UC Berkeley
Research Reports

Title

Modeling, Design and Implementation of Longitudinal Control Algorithm for Automated Vehicle Merging

Permalink

https://escholarship.org/uc/item/9sn473bg

Authors

Lu, Xiao-Yun
Tan, Han-Shue
Shladover, Steven E.
et al.

Publication Date

2000-09-01

This paper uses Postscript Type 3 fonts. Although reading it on the screen is difficult it will print out just fine.

Modeling, Design and Implementation of Longitudinal Control Algorithm for Automated Vehicle Merging

Xiao-Yun Lu, Han-Shue Tan, Steven E. Shladover, J. Karl Hedrick

The contents of this report reflect the views of the authors who are responsible for the facts and the accuracy of the data presented herein. The contents do not necessarily reflect the official views or policies of the State of California. This report does not constitute a standard, specification, or regulation.

CALIFORNIA PARTNERS FOR ADVANCED TRANSIT AND HIGHWAYS

Modeling, Design and Implementation
of Longitudinal Control Algorithm
for Automated Vehicle Merging

Xiao-Yun Lu

Han-Shue Tan

Steven E. Shladover

J. Karl Hedrick

PATH, ITS, U. C. Berkeley

July, 2000

Acknowledgment

Abstract

This part of the report presents the work on the development of the regulation layer for automated merging. It includes system modeling, control system synthesis, theoretical analysis and real-time implementation and field test. The essential part is a rather general adaptive merging algorithm.

The essence of the merging algorithm is to design a reference trajectory for the longitudinal controller of the merging vehicle. The controller must include both speed and distance control. Particularly, the response the controller to distance error must be fast enough because the merging maneuver is time and distance critical. The reference trajectory must be speed-based so that it is realtime implementable and so that when it is fed into the controller, the closed-loop system stability is guaranteed. This algorithm must have a solid mathematical foundation. Based on this consideration, a new approach has been adopted which includes detailed mathematical modeling for automated vehicle merging, system analysis, control design and synthesis, and realtime implementation.

In this report, a merging control problem with mathematical modeling is proposed first. A rather general real-time adaptive closed-loop algorithm is then presented, which is used to calculate a smooth reference speed trajectory for the merging vehicle based on the speed of the main lane vehicle. This algorithm can also be applied even when the main lane vehicles change speed. To make the algorithm adapt to different road layouts and to increase safety, the concept of virtual platooning is proposed. It effectively shifts the time of platoon formation forward prior to the start of real merging. Mathematical proof of the algorithm is in the Appendix. Aspects closely related to real-time implementation are discussed, such as the controller adopted, the use of magnetometer based distance measurement and information passing by communication. Test results are presented and briefly analyzed.

Keywords

Nomenclature

Variables and parameters used in the longitudinal control algorithm for merging are listed below:

P_i, $i = 1, 3-$ vehicle ID on main lane

P_2- vehicle ID on merging lane

$v(t)-$ merging vehicle speed, measurable

$v_{md}(t)-$ desired speed of merging vehicle, to be determined

$v_p(t)-$ speed of the leader vehicle in the platoon on main lane, measurable

$x(t), x_{md}(t)-$ relative distance and reference relative distance between two vehicles, which means the virtual distance between the leader vehicle and the merging vehicle here

$t_{10}-$ time instant for P_1 passing the virtual starting point on main lane

$t_{20}-$ time instant for P_2 passing the virtual starting point on merging lane

$t_{merg} = \max(t_{10}, t_{20})-$ time instant for longitudinal merging control algorithm to start

$T_{merg}-$ time instant when merging is completed

$T_{virt}-$ time instant when a virtual platoon is formed but merging is not completed

$Q_{start\_1}, Q_{start\_2}-$ vehicle positions on main lane and merging lane when $t = t_{merg}$, respectively

$Q_{virt\_1}, Q_{virt\_2}-$ virtual positions in main lane and merging lane at which virtual platoon is formed before practical vehicle merging happens

l_0- desired distance between the two consecutive vehicles in the main lane (for the merging vehicle to enter)

$l_{des\_follow}-$ desired distance between consecutive vehicles in platoon after merging

l_i- length of vehicle P_i, $i = 1, 2, 3$.

Q_1- a point in main lane marked by infrastructure of magnets using special coding

Q_2- a point in merging lane marked by infrastructure of magnets using special coding

Q_0- real crossing (merging) point of main lane and merging lane

$|Q_1 Q_0|-$ the distance between Q_1 and Q_0.

1 Introduction

Automated highway systems (AHS) offer the potential of significant improvements in traffic flow volume and stability, as well as safety, when compared with current manually-controlled highway driving [16]. Implementation of AHS requires completely automated lateral and longitudinal control of vehicle motions as well as coordination of the maneuvers of different vehicles. Considerable success has already been demonstrated in achieving high-performance automatic lateral [14, 15, 18] and longitudinal [5, 12] control of full-scale road vehicles on test track. However, the literature is not well supplied with documentation of successful tests of coordinated higher-level maneuvers such as those required to merge two streams of traffic at a highway entry point. This paper reports on implementation of a fully automated merge maneuver, in which the entering vehicle is inserted between two other vehicles already traveling in a lane.

The merge maneuver requires a combination of decisions and control actions at both the regulation layer and the coordination layer of the five-layer control hierarchy that Varaiya defined for AHS [20]. Current thinking about AHS operations favors the addition of vehicles entering at a merge junction on to the back end of a passing platoon of vehicles for reasons of safety and operating simplicity. However, a more general operating condition, which could provide higher efficiency in high-density traffic conditions, would permit entering vehicles to be inserted into the middle of a passing platoon. This more challenging case has been implemented here to show that it is feasible. The simpler case (attaching to the back of a passing platoon) requires only a subset of the capability demonstrated here.

Most generally, vehicle merging can be abstracted as the problem of one vehicle from the entry lane merging between two vehicles traveling in a platoon in the main lane. Other situations are all special cases of this general case. Generally, a roadside control computer at the merge junction would perform the coordination layer tasks, but in the experimental implementation described here those functions were performed by the lead vehicle in the main lane. These tasks involve:

(a) the determination of vehicle ID of the vehicles in the platoon before and after the merging maneuver

(b) the selection of two vehicles in the main lane between which the merging vehicle is to enter (based on the traffic situation in both main lane and merging lane and the road geometry)

(c) the coordination between the platoons in the main lane

(d) passing relevant information to each vehicle.

The regulation layer tasks are fulfilled by each vehicle itself, and always involve the lower-level control of throttle, brake and steering actuation. In some cases, they may also involve

generating higher-level control commands based on reference trajectories (although these may be considered as coordination-layer activities in some implementations).

Prior research on longitudinal control for vehicle merging has addressed the design of the coordination layer protocols [2] and the simulation of the macroscopic effects of merging on the capacity of an automated lane [1]. However, neither of those papers has addressed the regulation-layer control implementation, which must include consideration of the detailed vehicle dynamics in control design. Related research works are referred to [13]. However, the problem addressed and the solution obtained here are different from those discussed in [13].

This paper concentrates on the implementation of a newly developed algorithm for general merging of vehicles into a highway [8]. This completely new *real-time closed-loop adaptive merging algorithm* generates a smooth reference trajectory for the merging vehicle according to the speed of the leader vehicle in the platoon in the main lane. It is not only suitable for two typical road layouts which represent most practical cases but is also applicable to the difficult situation when the speed of the platoon $v_p(t)$ in the main lane is changing with respect to time. This algorithm is likely to be robust in practice because no additional requirement is imposed on the regulation layer longitudinal controller.

Part of the main idea here is to introduce the concept of *virtual platooning* which effectively shifts the time of platoon formation forward prior to the start of real merging. This means that a virtual platoon control strategy is actually activated some time before the merging vehicle arrives at the merging point. This gives the merging vehicle more flexibility to adjust its speed and acceleration to $v_p(t)$ and $a_p(t)$, which are speed and acceleration of the leader vehicle in the platoon, as well as its virtual relative distance. This is the highlight of the algorithm, and is extremely important for real-time implementation and safety.

Vehicles considered here include all the full-size fully automated vehicles that could run on automated highways, which may include cars, vans, buses and trucks, etc. [20]. One vehicle merging in between two other vehicles requires that at the time instant of merging T_{merg}, the following two conditions are to be satisfied:

(i)

$$v\left(T_{merg}\right) = v_p\left(T_{merg}\right)$$
$$a\left(T_{merg}\right) = a_p\left(T_{merg}\right)$$

(ii) The relative distance between the merging vehicle and either of the two vehicles of the platoon in the main lane is approximately the desired following distance l_{des_follow}.

Obviously, this is not just an acceleration control problem, but is also a speed and distance control problem.

This algorithm has been successfully implemented and tested on the Richmond Field Station (RFS) low speed track with magnets installed every one meter apart at PATH, U. C. Berkeley. This track, which was designed for different test purposes, poses several geometric challenges, making the merge implementation considerably more difficult than it would be in a normal highway environment. The track has very limited length available for vehicles to adjust their speed and its curvatures make it nearly impossible for vehicles to detect the locations of the other vehicles using their current ranging sensors, leaving them entirely dependent on absolute position measurements based on magnetic marker and communications for determination of their relative positions. It has also been tested on the test track at Crows Landing at higher speed. The test track also has a fixed merging point and thus quite similar to the RFS track from a control point of view. Thus, the success of the implementation on these tracks provides encouragement regarding the feasibility of implementation elsewhere.

This paper is structured as follows. Section 2 presents a rather general longitudinal control problem for vehicle merging and an adaptive algorithm. Section 3 addresses real-time implementation issues. Section 4 presents test results with brief analysis. Section 5 is for concluding remarks.

2   Adaptive Merging Algorithm

This section proposes the longitudinal control problem for vehicle merging and an adaptive algorithm.

For safety, a deterministic approach is adopted in the merging algorithm. From this viewpoint, the problem of one vehicle merging with a platoon of vehicles in the main lane can always be abstracted as the entrance of the merging vehicle between two pre-fixed vehicles in the platoon in the main lane. The leader vehicle in the platoon has a speed $v_p(t)$ and acceleration $a_p(t)$, which also represent *platoon speed* and *platoon acceleration* respectively at the moment of real merging happens. Only these two vehicles are directly relevant to the merging vehicle. Several points need to be made clear.

(1) The choice of the two relevant vehicles is determined by a roadside manager. After making the decision, the roadside manager passes the rest of the merging task to the relevant vehicles (the two pre-fixed vehicles in the main lane and the merging vehicle in the merging lane).

(2) A separate algorithm is used for a splitting maneuver of the two relevant vehicles and those following them in the platoon such that there is a proper gap for the merging vehicle to ente when the merging vehicle reach the merging point. The splitting maneuver trajectory

planning can be fond in [3]. The constant spacing platoon forming and platoon keeping can be found in [5, 7, 12, 17].

(3) Most importantly, this algorithm is actually a trajectory planning for the merging vehicle, which determines the desired speed for the merging vehicle such that once it arrives at the merging point, a new platoon of $n + 1$ vehicles is formed. To provide a reference speed is better than to provide a reference acceleration in that the former is suitable for both speed and distance control which are crucial for vehicle platooning. This also makes it impossible to incorporate the vehicle dynamics and different methods into the controller design. In fact, any reasonably good controller can be used to track it to form a closed-loop system.

Let $v(t)$ and $a(t)$ denote the speed and acceleration of the merging vehicle (the first vehicle on the merging lane) respectively. For safety and passeneger's comfort, it is required that at the time instant T_{merg} of merging, the following conditions should be satisfied

(i)

$$
\begin{aligned}
v\left(T_{merg}\right) &= v_p\left(T_{merg}\right) \\
a\left(T_{merg}\right) &= a_p\left(T_{merg}\right)
\end{aligned}
$$

(ii) The relative distances between the merging vehicle and the two vehicles of the platoon in main lane are the same as the desired following distance l_{des_follow}.

Obviously, three factors should be taken into consideration: acceleration control problem , speed and distance.

There are two major difficulties to deal with in the merging car trajectory planning. One is the differences in road layouts. The other is main lane vehicle speed variation. In general, only two typical road layouts which can practically appear: (a) there is no parallel section between the merging lane and the main lane; (b) there is a parallel section between them.

Obviously, the control task for case (a) is more demanding because the merging is required to be fulfilled in a very short time period, usually one second or two. To cope with different road layouts, a new concept, i.e. *virtual platooning*, is introduced, which essentially shifts the time instant for platoon forming forward before real merging starts. This gives the merging vehicle more flexibility to adjust its speed and acceleration to $v_p(t)$ and $a_p(t)$ as well as the relative distances to l_{des_follow}. This is one highlight of the algorithm.

To overcome the difficulty caused by $v_p(t)$ changing, a *closed-loop adaptive merging algorithm* is proposed. Essentially, the reference speed of the merging vehicle changes adoptively according to the speed of the leader vehicle in the main lane. Meanwhile it takes into account the distance requirement for merging . It is less demanding for the longitudinal controller and is thus more practical and robust in practice. This is the second highlight of the algorithm.

The third highlight is that the generated reference speed trajectory $v_{md}(t)$ has the same smoothness property as that of $v_p(t)-$ the leader vehicle speed in the main lane. This is important for real time implementation and safety.

The real-time merging maneuver has two main requirements:

(1) a splitting maneuver is performed by the two vehicles in the main lane after merging starts $(t \geq T_{merg})$ so that the distance between these two vehicles $l_0(= 2l_{des\_follow} + l_2)$ is achieved and maintained;

(2) the speed of the merging vehicle is properly adjusted in real time according to the speed of the leader vehicle in the main lane and the relative distance of each vehicle from the merging point.

Pertinent factors for the merging maneuver can be divided into three parts: (a) geometric layout of the road, (b) absolute speed and relative positions of relevant vehicles in both main lane and merging lane, and (c) signal measurement on each vehicle and communication between relevant vehicles.

2.1 Geometric Layout of the Road

Two different geometric road layouts, which represent most of the possible road layouts in practice, lead to slightly different problem formulation.
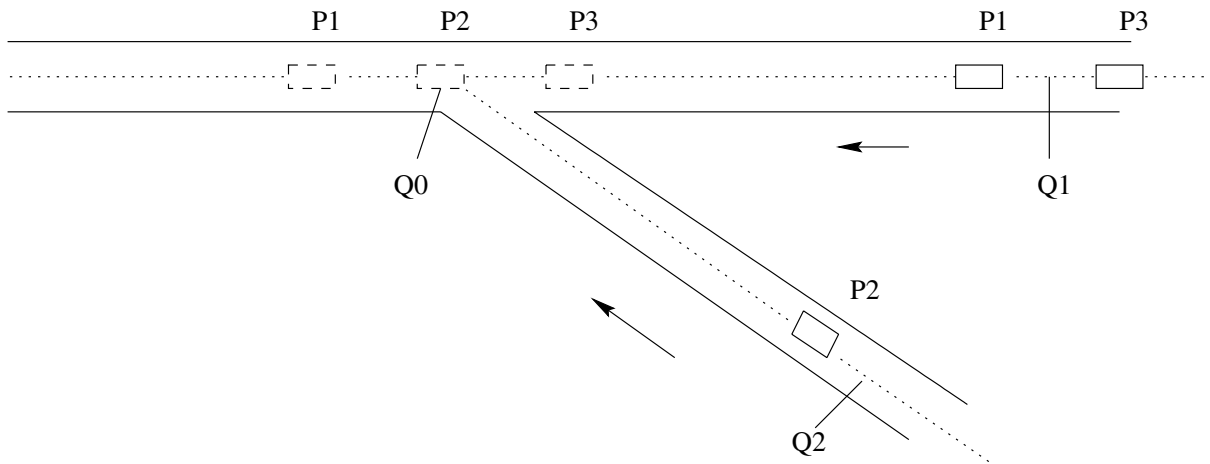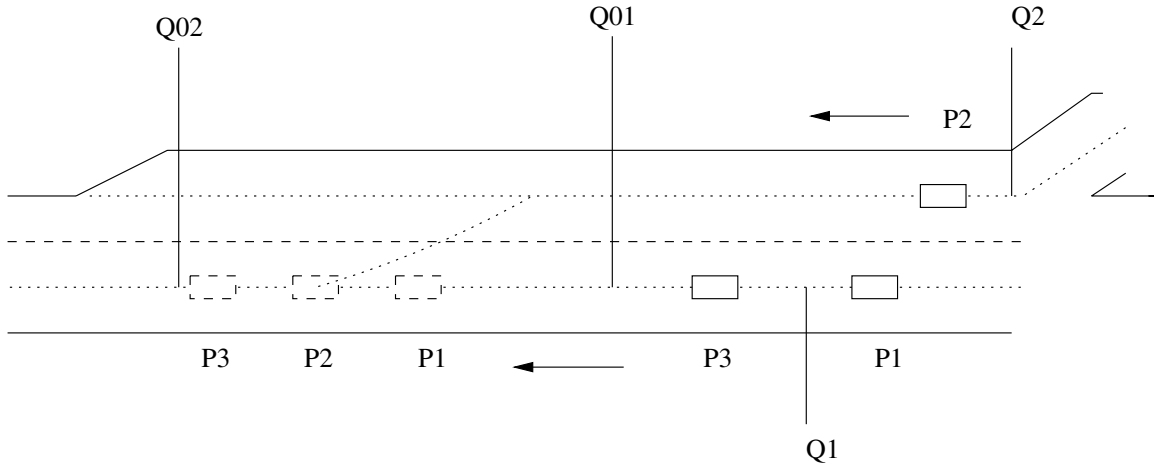

Figure 1: Road layout A

Figure 2: Road layout B

In the first layout, there is no parallel lane for the merging vehicle to accelerate, which is called *layout A* as in Fig. 1. The test track layout in Richmond Field Station at U. C. Berkeley and that at Crows Landing are of this type. In fact, as long as the merging point is fixed, which is the intersection of the merging lane and the main lane, the road layout always belongs to this type.

Q_1, Q_2, Q_0 are marked by infrastructure such as specially coded magnets or transponders. Once vehicle P_1 or P_2 passes them, it will communicate this to the corresponding vehicle, which indirectly tells time and position. Once P_1 and P_2 have passed Q_1 and Q_2 respectively, merging starts. For such scenario, since the merging point Q_0 is fixed, the merging vehicle needs to arrive this point at the right time instant and at proper speed and acceleration.

In a freeway, there is usually a short lane parallel to the main lane for the merging vehicle to accelerate, which is called *layout B* as in Fig. 2.

Here, Q_1 and Q_{01} on the main lane as well as Q_2, Q_{01} and Q_{02} on the merging lane are again marked by infrastructure such as specially coded magnets. This road layout has more flexibility because merging can be carried out at any point between Q_{01} and Q_{02} in a longer time period compared to the previous road layout. It is thus more flexible to adjust the merging time instant as well as the speed and acceleration of the merging vehicle in this layout.

It is clear that a control algorithm suitable for road layout A will be automatically applicable to the less challenging road layout B.

2.2 Merging and Platooning

(1) Merging in front of a platoon of size n: No vehicle is closely in front of the platoon. This case can be simplified as merging with a single vehicle (the first one in the platoon) because other vehicles do not need to be considered.

(2) Merging behind a platoon of vehicles of size n: No vehicle is closely behind the platoon. This case can be described as merging with only one vehicle in the main lane from behind because other vehicles in the platoon other than the last one are controlled by a separate controller and thus will not affect the algorithm.

(3) Merging in between two vehicles in a platoon of size n: The relative position of the merging vehicle in the platoon after merging is determined before the merging process starts. This can be arranged by a roadside manager using communication between vehicles on the main lane and those on merging lane. After merging an $n + 1$ vehicle platoon has been formed.

In this paper, case (3) is reduced to the situation that one vehicle in the merging lane is to merge in between two vehicles in a platoon on the main lane. These two vehicles are the relevant two determined in advance. This is also the most general situation. Case (1) and (2) are obviously special cases for when either the first car or the second car in the main lane is absent. The algorithm presented here is for the general case.

2.3 Mathematical Modelling for Merging

The main longitudinal control problem for vehicle merging from a control viewpoint is to determine a desired reference speed $v_{md}(t)$ of the merging vehicle. Any robust controller with reasonably good performance can be used to track this reference speed. The merging problem becomes a trajectory planning problem for the merging vehicle. $v_{md}(t)$ is determined by the speed of the platoon $v_p(t)$ in the main lane as well as the positions of the relevant vehicles when merging starts.

2.3.1 Longitudinal Control Problem for Merging

Design a reference trajectory $v_{md}(t)$ for merging vehicle P_2 such that
 (1)

$$
\begin{aligned}
v_{md}(t_{merg}) &= v(t_{merg}) \\
v_{md}(T_{merg}) &= v_p(T_{merg}) \\
a_{md}(T_{merg}) &= a_p(T_{merg})
\end{aligned}
$$

(2) three vehicles P_1, P_2, P_3 form a platoon of speed $v_p(t)$ for $t \geq T_{merg}$

(3) the relative distance between two consecutive vehicles is the same as the prescribed distance l_{des_follow} for $t \geq T_{merg}$.

The following conditions are assumed:

(a) The platoon of vehicles P_1 and P_3 will split to a prescribed distance $l_0 (= 2l_{des\_follow} + l_2)$ and maintain this distance by a separate controller around the time instant of real merging;

(b) v_p is measured by P_1 and is passed to P_2 and P_3 by communication;

(c) The distances $|Q_1 Q_0|$, $|Q_2 Q_0|$ and $|Q_1 Q_{02}|$ and $|Q_2 Q_{02}|$ are known in advance and

$$|Q_{start\_1} Q_0| - |Q_{start\_2} Q_0| + l_1 + l_{des\_follow} > 0.$$

(d) The time instant t_{01} when P_1 passes Q_1 is measured by P_1 and is passed to P_2 by communication. The time instant t_{02} when P_2 passes Q_2 is measured P_2 itself. Thus the merging starting time instant t_{merg} and position Q_{start_1} and Q_{start_2} can be calculated.

2.3.2 Unified Modeling

It is clear that, if a merging algorithm works for road layout A, it is also applicable to road layout B. For road layout A, the concept of the *virtual platoon* is proposed. The main idea is to form a virtual platoon before P_1 and P_2 arrive at the merging point Q_0. A virtual platoon means that at some time instant T_{virt}

$$t_{merg} \leq T_{virt} \leq T_{merg}$$

P_i arrives at Q_{virt_i} as shown in Fig. 3.

At this point, the following conditions are satisfied.

$$
\begin{aligned}
v_{md}(T_{virt}) &= v_p(T_{virt}) \\
a_{md}(T_{virt}) &= a_p(T_{virt}) \\
|Q_{virt\_1} Q_0| + l_1 + l_{des\_follow} &= |Q_{virt\_2} Q_0|
\end{aligned}
$$

It is noted that the distances of P_1 and P_3 with respect to the merging point Q_0 should satisfy the following condition

$$|Q_{virt\_3} Q_0| - |Q_{virt\_1} Q_0| = l_1 + l_2 + 2l_{des\_follow}$$

at time instant $t = T_{virt}$. This is supposed to be achieved by a separate controller.
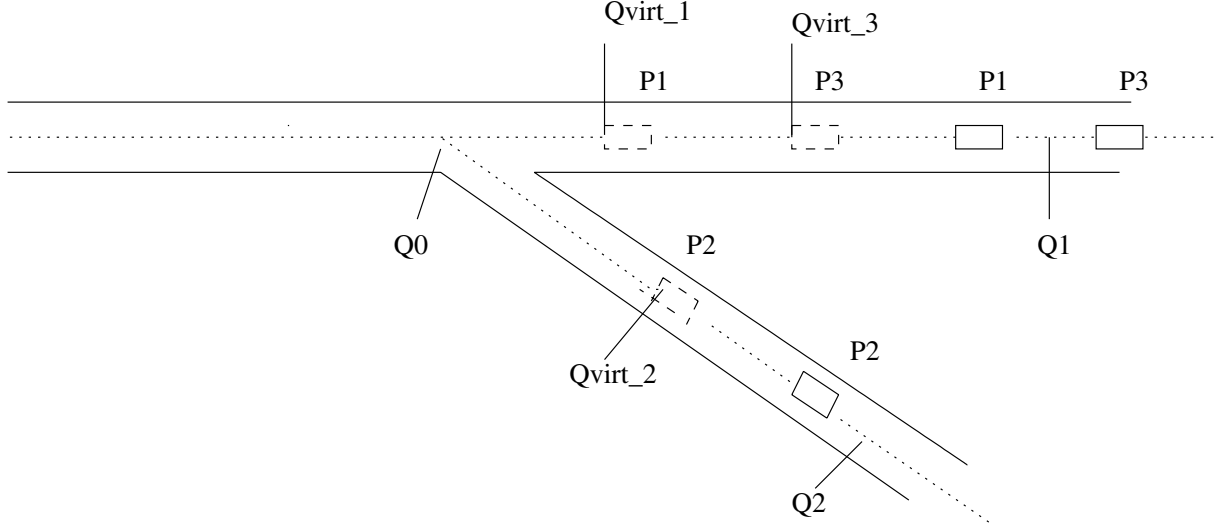
11

Figure 3: A unified road layout

Furthermore, the following sufficient conditions for $t \in [T_{virt}, T_{merg}]$ guarantee the real platoon can be formed

$$
\begin{aligned}
v_{md}(t) &= v_p(t) \\
a_{md}(t) &= a_p(t)
\end{aligned}
$$

Although finding a solution for the merging problem with respect to road layout B is more flexible, a unified control problem can be defined based on the virtual platoon concept. The mathematical model for merging can be stated as follows:

There exists time instant T_{virt} and point Q_{virt_1} between Q_1 and Q_0 (Q_{01} for road layout B) and Q_{virt_2} between Q_2 and Q_0 (Q_{02} for road layout B) such that

$$
\begin{aligned}
v_{md}(t_{merg}) &= v(t_{merg}) \\
\int_{t_{merg}}^{T_{virt}} v_p(t)dt &= |Q_{start\_1}Q_{virt}| \\
\int_{t_{merg}}^{T_{virt}} v_{md}(t)dt &= |Q_{start\_2}Q_{virt}| \\
|Q_{virt\_1}Q_0| + (l_1 + l_{des\_follow}) &= |Q_{virt\_2}Q_0| \\
v_{md}(T_{virt}) &= v_p(T_{virt}) \\
a_{md}(T_{virt}) &= a_p(T_{virt})
\end{aligned}
\tag{2.1}
$$

The physical meaning is briefly explained as follows. The first equation is required by the smoothness of the trajectory. The second, the third and the fourth equations are related to the

12

compatibility conditions of speed and distance. The last two conditions are virtual platooning requirements. There is no doubt that a human-driven vehicle does not need to satisfy this complete set of conditions in order to merge successfully. It is suggested that, for merging maneuvers, the v_{md} of an automated vehicle should satisfy these conditions for safety and smoothness.

2.4 Closed-loop Adaptive Solution

The following is a real-time closed-loop adaptive solution of v_{md} in (2.1). The solution is of a variable structure and has the same smoothness property as $v(t)$. This point is important in implementation when calculating $\frac{d}{dt}(v_{md}(t))$ in realtime.

Theorem Suppose that

(1) Time response of the known longitudinal controller is fast enough and speed and distance tracking error is small enough, i.e.

$$
\begin{aligned}
v(t) &\approx v_{md}(t) \\
x(t) &\approx x_{md}(t)
\end{aligned}
$$

for $t \in [t_{merg}, T_{merg}]$;

(2) $|Q_1 Q_0|$ and $|Q_2 Q_0|$ are large enough, or equivalently, the time interval $[t_{merg}, T_{merg}]$ is sufficiently large to adjust the speed and distance of the merging vehicle;

(3) The starting point for merging and vehicle length have the following relation

$$
dist\_para = |Q_{start\_1} Q_0| - |Q_{start\_2} Q_0| + l_1 + l_{des\_follow} > 0
$$

(4) There exists $\delta > 0$ such that the following condition is true for most of the time

$$
v_p(t) \geq v(t_{merg)} + \delta, \ t \in [t_{merg} + \eta, T_{virt}]
$$

where $0 < \eta << 1$; This means that there exists a constant $\epsilon > 0$ such that

$$
\int_t^{t+\epsilon} \left(v_p(t) - v(t_{merg)} \right) dt \geq \epsilon \delta
$$

as long as $[t, t + \epsilon] \subset [t_{merg} + \eta, T_{virt}]$.

(5) The following reference speed is used for the longitudinal control of P_2

$$
v_{md}(t) = \begin{cases} (1 - \alpha(t)) \ v(t_{merg}) + \alpha(t) \ v_p(t) \,, \ t_{merg} \leq t < T_{virt} \\ \qquad v_p(t), \qquad T_{virt} \leq t \leq T_{merg} \end{cases}
$$

$$
\alpha_0(t) = \frac{\int_{t_{merg}}^t v_p(s)ds}{\int_{t_{merg}}^t v(s)ds + |Q_{start\_1}Q_0| - |Q_{start\_2}Q_0| + l_1 + l_{des\_follow}}
$$

$$
\alpha(t) = \alpha_0^\beta(t), \quad \beta > 0
$$

13

Then virtual platooning is guaranteed to be formed for some $\beta > 0$. i.e. there exist a constant $\beta > 0$, a time instant $T_{virt} \in [t_{merg}, T_{merg}]$ and points Q_{virt_1} and Q_{virt_2} such that

$$|Q_{virt\_i}Q_0| \leq |Q_iQ_0|, \quad i = 1, 2$$

and all the conditions in (2.1) are satisfied.

Proof. See Appendix.

Remark 2.1 The physical meaning of the algorithm and the conditions in the theorem are briefly explained as follows.

(a) The reference trajectory can be divided into two phases according to time:

Phase 1. $t_{merg} \leq t < T_{virt}$, this is the essential part of the merging trajectory planning. The purpose is to adjust both speed $v(t)$ and distance $|Q_{virt\_1}Q_0|$ such that the formation of the virtual platoon is completed at the end of this phase.

Phase 2. $T_{virt} \leq t \leq T_{merg}$, virtual platooning control. As long as the merging vehicle follows the speed and acceleration of the platoon in the main lane, real merging will be guaranteed at time instant T_{merg}.

(b) The assumptions in the theorem are basically the physical constraints which can be set as required in practice. Condition (1) depends on the controller adopted. Condition (2) and (3) can be implemented by proper layout of the infrastructure (such as magnet coding, roadside transponder, or GPS and map signals for vehicle locations) in both merging lane and main lane. Condition (4) indicates that the overall platoon speed should be higher than the speed of the merging vehicle during phase 1 of merging, which is usually true or can be made true by suitably controlling the speed of the merging vehicle.

Remark 2.2 The parameter β determines the length of time period in phase 1. In general, larger β will lead to shorter phase 1. i.e., the virtual platoon is formed earlier. However, this will lead to higher acceleration demands. Usually, choosing $\beta \in [1.5, 5.0]$ will be sufficient. The larger the speed difference

$$v_p(t) - v(t_{merg}), \ t \in [t_{merg} + \eta, T_{merg}]$$

is, the sooner the virtual platoon will be formed.

3   Real-time Implementation

This section addresses some practical issues in the real-time implementation of longitudinal control for vehicle merging. In particular, the magnet-based infrastructure reference system [18] is used for the example in this paper.

3.1 Magnet Observer Based Longitudinal Position Information

For real-time implementation of the merging maneuver, a key point is the knowledge of the relative distance between vehicles and the distance to the merging point. In this paper, magnetometer and speedometer measurements are used to estimate the absolute moving distance of the merging vehicle and the platoon vehicles. The merging vehicle needs to acquire at any time instant $t \in [t_{merg}, T_{merg}]$ its absolute speed and relative distance with respect to the merging point. The two relevant vehicles in the main lane need to obtain their absolute speed and distance with respect to the merging point as well as their relative distance at any time instant $t \in [t_{merg}, T_{merg}]$. Radar measurement is obviously not suitable for the merging vehicle. Limitations to radar measurement of the distance between the two cars in the platoon will be discussed later.

The magnets in both main lane and merging lane are installed at a regular distance mag_dist with maximum error bound err_{mag}. A simple approach to get a continuous position measurement from the magnets is to fuse the discrete magnet measurement with the continuous speed measurement. Considering the possibility that the magnetometer could miss some magnets, the fusion of magnet distance is only carried out when such a miss does not occur. After each magnet updating, the time parameter is reset to 0. From this time instant, the temporary moving distance of the vehicle is

$$temp\_move\_dist = \int_0^{t_1} v(s)\, ds$$

where the time instant t_1 is when a new magnet is measured. Clearly, if no magnet is missed during this period, then there should be

$$temp\_move\_dist \approx mag\_dist + err_{mag}.$$

Otherwise, it should be

$$temp\_move\_dist \approx m\,(mag\_dist + err_{mag})$$

for some integer m which is the number of magnets missed. Now the absolute distance is

$$\sum temp\_move\_dist.$$

This estimation can also be used to calculate the relative distance between P_1 and P_3 in the main lane when radar reading is not appropriate and the virtual relative distance between P_1 and P_2.

Real-time application of this algorithm in the real-time test was successfully tested at low speed at RFS. Even if some magnets are missed, the distance measurement is still reasonably good. However, test data shows that the error between the ground measurement and the estimation based on vehicle speed and magnetometer varies a little bit with respect to speed:

Max speed (km/h)	21.0	24.3	27.6
Average distance measurement error (m)	0.255	0.286	0.328
Max distance measurement error (m)	0.450	0.518	0.828

The total testing distance varies within $237.0m \sim 278.0m$.

From the table, it can be seen that larger distance measurement errors of the observer are found for the higher speeds. Further research into quantitative description for this error with respect to speed variation is necessary to compensate for the error, which still remains a small fraction of the actual vehicle separation distance.

3.2 Signal Measurement and Communication

3.2.1 Speedometer

Vehicle speed measurement relies on fusion of data from 4 wheel speed sensors on each vehicle. Moving distance and acceleration are all calculated from speed by integration and differentiation. A new type of integral filter is used to calculate the vehicle acceleration. The advantages are that the measurement noise, estimation error and time delay are greatly reduced.

3.2.2 Communication

Real-time communication between vehicles is crucial for speed profile calculation and synchronization. This is different from the basic platoon operation where all the distances between consecutive vehicles can also be measured by radar. For merging, the virtual relative positions are calculated from the data passed by communication. WaveLan radio is used which is strong enough for communication between vehicles over $200m$ apart and thus suitable for merging communication at the PATH RFS test site. Medium access control of the WaveLan radio is by token ring passing.

The information passed by the WaveLan radio from the lead vehicle to each vehicle in the platoon and the merging vehicle as well as from each vehicle other than the lead vehicle to the one immediately behind it is the following:

$v-$ speed which is the most important data for longitudinal control;

Magnet counter - the number of magnets encountered;

Special magnet coding - tells special time instant (position) of the vehicles in both the main lane and the merging lane, which is used to determine the time instant and position of each vehicle when merging starts and to determine if the merging is successful.

Maneuver ID - determines maneuvers for each vehicle, such as start moving, acceleration, cruising, splitting, starting merging control, decelerating, braking to stop, etc.

3.2.3 Radar Distance Observer

Except during the time instant when real merging occurs, the second vehicle (P_3) in the main lane can use radar for the relative distance measurement as long as the measurement is valid. However, in this experiment, radar is used in the following car on the main lane for only a very short period of time after merging starts, for distance initialization. Afterwards, the distance between the two cars in the main lane is estimated entirely by the magnet observer. Radar is then switched on a short time after real merging is finished and the real platoon has been formed. This is because of the following limitations in the particular radar sets used for the test:

(a) The two vehicles in the main lane are to separate to a proper safe distance for the merging vehicle to come in, usually over $20.0m$. At this distance, a curve in part of the main lane track can cause the radar to miss the target (the preceding vehicle);

(b) The radar range rate will become infinite at the time instant when the merging vehicle comes in between the two vehicles in the main lane.

3.3 Safety Consideration

Safe merging depends on many reliability factors, including both software and hardware. Hardware reliability is related to sensors, actuators, physical condition of the vehicle, communication facilities, road environment conditions, etc., which are not discussed here. Some important safety factors closely related to control design and implementation are:

(1) Accuracy of controller: The controller used in the regulation layer to make the vehicle track a given reference trajectory is relatively mature and has been well tested. Maximum transient distance tracking error is within $0.5m$ while steady-state error is within $0.1m$ at low speed and $0.03m$ at high speed. Maximum speed tracking error is within $0.3m/s$ in steady state.

(2) Reliability of the splitting maneuver for the second vehicle in the main lane: This depends on the trajectory planning for splitting and the controller adopted. The distance

between the two vehicles in the main lane can reach the desired distance before the merging vehicle enters.

(3) Reliability of the merging algorithm for the merging vehicle: This is crucial because each controller as well as vehicle has certain physical restrictions, such as maximum acceleration and deceleration. The trajectory planning of the merging vehicle needs to consider these restrictions in practice. Theoretically, one can prove that the merging vehicle can follow the reference trajectory $v_{md}(t)$ to fulfil the merging maneuver. In practice, due to the road layout, the infrastructure arrangement such as magnet coding in both main and merging lanes, and the speed variation of the platoon, the desired acceleration.

$$a_{md}(t) = \dot{v}_{md}(t)$$

could be too large for the merging vehicle to reach. Therefore so that the stability could be destroyed. To avoid this, some safety measure should be taken:

(3-a) Forming a virtual platoon before the merging vehicle actually arrives at the merging point is the most important measure. To ensure this physically, one can install a specially magnet code for a sufficient distance before the merging point in each lane to tell the relevant vehicles in advance if the virtual platoon is actually formed as expected. If not, emergency steps should be taken to avoid crashes. Suppose the distance between the magnet and the merging point on the main lane is l_{safe}. The distance between the special magnet to the merging point on the merging lane should be set as $l_{safe} + l_{des\_follow} + l_1$. For safe merging, around the time instant when the leader vehicle passes the specially coded magnet on the main lane, the merging vehicle should pass the corresponding magnet on the merging lane. The leader car can pass this message to the merging vehicle, which will decide if the merging is successful by comparing the two time instants.

(3-b) Physical arrangements of the special coded magnets in both lanes which tell relevant vehicles when merging starts should be arranged to make $\frac{d}{dt}(v_{md}(t))$ as small as possible.

(3-c) In principle, larger β will lead to larger $\frac{d}{dt}v_{md}(t)$. It should also be chosen such that $\frac{d}{dt}(v_{md}(t))$ is sufficiently small.

(3-d) The maximum speed for relevant vehicles in the main lane is restricted within certain range.

(4) Fault management: Merging faults may come from two aspects: (a) The main lane vehicles fail to split properly; (b) The merging vehicle fails to satisfy any of the requirements in distance, speed and acceleration. The faultion detect and management techniques for platooning maneuvers, such as splitting, developed previously and under development for vehicle platooning can still be used for the main lane vehicles. Thorough consideration of fault man-

agement for the merging maneuver will be addressed in future work. Primary suggestion would include the following strategies.

(4-a) If communication range permits, one can install Q_1 and Q_2 further away from the merging point. Now the controller can be designed such that the virtual platoon is to be formed before P_1 reaches the middle point $\frac{1}{2}|Q_1Q_0|$ and P_2 reaches the middle point $\frac{1}{2}|Q_2Q_0|$. i.e.

$$
\begin{aligned}
|Q_{virt\_1}Q_0| &\leq \frac{1}{2}|Q_1Q_0| \\
|Q_{virt\_2}Q_0| &\leq \frac{1}{2}|Q_2Q_0|
\end{aligned}
$$

If the merging maneuver fails in the first half of the roads, one can repeat the merging maneuver in the second half. This will increase the chance of success.

(4-b) If the geographical situation of the road infrastructure does not permit (4-a), the merging vehicle can be stopped manually or automatically in emergency before P_2 reaches Q_0 to abort the merging maneuver.

(4-c) Similarly, one can design a splitting controller such that there is a distance of $l_0 (= 2l_{des\_follow} + l_2)$ between P_1 and P_3 before P_1 reaches the middle point $\frac{1}{2}|Q_2Q_0|$. If this fails, one can repeat the splitting maneuver before P_1 reaches the merging point. If the splitting maneuver still fails, the merging maneuver needs to be aborted by emergency stopping the merging vehicle.

3.4 Longitudinal Controller

In principle, any robust longitudinal controller can be used by the merging vehicle to track the reference trajectory. However, the controller must satisfy the following conditions:

(a) It is a speed based controller;

(b) It controls both speed and distance;

(c) Distance control is precise enough;

(d) Controller response to distance error is fast enough.

The longitudinal controller adopted in vehicle merging is the optimal dynamic back-stepping sliding surface control [12], which is particularly good for distance control. This method combines the following features in nonlinear control design:

(1) Sliding mode method: in ideal sliding mode, dynamics of the closed-loop system are restricted to a sliding manifold, essentially a lower dimensional sub-manifold, on which a stable steady state will be reached in finite time or asymptotically [4, 10, 19]. This approach is generally recognized to be robust to external uncertainties on this sub-manifold [9, 11].

(2) Back-stepping is used in control design logic. In each step, a proper sliding surface is chosen. Thus multiple sliding surfaces naturally result in the end. This is a promising way of dealing with additive unmatched uncertainties in nonlinear models [17].

(3) The sliding gain choice is based on nonlinear H_∞ approach, which can be characterized as disturbance attenuation with internal stability and with certain performance index optimized [7, 12].

(4) Integral filters are used to calculate the derivatives of the reference state (signal) at each time step. Thus analytic differentiation is avoided, which also effectively avoids the explosion of the number of terms [6]. Using integral filters has several advantages in both theory and real-time implementation. In theory, the differentiability of the reference state can be removed. What is required is the existence of a solution. In practice, integral filters are good in noise rejection when calculating the derivative of the signal corrupted by measurement noise, which is also an advantage over numerical differentiation.

3.5   Transition in Control

It is reminded that, during the merging maneuver, magnetometer is used to measure the distance between $P1$ and $P3$ in the main lane and the virtual distance between $P1$ in the main lane and the merging vehicle $P2$ in the merging lane. After merging is complete, it is necessary to transfer to using both radar and magnet measurement. This measurement redundancy will definitely increase reliability. This phase is called transition control which is carried out on no fault basis. Generally, radar distance measurement is slightly more exact than magnet distance measurement based the following arguments:

(1) Magnet measurement depends on the tire pressure and thus the temperature and the charge or the number of passengers, which would affect the diameter of the tires.

(2) It also depends on the exactness of the magnet infrastructure on the ground.

Radar measurement would not have such problems. However, magnet measurement is more reliable in general and radar sometimes appears to have abrupt problems. Proper fusion of data from these two measurement seems crucial for safety. It is thus necessary to have such a transition phase.

Suppose the maximum radar measurement error is err_r and the maximum magnet measurement error is err_m. Then the maximum passible error at the time when the transition starts would be

$$err\_r + err\_m$$

The transition control algorithm is basically a trajectory planning which should satisfy the

following conditions based system stability and ride quality considerations.

(a) These two measurement should be properly fused to reflect the real ground distance;

(b) Transition only activated on no fault basis;

(c) During the transition phase, if one measurement is detected to have intolerable fault, the trajectory should smoothly approach the other;

(d) The transition procedure should be smooth and fast enough but not too fast that the desired acceleration goes over the bandwidth of the controller which will cause instability of the system.

4   Test Results

This section presents the results of tests conducted on the RFS test track at U. C. Berkeley and on the test track at Crows Landing.

4.1   Test at RFS

The track layout has the general topology of Fig. 1, but the merge lane includes a 90 degree curve and the main lane includes multiple reverse curves. The starting positions for the leader car on the main lane can be chosen as $170.0m \sim 190.0m$ away from merging point, and the starting position for the merging car in the merging lane is fixed as $153m$ away from the merging point. The second car in the main lane starts from any point between $6.0m$ to $15.0m$ behind the leader car as long as its radar can catch the leader car for distance initialization (on a section of track with multiple curves). All cars start to move at the same time, synchronized by communication. This is an artifact of the limited space available at the test track, which necessitates minimizing the uncertainties in the initial conditions. The merging algorithm could be implemented in a real roadway environment with sufficient maneuvering space without this start-up synchronization.

The desired gap between the two cars in main lane when the merging vehicle enters is $20.0m$. The maximum speed for the leader vehicle was set to $21.0 \ km/h$ (Fig. 4), $24.3 \ km/h$ (Fig. 5) and $27.6 \ km/h$ (Fig. 6) respectively for three different test cases.

It can be observed from the figures that the three tests have the same qualitative behavior. Fig 4. is used as an example for brief explanation.

(1) Car reference speed for the merging car is quite different from the leader car reference speed profile. For $t \in [0, 37]$, all the cars are stationary. For $t \in [37, 40]$, merging car follows the leader car. During this period of time, both distance control and speed are involved. This
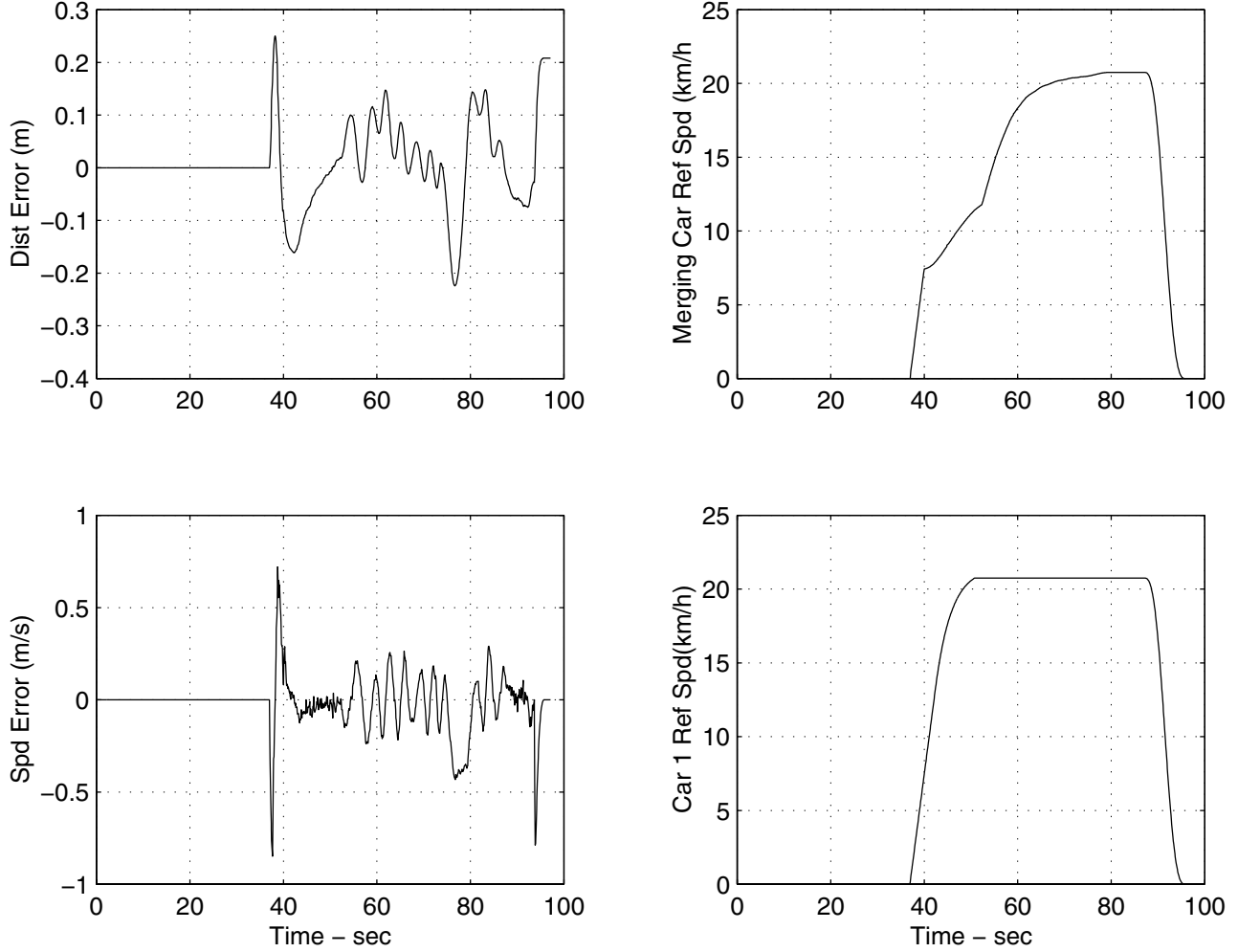
Figure 4: Maximum speed $21.0km/h$

ensures that when the merging algorithm is activated at $t = 40$ the speed profile for the merging car is continuous. For $t \in [40, 55]$, the merging vehicle has much lower acceleration compared to the leader car. For $t \in [55, 75]$, the merging car gradually increases its speed to catch up to the speed of the leader car. Around $t = 75\,\mathrm{sec}$, virtual merging is accomplished. During the time period $t \in [40, 75]$, only speed control is involved in the algorithm, while distance is used as logical guard. The distance error in this period is simply the integration of the speed error. Afterwards, the merging car simply follows the leader car. In the last period, both distance control and speed control are involved.

(2) There are in fact three transient phases for speed and distance errors. This is due to the change of control strategies as stated in (1). So the steady state behavior is not prominent.

Comparing Fig. 5 and Fig. 6 to Fig. 4, there are some similarities. However, a significant
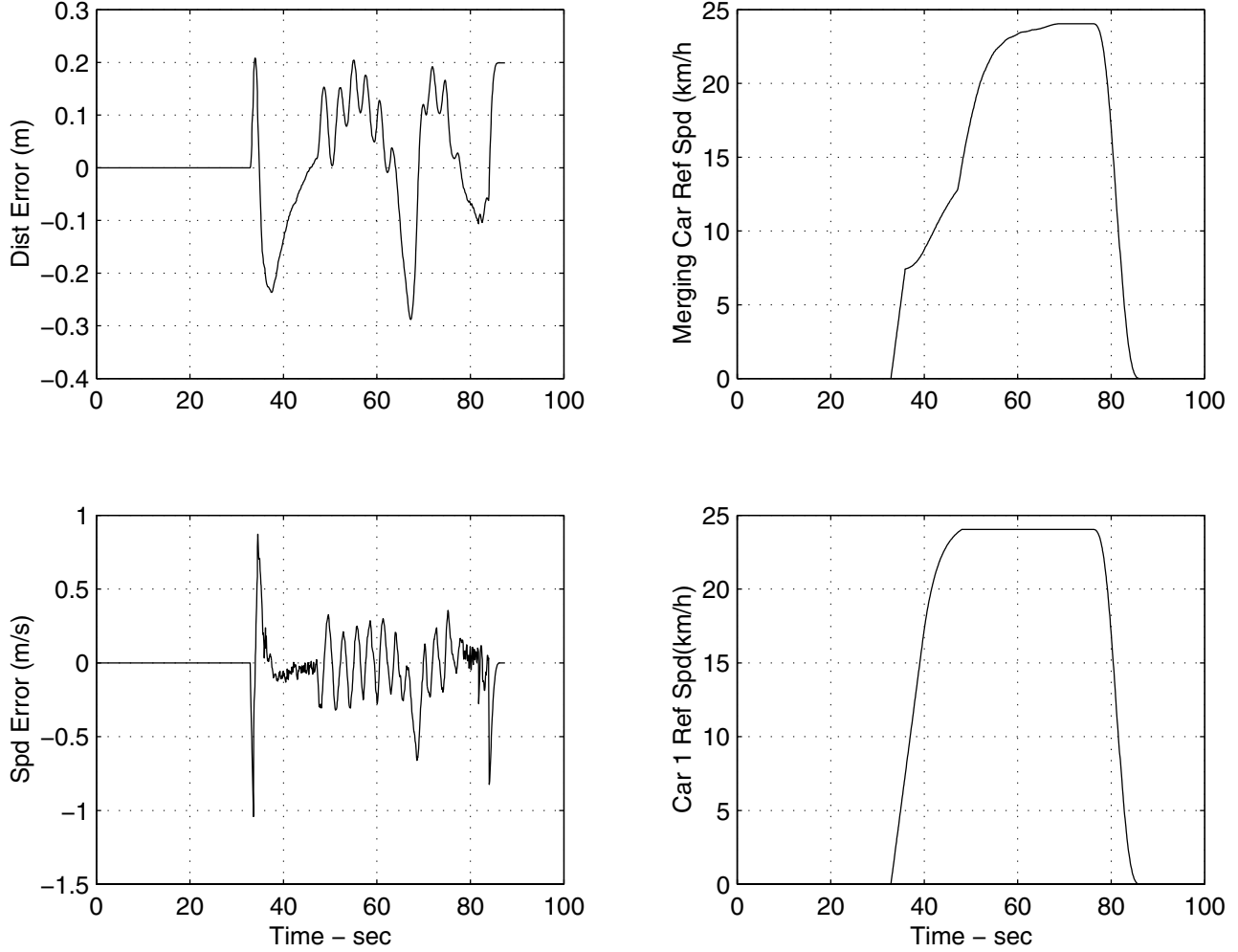
22

Figure 5: Maximum speed $24.30mk/h$

difference is that, due to the limited length of the test track, run time decreases with the increase of the maximum speed of the leader car P_1. This implies that with higher maximum speed of P_1, merging maneuver need to be accomplished in a shorter period of time. This is listed in the following table.

maximum speed	t_{merg}	t_{virt}	time used to form virtual platoon
$21.0kmh$	40	78	38 sec
$24.3km/h$	36	67	31 sec
$27.6km/h$	32	60	28 sec

This also shows the flexibility and capability of the merging algorithm.

Maneuverability of merging controller: Despite the difficulty caused by the road layout at RFS, the merging controller can achieve the following performance.
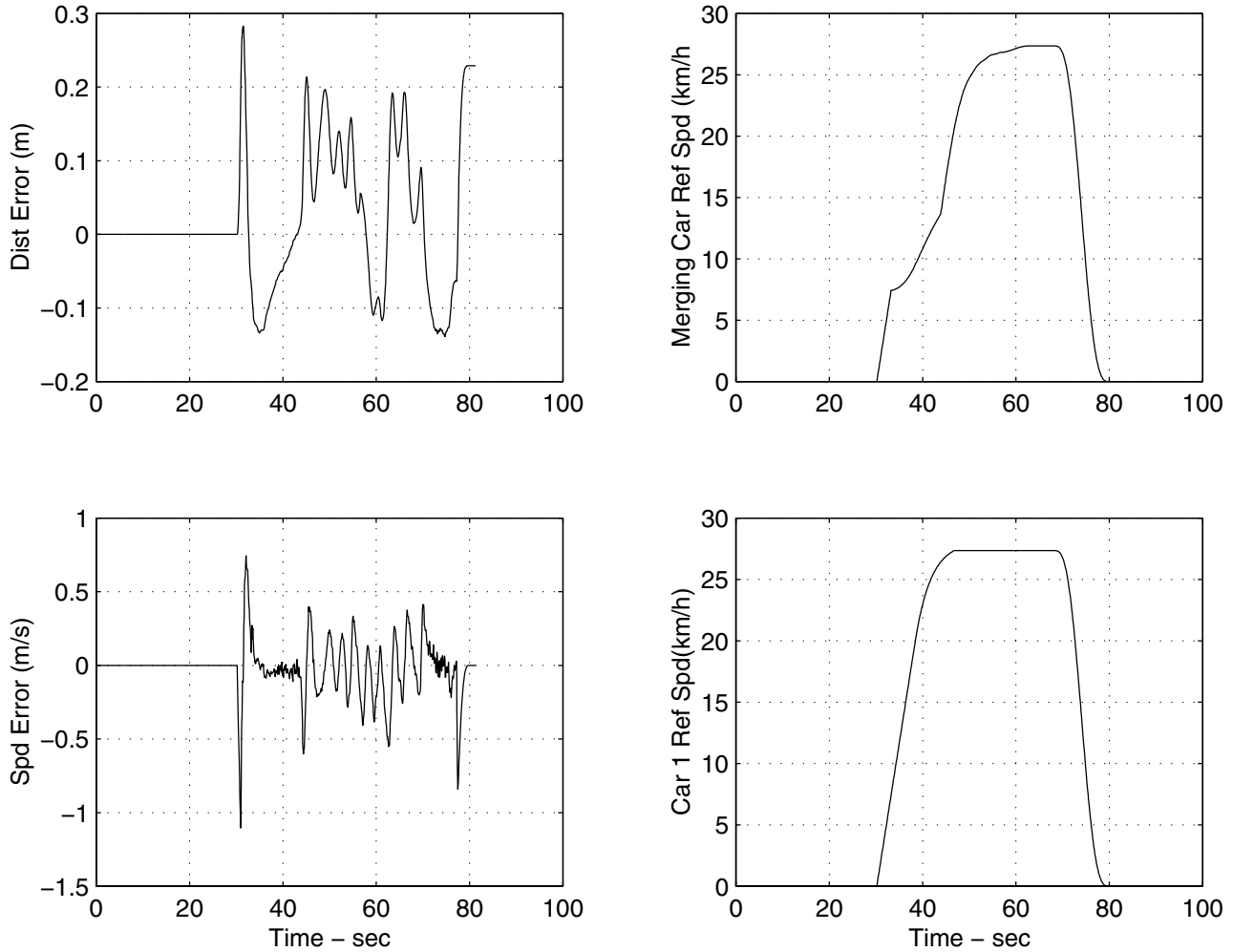
Figure 6: Maximum speed $27.6mk/h$

Distance from leader car to the merging point: $170.0m \sim 190.0m$

Distance from merging car to the merging point: $153m$

Maximum speed of leader vehicle: $21.0 \sim 27.6 \ km/h$

Desired inter-vehicle distance after merging: $8.0m$

4.2 Test at Crows Landing

The test track at Crows Landing also has a fixed merging point. However, it is a longer and thus the maximum speed of the leader vehicle can be set higher. The distance of the leader car (car 1) to the merging point is set to $492m$ and the distance of the merging car (car 2) to the merging point is $468m$. The desired following distance between two consecutive cars

after merging is $12m$. The maximum speed of the leader car was set to different value within $21.0km/h \sim 56.4km/h$. Here a test result is presented in Fig. 7 with the maximum speed for the leader vehicle set to $56.4km/h$. Due to the initial positions, the merging car was to slow down in phase 1 and then speed up to satisfy the virtual platoon condition. This process last for about 30 sec after start moving. The virtual platoon was formed after this phase. The distance error was the difference between the measured virtual distance and the desired virtual distance between car1 and car 2 (merging car). It would be interesting to have a look at the acceleration of the merging car. The maximum acceleration measured was about $1.5m/s^2$. It is noted that the maximum acceleration of merging car largely depends on the choice of β.

Distance measurement based on magnets for merging car and that for the second car in the main lane at Crows Landing test track seemed different as shown in the following table. In the table, the errors were based on the ground measurement after car stopped.

car ID	spd	number of runs	max err (m)	average err (m)
car 2	$56.4km/h$	10	$1.8m$	1.3
car 3	$56.4km/h$	10	$1.2m$	0.8

The error for merging car is obviously larger. This might be caused by the following factors:

(a) The errors of practical magnet distance on the ground along the merging track were larger, which was due to the infrastructure of the track;

(b) The merging car (car 2) had used more brake than car 3 to adjust its speed during the first phase before virtual platoon was formed, which might have some effects on the measurement precision of the speedometer;

(c) Slightly different wheel treads for different cars may lead to slightly different practical wheel diameters even when the wheel pressure for each car was set as the same. This small difference might cause some additional error when cars ran over a longer distance.

All these facts suggest that it is absolutely necessary to return to a distance measurement which is a proper fusion of both magnet distance and radar distance.

5   Concluding Remarks

This paper describes the implementation of a general adaptive closed-loop algorithm for vehicle merging. For safety and for the unification of merging algorithms for different road layouts, the concept of virtual merging is proposed. A virtual platoon in the following sense is formed before the merging vehicle arrives at the real merging point:
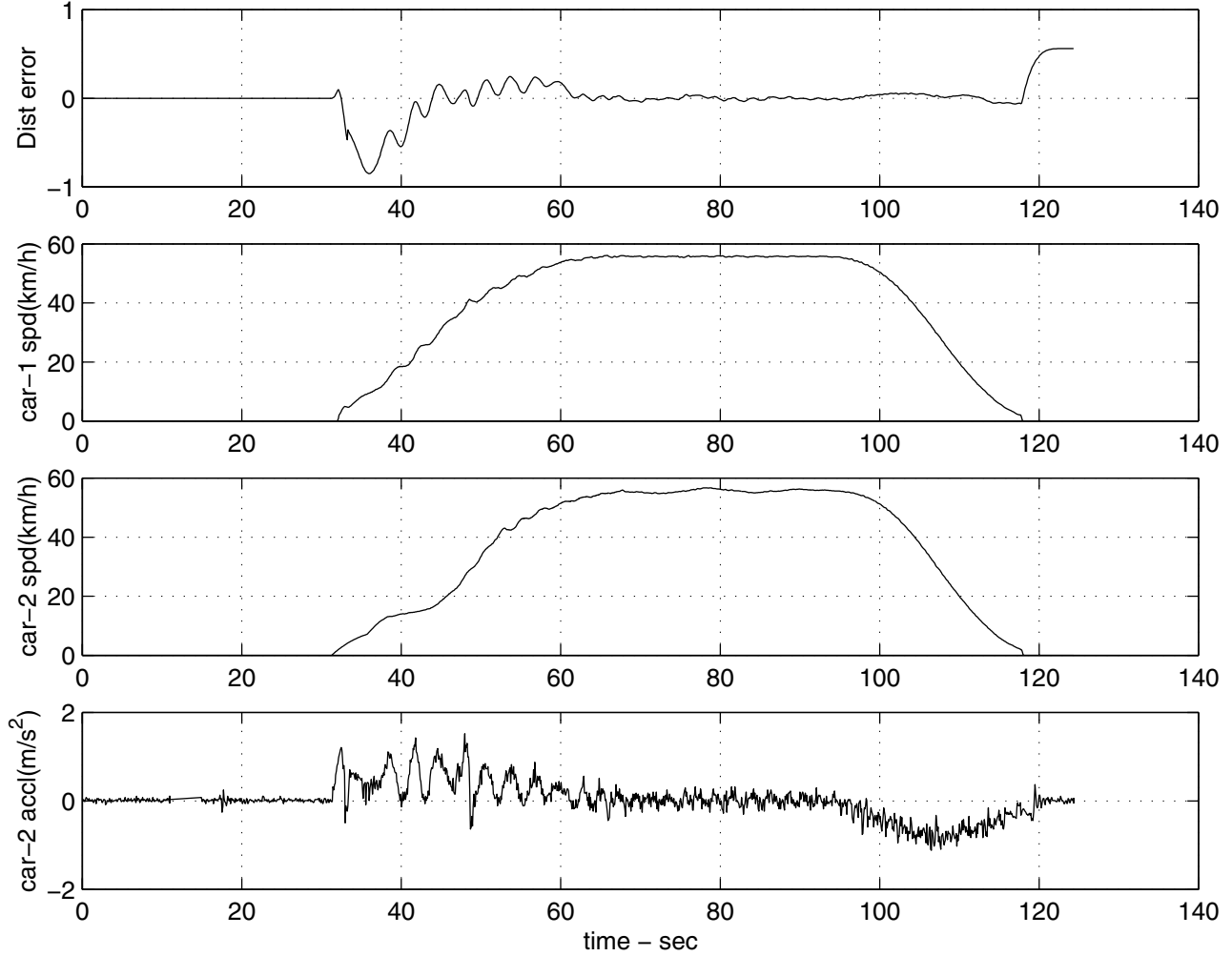
Figure 7: Crows Landing test, maximum speed $56.4 km/h$

(1) Speed and acceleration of the merging vehicle in the merging lane are the same as those of the platoon vehicles in the main lane;

(2) The distance from the merging point to the merging vehicle in the merging lane is the same as distance from the "merging slot" in the platoon in the main lane to the merging point.

There are two main tasks in the implementation of safe vehicle merging: (a) to split the two relevant vehicles in the main lane to a prescribed safe distance for the merging vehicle to enter; (b) to adaptively generate a smooth reference speed $v_{md}(t)$ for the merging vehicle, based on the speed of the leader vehicle in the platoon, the time instant and the relative positions with respect to the merging point of each vehicle when merging starts, and the positions of specially coded magnets in both main lane and merging lane. This algorithm has been successfully tested using automated cars with magnet- based speed and steering control.

26

References

[1] Antoniotti, M., Deshpande, A. and Girault, A., 1997, Microcirculation analysis of multiple merge junctions under autonomous AHS operation, Proc. of the IEEE Conference on Intelligent Transportation Systems (ITSC97), IEEE, November

[2] Antoniotti, M., Deshpande, A., and Girault, A., Microcirculation analysis of automated vehicles on multiple merge junction highways

[3] Connolly, T. R. and Hedrick, J. K., 1999, Longitudinal transition maneuver in an automated highway system, *Trans. of the ASME, J. Dynamic Syst.., Measurement , and Control,* **Vol. 121**, p471-478

[4] DeCarlo, R. A., Zak, S. H. , and Matthews, G. P., 1988, Variable structure control of nonlinear multivariable systems: A tutorial , *Proc., IEEE*, **Vol. 76, No.3**, p212-232

[5] Hedrick, J. K., 1998, Nonlinear controller design for automated vehicle applications, *Proc. UKACC Int. Conf. on Contr.'98, 1-4 Sept., Swansea, U. K.*, p23-31

[6] Lu, X.-Y., and Hedrick, J. K., 2000, Integral filters from a new viewpoint and their application in nonlinear control design, submitted

[7] Lu, X.-Y., and Hedrick, J. K., 2000, H_∞ Sub-optimization in dynamic back-stepping multiple surface control, submitted

[8] Lu, X.-Y., and Hedrick, J. K., 2000, Adaptive longitudinal control algorithm for automated vehicle merging, submitted

[9] Lu, X.-Y., and Spurgeon, S. K., 1997, Robust sliding mode control of uncertain nonlinear systems, *Systems & Control Letters*, **Vol. 32, no. 2,** p75-90

[10] Lu, X.-Y. and Spurgeon, S. K., 1998, Asymptotic stabilization of multiple input nonlinear systems via sliding modes, *Dynamics and Control*, **Vol. 8**, p231-254

[11] Lu, X.-Y. and Spurgeon, S. K., 1999, Robustness of static sliding mode control for nonlinear systems, *Int. J. Control*, **Vol. 72, no. 5**, p1343-1353

[12] Lu, X.-Y., Tan, H.-S., Shladover, S. E. and Hedrick, J. K., 2000, Implementation and comparison of nonlinear longitudinal controller for car platooning, to be presented at AVEC 2000

[13] Narendran, V. K., Hedrick, J. K., Chang, K. S., 1992, Merge control of vehicle in an automated highway system,Transportation Systems, ASME, DSC-Vol. 44, p269-279

[14] Peng, H. amd Tomitzuka, M., 1990, Vehicle lateral control for highway automation, *Proc. of American Control Conference*

[15] Peng, H., 1992, A theoretical and experimental study on vehicle lateral control, *Proc. of American Control Conference*

[16] Shladover, S. E., 1991, Potential freeway capacity effects of advanced vehicle control systems, *Proc. of Second Int. Conf. on Appl. of Advanced Tech. in Transportation Engineering*

[17] Swaroop, D., Gerdes, J. C., Yip, P. P., and Hedrick, J. K., 1997, Dynamic surface control of nonlinear systems, *Proc. of ACC-97*, p3028-3034

[18] Tan, H.-S., J. Guldner, Patwardhan, S., Chen, C. and Bougler, B., 1999, Development of an Automated Steering Vehicle Based on Roadway Magnets - A Case Study of Mechatronic System Design, *IEEE/ASME Transactions on Mechatronics*, **vol. 4, no. 3**, pp. 258-272.

[19] Utkin, V. I., 1992, *Sliding Modes in Control and Optimisation*, Springer-Verlag, Berlin

[20] Varaiya, P., 1993, Smart cars on smart roads: problems of control, *IEEE Trans.* **AC-38, no. 2,** p195-207

6   Appendix

The appendix presents the proof of the main theorem.

Proof of Theorem: Clearly $\alpha(t_{merg}) = 0$ and so $v_{md}(t_{merg}) = v(t_{merg})$. Thus the first condition in (2.1) is satisfied.

It is sufficient to prove that T_{virt} and Q_{virt} exist such that the rest of the conditions in (2.1) are satisfied. Equivalently, it is necessary and sufficient to prove that there exists a time instant $T_{virt} \in [t_{merg}, T_{merg}]$ such that

$$\alpha\left(T_{virt}\right) = 1$$

or equivalently

$$\alpha_0\left(T_{virt}\right) = 1$$

In fact, $\alpha_0\left(T_{virt}\right) = 1$ if and only if

$$\int_{t_{merg}}^{T_{virt}} v_p(s)ds = \int_{t_{merg}}^{T_{virt}} v(s)ds + |Q_{start\_1}Q_0| - |Q_{start\_2}Q_0| + l_1 + l_{des\_follow}$$

Or equivalently

$$|Q_{start\_1}Q_{vert\_1}| - |Q_{start\_1}Q_0| - (l_1 + l_{des\_follow}) = |Q_{start\_2}Q_{vert\_2}| - |Q_{start\_2}Q_0|$$

i.e.

$$|Q_{virt\_1}Q_0| + l_1 + l_{des\_follow} = |Q_{virt\_2}Q_0|$$

This is exactly the distance compatibility condition, the $4th$ in (2.1).

The $5th$ condition in (2.1) is clearly satisfied. To prove the $6th$ condition in (2.1), i.e. the continuous of acceleration, is satisfied, differentiae $v_{md}(t)$ to obtain

$$\dot{v}_{md}\left(t\right) = -\dot{\alpha}\left(t\right)v(t_{merg}) + \dot{\alpha}\left(t\right)v_p(t) + \alpha(t)a_p(t)$$

Because

$$\alpha(T_{virt}) = 1.$$

It is sufficient to prove that

$$\left[-\dot{\alpha}\left(t\right)v(t_{merg}) + \dot{\alpha}\left(t\right)v_p(t)\right]_{t=T_{virt}} = 0$$

or equivalently

$$\dot{\alpha}\left(t\right)\big|_{t=T_{virt}} = 0$$

because

$$v_p(t) - v(t_{merg}) \geq \delta$$

by assumption. Now

$$\dot{\alpha}\,(t) = \alpha_0^{\beta-1}(t)\ \dot{\alpha}_0\,(t)$$

and

$$
\begin{aligned}
\dot{\alpha}_0\,(t)\big|_{t=T_{virt}} &= \frac{v_p(t)\left(\int_{t_{merg}}^t v(s)ds + dist\_para\right) - v(t)\int_{t_{merg}}^t v_p(s)ds}{\left(\int_{t_{merg}}^t v(s)ds + dist\_para\right)^2}\Bigg|_{t=T_{virt}} \\
&\approx \frac{v_p(t)\left[\int_{t_{merg}}^t v(s)ds + dist\_para - \int_{t_{merg}}^t v_p(s)ds\right]}{\left(\int_{t_{merg}}^t v(s)ds + dist\_para\right)^2}\Bigg|_{t=T_{virt}} = 0
\end{aligned}
$$

because

$$
\begin{aligned}
v(T_{virt}) &\approx v_{md}(T_{virt}) = v_p(T_{virt}) \\
\alpha_0(T_{virt}) &= 1.
\end{aligned}
$$

Now it is to prove the existence of T_{virt} such that $\alpha\,(T_{virt}) = 1$ by contradiction. Suppose that such a T_{virt} does not exist in $[t_{merg}, T_{merg}]$.

First, it is noted that the following facts are true:

(a) $\alpha_0(t_{merg}) = 0$.

(b) $\alpha_0(t)$ is continuous for $t \in [t_{merg}, T_{merg}]$ because $\int_{t_{merg}}^t v_p(s)ds$ and $\int_{t_{merg}}^t v(s)ds$ are continuous.

Condition (3) implies that $0 \le \alpha_0(t)$. Thus if there exists $t_1 \in [t_{merg}, T_{merg}]$ such that $\alpha_0\,(t_1) > 1$, then there exists $T_{virt} \in [t_{merg}, T_{merg}]$ such that $\alpha_0(T_{virt}) = 1$ because $\alpha_0(t)$ is continuous. By assumption, it must be true that $\alpha_0(t) < 1,\ t \in [t_{merg}, T_{merg}]$. Then there exists a $\mu > 0$ such that $\alpha_0(t) \le 1 - \mu$ because $\alpha_0(t)$ is continuous and $[t_{merg}, T_{merg}]$ is a closed interval. Thus for $\varepsilon > 0$ sufficiently small, there exists β such that

$$\alpha(t) = \alpha_0^{\beta}(t) \le \varepsilon$$

Now

$$v_p\,(t) - v(t_{merg}) \ge \delta > 0,\ t \in [t_{merg} + \eta, T_{virt}]$$

Because $\eta > 0$ is small, it can be ignored for convenience without loss of generality.

$$
\begin{aligned}
v_{md}(t) &= (1 - \alpha(t))\ v(t_{merg}) + \alpha(t)\ v_p\,(t) \\
&= v(t_{merg}) + \alpha(t)\,(v_p\,(t) - v(t_{merg})) \\
&\le v(t_{merg}) + \varepsilon v_p\,(t)
\end{aligned}
$$

Compare the numerator and the denominator of $\alpha_0(t)$ as follows.

$$\int_{t_{merg}}^{t} v(s)ds \;\approx\; \int_{t_{merg}}^{t} v_{md}(s)ds$$

$$\leq\; \int_{t_{merg}}^{t} \left(\, v(t_{merg}) + \varepsilon v_p\left(s\right)\right) ds$$

$$0 \;\leq\; \int_{t_{merg}}^{t} v(s)ds + |Q_{start\_1}Q_0| - |Q_{start\_1}Q_0| + l_1 + l_{des\_follow}$$

$$\leq\; \int_{t_{merg}}^{t} \left(v(t_{merg}) + \varepsilon v_p\left(s\right)\right) ds + dist\_para$$

The difference between the denominator and the numerator of $\alpha_0(t)$ is

$$\int_{t_{merg}}^{t} \left(v(t_{merg}) + \varepsilon v_p\left(s\right)\right) ds + dist\_para - \int_{t_{merg}}^{t} v_p(s)ds$$

$$\int_{t_{merg}}^{t} \left(v(t_{merg}) + (\varepsilon - 1)v_p\left(s\right)\right) ds + dist\_para$$

$$\leq\; \int_{t_{merg}}^{t} \left(v(t_{merg}) + (\varepsilon - 1)\left(v(t_{merg}) + \delta\right)\right) ds + dist\_para$$

$$=\; \int_{t_{merg}}^{t} \left(\varepsilon v(t_{merg}) + (\varepsilon - 1)\,\delta\right) ds + dist\_para$$

$$=\; \left(\varepsilon\left[v(t_{merg}) + \delta\right] - \delta\right)(t - t_{merg}) + dist\_para$$

Thus when ε is small enough and $t - t_{merg}$ is large enough, it is negative. This is a contradiction. i.e. if T_{merg} is large enough, $\alpha_0(T_{merg}) < 1$ is violated. This contradiction implies the existence of T_{merg} such that $\alpha_0(T_{merg}) = 1$.

Furthermore, if $v_p(t)$ is continuously differentiable n times, them $\alpha(t)$ is continuously differentiable $n + 1$ times. Thus $v_{md}(t)$ and $v_p(t)$ have the same smoothness property.

This completes the proof. $\Diamond$