertices. However, the proper clique covering of H with c_2 and c_3 sharing a clique, c_1 and c_5 sharing a clique, and c_4 in a clique with some vertex of type $\{c_1, c_4\}$ contains fewer cliques. Instead, we conclude H has no c_1-distinct or c_3-distinct clique covering, and Corollary 2.8 implies G is not a forbidden induced subgraph of 1-HNG.

Case 4: D **contains a vertex of type** $\{c_1, c_2\}$**.**

Here, D contains only vertices of types $\{c_1, c_2\}$, $\{c_1, c_5\}$, $\{c_2, c_3\}$, $\{c_3, c_4\}$, $\{c_4, c_5\}$, $\{c_1, c_2, c_4\}$, $\{c_1, c_3, c_4\}$, $\{c_1, c_3, c_5\}$, $\{c_2, c_3, c_5\}$ and $\{c_2, c_4, c_5\}$. Without loss of generality, suppose that D contains a vertex v of type $\{c_1, c_2\}$. Thus D can also contain only vertices of types $\{c_1, c_2\}$ and $\{c_1, c_2, c_4\}$, which may or may not be adjacent to one another. With D so limited, we move directly to a clique covering.

The set $N(c_3) = \{c_2, c_4\}$ induces a stable set, and we claim that neither c_2 nor c_4 is θ-distinct in $H = G - \{c_3\}$. Since c_2 is complete to $\{v\} \cup N(v)$, no clique covering of H is c_2-distinct, else c_2 could be joined to v's clique. If a clique covering is c_4-distinct, then c_5 and c_1 share a clique. Since c_1 is complete to $\{v\} \cup N(v)$, add c_1 to v's clique and combine c_4 and c_5 into one clique to produce a proper clique covering of H with fewer cliques, for a contradiction. We conclude by Corollary 2.8 that G is not a forbidden induced subgraph of 1-HNG. $\square$

2.4 Graphs in 1-HNG are Apex-Perfect

We prove that graphs in 1-HNG are apex-perfect, that is, that the deletion of some vertex yields a perfect graph. This is sufficient to show that 1-HNG is χ-bounded by the function $f(x) = x + 1$.

Theorem 2.12. *If $G \in$ 1-HNG, then there exists $v \in G$ such that $G - \{v\}$ is a perfect graph.*

Proof. Let $G \in$ 1-HNG. The graphs $C_6, \overline{C_6}, P_6$, and $\overline{P_6}$ are elements of $\mathcal{F}$, so G contains none of these, and hence G contains no cycle C_m or cycle-complement $\overline{C_m}$ for $m \geq 6$. Thus G is perfect if and only if G contains no induced C_5. From here, we consider the structure of induced copies of C_5 in G to show that all induced copies of C_5 must intersect at a single vertex.

We may assume that G is imperfect (i.e., not a perfect graph) and let $C = \{c_1, c_2, c_3, c_4, c_5\} \in V(G)$ induce C_5 with edges $c_1c_2, c_2c_3, c_3c_4, c_4c_5$, and c_1c_5. If G has exactly one induced copy of C_5, then the theorem holds, so we may assume another exists. In the remainder of the proof, we consider vertex types that can be included in another C_5, and show that in each case G is apex-perfect.

The neighbor set of a vertex of type $\emptyset$ induces a clique, as shown in the proof of Case 1 of Theorem 2.9. Hence no vertices of type $\emptyset$ or, by complement, of type $\{c_1, c_2, c_3, c_4, c_5\}$ are elements of any induced C_5.

Suppose there exists a vertex v of type $\{c_1\}$, up to symmetry and complementation, contained in an induced subgraph X isomorphic to C_5. The only neighbors of v in G are c_1 and possibly some vertices of type $\{c_3, c_4\}$. Since our work in the proof of Case 2 of Theorem 2.9 showed that a vertex of type $\{c_1\}$ cannot have two (nonadjacent) neighbors of type $\{c_3, c_4\}$, we conclude that v's neighbors in X are c_1 and a vertex w of type $\{c_3, c_4\}$. If, for a contradiction, there exists a subset $Y \subseteq V(G)$ inducing C_5 with $c_1 \notin Y$, then by Proposition 2.11, the elements of Y must be some of c_2, c_3, c_4, c_5 and vertices of types $\{c_3, c_4\}$ and $\{c_1, c_3, c_4\}$. Let U denote the set of these possible elements. Since c_2 and c_5 have one neighbor in U, they cannot be in Y, and since c_3 and c_4 are adjacent to all but one element of U, they also cannot be in Y. Hence Y is comprised entirely of vertices of types $\{c_3, c_4\}$ and $\{c_1, c_3, c_4\}$. Suppose up to complementation that at least three vertices of Y are of type $\{c_3, c_4\}$. If three consecutive vertices of Y, say v_1, v_2, v_3 are of type $\{c_3, c_4\}$, then the induced subgraph of G on vertex set $\{v_1, v_2, v_3, c_1, c_2, c_5\}$ is forbidden by Theorem 2.9. Otherwise, up to symmetry v_1, v_2, and v_4 are of type $\{c_3, c_4\}$ and v_3, v_5 are of type $\{c_1, c_3, c_4\}$. Here, the induced subgraph on $\{v_2, v_3, v_4, c_1, c_2, c_5\}$ is forbidden by Theorem 2.9. In either case, we obtain the contradiction that $G \notin$ 1-HNG, so conclude instead that all induced copies of C_5 in G contain vertex c_1. Thus its deletion renders the graph perfect. Hence if G contains a vertex of types $\{c_1\}$ or $\{c_1, c_2, c_3, c_4\}$, up to symmetry, contained in an induced C_5, the graph is apex-perfect. We may assume for the remainder of the proof that G contains no such vertex.

If a vertex v of type $\{c_1, c_3\}$, up to symmetry and complementation, is in an induced C_5, then by Proposition 2.11, its two neighbors in the C_5 must be c_1 and c_3. If, for a contradiction, there exists a subset $Y \subseteq V(G)$ inducing C_5 with $c_1 \notin Y$, then by Proposition 2.11, the elements of Y must be some of c_2, c_3, c_4, c_5 and vertices of types $\{c_1, c_3\}$, $\{c_1, c_4\}$, $\{c_1, c_5\}$, $\{c_3, c_4\}$, $\{c_3, c_5\}$, $\{c_1, c_3, c_4\}$, and $\{c_1, c_3, c_5\}$. Let U denote the set of these possible elements. Since c_2 has one neighbor in U, it cannot be in Y. Since we've shown that any vertex of type $\{c_x, c_{x+2}\}$, with subscripts $\mod 5$ must have as its C_5 neighbors c_x and c_{x+2}, it follows that Y does not contain vertices of type $\{c_1, c_3\}$ or $\{c_1, c_4\}$. If Y contains a vertex of type $\{c_3, c_5\}$, then Y contains c_3, c_5, and vertices of types Proposition 2.11 restricts Y to containing c_3, c_5, and vertices of types $\{c_1, c_5\}$ and $\{c_1, c_3, c_5\}$. Since c_5 is adjacent to all but c_3, Y does not induce C_5. Thus Y does not contain a vertex of type $\{c_3, c_5\}$. If instead Y contains a vertex of type $\{c_3, c_4\}$ or $\{c_1, c_3, c_4\}$, then Y is restricted to containing c_3, c_4 and vertices of types $\{c_3, c_4\}$ or $\{c_1, c_3, c_4\}$, which was shown above to not be possible. Lastly, if Y contains a vertex of type $\{c_1, c_5\}$ or $\{c_1, c_3, c_5\}$, then in order to induce C_5, Y must contain two vertices v_1, v_2 of type $\{c_1, c_5\}$, two vertices v_3, v_5 of type $\{c_1, c_3, c_5\}$, and c_3. However, G then contains the induced subgraph $\overline{F_{22}}$, for a contradiction. Instead, all induced copies of C_5 in G contain vertex c_1, and its deletion renders the graph perfect. Hence if G contains a vertex of types $\{c_1, c_3\}$ or $\{c_1, c_2, c_3\}$, up to symmetry, contained in an induced C_5, the graph is apex-perfect. We may assume for the remainder of the proof that G contains no such vertex.

Lastly, if there is a vertex v of type $\{c_1, c_2\}$, up to symmetry and complementation, in an induced C_5, then by Proposition 2.11, the other vertices in that induced C_5 must be some of c_1, c_2, and vertices of types $\{c_1, c_2\}$ and $\{c_1, c_2, c_4\}$. Since c_1 and c_2 are adjacent to all but c_4, they do not occur in this C_5. As shown above, the C_5 cannot contain only vertices of types $\{c_1, c_2\}$ and $\{c_1, c_2, c_4\}$, so c_4 must be included. This C_5 then contains two vertices each of types $\{c_1, c_2\}$ and $\{c_1, c_2, c_4\}$, and G contains $\overline{F_{22}}$ as an induced subgraph, for a contradiction. □

It follows that graphs in 1-HNG are χ-bounded by the function $f(x) = x + 1$, which is best possible.

Theorem 2.13. *If $G \in$ 1-HNG, then $\chi(G) \leq \omega(G) + 1$.*

Proof. Let $G \in$ 1-HNG. By Theorem 2.12, there exists $v \in G$ such that $\chi(G - \{v\}) = \omega(G - \{v\})$. By Remark 2.5, $\chi(G) \leq \chi(G - \{v\}) + 1$. Either $\omega(G - \{v\}) = \omega(G)$ or $\omega(G - \{v\}) = \omega(G) - 1$. In the first case, $\chi(G) \leq \chi(G - \{v\}) + 1 = \omega(G - \{v\}) + 1 = \omega(G) + 1$, and in the second case, $\chi(G) \leq \chi(G - \{v\}) + 1 = \omega(G - \{v\}) + 1 = \omega(G)$. Both yield the desired result. □

2.5 The Intersection of 1-HNG with Several Common Graph Classes

In this section we provide equivalent structural and forbidden (induced) subgraph characterizations of 1-HNG intersected with line graphs, claw-free graphs, and triangle-free graphs.

Beginning with line graphs, we define two sets that will provide our structural and forbidden subgraph characterizations. Given graphs A_1 to A_6 shown in Figure 2.2, let $\mathcal{A} = \{A_1, \ldots, A_6, 3P_3, P_5+P_3, P_7, C_6, K_4, C_4+P_3, 2K_3, 2K_{1,3}, K_{2,3}, K_3+K_{1,3}\}$. Given families L_1 to L_{11} in Figure 2.2, let $\mathcal{L}$ be the set of all graphs in one of families L_1 to L_{11}. We characterize the set of graphs whose line graphs are in 1-HNG as the set of graphs containing none of $\mathcal{A}$ as a (not necessarily induced) subgraph, and equivalently as the set of graphs contained in an element of $\mathcal{L}$.

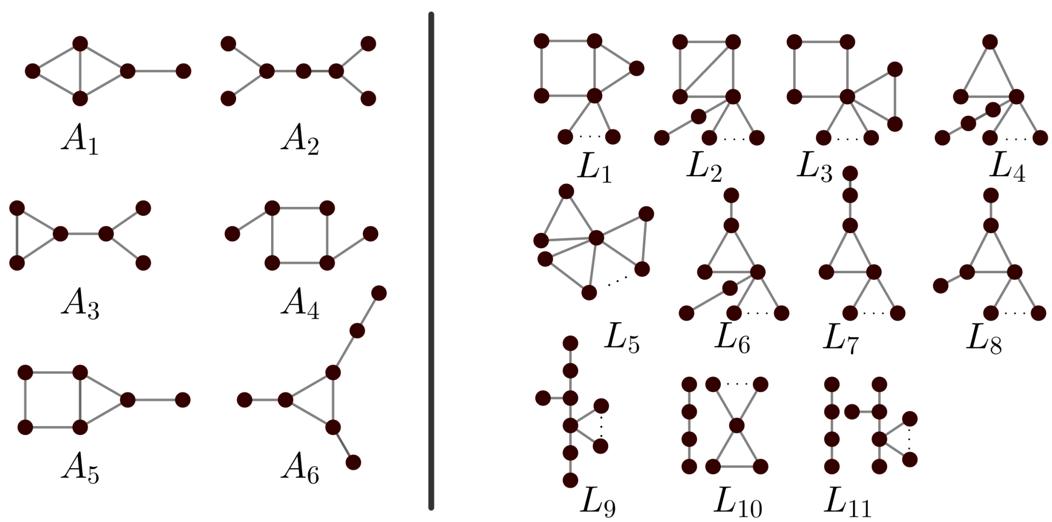

Figure 2.2: The set $\mathcal{A}$ comprises A_1 to A_6 as well as ten other small graphs. The set $\mathcal{L}$ comprises all graphs in families L_1 to L_{11}.

Theorem 2.14. *Given a graph G, the following are equivalent:*

(i) $L(G) \in$ 1-HNG.

(ii) *G contains none of $\mathcal{A}$ as a subgraph.*

(iii) G is a subgraph of a graph in $\mathcal{L}$.

Proof. (i) $\rightarrow$ (ii) It is straightforward to verify that for all $A \in \mathcal{A}$, the line graph $L(A) \in \mathcal{F}$. Thus if a graph G contains A as a subgraph, $L(G)$ contains the forbidden $L(A)$ as an induced subgraph, and by Theorem 2.9, $L(G) \notin 1\text{-}HNG$.

(ii) $\rightarrow$ (iii) Let G be a graph containing none of $\mathcal{A}$ as a subgraph. Since $C_6, P_7 \in \mathcal{A}$, it follows that G's largest cycle has 5 or fewer vertices, should one exist. It is clear that G can contain arbitrarily many connected components isomorphic to K_2, and so we restrict the subsequent analysis to edgewise nontrivial components. We split into cases according to the number of components containing at least two edges and the size of the largest cycle.

Case 1: The graph G has one edgewise nontrivial component, which contains a 5-cycle. Any added vertex with two neighbors in the 5-cycle either produces $C_6 \in \mathcal{A}$ as a subgraph if the neighbors are adjacent or $A_4 \in \mathcal{A}$ if the neighbors are not adjacent. Hence any vertex beyond the 5-cycle must have at most one neighbor in the 5-cycle. If two different vertices in the 5-cycle have added neighbors, then G contains either $P_7 \in \mathcal{A}$ (if the vertices are adjacent) or $A_2 \in \mathcal{A}$ (if not), so at most one vertex in the 5-cycle has an added neighbor. If there exists an edge e not incident to the 5-cycle, then e together with its path to the 5-cycle and the 5-cycle itself contain $P_7 \in \mathcal{A}$, so the only possible addition to the 5-cycle is a set of pendant vertices attached to one vertex. If the 5-cycle has two chords, then whether or not they are crossing chords, G contains $A_1 \in \mathcal{A}$ as a subgraph. It follows that G has at most one chord. A chord can be added if and only if it is incident to the vertex with pendant vertices, else G contains A_2 or $A_4 \in \mathcal{A}$. It can be added freely and we conclude that G is a subgraph of a graph in family L_1.

Case 2: The graph G has one edgewise nontrivial component, which contains no 5-cycle but does contain a 4-cycle. Any added vertex with two neighbors in the 4-cycle either produces C_5 (handled in Case 1) as a subgraph if the neighbors are adjacent or $K_{2,3} \in \mathcal{A}$ if the neighbors are not adjacent. Hence any added vertex has at most one neighbor in the 4-cycle. If two different vertices in the 4-cycle have added neighbors, then those two vertices must be adjacent, else G contains $A_4 \in \mathcal{A}$. If two adjacent vertices in the 4-cycle each have two added neighbors, then G contains $2K_{1,3} \in \mathcal{A}$, so instead only one vertex in the 4-cycle can have more than one additional neighbor. If there exists an edge e not incident to the 4-cycle and two vertices in the 4-cycle have additional neighbors, then e together with its path to the 4-cycle, the 4-cycle itself, and the additional neighbor contain $P_7 \in \mathcal{A}$. Hence either all additions to the 4-cycle are pendant vertices and all but one are appended to the same vertex, creating a subgraph of a graph in family L_1, or all additions to the 4-cycle are appended to the same vertex, which we call v.

However, we are limited in the subgraphs we can append to v: the set of vertices not included in the 4-cycle cannot contain P_3, else G contains $C_4 + P_3 \in \mathcal{A}$. Hence we are limited to appending isolated copies of P_3, K_3, and pendant vertices to v. With two copies of P_3 appended to v, G contains $P_5 + P_3 \in \mathcal{A}$, so we can append either one K_3 or one P_3 (and unlimited pendant vertices) to v. If the 4-cycle contains two chords, then G contains $K_4 \in \mathcal{A}$, so it has at most one chord. This chord must be incident to v, else G contains A_1. If there is a chord, then K_3 cannot be appended to v, else G contains $A_3 \in \mathcal{A}$. With a chord, G is thus a subgraph of a graph in family L_2, and without a chord, G is a subgraph of a graph in family L_3.

Case 3: The graph G has one edgewise nontrivial component, which contains K_3 but no larger cycle. Let $C = \{v_1, v_2, v_3\} \subseteq V(G)$ induce K_3. There is exactly one path from any other vertex to C, else G contains $2K_3 \in \mathcal{A}$ or some larger cycle, addressed above. Hence the subgraphs appended to each vertex in C are isolated from one another. If P_4 is appended with a vertex in C as an endpoint and G contains any other edge not incident to that vertex, then G contains either P_7 or $P_5 + P_3$. Hence if P_4 is appended to a vertex in C, G is a subgraph of a graph in family L_4.

A claw cannot be appended at a spoke to a vertex in C, since this would create $A_3 \in \mathcal{A}$ as a subgraph. Hence the subgraph appended to any vertex of C contains only isolated copies of P_3, K_3, and pendant vertices. If K_3 is appended to a vertex in C, then appending neighbors to any other vertex in (either copy of) K_3 yields A_3 as a subgraph. Any number of copies of K_3 can be appended at the same vertex, and we conclude that G is a subgraph of a graph in family L_5. If two copies of P_3 are appended to C at the same vertex, then no other vertex in C has an additional neighbor, else G contains $P_5 + P_3$. If two copies of P_3 are appended to different vertices, then G contains P_7. If two vertices in C each have two additional neighbors, then G contains A_2. If one vertex has a copy of P_3 appended and the other two vertices in C have additional neighbors, then G contains $A_6 \in \mathcal{A}$. Thus with a P_3 appended to a vertex in C, either that vertex can have a cluster of pendant vertices and G is a subgraph of a graph in L_6 or a different vertex has a cluster of pendant vertices and G is a subgraph of a graph in L_7. With no P_3 appended to a vertex in C, G is a subgraph of a graph in L_8.

Case 4: The graph G has one edgewise nontrivial component, which contains no cycles. If G contains P_6 on vertices $v_1, \ldots, v_6$ in that order, then it can have a cluster of pendant vertices appended to any non-endpoint vertex. No P_3 can be appended, else G contains P_7 or $P_5 + P_3$. Furthermore, two non-adjacent vertices cannot have appended vertices, else G contains A_2. If v_2 has a pendant vertex, then v_3 cannot, else G contains $P_5 + P_3$. If v_3 and v_4 each have two or more pendant vertices, then G contains $2K_{1,3}$. Thus G is isomorphic to a subgraph of P_6 together with a collection of pendant vertices to v_2, which is a subgraph of a graph in L_7; or G has a collection of pendant vertices to v_3 and a pendant vertex attached to v_4 (up to symmetry), and hence is a subgraph of a graph in L_9.

If G contains no induced P_6 but does contain an induced P_5 on vertices $v_1, \ldots, v_5$ in that order, then it can have adjoined copies of P_3, but only to v_3. The resulting graph, with a cluster of copies of P_3 adjoined to v_3, is a subgraph of a graph in L_5. Without an adjoined P_3, again G can only have one cluster of pendant vertices, and only neighboring vertices of the P_5 can have pendant vertices. The result is a subgraph a graph in L_9.

Case 5: The graph G has at least two edgewise nontrivial components. Since G cannot contain $3P_3 \in \mathcal{A}$ as an induced subgraph, G has at most two components with at least two edges. Components containing a single edge or vertex are trivial, and ignored henceforth. The nontrivial components cannot contain C_4 or P_5, since $C_4 + P_3$ and $P_5 + P_3$ are in $\mathcal{A}$. If one component has a triangle, there cannot be a copy of P_3 appended to one triangle vertex, additional neighbors appended to two triangle vertices, or a vertex of degree two or more appended to the triangle, since each would create a C_4 or P_5 subgraph. Hence a component with a triangle consists of, at most, a triangle with a cluster of pendant vertices adjoined to one vertex. The other component cannot contain a triangle or a claw since $2K_3, K_3 + K_{1,3} \in \mathcal{A}$, so H must be a subgraph of P_4. Thus G is a subgraph of a graph in L_{10}. Otherwise, both components are trees. If both components are subgraphs of P_4, G is a subgraph of a graph in L_{10}. Otherwise, exactly one component contains a claw. This component contains P_4 with a pendant vertex attached to one of the center vertices. Since both center vertices cannot have two or more pendant vertices simultaneously without producing a $2K_{1,3}$ subgraph, the component is a subgraph of P_4 together with a cluster of pendant vertices at one center vertex and a single pendant vertex attached to the other center vertex. The other nontrivial component is again a subgraph of P_4, and we conclude that G is a subgraph of a graph in L_{11}.

(iii) $\rightarrow$ (i) Compute the line graph of each graph in $\mathcal{L}$. By Theorem 2.9, each contains no forbidden induced subgraph, so is in 1-HNG. The same is true of its subgraphs. $\square$

In preparation for characterizing claw-free and triangle-free subsets of 1-HNG, we show that three families of graphs built from a copy of C_5 are in 1-HNG. The three families, D_1, D_2, and D_3, are shown in Figure 2.3. Each consists of an induced C_5 on vertices labeled $\{c_1, c_2, c_3, c_4, c_5\}$ together with a stable set of added vertices. In D_1, the added vertices may be of types $\{c_1, c_3\}$, $\{c_1, c_4\}$, and $\{c_1, c_3, c_4\}$. In D_2, the added vertices may be of types $\{c_4\}$, $\{c_1, c_4\}$, and $\{c_1, c_3, c_4\}$. In D_3, the added vertices may be of types $\{c_1\}$, $\{c_1, c_3\}$, and $\{c_1, c_4\}$.

Lemma 2.15. *All graphs in families D_1, D_2, or D_3 are in* 1-HNG.

Proof. Let G be a graph in one of these families, with a fixed induced C_5 on vertex set $C = \{c_1, c_2, c_3, c_4, c_5\}$ in that order. By Theorem 2.1 the addition to C of any three vertices from D does not induce an element of F_C. The choice of C_5 is immaterial: all induced copies of C_5 contain c_1, c_3, c_4, either c_2 or a vertex of type $\{c_1, c_3\}$, and either c_5 or a vertex of type

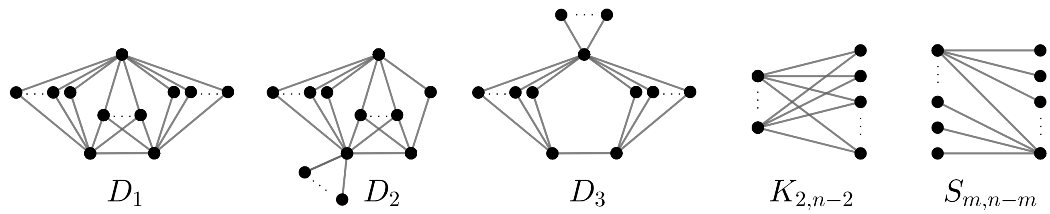

Figure 2.3: A claw-free graph in 1-HNG containing an induced C_5 is an induced subgraph of a graph in family $\overline{D_1}, \overline{D_2}$, or $\overline{D_3}$, possibly with dominating vertices. A triangle-free graph in 1-HNG is a subgraph of $K_{2,n-2}$ or $S_{m,n-m}$ for some $3 \leq m \leq n$, or an induced subgraph of D_3, possibly with isolated vertices.

$\{c_1, c_4\}$. With respect to any of these induced C_5, the types of vertices in $D = V(G) - C$ remain constant. Therefore G contains no element of F_C on any vertex subset.

Graphs in D_1 and D_2 may contain induced triangles, but all triangles share two common vertices, c_3 and c_4. Graphs in D_3 contain no induced triangles. In either case, G cannot contain an induced $\overline{F_1}, \ldots, \overline{F_{12}}, F_{13}$, or $\overline{F_{13}}$. See Figure 2.1 for these graphs.

If G is an element of D_1 or D_3, then the largest stable set containing c_1 has size 2. Hence if G contains an induced $F_1, \ldots, F_{11}$, or F_{12}, it cannot include c_1. Without c_1, however, every edge in G is incident to c_3 or c_4, so $\nu(G - \{c_1\}) = 2$. Hence G contains no induced $F_1, \ldots, F_{12}$. If G is an element of D_2, an analogous argument holds with respect to c_4.

Therefore G contains no induced element of $\mathcal{F}$, and Theorem 2.9 implies that $G \in$ 1-HNG. $\square$

Given the graphs F_i shown in Figure 2.1, define the set

$$\mathcal{B} = \{K_{1,3}, F_1, F_2, F_3, F_5, F_8, F_{13}, \overline{F_{21}}\} \cup \{\overline{F_1}, \overline{F_2}, \ldots, \overline{F_{13}}\}.$$

We use $\mathcal{B}$ as well as Lemma 2.15 to characterize the claw-free graphs in 1-HNG.

Theorem 2.16. *Given a graph G, the following are equivalent:*

(i) $G \in$ 1-HNG *and* G *is claw-free.*

(ii) G *contains no element of* $\mathcal{B}$ *as an induced subgraph.*

(iii) Either G *is perfect, claw-free, and in* 1-HNG*; or* $\overline{G}$ *is a graph in family* D_1, D_2*, or* D_3*, possibly with the addition of some dominating vertices.*

Proof. (i) $\to$ (iii) Let $G \in 1\text{-}HNG$ be claw-free. We may assume that G contains no isolated vertices. By Theorem 2.12, G is perfect if and only if it is C_5-free. If so, we are done, and if not, suppose G contains an induced C_5 on vertex set $C = \{c_1, c_2, c_3, c_4, c_5\}$ in that order. Let $D = V(G) - C$. Since G is claw-free, D cannot contain any vertex v of type $\{c_1\}, \{c_1, c_3\}$, or $\{c_1, c_3, c_4\}$, up to symmetry, else $\{c_1, c_2, c_5, v\}$ induces a claw. Additionally, D contains no vertex v of type $\emptyset$, since then by Proposition 2.11, its neighbor set contains a vertex w with two non-adjacent neighbors in C, inducing a claw.

By virtue of the remaining permissible types, Proposition 2.11 implies D must induce a clique. If D contains a vertex v of type $\{c_1, c_2\}$, up to symmetry, then Proposition 2.11 allows D contain vertices of types $\{c_1, c_2\}$, $\{c_1, c_2, c_3\}$, $\{c_1, c_2, c_5\}$, $\{c_1, c_2, c_3, c_4\}$, $\{c_1, c_2, c_3, c_5\}$, $\{c_1, c_2, c_4, c_5\}$ or $\{c_1, c_2, c_3, c_4, c_5\}$. However, if D contains a vertex w of type $\{c_1, c_2, c_3, c_5\}$ or $\{c_1, c_2, c_3, c_4, c_5\}$, then $\{v, w, c_3, c_5\}$ induces a claw. Since D cannot simultaneously contain vertices of types $\{c_1, c_2, c_5\}$ and $\{c_1, c_2, c_3, c_4\}$ or $\{c_1, c_2, c_4, c_5\}$ and $\{c_1, c_2, c_3, c_4\}$ by Proposition 2.11, it follows that D contains vertices of types from set $\{\{c_1, c_2\}, \{c_1, c_2, c_3\}, \{c_1, c_2, c_5\}\}$ or $\{\{c_1, c_2\}, \{c_1, c_2, c_3\}, \{c_1, c_2, c_3, c_4\}\}$. If the former, then $\overline{G}$ is in D_1, and if the latter, $\overline{G}$ is in D_2. If D contains no vertices of type $\{c_1, c_2\}$ up to symmetry, then the vertices of D are of types from the set $\{\{c_1, c_2, c_3\}, \{c_1, c_2, c_5\}, \{c_1, c_2, c_3, c_4\}, \{c_1, c_2, c_3, c_5\}, \{c_1, c_2, c_3, c_4, c_5\}\}$, up to symmetry. However, if D contains a vertex of type $\{c_1, c_2, c_3, c_4\}$, it cannot contain vertices of types $\{c_1, c_2, c_5\}$ or $\{c_1, c_2, c_3, c_5\}$. Up to symmetry, we conclude the vertices of D are from set $\{\{c_1, c_2, c_3\}, \{c_1, c_2, c_5\}, \{c_1, c_2, c_3, c_5\}, \{c_1, c_2, c_3, c_4, c_5\}\}$, and so G is in family D_3.

(iii) $\to$ (ii) Let G be a graph such that $\overline{G}$ is in family D_1, D_2, or D_3, possibly with dominating vertices. Since dominating and isolated vertices do not affect inclusion in $1\text{-}HNG$, suppose G has none. Lemma 2.15 implies G contains no element of $\mathcal{B}$ as an induced subgraph, except possibly the claw.

We now show that if G contains an induced C_5, it is claw-free. Suppose $\overline{G}$ is in D_1 or D_2. Since D induces a clique, the only stable set of at least three vertices is $\{v, c_3, c_5\}$, and no vertex is adjacent to all three of these vertices. Thus G contains no induced claw. Otherwise, if $\overline{G}$ is in D_3, then since D induces a clique, each vertex in D has at most two non-neighbors, which are adjacent. Thus $\alpha(G) = 2$ and G contains no induced claw.

(ii) $\to$ (i) Since $\mathcal{B}$ is precisely the subset of graphs in $\mathcal{F}$ not containing an induced claw, together with the claw, the result follows by Theorem 2.9. $\square$

In anticipation of characterizing triangle-free graphs of $1\text{-}HNG$, we provide a lemma about the structure of bipartite graphs with no $3K_2$ subgraph. By Theorem 2.9, this result also characterizes bipartite graphs in $1\text{-}HNG$. Let $S_{m,n-m}$ denote the double star graph with center vertices of degree m and $n - m$; see Figure 2.3.

Lemma 2.17. *A graph G is bipartite and has no $3K_2$ subgraph if and only if G is a subgraph of $K_{2,n-2}$ or $S_{m,n-m}$.*

Proof. Suppose G is a subgraph of a double star graph with non-pendant vertices v and w; hence G is bipartite. Since all edges of G are incident to v or w, G does not contain $3K_2$ as a subgraph.

Alternatively, suppose G is bipartite and does not contain $3K_2$ as a subgraph. If one side of any possible bipartition has two or fewer vertices, then G is a subgraph of $K_{2,n-2}$. Otherwise, in any bipartition $V(G) = A \cup B$ of G, both sides have three or more vertices, and suppose for a contradiction that G is not a subgraph of a double star graph. Thus two vertices in A must have different neighbors in B; suppose $E(G)$ contains a_1b_1 and a_2b_2. If there exists an edge with endpoints outside a_1, a_2, b_1, b_2, G must contain $3K_2$; else, all edges are incident to one of these vertices. If there exists a_3 adjacent to b_1 and b_3 adjacent to a_1, then a_1b_3, a_2b_2, a_3b_1 form $3K_2$, and analogously if there exists a_4 adjacent to b_2 and b_4 adjacent to a_2. Otherwise, without loss of generality, all vertices in A besides a_1 and a_2 are adjacent at most to b_1, and all vertices in B besides b_1 and b_2 are adjacent at most to a_2. This is a subgraph of a double star graph unless $E(G)$ contains a_1b_2. In that case, since A and B each contain a third non-isolated vertex a_3 and b_3, the edges a_1b_2, a_2b_3, a_3b_1 form $3K_2$. □

To end this section, we characterize the triangle-free graphs in 1-HNG. The structural characterization in (iii) is illustrated in Figure 2.3.

Theorem 2.18. *Given a graph G, the following are equivalent:*

(i) $G \in$ 1-HNG *and* $\omega(G) = 2$.

(ii) G *contains no induced* $K_3, F_1, \ldots, F_{12}$.

(iii) G *is either (1) a subgraph of* $K_{2,n-2}$, *(2) a subgraph of* $S_{m,n-m}$, *or (3) a graph in family* D_3, *possibly with isolated vertices.*

Proof. (i) → (iii) Let $G \in$ 1-HNG with $\omega(G) = 2$. By Theorem 2.12, G is perfect if and only if it is C_5-free. If so, $\omega = \chi = 2$ and so G is bipartite. By Theorem 2.9, G has no $3K_2$ subgraph, and so by Lemma 2.17, G is a subgraph of $K_{2,n-2}$ or a double star graph. Otherwise, G is imperfect and contains an induced C_5 on some vertex subset $\{c_1, c_2, c_3, c_4, c_5\}$. Any other vertices must be of types $\emptyset, \{c_1\}$, and $\{c_1, c_3\}$, up to symmetry, since other types would induce K_3. The result follows from Proposition 2.11.

(iii) → (ii) If G is a subgraph of $K_{2,n-2}$ or the double star graph, it clearly is bipartite and contains no induced K_3, and Lemma 2.17 implies $\nu(G) \leq 2$, so G contains no induced

$F_1, \ldots, F_{12}$. If G is a graph in family D_3, possibly with isolated vertices, then G contains no induced K_3. By Lemma 2.15, $G \in 1\text{-}HNG$.

(ii) $\to$ (i) Clearly $\omega(G) = 2$. All forbidden induced subgraphs of $1\text{-}HNG$, as laid out in Theorem 2.9, are given in (ii) or contain K_3, so we conclude that $G \in 1\text{-}HNG$. □

2.6 Optimization

In this section we show that membership in $1\text{-}HNG$, and many of its graphs' key invariants, can be determined in polynomial time. These results largely follow from the forbidden induced subgraph characterization and work in Section 4 on apex-perfection.

Theorem 2.19. *Graphs in* $1\text{-}HNG$ *can be recognized in polynomial time, specifically* $O(n^8)$.

Proof. It suffices to test whether the given graph contains one of the 52 graphs in the list, each of which has at most 8 vertices. □

Furthermore, the clique number and independence number of graphs in $1\text{-}HNG$ can be determined in polynomial time. Where the graphs are perfect, the results are immediate from the result by Chudnovsky et al. [15] that perfect graphs can be identified in polynomial time and from Grötschel, Lovász, and Schrijver's earlier work [34] that perfect graphs can be given a minimum proper coloring in polynomial time. For imperfect graphs, we begin with a lemma refining the proof of Theorem 2.12 to add an additional requirement to the deleted vertex, then continue with the full result in Theorem 2.21.

Lemma 2.20. *If* $G \in 1\text{-}HNG$ *and is imperfect, then there exists a vertex* v *such that* $G - \{v\}$ *is perfect and* $\omega(G - \{v\}) = \omega(G)$.

Proof. Let $G \in 1\text{-}HNG$ and suppose G is imperfect. By Theorem 2.12, G is apex-perfect, and contains at least one induced C_5 (note again that G cannot contain any of the other forbidden induced subgraphs of perfect graphs). Let $C = \{c_1, c_2, c_3, c_4, c_5\}$ induce a copy of C_5 in G with edge set $\{c_1c_2, c_2c_3, c_3c_4, c_4c_5, c_1c_5\}$ and let $D = V(G) - C$. If $D = \emptyset$, then G is isomorphic to C_5 and $\omega(G) = \omega(G - \{v\})$ for all $v \in V(G)$. If $\omega(G) = 2$, then since G contains C_5, $\omega(G - \{v\}) = 2$ for all $v \in V(G)$. Otherwise, suppose that D is nonempty and $\omega(G) \geq 3$.

If D only has vertices of types $\{c_1, c_2\}$ and $\{c_1, c_2, c_4\}$, up to symmetry, then by the proof of Theorem 2.12, G cannot contain an induced C_5 other than C. Since c_3 is contained in no triangles, its deletion verifies the theorem. We may assume going forward that G contains an induced C_5 other than C.

Suppose next that D may also have vertices of types $\{c_1, c_3\}$ and $\{c_1, c_2, c_3\}$, up to symmetry. If an induced C_5 contains a vertex of type $\{c_1, c_3\}$, then its neighbors in the copy of C_5 must be c_1 and c_3, by Proposition 2.11. The deletion of either verifies apex-perfection, so delete whichever of the two is not contained in a unique maximum clique. If an induced C_5 contains a vertex of type $\{c_1, c_2, c_3\}$, then its non-neighbors must analogously be c_4 and c_5. By Proposition 2.11, D cannot contain a vertex adjacent to both, so each is in a clique of size at most two, which is non-maximum by assumption. Delete either to reach the desired conclusion.

Suppose next that D may also have vertices of types $\{c_1\}$ and $\{c_1, c_2, c_3, c_4\}$, up to symmetry. If an induced C_5 contains neither, then the above argument holds to verify the theorem. If an induced C_5 contains a vertex of type $\{c_1\}$, then its neighbors in the C_5 must be c_1 and a vertex of type $\{c_3, c_4\}$. By Proposition 2.11, c_1 may neighbor vertices of types $\{c_1\}, \{c_1, c_3\}, \{c_1, c_4\}$ and $\{c_1, c_3, c_4\}$. All of these are isolated to one another, c_2, and c_5, with the exception of vertices of type $\{c_1, c_3, c_4\}$, which may be adjacent to one another. Hence if c_1 is in a maximum clique, the clique must comprise c_1 and at least two adjacent vertices of type $\{c_1, c_3, c_4\}$. Replace c_1 with c_3 and c_4 to obtain a strictly larger clique, for a contradiction. We conclude that c_1 is in no maximum clique, so its deletion verifies apex-perfection, by the proof of Theorem 2.12, and $\omega(G - \{c_1\}) = \omega(G)$.

If an induced C_5 contains a vertex of type $\{c_1, c_2, c_3, c_4\}$, then its non-neighbors in the C_5 must be c_5 and a vertex of type $\{c_2, c_3, c_5\}$. Since this case is the complement of the case in the proof of Theorem 2.12 where an induced C_5 contains a vertex of type $\{c_1\}$, it follows that $G - \{c_5\}$ is perfect. By Proposition 2.11, c_5 is in a clique of size at least 3 only if all other clique vertices are of type $\{c_2, c_3, c_5\}$, but these vertices together with c_2 and c_3 form a strictly larger clique. Hence $\omega(G - \{c_5\}) = \omega(G)$.

Lastly, suppose D may also contain vertices of types $\emptyset$ and $\{c_1, c_2, c_3, c_4, c_5\}$. Neither can be in any induced copies of C_5. Any vertex of type $\{c_1, c_2, c_3, c_4, c_5\}$ is included in all maximum cliques in G and its induced subgraphs $G - \{v\}$ for all other $v \in V(G)$, since its only possible non-neighbors, vertices of types $\{c_1\}$ or $\{c_2, c_5\}$, up to symmetry, are never included in any maximum cliques. (Any two adjacent neighbors of a vertex of these types are adjacent to both c_3 and c_4, which would produce a strictly larger clique). Hence a vertex of type $\{c_1, c_2, c_3, c_4, c_5\}$ cannot impact the relative size of a maximum clique, and we proceed assuming none exist in G.

Any vertex of type $\emptyset$ is included in no maximum cliques in G or its induced subgraphs $G - \{v\}$ for all other $v \in V(G)$, since its neighbor set (if nonempty) forms a clique dominating two adjacent vertices of C. Hence a vertex of type $\emptyset$ cannot change the clique number in G, and we can assume no such vertex exists in G. $\square$

Note also the complementary result pertaining to $\alpha(G)$ holds since 1-HNG, perfect graphs, and all processes used in the proof are invariant under complementation.

Theorem 2.21. *If $G \in 1$-HNG, then $\omega(G)$ and $\alpha(G)$ can be determined in polynomial time.*

Proof. If G is perfect, the result holds immediately, as shown in [34]. Otherwise, assume G is imperfect. We show $\omega(G)$ can be determined in polynomial time; by complementation, the same holds for $\alpha(G)$. As established in the proof of Theorem 2.12, G contains an induced copy of C_5. In $O(n^5)$ time, identify a subset of vertices $\{v_1, v_2, v_3, v_4, v_5\}$ inducing C_5. For each v_i, $1 \leq i \leq 5$, determine for which $G - \{v_i\}$ is perfect, which can be done in polynomial time by [15], and if so compute $\omega_i = \omega(G - \{v_i\})$, which is again possible in polynomial time. In all cases, $\omega_i = \omega(G)$ or $\omega(G) - 1$. By Lemma 2.20, this must hold with equality in the case of at least one ω_i, so we conclude that $\omega(G) = \max\{\omega_i\}$. □

We now perform a similar analysis to exactly determine the chromatic number of a graph in 1-HNG, and a complementary result allows us to calculate $\theta(G)$. These allow for $\chi(G)$ and $\theta(G)$ to be determined in polynomial time; see Corollary 2.23.

Theorem 2.22. *If $G \in 1$-HNG, then $\chi(G) = \omega(G)$ unless G comprises an induced C_5 and an isolated set of vertices of types either (i) a subset of $\{c_1\}$, $\{c_1, c_3\}$, $\{c_1, c_4\}$, and $\{c_1, c_2, c_3, c_4, c_5\}$, or (ii) both $\{c_1, c_2, c_4\}$ and $\{c_1, c_2, c_3, c_5\}$, up to symmetry. If either of these hold, $\chi(G) = \omega(G) + 1$.*

Proof. Let $G \in 1$-HNG and suppose G is imperfect. By Theorem 2.12, G is apex-perfect, and contains at least one induced C_5 (note again that G cannot contain any of the other forbidden induced subgraphs of perfect graphs). Let $C = \{c_1, c_2, c_3, c_4, c_5\}$ induce a copy of C_5 in G with edge set $\{c_1c_2, c_2c_3, c_3c_4, c_4c_5, c_1c_5\}$ and let $D = V(G) - C$. Suppose D matches none of the conditions given in the theorem and $|D| \geq 2$ (the result is straightforward to check if $|D| = 1$). If $\omega(D) = 1$ (i.e., D induces a stable set), then by Proposition 2.11, it follows that D comprises vertices of types $\{c_1, c_2\}$ and $\{c_1, c_2, c_4\}$, up to symmetry. The argument in the following paragraph holds. Note that by the proof of Theorem 2.12, D induces a perfect graph.

First, suppose D has only vertices of types $\{c_1, c_2\}$ and $\{c_1, c_2, c_4\}$, up to symmetry. If so, a minimum proper coloring of D can be extended to a proper coloring of G using $\chi(D) + 2$ colors by giving vertices c_1 and c_2 new colors and coloring c_4 with c_1's color, c_5 with c_2's color, and c_3 with any color used in D. Hence $\chi(G) \leq \chi(D) + 2$. However, since D is nonempty and complete to c_1 and c_2, it follows that $\omega(G) = \omega(D) + 2$. Since D is perfect, we have $\chi(G) \geq \omega(G) = \omega(D) + 2 = \chi(D) + 2$, and conclude that $\chi(G) = \omega(G)$.

Second, suppose D may also have vertices of types $\{c_1, c_3\}$ and $\{c_1, c_2, c_3\}$, up to symmetry. If D contains a vertex of type $\{c_1, c_3\}$, then D contains only vertices of types

$\{c_1, c_3\}$, $\{c_1, c_4\}$, $\{c_1, c_3, c_4\}$, and $\{c_3, c_4\}$, of which there must be at least two vertices of types $\{c_1, c_3, c_4\}$, and $\{c_3, c_4\}$ to produce a clique number at least two. Vertices of types $\{c_1, c_3\}$ and $\{c_1, c_4\}$ can be freely colored with the same color as a vertex of type $\{c_1, c_3, c_4\}$ or $\{c_3, c_4\}$, so the above argument holds to show $\chi(G) = \omega(G)$.

If D contains a vertex of type $\{c_1, c_2, c_3\}$, then D comprises either (i) vertices of types $\{c_1, c_2\}$, $\{c_1, c_2, c_3\}$, $\{c_1, c_2, c_4\}$, $\{c_1, c_2, c_5\}$ or (ii) vertices of types $\{c_2, c_3\}$, $\{c_1, c_2, c_3\}$, $\{c_2, c_3, c_4\}$, $\{c_2, c_3, c_5\}$. Assuming the former, without loss of generality, D is complete to c_1 and c_2, so a minimum proper coloring of D extends to G with two more colors: one is used for c_1 and c_3, one is used for c_2 and c_4, and c_5 can be colored with the same color as a vertex of type $\{c_1, c_2, c_3\}$. Note the coloring is proper since D induces a clique, so no neighbor of c_5 in D shares the same color as a vertex of type $\{c_1, c_2, c_3\}$. The above argument holds to show that $\chi(G) = \omega(G)$.

Third, suppose D may also have vertices of types $\{c_1\}$ and $\{c_1, c_2, c_3, c_4\}$, up to symmetry. If D contains a vertex of type $\{c_1\}$ not in a maximum clique, then D without its vertices of type $\{c_1\}$ is as described in one of the above three paragraphs. Since the vertices of type $\{c_1\}$ are not adjacent to one another and not in a maximum clique in D, they can be added back into any proper coloring without using new colors. If a vertex of type $\{c_1\}$ is in a maximum clique, then since $\omega(D) \geq 2$, this maximum clique must also have one vertex of type $\{c_3, c_4\}$. Recall from the proof of Theorem 2.9 that this maximum clique cannot contain two vertices of type $\{c_3, c_4\}$. Hence $\omega(D) = 2$, so all vertices of types $\{c_3, c_4\}$ and $\{c_1, c_3, c_4\}$ are isolated from one another. Here G can be properly colored with three colors: the first is used on c_2, c_5, and all vertices of types $\{c_3, c_4\}$ and $\{c_1, c_3, c_4\}$; the second is used on c_3 and all vertices of type $\{c_1\}$; and the third is used on c_1 and c_4.

If D contains a vertex of type $\{c_1, c_2, c_3, c_4\}$ and no vertex of type $\{c_2, c_3, c_5\}$, then by Proposition 2.11 it follows that c_5 has no neighbors in D. Hence any proper coloring of D can be extended to a proper coloring of G with two new colors, one for c_1 and c_3 and the other for c_2 and c_4, with a color from D given to c_5. Furthermore, Proposition 2.11 implies that $\omega(G) = \omega(D) + 2$, so it follows that $\chi(G) = \omega(G)$, as reasoned above. If D does have a vertex of type $\{c_2, c_3, c_5\}$, then by Proposition 2.11 all vertices in D are adjacent to c_2 and c_3. As a result $\omega(G) = \omega(D) + 2$. Moreover, any minimum proper coloring of D can be extended to a proper coloring of G by giving c_2 and c_4 one new color and c_1 and c_3 a second new color. If there exists a vertex v of type $\{c_1, c_2, c_3, c_4\}$ adjacent to all vertices of type $\{c_2, c_3, c_5\}$, then v dominates D and its color can be given to c_5 so that the coloring on G is proper. Otherwise every vertex of type $\{c_1, c_2, c_3, c_4\}$ has a non-neighbor of type $\{c_2, c_3, c_5\}$. If there are two vertices of type $\{c_1, c_2, c_3, c_4\}$, then G contains $\overline{F_{25}}$ or $\overline{F_{26}}$ as an induced subgraph. If there are two adjacent vertices of type $\{c_2, c_3, c_5\}$, then these two vertices together with c_1, c_2, c_5, and the vertex of type $\{c_1, c_2, c_3, c_4\}$ induce an element of $\mathcal{F}$. Otherwise all vertices of type $\{c_2, c_3, c_5\}$ are non-adjacent to one another, and the vertex of type $\{c_1, c_2, c_3, c_4\}$ is adjacent to at least one of them. Here, $\omega(D) = 2, \omega(G) = 4, \chi(D) = 2$, and $\chi(G) = 4$.

Lastly, suppose D also has vertices of types $\emptyset$ and/or $\{c_1, c_2, c_3, c_4, c_5\}$. We showed in the proof of Theorem 2.20 that a vertex of type $\emptyset$ is never in a maximum clique in G or D, and a vertex of type $\{c_1, c_2, c_3, c_4, c_5\}$ is in all maximum cliques. Let G' be the induced subgraph with no vertices of types $\emptyset$ or $\{c_1, c_2, c_3, c_4, c_5\}$, so by the above arguments, $\chi(G') = \omega(G')$. Suppose D has k vertices of type $\{c_1, c_2, c_3, c_4, c_5\}$. A minimum proper coloring of G' can be extended to a proper coloring of G with $\chi(G') + k$ colors, and $\omega(G) = \omega(G') + k$. Thus $\chi(G) \geq \omega(G) = \omega(G') + k = \chi(G') + k \geq \chi(G)$, so we conclude $\chi(G) = \omega(G)$.

For the second claim, let G comprise an induced C_5 and a stable set D of vertices of types either (i) a subset of $\{c_1\}$, $\{c_1, c_3\}$, $\{c_1, c_4\}$, and $\{c_1, c_2, c_3, c_4, c_5\}$, or (ii) both $\{c_1, c_2, c_4\}$ and $\{c_1, c_2, c_3, c_5\}$, up to symmetry. In either case, if D is nonempty, then $\omega(G) = 3$ and $\chi(G) = 4$. If D is empty, then G is isomorphic to C_5 and $\chi(G) = 3$. □

Corollary 2.23. *If $G \in 1\text{-}HNG$, then $\chi(G)$ and $\theta(G)$ can be determined in polynomial time.*

Proof. By Corollary 2.21, we can determine $\omega(G)$ in polynomial time. Subsequently, in $O(n^5)$ time, identify a subset of vertices $\{v_1, v_2, v_3, v_4, v_5\}$ inducing C_5, and check the condition in Theorem 2.22 to determine if $\chi(G) = \omega(G)$ or $\chi(G) = \omega(G) + 1$. □

Chapter 3

Generalized Nordhaus-Gaddum Graphs and Polyomino Realizations

3.1 Introduction

All graphs are finite and simple. Given a graph G, we consider a number of invariants associated to G. Let $n = |V(G)|$ be the order of the graph, which we often also denote by $|G|$, and let $\overline{G}$ be the complement of G. We use $\omega(G)$ and $\alpha(G)$ to denote the clique and independence numbers of G, respectively. Let $\chi(G)$ denote the *chromatic number* of G, that is, the minimum number of sets needed to partition $V(G)$ into stable sets. We call this process coloring and often refer to the sets in such a partition as *color classes*. Similarly, let $\theta(G)$ denote the *clique cover number* of G, which is the minimum number of sets needed to partition $V(G)$ into cliques. We refer to these sets as *clique classes*. It follows that $\theta(G) = \chi(\overline{G})$. Frequently, we apply a proper coloring and clique covering to G simultaneously, and as an abuse of language we refer to the combined set of subsets of $V(G)$ of color classes and clique classes as *classes*, not to be confused with graph classes.

Many important graph classes are defined by relationships among these invariants and others. For instance, bipartite graphs are exactly the graphs where $\chi(G) \leq 2$. Perfect graphs are those where $\chi(H) = \omega(H)$ for all induced subgraphs H of G. We also use inequalities among invariants to define graph classes. Perfect graphs, for instance, are those satisfying the known inequality $\omega(H)\alpha(H) \geq |H|$ for all induced subgraphs H of G [51]. Adding these two invariants produces the inequality $\omega(G)+\alpha(G) \leq n+1$, proven by Litjens and Sivaraman in [50]. They weaken this bound slightly to produce the class of *sum-perfect graphs*: graphs G where all induced subgraphs H satisfy the inequality $\omega(H) + \alpha(H) \geq |V(H)|$. We will also require our definitions to apply to all induced subgraphs: hence our generalized classes will be hereditary. Any hereditary graph class $\mathcal{A}$ can be characterized by a unique set of forbidden induced subgraphs, none of which are induced in one another. We refer to such graphs as *forbidden induced subgraphs*, implying minimality.

Similarly, Nordhaus and Gaddum [54] show that all graphs satisfy $\chi(G) + \theta(G) \leq n + 1$. The well-studied class of Nordhaus-Gaddum graphs (see [27], [63], [20], and [14] for characterizations) are those graphs where this inequality holds with equality. In Chapter 2, we introduced the generalized classes of *a-Nordhaus-Gaddum graphs* (*a-NG*) and *a-hereditary-Nordhaus-Gaddum graphs* (*a-HNG*). These respectively generalize the Nordhaus-Gaddum graphs (denoted 0-NG by this schematic) and their hereditary subclass (0-HNG).

Definition 3.1. *For $a \geq 0$, a graph G is a-*Nordhaus-Gaddum *if $\chi(G) + \theta(G) \geq |G| + 1 - a$. A graph G is a-*hereditary-Nordhaus-Gaddum *if for all induced subgraphs H of G, $\chi(H) + \theta(H) \geq |H| + 1 - a$. Let a-$NG$ denote the set of a-Nordhaus-Gaddum graphs and let a-HNG denote the set of a-hereditary-Nordhaus-Gaddum graphs.*

In this chapter we extend our results on 1-HNG to a-HNG for all $a \geq 0$. In Section 3.2, we show that for all $a \in \mathbb{N}$, the class a-HNG has finitely many forbidden induced subgraphs, and we identify large swaths of these graphs. We conclude Section 3.2 with optimization results connected to a-HNG, proving that a-HNG is linearly χ-bounded and that for all $G \in a$-HNG, $\omega(G)$ can be determined in polynomial time. The parallel result for $\chi(G)$ remains open. Many of our results in Section 3.2 apply equally to a family of generalizations of sum-perfect graphs and we make this connection explicit in Section 3.3.

Before proceeding, we conclude the introduction with three subsections detailing various aspects of our work. Subsection 3.1.1 gives a more thorough overview of results from Chapter 2 used in this chapter. In Subsection 3.1.2, we give definitions for graph classes used throughout the chapter. Lastly, the proofs in Subsection sec:finiteness rely on the study of polyominoes, which we introduce in Subsection 3.1.3.

3.1.1 Prior Work on Hereditary Nordhaus-Gaddum Graphs

In Chapter 2, we showed that the classes under discussion are strictly contained within each other: for all $a \geq 0$, it holds that a-$HNG \subset a$-$NG \subset (a+1)$-HNG. We gave a forbidden induced subgraph characterization of 1-HNG with 52 graphs on at most eight vertices, then leverage this characterization to show that for all graphs $G \in 1$-HNG, we have $\chi(G) \leq \omega(G) + 1$ and that $\omega(G)$ and $\chi(G)$ can be determined in polynomial time. We say that a hereditary class $\mathcal{A}$ of graphs is χ-*bounded* by a function $f : \mathbb{N} \to \mathbb{R}$ if for all $G \in \mathcal{A}$, we have $\chi(G) \leq f(\omega(G))$. Thus 1-$HNG$ is χ-bounded by the linear function $f(x) = x + 1$.

We also include here Theorem 2.7 which is used throughout this chapter for limiting which graphs can be forbidden induced subgraphs of a-HNG. Given a graph G and a vertex $v \in V(G)$, let $G - \{v\}$ be the induced subgraph of G with vertex set $V(G) - \{v\}$. We say a vertex of a graph G is χ-*distinct* if its deletion changes the chromatic number of the graph: i.e., if $\chi(G - \{v\}) < \chi(G)$. An analogous definition exists for θ-distinct vertices.

Theorem 3.1. *If G is a forbidden induced subgraph of a-HNG, then $G \in (a+1)$-HNG, and no vertex of G is χ-distinct or θ-distinct.*

With no χ-distinct vertices, all color classes in a minimum proper coloring contain at least two vertices. Therefore, in a forbidden induced subgraph, all color and clique classes (in minimum proper colorings and clique coverings) have at least two vertices.

3.1.2 Related Graph Classes

Within the chapter, we make reference to *split graphs*, which are graphs whose vertex sets are partitionable into a clique and a stable set. It is possible for this partition to be unique (for example, for P_4) or non-unique (for example, for K_4). The class of split graphs is perfect, hereditary, and defined by forbidding $2K_2, C_4$, and C_5, the cycle on five vertices [28]. We also prove a result about a generalization of split graphs. A graph G is *apex-split* if there exists $v \in V(G)$ such that $G - \{v\}$ is split; we say that G is *c-apex-split* if we can delete c vertices from G to produce a split graph. A class $\mathcal{A}$ of graphs is *c-apex-split* if all graphs $G \in \mathcal{A}$ are c-apex-split. Following the nomenclature of [41], we say $\mathcal{A}$ is *bounded-apex-split* if it is c-apex-split for some c. At the end of the chapter, we show an equivalence between hereditary Nordhaus-Gaddum graphs, bounded-apex-split graphs, and generalized sum-perfect graphs based on their shared bipartite forbidden induced subgraphs.

The class of split graphs is closely related to 0-NG (the standard Nordhaus-Gaddum graphs). Cheng, Collins, and Trenk show that 0-NG contains exactly the split graphs that do not have a unique vertex partition into a clique and a stable set [14]. The remainder of the class 0-NG comprises all $(2K_2, C_4)$-free graphs containing an induced C_5. Note that $(2K_2, C_4)$-free graphs are exactly the *pseudosplit graphs*, which are known to be 1-NG by [12]. Within 0-NG , we showed in Theorem 2.3 that the hereditary subclass 0-HNG is exactly the threshold graphs. Recall that a graph is threshold if it contains none of $2K_2, C_4$, or P_4 (all shown in Figure 3.2) as an induced subgraph [18] [19].

3.1.3 Dot Grids and Polyominoes

As mentioned earlier, many structural characterizations of 0-NG exist in the literature. Of these, the most interesting to our point of view is Finck's 1966 characterization of 0-NG with grids of dots [27]. Since Finck's paper is not widely available online, we include a copy of the result here. Arrange n dots in a rectangular grid. The first row and column must be filled with dots, but the placement of other dots is chosen freely. To transform this collection of dots into a graph, we consider each dot to be a vertex, and say that two vertices sharing a row must be adjacent, two vertices sharing a column cannot be adjacent, and all other pairs of vertices can freely be chosen to be adjacent or not. Critically each dot grid produces multiple possible graphs. The dot in the first row and column might also be replaced by a set of five dots inducing a cycle. Finck shows that it is possible to characterize inclusion in

0-NG (and a number of other classes based on Nordhaus-Gaddum inequalities) exclusively from the underlying dot grid.

Graphs of type $T_1(n; \alpha, n-\alpha+1)$ consist of n vertices which can be partitioned into α cliques of size less than or equal to $n-\alpha+1$ and $n-\alpha+1$ stable sets of size less than or equal to α, where both of these maximum sizes are attained by some clique and stable set. Graphs of type $T_2(n, \alpha)$ consist of a stable set of α vertices, a clique of $n-\alpha-5$ vertices, and a five-cycle whose vertices are adjacent to all clique vertices and not adjacent to any stable set vertices. We illustrate graphs of types T_1 and T_2 in Figure 3.1.

Theorem 3.2. *[27] The equality $\chi(G)+\theta(G) = n+1$ holds if and only if either G is of type $T_1(n; \alpha, n-\alpha+1)$, for $1 \leq \alpha \leq n$, or G is of type $T_2(n, \alpha)$, for $1 \leq \alpha \leq n-5$.*

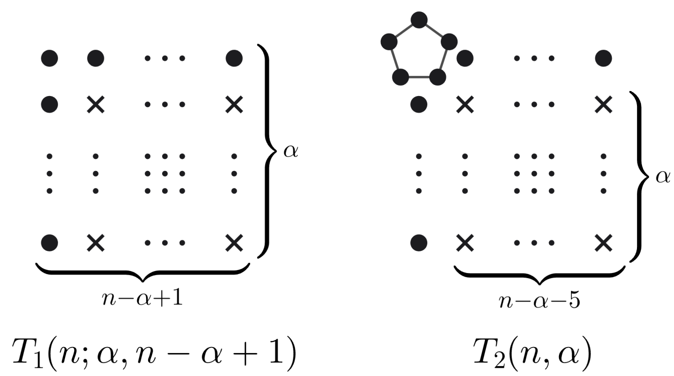

Figure 3.1: Copies of Finck's dot grids characterizing 0-NG. The solid vertices of row 1 and column 1 must be included in a graph of either type T_1 or T_2, and the remaining vertices are distributed arbitrarily, with each x potentially representing a vertex.

Although outside the scope of this chapter, we suspect a generalization of Theorem 3.2 to a-NG would be feasible and interesting. For which graphs would we need to make special cases, as with C_5 in $T_2(n, \alpha)$?

In this chapter, we extend Finck's observations about dot grids to the related and more well-known object, polyominoes. A polyomino is a collection of a squares of equal side length arranged with coincident sides. From each polyomino it is possible to construct multiple non-isomorphic graphs whose vertices are the corners of the squares. We use Finck's rules for edge relationships: two vertices are adjacent if they share a row and only if they do not share a column. In Figure 3.2 the smallest polyomino and its three realizations are shown.

We begin by finding the size of the smallest forbidden induced subgraphs for a-HNG. We are able to use direct counting arguments to establish this minimum, but the sequence of

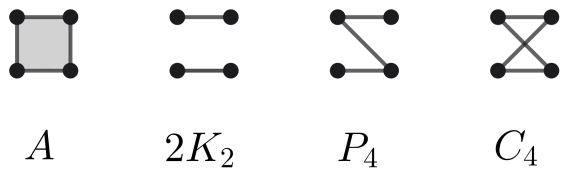

Figure 3.2: The smallest non-trivial polyomino, labeled A, and its three possible realizations, up to isomorphism, which are the forbidden induced subgraphs of the class of threshold graphs.

minimum orders is identical to a sequence related to polyominoes. Subsequently, we establish large classes of polyominoes that must produce at least one forbidden induced subgraph for a-HNG as well as a characterization of which convex polyominoes admit exclusively forbidden induced subgraph constructions (we define *convex* in Section 3.2). The associated forbidden induced subgraphs permit us to show that there are finitely many forbidden induced subgraphs for each class a-HNG. This answers an open question from [50].

3.2 a-HNG and Its Forbidden Induced Subgraphs

While we are not able to identify all forbidden induced subgraphs of each class a-HNG, we are able to make substantial progress towards a characterization. In particular, we prove there are finitely many such graphs and are able to give large subfamilies of the characterization, including all minimum order forbidden induced subgraphs. Understandably, the forbidden induced subgraphs of a-HNG appear to grow larger and more numerous as a gets larger. It is not too difficult to deduce the minimum order of a forbidden induced subgraph of a-HNG. We show later in Theorem 3.7 that such graphs actually exist and provide a means of determining them all.

Theorem 3.3. *Given $a \geq 0$, the minimum order of a forbidden induced subgraph of a-HNG is given by $\lceil a + 2 + 2\sqrt{a+1} \rceil$.*

Proof. Let G be a forbidden induced subgraph of a-HNG of order n and suppose that $\theta(G) = t$. In any proper coloring of G, two vertices in the same color class must be in different clique classes, so $n \leq \chi(G)t$. Since G is a forbidden induced subgraph, we have $\chi(G) + \theta(G) \leq n - a$, so by substitution $\frac{n}{t} + t \leq n - a$. Solving for n, we get $n \geq \frac{t(t+a)}{t-1}$. We would like to minimize n, so we differentiate to get $\frac{dn}{dt} = \frac{t^2-2t-a}{(t-1)^2}$. This is equal to 0 where $t^2 - 2t - a = 0$, so $t = 1 + \sqrt{a+1}$. The minimum occurs at $a + 2 + 2\sqrt{a+1}$, so we conclude that $n = \lceil a + 2 + 2\sqrt{a+1} \rceil$. $\square$

The sequence $n(a)$ of minimum orders of a forbidden induced subgraph of a-HNG is given by sequence A080037 in the Online Encyclopedia of Integer Sequences (see also the identical sequence A121150). Sequence A080037 gives the equivalent formula $n(a) = \lfloor a+3+\sqrt{4a-3}\rfloor$. We note that this bound is equal to our result in Theorem 3.3 as follows. We have $\lceil a+2+2\sqrt{a+1}\rceil = a+2+\lceil\sqrt{4a+4}\rceil$ and $\lfloor (a+1)+2+\sqrt{4(a+1)-3}\rfloor = a+3+\lfloor\sqrt{4a+1}\rfloor$. The quantities $\lceil\sqrt{4a+4}\rceil$ and $1+\lfloor\sqrt{4a+1}\rfloor$ are equal unless $4a+2$ or $4a+3$ is a perfect square. However, all perfect squares are congruent to $0 \pmod 4$ or $1 \pmod 4$, so this cannot arise.

By OEIS A080037, these equivalent formulas count the minimum number of vertices (corners of squares) of an $(a+1)$-polyomino. Recall that an *a-polyomino* is defined to be a set of a squares of equal side length arranged with coincident sides. The corners of each square are the vertices of the polyomino. We study these polyominoes at some length in order to show a link between polyominoes and forbidden induced subgraphs, and to better understand the forbidden induced subgraphs of a-HNG.

We may assume that every polyomino is given an orthogonal orientation in the plane, so that we can refer to rows and columns of vertices in the polyomino. We consider polyominoes invariant under reflection or 180-degree rotation, but not 90-degree rotation. Nonetheless, the graphs associated to a polyomino are the complements of graphs associated to the 90-degree rotation of that polyomino. Since a-HNG is closed under complementation, such rotations are insignificant for determining forbidden induced subgraphs. A polyomino is *convex* if its perimeter is equal to the perimeter of its minimum bounding box, thus it has neither indents nor missing squares from the center. In Figure 3.3, for example, A is not a polyomino and B is not convex, but C and D are examples of convex polyominoes. For B, C, and D, the minimum bounding box is outlined in a dashed line. By definition, all polyominoes do not have vertex holes, and convex polyominoes cannot have discontinuous rows or columns, nor a row or column with only one vertex.

Let $V(A)$ be the set of vertices (corners) in the polyomino. Given polyomino A, we say that a graph G *realizes* A if there is a bijection from $V(A)$ to $V(G)$ such that two vertices in the same row of A correspond to adjacent vertices in G and two vertices in the same column of A correspond to non-adjacent vertices in G. We will omit formal discussion of bijections moving forward, and generally consider a polyomino and its graph realizations to have the same vertex set. Hence each row of A corresponds to a clique in G and each column of A corresponds to a stable set in G. A polyomino A typically has numerous realizations, since each diagonal pair of vertices can be chosen arbitrarily to be adjacent or not in G. In Figure 3.2 we see the three possible realizations of the single-square polyomino, up to isomorphism. These realizations are constructed by choosing subsets of the possible diagonal edges; there are four such subsets, but both graphs with one diagonal edge are isomorphic.

Given a polyomino A, let s be the number of columns of vertices in A and t be the

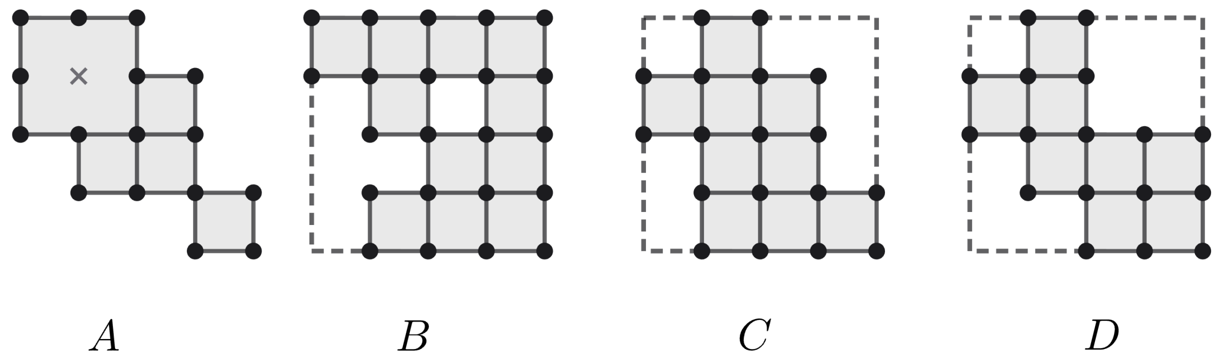

Figure 3.3: The figure A is not a polyomino because not all squares have equal side length (see missing vertex, marked with an x) and not all squares have coincident sides. The figure B is a non-convex polyomino, whereas C and D are convex polyominoes.

number of rows of vertices. Given a realization G of a polyomino A, we can always give a proper coloring of G by assigning a distinct color to each column's vertices, and can give a proper clique covering of G by considering each row as a distinct clique. While this coloring and clique covering may not be minimum, it follows that $\chi(G) + \theta(G) \leq s + t$. These realizations allow us to draw connections between polyominoes and forbidden induced subgraphs of a-HNG. We are able to show that a generic polyomino admits at least one a forbidden induced subgraph realization, provided that the polyomino does not contain both a strictly longest row of vertices and a strictly longest column. See Figure 3.3, polyomino D for an example of a polyomino failing this criterion.

Theorem 3.4. *If A is a polyomino that does not contain both a strictly longest row and a strictly longest column, then there exist $a \geq 0$ and a graph G realizing A such that G is a forbidden induced subgraph G of a-HNG.*

Proof. Let A be a polyomino not containing both a strictly longest row and a strictly longest column, and let G_0 be the realization of A such that two vertices are adjacent in G_0 if and only if their corresponding vertices in A are in the same row. Accordingly, no edges are added between diagonal pairs of vertices. See Figure 3.6 for an example of a polyomino A and its edge-minimum realization G_0. Since polyominoes are built from orthogonally adjacent squares, it follows that every row has at least two vertices. Let the rows of A have lengths $r_1, r_2, \ldots, r_t$, written in nonincreasing order. Thus, G_0 is isomorphic to a disjoint union of cliques, and we can write $G_0 \cong K_{r_1} \cup K_{r_2} \cup \ldots \cup K_{r_t}$. We note that t is greater than or equal to the length of the longest column. As a disjoint union of cliques, we know that $\chi(G_0) = r_1$ and $\theta(G_0) = t$. Hence let a be equal to $|G_0| - (r_1 + t) = r_2 + \ldots + r_t - t$.

Suppose there exists a forbidden induced subgraph H of a-HNG that is properly induced in G_0. Therefore, H is isomorphic to a disjoint union $K_{w_1} \cup K_{w_2} \cup \ldots \cup K_{w_p}$ of cliques, with $w_1 \geq w_2 \geq \ldots \geq w_p$. For convenience, let $w_i = 0$ for all $i > p$. We can write $H \cong K_{w_1} \cup K_{w_2} \cup \ldots \cup K_{w_t}$. Since H is induced in G_0, it follows that $w_i \leq r_i$ for all $i \leq t$, and that at least one of these inequalities is strict.

We know that $\chi(H) = w_1$ and $\theta(H) = p$. Since H is a forbidden induced subgraph of a-HNG, it follows from Theorem 3.1 that $a = |H| - \chi(H) - \theta(H) = w_2 + w_3 + \ldots + w_p - p$. Suppose $p = t - i$ for some constant $i > 0$. Recall that $r_i \geq 2$ for all i. Therefore, for all j such that $p < j \leq t$, we have $r_j \geq w_j + 2$. Then $a = r_2 + \ldots + r_t - t = r_2 + \ldots + r_p + 2i + (r_{p+1} - 2) + \ldots + (r_t - 2) - (p - i) \geq w_2 + \ldots + w_t + i - p = a + i$, for a contradiction. We instead conclude that G_0 is a forbidden induced subgraph.

Otherwise $p = t$, then we have $r_2 + r_3 + \ldots + r_t = w_2 + w_3 + \ldots + w_t$. This implies vertices are only deleted from the longest row of G_0. Since $\theta(G_0) = \theta(H)$, we have $\chi(G_0) > \chi(H)$ and accordingly $r_1 > w_1$. It follows that $r_1 > r_2$ in A, else the vertex deletion would not change the chromatic number.

While it is possible that G_0 is not a forbidden induced subgraph realizing A, we construct an alternate forbidden induced subgraph. Let G_1 be the realization of A where all vertices are adjacent unless they share a column. Thus the complement of G_1 is isomorphic to a disjoint union of cliques, where each column of A induces such a clique in G_1. Let the columns of A have lengths $x_1, x_2, \ldots, x_s$, written in nonincreasing order. The above argument holds to show that G_1 is a forbidden induced subgraph unless $x_1 > x_2$. If $x_1 > x_2$, then A has both a strictly longest row (of length r_1) and a strictly longest column (of length c_1), for a contradiction. □

Since the proof of the theorem is constructive, for every polyomino meeting this criterion we gain a forbidden induced subgraph of a-HNG for some $a \geq 0$, although we caution that it is possible for different polyominoes to produce the same graph realizations. Nonetheless, this produces a huge family of known forbidden induced subgraphs. On the other hand, where a polyomino A has both a strictly longest row and a strictly longest column, we can generally merge the classes associated to two rows/columns (if they do not intersect) to produce fewer classes, or we can delete the intersection vertex to produce fewer classes, no matter the choice of edges and non-edges.

We now shift our focus to studying convex polyominoes where all realizations are forbidden induced subgraphs. To do this, we first consider a canonical form for convex polyominoes. Given two convex a-polyominoes A and B with the same number of vertices, we say that A and B are realization-equivalent if there exists a bijection $f : V(A) \to V(B)$ that preserves rows and columns: $v, w \in V(A)$ share a row A if and only if $f(v)$ and $f(w)$ share a row in B, and v and w share a column in A if and only if $f(v)$ and $f(w)$ share a column in B. As a result, A and B admit the same set of realizations, up to isomorphism. This operation defines an equivalence class on the set of polyominoes. We consider the canonical representative of each class to be the convex polyomino with lexicographically largest list of row lengths, then column lengths. Figure 3.4 shows a convex polyomino A and the canonical representative B of its class of realization-equivalent polyominoes.

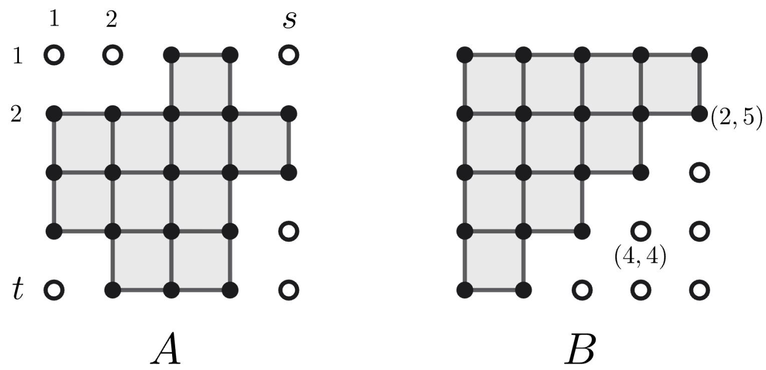

Figure 3.4: Convex polyominoes A and B are realization-equivalent. Rows and columns are as labeled, and we show open spots within the $s \times t$ grid with open circles.

We label the columns of our polyominoes left-to-right from 1 to s. The rows are numbered top-to-bottom from 1 to t. Hence a polyomino is a subset of the vertices in an $s \times t$ grid. Within that grid, let (i, j) be the spot where row i intersects column j: this may or may not contain a vertex. Should such a vertex exist, we call the spot *occupied* and denote its vertex by $v_{(i,j)}$. Otherwise, we call the spot *open*. This labeling system is shown in Figure 3.4; for example, spot $(2, 5)$ is occupied but spot $(4, 4)$ is open.

In general, since each vertex in a row must receive a different color, $\chi(G)$ is at least the length of the longest row of a polyomino it realizes. A spanning row is a row meeting every column; alternatively, a row containing s vertices, such that all spots are occupied. If a polyomino A has at least one spanning row, then $\chi(G) = s$ for all realizations G of A, since coloring by columns is always proper. By similar logic, $\theta(G)$ is at least the length of all columns in the associated polyomino. If A has at least one spanning column, then $\theta(G) = t$ for all realizations G of A.

Given a graph G, $a(G) = |G| - \chi(G) - \theta(G)$ is the minimum integer a such that G is potentially a forbidden induced subgraph for a-HNG. Where A has both a spanning row and a spanning column, we define $a(A) = |V(A)| - s - t$ and note that $a(A) = a(G)$ for all realizations G of A. Regardless of the presence of spanning rows or columns, by Theorem 3.1, G is not a forbidden induced subgraph of a-HNG if $a \neq a(G)$. Moreover, it must hold that for all proper induced subgraphs H of G, $a(H) < a(G)$. We record these observations as a remark.

Remark 3.5. *Given $a \geq 0$, a graph G is a forbidden induced subgraph of a-HNG if and only if $a = a(G)$ and for all proper induced subgraphs H, $a(H) < a(G)$.*

We now characterize which convex polyominoes admit exclusively forbidden induced subgraph realizations. Generally, these polyominoes contain numerous spanning rows and columns and are quite dense.

Theorem 3.6. *Let A be a convex polyomino with s columns and t rows. Every realization G of A is a forbidden induced subgraph of $a(G) - HNG$ if and only if there are at least two spanning rows, at least two spanning columns, and either of the following conditions hold:*

(i) A has at least three spanning rows or three spanning columns.

(ii) A has least $(s-1)(t-1)+3$ vertices.

Proof. Let A be a convex polyomino with s rows and t columns. The structure of the proof is as follows. We first show if A satisfies (i), then all its realizations are forbidden induced subgraphs (Claim 1), then that if A satisfies (ii), all its realizations are forbidden induced subgraphs (Claim 2). Note that the presence of both spanning rows and spanning columns implies that all realizations of A are forbidden induced subgraphs of a-HNG for the same value of a. Next we consider when A fails to meet the conditions of the theorem: if it has at most one spanning row or at most one spanning column, we can produce a realization that is not a forbidden induced subgraph (Claim 3). Finally, if A has two spanning rows and two spanning columns but fewer than $(s-1)(t-1)+3$ vertices, we prove a claim about where we can assume those vertices are (Claim 4), then use that claim to show that A has a non-forbidden induced subgraph realization.

Suppose that A has at least two spanning rows and at least two spanning columns. Thus $a(A)$ is well-defined, so let $a = a(A)$. Since A is convex, the set of spanning rows is contiguous and the set of spanning columns is contiguous. We prove that either condition of Theorem 3.6 implies that all realizations of A are forbidden induced subgraphs.

Claim 1. *If there are at least three spanning rows or at least three spanning columns, then all realizations of A are forbidden induced subgraphs.*

Proof. Suppose for a contradiction that some realization G of A is not a forbidden induced subgraph of a-HNG. By Theorem 3.5, G contains a proper induced subgraph H with $a = a(H)$. Let $k = \chi(G) - \chi(H)$ and let $l = \theta(G) - \theta(H)$. Suppose A has p spanning columns and q spanning rows. With p spanning columns, there are at least p disjoint stable sets of size t in G. Since $\alpha(H) \leq \theta(H) = t - l$, it follows that each of the p disjoint stable sets has at least l fewer vertices in H. Thus $|G| - |H| \geq pl$, and by a parallel argument we have $|G| - |H| \geq kq$. Since $a = a(H)$, it follows that $|G| - |H| = k + l$. Thus, $k + l \geq kq$ and $k + l \geq pl$, so $k \geq l(p - 1)$ and $l \geq k(q - 1)$. By substitution, we have $(p - 1)(q - 1) \leq 1$. Since one of p and q is at least three, the other must be at most one, for a contradiction. □

Otherwise, suppose A has exactly two spanning rows and exactly two spanning columns. If A has at most three rows or at most three columns, then either A has too many spanning rows or columns, or A is exactly as shown in Figure 3.5, up to realization equivalence and 90-degree rotation. If the latter, we argue that A is a forbidden induced subgraph of a-HNG as follows. With two spanning rows and two spanning columns in A, a proper induced subgraph H of G would have to contain at least two fewer vertices than G. Hence H has at least two fewer classes than G, so must have at least four fewer vertices than G. If we remove k color classes, then we must remove at least $2k$ vertices. If we remove $l \leq 1$ clique classes, then we must remove at least $2l$ vertices. There is no overlap between vertices removed when deleting color classes of size 2 and vertices removed when deleting the clique class of size 2; hence $|G| - |H| = 2k + 2l$. As in Claim 1, $|G| - |H| = k + l$, so we conclude that $k = 0$ and $l = 0$, for a contradiction. Accordingly we may assume there are at least four rows and at least four columns in A.

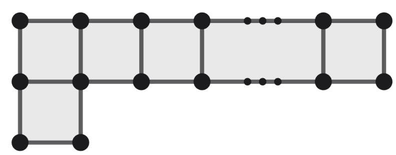

Figure 3.5: The family of polyominoes with two spanning rows, two spanning columns, and three total rows. All realizations of these polyominoes are forbidden induced subgraphs.

Claim 2. *If $|V(A)| \geq (s - 1)(t - 1) + 3$, then all realizations of A are forbidden induced subgraphs.*

Proof. Suppose for a contradiction that some realization G of A is not a forbidden induced subgraph of a-HNG, where $a = a(G)$. By Theorem 3.5, G contains a proper induced subgraph H with $a(G) = a(H)$. Let $k = \chi(G) - \chi(H)$ and let $l = \theta(G) - \theta(H)$. As above, we have $|G| - |H| = k + l \geq 2k$ and $k + l \geq 2l$. Therefore, if $k = 1$, the inequality $k + l \geq 2l$ implies $l = 1$. In that case, H has at most $(s - 1)(t - 1)$ vertices, so G has at most $(s - 1)(t - 1) + 2$ vertices. If $l = 1$ then the argument is symmetric. Otherwise $k, l \geq 2$. Since only four vertices are simultaneously in a spanning row and a spanning column, we have $|G| - |H| \geq 2k + 2l - 4$. Thus $k + l \geq 2k + 2l - 4$, implying $k = l = 2$. Hence H has at most $(s - 2)(t - 2)$ vertices and G has at most $(s-2)(t-2)+4$ vertices. Since $s \geq 4$ and $t \geq 4$, it holds that $(s-2)(t-2)+4 \leq (s-1)(t-1)+2$, for a contradiction. $\square$

For the reverse direction, suppose that A fails to meet the conditions laid out in Theorem 3.6. First, we show that if A has at most one spanning row or at most one spanning column, then its realizations are not all forbidden induced subgraphs. We then show that if A has two spanning rows and two spanning columns but fails the vertex enumeration, then its realizations are not all forbidden induced subgraphs. Let G_0 be the realization of A with edges only between vertices sharing a row in A. Hence G_0 is isomorphic to $\bigcup_{1 \leq i \leq t} K_{j_i}$ for some choices of $j_i \in [s]$.

Given a vertex $v = v_{(i,j)}$, when we *reassign* v to an open spot (k,l), we add edges from v to all vertices in row k and include v in the clique class corresponding to row k and the color class corresponding to column l. The reassignment is legal if and only if v is not adjacent to any vertices currently in column l, such that there exist realizations post-reassignment. From here on out, $v = v_{(k,l)}$ for the purpose of further reassignments (a vertex next reassigned to row k will be made adjacent to v), but of course v retains its past adjacencies and non-adjacencies. Insofar as the polyomino provides a visual representation of the class structure, this analogy of reassigning vertices creates a simpler visual understanding of the underlying convoluted process of adding edges to create a new and smaller set of clique classes.

Claim 3. *If there is at most one spanning row or at most one spanning column, then A has some realization that is not a forbidden induced subgraph.*

Proof. Suppose that A has at most one spanning column. Consider the top row of the polyomino. We will construct a graph G_1 realizing A that contains a proper induced subgraph G_2 such that $a(G_2) \geq a(G_1)$.

Let G_1 and G_2 be graphs initially isomorphic to G_0. If A has a spanning column, delete its vertex in row 1 from G_2 only. For each non-spanning column with spot $(1,i)$ occupied, let j_i be the largest row such that (j_i, i) is occupied. Accordingly, $j_i < t$. Reassign $v_{(1,i)}$ to spot (j_i+1, i) in both G_1 and G_2. That is, we add to $E(G_1)$ and to $E(G_2)$ edges from $v_{(1,i)}$ to all vertices in row j_i+1. We give an example of constructing these graphs from a polyomino A in Figure 3.6. The first rendering of G_2 shows the vertices in row 1 with the added edges, and the second rendering (G_2 (redrawn)) shows the reassigned vertices in their new spots and cliques.

Since there is at most one spanning column, G_2 has at most one fewer vertex than G_1, whose vertex set was not modified. However, $\theta(G_2) < \theta(G_1)$: we can give a clique partition of G_2 where each vertex $v_{(1,i)}$ is included in the clique class generated by row j_i+1, by virtue of the added edges. Recall that $j_i < t$, so this is not a new clique class. With no clique class corresponding to row 1, there are just $\theta(G)-1$ clique classes, so

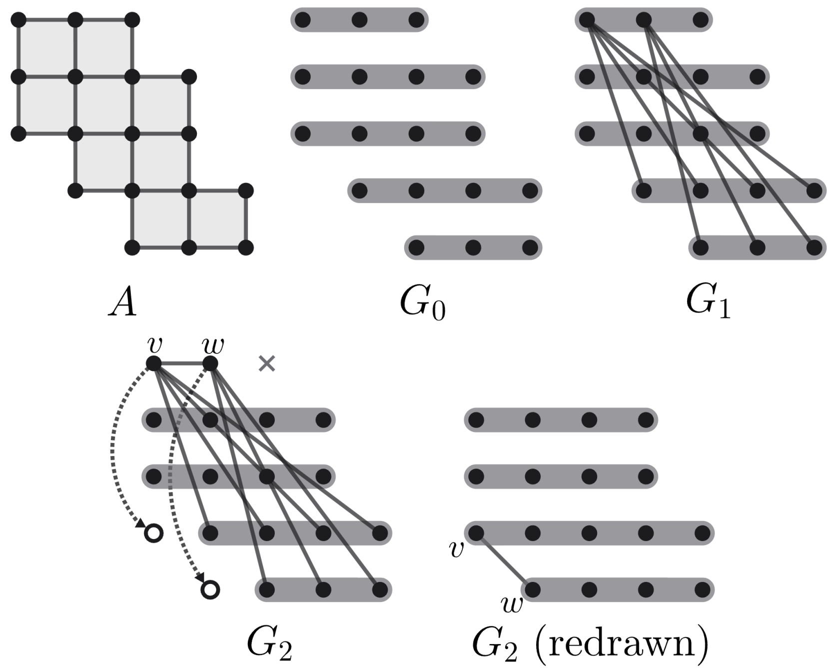

Figure 3.6: A polyomino A and two realizations (G_0 and G_1). The graph G_2 is a proper induced subgraph of G_1, drawn first to show this relationship, and redrawn to show that $\theta(G_2) = 4$. Thick gray edges denote that the highlighted vertices induce a clique; x marks the spot of the deleted vertex in row 1.

$\theta(G_2) \leq \theta(G) - 1 = t - 1$. Since there is a spanning column of t vertices in G_1, we have $\theta(G_1) = t$. Thus $\theta(G_2) < \theta(G_1)$, so $|G_2| - \theta(G_2) \geq |G_1| - \theta(G_1)$. Since G_2 is induced in G_1, it follows that $\chi(G_2) \leq \chi(G_1)$. Thus $a(G_2) = |G_2| - \chi(G_2) - \theta(G_2) \geq |G_1| - \chi(G_1) - \theta(G_1) = a(G_1)$. It follows from Theorem 3.5 that G_1 is not a forbidden induced subgraph realization of A. □

Otherwise, suppose that A has exactly two spanning rows, exactly two spanning columns, and at most $(s-1)(t-1)+2$ vertices. We will produce a graph G realizing A along with an induced subgraph H of G with two fewer vertices, one fewer color class, and one fewer clique class. To do this, we will use and expand on the same reassignment technique as in the proof of Claim 3. We dissolve the bottom row and rightmost column of A, reassigning all but two of these vertices into other columns and rows of the polyomino. We then delete these last two vertices, producing an induced subgraph with two fewer vertices and two fewer classes.

Before finishing the proof of Theorem 3.6, we prove a claim about reassigning vertices to new classes in a polyomino with vertices in exactly two corner spots $(1, s)$ and $(t, 1)$.

Claim 4. *Suppose A has spanning rows i and $i+1$, spanning columns j and $j+1$, and vertices in spots $(1,s)$ and $(t,1)$ but not $(1,1)$ or (t,s). There exists a realization G of A where $\{v_{(i,s)}, v_{(i+1,s)}\}$ is a color class in a minimum coloring of G and $\{v_{(t,j)}, v_{(t,j+1)}\}$ is a class in a minimum clique covering of G.*

Proof. This is equivalent to saying that we can make a series of reassignments until column s and row t each have just two vertices. If spots $(2,2)$ and $(t-1,s-1)$ are occupied, then A has at least $(s-2)(t-2)+s+2+t+2-2=(s-1)(t-1)+5$ vertices, for a contradiction. Hence we may assume up to symmetry that spot $(2,2)$ is open.

Define the upper-left quadrant of a polyomino to be the set of vertices (x,y) with $x \leq i$ and $y \leq j$. We say that the upper-left quadrant of a polyomino is convex if all vertices in the quadrant have a neighboring vertex below and to their right. (It is possible to make an equivalent definition on the squares of the polyomino.) In A, since we need column s to correspond to the color class $\{v_{(i,s)}, v_{(i+1,s)}\}$, we must reassign all other vertices in column s above row i into the upper-left quadrant. Since $(1,s)$ is occupied, there are no vertices in column s below row $i+1$, so it suffices to relocate the vertices above row i. Similarly, we must reassign vertices in row t to the left of (t,j) into the upper-left quadrant. After these reassignments, the resulting polyomino will not be realization-equivalent to A, but will share at least one realization with A.

Given a polyomino A', let i be the smallest index such that $(i,1)$ is occupied, j is the smallest index such that $(1,j)$ is occupied, p be the smallest index such that (p,t) is occupied, and q be the smallest index such that (q,s) is occupied. Let Z be the set of spots (x,y) such that $(p,q) \leq (x,y) \leq (i,j)$. Our claim is equivalent to the following: suppose in A' the following six conditions hold:

1. the upper-left quadrant is convex,
2. the only occupied spot in Z in column q is (i,q),
3. the only occupied spot in Z in row p is (p,j),
4. spot $(p+1,q+1)$ is open,
5. spots $(p,s), \ldots (i-1,s)$ are occupied,
6. and spots $(t,q), \ldots, (t,j-1)$ are occupied.

Given these myriad conditions, we claim we can simultaneously reassign vertices $v_{(t,q)}$ to $v_{(t,j-1)}$ into rows with index less than i without changing their column, and vertices

$v_{(p,s)}$ to $v_{(i-1,s)}$ into columns with index less than j without changing their row, and that the upper-left quadrant of the resulting polyomino is convex. We prove this by induction on the ordered pair (i, j), which we call the *pivot spot* (shown in gray in our figures).

Since spot $(2, 2)$ is open, our base case is pivot spot $(3, 3)$ (see Figure 3.7, A). With $(p, q) = (1, 1)$, we reassign $v_{(1,s)}$ to $(1, 1)$, $v_{(2,s)}$ to $(2, 2)$, $v_{(t,1)}$ to $(2, 1)$, and $v_{(t,2)}$ to $(1, 2)$.

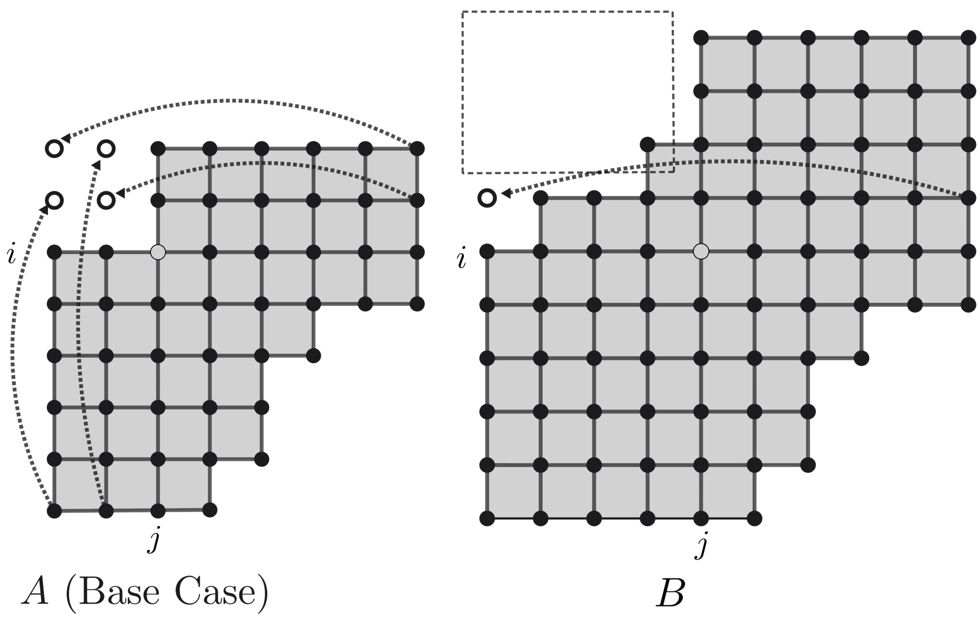

Figure 3.7: A is a representative polyomino for the base case of the proof of Claim 4. B is a representative polyomino for the case where spot $(i-1, 2)$ is occupied.

Let $x, y \geq 3$ and suppose the claim holds for all pivot spots (i, j) with $i \leq x$, $j \leq y$, and $(i, j) \neq (x, y)$. Now, suppose $i = x$ and $j = y$. If spot $(i-1, 2)$ is occupied, then reassign $v_{(i-1,s)}$ to spot $(i-1, 1)$ (see Figure 3.7, B). Thus row $i-1$ is now a spanning row. Note that since $(2, 2)$ is open, we have $i \geq 4$. From here, our induction hypothesis is satisfied with pivot spot $(i-1, j)$ to finish the argument. A symmetric argument holds if spot $(2, j-1)$ is occupied.

Otherwise, both spots $(i-1, 2)$ and $(2, j-1)$ are open. Suppose that $(i-1, z)$ is the leftmost occupied spot in row $i-1$. If $z < j$, we proceed as follows (see Figure 3.8, C). For $d \leq z-2$, reassign vertex $v_{(t,d)}$ to spot $(i-1, d)$, and reassign vertex $v_{(i-1,s)}$ to spot $(i-1, z-1)$, filling row $i-1$ in Z. From here, pivot spot $(i-1, j)$

satisfies our induction hypothesis with $(p, q) = (1, z-1)$ (note that all of columns 1 and 2 above row $i-1$ remain empty).

Else $z = j$ and as such the only occupied spots in the upper-left quadrant are in row i or column j. We handle this case directly (see Figure 3.8, D). For $1 \leq d \leq j-2$, reassign $v_{(t,d)}$ to spot $(i-1, d)$, and reassign $v_{(t,j-1)}$ to spot $(i-2, j-1)$. For $1 \leq d \leq i-3$, reassign $v_{(d,s)}$ to spot $(j-1, s)$. Lastly, reassign $v_{(i-2,s)}$ to spot $(i-2, j-2)$ and reassign $v_{(i-1,s)}$ to spot $(i-1, j-1)$, completing the case. □

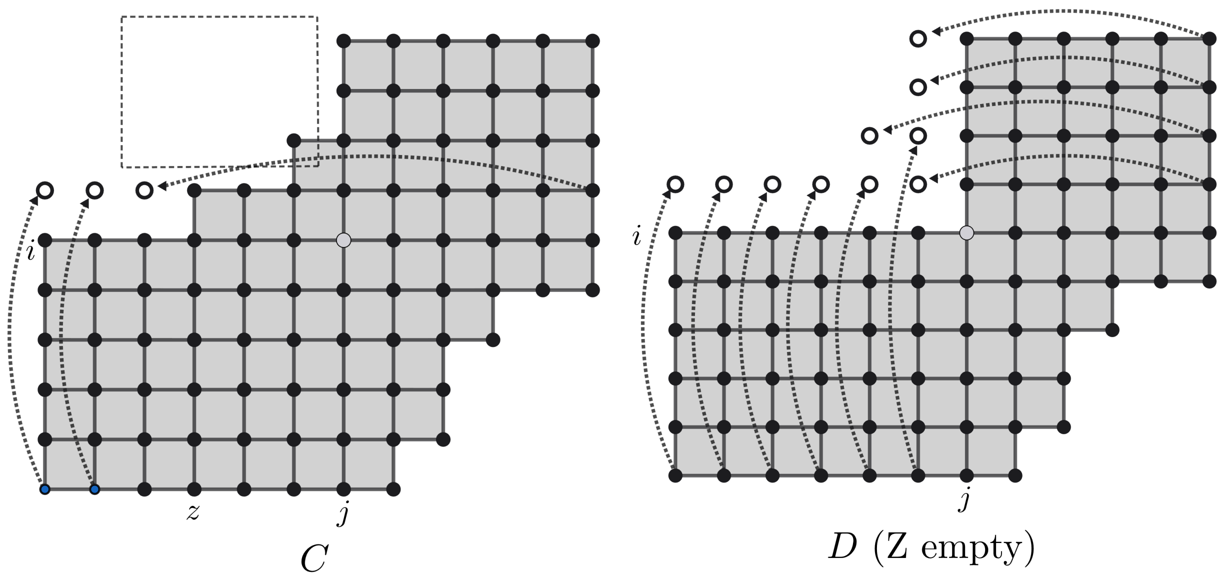

Figure 3.8: C is a representative polyomino for the case where $z < j$, and D gives an example of reassignment for polyominoes with $z = j$.

We leverage Claim 4 to finish the proof of the theorem. Recall we have assumed A has exactly two spanning rows, exactly two spanning columns, and at most $(s-1)(t-1)+2$ vertices. We will produce a graph G_1 realizing A along with an induced subgraph G_2 of G_1 with two fewer vertices, one fewer color class, and one fewer clique class. By Theorem 3.5, $a(G_2) = a(G_1)$, so G_1 is not a forbidden induced subgraph.

To begin, let $G_2 = G_0$ and let $G_1 = G_0$. We consider three cases based on which of the four corner spots $(1, 1), (1, s), (t, 1)$, and (t, s) are occupied, and show that for each, either the polyomino is realization-equivalent to one with only two vertices in column s and two vertices in row t, or we can reassign vertices such that column s and row t have just two vertices each.

First, if any two corner spots sharing a column or row are occupied, then that column or row is spanning. Furthermore, A is realization-equivalent to a polyomino where the rows and columns are in non-increasing order. We may assume A has this canonical form, with spanning rows 1 and 2 and spanning columns 1 and 2. It follows that row t and column s each contain only two vertices, in the respective spanning rows and columns.

Second, if there are exactly two diagonally-opposite corner-spot vertices that are occupied, then suppose up to reflection that they are $v_{(1,s)}$ and $v_{(t,1)}$. Note that reflection preserves realization-equivalence. With these opposite corner-spot vertices, we use the recursive algorithm given in the proof of Claim 4 to reassign all non-spanning vertices in the bottom row and right column out of this row and column. For each vertex moved to a new row, add all edges between that vertex and vertices in its new row to $E(G_2)$ and $E(G_1)$. Now, row t and column s contain exactly two vertices and the polyomino remains convex.

Third, if A began with at most one corner-spot vertex, for each i such that (t, i) is occupied and column i is not a spanning column, let j_i be the smallest index such that (j_i, i) is occupied. Hence $j_i \geq 2$. We reassign $v_{(t,i)}$ up to spot $(j_i - 1, i)$. Add to $E(G_2)$ and $E(G_1)$ edges from $v(t, i)$ to all vertices in row $j_i - 1$. By symmetry, for each i such that $v_{(i,s)} \in V(G)$ and row i is not a spanning row, let j_i be the smallest index such that $v_{(i,j_i)} \in V(G)$. Hence $j_i \geq 2$. We reassign $v_{(i,s)}$ leftward to spot $(i, j_i - 1)$. Hence at this point we have a clique partition and a coloring where the bottom row and the rightmost column each contain only the two vertices from the spanning rows and columns. In fact, we have shown this holds regardless of the starting arrangement of corner-spot vertices in A.

In all cases, since there are at most $(s - 1)(t - 1) + 2$ vertices, and exactly four of these are now in row t or column s, it follows that within the upper $t - 1$ rows and $s - 1$ columns, there are at least two open spots. Suppose that rows i and $i + 1$ and columns j and $j + 1$ are the spanning rows and columns, respectively. Among the open spots in the upper $t - 1$ rows and $s - 1$ columns ranked by taxicab distance to $v_{(i,j)}$, breaking ties arbitrarily, let (h_1, h_2) and (h_3, h_4) be the two smallest such spots.

We now provide an algorithm for deleting in G_2 two of the four vertices from row t and column s, and reassigning in G_1 and G_2 the other two vertices into the open spots (h_1, h_2) and (h_3, h_4). The algorithm is illustrated in Figure 3.9. First, reassign vertex $v_{(i,h_2)}$ to (h_1, h_2), adding the requisite edges to all other vertices in row h_1. Second, reassign vertex $v_{(t,j)}$ to (i, h_2), making it adjacent to all vertices in row i except $v_{(i,j)}$. Third, reassign $v_{(i+1,h_4)}$ to (h_3, h_4), adding all requisite edges except to $v_{(h_3,j+1)}$ to $E(G_1)$ and $E(G_2)$. Fourth, reassign vertex $v_{(h_3,j+1)}$ to now-open spot $(i + 1, h_4)$, adding all requisite edges except to $v_{(i+1,j+1)}$. Fifth, reassign vertex $v_{(t,j+1)}$ to now-open spot $(h_3, j + 1)$, making it adjacent to all vertices now in row h_3. With respect to the original locations of vertices, these are all diagonal edges, so the original columns and rows still provide a minimum coloring and clique covering of G_1 in s colors and t clique classes.

In G_2, now, we delete vertices $v_{(i,j)}$ and $v_{(i+1,j+1)}$. We reassign vertex $v_{(i,s)}$ to now-open spot (i, j) and vertex $v_{(i+1,s)}$ to now-open spot $(i+1, j+1)$. Since their rows remain constant, no new edges are required. Then G_2 has two fewer vertices than G_1, whose vertex set was not modified. By the given edge sets, G_2 is an induced subgraph of G_1.

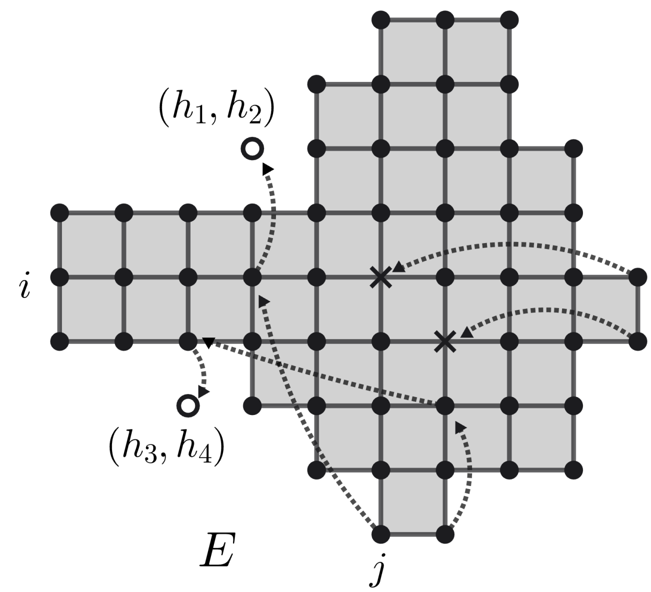

Figure 3.9: E is a representative polyomino for the reassignment algorithm described at the end of the proof of Theorem 3.6. The vertices at spots (i, j) and $(i + 1, j + 1)$ are deleted at the beginning of the algorithm.

We can show that $\chi(G_2) + \theta(G_2) \leq \chi(G_1) + \theta(G_1) - 2$ as follows. We are able to give a proper coloring of G_2 using columns 1 to $s - 1$, since every vertex in column s has been reassigned to a new column, and each of the $s - 1$ columns still induces an independent set. Hence $\chi(G_2) \leq s - 1$. Similarly, we can give a clique covering of G_2 using rows 1 to $t-1$. Each of these rows, together with their reassigned vertices from row t, induces a clique. Hence $\theta(G_2) \leq t - 1$. Thus $a(G_2) = |G_2| - \chi(G_2) - \theta(G_2) \geq (|G_1| - 2) - (s - 1 + t - 1) = |G_1| - s - t = a(G_1)$. Therefore by Theorem 3.5, G_1 is not a forbidden induced subgraph realization of A.

We have shown that any polyomino with exactly two spanning rows and exactly two spanning columns but fewer than $(s - 1)(t - 1) + 3$ vertices admits a realization G_1 that is not a forbidden induced subgraph of a-HNG for any $a \geq 0$, completing our proof of Theorem 3.6. $\square$

The remainder of this section consists of applications of this theorem. We will use Theorem 3.6 to characterize the relationship between minimum-order polyominoes and minimum forbidden induced subgraphs as well as to establish an upper bound on the order of forbidden induced subgraphs.

As shown above, the minimum order of a forbidden induced subgraph of a-HNG is equal to the minimum number of vertices of an $(a+1)$-polyomino. In fact, all minimum forbidden induced subgraphs of a-HNG are realizations of an $(a+1)$-polyomino with minimum vertex count. Six examples of these minimum polyominoes are given in Figure 3.10. The minimum polyomino for 0-HNG and its three realizations, which are the three forbidden induced subgraphs of 0-HNG , are shown in Figure 3.2. Recall from Theorem 3.3 that the minimum order of a forbidden induced subgraph of a-HNG is $\lceil a+2+2\sqrt{a+1}\rceil$.

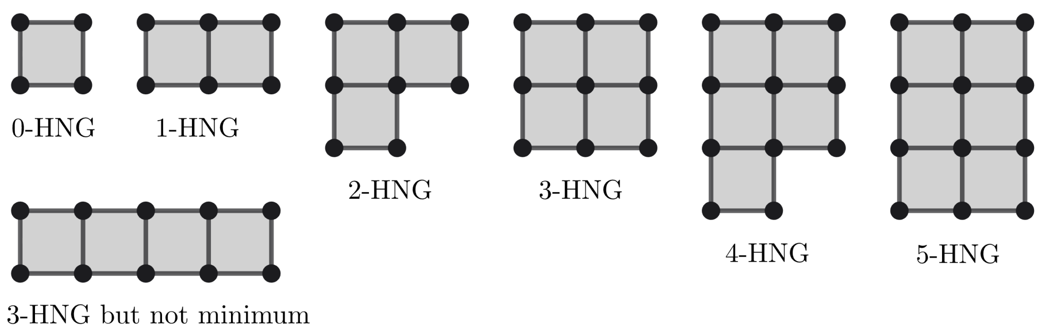

Figure 3.10: The polyominoes from which minimum forbidden induced subgraphs of 0-HNG to 5-HNG are realized. The bottom polyomino admits realizations that are all minimal **but not minimum** forbidden induced subgraphs of 3-HNG

Theorem 3.7. *A graph G is a* minimum *forbidden induced subgraph of a-HNG if and only if it is a realization of a minimum-order $(a+1)$-polyomino.*

Proof. We show that minimum-order $(a+1)$-polyominoes are convex and satisfy the constraints of Theorem 3.6. We ultimately leverage this theorem to imply all realizations of such polyominoes are forbidden induced subgraphs.

Suppose that A is a non-convex $(a+1)$-polyomino. Thus A is non-empty. Rather than discussing rows and columns of vertices, as in the prior theorems, we spend the next few paragraphs considering rows and columns of squares. Since A is non-convex, there exists at least one row or column of squares where the set of squares are disconnected. Up to symmetry, suppose a row has this property.

Let A' be the left-justified configuration with the same number of squares in each row as has A: that is, if A has x squares in row i, then A' also has x squares in row i, but specifically in columns 1 to x. Since A is connected and non-empty, every row of A contains at least one square. Thus column 1 in A' contains squares in all rows, and A' is connected and hence an $(a+1)$-polyomino. Each row of A' is a connected unit, without gaps. Furthermore, $|V(A')| \leq |V(A)|$, since each row of vertices in A' has at most as many vertices as the corresponding row of A. Since A has a disconnected row, there are now strictly fewer vertices in that row of A'. Thus $|V(A')| < |V(A)|$.

Let A'' be the top-justified configuration with the same number of squares in each column as has A', constructed as above. Thus every column of A'' contains at least one square, and the leftmost column is unchanged from A'. Since the columns of A'' have a non-increasing number of squares from left to right, and the rows of A'' now have a non-increasing number of squares from top to bottom, it follows that A'' is convex. Furthermore, $|V(A'') \leq V(A')|$, since each column of vertices in A' has at most as many vertices as the corresponding column of A'. Thus $|V(A'')| < |V(A)|$. Since A'' is a convex $(a+1)$-polyomino with fewer vertices than A but the same number of squares, we conclude that any minimum $(a+1)$-polyomino must be convex.

Suppose that A is a convex $(a+1)$-polyomino. We construct A' and A'' as above; they hence are convex. In general, a row of vertices must be at least one longer than either of its neighboring rows of squares. This minimum length is achieved if and only if the squares directly above the vertex row are in a columnar subset of the squares directly below the vertex row, or vice versa. If A has no spanning row of squares, then there exists a pair of consecutive rows of squares where one is not a subset of the other. The row of vertices in between those two rows, then, is strictly longer in A than in A'. Since no row is shorter in A than in A', it follows that $|V(A')| < |V(A)|$.

Let A be a minimum $(a+1)$-polyomino; it is hence convex and contains at least two spanning vertex rows and two spanning vertex columns, as shown above, which must be contiguous. If A contains at least three spanning rows or three spanning columns of vertices, then by Theorem 3.6, all its realizations are forbidden induced subgraphs of a-HNG. Otherwise, A has only two spanning rows and two spanning columns of vertices. Suppose that i is the index of the spanning row of squares and j is the index of the spanning column of squares. Let $s-1$ be the number of columns of squares and $t-1$ be the number of rows of squares. We can safely assume that each is at least 2, else the polyomino consists of one square only or has at least three spanning rows/columns of vertices.

Consider the first and last rows of squares; they can overlap by at most one column of squares, else A would have more than two spanning columns of vertices. The squares of the polyomino each live in the $s-1 \times t-1$ rectangle formed by the spanning row and column of squares. For each unit square within that $s-1 \times t-1$ rectangle, we call the square an *open position* if it is not included in A. Construct polyomino A^* as follows (see Figure 3.11 for an example): for each square in the bottom row not in column j, move the square to the open position in its column with row index closest to i. Since the column and originally contains a square in the bottom row, there must exist an open position with index less than i. The result, A^*, is column-convex by definition and is row-convex because a gap in row $k < i$ could be created only if a square is sent past the lowest open position in its column.

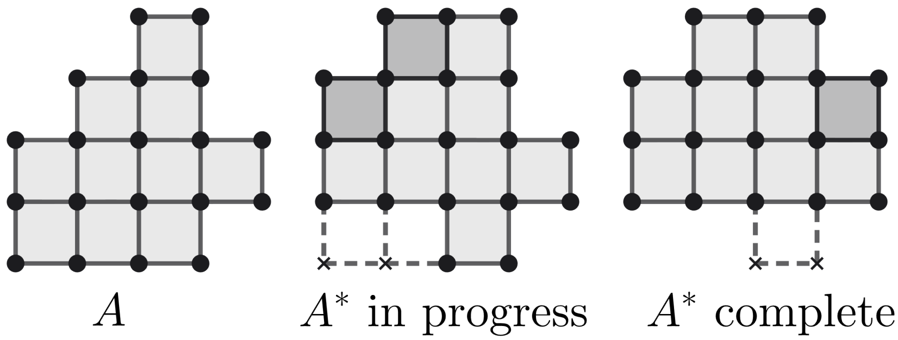

Figure 3.11: An example of constructing A^* from a convex polyomino A with two spanning rows and two spanning columns. At each step, the added squares are darkened and the removed squares are shown with a dashed border.

Suppose there are c squares in the bottom row of A: accordingly, $c-1$ vertices are deleted in the construction of A^*. However, $c-1$ new vertices are added in its construction, since each square's new position in A^* is incident to a square directly below it and to its left.

If there are any more open positions not in the bottom row of A^*, delete the final square from the spanning column of A^* and move it to an open position with least taxicab distance to (i, j); this move ensures that A^* has strictly fewer vertices than A, contradicting the minimality of the original polyomino.

Otherwise, there are no more open positions above the bottom row. We compute the number of vertices in A. Since A contains two spanning rows and two spanning columns, the number of vertices in all but the bottom row of A is equal to $s(t-1)-b$, where b is the number of open positions in all but the bottom row of A. Suppose there are c squares in the bottom row of A; thus $b < c$. Those c squares contribute $c+1$ extra vertices to A. Hence $|V(A)| = s(t-1) - b + c = (s-1)(t-1) + (c-b) + (t-1)$. Since $t-1 \geq 2$ and $c-b \geq 1$, we conclude that $|V(A)| \geq (s-1)(t-1)+3$. Therefore by Theorem 3.6, all realizations of A are forbidden induced subgraphs of a-HNG.

Hence: any realization of a minimum-order $(a+1)$-polyomino is necessarily a realization of a polyomino satisfying Theorem 3.6, so is a forbidden induced subgraph of a-HNG. Because it has $\lceil a+2+2\sqrt{a+1}\rceil$ vertices by OEIS A080037, it follows from Theorem 3.3 that it is a minimum forbidden induced subgraph of a-HNG. Conversely, a minimum forbidden induced subgraph of a-HNG, like all graphs, is a realization of some polyomino; by Theorem 3.3 and OEIS A080037, this polyomino is of minimum order. □

Theorem 3.6 applies to far more than minimum polyominoes: lots of types of polyominoes satisfy its requirements. These include all rectangles, all rectangles with squares missing from at most one row and one column, and rectangles with squares missing from two rows (if $s \leq t$) or two columns (if $t \leq s$). The $2 \times t$ rectangle polyominoes are particularly useful because their realizations are all bipartite graphs on $2a + 4$ vertices. We record information about these forbidden induced subgraphs as a corollary. A *perfect matching* in a bipartite graph on $2m$ vertices is a set of m edges with distinct endpoints.

Corollary 3.8. *Any bipartite graph with $2a + 4$ vertices that contains a perfect matching or its complement is a forbidden induced subgraph for a-HNG.*

Proof. Suppose that G is a bipartite graph with $2a + 4$ vertices and containing a perfect matching. Therefore, G is a realization of the polyomino A with 2 columns and $a + 2$ rows, containing $2(a + 2) = 2a + 4$ vertices. The polyomino A is shown in Figure 3.12. Since $2a + 4 \geq a + 2$, it holds from Theorem 3.4 that all realizations of A are forbidden induced subgraphs of a-HNG. Thus G is a forbidden induced subgraph of a-HNG. Since a-HNG is closed under complementation, the corollary holds. □

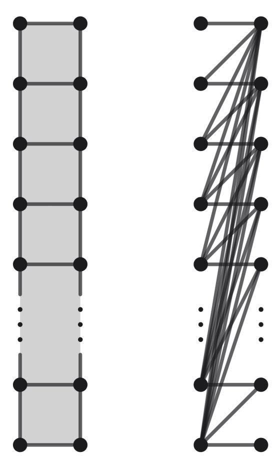

Figure 3.12: Two-column polyominoes admit realizations that are bipartite graphs containing perfect matchings. The graph shown at right is a half-graph that realizes the polyomino at left.

We use these bipartite forbidden induced subgraphs to give an upper bound on the order of forbidden induced subgraphs for a-HNG.

Theorem 3.9. *For all $a \geq 0$, the class a-HNG has finitely many forbidden induced subgraphs. Specifically, if a graph G has order at least $12a^2 + 24a$, it cannot be a forbidden induced subgraph.*

Proof. The cases $a = 0$ and $a = 1$ are proven in Theorem 2.3 and Theorem 2.9.

For $a \geq 2$, we prove that all forbidden induced subgraphs of a-HNG are of order less than $12a^2 + 24a$. We proceed with a combinatorial argument to show that a forbidden induced subgraph on a large number of vertices relative to a must in any proper coloring and clique covering contain a large number of color and clique classes of size 2. A set of these classes, in turn, either form a smaller forbidden induced subgraph or can be removed to leave behind a smaller induced subgraph H with $a(H) = a(G)$, contradicting Theorem 3.5.

Suppose that a graph G with n vertices is a forbidden induced subgraph of a-HNG for $a = a(G)$. Fix a minimum proper coloring and clique covering of G and let $\mathcal{C}$ be the union of the set of classes of each. Hence $\mathcal{C}$ is a set of stable sets and cliques in $V(G)$, and each vertex in G is contained in exactly two classes of $\mathcal{C}$. By Theorem 3.1, each class C in $\mathcal{C}$ contains at least two vertices, and $\mathcal{C}$ has $n - a$ classes in total. Thus the number of instances of vertices in excess of 2 in classes of $\mathcal{C}$ is counted by

$$\sum_{C \in \mathcal{C}} (|C| - 2) = |\mathcal{C}|(-2) + \sum_{C \in \mathcal{C}} |C| = (n - a)(-2) + 2n = 2a.$$

Even if each of these $2a$ elements belongs to a different class, there are at least $n - 3a$ classes of size 2. With so many classes of size 2, most intersect only other classes of size 2. Define the distance of a vertex in the configuration to be the number of classes separating it from a class of size at least 3: a vertex v_0 in a class of size at least 3 has distance 0, any vertex v_1 in a size 2 class with v_0 has distance at most 1, any vertex v_2 sharing a size 2 class with v_1 has distance at most 2, and so on. Any vertices without finite distance are said to have distance ∞.

At most $2a$ vertices can be in excess of 2 in a class, so at most $6a$ vertices are in a class of size at least 3. Thus at most $6a$ vertices have distance 0. Since each can be in at most one class of size 2, there are at most $6a$ vertices with distance 1. In general, there are at most $6ka$ vertices of distance at most k. Hence there is at least one vertex of distance $\lceil \frac{n}{6a} \rceil - 1$.

Define a chain of classes of size 2 to be a set of classes of size 2 where each shares one vertex with the next. Given a chain of length k, we can label the vertices of the chain by $\{v_j, v_{j+1}, \ldots, v_{j+k-1}\}$ for j equal to 0 or 1, such that vertices $\{v_i, v_{i+1}\}$ form a clique class where i is even and a color class where i is odd. Let $\mathcal{S}$ be a chain with at least 6 vertices. We will determine the edge relations in $\mathcal{S}$ except for between vertices v_i, v_{i+2} for any i. First, given vertices $v_i, v_j \in S$ such that i is even, j is odd, and $j \geq i + 3$, we claim that v_i and v_j

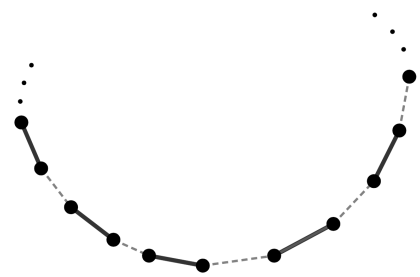

Figure 3.13: A chain in a graph is a series of clique classes (shown with solid edges) and color classes (shown with dotted lines), all of size 2.

are non-adjacent. Suppose for a contradiction that they are adjacent and consider the graph G' induced on vertex set $V(G) - \{v_{i+1}, \dots, v_{j-1}\}$, with a fixed coloring and clique covering inherited from G. Merge the clique classes $\{v_i\}$ and $\{v_j\}$ into a single clique class. Now G' has $j-i-2$ fewer vertices than G and $j-i-2$ fewer clique and color classes, so G is not a forbidden induced subgraph by Theorem 3.5, for a contradiction. By complementation, if i is odd and $j \geq i+3$ is even, v_i and v_j must be adjacent.

Next, we show that that for all clique classes $\{v_i, v_{i+1}\}$ in the chain, any other vertex v_j cannot be adjacent to both. This is trivially true for $j = i-1$ or $j = i+2$, and the above paragraph implies the result for $j = i-2$ or $j = i+3$. Suppose for a contradiction that there exists $i < j$ such that i is even, $j \leq i-3$ or $j \geq i+4$, and v_j is adjacent to both v_i and v_{i+1}. Consider the graph G' induced on vertex set $V(G) - \{v_{i+2}, \dots, v_{j-1}\}$, with a fixed coloring and clique covering inherited from G. Merge the clique classes $\{v_i, v_{i+1}\}$ and $\{v_j\}$ into a single clique class. Now G' has $j-i-3$ fewer vertices than G and $j-i-3$ fewer clique and color classes, so G is not a forbidden induced subgraph by Theorem 3.5, for a contradiction. An analogous proof holds for $j \leq i-3$, and we conclude that vertices sharing a clique class have no common neighbors. Thus, given vertices $v_i, v_j \in S$ such that i and j have the same parity and $j \geq i+4$, we claim that v_i and v_j are non-adjacent. If i and j are even, then v_{i+1} is adjacent to v_j, so v_i cannot be. If i and j are odd, then v_i is adjacent to v_{j-1}, so it cannot be adjacent to v_j.

Suppose for a contradiction that $\mathcal{S}$ contains at least $4a+7$ vertices. Then there exists a subset $S \in V(G)$ equal to $\{v_i, v_{i+1}, v_{i+4}, v_{i+5}, v_{i+8}, v_{i+9}, \dots, v_{i+4a+4}, v_{i+4a+5}\}$ for some even $i \in \mathbb{N}$ such that all vertices in S are contained in the chain $\mathcal{S}$. By the above arguments, S induces a subgraph of G whose edge relations are completely determined: the only adjacencies are from each odd-indexed vertex v_j to all vertices with even indices at least $j-1$. Thus the induced subgraph with vertex set S is isomorphic to the half graph on $2a+4$ vertices, which is known to be a forbidden induced subgraph for a-HNG by Theorem 3.8.

Since G does not contain any proper forbidden induced subgraphs of a-HNG, it follows that G cannot contain a chain with $4a+7$ vertices in it. A chain with at least $4a+7$ vertices has a maximum vertex distance of at least $2a+3$. Thus G cannot contain such a vertex. We already showed that $\mathcal{C}$ must contain a vertex of distance at least $\lceil \frac{n}{6a} \rceil - 1$. Hence $\lceil \frac{n}{6a} \rceil - 1 \leq 2a+2$. Therefore $\frac{n}{6a} < 2a+4$, and we conclude that $n < 12a^2 + 24a$. □

We conclude this section with two optimization results on a-HNG, which make use of Ramsey's and Hall's well-known theorems.

Theorem 3.10 (Ramsey [57]). *Given positive integers x and y, there exists a least positive integer $R(x,y)$ such that all graphs G with at least $R(x,y)$ vertices contains either a clique of size x or a stable set of size y.*

Let $N(v)$ denote the neighborhood of vertex v (that is, the set of vertices adjacent to v). Given a subset $W \subseteq V(G)$, define $N(W) = \bigcup_{v \in W} N(v)$.

Theorem 3.11 (Hall [38]). *Let G be a bipartite graph with bipartite sets Y and Z. The graph G contains a perfect matching if and only if for all $W \subseteq Y$ we have $|W| \leq |N(W)|$ and for all $X \subseteq Z$ we have $|X| \leq |N(X)|$.*

If G does not have a perfect matching, then without loss of generality for some subset W of Y we have $|W| > |N(W)|$. Then $W \cup (Y - N(W))$ is a stable set with cardinality strictly greater than Y.

We are able to show that a-HNG is linearly χ-bounded and that the clique and stable set numbers of graphs in a-HNG can be found in polynomial time. The problem of finding the chromatic and clique cover numbers in polynomial time, as is possible in 1-HNG, remains open.

Theorem 3.12. *The graph class $a-HNG$ is linearly χ-bounded by $f(x) \leq x + R(a+1, a+1)$.*

Proof. Let $G \in a$-HNG, let K be a maximum clique in G, and let S be a maximum stable set in G. Let $C = V(G) - K - S$ be the set of remaining vertices. Hence $K \cup S$ induces a split graph H. Since split graphs are perfect, $\chi(H) = \omega(H) \leq |K| + 1$.

We claim that $\omega(C) \leq a+1$ and $\alpha(C) \leq a+1$. Without loss of generality, suppose $\alpha(C) \geq a+2$, so $|S| \geq \alpha(C) \geq a+2$ as well. Let X be a maximum stable set in C. If $S \cup X$ contains a perfect matching, then it induces a forbidden induced subgraph by Theorem 3.8. Otherwise, $S \cup X$ is a bipartite graph with no perfect matching, so its largest stable set is larger than S by Hall's Theorem, for a contradiction. Hence $\omega(C) \leq a+1$ and $\alpha(C) \leq a+1$, so $\chi(C)$ is bounded by the Ramsey number $R(a+1, a+1)$. Thus $\chi(G) \leq \omega(G) + R(a+1, a+1)$. □

Theorem 3.13. *Let $G \in$a-HNG. Then $\omega(G)$ and $\alpha(G)$ can be determined in polynomial time.*

Proof. Let $G \in$a-HNG. Since $\chi(G)$ is bounded linearly in terms of $\omega(G)$ by Theorem 3.12, it is possible to delete at most $R(a+1, a+1)$ vertices to produce a perfect induced subgraph of G. Furthermore, some such perfect induced subgraph H of G must have $\omega(H) = \omega(G)$, as in the construction in the proof of Theorem 3.12. There are polynomially many (in $n = |G|$) subsets W of $V(G)$ of size at most $R(a+1, a+1)$. For each of these subsets, check if the subgraph induced on $V(G) - W$ is perfect, which can be done in polynomial time [15]. If it is perfect, compute the clique number in polynomial time (Grötschel, Lovász, and Schrijver show this is possible [34]). Again, one of these induced subgraph has clique number equal to $\omega(G)$, so among all the candidate clique numbers, the maximum must be $\omega(G)$. By complementation we can find $\alpha(G)$ in polynomial time. □

The related problems of finding $\chi(G)$ and $\theta(G)$ for graphs in *a*-HNG remain open.

3.3 Sum-Perfect and Apex-Split Graphs

In this section we discuss two related classes. First, we study the sister-class of sum-perfect graphs, which are defined by a similar inequality as the Nordhaus-Gaddum graphs and share many forbidden induced subgraphs. The results derived from polyominoes in Section 3.2 apply in large part to hereditary generalizations of sum-perfect graphs. Second, we relate those two classes to the bounded-apex-split graphs. All three of these classes are related by their similar definitions and by that bipartite graphs with perfect matchings are forbidden induced subgraphs for them all.

Recall that for all graphs G, the inequality $\omega(G) + \alpha(G) \leq |G| + 1$ holds to describe the relationship between the clique number of a graph and that of its complement. Of course, $\alpha(G) = \omega(\overline{G})$. Litjens, Polak, and Sivaraman [50] defined the *sum-perfect graphs* to be the set of graphs G such that for all induced subgraphs H of G, we have $\omega(H) + \alpha(H) \geq |H|$. We generalize these classes in a fashion akin to *a*-HNG.

Definition 3.2. *For $a \geq 1$, a graph G is a-sum-perfect if for all induced subgraphs H of G, $\omega(H) + \alpha(H) \geq |H| + 1 - a$. Let a-$SP$ denote the set of a-sum-perfect graphs.*

The class 1-SP is exactly the generic class of sum-perfect graphs studied in [50]. By definition, the class is hereditary, and it shares many forbidden induced subgraphs with 1-HNG. In fact, all classes *a*-SP are hereditary by definition, and have many shared properties with *a*-HNG. There exists an analogous lemma to Theorem 3.1 for generalized sum-perfect graph classes. The proof is adapted from that of Theorem 2.7.

Lemma 3.14. *For all $a \geq 1$, if G is a forbidden induced subgraph of a-SP, then $G \in (a+1)$-SP and no vertex's deletion from $V(G)$ changes the clique number or independence number.*

Proof. Let G be a forbidden induced subgraph of a-SP and let $v \in V(G)$. By definition, $G - \{v\} \in a$-SP. Thus we have $\omega(G) + \alpha(G) \leq |G| - a$, but $\omega(G - \{v\}) + \alpha(G - \{v\}) \geq (|G|-1)+1-a = |G|-a$. It holds true that $\omega(G) \geq \omega(G-\{v\})$ and $\alpha(G) \geq \alpha(G-\{v\})$. We therefore conclude that both $\omega(G)+\alpha(G) = |G|-a$ and $\omega(G-\{v\})+\alpha(G-\{v\}) = |G|-a$, so $\omega(G) = \omega(G - \{v\})$, $\alpha(G) = \alpha(G - \{v\})$, and $G \in (a+1)$-SP. ☐

For perfect graphs, the classes a-SP and a-HNG are identical.

Theorem 3.15. *A perfect graph G is a forbidden induced subgraph of a-SP if and only if it is a forbidden induced subgraph of a-HNG.*

Proof. Suppose that G is a perfect graph. Since G and all proper induced subgraphs H are perfect, we have $\chi(G) = \omega(G)$, $\chi(H) = \omega(H)$, $\theta(G) = \alpha(G)$, and $\theta(H) = \alpha(H)$. Then G is a forbidden induced subgraph of a-HNG if and only if by Theorem 3.1, $G \in (a+1)$-HNG and all proper induced subgraphs H of G are in a-HNG. By substitution, $G \in (a+1)$-HNG and all proper induced subgraphs H of G are in a-HNG if and only if $\omega(G)+\alpha(G) = |G|-1$ and $\omega(H) + \alpha(H) \geq |H| + 1 - a$. These hold if and only if $G \in (a+1)$-SP and $H \in a$-SP, i.e., by Theorem 3.14 we have that G is a forbidden induced subgraph of a-SP. ☐

Therefore, all bipartite forbidden induced subgraphs of a-HNG from Theorem 3.8 are also forbidden induced subgraphs of a-SP. This implies that a-SP is linearly χ-bounded by an identical argument to Theorem 3.12, which we record as a corollary.

Corollary 3.16. *The class a-SP is linearly χ-bounded by $f(x) \leq x + R(a+1, a+1)$.*

We suspect that the link between forbidden induced subgraphs of a-SP and a-HNG is far deeper than perfect graphs. The only imperfect forbidden induced subgraph of 1-SP is C_5, which is induced in many forbidden induced subgraphs of 1-HNG(see Theorem 2.9). This inspires the following conjecture.

Conjecture 3.17. *Each forbidden induced subgraph of a-SP is induced in a forbidden induced subgraph of a-HNG.*

We might also be able to say that where a polyomino has a spanning row and a spanning column, the forbidden induced subgraphs are the same.

Furthermore, we can use the set of bipartite forbidden induced subgraphs to show that a-SP has finitely many forbidden induced subgraphs.

Theorem 3.18. *For all $a \geq 1$, the class a-SP has finitely many forbidden induced subgraphs. Specifically, if a graph G has order at least $6a^3 + 36a^2 + 71a + 46$, it cannot be a forbidden induced subgraph.*

Proof. Fix $a \geq 1$ and suppose for a contradiction that G is a forbidden induced subgraph of order at least $6a^3+36a^2+71a+46$. Let K be a maximum clique in G and let S be a maximum stable set in G. By Theorem 3.14, we have that $G \in (a+1)$-SP, so $|K|+|S| = |G|-a$. Since $|K \cap S| \leq 1$, it follows that $|V(G) - K - S|$ is equal to a or $a+1$. Let $C = V(G) - K - S$. Since $|C| \leq a+1$, it follows that $|K \cup S| \geq 6a^3 + 36a^2 + 70a + 45$. The pigeonhole principle implies that at least one of K and S has order at least $3a^3 + 18a^2 + 35a + 23$. Suppose without loss of generality that this is K.

Let A be the set of vertices in $C \cup S$ that occur in some maximum clique of G, and let $S' = A \cap S$. Thus $|A| \leq |C| + |S'| \leq |C| + |S|$. Each maximum clique in G uses at most one vertex from S' and at most $|C| \leq a+1$ vertices from C, so excludes at most $a+2$ vertices from K. Let $v \in A$. Since v is in some maximum clique, it has at most $a+2$ non-neighbors in K. Thus the vertices of A collectively have at most $|A|(a+2)$ non-neighbors in K. Similarly, we define B to be the set of vertices in $K \cup C$ in a maximum stable set of G, and let $K' = K \cup B$.

Suppose $|S'| \geq 2a+5$ and $|K'| \geq 2a+5$. Just as each vertex in S' has at most $a+2$ non-neighbors in K, each vertex in K' has at most $a+2$ neighbors in S. Hence each vertex in K' has at least $|S'|-(a+2) \geq 2a+5-(a+2) = a+3$ non-neighbors in S'. Thus K' and S' have at least $|K'|(a+3) \leq (2a+5)(a+3)$ non-edges between them. However, since each vertex in S' has at most $a+2$ non-neighbors in K, there are at most $|S'|(a+2) \geq (2a+5)(a+2)$ non-edges between S' and K'. Hence:

$$|K'|(a+3) \leq |S'|(a+2).$$

By a parallel argument, there are at most $|K'|(a+2)$ edges between K' and S', and at least $|S'|(|K'| - (a+2)) \leq |S'|(a+3)$ edges between K' and S'. Thus:

$$|S'|(a+3) \leq |K'|(a+2).$$

The centered inequalities are irreconcilable for nonempty K' and S', so we reach a contradiction.

Otherwise one of S' and K' has at most $2a+4$ vertices. Suppose $|S'| \leq 2a+4$. Hence $|A| \leq a+1+2a+4 = 3a+5$, so at most $3a+5$ vertices outside of K are in any maximum clique. Thus the vertices of A collectively have at most $(3a+5)(a+2) = 3a^2 + 11a + 10$

non-neighbors in K. In other words, at most $3a^2+11a+10$ vertices in K have non-neighbors in A. Since $|K| > 3a^2+11a+10$, it follows that some vertex $k \in K$ is complete to A and therefore in all maximum cliques of G. Its deletion changes the clique number, contradicting the assumption that G is a forbidden induced subgraph by Theorem 3.14.

Otherwise, $|K'| \leq 2a+4$. By the above argument, if $|S| > 3a^2+11a+10$, we reach a contradiction, so $S \leq 3a^2+11a+10$. Hence $|A| \leq |C|+|S| \leq (a+1)+(3a^2+11a+10) = 3a^2+12a+11$. There are at most $|A|(a+2) \leq (3a^2+12a+11)(a+2) = 3a^3+18a^2+35a+22$ non-edges between A and K. Hence at most $3a^3+18a^2+35a+22$ vertices in K have non-neighbors in A. Since $|K| \geq 3a^3+18a^2+35a+23$, it follows that some vertex $k \in K$ is complete to A and therefore in all maximum cliques of G. Its deletion changes the clique number, contradicting Theorem 3.14. This completes the proof. □

We end the chapter with a further link between generalized hereditary-Nordhaus-Gaddum graphs, generalized sum-perfect graphs, and bounded-apex-split graphs. Recall that a graph family is bounded-apex-split if there exists a positive integer c such that each graph G in the class contains a split induced subgraph with $|G|-c$ vertices. All classes in these families have bipartite forbidden induced subgraphs, and therefore any hereditary class contained within a class in one such family must be contained within a class in all these families. While our previous characterizations (Theorem 3.8 for *a-HNG* and Theorem 3.15 for *a-SP*) pertain to all bipartite graphs with perfect matchings, we show we need only consider complete bipartite graphs, perfect induced matchings, and their complements.

Let $K_{d,d}$ denote the complete bipartite graph with d vertices in each partite set, and let dK_2 denote the graph composed of d disconnected edges.

Theorem 3.19. *The following are equivalent for a hereditary class $\mathcal{F}$:*

(i) There exists $a \geq 0$ such that $\mathcal{F}$ is a subset of the a-hereditary-Nordhaus-Gaddum graphs.

(ii) There exists $b \geq 0$ such that $\mathcal{F}$ is a subset of the b-sum-perfect graphs.

(iii) There exists $c \geq 0$ such that $\mathcal{F}$ is a subset of the c-apex-split graphs.

(iv) There exists $d \geq 0$ such that no graph in $\mathcal{F}$ contains $K_{d,d}$, dK_2, or the complements of these graphs as induced subgraphs.

Proof. Let $\mathcal{F}$ be a hereditary class. Let $\mathcal{B}_d = \{K_{d,d}, dK_2, \overline{K_{d,d}}, \overline{dK_2}\}$ be the set of graphs of order $2d$ forbidden in (iv). Let $\mathcal{B} = \bigcup_{d\geq 0} \mathcal{B}_d$ be the set of graphs of any order forbidden in (iv). We first assume (iv) is false and prove each of (i), (ii), and (iii) are false, then assume (iv) is true and show the same holds of (i), (ii), and (iii).

Suppose (iv) is false, i.e., that for all $d \geq 0$ there exists G_d in $\mathcal{F}$ containing an element of $\mathcal{B}_d$ as an induced subgraph. By Theorem 3.8, all elements of $\mathcal{B}_d$ are forbidden induced subgraphs of $(d-2)$-HNG. Hence $G_d \notin (d-2)$-HNG. Thus given $a \geq 0$, $G_{a+2} \notin a$-HNG, and we conclude that (i) is false.

By Theorem 3.15, all elements of $\mathcal{B}_d$ are also forbidden induced subgraphs of $(d-2)$-SP, so an identical argument shows that (ii) is false.

We show directly that the elements of $\mathcal{B}_d$ are also forbidden induced subgraphs of the $(d-1)$-apex-split graphs. Consider dK_2. Since $2K_2$ is a forbidden induced subgraph of the class of split graphs, we must delete at least one vertex from every copy of $2K_2$ induced in dK_2. Deleting one vertex from every edge save one is necessary, hence dK_2 is not $(d-2)$-apex-split. Similarly, $\overline{K_{d,d}}$, which is isomorphic to $2K_d$, has a necessary and sufficient deletion of $d-1$ vertices from either copy of K_d to yield a split graph. Since b-apex-split graphs and their forbidden induced subgraphs are invariant under complementation, it follows that all graphs in $\mathcal{B}_d$ are not included in the class of $(d-2)$-apex-split graphs. Hence G_d is not a $(d-2)$-apex-split graph, so G_{c+2} is not a c-apex-split graph. We conclude that (iii) is false.

We now analyze the relationship between bipartite graphs with perfect matchings and the graphs of $\mathcal{B}$. Because Theorem 3.8 pertains to all bipartite graphs with perfect matchings, not just $K_{d,d}$ and dK_2, we account for the disparity. We argue that all sufficiently large bipartite graphs with perfect matchings contain an induced complete bipartite graph or a perfect induced matching. Suppose that a graph G contains a induced bipartite graph H with a perfect matching on $2e$ vertices. We show that if $e \geq R(d,d,2d,2d)$ for any $d \geq 0$, then G must contain either a complete bipartite graph or a perfect induced matching of order $2d$. Let $A \cup B$ be a bipartition of $V(H)$ with $A = \{a_1, a_2, \ldots, a_e\}, B = \{b_1, b_2, \ldots, b_e\}$, and $a_ib_i \in E(G)$ for all i, $1 \leq i \leq e$. Consider the complete graph $K = K_e$ with vertex set A. For each unordered pair i, j with $i \leq j$, we color $a_ia_j \in E(K)$ with color 1 if $\{a_ib_j, a_jb_i\} \subseteq E(G)$, with color 2 if $a_ib_j \notin E(G)$ and $a_jb_i \notin E(G)$, with color 3 if $a_ib_j \in E(G)$ but $a_jb_i \notin E(G)$, and with color 4 if $a_ib_j \notin E(G)$ but $a_jb_i \in E(G)$. Let K^i be the subgraph of edges of color i. By Ramsey's Theorem, if $e \geq R(d,d,2d,2d)$, then at least one of K^1, K^2, K^3, or K^4 must induce a clique of size d, d, $2d$, or $2d$, respectively. If KK^1 induces a clique in K, then the induced subgraph of G with vertex set $V(K^1) \cup \{b_i | a_i \in V(K^1)\}$ is isomorphic to a complete bipartite graph of order at least $2d$. If K^2 induces a clique, then the respective subgraph of G is isomorphic to a perfect induced matching of order at least $2d$. If K^3 or K^4 induce cliques in K, then the respective induced subgraphs of G are isomorphic to a half graph of order at least $4d$. For $x \geq 1$, the half graph of order $4x$ contains a complete bipartite induced subgraph on $2x$ vertices, so we conclude that G contains an induced complete bipartite graph of order at least $2d$.

Suppose (iv) is true and fix d accordingly. Let $x = R(d,d,2d,2d)$. By the above argument, no graph in $\mathcal{F}$ contains a bipartite induced subgraph with a perfect matching of order $2x$.

Let $G \in \mathcal{F}$ and fix a maximum clique K and a maximum stable set S in G. Let C be the set of remaining vertices. We claim $\alpha(C) < x$. If not, then a maximum stable set X in C, taken opposite its larger counterpart S, induces a bipartite graph. By Theorem 3.11, that graph contains either a perfect matching of order at least $2x$ or a maximum stable set larger than S, for a contradiction. Similarly, $\omega(C) < x$. Thus $|C| \leq R(x,x)$. Accordingly, G is $R(x,x)$-apex-split, proving (iii).

We also have $|K|+|S|+|C| \geq |G|$, so $\omega(G)+\alpha(G) \geq |G|-|C| \geq |G|-R(x,x)$. Therefore G is $(R(x,x)+1)$-sum-perfect, proving (ii). Note that the $+1$, relative to the c-apex-split case above, accounts for the fact that K and S might have non-empty intersection.

Lastly, we have $\chi(G) \geq |K|$ and $\theta(G) \geq |S|$, so $\chi(G)+\theta(G) \geq |K|+|S| \geq |G|-|C| \geq |G|-R(x,x)$. Therefore G is $(R(x,x)+1)$-HNG, proving (i). □

Chapter 4

The Hereditary Closure of the Unigraphs

The majority of this chapter is published with Michael Barrus and Ann Trenk in [6], with the exception of the material in Section 4.3 and Section 4.7. That content can be found in part at [7] and is otherwise first included here.

4.1 Introduction

In this chapter we consider two classes (which we will call *HU* and *HCU*) of finite, simple graphs that are characterized in terms of possible realizations of their degree sequences. Recall that the *degree sequence* of a graph is the list of degrees of the graph's vertices, written in non-increasing order. A graph G with a degree sequence π is said to *realize* π or to be a *realization* of π. When a realization exists for a finite list ρ of non-negative integers, then we call ρ *graphic*.

A *unigraph* is a graph that is the unique graph up to isomorphism having its specific degree sequence. For example, each graph on four or fewer vertices is a unigraph. On the other hand, neither P_5 nor $K_2 + K_3$ is a unigraph, since these non-isomorphic graphs share the degree sequence $(2, 2, 2, 1, 1)$. (Here we use $+$ to denote a disjoint union, and P_n or K_n to denote a path or complete graph, respectively, on n vertices.) Degree sequences will be called *unigraphic* or *non-unigraphic* depending on whether they have only one realization or multiple, respectively.

Unigraphs are rare (see [65, 67] for counts due to Tyshkevich) and special. The ability to precisely describe each unigraph by listing its vertex degrees simplifies some combinatorial problems, and unigraphs can be used to examine the effect of degree sequence-based bounds on graph parameters (see, for example, [3], where unigraphs highlight the tightness of the degree sequence residue as a bound on the independence numbers of graph realizations).

Early work on unigraphs and their degree sequences was done by many authors (see [42, 43, 45, 47, 48, 49], for several important works), culminating in a complete characterization by Tyshkevich and Chernyak (see [68, 69, 70, 71] and [67]).

The class of unigraphs contains several families of graphs that are remarkable in their own right. These include the families of threshold graphs [37], matroidal graphs, and matrogenic graphs [53, 66]. See the references for these families' definitions. With one exception, each of these families of unigraphs, including the family of unigraphs itself, has three distinct types of characterizations. To begin with, each has a characterization in terms of degree sequences (see [37, 45, 47, 48, 49, 53, 66]). This may not be surprising, given that all graphs involved are unigraphs.

As a second type of characterization, each of the classes of threshold graphs, matroidal graphs, matrogenic graphs, and general unigraphs has a characterization in terms of graph structure. The members of each family can be recognized as having subgraphs of certain prescribed types, connected by rigidly prescribed patterns of adjacency; see [19, 29, 52, 56, 67].

Finally, the threshold graphs, matroidal graphs, and matrogenic graphs each can be characterized in terms of a list of forbidden induced subgraphs; see [19, 29, 56]. This is possible because each of these families of graphs is a *hereditary* family, i.e., one that is closed under taking induced subgraphs of its members.

The full class of unigraphs, however, is not hereditary. As one example, observe that though $(3, 3, 3, 3, 3, 1)$ has a unique realization G, the graph G contains two non-unigraphs as induced subgraphs (namely, the two realizations of $(3, 2, 2, 2, 1)$).

Given this difference between the family of unigraphs and those of its remarkable subfamilies, in [2] and [4] Barrus defined the family of *hereditary unigraphs* to be the largest subfamily of the unigraphs that is closed under taking induced subgraphs. In this chapter we denote this family by HU. By definition, HU does not contain the unigraph with degree sequence $(3, 3, 3, 3, 3, 1)$ mentioned above, but it does contain each of the threshold, matroidal, and matrogenic graphs. Barrus showed that like these families, HU has characterizations in terms of the degree sequences and structure of its elements, and HU has a characterization in terms of a finite number of forbidden induced subgraphs.

We now ask about a complementary question. The family HU is the largest hereditary family contained by the (non-hereditary) family of unigraphs; what does the *smallest* hereditary family *containing* the family of unigraphs look like?

We define the *hereditary closure of the unigraphs*, which we denote henceforth by HCU, as the collection of all graphs that occur as an induced subgraph of some unigraph. Under this definition the two realizations of $(3, 2, 2, 2, 1)$ belong to HCU, as do all unigraphs. In this chapter we show that HCU, like its subfamilies described above, has characterizations in terms of degree sequences, structural conditions, and a finite list of forbidden induced subgraphs. Along the way, we discuss a new type of characterization for graphs in HU and HCU. This characterization relies on a partial order on degree sequences introduced by S.B. Rao in [58]. The result is a hybrid incorporating elements of degree sequences and forbidden induced subgraphs.

The chapter is structured as follows. In Section 2 we discuss tools that aid in our characterizations; these include ideas about split graphs and the scheme of graph composition introduced by Tyshkevich in [67]. We use this scheme in Section 3 to show that any graph G in HCU can be colored with at most $\omega(G) + 1$ colors, where $\omega(G)$ is the clique number of the graph. In Section 4 we provide a structural characterization for graphs in HCU, extending the collection of "building blocks" introduced in [67] to allow for non-unigraph members in HCU. In Section 5 we provide the first of our characterizations of HCU that involve the degree sequences of its elements. In preparation for another degree sequence characterization, in Section 6 we introduce Rao's degree sequence poset and develop some preliminary results; we also give a new characterization of HU involving the Rao poset. In Section 7 we establish a degree sequence characterization of HCU in terms of the Rao poset. Finally, in Section 8 we describe a computational search resulting in the list of minimal forbidden induced subgraphs for the family HCU.

4.2 Preliminaries and Tyshkevich decomposition

In this section we collect preliminary notions and describe the Tyshkevich decomposition of graphs, which will serve as a structural framework in all of our characterizations of graphs in HCU. In our approaches to characterizing HCU we will often deal separately with its split and non-split elements. Split graphs were first defined by Gyarfas and Lehel in [36], and independently by Földes and Hammer in [28]. We recall some known preliminaries here; for a more comprehensive treatment, see [22].

Definition 4.1. *A* split graph *is a graph in which the vertex set can be partitioned into a clique and a stable set (also known as an independent set). For a split graph G, a partition $V(G) = K \cup S$ is called a KS-partition if K is a clique and S is a stable set.*

The class of split graphs is hereditary, since any induced subgraph inherits a KS-partition from the original graph. Membership in the class of split graphs can be characterized by degree sequence [39]. Accordingly, we call a degree sequence *split* if its realizations are split graphs, and *non-split* otherwise.

In a split graph G, we call a vertex v a *swing vertex* if there exist two KS-partitions of G such that v is in the clique of one and the stable set of the other (with the idea that v can "swing" back and forth between the clique and the stable set). Equivalently, v is a swing vertex if its degree is one smaller than the clique number of G. A split graph has a unique KS-partition if and only if it has no swing vertices.

Given a split graph G with a fixed KS-partition, the *inverse* of G, denoted G^I, is the graph obtained by adding to the graph all possible edges with both endpoints in S and removing from the graph all edges with both endpoints in K. In this way G^I is also a split graph, with S and K partitioning its vertex set into a clique and a stable set, respectively. Note that our definition depends on a specified KS-partition for G; if G has more than one such partition, more than one inverse will exist. Context should make the partition clear, though we will often deal with split graphs with unique KS-partitions, and in this instance the inverse is unique.

We shall denote the complement of a graph G by $\overline{G}$. Note that if G is a split graph with KS-partition $K \cup S$ then the complement $\overline{G}$ is also a split graph, with S and K serving as the clique and stable set, respectively, in a KS-partition of $\overline{G}$. If G is a split graph with fixed KS-partition, then the complement of the inverse of G and the inverse of the complement of G are equal, and we denote this graph by $\overline{G^I}$. Tracing the effects of complementation and taking inverses of distinct realizations of a degree sequence shows the following result.

Remark 4.1. *If U is a unigraph, then $\overline{U}$ is also a unigraph. If U is also a split graph and $K \cup S$ is a KS-partition of U, then both U^I and $\overline{U^I}$ are unigraphs.*

Tyshkevich [67] introduced a way to express any graph as a composition of split graphs and at most one non-split graph. We first define the composition of a split graph with another graph. The composition of a split graph G with fixed KS-partition and another graph H is achieved by taking the disjoint union of G and H and joining each vertex of H to each vertex of K. We make this precise in the next definition.

Definition 4.2. *Let G be a split graph with KS-partition $K \cup S$, and let H be any graph. The (Tyshkevich) composition of G and H is the graph on $V(G) \cup V(H)$ with edge set $E(G) \cup E(H) \cup \{uv | u \in K, v \in V(H)\}$. We denote this by $(G, K, S) \circ H$, and write $G \circ H$ when the KS-partition of G is clear.*

It is not hard to show that the composition $G \circ H$ is a split graph if and only if H is a split graph. Using this, it is straightforward to extend the composition to n split graphs and one graph chosen freely, and to observe that this operation is associative. Figure 4.1 shows the composition of P_4, which has a unique KS-partition; the single vertex labeled c, which is placed in the S part of its KS-partition; and the non-split graph C_4. (We use C_n to denote a cycle graph on n vertices, and other standard graph naming notation throughout). Throughout the chapter, we draw clique vertices of split graphs with a solid circle, and stable set vertices with an open circle.

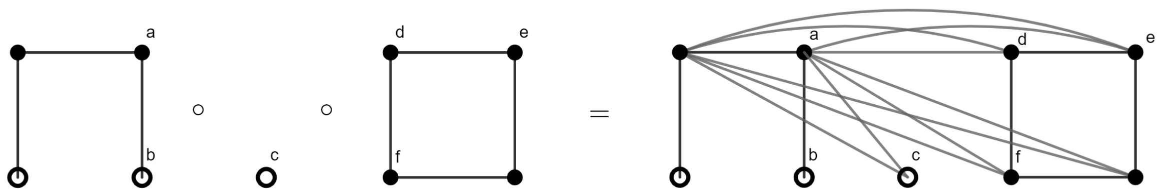

Figure 4.1: The composition $P_4 \circ \overline{K_1} \circ C_4$. Vertices of P_4 and $\overline{K_1}$ drawn on the top row are in the clique portion of the partitions and those on the bottom row are in the stable set. We write $\overline{K_1}$ to indicate that the KS-partition of the graph for the composition is with an empty clique.

Graphs that cannot be expressed as a non-trivial graph composition play a fundamental role in our characterization theorems.

Definition 4.3. *A graph G is decomposable if there exist nonempty graphs G_1, G_0, where G_1 is a split graph and $G = G_1 \circ G_0$, and indecomposable otherwise.*

Notice that a graph G on at least two vertices with an isolated vertex v is decomposable, with vertex v as a leftmost component that has $K = \emptyset$ and $S = \{v\}$. With the opposite KS-partition, graphs with a dominating vertex are similarly decomposable, as are split graphs with a swing vertex, where the rightmost component is a singleton vertex. A consequence of this last observation is that indecomposable split graphs have no swing vertices and thus have exactly one KS-partition.

Remark 4.2. *Indecomposable graphs with at least two vertices contain no dominating, isolated, or swing vertices.*

We state the existence and uniqueness of Tyshkevich decomposition; a proof is found in [67].

Theorem 4.3. *[67] Every graph G can be written as a composition of indecomposable components, where each of $G_n, \ldots, G_1$ has a fixed KS-partition:*

$$G = G_n \circ \ldots \circ G_1 \circ G_0.$$

Furthermore, when each G_i is nonempty, this decomposition is unique up to isomorphism, where neither the order of the components nor the choice of KS-partitions can vary.

When we refer to the *decomposition* $G = G_n \circ \ldots \circ G_0$, we mean the Tyshkevich decomposition of G into its *indecomposable* parts as guaranteed by Theorem 4.3, whereas a (Tyshkevich) composition $H = H_n \circ \ldots \circ H_0$ is defined even when not all H_i are indecomposable.

We note that both complementation and inversion commute with composition, as well as with one another. Whereas the complement of a composition is taken component-wise, the inverse of a composition flips the order of the components. Therefore, the property of being indecomposable is shared by a graph and its complement, and if the graph is a split graph, it is also shared by its inverse and inverse-complement. We record these as remarks.

Remark 4.4. *If $G = G_n \circ \ldots \circ G_0$, then $\overline{G} = \overline{G_n} \circ \ldots \circ \overline{G_0}$, with cliques and stable sets reversed. If G_0 is a split graph with fixed KS-partition, then the inverse G^I with respect to the KS-partition implied by $G_n \circ \ldots \circ G_0$ is given by $G^I = G_0^I \circ \ldots \circ G_n^I$, reversing the order of the components.*

Remark 4.5. *The complement of an indecomposable graph is indecomposable, and the inverse of an indecomposable split graph is indecomposable.*

Although the class of unigraphs has no forbidden induced subgraph characterization, it can be characterized via decomposition.

Theorem 4.6. *[67] A graph G is a unigraph if and only if each component of its decomposition is a unigraph.*

To understand the set of unigraphs, using Theorem 4.6, it suffices to determine which indecomposable graphs are unigraphs. Let $\mathcal{U}$ be the set of indecomposable unigraphs. Tyshkevich characterized $\mathcal{U}$ as a set of two graphs and seven families; we state this characterization in Theorem 4.7, using streamlined notation for each graph family, then give a brief description of each family. The individual graphs and graph families are shown in Figures 4.2 and 4.3. We also list the names of several graph classes in Table 4.1; the definitions of some of these classes are forthcoming in this and the next section.

For $m \geq 2$, the graph $T_1(m)$ is the disjoint union of m copies of K_2. For $m \geq 1$ and $n \geq 2$, the graph $T_2(m, n)$ is the disjoint union of $T_1(m)$ and the n-star $K_{n,1}$, for $m \geq 1$ and $n \geq 2$. The graph $T_3(m)$ is formed by attaching one C_4 and $m \geq 1$ copies of C_3 at a shared vertex.

The graph $T_4(p, q)$ consists of $q \geq 2$ clique vertices, each of which is adjacent to exactly $p \geq 1$ leaves. For $p_i \geq 1, q_i \geq 1$, and $r \geq 2$, the graph $T_5(p_1, q_1, \ldots, p_r, q_r)$ has $q_1 + q_2 + \ldots + q_r$ clique vertices, and each of q_i clique vertices are adjacent to exactly p_i leaves. For $p \geq 1, q_1 \geq 1$, and $q_2 \geq 1$, the graph $T_6(p, q_1, q_2)$ is a split graph with a clique containing $q_1 + q_2$ vertices.

Each clique vertex has $p+1$ neighbors in the stable set; q_2 of them are adjacent to $p+1$ leaves, and q_1 of them are adjacent to p leaves and a shared stable set vertex, e. For $p \geq 1$ and $q \geq 1$, the graph $T_7(p,q)$ is formed from $T_6(p,2,q)$ by adding a vertex f to the clique, adjacent to all stable set vertices except e. Let x_1 and x_2 be the two (clique) neighbors of e.

For $1 \leq i \leq 7$, the family $\mathcal{T}_i$ contains exactly the graphs T_i with acceptable choices of parameters.

Theorem 4.7. *[67] The indecomposable non-split unigraphs are C_5 and the families $\mathcal{T}_1$, $\mathcal{T}_2$, and $\mathcal{T}_3$, as well as the complements of each graph in these families. The indecomposable split unigraphs are K_1 and the families $\mathcal{T}_4$, $\mathcal{T}_5$, $\mathcal{T}_6$, and $\mathcal{T}_7$, as well as the complements, inverses, and inverse-complements of each graph in these families.*

4.3 Coloring *HCU*

We include a small discussion of coloring graphs in HCU that makes use of the connection between Tyshkevich decomposition and unigraphs. In fact, these graphs have excellent graph coloring properties. Let $\omega(G)$ and $\chi(G)$ denote, respectively, the clique number and chromatic number of a graph G. For a hereditary class $\mathcal{A}$, $f : \mathbb{N} \to \mathbb{N}$ is a χ-bounding function on $\mathcal{A}$ if for all $G \in \mathcal{A}$, $\chi(G) \leq f(\omega(G))$. For perfect graphs, the function $f(x) = x$ is χ-bounding, and this is of course best possible. See [61] for a survey of hereditary classes with known χ-bounding functions. We show that even HCU is bounded by $f(x) = x+1$, pointing to a strong relationship between graph realization and graph coloring. Recall that a graph is *apex-perfect* if and only if there exists $v \in V(G)$ such that the induced subgraph $G - \{v\}$ is perfect.

In general, clique numbers and chromatic numbers behave predictably with respect to graph composition.

Proposition 4.8. *Let G_i, $1 \leq i \leq n$ be a split graph with fixed KS-partition $K^i \cup S^i$ and G_0 be a nonempty graph. Given $G = (G_n, K^n, S^n) \circ \ldots \circ (G_1, K^1, S^1) \circ G_0$, it holds that $\omega(G) = |K^n| + \ldots + |K^1| + \omega(G_0)$ and $\chi(G) = |K^n| + \ldots + |K^1| + \chi(G_0)$.*

Proof. We prove the base case $n = 1$; the full result holds by induction. We denote K^1 by K and S^1 by S.

Let A be a maximum clique in G_0. The set $K \cup A$ induces a clique in G, so $\omega(G) \geq |K| + \omega(G_0)$. Any maximum clique including $s \in S$ contains at most $|K|+1$ vertices, which is not greater than $|K| + \omega(G_0)$, and any maximum clique including $v \in V(G_0) - A$ still contains at most $|K| + \omega(G_0)$ vertices. Hence $\omega(G) = |K| + \omega(G_0)$.

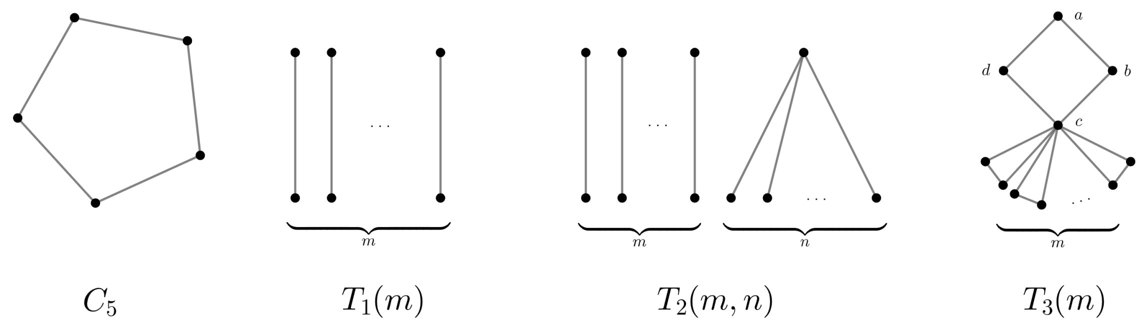

Figure 4.2: The non-split graphs in $\mathcal{U}$.

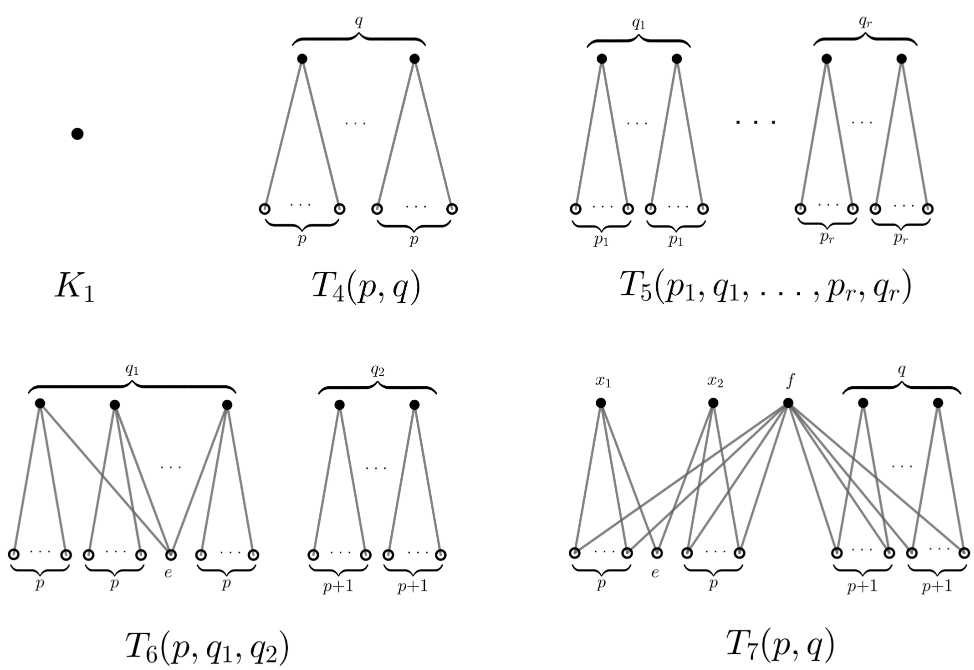

Figure 4.3: The split graphs in $\mathcal{U}$. The solid vertices in each graph's upper row form the graph's clique, although clique edges are not shown. The lower row of hollow vertices in each graph are the stable set vertices.

HU	Unigraphs for which all induced subgraphs are unigraphs
HCU	Smallest hereditary class containing all unigraphs; equivalently, graphs induced in a unigraph
$\mathcal{U}$	Indecomposable unigraphs
$\mathcal{I}$	Indecomposable graphs induced in a unigraph
$\mathcal{N}$	Non-unigraphs in $\mathcal{I}$

Table 4.1: Several subclasses of HCU.

A proper coloring of G_0 must use at least $\chi(G_0)$ colors, and a proper coloring of G must give every vertex in K a color distinct from one another and from the colors used on G_0. Hence $\chi(G) \geq |K| + \chi(G_0)$. All vertices in S can be colored with any color used on G_0, since G_0 is non-empty. This produces a proper coloring of G using $|K| + \chi(G_0)$ colors, so we conclude that $\chi(G) = |K| + \chi(G_0)$. □

Where $G_n, \ldots, G_1$ each have exactly one split partition, $\omega(G) = \omega(G_n) + \ldots + \omega(G_0)$ and $\chi(G) = \chi(G_n) + \ldots + \chi(G_0)$.

Theorem 4.9. *The hereditary closure of the unigraphs is a subset of the apex-perfect graphs; hence the class is χ-bounded by the function $f(x) = x + 1$.*

Proof. By Theorem 4.7, all indecomposable unigraphs are perfect except for C_5, which is apex-perfect. Let G be a unigraph, and per Theorem 4.6 write G as the composition $G_n \circ \ldots \circ G_1 \circ G_0$ for some set of indecomposable unigraphs $G_0, \ldots, G_n$. Per Theorem 4.2, all indecomposable split unigraphs have exactly one split partition, so the composition is well-defined. As split graphs, all of $G_1, \ldots, G_n$ are perfect. The graph G_0 is either perfect or isomorphic to C_5, such that $\chi(G_0) \leq \omega(G_0) + 1$. Thus $\chi(G) = \chi(G_n) + \ldots + \chi(G_0) \leq \omega(G_n) + \ldots + \omega(G_1) + \omega(G_0) + 1$, by Proposition 4.8.

Since the set of apex-perfect graphs is hereditary, we conclude that any graph induced in a unigraph is apex-perfect. Hence the hereditary closure of the unigraphs is a subset of the apex-perfect graphs. □

Thus all unigraphs and all graphs in HCU are apex-perfect.

4.4 A structural characterization of HCU

In this section, we show that to characterize the class HCU, it suffices to characterize the indecomposable graphs induced in a unigraph. We begin with a lemma describing the interaction of induced subgraphs and compositions.

If G is written as a composition $G = G_n \circ \ldots \circ G_0$ and H is an induced subgraph of G, then removing the vertices of G not in H and maintaining the same KS-partitions in each split graph component provides a representation of H as a composition. This is recorded in Lemma 4.10 below and illustrated in Figure 4.1, where the graph induced in $G = P_4 \circ \overline{K_1} \circ C_4$ by the labeled vertices a-f is equal to the composition $K_2 \circ \overline{K_1} \circ P_3$, with KS-partitions inherited from each component on the left-hand side.

Lemma 4.10. *Let G_0 be a graph, let $G_1, \ldots, G_n$ be split graphs and let $K_i \cup S_i$ be a KS-partition of G_i for $1 \leq i \leq n$. Further, let H be an induced subgraph of G. Define H_i to be the graph induced in H by the vertices of G_i, and if $i \geq 1$, assign H_i the KS-partition inherited from G_i. Then $H = H_n \circ \ldots \circ H_0$.*

Lemma 4.10 holds regardless of whether any components G_i or H_i are decomposable.

Let $\mathcal{I}$ be the set of indecomposable graphs induced in a unigraph, that is, $H \in \mathcal{I}$ if and only if H is indecomposable and there exists a unigraph G for which H is induced in G. We first characterize HCU in terms of $\mathcal{I}$, then return to a characterization of $\mathcal{I}$.

Theorem 4.11. *A graph is in HCU if and only if each graph in its decomposition is in $\mathcal{I}$.*

Proof. First, let G be a graph with decomposition $G_n \circ \ldots \circ G_0$ where $G_i \in \mathcal{I}$ for $0 \leq i \leq n$. By definition of $\mathcal{I}$, there exists a unigraph U_i containing G_i as an induced subgraph. By the definition of composition, $G = G_n \circ \ldots \circ G_0$ is induced in $U_n \circ \ldots \circ U_0$, which is a unigraph by Theorem 4.6. Hence G is induced in a unigraph and therefore $G \in HCU$.

For the reverse direction, let $G \in HCU$; thus there exists a unigraph U with G induced in U. Let $U = U_n \circ \ldots \circ U_0$ be the decomposition of U, so each U_i is an indecomposable unigraph by Theorem 4.6. Let $G_i = G \cap U_i$, thus each G_i is induced in U_i and by Lemma 4.10, $G = G_n \circ \ldots \circ G_0$. Since the graph G_i is induced in U_i, we know for all i that G_i is an induced subgraph of an indecomposable unigraph. However, G_i might not be indecomposable. For each i, let $G_{i,m_i} \circ \ldots \circ G_{i,0}$ be the decomposition of G_i. Now, for all $j \leq m_i$, $G_{i,j}$ is indecomposable and induced in G_i, which in turn is induced in U_i. Therefore $G_{i,j}$ is induced in U_i, and so $G_{i,j} \in \mathcal{I}$ for all (i, j). By substitution, the decomposition of G into indecomposable parts is $G = (G_{n,m_n} \circ \ldots \circ G_{n,0}) \circ \ldots \circ (G_{0,m_0} \circ \ldots \circ G_{0,0})$. Since the composition relation $\circ$ is associative, we conclude that the decomposition of G is $G = G_{n,m_n} \circ \ldots \circ G_{0,0}$, and each of these components is in $\mathcal{I}$. $\square$

Recall from Remark 4.4 that complements and inverses both commute with composition, albeit with reordered components. In conjunction with this observation, we conclude from Theorem 4.11 that HCU is closed under complementation, and the subset of split graphs within HCU is closed under inversion.

Corollary 4.12. *A graph G is in HCU if and only if $\overline{G}$ is in HCU. If G is a split graph and $K \cup S$ is a KS-partition of G, then either all or none of $G, \overline{G}, G^I, \overline{G^I}$ are in HCU.*

This implies the same of minimal forbidden induced subgraphs of HCU. Furthermore, such graphs must be indecomposable, else some proper indecomposable component of a forbidden graph must also be forbidden.

Corollary 4.13. *If G is a minimal forbidden induced subgraph of HCU, then G is indecomposable, and $\overline{G}, G^I$, and $\overline{G^I}$ are each also minimal forbidden induced subgraphs.*

We now present a characterization of $\mathcal{I}$. First we show we need only consider graphs induced in *indecomposable* unigraphs.

Lemma 4.14. *If an indecomposable graph is induced in a unigraph, then it is also induced in an indecomposable unigraph.*

Proof. Let G be an indecomposable graph induced in a unigraph U. By Theorem 4.6, each component U_i of the decomposition $U = U_n \circ \ldots \circ U_0$ is also a unigraph. Now by Lemma 4.10, we can write G as a composition $G = G_n \circ \ldots \circ G_0$ where G_i is the graph induced in G by $V(U_i)$. Since G is indecomposable, Definition 4.3 implies that there is only one nonempty component G_j in this composition, so $G = G_j$. Thus G is induced in the indecomposable unigraph U_j. □

Let $\mathcal{N}$ be the set of non-unigraphs in $\mathcal{I}$. Thus $\mathcal{I} = \mathcal{U} \cup \mathcal{N}$, and by Lemma 4.14, $\mathcal{N}$ is the class of indecomposable non-unigraphs induced in a graph in $\mathcal{U}$. The class $\mathcal{N}$ consists of five graph families and was characterized in [73]. We present this result in Theorem 4.16 and give a sketch of the proof.

The proof of Theorem 4.16 uses the next lemma to identify indecomposable graphs. Lemma 4.15 follows from the observation that in a composition, vertices inducing the indecomposable graphs P_4, C_4, and $2K_2$ cannot be separated into multiple indecomposable components. A more general result can be found in Barrus and West [8].

Lemma 4.15. *[8] If G is a graph and there exists a vertex $v \in V(G)$ such that for all $w \in V(G)$, there exists an induced P_4, C_4, or $2K_2$ containing both v and w, then G is indecomposable.*

Theorem 4.16. *A graph $G \in \mathcal{N}$ if and only if G contains no dominating, isolated, or swing vertices, and at least one of the following holds for G, its complement, or, if G is a split graph, its inverse or inverse-complement: (See Figures 4.2 and 4.3 and definitions preceding Theorem 4.7.)*

(1) G is induced in a graph in $\mathcal{T}_3$; $a, b, c, d \in V(G)$; and c has a neighbor of degree 1.

(2) G is induced in a graph in $\mathcal{T}_3$; $a, c \in V(G)$; exactly one of b and d is in $V(G)$; and G contains a K_3 subgraph.

(3) G is induced in a graph in $\mathcal{T}_6$; $e \in V(G)$ with $\deg e \geq 2$; and there exists a pair of clique vertices of unequal degree, exactly one of which is adjacent to e.

(4) G is induced in a graph in $\mathcal{T}_7$; $e \in V(G)$ with $\deg e = 2$; and there exists a pair of clique vertices of unequal degree, exactly one of which is adjacent to e.

(5) G is induced in a graph in $\mathcal{T}_7$; $e, f, x_1 \in V(G)$; there exists a vertex of degree 1 in G; and some vertex of degree 2 has a neighbor other than x_1 and f.

Proof. If $G \in \mathcal{N}$, then by Lemma 4.14, G is an indecomposable non-unigraph and G is induced in a graph listed in Theorem 4.7. It is not hard to show that the graphs C_5, K_1, and those in $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_4$, and $\mathcal{T}_5$ are in HU, so we need only consider when G, its complement, and, if G is a split graph, its inverse or inverse-complement, are induced in graphs in $\mathcal{T}_3$, $\mathcal{T}_6$, or $\mathcal{T}_7$. Without loss of generality, suppose G itself is induced in one of these families.

We provide a detailed argument in the case that G is induced in $U = T_3(m)$. We refer to the vertex labels that are in Figure 4.2. If $c \notin V(G)$, then G is induced in a graph in $\mathcal{T}_2$ and so is a unigraph. Thus $c \in V(G)$. If $a \notin V(G)$, then c is a dominating vertex and G is decomposable, contradicting $G \in \mathcal{N}$. Thus $a \in V(G)$. Similarly, vertex a cannot be an isolated vertex, so without loss of generality we assume $b \in V(G)$. From here, there are four possibilities. First, if G contains neither C_4 nor any C_3, then $G \in \mathcal{T}_5$ and G is an indecomposable unigraph by Theorem 4.7. Next, if G contains some C_3, but not C_4, then G satisfies (2). If G contains C_4, but c has no degree 1 neighbors, then either $G = C_4$ or $G = T_3(k)$ for some k, and is a unigraph in either case. Lastly, if G contains C_4 and c has at least one degree 1 neighbor, then G satisfies (1).

We omit the cases where G is induced in a graph in $\mathcal{T}_6$ or $\mathcal{T}_7$.

For the reverse direction, let G be a graph without dominating, isolated, or swing vertices, and so that one of G, its complement, and, if G is a split graph, its inverse and inverse-complement satisfy one of (1) to (5). Suppose without loss of generality that G itself satisfies one of these. We provide a detailed argument in the case that G satisfies (1). In this case, G is induced in $U \in \mathcal{T}_3$, the subset $\{a, b, c, d\}$ induces C_4, and c has at least one degree 1 neighbor v. It suffices to show G is a non-unigraph and indecomposable. Create a graph G' from G by deleting edge ad and adding edge vd. Note that G and G' have the same degree sequence but are not isomorphic, since G' has no induced C_4. We conclude that G is a non-unigraph.

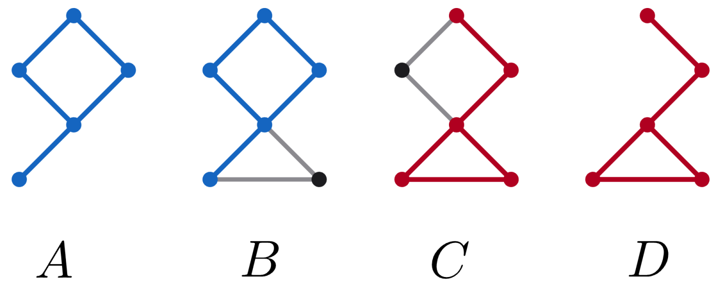

Figure 4.4: Given a non-unigraph A induced in a unigraph B, we can replace it with a different realization D of its degree sequence to produce a graph C isomorphic to B.

Next, we show that G is indecomposable. The set of vertices $\{a, b, c, d\}$ induces C_4 in G and any other vertex $w \in V(G)$ together with vertices a, b, c induces P_4. Thus every vertex of G is part of an induced P_4 or C_4 containing vertex c, so by Lemma 4.15, G is indecomposable.

We omit proofs of the remaining cases, which are similar. $\square$

As stated above, $\mathcal{I} = \mathcal{U} \cup \mathcal{N}$, so our characterization of $\mathcal{N}$ in Theorem 4.16 and Tyshkevich's characterization of $\mathcal{U}$ in Theorem 4.7 together give a complete list of the graphs in $\mathcal{I}$. By Theorem 4.11, the composition of elements in $\mathcal{I}$ gives us exactly the class HCU.

4.5 A degree sequence characterization of HCU

In this section, we will characterize the degree sequences of graphs in HCU. We begin by showing that membership in the class HCU is determined by degree sequence, ensuring the existence of such a characterization.

Lemma 4.17. *If graphs G and G' have the same degree sequence and $G \in HCU$, then $G' \in HCU$.*

Proof. Suppose G and G' are graphs with the same degree sequence and $G \in HCU$. Then there exists a unigraph U with G induced in U. Fix a bijection $f : V(G) \to V(G')$ that preserves vertex degree. Then, let U' be the graph with vertex set $V(U') = V(U)$ and edge set $E(U') = \{xy | x \notin V(G)$ or $y \notin V(G)$ or $f(x)f(y) \in E(G')\}$. Accordingly, we have the edges in U that have both endpoints in G with the edges in G'. The result is a graph U' that contains G' as an induced subgraph, and each vertex of U' has the same degree as in U. Thus U' has the same degree sequence as U. Since U is a unigraph, it is isomorphic to U' and thus $G' \in HCU$. This process is illustrated in Figure 4.4. $\square$

The next result from Tyshkevich [67] shows that when two graphs have the same degree sequence, so do the corresponding components of their decomposition.

Theorem 4.18. *[67] Suppose G is a graph with decomposition $G_n \circ \ldots \circ G_0$ and H is a graph with decomposition $H_m \circ \ldots \circ H_0$. If G and H have the same degree sequence, then $n = m$ and the degree sequence of G_i equals the degree sequence of H_i for all i, $0 \le i \le n$.*

Following Tyshkevich, Theorem 4.18 allows us to define the decomposition of a graphic sequence irrespective of the choice of realization, and to define decomposable and indecomposable degree sequences.

Definition 4.4. *Let π be a graphic sequence, let G be a graph realizing π, and let $G_n \circ \ldots \circ G_0$ be the decomposition of G. Define the (Tyshkevich) decomposition of π to be $\pi^n \circ \ldots \circ \pi^0$ where π^i is the degree sequence of G^i for each i. If $n = 0$ then π is indecomposable and if $n \ge 1$ then π is decomposable.*

Furthermore, we can discern from the decomposition of a degree sequence whether its realizations are in HCU.

Lemma 4.19. *Let π be a degree sequence with decomposition $\pi^n \circ \ldots \circ \pi^0$. Each realization of π is in HCU if and only if each of the components π^i is realized by a graph in HCU.*

Proof. Suppose π is a graphic sequence with realization G. Let $G_n \circ \ldots \circ G_0$ and $\pi^n \circ \ldots \circ \pi^0$ be the decompositions of G and π, respectively. By Definition 4.4, π^i is the degree sequence of G_i for each i. For the forward direction, assume that $G \in HCU$. Since HCU is a hereditary class and G_i is induced in G, it follows that $G_i \in HCU$ for each i. Conversely, suppose each π^i is realized by a graph $H_i \in HCU$. Since G_i and H_i have the same degree sequence for each i and $H_i \in HCU$, we know $G_i \in HCU$ by Lemma 4.17. Each G_i is indecomposable, so $G_i \in \mathcal{I}$ by definition of $\mathcal{I}$. Now Theorem 4.11 implies $G \in HCU$ as desired. □

The indecomposable components of a degree sequence can be efficiently determined from the degree sequence itself (see [67] and also [4]). For convenience we record here a helpful fact about Tyshkevich decomposition from [4]. Recall that a non-increasing sequence $\pi = (d_1, \ldots, d_n)$ of non-negative integers is graphic if and only if it has even sum and satisfies the kth Erdős-Gallai inequality

$$\sum_{i=1}^{k} d_i \le k(k-1) + \sum_{i=k+1}^{n} \min\{k, d_i\}$$

with equality for all $k, 1 \le k \le n$ [25].

Lemma 4.20 (See [4, Observation 5.2] and [4, Theorem 5.6])**.** *Let $\pi = (d_1, \ldots, d_n)$ be a graphic sequence with $t = \max\{i|d_i \geq i-1\}$. Then π is decomposable if and only if $d_n = 0$, $d_n = n-1$, or there exists $k < t$ with $d_k > d_{k+1}$ such that the kth Erdős-Gallai inequality holds with equality.*

We characterize when indecomposable sequences are realized by graphs in HCU separately for split and non-split sequences, beginning with the latter. Theorems 4.21 and 4.23 both show that the degree sequences of graphs in HCU are extreme cases: for instance, degree sequences satisfying (1) of the following theorem have all but one term equal to 1, the smallest possible value in an indecomposable sequence.

Theorem 4.21. *All realizations of a non-split, graphic sequence $\pi = (d_1, \ldots, d_n)$ are indecomposable and in HCU if and only if $d_n \geq 1, d_1 \leq n-2$, and at least one of the following holds:*

(1) $d_2 = 1$

(2) $d_{n-1} = n-2$

(3) $d_1 = n-2$ *and* $d_2 = 2$

(4) $d_n = 1$ *and* $d_{n-1} = n-3$

(5) $d = (2, 2, 2, 2, 2)$

Proof. For the forward direction, let G be an indecomposable non-split graph in HCU with degree sequence $\pi = (d_1, \ldots, d_n)$. Since G is indecomposable, it has no dominating or isolated vertices, so $d_n \geq 1$ and $d_1 \leq n-2$. By Lemma 4.14, the graph G is induced in an indecomposable unigraph U, and since G is a non-split graph, so is U. The indecomposable non-split unigraphs are characterized in Theorem 4.16 and depicted in Figure 4.2. We consider each possibility for U. If $U = C_5$, then the only possibility for G is C_5 itself, and π satisfies (5). It is not hard to verify that if G is induced in a graph in $\mathcal{T}_1$ or $\mathcal{T}_2$ then π satisfies (1), and if G is induced in their complements, then π satisfies (2). We can similarly see that if G is induced in a graph in $\mathcal{T}_3$ or its complement, for some $m \geq 1$, then π satisfies (3) or (4), respectively.

For the backwards direction, suppose $\pi = (d_1, \ldots, d_n)$ is a non-split, graphic sequence with $d_n \geq 1$, $d_1 \leq n-2$, and satisfying at least one of conditions (1) to (5). Let G be a graph realizing π with $V(G) = \{v_1, \ldots, v_n\}$ and $\deg v_i = d_i$ for all i, $i \leq n$. We will show that G is an indecomposable graph in HCU.

First suppose π satisfies (1), so $\pi = (d_1, 1, 1, \ldots, 1)$. If $d_1 = 1$, then n is even and G is isomorphic to $\frac{n}{2}K_2$. If $d_1 \geq 2$, then G is isomorphic to $T_2(\frac{n-1-d_1}{2}, d_1)$, where $n-1-d_1$ is an even integer (because π is graphic) and at least two (because G is not a split graph). In either case, G is an indecomposable unigraph by Theorem 4.7.

Next, suppose π satisfies (2), so $\pi = (n-2, \ldots, n-2, d_n)$. In this case, the degree sequence of $\overline{G}$ satisfies (1), so $\overline{G}$ is an indecomposable unigraph by the above. By Remarks 4.1 and 4.5, G is an indecomposable unigraph.

Now, suppose π satisfies (3), so $\pi = (n-2, 2, \ldots, 2, 1, \ldots, 1)$. Let a be the number of terms in π equal to 1. We will show that G is induced in the unigraph $U = T_3(\frac{n+a-4}{2})$, and it may be helpful to refer to Figure 4.2. Since $d_1 = n-2$, it follows that v_1 (labeled c in Figure 4.2) is adjacent to all other vertices except one, v_s (labeled a). If $\deg v_s = 2$, then v_s, its neighbors, and v_1 induce C_4. Otherwise, $\deg v_s = 1$, and then v_s, v_1, and their mutual neighbor induce P_3. In either case, the remaining degree two vertices must each be adjacent to v_1, so in pairs they form induced C_3 graphs with v_1 in G. The vertices of degree 1 (other than v_s) must be adjacent to v_1 and for each degree 1 vertex v_j, add a new vertex w_j of degree 2 adjacent to v_1 and v_j. If $\deg(v_s) = 1$, then w_s completes an induced C_4 and in either case, the remaining w_j form induced C_3 graphs with v_1 and v_j. The result is the unigraph U containing G as an induced subgraph, so $G \in HCU$.

It remains to show G is indecomposable when π satisfies (3). If $d_3 = 1$ then $t = \max\{i | d_i \geq i-1\} = 2$ and if $d_3 = 2$ then $t = 3$. In either case, the only index less than t for which $d_k > d_{k+1}$ is $k = 1$. However, $n-2 = d_1 < 1(0) + \sum_{i=2}^{n} \min\{1, d_i\} = n-1$, so the 1st Erdős-Gallai inequality holds strictly. By assumption, $1 \leq d_n \leq n-2$, and we conclude from Lemma 4.20 and Remark 4.2 that d is indecomposable, implying G is as well.

When π satisfies (4) the degree sequence of $\overline{G}$ satisfies (3), so as before, G is an indecomposable graph in HCU. Finally, when π satisfies (5), then $G = C_5$, which is an indecomposable unigraph. □

We now turn to the characterization of degree sequences of split graphs in HCU. For split graphs with a fixed KS-partition, it will be helpful to consider only the edges with one endpoint in K and the other in S, and for each vertex, the number of such edges incident to it.

Definition 4.5. *If G is a split graph with a fixed KS-partition, the cross-degree of vertex v, denoted by* $\operatorname{cdeg}(v)$*, is defined as follows:*

If $v \in K$ then $\operatorname{cdeg}(v) = \deg(v) - (|K| - 1)$.

If $v \in S$ then $\operatorname{cdeg}(v) = \deg(v)$.

For example, the split graph P_4 has a unique KS-partition and each vertex has cross-degree 1. Generally, the vertices of any split graph with no swing vertices will have unique cross-degrees. For split graphs with a swing vertex, the cross-degrees depend on the KS-

partition. We introduce the cross-degree sequence pair of a split graph, which we define with respect to a given KS-partition of the graph.

Definition 4.6. *Suppose G is a split graph and fix a KS-partition of G. Let $K = \{v_1, \ldots, v_n\}$ with $\operatorname{cdeg}(v_i) \geq \operatorname{cdeg}(v_{i+1})$ for all $i \leq n-1$, and let $S = \{w_1, \ldots, w_m\}$ with $\operatorname{cdeg}(w_i) \geq \operatorname{cdeg}(w_{i+1})$ for all $i \leq m-1$. Let $a_i = \operatorname{cdeg}(v_i)$ and $b_j = \operatorname{cdeg}(w_j)$ for all $i \leq n$ and $j \leq m$. The cross-degree sequence pair of G with the given KS-partition is given by the unordered pair $\{\alpha, \beta\}$ with $\alpha = (a_1, \ldots, a_n)$ and $\beta = (b_1, \ldots, b_m)\}$.*

For example, the cross-degree sequence pair of P_4 is $\{(1,1),(1,1)\}$. Using Definition 4.5, we can easily convert between cross-degree sequence pairs and split graphic (degree) sequences, although this correspondence is not generally bijective. The most useful property of the cross-degree of a vertex and the cross-degree sequence pair of a split graph is that these are invariant under inversion, with respect to a given KS-partition. As a result, graphs with a unique KS-partition, including all indecomposable split graphs, have a unique cross-degree sequence pair. By conversion to degree sequences, it is straightforward to show that split graphs in HCU can be characterized by their cross-degree sequence pairs, and to give well-defined notions of decomposition and indecomposability.

Lemma 4.20 is also easily translated to cross-degree sequence pairs by using a second inequality, proven independently by Gale [31] and Ryser [60], in lieu of the Erdős-Gallai inequality. We leave the proof to the reader.

Lemma 4.22. *A cross-degree sequence pair $\{\alpha, \beta\} = \{(a_1, \ldots, a_n), (b_1, \ldots, b_m)\}$ is decomposable if and only if $a_n = 0, a_n = m$, or there exists k such that $a_k > a_{k+1}$ and $\{\alpha, \beta\}$ satisfies with equality the kth Gale-Ryser inequality*

$$\sum_{i=1}^{k} a_i \leq \sum_{i=1}^{m} \min\{k, b_i\}.$$

Moreover, since $\{\alpha, \beta\}$ is unordered, $\{\alpha, \beta\}$ is decomposable if and only if $b_m = 0$, $b_m = n$, or there exists k such that $b_k > b_{k+1}$ and $\sum_{i=1}^{k} b_i \leq \sum_{i=1}^{n} \min\{k, a_i\}$ holds with equality.

We characterize all indecomposable split graphs in the hereditary closure by the cross-degrees of the vertices in a given partition. The use of the cross-degree, rather than ordinary degree sequences, draws out the almost identical relationship between the cases of Theorems 4.21 and 4.23. Here, (1a) and (1b) mirror (1) of Theorem 4.21, although the non-split graph C_5 has no parallel. In addition, (3a) and (3b) describe cross-degree sequence pairs where one term is as large as possible, and the terms on the other side are all 1 or 2, just as (3) in Theorem 4.21 characterizes degree sequences where one term is as large as possible and all remaining terms are 1 or 2. In both theorems, the complements behave similarly: (2a) and (2b) together mirror (2), and (4a) and (4b) mirror (4).

Theorem 4.23. *All realizations of a cross-degree sequence pair $\{\alpha, \beta\} = \{(a_1, \ldots, a_n), (b_1, \ldots, b_m)\}$ are indecomposable and in HCU if and only if $a_n, b_m \geq 1$, $a_1 \leq m-1$, $b_1 \leq n-1$, and at least one of the following holds:*

(1a) $a_2 = 1$.

(1b) $b_2 = 1$.

(2a) $a_{n-1} = m - 1$.

(2b) $b_{m-1} = n - 1$.

(3a) $b_1 = n - 1$ *and* $a_1 \leq 2$.

(3b) $a_1 = m - 1$ *and* $b_1 \leq 2$.

(4a) $b_m = 1$ *and* $a_n \geq m - 2$.

(4b) $a_n = 1$ *and* $b_m \geq n - 2$.

Proof. For the forward direction, let G be an indecomposable split graph in HCU. By Remark 4.2 we know G has no dominating, isolated, or swing vertices, hence G has a unique KS-partition and unique cross-degree sequence pair $\{\alpha, \beta\}$. The same remark tells us that α and β each have at least two terms, and $a_1 \leq m - 1$, $b_1 \leq n - 1$, and $a_n, b_m \geq 1$. Since the conditions we seek to prove are symmetric with respect to α and β, we may assume without loss of generality that $K = \{v_1, \ldots, v_n\}$ and $S = \{w_1, \ldots, w_m\}$ where $\text{cdeg}(v_i) = a_i$ and $\text{cdeg}(w_j) = b_j$. By Lemma 4.14, G is induced in an indecomposable unigraph $U \in \mathcal{U}$. We know from Theorem 4.16 that graphs in $\mathcal{N}$ that are induced in non-split graphs in $\mathcal{U}$ are themselves non-split graphs. We can therefore choose U to be a split graph. The split graphs in $\mathcal{U}$ are characterized in Theorem 4.7 and depicted in Figure 4.3. We consider each possibility for U.

First, suppose U is in $\mathcal{T}_4$ or $\mathcal{T}_5$. In this case, every vertex in the stable set S has degree 1, so $b_2 = 1$ and $\{\alpha, \beta\}$ satisfies (1b). If U is the inverse of a graph in $\mathcal{T}_4$ or $\mathcal{T}_5$, then $\{\alpha, \beta\}$ satisfies (1a). If U is the complement of a graph in $\mathcal{T}_4$ or $\mathcal{T}_5$ then $\{\alpha, \beta\}$ satisfies (2b), and finally if G is the inverse-complement of a graph in $\mathcal{T}_4$ or $\mathcal{T}_5$, then $\{\alpha, \beta\}$ satisfies (2a). Next suppose U is in $\mathcal{T}_6$. Then every vertex in S has degree 1, except possibly w_1, so $b_2 = 1$ and $\{\alpha, \beta\}$ satisfies (1b). As above, if U is the inverse, complement, or inverse-complement of a graph in $\mathcal{T}_6$, then $\{\alpha, \beta\}$ satisfies (1a), (2b), or (2a), respectively.

Finally, suppose U is in $\mathcal{T}_7$ and refer to Figure 4.3. If $f \notin V(G)$, then G is induced in a graph in $\mathcal{T}_6$, and the previous case applies. If $f \in V(G)$ and $e \notin V(G)$, then f is a dominating vertex in G, a contradiction. It remains to consider when $e, f \in V(G)$. All vertices in S have degree 1 or 2 and $f \in K$ is adjacent to all vertices in S except e. Thus $a_1 = m-1$ and $b_1 \leq 2$, and $\{\alpha, \beta\}$ satisfies (3b). As before, if U is the inverse, complement, or inverse-complement of a graph in $\mathcal{T}_7$, then $\{\alpha, \beta\}$ satisfies (3a), (4b), or (4a), respectively.

For the reverse direction, suppose $\{\alpha, \beta\} = \{(a_1, \ldots, a_n), (b_1, \ldots, b_m)\}$ is a cross-degree sequence pair with $a_n, b_m \geq 1, a_1 \leq m - 1$, $b_1 \leq n - 1$, and satisfying at least one of the eight conditions. Let G be a graph realizing $\{\alpha, \beta\}$, with split partition $V(G) = V \cup W$, $V = \{v_1, \ldots, v_n\}$ and $W = \{w_1, \ldots, w_m\}$ such that $\text{cdeg}(v_i) = a_i$ and $\text{cdeg}(w_j) = b_j$ for

$i \le n$ and $j \le m$. One of V, W is a clique and the other is a stable set. We will show that G is in HCU and is indecomposable.

First suppose that $\{\alpha, \beta\}$ satisfies (1a), thus vertices $v_2, \ldots, v_n$ each have cross-degree 1 in G. If $a_1 = 1$ then G is a graph in the family $\mathcal{T}_5$, hence is an indecomposable unigraph in HCU. Otherwise we will show that G is induced in a unigraph $U \in \mathcal{T}_6$, where W is the clique in the split partition of $W(G)$. It may be helpful to refer to Figure 4.3. For each $w_j \in W$, add $b_1 - b_j$ vertices to V, each of cross-degree 1 and adjacent to w_j. This increases the cross-degree of every vertex in W to b_1. Each vertex of V other than v_1 (labeled e in Figure 4.3) has cross-degree 1 and the result is $U = T_6(b_1 - 1, a_1, m - a_1)$. The only descent in α occurs at index 1, but by assumption $a_1 \le m - 1$ so the 1st Gale-Ryser inequality is strict: $a_1 < \sum_{i=1}^{m} \min\{1, b_i\} = m$. Since $1 \le a_n \le m-1$ by assumption, it follows by Lemma 4.22 that $\{\alpha, \beta\}$ is indecomposable and hence G is as well. If $\{\alpha, \beta\}$ satisfies (2a), then $\overline{G}$ satisfies (1a), so G is indecomposable and contained in the indecomposable unigraph $\overline{U}$.

Next suppose $\{\alpha, \beta\}$ satisfies (3a). The case $a_2 = 1$ was considered in (1a), so we may assume $a_2 \ge 2$ and thus $\alpha = (2, 2, \ldots, 2, 1, \ldots, 1)$. Let a be the larger of b_2 and the number of 1s in α. Since $b_1 = n - 1$, there exists a unique vertex $v_j \in V$ such that w_1 is adjacent to all vertices in V except v_j. We will show G is induced in the unigraph $U = T_7(a-1, m-2)$, where W is the clique in the split partition of $V(G)$. Again, it is helpful to refer to Figure 4.3. First, add an additional vertex w_{m+1} to W with w_{m+1} adjacent to every vertex in V that has cross-degree equal to 1. As a result, every vertex in V now has cross-degree equal to 2. Let $W' = W \cup \{w_{m+1}\}$. For each $w_i \in W'$ with $2 \le i \le m + 1$, add $a - b_i$ vertices to V, where each added vertex has cross-degree 2 and is adjacent to w_i and w_1. As a result, each vertex in W' other than w_1 has cross-degree equal to a. Let V' be the set V together with these added vertices. Since the cross-degree of w_1 is $|V'| - 1$, the resulting split graph U (where V' is a stable set and W' is a clique) contains G as an induced subgraph and is equal to $T_7(a - 1, m - 2)$. Here, v_j is the vertex labeled e and w_1 is the vertex labeled f in Figure 4.3. Thus $G \in HCU$.

We show that G is indecomposable. Since all terms in α are less than or equal to two, the kth Gale-Ryser inequality is strict for $2 \le k \le m - 1$, as we have $\sum_{i=1}^{n} \min\{k, a_i\} = \sum_{i=1}^{n} a_i = \sum_{i=1}^{m} b_i > \sum_{i=1}^{k} b_i$. The assumption $b_1 \le n - 1$ implies that the 1st Gale-Ryser inequality also holds strictly, so we conclude from Lemma 4.22 that $\{\alpha, \beta\}$ is indecomposable, thus G is indecomposable. If $\{\alpha, \beta\}$ satisfies (4a) then $\overline{G}$ satisfies (3a), so G is contained in the indecomposable unigraph $\overline{U}$.

Finally, if $\{\alpha, \beta\}$ satisfies (1b), (2b), (3b), or (4b), the above arguments apply with the roles of V and W reversed and the graphs and unigraphs each replaced by their inverses. $\square$

It is straightforward to determine which degree sequences produce split graphs as realizations [39]. Thus together, Theorems 4.21 and 4.23 provide a degree sequence characterization of graphs in HCU.

4.6 The Rao poset on degree sequences

In Section 4.7 we will provide another characterization of the class HCU in terms of degree sequences. This characterization will be strongly linked to a forbidden induced subgraph characterization, which will follow in Section 4.8. The context that reveals this link is a partially ordered set on degree sequences introduced by S.B. Rao that incorporates information about degree sequences of induced subgraphs.

In [58], Rao defined a partial order on the set of degree sequences of simple graphs with the following relation: degree sequences π, ρ satisfy $\pi \preceq \rho$ if there exists a realization G of π and a realization H of ρ such that G is an induced subgraph of H. We refer to $\preceq$ as the *Rao relation*, and we may also write $\rho \succeq \pi$. We say that π *is Rao-contained in* ρ or that ρ *Rao-contains* π if $\pi \preceq \rho$. Note that if ρ does not Rao-contain π, then no realization of ρ contains any realization of π as an induced subgraph.

As a simple example, observe that $(2, 2, 2) \preceq (2, 2, 2, 1, 1)$, since $(2, 2, 2)$ is realized by K_3, and $(2, 2, 2, 1, 1)$ has $K_3 + K_2$ as one of its realizations. (The fact that the other realization, a path on five vertices, does not contain K_3 as an induced subgraph is irrelevant.) Hence the Rao poset captures, in the context of degree sequences, some of the relationships that the induced subgraph order does for graphs.

In [58] Rao showed that $\preceq$ is indeed a partial order and described some of its properties and related questions. Of note, Rao asked whether the poset so defined is a well-quasi-order (WQO), i.e., whether for any infinite sequence $\pi^1, \pi^2, \ldots$ of degree sequences, there would exist indices $i, j \in \mathbb{N}$ such that $i < j$ and $\pi^i \preceq \pi^j$. Equivalently, the partial order would permit no infinite set of pairwise incomparable degree sequences. This question was answered many years later in [17], where Chudnovsky and Seymour proved that Rao's poset is in fact a WQO.

We will show in the next section that graphs in HCU can be characterized as those graphs whose degree sequences avoid certain "minimal obstructions" in the Rao poset. To that end, we develop a few results here about Rao-containment.

Our first result shows that we can recognize Rao-containment without having to examine the realizations of the degree sequences involved. Given a degree sequence π of length n and an integer $k \in \{0, \ldots, n\}$, let us say that to *augment* π *by the term* k means to create a new degree sequence by increasing k distinct terms of π by 1 and inserting a new term equal to k. Because of the choices available in which terms of π to increase, there are often several ways to augment π by an integer k.

For example, augmenting the degree sequence $\pi = (3,2,2,1)$ by the terms 0, 1, 2, 3, or 4 can yield any of the following degree sequences:

Augmenting by:	0	1	2	3	4
	$(3,2,2,1,0)$	$(4,2,2,1,1)$	$(4,3,2,2,1)$	$(4,3,3,3,1)$	$(4,4,3,3,2)$
		$(3,3,2,1,1)$	$(4,2,2,2,2)$	$(4,3,3,2,2)$	
		$(3,2,2,2,1)$	$(3,3,3,2,1)$	$(3,3,3,3,2)$	
			$(3,3,2,2,2)$		

The act of augmenting a degree sequence by an integer k mimics the effect of adding a new vertex of degree k to a graph realizing π. We can similarly define an operation that mimics vertex deletion. To *decrement π by the term k* will mean to remove a term of π equal to k and to decrease k of the remaining positive terms by 1. This requires both that π have k as one of its terms and that π have at least $k+1$ positive terms to begin with. Note that each degree sequence in the lists above can be decremented by the number heading its column in such a way that $(3,2,2,1)$ is the result. We call the resulting sequences augmentations and decrementations of π, respectively.

Lemma 4.24. *Let π and ρ be degree sequences of graphs. The following are equivalent.*

(1) *The relation $\pi \preceq \rho$ holds.*

(2) *The sequence ρ can be formed by iteratively augmenting π and each resulting sequence by non-negative integers.*

(3) *The sequence π can be formed by iteratively decrementing ρ and each resulting sequence by a term from the sequence.*

Proof. To see that (1) implies (3), let G and H be realizations of π and ρ, respectively, such that G is an induced subgraph of H. Denote the vertices in $V(H) - V(G)$ by $v_1, \dots, v_p$. Letting $H_i = H_{i-1} - v_i$ for $1 \le i \le p$ yields a sequence $H = H_0, H_1, \dots, H_p = G$ of induced subgraphs of H. Here the degree sequence of each H_i is obtained by decrementing the degree sequence of H_{i-1} by the degree (in H_{i-1}) of v_i.

The equivalence of (2) and (3) can be verified by paying attention to the inserted/removed term and the increased/decreased terms at each iterative step.

To see that (2) implies (1), let $\pi = \pi^0, \pi^1, \dots, \pi^q = \rho$ be the degree sequences obtained at each stage of the augmenting that creates ρ from π, and for $i \in \{0, \dots, q-1\}$ let k_i denote the integer used in augmenting π^i to form π^{i+1}. Now let G_0 be any realization of $\pi = \pi^0$. Form a realization G_1 of π^1 by adding a new vertex v to G_0 and joining it by edges to k_0

vertices having degrees in G_0 that exactly match the terms of π^0 augmented to form π^1. The resulting realization G_1 contains G_0 as an induced subgraph. Proceeding inductively, we may construct $G_2, \ldots, G_q = H$ in this same way. Since H will be a realization of ρ containing G as an induced subgraph, we have $\pi \preceq \rho$. □

Lemma 4.24 allows us to more conveniently automate the process of determining whether $\pi \preceq \rho$ for given degree sequences π, ρ.

We now show that the notion of Rao-containment allows us to characterize the graphs in HCU, along with those in HU from [2, 4].

Lemma 4.25. *Suppose that $\pi \preceq \rho$. If ρ is the degree sequence of a graph in HU, then so is π; likewise, if ρ is the degree sequence of a graph in HCU, then so is π.*

Proof. Suppose first that $\pi \preceq \rho$ and that ρ is the degree sequence of a graph $H \in HU$. Since H is a unigraph, it is the only realization of ρ up to isomorphism. By the definition of Rao-containment, there exists a realization G of π that is an induced subgraph of H. We know that G and all its induced subgraphs are unigraphs because $H \in HU$; hence π is the degree sequence of a graph $G \in HU$.

Suppose instead that $\pi \preceq \rho$ and that ρ is the degree sequence of a graph in HCU. Since $\pi \preceq \rho$, there exists a realization G of π that is an induced subgraph of a realization H of ρ. Since the degree sequence of H is that of a graph in HCU, by Lemma 4.17 we conclude that H is an induced subgraph of a unigraph J. Since the induced subgraph relation is transitive, G is an induced subgraph of J, and π is the degree sequence of a graph in HCU. □

We will present the characterization of the graphs in HCU via Rao-containment in the next section. We use this later in Section 7 to find a forbidden induced subgraph characterization, so we conclude this section by developing the relationship between Rao-containment in degree sequences and containment of induced subgraphs in graphs.

Given a collection $\mathcal{D}$ of degree sequences, let $\mathcal{U}(\mathcal{D})$ denote the set of all degree sequences ρ such that $\rho \succeq \pi$ for some $\pi \in \mathcal{D}$. Now consider the infinitely many graphs having an element of $\mathcal{U}(\mathcal{D})$ as their degree sequence. Many of these graphs are induced subgraphs of others. Let $\mathcal{M}(\mathcal{D})$ denote the minimal such graphs with respect to the induced subgraph order. Alternatively, we recognize that $G \in \mathcal{M}(\mathcal{D})$ if the degree sequence of G belongs to $\mathcal{U}(\mathcal{D})$ but no proper induced subgraph of G has its degree sequence in $\mathcal{U}(\mathcal{D})$.

For example, suppose that $\mathcal{D} = \{(2, 2, 2)\}$. Then $\mathcal{M}(\mathcal{D})$ contains the graphs K_3, P_5, and $K_{2,3}$, among others. Note that neither P_5 nor $K_{2,3}$ contains K_3 as an induced subgraph, but each has a degree sequence that Rao-contains $(2, 2, 2)$, and no proper induced subgraph of these graphs has that same property.

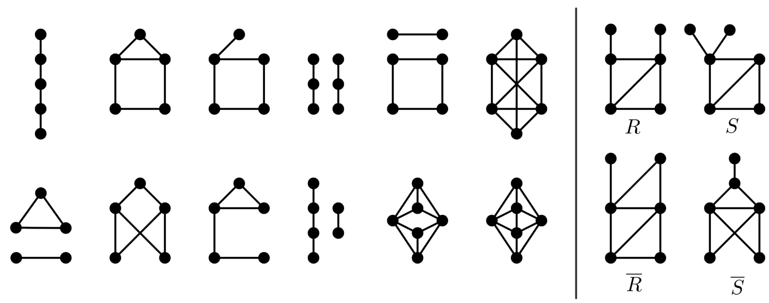

Figure 4.5: Minimal forbidden induced subgraphs for the family HU

Lemma 4.26. *Let $\mathcal{D}$ be a collection of degree sequences. For any graph G, the following are equivalent.*

(1) *The degree sequence of G Rao-contains some element of $\mathcal{D}$.*

(2) *The graph G contains an element of $\mathcal{M}(\mathcal{D})$ as an induced subgraph.*

Proof. Given a fixed collection $\mathcal{D}$ of degree sequences, in the following let G be a graph, and let ρ be its degree sequence.

Suppose first that ρ Rao-contains some element of $\mathcal{D}$. Then $\rho \in \mathcal{U}(d)$, and by definition, G contains an element of $\mathcal{M}(\mathcal{D})$ as an induced subgraph. This shows that (1) implies (2).

To prove the converse, suppose that G contains some element H of $\mathcal{M}(\mathcal{D})$ as an induced subgraph. If π denotes the degree sequence of H, then we have $\rho \succeq \pi$. Now let τ denote a degree sequence in $\mathcal{D}$ such that $\pi \succeq \tau$. By the transitivity of Rao's relation, we see that $\rho \succeq \tau$. This shows that (2) implies (1) and completes our proof. $\square$

We illustrate Lemma 4.26 with a comment on the family HU of hereditary unigraphs. In [2], Barrus presented a list $\mathcal{F}$ of 16 minimal forbidden induced subgraphs for HU; the elements of $\mathcal{F}$ are displayed in Figure 4.5. We add a new characterization of the class HU in terms of Rao-containment.

Theorem 4.27. *The following are equivalent for a graph G having degree sequence π.*

(1) *The graph G belongs to HU.*

(2) *The graph G contains no element of $\mathcal{F}$ as an induced subgraph.*

(3) *The sequence π does not Rao-contain any element of $\mathcal{L}_1$, where*

$$\mathcal{L}_1 = \{(2,2,2,1,1),(3,2,2,2,1),(3,3,2,2,2),\\ (2,2,1,1,1,1),(4,3,3,2,1,1),(4,4,3,2,2,1),(4,4,4,4,3,3)\}.$$

Proof. The equivalence of (1) and (2) was shown in [2]. Observe that no degree sequence in $\mathcal{L}_1$ is unigraphic because at least two realizations of each appear in Figure 4.5. Hence by Lemma 4.25 the statement (1) implies (3).

We now show that (3) implies (2). Suppose that a graph G having degree sequence π contains an element F of $\mathcal{F}$ as an induced subgraph. Then π Rao-contains the degree sequence of F by definition. If F is the graph K_2+C_4 or its complement, then π Rao-contains $(2,2,2,2,1,1)$ or $(4,4,3,3,3,3)$, which respectively Rao-contain the sequences $(2,2,2,1,1)$ and $(3,3,2,2,2)$ from $\mathcal{L}_1$. Otherwise, the degree sequence of F is listed in $\mathcal{L}_1$, so π Rao-contains an element of $\mathcal{L}_1$. □

4.7 A Rao-minimal degree sequence characterization of HCU

Lemma 4.25 implies that graphs in HU and HCU can be characterized as those whose degree sequences do not Rao-contain certain minimal "forbidden" sequences. This is illustrated for the family HU in part (3) of Theorem 4.27, where we give a list $\mathcal{L}_1$ of seven sequences that are not Rao-contained by the degree sequence of any graph in HU. Conversely, any graph not in HU has a degree sequence that Rao-contains at least one element of $\mathcal{L}_1$.

In this section we characterize graphs in HCU in a similar manner. We determine a minimal list $\mathcal{L}_2$ of degree sequences such that a graph G belongs to HCU if and only if the degree sequence of G does not Rao-contain any element of $\mathcal{L}_2$. We will do this in two parts, handling the degree sequences of non-split and split graphs separately. The resulting elements of $\mathcal{L}_2$ are listed in Theorems 4.32 and 4.34.

In light of Lemma 4.17, a degree sequence is *forbidden* if it is not the degree sequence of a graph in HCU. A forbidden degree sequence is *Rao-minimal* if it does not Rao-contain any other forbidden degree sequence. (Note that the degree sequences comprising $\mathcal{L}_1$ in Theorem 4.27(3) are each Rao-minimal with respect to the corresponding notion of "forbidden" for HU.) From the definition of Rao-containment, it follows that Rao-minimal sequences are related to minimal forbidden induced subgraphs. We record this, along with several observations derived from Corollary 4.13, as a remark.

Remark 4.28. *A forbidden degree sequence π is Rao-minimal if and only if its realizations are all minimal forbidden induced subgraphs for the class HCU. Moreover, each Rao-minimal*

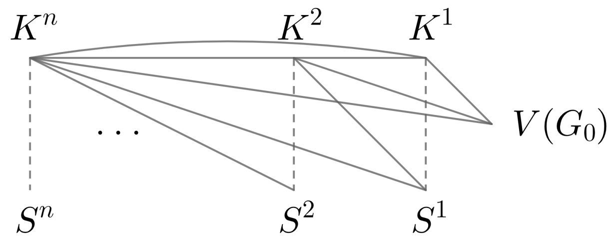

Figure 4.6: The structure of the composition $G_n \circ \ldots \circ G_0$, where $V(G_i)$ is given the KS-partition $K_i \cup S_i$. Solid lines indicate that all edges are present between two vertex sets, dotted lines indicate that some edges may exist, and two sets not connected with a line have no edges between them.

sequence π is indecomposable, and the degree sequence of the complement, and, if split, the inverse and inverse-complement, of any realization of π is also Rao-minimal.

Our approach will often turn to decrementations of degree sequences. The idea behind these decrementations arises from a result of Kleitman and Wang.

Theorem 4.29 ([46, Theorem 2.1])**.** *Let $\pi = (d_1, \ldots, d_n)$ be a graphic sequence. For all $k, 1 \leq k \leq n$, there exists a realization G_k of π with $V(G_k) = \{v_1, \ldots, v_n\}$ in which $\deg_{G_k} v_i = d_i$ for all i, $1 \leq i \leq n$, and the vertex v_k is adjacent to the first d_k vertices other than itself.*

The result of Kleitman and Wang implies that we may form another graphic sequence from $(d_1, \ldots, d_n)$ by imagining the impact of deleting a vertex v_k whose neighbors are as described in the theorem.

Definition 4.7. *Let π be a degree sequence with $\pi = (d_1, \ldots, d_n)$. For $1 \leq i \leq n$, the KW-reduction at index i is the degree sequence π' obtained by deleting d_i from π and decrementing d_i remaining largest terms by* 1.

For example, given $\pi = (4, 3, 2, 2, 2, 1)$, the KW-reduction at index 6 is $(3, 3, 2, 2, 2)$, and at index 2 is $(3, 2, 1, 1, 1)$. Since $d_4 = d_5$, we finished the KW-reduction at index 2 by reordering and reindexing the terms.

Remark 4.30. *The KW-reduction at index i of a graphic sequence is a special case of decrementing a sequence by its ith term as described in the previous section. By Lemma 4.24 this implies that if τ is a KW-reduction of π at any index, then π Rao-contains τ.*

It will be helpful to us to examine KW-reductions of degree sequences in light of Tyshkevich decomposition. The following analysis slightly generalizes the ideas in [3], which dealt with the Havel-Hakimi algorithm.

Given degree sequences $\pi^0, \ldots, \pi^k$ and a sequence π that can be written as $\pi^k \circ \cdots \circ \pi^1 \circ \pi^0$, let π' be the KW-reduction of π at an index in π^j. It is straightforward to see that

$$\pi' = \pi^k \circ \cdots \circ \pi^{j+1} \circ \rho \circ \pi^{j-1} \circ \cdots \circ \pi^0,$$

where ρ is the corresponding KW-reduction on π^j. Note that no assumption is made that $\pi^0, \ldots, \pi^k$ are indecomposable.

Indeed, whether a degree sequence is decomposable may change (in either direction) between a degree sequence and its KW-reduction. For example, if a graph G contains a single dominating or isolated vertex, and the decomposition of its degree sequence π can be written as $(0) \circ \rho$, where ρ is indecomposable, then the KW-reduction at index 1 of the decomposable sequence π is the indecomposable sequence ρ. Likewise, a KW-reduction on an indecomposable sequence may be a decomposable sequence. This arises exactly from dominating, isolated, and swing vertices.

Theorem 4.31. *If π is an indecomposable sequence of length n, and π' is a* decomposable *sequence obtained as a KW-reduction of π, then π' may be written as $(0) \circ \rho$ or $\rho \circ (0)$ for some graphic sequence ρ.*

Proof. Let π be an indecomposable sequence of length n and let π' be the KW-reduction of π indexed at a term with value t. Let G be a realization of π having a vertex v of degree t with the property (guaranteed by Theorem 4.29) that the neighbors of v have degrees as high as possible. Thus π' is the degree sequence of $G' = G - v$, i.e., deleting v from G has the same effect on the degree sequence of G that a KW-reduction does changing π into π'. Suppose that π' cannot be written as $(0) \circ \rho$ for any sequence ρ. We show π can be written as $\rho \circ (0)$ for some sequence ρ.

Our proof proceeds by showing that if the leftmost indecomposable part in the Tyshkevich decomposition of a graph G' has more than one vertex, then restoring vertex v to G' (forming the graph G) so as to make G indecomposable requires that G' have a single vertex in the rightmost component of its Tyshkevich decomposition. Throughout, we notate the induced subgraph of G on a set V of vertices by $G[V]$ and let $\deg_G V$ denote the set $\{\deg_G(v) | v \in V\}$. For any natural number k, we write $\deg_G V \leq k$ (respectively, $<$) if for all $v \in V$, $\deg_G(v) \leq k$ (resp., $<$).

Since G' is decomposable, it can be written as $(G'[A \cup B], A, B) \circ G'[C]$, where $G'[A \cup B]$ is indecomposable and has more than one vertex, and $G'[C]$ has at least one vertex. (As in Definition 4.2, we are specifying the KS-partition of $G'[A \cup B]$). This forces both A and B

to be nonempty. Now since G is indecomposable, the vertex v must have a neighbor in B or a non-neighbor in A (otherwise, G could be written as $(G[A \cup B], A, B) \circ G[C \cup \{v\}]$ for a contradiction). Suppose first that v has a neighbor in B. Let b be a neighbor of v in B.

For any $w \in V(G)$ with $\deg_{G'}(w) \geq \deg_{G'}(b) + 2$, we have $\deg_G(w) > \deg_G(b)$, and since the deletion of V of G corresponds to a KW-reduction on the degree sequence of G, we conclude that $vw \in E(G)$. Thus v is adjacent to each vertex whose degree in G' is at least two greater than $\deg_{G'}(b)$. Now $G'[A \cup B]$ is indecomposable, so each vertex of A has a neighbor and a non-neighbor in G and vice-versa. This means $\deg_{G'}(b) \leq |A| - 1$. Any $a \in A$ has a neighbor in B and is adjacent to every vertex in A and C, excepting itself, so $\deg_{G'}(a) \geq |A| + |C| \geq \deg_{G'}(b) + 2$. It follows that v is adjacent to every vertex of A.

We now consider v's adjacencies in C. If v is adjacent to all of C, then we may write G as $(G[A \cup B \cup \{v\}], A \cup \{v\}, B) \circ G[C]$, contradicting the indecomposability of G. Hence v has at least one non-neighbor in C; let C' denote the set of such vertices. Since $\deg_{G'}(b) \leq |A| - 1$, if any vertex in C' has any adjacency in C, then its degree is at least $|A| + 1$, thus at least two greater than $\deg_{G'}(b)$, and thus the vertex in C' would be adjacent to v, for a contradiction. Hence each vertex of C' is isolated in $G[C]$. At this point we may express G as $(G[A \cup B \cup C' \cup \{v\}], A \cup \{v\}, B \cup C') \circ G[C \setminus C']$. Since G is indecomposable, we conclude that $C = C'$. Therefore $G[C]$ is a set of isolated vertices, and the rightmost component of the Tyshkevich decomposition of G is trivial.

Now, suppose v has no neighbor in B, and instead must have a non-neighbor a in A. It is straightforward to verify in this case that the complements of G and G', along with their connection via a KW-reduction on the degree sequence of the complement of π indexed at a term equal to $n - 1 - t$, satisfy the conditions of the theorem. The result follows because swing vertices are preserved by KW-reductions and complementation. ☐

4.7.1 Non-split Rao-minimal sequences

As above, $\mathcal{L}_2$ denotes the complete list of Rao-minimal forbidden degree sequences for HCU. We characterize the non-split Rao-minimal sequences first.

Theorem 4.32. *The degree sequences* $(3,3,2,2,2), (2,2,2,1,1), (2,2,1,1,1,1), (4,4,4,4,3,3), (5,4,3,3,3,1,1),$ *and* $(5,5,3,3,3,2,1)$ *are elements of* $\mathcal{L}_2$*; these are exactly the Rao-minimal forbidden degree sequences of non-split graphs.*

Proof. In this proof, we call a degree sequence a *target sequence* if it is indecomposable, forbidden, and non-split. By Lemma 4.20 and Theorem 4.21, the degree sequences listed in the theorem are target sequences. By Remark 4.28, non-split Rao-minimal sequences must be target sequences, so it suffices to prove that of all target sequences, exactly those listed in Theorem 4.32 are Rao-minimal. The graphs C_4 and $2K_2$ are the only two non-split graphs on four or fewer vertices, and both are included in HCU. This implies that $(3, 3, 2, 2, 2)$ and $(2, 2, 2, 1, 1)$ are Rao-minimal. The other four sequences listed in the theorem do not Rao-contain any smaller target sequences, so they are also Rao-minimal.

We proceed inductively, showing that for any target sequence π on six or more vertices not listed in Theorem 4.32, there exists a degree sequence π' on one or two fewer vertices such that π' is a target sequence and some realization of π' is induced in some realization of π. Hence π is not Rao-minimal.

Let π be a target sequence of length $n \geq 6$ other than those listed in Theorem 4.32. First suppose that there exists $i \in \{1, \ldots, n\}$ for which the KW-reduction at index i satisfies none of the five conditions in Theorem 4.21 and contains no terms equal to 0 or $n - 2$. Let π' be this KW-reduction. By Remark 4.30, π' is Rao-contained by π, and by Theorem 4.31, π' is indecomposable.

Suppose for a contradiction that there exists an (indecomposable) split graph G' realizing π', with KS-partition $K' \cup S'$. Let G be the graph created from G' by adding a vertex v adjacent to d_i vertices $v_1, \ldots, v_{d_i}$ of highest degree in G'. Since π' is a KW-reduction of π at index i, it follows that G is a realization of π. In an indecomposable split graph, the clique vertices have greater degree than the stable set vertices. If $d_i \leq |K'|$, then v is adjacent only to clique vertices, so G is a split graph with clique K' and stable set $S' \cup \{v\}$. If $d_i > |K'|$, then v is adjacent to all clique vertices and some stable set vertices, so G is a split graph with clique $K' \cup \{v\}$ and stable set S'. Either way, we conclude that G is split.

Hence π' is a target sequence Rao-contained by π, and we conclude that π is not Rao-minimal. Therefore, as claimed, if for some $i \in \{1, \ldots, n\}$ the KW-reduction π' at index i has no terms equal to 0 or $n - 2$ and satisfies none of the five conditions in Theorem 4.21, then π is not Rao-minimal.

We now consider the remaining possibility, i.e., what occurs if for all i, $1 \leq i \leq n$, the KW-reduction of π at index i satisfies one of the five conditions in Theorem 4.21 or contains a term equal to 0 or $n - 2$. We will show that π Rao-contains an element of the list $\mathcal{L}_2$ in the statement of Theorem 4.32.

Our assumption about the KW-reductions certainly pertains to the KW-reduction at index 2, and we begin our analysis here. Since π is indecomposable, $1 \leq d_j \leq n - 2$ for all $j \in \{1, \ldots, n\}$. Let π' be the KW-reduction at index 2, and let $x = d_2$ denote the removed term. If $x = 1$, then π satisfies condition (1) of Theorem 4.21, a contradiction; hence $x \geq 2$, and the last term of π to be reduced by 1 is always equal to d_{x+1} (because of the removal of the term d_2 in the KW-reduction). Recall that $n \geq 6$ and that the terms of π are assumed to be in non-increasing order.

By the present assumptions, π' satisfies one of the conditions (1)-(5) from Theorem 4.21, or π' has a term equal to 0 or $n-2$. We claim that statements (i) through (vii) below convert these possibilities into conditions on the sequence π.

(i-a) $d_3 = \cdots = d_{x+1} = 2$ and $d_{x+2} = \cdots = d_n = 1$.

(i-b) $d_1 = \cdots = d_4 = 2$ and $d_5 = \cdots = d_n = 1$.

(ii-a) $d_1 = \cdots = d_{x+1} = n-2$ and $d_{x+2} = \cdots = d_{n-1} = n-3$.

(ii-b) $d_1 = \cdots = d_{n-2} = n-2$ and $d_{n-1} = d_n = n-3$.

(iii-a) $d_1 = n-2, d_3 = 3$, and $d_{x+2} \leq 2$.

(iii-b) $d_1 = n-2, d_2 \geq 3$ and $d_3 = \cdots = d_{x+2} = 2$.

(iv) $d_2 = \cdots = d_{n-2} = n-3$, $d_{n-1} = n-4$, and $d_n = 1$.

(v) $\pi = (3,3,3,3,2,2)$

(vi) $d_{x+1} = \cdots = d_n = 1$.

(vii) $d_1 = \cdots = d_{x+2} = n-2$.

It is straightforward to verify that conditions (vi) and (vii) are respectively equivalent to the sequence π' containing a term equal to 0 or to $n-2$. We further claim that each of the conditions (1)–(5) from Theorem 4.21 implies at least one of the conditions (i-a)–(vii), most often one having a corresponding Roman numeral. Conversely, we claim that (1)–(5) are each implied by the statement or disjunction of statements listed above having the same Roman numeral.

We prove a representative case and leave verification of the others to the reader. Observe that if π satisfies (i-a) or (i-b), then π' satisfies condition (1). Conversely, suppose that π' satisfies (1). Since forming π' involves removing the second term of π, the second term of π' is equal to

$$\begin{cases} d_3 - 1 & \text{if } d_3 > d_{x+2} & \text{Example: } \pi = (2,2,2,1,1); \\ d_3 & \text{if } d_1 > d_3 = d_{x+2} & \text{Example: } \pi = (2,2,1,1,1,1); \\ d_3 - 1 & \text{if } d_1 = d_3 = d_{x+2} > d_{x+3} & \text{Example: } \pi = (2,2,2,2,1,1); \\ d_3 & \text{if } d_1 = d_3 = d_{x+2} = d_{x+3} & \text{Example: } \pi = (1,1,1,1,1,1). \end{cases}$$

Condition (1) requires that the second term of π' be 1, so in the first of the cases above, $d_3 = 2$ and $d_{x+2} = 1$, yielding (i-a) or (vi). In the second case, $d_3 = \cdots = d_n = 1$, so (vi) holds, and in the third case, $d_1 = x = d_3 = 2$, and (i-b) holds. The fourth case leads to a contradiction, since then $d_2 = 1$ and π is not a target sequence by Theorem 4.21. The other verifications are similar.

Hence π satisfies one of the ten conditions (i-a) through (vii) listed above, and we will consider each one. In most cases a condition from this list quickly leads to a contradiction or to the conclusion that π Rao-contains an element of the list $\mathcal{L}_2$.

If π satisfies (i-a), then it has the form $(d_1, x, 2, ..., 2, 1, ...1)$, where there are $x-1$ terms equal to 2 following x. We show by induction that all such sequences of length at least 5 Rao-contain $(2,2,2,1,1)$. When the length is 5, parity considerations show that $(2,2,2,1,1)$ is the only sequence satisfying (i-a). Now assume that $n > 5$ and that every target sequence of length at most $n-1$ that satisfies (i-a) Rao-contains $(2,2,2,1,1)$. If $d_1 = 2$, then π clearly Rao-contains $(2,2,2,1,1)$. If $d_1 > 2$ and $x = 2$, then π has multiple terms equal to 1, and the KW-reduction at index n on π does not change any term from π but the first term and the omitted last term; then this KW-reduction satisfies (i-a), and by the induction hypothesis and the transitivity of Rao-containment, π Rao-contains $(2,2,2,1,1)$. Finally, if $x > 2$, then the fourth term of π equals 2; here the KW-reduction at index 3 reduces the first two terms of π by 1 and leaves later remaining terms unchanged, so this KW-reduction again satisfies (i-a), and we conclude that π Rao-contains $(2,2,2,1,1)$.

If π satisfies (i-b), then it has the form $(2,2,2,2,1,\dots,1)$, where we may assume that there are at least two 1 terms. It is straightforward to find a realization of π that contains P_5 as an induced subgraph, so π Rao-contains $(2,2,2,1,1)$.

The sequence π cannot satisfy (ii-a): here $x = n-2$, forcing d_{n-1} to equal both $n-2$ and $n-3$.

Suppose that π satisfies (ii-b). Note that n must be even, and π has as one of its realizations the complement of $P_4 + (n/2-2)K_2$, which has the complement of $P_4 + K_2$ as an induced subgraph. It follows that π Rao-contains $(4,4,4,4,3,3)$.

We will return to conditions (iii-a) and (iii-b) later. We next show that if any target sequence of length at least 5 satisfies (iv), then the sequence Rao-contains $(2,2,2,1,1)$. We use induction on the length, noting that $(2,2,2,1,1)$ itself satisfies (iv). Fix $n \geq 6$, and assume that this claim holds for all sequences satisfying (iv) with length between 5 and $n-1$. If n is even, then parity requires that $d_1 = n-2$, and performing the KW-reduction at index 1 on π results in a sequence of smaller length that also satisfies (iv) and hence Rao-contains $(2,2,2,1,1)$. If n is odd, then $d_1 = n-3$. Consider the degree sequence ρ of length $n-1$ satisfying (iv). A realization of π may be formed by adding a vertex to a realization G' of ρ that is adjacent exactly to all vertices of G' except for the vertex of maximum degree and the vertex of minimum degree. It follows that π Rao-contains ρ, and by transitivity and the induction hypothesis, π Rao-contains $(2,2,2,1,1)$.

If π satisfies (v) – that is, $\pi = (3, 3, 3, 3, 2, 2)$ – then π Rao-contains $(3, 3, 2, 2, 2)$, and our conclusion holds.

We also delay handling condition (vi) momentarily. If π satisfies (vii), then $x = n - 2$ and in fact every term of π is $n - 2$. This is a contradiction, since by condition (2) of Theorem 4.21, π would not be forbidden.

It remains to show that our target sequence π Rao-contains an element of $\mathcal{L}_2$ when π satisfies one of the conditions (iii-a), (iii-b), or (vi). To assist us, we consider the "complementary" degree sequence $\overline{\pi} = (n - 1 - d_n, n - 1 - d_{n-1}, \ldots, n - 1 - d_1)$ with ith term $n - 1 - d_{n+1-i}$. Since the graph properties of being indecomposable, forbidden, and non-split are all preserved under graph complementation, the arguments thus far in the proof also apply to $\overline{\pi}$. If the forbidden sequence $\overline{\pi}$ Rao-contains an element of $\mathcal{L}_2$, then the same must be true for π by Remark 4.28. Hence we may assume that $\overline{\pi}$ satisfies a requirement analogous to conditions (iii-a), (iii-b), or (vi). This implies that π satisfies one of the additional conditions listed below, where $y = d_{n-1}$. Note that the second term $\overline{x}$ of $\overline{\pi}$ equals $n-1-y$ (so for instance the term of $\overline{\pi}$ with index $\overline{x}+2$ is equal to $n-1-d_{n+1-(\overline{x}+2)} = n-1-d_{n+1-(n-1-y)-2} = n-1-d_y$). We may assume that $y \leq n - 3$ since otherwise condition (2) of Theorem 4.21 would imply that π is not forbidden.

(iii-a′) $d_n = 1$, $d_{n-2} = n - 4$, and $d_y \geq n - 3$;

(iii-b′) $d_n = 1$, $d_{n-1} \leq n - 4$, and $d_y = \cdots = d_{n-2} = n - 3$;

(vi′) $d_1 = \cdots = d_{y+1} = n - 2$.

Suppose first that π satisfies condition (iii-a). If π also satisfies (iii-a′), then $3 = d_3 \geq d_{n-2} = n-4$, so $n \leq 7$. The only non-increasing, graphic sequences of length 6 or 7 satisfying these conditions are $(4, 3, 3, 2, 1, 1)$, $(4, 4, 3, 2, 2, 1)$, $(5, 4, 3, 3, 3, 1, 1)$, and $(5, 5, 3, 3, 3, 2, 1)$. The first two are split sequences, contrary to our assumption on π. The last two are elements of $\mathcal{L}_2$.

If instead π satisfies (iii-a) and (iii-b′), then since $d_3 = 3 \leq n - 3 = d_{n-2} \leq 3$, we have $n = 6$. The only non-increasing, graphic sequence of length 6 satisfying these conditions is $(4, 3, 3, 3, 2, 1)$, which Rao-contains $(2, 2, 2, 1, 1)$.

Suppose that π satisfies (iii-a) and (vi′). If $y > 1$ then $3 = d_3 = n-2$ and $n = 5$, contrary to our assumption. With $y = 1$ we see that $d_1 = d_2 = n - 2$ and $d_{n-1} = d_n = 1$. If G is any realization of π, then each vertex of G of degree $n-2$ must be adjacent to at least one vertex of degree 1; the only way for this to happen is if the two vertices with degrees d_1 and d_2 are both adjacent to all other vertices in the graph. This is a contradiction, for then G and hence π are decomposable in the Tyshkevich decomposition; we may write $\pi = (2, 2, 1, 1) \circ \pi^*$, where π^* is the degree sequence of the induced subgraph resulting from deleting two vertices of maximum degree and two of minimum degree from G. Hence π cannot satisfy (iii-a) and (vi′).

If π satisfies (iii-b) and (iii-b′), then a contradiction arises, as $2 = d_3 \geq d_{n-2} = n - 3$.

Suppose that π satisfies (iii-b) and (vi′). Since $d_{y+1} = n - 2 > 2 = d_3$, we would need $y = 1$ and hence $x = d_2 = n - 2$, but then $d_n = d_{x+2} = 2 > 1 = d_{n-1}$, a contradiction.

Finally, suppose that π satisfies (vi) and (vi′). Here $x = n-2$, so π satisfies $d_1 = d_2 = n-2$ and $d_{n-1} = d_n = 1$. As before, this is a contradiction, since π is then decomposable.

Note now that if π satisfies (iii-b) and (iii-a′), (vi) and (iii-a′), or (vi) and (iii-b′), then the arguments of the preceding paragraphs, applied appropriately to the degree sequence $\overline{\pi}$, either end in contradiction or show that $\overline{\pi}$ (and hence π) Rao-contains a degree sequence from $\mathcal{L}_2$. This completes the proof of Theorem 4.32. □

In the proof of the theorem, we show that every non-minimal target sequence can be reduced to another *non-split* sequence, and none Rao-contain only a split Rao-minimal sequence. We record this as a corollary.

Corollary 4.33. *Let π be an indecomposable, forbidden, non-split degree sequence. Then π Rao-contains a non-split element of $\mathcal{L}_2$.*

4.7.2 Split Rao-minimal sequences

We now characterize the Rao-minimal split degree sequences, which are all of length 8.

Theorem 4.34. *The degree sequences*

$$\begin{array}{llll}
(5,5,4,4,2,2,1,1), & (5,4,4,2,2,1,1,1), & (5,5,4,2,2,2,1,1), & (5,5,5,4,2,2,2,1), \\
(5,5,5,5,2,2,2,2), & (6,5,4,4,3,2,1,1), & (6,5,5,4,3,2,2,1), & (6,5,5,4,3,3,1,1), \\
(6,5,5,5,3,2,2,2), & (6,6,4,4,3,2,2,1), & (6,6,5,4,3,3,2,1), & (6,6,5,5,3,3,2,2), \\
(6,6,5,5,5,3,2,2), & (6,6,6,5,5,3,3,2) & &
\end{array}$$

are elements of $\mathcal{L}_2$; these are exactly the Rao-minimal forbidden degree sequences of split graphs.

Our arguments proving Theorem 4.34 will be similar to those in Section 4.7.1, though here we will replace degree sequences by cross-degree sequence pairs in order to take advantage of Theorem 4.23. Given two cross-degree sequence pairs $\{\alpha, \beta\}$ and $\{\alpha', \beta'\}$, we define $\{\alpha, \beta\}$ to be *Rao-contained by* $\{\alpha', \beta'\}$ if some realization of $\{\alpha, \beta\}$ is an induced subgraph of some realization of $\{\alpha', \beta'\}$. Equivalently, $\{\alpha, \beta\}$ is Rao-contained by $\{\alpha', \beta'\}$ if any degree sequence corresponding to $\{\alpha, \beta\}$ is Rao-contained by some degree sequence corresponding to $\{\alpha', \beta'\}$. A cross-degree sequence pair $\{\alpha, \beta\}$ is *forbidden* or *Rao-minimal* if a degree sequence corresponding to $\{\alpha, \beta\}$ is forbidden or Rao-minimal, respectively, and $\{\alpha, \beta\}$ is a *target pair* if any (and hence all) of its corresponding degree sequences is indecomposable and forbidden. Finally, if $\alpha = (a_1, \ldots, a_n)$ and $\beta = (b_1, \ldots, b_m)$, define the *length* of $\{\alpha, \beta\}$ as $n + m$, the total number of terms within the two sequences.

The degree sequences in Theorem 4.34 correspond to the following cross-degree sequence pairs. Since cross-degree sequence pairs are unordered, there exist three instances of pairs of degree sequences above mapping to the same cross-degree sequence pair.

$$\begin{array}{lll}\{(2,2,1,1,1),(3,2,2)\}, & \{(2,2,2,1,1),(3,3,2)\}, & \{(2,2,1,1),(2,2,1,1)\},\\ \{(3,2,1,1),(3,2,1,1)\}, & \{(2,2,2,1),(2,2,2,1)\}, & \{(2,2,2,2),(2,2,2,2)\},\\ \{(3,2,2,1),(3,2,2,1)\}, & \{(3,3,1,1),(3,2,2,1)\}, & \{(3,2,2,2),(3,2,2,2)\},\\ \{(3,3,2,1),(3,3,2,1)\}, & \{(3,3,2,2),(3,3,2,2)\}. & \end{array} \tag{$*$}$$

It is not hard to show by Lemma 4.22 and Theorem 4.23 that each of these is a target pair.

In preparation for the proof of Theorem 4.34, we note when an unordered pair of sequences is the cross-degree sequence pair of a split graph. This lemma, due independently to Gale and Ryser, was originally proven in the context of bipartite graphs.

Lemma 4.35 ([31, 60])**.** *Let $\alpha = (a_1, \ldots, a_n)$ and $\beta = (b_1, \ldots, b_m)$ be two non-increasing, non-negative sequences. The unordered pair $\{\alpha, \beta\}$ is the cross-degree sequence pair of a split graph if and only if $\{\alpha, \beta\}$ satisfies the kth Gale–Ryser inequality given in Lemma 4.22 for all $1 \leq k \leq n$.*

We also find an analogue of Theorem 4.29 for cross-degree sequence pairs. Recall that in each of α and β the terms are assumed to be in non-increasing order.

Lemma 4.36. *Let $\{\alpha, \beta\}$ be a cross-degree sequence pair of a graph with $\alpha = (a_1, \ldots, a_n)$ and $\beta = (b_1, \ldots, b_m)$. For any term a_i (resp. b_i), there exists a realization of $\{\alpha, \beta\}$ such that the vertex corresponding to the term a_i (resp. b_i) is adjacent to the vertices corresponding to terms $b_1, \ldots, b_{a_i}$ (resp. $a_1, \ldots, a_{b_i}$).*

Proof. For any $a_i \in \alpha$, the sequence $\pi = (b_1 + m - 1, b_2 + m - 1, \ldots, b_m + m - 1, a_1, \ldots, a_n)$ is a degree sequence associated with $\{\alpha, \beta\}$. Since $\{\alpha, \beta\}$ corresponds to a split graph having a clique of size $|\beta| = m$, we have $a_1 \leq m$, with equality only if $b_m \geq 1$. Hence π is non-increasing, and to finish the proof we apply Theorem 4.29. $\square$

We now begin our proof of Theorem 4.34. We first show that the sequence pairs in $(*)$ are Rao-minimal. Proceeding as in the proof of Theorem 4.32, we define Kleitman-Wang-style reductions and apply them to all larger target pairs, concluding that every target pair other than the ones listed in $(*)$ may be successfully reduced to a shorter target pair, so the result follows inductively.

Proof of Theorem 4.34. As noted above, each pair in $(*)$ is a target pair. To see that the sequence pairs in $(*)$ are Rao-minimal, we show that sequence pairs $\{\alpha, \beta\}$ with length at most 7 correspond to degree sequences of graphs in HCU. Let $\alpha = (a_1, \dots, a_n)$ and $\beta = (b_1, \dots, b_m)$, and assume that $n + m \leq 7$. If $m \leq 2$ (or $n \leq 2$, respectively), then since $\{\alpha, \beta\}$ is indecomposable, every term in α (in β) must be 1, so the sequence pair satisfies (1a) or (1b) of Theorem 4.23. If instead $m = 3$ and $n = 3$, then all terms in α and β are 1 or 2. Since at least two terms of α must be equal, condition (1a) or (2a) from Theorem 4.23 applies. If $m = 3$ and $n = 4$, then all terms in α are 1 or 2. This yields condition (1a) or (2a) as before unless $\alpha = (2, 2, 1, 1)$; with that choice of α, conditions (3a) or (4a) apply unless $b_1 \neq 3$ and $b_m \neq 1$, i.e., $\beta = (2, 2, 2)$. Finally, the pair $\{(2, 2, 1, 1), (2, 2, 2)\}$, fits conditions (3b) and (4b). The case $m = 4$ and $n = 3$ is similar. In every case, Theorem 4.23 shows that $\{\alpha, \beta\}$ belongs to a graph in HCU, as claimed.

Hence target pairs with length 8 are necessarily Rao-minimal. It is straightforward to verify that the sequence pairs in $(*)$ are precisely these target pairs.

From here, we proceed by induction on the length of a target pair, showing that all target pairs with length at least 9 must Rao-contain a target pair with length that is 1 or 2 less. Let $\{\alpha, \beta\}$ be a target pair where $\alpha = (a_1, \dots, a_n)$ and $\beta = (b_1, \dots, b_m)$ and α, β satisfy $m + n \geq 9$ with $m, n \geq 3$. (We lose no generality here, for if m or n is at most 2, then Lemma 4.20 or Theorem 4.23 implies that any realization of $\{\alpha, \beta\}$ is decomposable or an element of HCU.) For each term $a = a_i$ of α, define a reduction $R_{a,\alpha}$ to mean removing a_i from α and decrementing each of $b_1, \dots, b_a$ by 1, reordering terms in the resulting sequence as necessary to keep them in non-increasing order. We denote the reduced sequence pair by $\{\alpha', \beta'\}_{a,\alpha}$. Reductions $R_{b,\beta}$ may be similarly defined for terms $b = b_j$ belonging to β; the resulting cross-degree pair is denoted $\{\alpha', \beta'\}_{b,\beta}$. We may omit subscripts on reduced sequence pairs $\{\alpha', \beta'\}$ when context makes it clear what is meant. Note that all of these reductions are cross-degree analogs of KW-reductions, and in each the reduced pair $\{\alpha', \beta'\}$ corresponds to a degree sequence resulting from a KW-reduction on a degree sequence corresponding to $\{\alpha, \beta\}$.

To help in recognizing the indecomposability of sequence pairs after reductions, we specialize Theorem 4.31 to the context of split graphs.

Corollary 4.37. *Let $\{\alpha, \beta\}$ be an indecomposable sequence pair, and let $\{\alpha', \beta'\}$ be the result of either $R_{a,\alpha}$ or $R_{b,\beta}$ for some term a or b. If α' contains no term equal to 0 or $|\beta'|$, and if*

β' contains no term equal to 0 *or* $|\alpha'|$*, then* $\{\alpha', \beta'\}$ *is an indecomposable sequence pair.*

Given $\{\alpha, \beta\}$, if after a reduction $R_{a,\alpha}$ or $R_{b,\beta}$ the pair $\{\alpha', \beta'\}$ satisfies none of the eight conditions in Theorem 4.23 and contains no terms in α' equal to 0 or $|\beta'|$ and no terms in β' equal to 0 or $|\alpha'|$, then Theorem 4.23 implies that $\{\alpha', \beta'\}$ is forbidden, Lemma 4.36 implies that $\{\alpha', \beta'\}$ is Rao-contained by $\{\alpha, \beta\}$, and Corollary 4.37 implies that $\{\alpha', \beta'\}$ is indecomposable. In such a case, $\{\alpha, \beta\}$ is not Rao-minimal.

We now consider what happens if every reduction $R_{a,\alpha}$ and $R_{b,\beta}$ yields a pair $\{\alpha', \beta'\}$ that satisfies one of the eight conditions in Theorem 4.23, or contains some term in α' equal to 0 or $|\beta'|$, or contains some term in β' equal to 0 or $|\alpha'|$. Given $a = a_i$ for some i in $\{1, \ldots, n\}$, we claim that the reduction $R_{a,\alpha}$ yields one of these outcomes if and only if one or more of the conditions occurs from the list that follows this paragraph. As in the previous subsection, for most conditions the Roman numeral and accompanying Latin letter indicate which condition from Theorem 4.23 is met by $\{\alpha', \beta'\}$; for instance, condition (II.b.1) below is one case corresponding to condition (2b) in the earlier theorem. Suffixes with Latin numbers are used when a condition from Theorem 4.23 is equivalent to a disjunction of multiple conditions here. Since $\{\alpha, \beta\}$ is not the cross-degree sequence pair of a graph in HCU, by Theorem 4.23 we see that $a_2 \geq 2$ and $b_2 \geq 2$, and $a_{n-1} \leq m-2$ and $b_{m-1} \leq n-2$. More generally, in arriving at these conditions care has been taken to ensure that none of the conditions from Theorem 4.23 holds for $\{\alpha, \beta\}$.

(I.a) $i \leq 2$ and $a_3 = \cdots = a_n = 1$.

(I.b.1) $a \geq 2$ and $b_2 = 2$ and $b_{a+1} = \cdots = b_m = 1$.

(I.b.2) $b_1 = \cdots = b_{a+1} = 2$ and $b_{a+2} = \cdots = b_m = 1$.

(II.a) $i \geq n-1$ and $a_1 = \cdots = a_{n-2} = m-1$.

(II.b.1) $b_1 = \cdots = b_a = n-1$ and $b_{m-1} = n-2$.

(II.b.2) $b_1 = \cdots = b_{a-1} = n-1$ and $b_a = \cdots = b_m = n-2$.

(III.a.1) $i = 1$ or $a_1 = 2$; $a_2 = 2$; and $b_1 = \ldots = b_{a+1} = n-2$.

(III.a.2) $i = 1$, $a_1 \geq 3$, $a_2 = 2$, and $b_1 = n-1$.

(III.b) $i \geq 2$ or $a_2 = m-1$; $a_1 = m-1$; $b_1 = 3$; and $b_{a+1} \leq 2$.

(IV.a.1) $i = n$ or $a_n = m-2$; $a_{n-1} = m-2$; and $b_a = \cdots = b_m = 2$.

(IV.a.2) $i = n$, $a_{n-1} = m-2$, $a_n \leq m-3$, and $b_m = 1$.

(IV.b) $i \leq n-1$ or $a_{n-1} = 1$; $a_n = 1$; $b_a \geq n-2$; and $b_m = n-3$.

(V) $b_a = 1$.

(VI) $b_{a+1} = n - 1$.

As in the non-split argument in the previous subsection, for brevity we illustrate only a representative equivalence from the list above and leave verification of the others to the reader. Specifically, we show the equivalence of condition (2b) from Theorem 4.23, applied to $\{\alpha', \beta'\}_{a,\alpha}$, and the disjunction of conditions (II.b.1) and (II.b.2). It is straightforward to see that if $\{\alpha, \beta\}$ satisfies either (II.b.1) or (II.b.2) above, then the reduced pair $\{\alpha', \beta'\}_{a,\alpha}$ satisfies condition (2b) of Theorem 4.23. Conversely, if (2b) holds for $\{\alpha', \beta'\}_{a,\alpha}$, then the second-to-last term of β' is one less than the length of α', so this term equals $n-2$. Now the second-to-last term of β' must equal either b_{m-1} or $b_{m-1} - 1$, since even after decrementing and reordering, the value in no position decreases by more than 1 as β is changed to β'. Since $b_{m-1} \leq n-2$, we conclude that both b_{m-1} and the second-to-last term of β' equal $n-2$. We must now determine conditions under which the values in these positions of β and of β' agree. If no reordering of terms involves the $(m-1)$th position, then $a < m-1$ and $b_a > b_{m-1} = n-2$; this requires that $b_1 = \cdots = b_a = n-1$, and (II.b.1) holds. If reordering of terms involving the $(m-1)$th position does happen after terms of β are decremented, then some term of β that was decremented originally had value b_{m-1}, but any such term must be moved to a position after the $(m-1)$th during reordering; this requires that $b_{a-1} > b_a = b_{m-1} = b_m$, and (II.b.2) holds, as desired.

Given the assumptions outlined above, we assume henceforth the equivalence of the conditions from Theorem 4.23, as applied to $\{\alpha', \beta'\}_{a,\alpha}$, and conditions (I.a) through (VI) as applied to $\{\alpha, \beta\}$. To this point we have not had a reason to distinguish between α and β, and we have likewise assumed that every reduced pair $\{\alpha', \beta'\}_{b,\beta}$ satisfies a condition from Theorem 4.23 or contains a term in α' or β' that would imply that the reduced pair is decomposable. Hence by the arguments above, we may assume that $\{\alpha, \beta\}$ also satisfies one of fourteen conditions (I.a′), (I.b.1′), and so on through (VI′), where in each corresponding condition listed above n and m are exchanged, as are the roles played by terms of α and of β. The assumption here is that a reduction $R_{b,\beta}$ was performed (with $b = b_i$) rather than $R_{a,\alpha}$. For example, condition (III.a.1′) is the condition that $i = 1$ or $b_1 = 2$; $b_2 = 2$; and $a_1 = \cdots = a_{b+1} = m-2$.

We now turn to showing that each condition (I.a) through (VI) leads either to a contradiction or to Rao-containment by $\{\alpha, \beta\}$ of one of the cross degree sequence pairs listed in $(*)$. At times the conditions (I.a′) through (VI′) will help our analysis.

For ease of notation throughout, let

$$\sum \alpha = \sum_{t=1}^{n} a_t \text{ and } \sum \beta = \sum_{t=1}^{m} b_t.$$

Since each sum counts the same edges in a split graph, $\sum \alpha = \sum \beta$. When we compute upper and lower bounds on these sums, we notate them first in unsimplified form: (multiplicity of terms equal to largest term)$\cdot$ (value of largest term) + (multiplicity of terms equal to next lower term) $\cdot$ (value of that term) $+ \ldots$.

We handle sequence pairs with a"short" α or β first. Earlier we showed that $m, n \geq 3$; we begin with the case that $m = 3$ and hence $n \geq 6$. By assumption, $\{\alpha, \beta\}$ does not satisfy the hypothesis of Theorem 4.23; it follows that $a_1 = a_2 = 2$ and $a_{n-1} = a_n = 1$, and $3 \leq b_1 \leq n-2$ and $2 \leq b_3 \leq n-3$. On the other hand, the reduced pair $\{\alpha', \beta'\}_{a_3,\alpha}$ does satisfy one of the conditions (I.a) to (VI) above. Since we may take $i = 3$ and consequently $a \in \{1, 2\}$, our known values or bounds for the beginning/ending terms of α, β force the applicable condition to be one of the following: (III.a.1) if $a = 1$, (III.b), (IV.a.1), or (IV.b). If (III.a.1) holds with $a = 1$, then $b_2 = n - 2$ and $\sum \beta \geq 2(n-2) + 1 \cdot 2$ while $\sum \alpha = n + 2$, a contradiction since $n \geq 6$. If (III.b) holds, then $b_1 = 3$ and $b_3 \leq 2$, so $8 \leq n + 2 \leq \sum \alpha = \sum \beta \leq 8$, and the resulting equalities imply that $\alpha = (2, 2, 1, 1, 1, 1)$ and $\beta = (3, 3, 2)$; in this case $\{\alpha', \beta'\}_{a_3,\alpha} = \{(2, 2, 1, 1, 1), (3, 2, 2)\}$, which appears in $(*)$. If instead (IV.a.1) holds, then since $b_1 > 2$ we must have $a = 2$ and $b_2 = b_3 = 2$; but then $\sum \beta \leq n + 2$ while $\sum \alpha \geq n + 3$, a contradiction. Finally, if (IV.b) holds, then since $b_3 = n - 3$ we have $3n - 9 \leq \sum \beta = \sum \alpha \leq 2n - 2$, implying that $n \in \{6, 7\}$. Direct verification shows that if $n = 6$, the only feasible pairs $\{\alpha, \beta\}$ are $\{(2, 2, 2, 1, 1, 1), (3, 3, 3)\}$ and $\{(2, 2, 2, 2, 1, 1), (4, 3, 3)\}$; in both cases $\{\alpha', \beta'\}_{a_3,\alpha}$ appears in $(*)$. If $n = 7$, the only feasible pair is $\{(2, 2, 2, 2, 2, 1, 1), (4, 4, 4)\}$. Here $\{\alpha', \beta'\}_{a_3,\alpha} = \{(2, 2, 2, 2, 1, 1), (4, 3, 3)\}$, one of the sequences just considered; we see again that $\{\alpha, \beta\}$ Rao-contains an element of $(*)$.

To this point we have not distinguished between α and β in the discussion, so we may exchange the names of α, β and apply the preceding arguments if $n = 3$. Breaking the symmetry between α and β somewhat, we assume henceforth that $4 \leq m \leq n$. Since $m + n \geq 9$, we have $n \geq 5$.

To make the remaining proof more efficient, we consider the conditions from our list in the following order: (III.a.1), (IV.a.1), (III.a.2), (IV.a.2), (I.b.1), (II.b.1), (I.b.2), (II.b.2), (I.a), (II.a), (III.b), (IV.b), and lastly (V) and (VI) together.

Suppose first that $\{\alpha, \beta\}$ satisfies (III.a.1). Here $i = 1$ or $a_1 = 2$; $a_2 = 2$; and $b_1 = \ldots = b_{a+1} = n - 2$. (Recall that $a = a_i \geq 1$.) Since the terms of α and β are non-increasing,

$$(a + 1)(n - 2) + (m - a - 1) \cdot 1 \leq \sum \beta = \sum \alpha \leq (n - 1) \cdot 2 + 1 \cdot a. \tag{4.1}$$

Rearranging this inequality yields $(n-4)(a-1) \leq 5-m$. Since $m \geq 4$ and $n \geq 5$ and $a \geq 1$, both sides of this latter inequality are non-negative; hence $m \in \{4,5\}$, and we see that that $a = 1$ unless $a = 2$ and $n = 5$ and $m = 4$. In the latter case, the resulting equality forces $\{\alpha, \beta\} = \{(2,2,2,2,2),(3,3,3,1)\}$, which satisfies (4a) of Theorem 4.23, contradicting the assumption that $\{\alpha, \beta\}$ was forbidden. If instead $a = 1$, then we may write $\alpha = (2, 2, a_3, \ldots, a_{n-1}, 1)$ and $\beta = (n-2, n-2, b_3, \ldots, b_m)$. Consider the reduction R_{a_2}; we claim that $\{\alpha', \beta'\}$ is a target sequence pair with smaller length. Indeed, this pair is indecomposable by Corollary 4.37. To see that it is forbidden as well, we consult Theorem 4.23 as applied to $\alpha' = (2, a_3, \ldots, a_{n-1}, 1)$ and $\beta' = (n-3, n-3, b_3, \ldots, b_m)$, though the terms of β' may be reordered so as to be non-increasing. Observe that $|\alpha'| = n-1$, and conditions (1b), (2a), (3b), and (4a) clearly cannot hold due to our lower bounds on m and n. If (1a) were to hold for $\{\alpha', \beta'\}$, then $\alpha = (2, 2, 1, \ldots, 1)$ and the inequality (4.1) yields $2(n-2)+1(m-2) \leq n+2$ and hence $m+n \leq 8$, a contradiction. The condition (2b) cannot hold for $\{\alpha', \beta'\}$, as β' has at least two terms equal to $n-3$. If (3a) were to hold for $\{\alpha', \beta'\}$, then $b_3 = n-2$, and the inequality (4.1) yields $3(n-2)+1(m-3) \leq 2(n-1)+1$, again leading to the contradiction $m+n \leq 8$. Finally, if $\{\alpha', \beta'\}$ satisfied (4b) from Theorem 4.23, then $b_m \geq n-3$, so the inequality (4.1) implies that $(n-2)2+(n-3)(m-2) \leq 2(n-1)+1$, leading to $(n-3)(m-2) \leq 3$; this is a contradiction as $n \geq 5$ and $m \geq 4$. Hence $\{\alpha', \beta'\}$ is a target sequence pair, and $\{\alpha, \beta\}$ is not a Rao-minimal sequence pair.

Suppose instead that $\{\alpha, \beta\}$ satisfies condition (IV.a.1). The complement of a split graph realization of $\{\alpha, \beta\}$ has degree sequence pair $\{\overline{\alpha}, \overline{\beta}\}$, where $\overline{\alpha} = (m-a_n, \ldots, m-a_1)$ and $\overline{\beta} = (n-b_m, \ldots, n-b_1)$. Since $\{\alpha, \beta\}$ satisfies (IV.a.1), it is straightforward to verify that $\{\overline{\alpha}, \overline{\beta}\}$ satisfies (III.a.1). By the previous argument and Remark 4.28, we see that $\{\alpha, \beta\}$ is not Rao-minimal.

Suppose next that $\{\alpha, \beta\}$ satisfies (III.a.2). Here $a = a_1 \geq 3$, $a_2 = 2$, and $b_1 = n-1$. We will see that for any such $\{\alpha, \beta\}$, the reduction $R_{a_{n-1},\alpha}$ yields a target pair $\{\alpha', \beta'\}$. First, prior assumptions ensure that $\{\alpha', \beta'\}_{a_{n-1},\alpha}$ has no terms in β' equal to 0, nor terms in α' equal to 0 or $|\beta'|$. If $\{\alpha', \beta'\}$ contains a term in β' equal to $|\alpha'| = n-1$, then $b_{k+1} = n-1$, where $k = a_{n-1}$. We find $\sum \beta \geq (k+1)(n-1)+(m-k-1)\cdot 1$ and $\sum \alpha \leq 1\cdot(m-1)+(n-3)\cdot 2+2\cdot k$, and $\sum \beta = \sum \alpha$ leads to $(n-4)(k-1) \leq -1$, a contradiction. It follows from Corollary 4.37 that $\{\alpha', \beta'\}$ is indecomposable. Contrary to our claim that $\{\alpha', \beta'\}$ is a target pair, suppose that one of the other conditions from Theorem 4.23 holds for $\{\alpha', \beta'\}$. Our bounds $4 \leq m \leq n$ and the present case's starting conditions on a_1, a_2, and b_1 rule out (1a), (2a), (3a), and (3b). If (4a) or (4b) were to occur for $\{\alpha', \beta'\}$, then $\{\alpha, \beta\}$ would also satisfy (4a) or (4b), respectively, a contradiction. If $\{\alpha', \beta'\}$ satisfies (1b), then we must have $a_{n-1} = 2$ and $b_2 = 2$. Thus $\beta = (n-1, 2, 1, \ldots, 1)$ and $\alpha = (a_1, 2, \ldots, 2, a_n)$, which yields a contradiction to $\sum \alpha = \sum \beta$. If $\{\alpha', \beta'\}$ satisfies (2b), then $\beta' = (n-2, \ldots, n-2, b'_m)$ for some b'_m, so $\sum \beta \geq a_{n-1}(n-1)+(m-1-a_{n-1})(n-2)+1\cdot 1$. However, $\sum \alpha \leq 1\cdot(m-1)+(n-3)\cdot 2+2\cdot a_{n-1}$, and the relation $\sum \beta = \sum \alpha$ yields $(m-3)(n-3) \leq a_{n-1}-1$, a contradiction. Hence $\{\alpha', \beta'\}$ is a target sequence, as claimed, and $\{\alpha, \beta\}$ is not Rao-minimal.

If $\{\alpha, \beta\}$ satisfies condition (IV.a.2), then with an argument similar to that of condition (IV.a.1) above, we can apply condition (III.a.2) to the complementary pair $\{\overline{\alpha}, \overline{\beta}\}$ to conclude that $\{\alpha, \beta\}$ is not Rao-minimal.

At this point we comment on our process. As we have considered the conditions (III.a.1) and (III.a.2) above and deduced results about conditions (IV.a.1) and (IV.a.2), we have proceeded by taking an arbitrary target sequence pair $\{\alpha, \beta\}$ satisfying the given condition and examining a reduced pair $\{\alpha', \beta'\}$. After showing that that $\{\alpha', \beta'\}$ must be indecomposable, we show that $\{\alpha', \beta'\}$ is also forbidden (and hence a target sequence pair) by checking each of the conditions (1a) through (4b) from Theorem 4.23. As we continue the proof in the coming paragraphs, moving through other conditions from the list (I.a) through (VI), from this point on it will not be necessary to check conditions (3a) or (4a) from Theorem 4.23 while verifying that $\{\alpha', \beta'\}$ is forbidden, as long as $\{\alpha', \beta'\} = \{\alpha', \beta'\}_{a,\alpha}$ for some term $a \in \alpha$. This is because of the respective equivalences already mentioned between (3a) and (4a), applied to $\{\alpha', \beta'\}$, and the disjunctions "(III.a.1) or (III.a.2)" and "(IV.a.1) or (IV.a.2)" applicable to $\{\alpha, \beta\}$, as well as the arguments thus far in the proof.

Resuming our progress, suppose now that $\{\alpha, \beta\}$ satisfies (I.b.1) for some term a of α, so $b_2 = 2$ and $b_{a+1} = 1$. Here $a > 1$ and $\sum \beta \leq 1 \cdot (n-1) + (a-1) \cdot 2 + (m-a) \cdot 1 = n+m+a-3$. If $a_n \geq 2$, then $\sum \alpha \geq 1 \cdot a + (n-1) \cdot 2$ and we have $2n + a - 2 \leq n + m + a - 3$, which is a contradiction since $m \leq n$. Hence $a_n = 1$. We now show that the reduction $R_{a_n,\alpha}$ yields a target sequence. First, it is clear that $\{\alpha', \beta'\}_{a_n,\alpha}$ has no terms in α' equal to 0 or $|\beta'|$ (since α has none) and no terms in β' equal to 0 or $|\alpha'|$, since $b_1 \geq 2$ and $b_2 \leq n-2$. By Corollary 4.37, $\{\alpha', \beta'\}$ is indecomposable. Consulting conditions in Theorem 4.23 as they might apply to $\{\alpha', \beta'\}$, we easily rule out (1a) as it implies that $\{\alpha, \beta\}$ satisfies (1a) as well; (2b) and (4b) likewise cannot hold given our lower bound on n. Condition (1b) implies that $b_1 = 2$ and $b_3 = 1$, so $\sum \beta = m+2$. In this case, since (1a) does not hold, $a_2 \geq 2$, so $m+2 = \sum \alpha \geq n+2$. Since $n \geq m$ we conclude that $\{\alpha, \beta\} = \{(2, 2, 1, \ldots, 1), (2, 2, 1, \ldots, 1)\}$ with $m = n \geq 5$; this pair Rao-contains the pair $\{(2, 2, 1, 1), (2, 2, 1, 1)\}$ from $(*)$, contradicting our assumption that $\{\alpha, \beta\}$ is Rao-minimal. Thus (1b) cannot hold. If instead $\{\alpha', \beta'\}$ satisfies (2a), then $a_{n-2} = m - 1$, so $\sum \alpha \geq (n-2)(m-1) + 2 \cdot 1$, and $\sum \beta \leq 1 \cdot (n-1) + (m-2) \cdot 2 + 1 \cdot 1$. Combining these inequalities via $\sum \alpha = \sum \beta$ yields $(n-4)(m-2) \leq 0$, a contradiction, so (2a) does not hold for $\{\alpha, \beta\}$. Lastly, if $\{\alpha', \beta'\}$ satisfies (3b), then $b_1 \leq 3$; we proceed through the possibilities for b_1. If $b_1 \leq 2$, then since neither (1a) nor (1b) holds, we have $a_2 \geq 2$ and $b_1 = b_2 = b_3 = 2$, so $a \geq 3$ and the equality $\sum \alpha = \sum \beta$ implies that $n = m$, that $\alpha = (a, 2, 1, \ldots, 1)$, and that β has a terms equal to 2 and $n - a$ terms equal to 1. Here $\{\alpha, \beta\}$ Rao-contains $\{(2, 2, 1, 1), (2, 2, 1, 1)\}$, a contradiction. Hence we take $b_1 = 3$; here $1 \cdot (m-1) + 1 \cdot 2 + (n-2) \cdot 1 \leq \sum \alpha = \sum \beta \leq 1 \cdot 3 + (m-2) \cdot 2 + 1 \cdot 1$, yielding $n \leq m + 1$. If $n = m + 1$, then the equality forces $\alpha = (m-1, 2, 1, \ldots, 1)$ and $\beta = (3, 2, \ldots, 2, 1)$, while if $n = m$, then it is possible to have β as above and $\alpha = (m-1, 3, 1, \ldots, 1)$ or $\alpha = (m-1, 2, 2, 1, \ldots, 1)$, or to have $\beta = (3, 2, \ldots, 2, 1, 1)$ and $\alpha = (m-1, 2, 1, \ldots, 1)$. In these four cases, $\{\alpha, \beta\}$ respectively Rao-contains $\{(2, 2, 1, 1, 1), (3, 2, 2)\}$,

$\{(3,3,1,1),(3,2,2,1)\}$, $\{(3,2,2,1),(3,2,2,1)\}$, and $\{(3,2,1,1),(3,2,1,1)\}$, which appear in $(*)$. (To see this, consider the cross-degree sequence pairs formed when the high-degree term and $m-4$ terms equal to 1 in α are successively decremented by $m-4$ terms equal to 2 from β.) Hence no target cross-degree pairs satisfying (I.b.1) are Rao-minimal.

If $\{\alpha,\beta\}$ satisfies condition (II.b.1), then we can apply condition (I.b.1) to the complementary pair $\{\overline{\alpha},\overline{\beta}\}$ to see that $\{\alpha,\beta\}$ is not Rao-minimal.

Suppose next that $\{\alpha,\beta\}$ satisfies (I.b.2) for some term a in α, so β has $a+1$ terms equal to 2 and $m-a-1$ terms equal to 1. Since $\{\alpha,\beta\}$ is forbidden, it cannot satisfy condition (3b) from Theorem 4.23, so $a_1 \leq m-2$. Hence $a \leq m-2$ as well, and we compute $\sum\beta = m+a+1 \leq 2m-1$. Let $\{\alpha',\beta'\}$ be the sequence pair resulting from the reduction $R_{b_m,\beta}$; we will show that $\{\alpha',\beta'\}$ is a target sequence pair. Notice that $\{\alpha',\beta'\}$ has no terms in β' equal to 0 or $|\alpha'|$ (since β does not), and it has no term in α' equal to 0 or $|\beta'|$, since $2 \leq a_2 \leq a_1 \leq m-2$. By Corollary 4.37, $\{\alpha',\beta'\}$ is indecomposable. We now check $\{\alpha',\beta'\}$ against the conditions in Theorem 4.23. Note that we must check (3a) and (4a) here since the Kleitman-Wang reduction is indexed in β and not α. Clearly (1b), (2b), (3a), and (4b) cannot hold because of β's structure and our bounds on m and n. If $\{\alpha',\beta'\}$ satisfies (1a), then since $a_2 \geq 2$ we have $a_1 = a_2 = 2$ and $a_3 = 1$. This yields $\alpha = (2,2,1,\ldots,1)$ and either $\beta = (2,2,2,1,\ldots,1)$ or $\beta = (2,2,1,\ldots,1)$; the length of β is determined by the relation $\sum\alpha = \sum\beta$. In either case, it is straightforward to see that the sequence $\{(2,2,1,1),(2,2,1,1)\}$ from $(*)$ is Rao-contained by $\{\alpha,\beta\}$. If $\{\alpha',\beta'\}$ instead satisfies (2a), then $a_{n-1} \geq a'_{n-1} = m-2$, so we find $\sum\alpha \geq (n-1)(m-2)+1\cdot 1 > 2m-1 \geq \sum\beta$, a contradiction. If $\{\alpha',\beta'\}$ satisfies (3b), then $a_1 = a_2 = m-2$, so $2m-1 \geq \sum\beta = \sum\alpha \geq 2(m-2)+(n-2)\cdot 1 = 2m+n-6$. Hence $n=5$, and equality, together with the condition that $4 \leq m \leq n$, implies that $\{\alpha,\beta\}$ is either $\{(2,2,1,1,1),(2,2,2,1)\}$ or $\{(3,3,1,1,1),(2,2,2,2,1)\}$, both of which Rao-contain $\{(2,2,1,1),(2,2,1,1)\}$ from $(*)$. Finally, if $\{\alpha',\beta'\}$ satisfies (4a), then $a_n \geq m-3$ and $b_{m-1} = 1$, so $n(m-3) \leq \sum\alpha = \sum\beta \leq 2m-2$; we obtain $(m-3)(n-2) \leq 4$, which holds only when holds when $m=4$ and $n \leq 6$. However, the bounds on $\sum\alpha$ and $\sum\beta$ also force $\sum\alpha = 6$, which is impossible as $a_2 \geq 2$.

If $\{\alpha,\beta\}$ satisfies condition (II.b.2), then the complementary pair $\{\overline{\alpha},\overline{\beta}\}$ satisfies (I.b.2), showing that $\{\alpha,\beta\}$ is not Rao-minimal. Note also that the disjunctions "(I.b.1) or (I.b.2)" and "(II.b.1) or (II.b.2)," applicable to $\{\alpha,\beta\}$, have been seen to be equivalent to Theorem 4.23's conditions (1b) and (2b) on a reduced pair $\{\alpha',\beta'\}_{a,\alpha}$. We may then assume henceforth that whenever we check a reduced pair $\{\alpha',\beta'\}_{a,\alpha}$ to see if its realization is in HCU (as it should be for $\{\alpha,\beta\}$ to be Rao-minimal), it suffices to check conditions other than (1b) or (2b) (or (3a) or (4a), as established earlier).

Suppose now that $\{\alpha,\beta\}$ satisfies (I.a), so $a_3 = \ldots = a_n = 1$. We will handle this this possibility with two choices for $\{\alpha',\beta'\}$, depending on the size of b_1. Suppose first that $b_1 \geq 4$ or that $b_1 = b_2 = 3$. In this case let $\{\alpha',\beta'\} = \{\alpha',\beta'\}_{a_n,\alpha}$. Clearly no term of α' equals 0 or $|\beta'|$, and no term of β' equals 0; more care is needed to show that no term of β' equals $|\alpha'| = n-1$. If instead this were to happen, then $b_1 = b_2 = n-1$, and $2(n-1)+(m-2)\cdot 1 \leq \sum \beta = \sum \alpha \leq 2(m-1)+(n-2)\cdot 1$, which implies that $n \leq m$ and hence $m = n \geq 5$. Equality in the inequalities above requires that $\alpha = \beta = (n-1, n-1, 1, \ldots, 1)$, and $\{\alpha, \beta\}$ now fails the condition of Lemma 4.35 at $k = 2$, a contradiction. Thus β' contains no terms equal to $|\alpha'|$, and $b_2 \leq n-2$. Turning now to Theorem 4.23 as it applies to $\{\alpha', \beta'\}$, we verify that $\{\alpha', \beta'\}$ is forbidden by checking conditions (1a), (2a), (3b), and (4b). Indeed, (1a) cannot hold, since it does not hold for $\{\alpha, \beta\}$, and (2a) fails since $m \geq 4$. Since $b_1 \geq 4$ or $b_1 = b_2 \geq 3$, we have $b'_1 > 2$ and hence (3b) cannot hold. If $\{\alpha', \beta'\}$ satisfies (4b), then $b_m \geq n-3$ and $b_1 \geq n-2$, so $1\cdot(n-2)+(m-1)(n-3) \leq \sum \beta = \sum \alpha \leq 2(m-1)+(n-2)\cdot 1$, which yields $(m-1)(n-5) \leq 0$. Since $m \geq 4$ and $n \geq 5$, we then have $n = 5$ and equality in the degree sum inequalities forces $\alpha = (m-1, m-1, 1, 1, 1)$ and $\beta = (3, 2, \ldots, 2)$. This contradicts our hypothesis that $b_1 \geq 4$ or $b_1 = b_2 = 3$; we conclude that $\{\alpha', \beta'\}$ is forbidden, so $\{\alpha, \beta\}$ is not Rao-minimal.

Continuing with case (I.a), now suppose that $b_1 \leq 3$ and $b_2 \leq 2$, and let $\{\alpha', \beta'\} = \{\alpha', \beta'\}_{b_m, \beta}$. No term of β' equals 0 or $|\alpha'|$, and since $a_2 \geq 2$, no term of α' equals 0. To see that no term of α' is equal to $|\beta'|$, note that otherwise would require that $a_1 = a_2 = m-1$ and $b_m = 1$, yielding $2(m-1) + (n-2)\cdot 1 = \sum \alpha = \sum \beta \leq 1\cdot 3 + (m-2)\cdot 2 + 1\cdot 1$, a contradiction. Considering the conditions from Theorem 4.23 as applied to $\{\alpha', \beta'\}$, we see that (1b) cannot hold for $\{\alpha', \beta'\}$ since it does not hold for $\{\alpha, \beta\}$, and the sizes of m and n quickly preclude (2a), (2b), (3a), (4a), or (4b) from happening. It remains to check the cases when $\{\alpha', \beta'\}$ satisfies (1a) or (3b). If (1a) holds for $\{\alpha', \beta'\}$, then either $b_m = 1$ and $a_1 = a_2 = 2$, or $b_m = 2$ and $a_2 = 2$. In the former case, where $\alpha = (2, 2, 1, \ldots, 1)$, one can verify that $\{\alpha, \beta\}$ Rao-contains $\{(2, 2, 1, 1), (2, 2, 1, 1)\}$. In the latter case we have $\alpha = (a_1, 2, 1, \ldots, 1)$ and $\beta = (b_1, 2, \ldots, 2)$. Since $\{\alpha, \beta\}$ does not satisfy condition (3b) from Theorem 4.23, if $b_1 = 2$ then $a_1 \leq m-2$, and $\{\alpha, \beta\} = \{(a_1, 2, 1, ..., 1), (2, ..., 2)\}$, which again Rao-contains $\{(2, 2, 1, 1), (2, 2, 1, 1)\}$; this can be seen by induction on m, where the induction step involves taking a realization where a vertex with cross-degree $b_m = 2$ is adjacent to vertices with cross-degrees $a_1, 1$ (if $a_1 > 2$) or $1, 1$. Assume now that $b_1 = 3$, so $\alpha = (a_1, 2, 1, \ldots, 1)$ and $\beta = (3, 2, \ldots, 2)$; this cross-degree sequence pair Rao-contains $\{(2, 2, 1, 1, 1), (3, 2, 2)\}$, since $\sum \alpha = \sum \beta$ forces $m + 2 \leq n \leq 2m - 1$ and we may use $m-3$ terms of 2 from β to iteratively decrement from α both a term equal to 1 and the term of highest degree. This shows that $\{\alpha, \beta\}$ is not Rao-minimal in this case. If $\{\alpha', \beta'\}$ instead satisfies (3b), then $2 \geq b'_1 = b_1 \geq 2$; since $\{\alpha, \beta\}$ itself does not satisfy (3b), we must have $a_1 < m-1$, but since $a'_1 = m-2$, we find that $a_1 = a_2 = m-2$ and $b_m = 1$. Then $2m + n - 6 = \sum \alpha = \sum \beta \leq (m-1)\cdot 2 + 1\cdot 1$, and we obtain $n = 5$. The pair $\{\alpha, \beta\}$ is then $\{(3, 3, 1, 1, 1), (2, 2, 2, 2, 1)\}$, which Rao-contains $\{(2, 2, 1, 1), (2, 2, 1, 1)\}$, so $\{\alpha, \beta\}$ is not Rao-minimal.

Proceeding as in previous cases, we note that if $\{\alpha, \beta\}$ satisfies condition (II.a), then $\{\overline{\alpha}, \overline{\beta})$ satisfies the case (I.a) just considered and hence $\{\alpha, \beta\}$ is not Rao-minimal. Furthermore, the respective conditions (I.a) and (II.a), applicable to $\{\alpha, \beta\}$, are equivalent to the conditions (1a) and (2a) from Theorem 4.23 on a reduced pair $\{\alpha', \beta'\}_{a,\alpha}$, so we henceforth need not check conditions (1a) or (2a) (or the previously considered conditions (1b), (2b), (3a), (4a)) when considering whether a reduced pair $\{\alpha', \beta'\}_{a,\alpha}$ belongs to HCU.

Suppose that $\{\alpha, \beta\}$ satisfies (III.b) for some term $a = a_i$ of α, so $b_1 = 3$, $b_{a+1} \leq 2$, and $a_1 = m - 1$, while $i \geq 2$ or $a_2 = m - 1$. We may assume that $a_3 \geq 2$ since we have considered the case (I.a) above. We consider first the subcase in which $b_1 = \cdots = b_{a_n+1} = 3$. Let $\{\alpha', \beta'\} = \{\alpha', \beta'\}_{a_n,\alpha}$. Clearly no term of α' is equal to 0 or $|\beta'|$, and the values $b_1 = b_{a_n} = 3$ ensure that no term of β' is equal to $|\alpha|$ or 0. By Corollary 4.37, $\{\alpha', \beta'\}$ is indecomposable. Checking $\{\alpha', \beta'\}$ for the conditions of Theorem 4.23 not previously handled, we see that $b'_1 > 2$, so (3b) does not hold. If $\{\alpha', \beta'\}$ satisfies (4b), then $a_{n-1} = 1$ and $b_m \geq n - 3$; since (III.b) requires the last term of β to be at most 2, this forces $n = 5$ and $b_m = 2$. Likewise, $a_n = 1$, so $b_2 = b_{a_n+1} = 3$. Listing the possibilities shows that there are six cross-degree sequence pairs $\{\alpha, \beta\}$ with $\alpha = (m - 1, a_2, a_3, 1, 1)$ and $\beta = (3, 3, b_3, b_4)$ or $\beta = (3, 3, b_3, b_4, b_5)$, where $a_2 \geq 2$ and the last term of β is no more than 2, and $\sum \alpha = \sum \beta$. These pairs are $\{(3, 3, 2, 1, 1), (3, 3, 2, 2)\}$, $\{(3, 3, 3, 1, 1), (3, 3, 3, 2)\}$, $\{(4, 3, 3, 1, 1), (3, 3, 2, 2, 2)\}$, $\{(4, 4, 2, 1, 1), (3, 3, 2, 2, 2)\}$, $\{(4, 4, 3, 1, 1), (3, 3, 3, 2, 2)\}$, and $\{(4, 4, 4, 1, 1), (3, 3, 3, 3, 2)\}$. Observe that each of these Rao-contains $\{(3, 2, 1, 1)(3, 2, 1, 1)\}$ or $\{(3, 3, 1, 1)(3, 2, 2, 1)\}$, so in this subcase $\{\alpha, \beta\}$ is not Rao-minimal.

Still assuming that $\{\alpha, \beta\}$ satisfies (III.b), consider the subcase in which $b_{a_n+1} \leq 2$. Here $\sum \beta < 3m \leq 3n$, so $a_n \leq 2$ and hence $b_3 \leq 2$. Let $\{\alpha', \beta'\} = \{\alpha', \beta'\}_{b,\beta}$, where $b = b_m$. Clearly β' contains no terms equal to 0 or $|\alpha'|$, and since $b_m \leq 2$ and $a_2 \geq 2$, clearly α' contains no 0 term. If $a'_1 = m - 1 = |\beta'|$, then $a_1 = a_2 = m - 1$; we continue to assume $a_3 \geq 2$. This leads to $\sum \alpha \geq 2(m-1)+1\cdot 2+(n-3)\cdot 1 = 2m+n-3$ and $\sum \beta \leq 2\cdot 3+(m-2)\cdot 2 = 2m+2$; we conclude that $n = 5$ and $\{\alpha, \beta\} = \{(m-1, m-1, 2, 1, 1), (3, 3, 2, \ldots, 2)\}$, but then $a'_1 = m-2$. Hence we may assume by Corollary 4.37 that $\{\alpha', \beta'\}$ is indecomposable and check $\{\alpha', \beta'\}$ against the conditions in Theorem 4.23. Conditions (1a) and (3b) fail to hold in light of our known values of terms in $\{\alpha, \beta\}$, and because $n \geq 5$, conditions (2b), (3a) and (4b) similarly fail. Since $\{\alpha, \beta\}$ does not satisfy (1b), neither can $\{\alpha', \beta'\}$. If $\{\alpha', \beta'\}$ satisfies (4a), then knowing that $b_m \leq b_{m-1} = 1$ and $m - 3 \leq a_n \leq 2$ forces $m \leq 5$. Since $a_1 = m - 1$ and $a_3 \geq 2$, while $b_1 = 3$ and $b_3 \leq 2$ and $b_{m-1} \leq 1$, we find

$$m + n = 1 \cdot (m - 1) + 2 \cdot 2 + (n - 3) \cdot 1 \leq \sum \alpha = \sum \beta \leq 2 \cdot 3 + (m - 4) \cdot 2 + 2 \cdot 1 = 2m.$$

Since $n \geq m$ we see that $m = n = 5$ and equality must hold in each of the inequalities, forcing $a_n = 1$ and $b_2 = 3$, which contradicts the assumption that $b_{a_n+1} \leq 2$. Hence $\{\alpha', \beta'\}$ cannot satisfy (4a). If instead $\{\alpha', \beta'\}$ satisfies (2a), then $a_{n-1} \geq m - 2$; then $2m + 2 \geq \sum \beta = \sum \alpha \geq 1 \cdot (m - 1) + (n - 2)(m - 2) + 1 \cdot 1$. This would imply that $(m - 2)(m - 3) \leq 4$, which forces $m = 2$ and $n = 3$ and $\{\alpha, \beta\} = \{(3, 2, 2, 2, 1), (3, 3, 2, 2)\}$, contradicting $b_{a_n+1} \leq 2$.

If $\{\alpha, \beta\}$ satisfies condition (IV.b), then the complementary pair $\{\overline{\alpha}, \overline{\beta}\}$ satisfies (III.b), showing again that $\{\alpha, \beta\}$ is not Rao-minimal.

If none of conditions (I.a) through (IV.b) hold for any reduction $R_{a_i,\alpha}$, then for each term a of α, the reduction $R_{a,\alpha}$ must satisfy condition (V) or (VI). However, this leads to a contradiction. First, if the reduced pairs $\{\alpha', \beta'\}_{a_i,\alpha}$ satisfy (V) for all terms a_i in α, then $b_{a_n} = \cdots = b_m = 1$; since $b_2 \geq 2$, we have $a_n \geq 3$. Hence $\sum_{i=1}^{a_n-1} b_i = \sum \beta - (m - a_n + 1)$ and $\sum_{i=1}^{n} \min\{a_i, a_n - 1\} = n(a_n - 1)$. By Lemma 4.35, the $(a_n - 1)$th Gale–Ryser inequality implies that $\sum \beta \leq m - a_n + 1 + na_n - n \leq na_n - a_n + 1$. Since $\sum \beta = \sum \alpha \geq na_n$ and $a_n \geq 3$, this is a contradiction.

Suppose instead that condition (V) fails for some term a_i in α. Let t be the least subscript on a term a_t of α such that $b_{a_t} > 1$. By previous assumptions, condition (VI) then holds for $\{\alpha', \beta'\}_{a_t+1,\alpha}$. Thus $b_{a_t+1} = n-1$, $b_{a_{t-1}} = 1$, and $a_{t-1} \geq a_t + 2$. By minimality of t, $a_{t-1} > a_t$. Via the $(a_t + 1)$th Gale–Ryser inequality, Lemma 4.35 implies that

$$\sum_{i=1}^{a_t+1} b_i = (n-1)(a_t+1) \leq \sum_{i=1}^{n} \min\{a_t+1, a_i\} = (t-1)(a_t+1) + \sum_{i=t}^{n} a_i.$$

We manipulate this inequality to show that $\sum_{i=t}^{n} a_i \geq (n-t)(a_t+1)$. Moreover, the monotonicity of α implies $(n-t+1)a_t \geq \sum_{i=t}^{n} a_i$. It follows that $a_t \geq n - t$, that is, $n \leq t + a_t$. However, the $(a_{t-1} - 1)$th Gale–Ryser inequality implies

$$\begin{aligned} \sum \beta - (m - a_{t-1} + 1) &= \sum_{i=1}^{a_{t-1}-1} b_i \leq \sum_{i=1}^{n} \min\{a_{t-1} - 1, a_i\} \\ &= (t-1)(a_{t-1} - 1) + \sum_{i=t}^{n} a_i = \sum \alpha - \sum_{i=1}^{t-1} (a_i - a_{t-1} + 1). \end{aligned}$$

This is equivalent to $\sum_{i=1}^{t-1} (a_i - a_{t-1}) \leq m - a_{t-1} - t + 2$. Since the left side is non-negative, we have $a_{t-1} + t \leq m + 2$, or equivalently, $a_t + t \leq m \leq n$.

We have shown the (a_t+1)th Gale–Ryser inequality implies $n \leq t + a_t$ and the $(a_{t-1} - 1)$th Gale–Ryser inequality implies $n \geq t + a_t$. Hence $n = t + a_t$, both Gale–Ryser inequalities hold with equality, and so do all other bounds used in their computation. Specifically, $a_{t-1} = a_t + 2$, so $n - 1 = b_{a_t+1} > b_{a_t+2} = b_{a_{t-1}} = 1$. Thus Lemma 4.22, applied at index $a_t + 1$ of β, implies that $\{\alpha, \beta\}$ is decomposable, for a contradiction. This completes the proof. $\square$

4.8 A forbidden induced subgraph characterization of *HCU*

We use Barrus and Hartke's [5] theory of degree-sequence-forcing sets to establish a forbidden induced subgraph characterization of *HCU*. We first define a class of graphs to be *charac-*

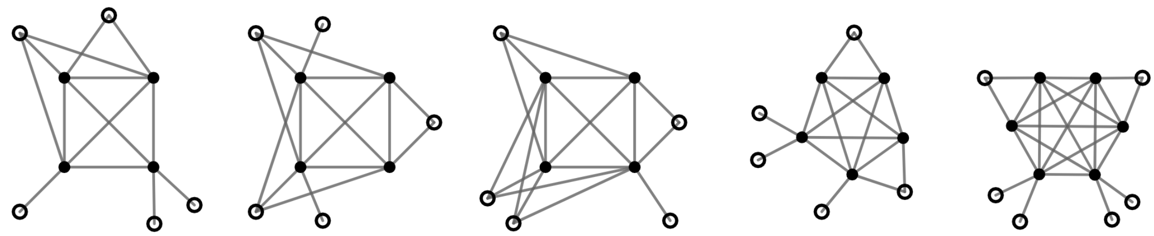

Figure 4.7: The five split graphs which, together with their complements, are the subset of $\mathcal{F}_{HCU}$ whose degree sequences are not Rao-minimal.

terized by its degree sequences if for every degree sequence d, if one realization belongs to the class, then every realization belongs to the class. A set $\mathcal{F}$ of graphs is *degree-sequence-forcing* (*DSF*) if the set of $\mathcal{F}$-free graphs (in the induced subgraph context) is characterized by its degree sequences. In other words, a graph G is $\mathcal{F}$-free if and only if every other realization of the degree sequence of G is $\mathcal{F}$-free. If a set $\mathcal{F}$ is not DSF, then there exists a degree sequence d with realizations G and G' such that G is $\mathcal{F}$-free and G' is not. As in [5], we call such a set an *$\mathcal{F}$-breaking pair*. When $\mathcal{F}$ is finite, it is possible to identify whether it is DSF without needing to check all degree sequences. We present this result in the following lemma.

Lemma 4.38 ([5], Lemma 2.1). *Let $\mathcal{F}$ be a set of graphs, each with at most n vertices. If the subset of $\mathcal{F}$-free graphs of order up to $n+2$ is characterized by its degree sequences, then the whole set of $\mathcal{F}$-free graphs is characterized by its degree sequences, that is, $\mathcal{F}$ is DSF.*

The proof of Lemma 4.38 relies on a graph operation called a *2-switch*: given a 4-vertex subset of a graph with edges ab, cd and non-edges ad, bc, a 2-switch deletes ab, cd and adds ad, bc. Critically, given two graphs G, G', there exists a finite sequence of 2-switches transforming G to G' if and only if G and G' have the same degree sequence [30].

Since HCU is hereditary by definition, there exists a set $\mathcal{F}_{HCU}$ of minimal forbidden induced subgraphs, such that $G \in HCU$ if and only if G is $\mathcal{F}_{HCU}$-free. Since Theorems 4.21 and 4.23 give a degree sequence characterization of HCU, it follows that $\mathcal{F}_{HCU}$ is a DSF set. By Remark 4.28, every realization of a Rao-minimal degree sequence is in $\mathcal{F}_{HCU}$. Let $\mathcal{Y}$ be the set of graphs shown in Figure 4.7 together with their complements and the realizations of each Rao-minimal sequence identified in Theorems 4.32 and 4.34. Using Lemma 4.38, we give a forbidden induced subgraph characterization of HCU.

Theorem 4.39. *The minimal forbidden induced subgraphs of HCU are the graphs shown in Figure 4.7, their complements, and the graphs realizing the Rao-minimal forbidden degree* *sequences identified in Theorems 4.32 and 4.34. That is, $\mathcal{F}_{HCU} = \mathcal{Y}$.*

Proof. First, we show $\mathcal{Y}$ is contained in $\mathcal{F}_{HCU}$. Using Lemmas 4.20 and 4.22 and Theorems 4.21 and 4.23, it is straightforward to verify from the degree sequences that each graph in $\mathcal{Y}$ is indecomposable and not in HCU. We can also see that any proper induced subgraph of these graphs is in HCU, implying that $\mathcal{Y} \subseteq \mathcal{F}_{HCU}$.

We now show $\mathcal{F}_{HCU}$ is contained in $\mathcal{Y}$. Suppose for a contradiction that there exists some graph G in $\mathcal{F}_{HCU}$ but not in $\mathcal{Y}$. This implies the existence of a $\mathcal{Y}$-breaking pair. Let d be the degree sequence of G, and since G is not in HCU, there exists a Rao-minimal forbidden sequence r that is Rao-contained by d. Thus there exists some realization H of r induced in a realization G' of d. The class $\mathcal{Y}$ already includes every realization of the Rao-minimal sequences of HCU, so $H \in \mathcal{Y}$. By assumption, G is $\mathcal{Y}$-free, but since H is induced in G', it follows that G' is not $\mathcal{Y}$-free. Therefore (G, G') is a $\mathcal{Y}$-breaking pair, so we know that $\mathcal{Y}$ is not DSF.

Hence if $\mathcal{Y}$ is DSF, then $\mathcal{Y} = \mathcal{F}_{HCU}$, so it remains to show that $\mathcal{Y}$ is DSF. By Lemma 4.38, it suffices to check degree sequences on 14 or fewer vertices. This is unwieldy, however – there are 29054155657235488 graphs on 14 vertices [55] – so we simplify the computational work in several ways.

First, we argue that for degree sequences of non-split graphs, it suffices to check graphs up to 9 vertices. Let $\mathcal{Y}_{\mathcal{N}}$ be the set of non-split graphs in $\mathcal{Y}$ and let $\mathcal{Y}_{\mathcal{S}}$ be the set of split graphs in $\mathcal{Y}$. Since the graphs in Figure 4.7 are split graphs, every graph in $\mathcal{Y}_{\mathcal{N}}$ realizes a Rao-minimal sequence on 7 or fewer terms. Let (G, G') be a $\mathcal{Y}$-breaking pair of non-split graphs with decompositions $G = G_n \circ \ldots \circ G_0$ and $G' = G'_n \circ \ldots \circ G'_0$, and recall from Theorem 4.18 that G_i and G'_i have the same degree sequence. Since all graphs in $\mathcal{Y}$ are indecomposable, there exists i, $0 \leq i \leq n$, such that (G_i, G'_i) is a $\mathcal{Y}$-breaking pair. We thus assume without loss of generality that (G, G') is an indecomposable pair of non-split graphs where G contains an induced element of $\mathcal{Y}$ and G' does not. Then the degree sequence π of G and G' is indecomposable, non-split, and not in HCU, so Corollary 4.33 implies there exists a realization G'' of π that contains an induced realization of a non-split Rao-minimal sequence. Therefore (G', G'') is a $\mathcal{Y}_{\mathcal{N}}$-breaking pair. By the proof of Lemma 4.38, there exists a $\mathcal{Y}_{\mathcal{N}}$-breaking pair on at most 9 vertices, which arises as we navigate via 2-switches from G' to G''.

We use Algorithm 2 shown at the end of the section with input $k = 0$ to verify that each non-split degree sequence on 9 or fewer vertices yields no $\mathcal{Y}_{\mathcal{N}}$-breaking pairs. If a $\mathcal{Y}_{\mathcal{N}}$-breaking pair were to exist, then it would contain as an induced subgraph an element of $\mathcal{F}_{HCU}$ but not $\mathcal{Y}_{\mathcal{N}}$. Hence it is equivalent to check that the all non-split graphs in $\mathcal{F}_{HCU}$ on at most 9 vertices are already elements of $\mathcal{Y}_{\mathcal{N}}$. We know that all indecomposable graphs in HCU on 5 vertices are Rao-minimal, so beginning with $n = 6$, we compute all forbidden degree sequences and check if their realizations include any graphs in $\mathcal{F}_{HCU}$. Every forbidden degree sequence on n vertices is either Rao-minimal or greater than some forbidden degree sequence on $n-1$ vertices in the Rao ordering. We efficiently compute the set of forbidden degree sequences on n vertices by taking the union of the non-split Rao-minimal sequences on n vertices with the set of sequences on n vertices that Rao-contain a forbidden sequence on $n-1$ vertices (lines 1 - 8). From this list, we retain only the non-split sequences (line 9). By Corollary 4.13, all elements of $\mathcal{F}_{HCU}$ are indecomposable, so we further test only the indecomposable sequences (line 10). Corollary 4.13 also implies that $\mathcal{F}_{HCU}$ is closed under complementation, so we keep only the first of these pairs of sequences (line 11). If a pair were $\mathcal{Y}_{\mathcal{N}}$-breaking, its complement would be as well, so this allows us to test close to half as many sequences as we would otherwise.

If a degree sequence π on n terms is the degree sequence of some $G \in \mathcal{F}_{HCU}$, then all induced subgraphs of G on $n-1$ vertices are contained in HCU. Let $v(\pi)$ be the number of different terms in π; thus G must have at least $v(\pi)$ non-isomorphic induced subgraphs on $n-1$ vertices in HCU (lines 12-13). Before computing the realizations of π, we check for this by computing the number of graphic decrementations of π in HCU . This is counted by comparison with the list of forbidden degree sequences on $n-1$ vertices, since the criteria in Theorems 4.21 and 4.23 apply only to indecomposable sequences. If there are fewer than $v(\pi)$ such sequences, then no realization of π is in $\mathcal{F}_{HCU}$ (lines 14-15).

If there are at least $v(\pi)$ such sequences, then we compute all realizations of π, using an algorithm developed by Zoltán Király [44] (line 16). For each realization G, we compute the degree sequence of each induced subgraph on $n-1$ vertices. If none of these are forbidden, then $G \in \mathcal{F}_{HCU}$. We add G and $\overline{G}$ to the set of minimal forbidden induced subgraphs on n vertices (lines 17-19).

Incrementing n, we repeat the entire above procedure until there are no minimal graphs for two consecutive values of n (lines 20-24). The code terminates, returning n and identifying all non-split graphs in $\mathcal{F}_{HCU}$ (line 25). We ran this algorithm in Mathematica in twenty minutes. Fortunately, it returned $n = 9$ and the minimal graphs were precisely the set $\mathcal{Y}_{\mathcal{N}}$.

We proceed to verify the same for $\mathcal{Y}_{\mathcal{S}}$. While we do need to check degree sequences of order up to 14, we know that any $\mathcal{Y}$-breaking pair of non-split graphs is a $\mathcal{Y}_{\mathcal{N}}$-breaking pair, as shown above. Therefore, we need only check split degree sequences of orders 8 to 14. There are only 67997750 split graphs on 14 vertices [59], so this is a more manageable task. Algorithm 2 with $k = 1$ has three small differences from the non-split version: first, we calculate only split forbidden degree sequences. If π is non-split, then all augmentations of π are non-split, so in order to calculate the split forbidden degree sequences on n vertices, we find the union of the set of augmentations of π over all split forbidden degree sequences π on $n-1$ vertices, then keep only the split sequences (lines 8-9). Second, by Lemma 4.12, a split graph G is in $\mathcal{F}_{HCU}$ if and only if $\overline{G}, G^I$, and $\overline{G^I}$ are as well. We group degree sequences with their inverses, complements, and inverse-complements, keeping one representative (lines 11-12). Third and finally, all induced subgraphs of split graphs are split graphs, so when counting the number of graphic decrementations of π in HCU, it suffices to compare to the list of forbidden split degree sequences (line 15). Since the algorithm is otherwise identical, it holds by the same proof. The program ran in two days and returned 14, confirming that there are no $\mathcal{Y}_{\mathcal{S}}$-breaking pairs of split graphs on 8 to 14 vertices. We therefore conclude that a graph G is in HCU if and only if it contains no element of $\mathcal{Y}$. □

Algorithm Minimal Forbidden Induced Subgraphs

Require: For $5 \leq n \leq 8$, the sets RAO_n of Rao-minimal forbidden degree sequences on n vertices, and $k \in 0, 1$.
Ensure: The minimal forbidden induced subgraphs of HCU.
1: **(if** $k = 0$**)** $n := 5$ **or (if** $k = 1$**)** $n := 8$
2: **(if** $k = 0$**)** $S_n := \{\pi \in RAO_n : \pi \text{ is non-split}\}$ **or (if** $k = 1$**)** $S_n := \{\pi \in RAO_n : \pi \text{ is split}\}$
3: $M_n := \{G : G \text{ realizes a degree sequences in } RAO_n\}$
4: $M_{n-1} := \emptyset$
5: **while** $M_n \neq \emptyset$ **or** $M_{n-1} \neq \emptyset$ **do**
6:     $n := n + 1$
7:     $M_n := \emptyset$
8:     $F_n := RAO_n \cup \{\pi : \exists \rho \in S_{n-1} \text{ such that } \pi \text{ is an augmentation of } \rho\}$
9:     **(if** $k = 0$**)** $S_n := \{\pi \in F_n : \pi \text{ is non-split}\}$ **or (if** $k = 1$**)** $S_n := \{\pi \in F_n : \pi \text{ is split}\}$
10:     $I_n := \{\pi \in S_n : \pi \text{ is indecomposable}\}$
11:     **choose** $E_n \subseteq I_n$ such that for all $\pi \in I_n$, exactly one of, π, its complement, and if $k = 1$ its inverse and inverse-complement are in E_n
12:     **for all** $\pi \in E_n$ **do**
13:         $V_\pi := \{x \in \mathbb{N} : x \in \pi\}$
14:         $D_\pi := \{\rho : \rho \text{ is a graphic decrementation of } \pi \text{ and } \rho \notin S_{n-1}\}$
15:         **if** $|D_\pi| \geq |V_\pi|$ **then**
16:             $H_\pi := \{G : G \text{ is a realization of } \pi\}$
17:             **for all** $G \in H_\pi$ **do**
18:                 **if** G contains no induced subgraphs with $n - 1$ vertices that have degree sequence in S_{n-1} **then**
19:                     $M_n := M_n \cup \{G, \overline{G}\}$
20:                 **end if**
21:             **end for**
22:         **end if**
23:     **end for**
24: **end while**
25: **return** n **and (if** $k = 0$**)** $M_5 \cup M_6 \cup \ldots \cup M_n$ **or (if** $k = 1$**)** $M_8 \cup M_9 \cup \ldots \cup M_n$

Chapter 5

Hereditary Intersections of Unigraphs and Subclasses of Perfect Graphs

5.1 Introduction

In this chapter we consider a series of hereditary families related to unigraphs. All graphs are finite and simple. Given a graph $G = (V(G), E(G))$, the degree sequence d of G is the list of the degrees of the graph's vertices, written in non-increasing order. Where terms are repeated, we often denote their multiplicity with an exponent; for instance, $(3, 2, 2, 2, 2, 1)$ and $(3, 2^4, 1)$ are the same degree sequence. We say that G *realizes* d or is a *realization* of d. Frequently, a degree sequence d has numerous realizations, such that knowing d is insufficient for determining much about a realization G. Where d has exactly one realization G up to isomorphism, we call d *unigraphic* and G a *unigraph*.

Unigraphs have been studied for about fifty years, with structural and degree sequence characterizations by Tyshkevich and Chernyak (see [68, 69, 70, 71] and the English translation [67]). The class of *split graphs* – graphs for which the vertex set $V(G)$ can be partitioned into a subset K inducing a clique and a subset S inducing a stable set – is an important tool in these characterizations of the class of unigraphs. Specifically, both characterizations rely on an operation defined by Tyshkevich that decomposes a graph into n split graph components and one unspecified component [65] [67]. The authors are then able to classify exactly the indecomposable components of a unigraph by structure and degree sequence. We summarize results pertaining to unigraphs here.

Theorem 5.1. *[67] A graph G is a unigraph if and only if each indecomposable component of its decomposition is a unigraph.*

Given a graph G, we denote its complement by $\overline{G}$. Given a split graph G, a *KS-partition* is a partition of $V(G)$ into subsets K inducing a clique and S inducing a stable set. We write (G, K, S) to indicate a split graph G with KS-partition $K \cup S$. Given (G, K, S), the *inverse* G^I is the split graph obtained from G by removing all edges with both endpoints in K and adding all edges with both endpoints in S (thus switching the clique and stable set of the graph). Note that the inverse depends on the choice of KS-partition.

Neither complementation nor inversion changes a graph's status as a unigraph. As a result, indecomposable unigraphs can be characterized as follows.

Theorem 5.2. *[67] A graph G is an indecomposable unigraph if and only if one of G or $\overline{G}$ (or G^I or $\overline{G^I}$, if split) is isomorphic to K_1 or C_5 or belongs to one of seven infinite families of graphs.*

Included in these seven families of unigraphs are all graphs isomorphic to the union of a star and nK_2, $n \geq 0$, and split graphs where all vertices in the stable set are of degree 1. We refer the reader to [67] for the complete list.

A class $\mathcal{A}$ of graphs is *hereditary* if it is closed under taking induced subgraphs of its members. All hereditary classes can be characterized by a set of forbidden induced subgraphs. For any class $\mathcal{A}$ of graphs, we define two closely related hereditary classes. First, let the *hereditary subclass of* $\mathcal{A}$ be the largest hereditary class contained within $\mathcal{A}$. For example, the perfect graphs are the hereditary subclass of the set of graphs with equal clique and chromatic numbers. The hereditary subclass of $\mathcal{A}$ is equivalently the subset of elements of $\mathcal{A}$ for which all induced subgraphs are also elements of $\mathcal{A}$. We call a graph G in the hereditary subclass of $\mathcal{A}$ a *hereditary $\mathcal{A}$-graph* (e.g., hereditary unigraph).

The class $\mathcal{U}$ of unigraphs contains a number of important hereditary classes, including threshold [39], matroidal [53], and matrogenic graphs [66], but is not itself hereditary. The smallest counterexample is the unigraphic degree sequence $(3, 3, 3, 3, 3, 1)$: its realization contains two non-isomorphic realizations of $(3, 2, 2, 2, 1)$. Barrus [2] [4] studied the hereditary subclass of the unigraphs (which we denote HU) and characterized its elements in terms of their degree sequences, their structure, and a finite list of forbidden induced subgraphs for the class. Specifically, there are sixteen forbidden induced subgraphs, each containing five to seven vertices, which are shown in Figure 5.1. Let $\mathcal{F}_{HU}$ be the set of these graphs. The graphs R, S, and their complements, defined as shown in Figure 5.1 are the four split graphs in this set.

Theorem 5.3. *[2] A graph G is a hereditary unigraph if and only if G contains no element of $\mathcal{F}_{HU}$ as an induced subgraph.*

In Chapter 4, we characterized the hereditary closure of the unigraphs, which is the set of graphs that can be induced in a unigraph. This class (HCU) was characterized in terms of degree sequences, structure, and a finite list of forbidden induced subgraphs. We

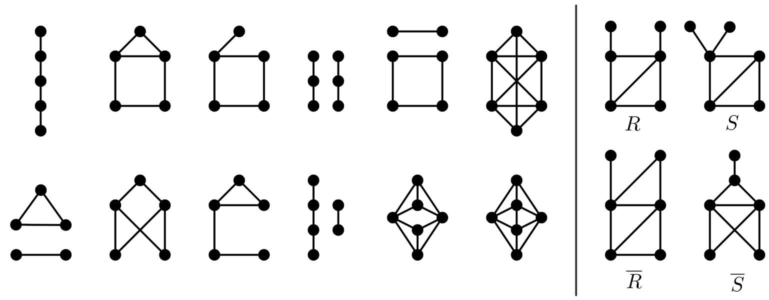

Figure 5.1: The 16 forbidden induced subgraphs of the hereditary unigraphs.

also provided a characterization of HU and HCU in terms of minimally forbidden degree sequences, with a partial ordering introduced by Rao [58].

In this chapter, we identify unique realizations within a number of important classes of graphs. In other words, what graphs – within the restricted universe of a particular graph class – are the only realizations of their degree sequences? Define an $\mathcal{A}$-*unigraph* to be a graph $G \in \mathcal{A}$ such that its degree sequence d has exactly one realization that is an element of $\mathcal{A}$. We call d $\mathcal{A}$-*unigraphic*. Note that $\mathcal{A}$-unigraphs and $\mathcal{A}$-unigraphic degree sequences are not necessarily unigraphs and unigraphic degree sequences without qualification. For example, P_5 is a bipartite-unigraph, since the only other realization of $d(P_5) = (2, 2, 2, 1, 1)$ is $K_3 + K_2$, which is not a bipartite graph, but P_5 is not a unigraph writ large.

Given a graph G, let $\omega(G)$ denote the *clique number* of G, that is, the size of the largest complete induced subgraph of G. Let $\chi(G)$ denote the *chromatic number* of G, that is, the size of the smallest partition of $V(G)$ into independent sets. Where for all induced subgraphs H of G, $\chi(H) = \omega(H)$, we say that G is *perfect*. Alternatively, perfect graphs are exactly those not containing C_{2n+1} or $\overline{C_{2n+1}}$ for all $n \geq 2$ as an induced subgraph [16].

For a graph G not in a hereditary class $\mathcal{A}$, we say that G is *apex*-$\mathcal{A}$ if there exists $v \in V(G)$ such that the induced subgraph $G - \{v\}$ is in $\mathcal{A}$. In Theorem 4.9, we showed that all unigraphs are apex-perfect. Hence here we move to consider subclasses of perfect graphs as $\mathcal{A}$-unigraphs, which will have even stronger coloring properties. Weakly chordal graphs are those not containing any induced cycle or its complement on five or more vertices, and chordal graphs are those not containing any induced cycle of four or more vertices.

In Section 2 we consider classes $\mathcal{A}$ defined by forbidden cycles: perfect graphs, weakly chordal graphs, chordal graphs, and forests. In Section 3, we consider bipartite graphs in depth, giving exact structural and forbidden induced subgraph characterizations. In particular, we show that hereditary bipartite-unigraphs are closely related to complete bipartite graphs. We also characterize a set of forbidden partitioned degree sequences for the class, inspired by Rao's techniques [58].

5.2 Classes defined from cycles

While all unigraphs are apex-perfect by Theorem 4.9, not all are perfect. To this, C_5 is the most notable counterexample, and the only counterexample among forbidden induced subgraphs. In this section we attempt to identify "better" related classes of graphs by considering hereditary $\mathcal{A}$-unigraphs for various subclasses $\mathcal{A}$ of the perfect graphs. For perfect graphs themselves, the characterization is straightforward.

Theorem 5.4. *A graph G is a hereditary perfect-unigraph if and only if G contains none of the graphs in $\mathcal{F}_{HU}$ nor C_5 as an induced subgraph.*

Proof. Suppose G is a hereditary perfect-unigraph. Of course it cannot contain C_5; moreover, all graphs in $\mathcal{F}_{HU}$ are perfect but not hereditary unigraphs, so G contains none of them as an induced subgraph.

For the reverse, suppose G contains no element of $\mathcal{F}_{HU}$ nor C_5 as an induced subgraph. Thus G is a hereditary unigraph, and since $\{P_5, \overline{P_5}\} \subset \mathcal{F}_{HU}$, it follows that G is perfect. Since $\mathcal{F}_{HU}$ is closed under degree sequence realizations, it follows that G is a hereditary perfect-unigraph. □

An identical characterization holds for the weakly-chordal-unigraphs, and hence the classes are equal.

Theorem 5.5. *The class of hereditary weakly-chordal-unigraphs is equal to the class of hereditary perfect-unigraphs.*

Proof. All hereditary weakly-chordal-unigraphs are hereditary perfect-unigraphs by definition.

Suppose G is a hereditary perfect-unigraph. Since it contains neither P_5 nor $\overline{P_5}$, it follows that neither C_{2n} nor $\overline{C_{2n}}$ is an induced subgraph for $n \geq 3$. Nor does G contain any odd cycle or its complement of length at least 5, so G is weakly-chordal. Any other realization of G is imperfect, and hence not weakly-chordal. Hence G is a weakly-chordal-unigraph. Since the argument applies to all induced subgraphs of G, we conclude that G is a hereditary weakly-chordal-unigraph. □

We now turn to the more complicated characterization of hereditary chordal-unigraphs.

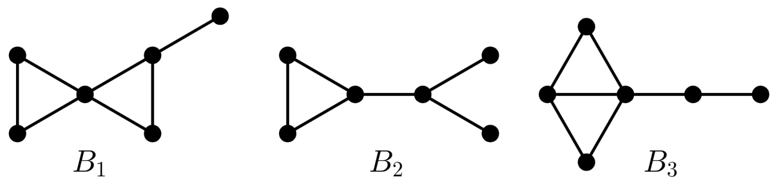

Figure 5.2: Three of the thirteen forbidden induced subgraphs for the hereditary chordal-unigraphs.

We provide two standard theorems about split graphs in advance of their use in the following theorem characterizing $H(\mathcal{U}_C)$.

Theorem 5.6. *[28] A graph G is a split graph if and only if it contains none of C_4, C_5, or $2K_2$ as an induced subgraph.*

Theorem 5.7. *[39] A graph G is a split graph if and only if all other realizations of its degree sequence are split graphs.*

Let R, S be as defined before Theorem 5.3. Let B_1, B_2, B_3 be as shown in Figure 5.2. Let $\mathcal{F}_{H(\mathcal{U}_C)} = \{C_4, C_5, P_5, K_3 + K_2, 2P_3, P_4 + K_2, R, \overline{R}, S, \overline{S}, B_1, B_2, B_3\}$.

Theorem 5.8. *A graph G is a hereditary chordal-unigraph if and only if G contains none of $\mathcal{F}_{H(\mathcal{U}_C)}$ as an induced subgraph.*

Proof. Suppose G is a hereditary chordal-unigraph. Since G is chordal, it contains neither C_4 nor C_5 as an induced subgraph. The graphs P_5, $K_3 + K_2$, $2P_3$, $P_4 + K_2$, R, $\overline{R}$, S, and $\overline{S}$ are all chordal and minimal non-unigraphs, so G can contain none of these. The graphs B_1 and B_2 are both chordal realizations of $(4, 3, 2, 2, 2, 1)$. The sequence $(3, 3, 2, 2, 1, 1)$ has two chordal realizations – one is B_3 and the other contains P_5 as an induced subgraph – so G cannot contain B_3 as an induced subgraph.

For the reverse direction, suppose G contains none of $\mathcal{F}_{H(\mathcal{U}_C)}$ as an induced subgraph. We may assume further that G contains no isolated vertices, since these do not impact graph realizations. We will show G is a chordal-unigraph. Since the proof applies equally to induced subgraphs of G, we can conclude it is a hereditary chordal-unigraph. First off, since G cannot contain C_4, C_5, or P_5, G is chordal. If G is a chordal-unigraph and contains an isolated or dominating vertex v, then $G - \{v\}$ is also a chordal-unigraph, since v is isolated or dominating in every realization, and cannot be induced in a cycle. Hence we may assume that G contains neither isolated nor dominating vertices. If G is a split graph, then all realizations of its degree sequence are split by Theorem 5.7, and G is a hereditary chordal-unigraph if and only if G is a hereditary unigraph. Since G contains none of R, $\overline{R}$, S, or $\overline{S}$, it follows that G is a split hereditary unigraph by Theorem 5.3. Hence we may assume further that G is not a split graph.

Since G is not a split graph, it therefore must contain a copy of $2K_2$ as an induced subgraph by Theorem 5.6. Suppose vertices a, b, c, d of G induce $2K_2$ with edges ab and cd. Of course, G has more than four vertices, so we consider the possible subsets of $A = \{a, b, c, d\}$ that can be contained in the neighbor sets of other vertices. Let $B = V(G) - A$.

The remainder of the proof proceeds as follows. First, we show that B cannot contain vertices with two neighbors in A. Second, we prove that B cannot simultaneously contain vertices with one and three neighbors in A. Third, we prove that all vertices in B with one neighbor in A must have the same neighbor, then show that B cannot consist exclusively of such vertices. Fourth, we prove that B cannot consist exclusively of vertices with three neighbors in A. Given this, B must have a vertex with either four or zero neighbors in A. Our fifth step is to show the former is impossible. We finish the proof by arguing that B can neither contain a vertex with zero neighbors in A.

First, we show it is impossible for any other vertex to neighbor exactly two of $\{a, b, c, d\}$. If a vertex $v \in B$ is adjacent, up to symmetry, to a and b but not c or d, then $\{a, b, c, d, v\}$ induces K_3+K_2, for a contradiction. If another vertex $v \in V(G)$ is adjacent, up to symmetry, to a and c but not b or d, then $\{a, b, c, d, v\}$ induces P_5, for a contradiction. Hence no vertex in G is adjacent to exactly two vertices out of a, b, c, and d.

Second, if there exist v adjacent to one of $\{a, b, c, d\}$ and w adjacent to three of $\{a, b, c, d\}$, then G must contain one of P_5, C_4, B_1, B_2, or B_3 as an induced subgraph. Hence G cannot contain vertices with both one and three edges to vertices in $\{a, b, c, d\}$, aside from a, b, c, and d.

Third, we consider vertices in B that neighbor exactly one vertex in A. Let $v_1, \ldots, v_k \in V(G)$ be the set of such vertices. If $k \geq 2$ and v_1 and v_2 have different neighbors in A, the set $\{a, b, c, d, v_1, v_2\}$ induces P_5 (if $v_1v_2 \in E(G)$) or $P_4 + K_2$ or $2P_3$ (if $v_1v_2 \notin E(G)$). Hence all vertices $\{v_1, \ldots, v_k\}$ have the same neighbor in A; furthermore, none are adjacent lest the graph contain $K_3 + K_2$ as an induced subgraph. Suppose that all vertices in B have exactly one neighbor in A. By the above, we can assume there exist pairwise-nonadjacent vertices $\{v_1, \ldots v_k\}$ that are all adjacent to some vertex $a \in A$. The graph G is isomorphic to the disjoint union of a star and K_2, and is unigraphic by Theorem 5.2.

Fourth, we consider vertices in B that neighbor exactly three vertices in A. Let $v_1, \ldots, v_k \in V(G)$ be the set of such vertices. We now show that $v_1, \ldots, v_k$ are all adjacent to one another and each is adjacent to both a, d and then one of b or c, up to symmetry; otherwise, G contains C_4 or C_5 as an induced subgraph. Suppose $k \geq 2$ and assume for a contradiction that for some $i, j < k$, v_i and v_j are nonadjacent. Without loss of generality we may assume $N(v_i) \cap A = \{a, b, c\}$. If $N(v_j) \cap A = \{a, b, c\}$, then $\{a, v_i, c, v_j\}$ induces a copy of C_4; if $N(v_j) \cap A = \{a, b, d\}$, then $\{a, v_i, c, d, v_j\}$ induces a copy of C_5; and if $N(v_j) \cap A = \{a, c, d\}$ or $\{b, c, d\}$, then $\{a, v_i, c, v_j\}$ (or respectively $\{b, v_i, c, v_j\}$) induces a copy of C_4, for a contradiction. Hence the vertices with three neighbors in A induce a clique. If for a contradiction some vertex v_i in $\{v_1, \ldots, v_k\}$ is not adjacent to c and another vertex v_j in $\{v_1, \ldots, v_k\}$ is not adjacent to d, then $\{v_i, v_j, d, c\}$ induces C_4. By symmetry we may assume that all vertices with three neighbors in A share two of those neighbors, a and d.

Suppose all vertices in B have exactly three neighbors in A. In this case, we can assume there exist $v_1, \ldots, v_k$ adjacent to a, b, and d, and $w_1, \ldots, w_m$ adjacent to a, c, and d. The degree sequence of G is thus $((k+m+2)^{k+m}, (k+m+1)^2, k+1, m+1)$. We claim this sequence has exactly one chordal realization, up to isomorphism. First, such a realization cannot contain C_4 or C_5. Since G is not a split graph, G' cannot be split, so it must contain $2K_2$ as an induced subgraph. Since vertices of degree $k+m+2$ are adjacent to all but one other vertex in G', none can be part of an induced $2K_2$, so the four lower-degree vertices must form this $2K_2$. Second, if two vertices p, q of degree $k+m+2$ are not adjacent to one another, then they must be complete to the remaining vertices. Together with any two non-adjacent vertices r, s, the set $\{p, q, r, s\}$ induces C_4 in G', for a contradiction. Hence the vertices of degree $k+m+2$ are complete to one another, and each is non-adjacent to exactly one more vertex. Third, if $k \geq 0$, $m \geq 0$, and the vertex of degree $k+1$ is adjacent to the vertex of degree $m+1$, then each is non-adjacent to some vertex of degree $k+m+2$, which must be distinct. These four vertices then induce C_4, for a contradiction. If $k = 0$ or $m = 0$, then there is only one possible induced $2K_2$, up to symmetry. Hence one vertex of degree $k+m+1$ is adjacent to the vertex of degree $k+1$, and the other is adjacent to the vertex of degree $m+1$, inducing $2K_2$. The vertices of degree $k+m+1$ are by necessity complete to the vertices of degree $k+m+2$, each of which is adjacent to either the vertex of degree $k+1$ or the vertex of degree $m+1$, but not both, producing G.

We have considered all possible cases where the vertices of B have between one and three neighbors in A. Hence there must exist a vertex $v \in B$ that is adjacent to all or none of the vertices in A.

Fifth, if there exists a vertex $v \in V(G)$ adjacent to all of $\{a, b, c, d\}$, then since G has no dominating vertices, there exists $w \in B$ not adjacent to v. If w is adjacent to one vertex from $\{a, b, c, d\}$, then G contains B_1 as an induced subgraph. If w is adjacent to three or all four vertices from $\{a, b, c, d\}$, then G contains C_4 as an induced subgraph. Hence w is adjacent to none of $\{a, b, c, d\}$, and lest w be isolated, we know there exists x adjacent to w and none of $\{a, b, c, d, v\}$. If x has any other neighbor in G, then G contains $2P_3$ or $K_3 + K_2$ as an induced subgraph, so instead $\{w, x\}$ is a disconnected component of G. Thus G contains $K_3 + K_2$ as an induced subgraph, for a contradiction. Hence we may assume B contains no vertex adjacent to all of A.

Sixth and accordingly, there exists a vertex $v \in V(G)$ adjacent to none of $\{a, b, c, d\}$. Since G has no isolated vertices, there exists $w \in V(G)$ adjacent to v. We split into three cases based on how many neighbors w has in A. First, if w is adjacent to one of $\{a, b, c, d\}$, then G contains $P_4 + K_2$ as an induced subgraph.

Second, suppose w is adjacent to three of $\{a, b, c, d\}$; up to symmetry, we may assume these vertices are a, b and c. We know that w must be the only vertex in B that is adjacent to three of $\{a, b, c, d\}$, else G contains R or $\overline{R}$ as an induced subgraph. As argued above, it is not possible for G to simultaneously contain vertices with one and three neighbors in A; thus, all other vertices in B have no neighbors in A. Suppose there is a vertex x of B not adjacent to w. If there exists a path from x to w, in which case the union of that path, c, and d contains P_5 as an induced subgraph. Otherwise, the graph is disconnected. Since the component containing w contains both K_3 and P_4 as induced subgraphs, it follows that x is an isolated vertex, for a contradiction.

Hence instead all vertices of $B - \{w\}$ are adjacent to w and no vertices of A. Since $\{a, b\}$ together with any induced copy of K_3 in $B - \{w\}$ together induce $K_3 + K_2$, it follows that $B - \{w\}$ induces a forest. Should that forest contain an induced copy of P_3, it together with $\{w, c, d\}$ would induce B_3. Hence $B - \{w\}$ consists of isolated vertices and copies of K_2. Accordingly, G satisfies condition (2) of Theorem 4.14, so it is not a unigraph. Importantly, by the proof of the theorem, the other realization of the degree sequence of G contains a copy of C_4 as an induced subgraph, so G is a chordal-unigraph.

Third and finally, suppose w is adjacent to none of $\{a, b, c, d\}$. If G is connected, then there exists another vertex in B neighboring one or three vertices in A, and so G contains P_5 as an induced subgraph. Otherwise, G is disconnected; the component containing a contains K_2, so G is isomorphic to $P_3 + nK_2$. It is a hereditary unigraph by Theorem 5.2, concluding the proof. □

Among subclasses of chordal graphs, by Theorem 5.7, split graphs are closed under degree sequences. Hence the hereditary split-unigraphs are just the subset of hereditary unigraphs that are split. Threshold graphs fare even worse – since their forbidden induced subgraphs ($2K_2, C_4$, and P_4) are induced in each of the graphs of $\mathcal{F}_{HU}$, it follows that all threshold graphs are hereditary unigraphs.

We lastly consider forests, which are graphs containing no cycles at all, not even K_3. Unlike trees, forests might be disconnected, and hence form a hereditary class.

Theorem 5.9. *A graph G is a hereditary forest-unigraph if and only if it contains none of K_3, C_4, C_5, C_6, C_7, $2P_3, P_4 + K_2, A_1$, or A_2 as an induced subgraph.*

Proof. Suppose G is a hereditary forest-unigraph. As a forest, it cannot contain an induced cycle. The graphs $2P_3$ and P_4+K_2 realize the same degree sequence, as do A_1 and A_2. Since all four are forests, none are forest-unigraphs, so none can be induced in G.

Suppose G contains none of $K_3, C_4, C_5, C_6, C_7, 2P_3, P_4 + K_2, A_1$, or A_2 as an induced subgraph. Since $2P_3$ is induced in all cycles with at least eight vertices, it follows that G is a forest. If G is disconnected, then it must be an induced subgraph of $P_3 + mP_2$, which is a unigraph by [67]. Hence G is a unigraph, and thus a forest-unigraph.

Otherwise, we may assume G is connected. We consider cases based on the longest induced path in G; note that all other vertices must be adjacent to exactly one path vertex. Since it does not contain $2P_3$, the longest induced path contains six vertices. The graph P_6 itself is verifiably a forest-unigraph. Nonetheless, if there exists a seventh vertex adjacent to some vertex along the path, then G must contain A_2 as an induced subgraph. Similarly, if the longest induced path in G contains five vertices, then any additional vertex in G adjacent to a path vertex induces A_1 or A_2. With at most four vertices in the longest induced path in G, the graph is induced in a double star graph and a unigraph by [67]. $\square$

5.3 Hereditary bipartite-unigraphs

We now turn to three equivalent characterizations of hereditary bipartite-unigraphs, HU_2: by graph structure, by forbidden induced subgraphs, and by forbidden partitioned degree sequence pairs, which we define shortly. Before presenting these characterizations in Theorem 5.11, we construct a partial ordering of degree sequences of bipartite graphs.

Given a graph G with vertex subset A, let $d(A)$ be the list of degrees of vertices in A. An unordered pair $\{d_1, d_2\}$ is a *bipartitioned pair* if there exists a bipartite graph G with bipartition $V(G) = A \cup B$ such that $d(A) = d_1$ and $d(B) = d_2$. Thus $d(G) = d_1 \cup d_2$.

Given bipartitioned pairs $\{d_1, d_2\}$ and $\{e_1, e_2\}$, we say that $\{d_1, d_2\}$ *bipartite-Rao-contains* $\{e_1, e_2\}$ (and write $\{d_1, d_2\} \succeq \{e_1, e_2\}$) if and only if there exist bipartite graphs G and H such that (i) $V(G) = G_1 \cup G_2$ and $V(H) = H_1 \cup H_2$ are bipartitions of G and H, respectively, with $d(G_1) = d_1$, $d(G_2) = d_2$, $d(H_1) = e_1$, and $d(H_2) = e_2$, and (ii) there exists an injective homomorphism $\phi : V(H) \to V(G)$ with $\phi(h) \in G_1$ if and only if $h \in H_1$. We also say that a degree sequence d bipartite-Rao-contains a bipartitioned pair $\{e_1, e_2\}$ if there exists a bipartition $\{d_1, d_2\}$ of d with $\{d_1, d_2\} \succeq \{e_1, e_2\}$.

A bipartitioned pair $\{d_1, d_2\}$ is a *forbidden pair* if any bipartite realization of $d_1 \cup d_2$ is not in $H(\mathcal{B}_2)$. Among forbidden pairs, a forbidden pair is *bipartite-Rao-minimal* if it does not bipartite-Rao-contain any other forbidden pair. Forbidden pairs and forbidden induced subgraphs are related per the following lemma.

Lemma 5.10. *A forbidden pair is bipartite-Rao-minimal if and only if all its realizations*

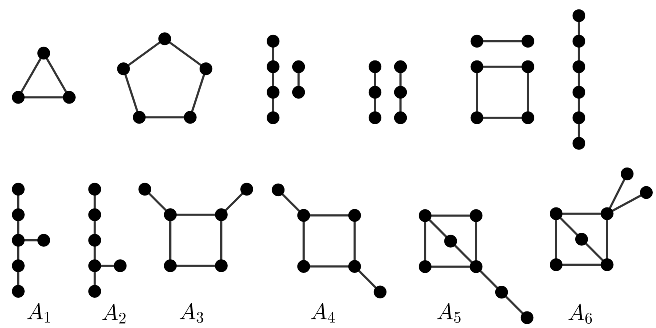

Figure 5.3: The class of hereditary bipartite-unigraphs has twelve forbidden induced subgraphs.

are forbidden induced subgraphs for $H(\mathcal{B}_2)$.

Proof. Suppose that $\{d_1, d_2\}$ is a forbidden pair. If $\{d_1, d_2\}$ is not bipartite-Rao-minimal, then there exists a forbidden pair $\{e_1, e_2\}$ bipartite-Rao-contained in $\{d_1, d_2\}$. Thus, there exist bipartite realizations E of $\{e_1, e_2\}$ and D of $\{d_1, d_2\}$ with E induced in D. We conclude that D is not a forbidden induced subgraph.

If instead $\{d_1, d_2\}$ has a realization D that is not a forbidden induced subgraph, it follows that D contains some forbidden induced subgraph E of $H(\mathcal{B}_2)$. The forbidden pair corresponding to E is thus bipartite-Rao-contained in $\{d_1, d_2\}$, which we conclude is not bipartite-Rao-minimal. □

We will thus be able to characterize HU_2 by its forbidden pairs. Let $\mathcal{F}_{HU_2} = \{K_3, C_5, P_4 + K_2, 2P_3, C_4 + K_2, P_6, A_1, A_2, A_3, A_4, A_5, A_6\}$, as shown in Figure 5.3. Let $\mathcal{R}_{HU_2} = \{\{(2,2),(1,1,1,1)\}, \{(2,1,1),(2,1,1)\}, \{(2,2,1),(2,2,1)\}, \{(3,1,1),(2,2,1)\}, \{(3,2,1),(3,2,1)\}, \{(4,3,1),(2,2,2,2)\}, \{(4,2,2),(3,3,1,1)\}\}$. We show, among other characterizations, that $\mathcal{F}_{HU_2}$ is the set of forbidden induced subgraphs for HU_2, and that $\mathcal{R}_{HU_2}$ is the set of bipartite-Rao-minimal pairs. Note that a graph G is a hereditary bipartite-unigraph if and only $G + K_1$ is a hereditary bipartite-unigraph. Hence we may assume in the following characterization that G contains no isolated vertices.

Theorem 5.11. *The following are equivalent for a graph G containing no isolated vertices:*

(i) G is a hereditary bipartite-unigraph.

(ii) G contains no element of $\mathcal{F}_{HU_2}$ as an induced subgraph.

(iii) G is bipartite and each connected component is apex-complete-bipartite.

(iv) G is bipartite and its degree sequence d does not bipartite-Rao-contain any element of $\mathcal{R}_{HU_2}$.

Proof. (i) $\rightarrow$ (ii):

Suppose $G \in HU_2$. Since G is bipartite, it contains no K_3 or C_5. The pairs of graphs $P_4 + K_2$ and $2P_3$, $C_4 + K_2$ and P_6, A_1 and A_2, and A_3 and A_4 realize the same degree sequences, so are not themselves bipartite-unigraphs and cannot be induced subgraphs of G. The degree sequences $(4, 3, 2, 2, 2, 2, 1)$ of A_5 and $(4, 3, 3, 2, 2, 1, 1)$ of A_6 each also admit multiple bipartite realizations, so neither A_5 nor A_6 is contained in G.

(ii) $\rightarrow$ (iii):

Suppose G contains no element of $\mathcal{F}_{HU_2}$ as an induced subgraph. Note that all indecomposable unigraphs which are bipartite, as identified in Theorem 5.2 and [67], satisfy (iii).

If G is disconnected, then since $2P_3$ and K_3 are forbidden, at most one connected component, which we call H, has 3 or more vertices. All other components are isomorphic to K_2, so H cannot contain P_4 or C_4 as an induced subgraph. If no component has three vertices, then G is isomorphic to nK_2 and is a unigraph by Theorem 5.2 and [67]. If there exists a component H with at least three vertices, then H contains an induced P_3, on, say, vertex set $\{a, b, c\}$ with edges ab and bc. All other vertices in H must have neighbor set exactly $\{b\}$, else P_4, C_4, or K_3 is produced as an induced subgraph. Thus H is isomorphic to a star, and G, comprising a star and nK_2, is a unigraph by Theorem 5.2 and [67]. Stars are complete-bipartite graphs and hence apex-complete-bipartite.

Otherwise, suppose G is connected, and has at least 8 vertices (it is straightforward to check that the result holds for all graphs on 7 or fewer vertices). If G contains no P_3 as an induced subgraph, then G is isomorphic to K_2, for a contradiction. We consider cases based on the longest induced path in G. It must contain at least three and at most five vertices, since G does not contain P_6 as an induced subgraph.

If the longest path in G has three vertices, let $\{a, b, c\}$ with edges ab and bc induce such a path. Either all additional vertices in G have neighbor set $\{b\}$ and G is a star and hence a unigraph by 5.2, or G contains some vertex adjacent to a and c but not b. Let AC be the set of vertices in $V(G) - \{b\}$ adjacent to a and c but not b, and B be the set of vertices in $V(G) - \{a, c\}$ adjacent to b but not a or c. If AC and B are complete to one another or missing at most one edge, then G meets condition (iii). If $|AC| \leq 1$ and B contains at least two vertices not adjacent to a vertex in AC, then G meets condition (iii). If $|AC| \geq 2$ and B contains at least two different vertices not adjacent to the same vertex in AC, then

G contains an induced A_6. If $|AC| \geq 2$ and B contains at least two different vertices not adjacent to different vertices in AC, then G contains an induced A_4. If $|AC| \geq 2$ and B contains only one vertex not complete to AC, but said vertex is non-adjacent to at least two vertices in AC, then G meets condition (iii). Hence if the longest path in G has three vertices, G is a bipartite-unigraph.

If the longest path in G has four vertices, let $\{a, b, c, d\}$ induce such a path with edges ab, bc, and cd. By excluding P_5, K_3, and C_5, G may contain a set B of vertices adjacent to b but not a, c, or d; a set C of vertices adjacent to c but not a, b, or d; a set AC of vertices adjacent to a and c but not b or d; and a set BD of vertices adjacent to b and d but not a or c. Since G contains no induced A_3, B is complete to AC and anti-complete to C, and C is complete to BD. If $|AC| \geq 2$, it is complete to BD and if $|BD| \geq 2$, it is complete to AC, else G contains an induced A_2 or A_4. If B and C are non-empty, then AC and BD are both empty lest G contain A_3. Now, however, G is a unigraph by Theorem 5.2. Otherwise we may assume without loss of generality that C is empty. Since $B \cup \{a, c\} \cup BD$ is complete to $AC \cup \{b\}$, G meets condition (iii).

If the longest path in G has five vertices, let $\{a, b, c, d, e\}$ induce such a path with edges ab, bc, cd and de. Since G contains no $K_3, C_5, P_6, A_1, A_2, A_3$, or A_4, G may contain a set AC of vertices adjacent to a and c but not b, d, or e; a set CE of vertices adjacent to c and e but not a, b, or d; a set AE of vertices adjacent to a and e but not b, c, or d; and a set ACE of vertices adjacent to a, c, and e but not b or d. The set $AC \cup CE \cup AE \cup ACE$ is a stable set. If $|AC| \geq 2$, then G contains an induced A_5. If $|CE| \geq 2$, then G contains an induced A_5. If $|AE| \geq 2$, then G contains an induced A_4. If $|ACE| \geq 2$, then G contains an induced A_4. If AC and AE are both non-empty, or CE and AE are both non-empty, then G contains an induced A_2. If AE and ACE are both non-empty, then G contains an induced A_4. If AC, CE, and ACE are all non-empty, then G contains an induced A_2. Hence $|G| \leq 8$, for a contradiction.

(iv) $\rightarrow$ (iii):

Suppose that G contains a graph from $\mathcal{F}_{HU_2}$ as an induced subgraph. If G contains K_3 or C_5, then it is not bipartite, for a contradiction. Otherwise, if G contains $P_4 + K_2$ or $2P_3$, then $d(G)$ bipartite-Rao-contains $\{(2, 2), (1, 1, 1, 1)\}$ and $\{(2, 1, 1), (2, 1, 1)\}$. If G contains $C_4 + K_2$ or P_6, then $d(G)$ bipartite-Rao-contains $\{(2, 2, 1), (2, 2, 1)\}$. If G contains A_1 or A_2, then $d(G)$ bipartite-Rao-contains $\{(3, 1, 1), (2, 2, 1)\}$. If G contains A_3 or A_4, then $d(G)$ bipartite-Rao-contains $\{(3, 2, 1), (3, 2, 1)\}$. If G contains A_5, then $d(G)$ bipartite-Rao-contains $\{(4, 3, 1), (2, 2, 2, 2)\}$. If G contains A_6, then $d(G)$ bipartite-Rao-contains $\{(4, 2, 2), (3, 3, 1, 1)\}$.

(iii) $\rightarrow$ (iv):

Suppose G is bipartite and its degree sequence $d(G)$ bipartite-Rao-contains a sequence in $\mathcal{R}_{HU_2}$. Thus there exists a bipartite graph H realizing a bipartitioned degree sequence in $\mathcal{R}_{HU_2}$, with H induced in a bipartite realization G' of $d(G)$. By Lemma 5.10, all realizations of a sequence in $\mathcal{R}_{HU_2}$ are forbidden induced subgraphs, which we showed above are precisely the graphs in $\mathcal{F}_{HU_2}$. Hence G' contains a graph from $\mathcal{F}_{HU_2}$ as an induced subgraph. Every realization of the degree sequence of a graph in $\mathcal{F}_{HU_2}$ itself contains an induced subgraph from $\mathcal{F}_{HU_2}$. This implies G contains an element of $\mathcal{F}_{HU_2}$ as an induced subgraph, completing the proof. $\square$

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