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ABSTRACT

A numerical solution for a rather general second order partial differential equation in two variables subject to general boundary conditions on a rectangle is obtained by use of cubic splines. The eventual bicubic spline approximation obtained is very amenable to interpolation.

INTRODUCTION

We consider the differential equation

$$au_{xx} + bu_{yy} + cu_x + du_y + eu = r \tag{1}$$

where a, b, c, d, e and r are known functions and u is an unknown function of x and y on a rectangle

$$R = [\underline{x}, \overline{x}] \times [\underline{y}, \overline{y}]$$

We make the usual assumptions of continuity for the "coefficient" functions and second order continuity for the solution. It is further assumed that a and b do not vanish simultaneously at any point of R.

We require that the solution satisfy the following conditions on the boundary of R:

$$\underline{g}u + \underline{q}u_x = \underline{\phi} \quad \text{for} \quad x = \underline{x},\ y \in [\underline{y}, \overline{y}] \tag{2}$$

$$\overline{g}u + \overline{q}u_x = \overline{\phi} \quad \text{for} \quad x = \overline{x},\ y \in [\underline{y}, \overline{y}] \tag{3}$$

$$\underline{f}u + \underline{p}u_y = \underline{\psi} \quad \text{for} \quad y = \underline{y},\ x \in [\underline{x}, \overline{x}] \tag{4}$$

$$\overline{f}u + \overline{p}u_y = \overline{\psi} \quad \text{for} \quad y = \overline{y},\ x \in [\underline{x}, \overline{x}] \tag{5}$$

It is assumed that the "coefficient" functions are continuous for the appropriate (single) variable and domain thereof. The functions, r, $\underline{\phi}$, $\overline{\phi}$, $\underline{\psi}$ and $\overline{\psi}$ cannot all be identically zero. (Otherwise a trivial solution results).

DISCRETIZATION

We partition the interval, $[\underline{x}, \overline{x}]$ into m-1 subintervals of equal length,

$$\delta = (\overline{x} - \underline{x})/(m-1)$$

obtaining

$$\underline{x} = x_1, \ldots, x_m = \overline{x}$$

and partition $[\underline{y},\bar{y}]$ into n-1 subintervals of length

$$\Delta = (\bar{y} - \underline{y})/(n - 1)$$

obtaining

$$\underline{y} = y_1, \ldots, y_n = \bar{y}$$

We propose to apply Equations (1), (2), (3), (4) and (5) at the pertinent points, (x_i, y_j), from the above and to obtain thereby a linear algebraic system with mn + 2m + 2n equations in a like number of unknowns consisting of approximations for u at all points, (x_i, y_j) and for its normal derivatives at pertinent boundary grid-points using a bicubic spline approximation for u.

BASIC CUBIC SPLINES

Since the solution, u, is to be approximated by a bicubic spline, v(x,y), defined on R with knots at the points, (x_i, y_j), for i = 1 to m and j = 1 to n, for any fixed y_j, v must be a cubic spline $S_j(x)$ with knots at the x_i. Likewise for any fixed x_i, v must be a cubic spline $T_i(y)$ with knots at the y_j.

The space of all cubic splines in x with knots at the x_i is linear with dimension m + 2. (See reference 1). A convenient basis for this space consists of the cubic splines, s_ℓ for $\ell = 0$ to μ, with $\mu = m + 1$, defined by

$$\begin{array}{lll} s'(x_1) = 1 & s_o(x_i) = 0 \text{ for all } i & s'(x_\mu) = 0 \\ s'_\ell(x_1) = 0 & s_\ell(x_i) = \delta_{i\ell} & s'_\ell(x_\mu) = 0 \text{ for } \ell = 1 \text{ to } m \\ s'_\mu(x_1) = 0 & s_\mu(x_i) = 0 \text{ all } i & s'_\mu(x_\mu) = 1 \end{array}$$

Now, any spline in the space is a linear combination of the above, and in particular, for any j, $S_j(x)$ is such a linear combination.

Similarly we define a basis for the cubic splines in y with knots at the y_j be

$$t_o'(y_1) = 1 \qquad t_o(y_j) = 0 \text{ all } j \qquad t_o'(y_\gamma) = 0$$

$$t_k'(y_1) = 0 \qquad t_k(y_j) = \delta_{jk} \qquad t_k'(y_\gamma) = 0 \text{ for } k = 1 \text{ to } n$$

$$t_\gamma'(y_1) = 0 \qquad t_\gamma(y_j) = 0 \text{ all } j \qquad t_\gamma'(y_\gamma) = 1$$

where $\gamma = n + 1$. For any i, the cubic spline, $T_i(y)$ is a linear combination of the above.

Values for $s_\ell'(x_i)$, $s_\ell''(x_i)$ and $t_k'(y_j)$, $t_k''(y_j)$ for pertinent values of the indices are readily computed from the definitions.

APPROXIMATION OF u IN TERMS OF THE s_ℓ and t_k

We now need to find the appropriate coefficients for the linear combinations mentioned in the previous section so that the bicubic spline defined thereby will satisfy the Equations (1), (2), (3), (4) and (5) at pertinent grid-points.

We have for fixed y_j

$$\left.\begin{aligned} v(x_i, y_j) &= S_j(x_i) = \sum_{o}^{\mu} \alpha_{\ell j}\, s_\ell(x_i) \\ v_x(x_i, y_j) &= S_j'(x_i) = \sum_{o}^{\mu} \alpha_{\ell j}\, s_\ell'(x_i) \\ v_{xx}(x_i, y_j) &= S''(x_i) = \sum_{o}^{\mu} \alpha_{\ell j}\, s''(x_i) \end{aligned}\right\} \text{ for } i = 1 \text{ to } m, \text{ derivatives with respect to } x. \tag{6}$$

and for fixed x_i

$$\left.\begin{aligned} v(x_i, y_j) &= T_i(y_j) = \sum_{o}^{\gamma} \beta_{ki}\, t_k(y_j) \\ v_y(x_i, y_j) &= T_i'(y_j) = \sum_{o}^{\gamma} \beta_{ki}\, t_k'(y_j) \\ v_{yy}(x_i, y_j) &= T_i''(y_j) = \sum_{o}^{\gamma} \beta_{ki}\, t_k''(y_j) \end{aligned}\right\} \text{for } j = 1 \text{ to } n, \text{ derivatives with respect to } y. \qquad (7)$$

The 2mn + 2n + 2m coefficients $\alpha_{\ell j}$ and β_{ki} are to be determined, yet we have only mn + 2n + 2m conditions to be met. In the next section we discover that mn of the β_{ki} are in fact equal to mn of the $\alpha_{\ell j}$ relieving this situation.

REDEFINITION AND REDUCTION OF THE COEFFICIENTS

In this section we not only reduce the number of coefficients to mn + 2n + 2m as required, but we redefine them in a way more meaningful to our original problem.

From our definition of the basis, s_ℓ, $\ell = 0$ to μ we recall that

$$s_\ell(x_i) = 1 \quad \text{for} \quad \ell = i \quad \text{and is zero otherwise}$$

consequently

$$v(x_i, y_j) = \alpha_{ij} \quad \text{for} \quad i = 1 \text{ to } m,\ j = 1 \text{ to } n.$$

and, similarly, from the definition of the t_k

$$v(x_i, y_j) = \beta_{ji} \quad \text{for} \quad j = 1 \text{ to } n,\ i = 1 \text{ to } m.$$

Hence we define

$$v_{ij} = \alpha_{ij} = \beta_{ji} \quad \text{for } i = 1 \text{ to } m,\ j = 1 \text{ to } n.$$

(Note that we have reduced the number of unknowns by the number mn).

Further we recall

$$s_\ell'(x_1) = 1 \quad \text{for } \ell = 0 \text{ and is zero otherwise}$$

consequently

$$v_\wedge(x_1, y_j) = \alpha_{oj} \qquad \text{for } j = 1 \text{ to } n$$

and, similarly

$$v_x(x_m, y_j) = \alpha_{\mu j} \qquad \text{for } j = 1 \text{ to } n$$

$$v_y(x_i, y_1) = \beta_{oi} \qquad \text{for } i = 1 \text{ to } m$$

$$v_y(x_i, y_\gamma) = \beta_{\gamma i} \qquad \text{for } i = 1 \text{ to } m$$

For convenience in indexing we define

$$v_{oj} \equiv \alpha_{oj} \equiv v_x(x_1, y_j) \qquad \text{for } j = 1 \text{ to } n$$

$$v_{\mu j} \equiv \alpha_{\mu j} \equiv v_x(x_m, y_j) \qquad \text{for } j = 1 \text{ to } n$$

$$v_{io} \equiv \beta_{oi} = v_y(x_i, y_1) \qquad \text{for } i = 1 \text{ to } m.$$

$$v_{i\gamma} = \beta_{\gamma i} = v_y(x_i, y_n) \qquad \text{for } i = 1 \text{ to } m.$$

It should be noted that these quantities are <u>not</u> values for v, but rather for normal derivatives.

However, we see that values for $v_{\ell k}$ $\ell = 0$ to μ and $k = 0$ to γ (except for double indices, 00, 0γ, $\mu 0$, and $\mu\gamma$ which do not appear) are precisely those desired to approximate the solution, u.

Substituting these new coefficients in Equations (6) and (7) we obtain for $i = 1$ to m and $j = 1$ to n:

$$v(x_i, y_j) = v_{ij}$$

$$v_x(x_i, y_j) = s'_o(x_i)v_{oj} + \sum_1^m s'_\ell(x_i)\, v_{\ell j} + s'_\mu(x_i)v_{\mu j}$$

$$v_{xx}(x_i, y_j) = s_o''(x_i)v_{oj} + \sum_1^m s''_\ell(x_i)v_{\ell j} + s_\mu''(x_i)v_{\mu j} \qquad (8)$$

$$v_y(x_i, y_j) = t'_o(y_j)v_{io} + \sum_1^n t'_k(y_j)v_{ik} + t'_\gamma(y_j)v_{i\gamma}$$

$$v_{yy}(x_i, y_j) = t_o''(y_j)v_{io} + \sum_1^n t''_k\,(y_j)v_{ik} + t''_\gamma\,(y_j)v_{i\gamma}$$

Note that the derivative values for the right hand side are known quantities, previously computed. We can now substitute the above (right-hand side) expressions in Equations (1) thru (5) and obtain $mn + 2n + 2m$ equations in $mn + 2n + 2m$ unknowns.

In solving the linear algebraic system above, we find it is more convenient to have a simply-indexed unknown rather than a doubly indexed one and further, to include formally defined quantities, ν_{oo}, $v_{o\gamma}$, $v_{\mu o}$ and $v_{\mu\gamma}$ in the lot.

We define a single index λ for 1 to $(mn + 2n + 2m + 4)$ by

$$\lambda = 1 + \ell + k(m+2) \quad \text{for } \ell = 0 \text{ to } \mu \text{ and } k = 0 \text{ to } \gamma$$

and define

$$w_\lambda = v_{\ell k} \text{ where defined}$$

$$w_\lambda = 0 \text{ otherwise (dummy unknown).}$$

We now seek values for w_λ for $\lambda = 1$ to $mn + 2n + 2m + 4$ (four of the values are arbitrarily zero from the above).

For convenience we let $\mu^* = \mu + 1 = m + 2$ and $\gamma^* = \gamma + 1 = n + 2$ and write Equations (8) in terms of the w_λ, obtaining for $i = 1$ to m, and $j = 1$ to n:

$$v(x_i,y_j) = w_{1+i+j\mu^*}$$

$$v_x(x_i,y_j) = s_o'(x_i)w_{1+j\mu^*} + \sum_1^M s_\ell'(x_i)w_{1+\ell+j\mu^*} + s_\mu'(x_i)w_{1+\mu+j\mu^*}$$

$$v_{xx}(x_i,y_j) = s_o''(x_i)w_{1+j\mu^*} + \sum_1^M s_\ell''(x_i)w_{1+\ell+j\mu^*} + s_\mu''(x_i)w_{1+\mu+j\mu^*}$$

$$v_y(x_i,y_j) = t_o'(y_j)w_{1+i} + \sum_1^N t_k'(y_j)w_{1+i+k\mu^*} + t_\nu'(y_j)w_{1+i+\gamma\mu^*}$$

$$v_{yy}(x_i,y_j) = t_o''(y_j)w_{1+i} + \sum_1^N t_k''(y_j)w_{1+i+k\mu^*} + t_\nu''(y_j)w_{1+i+\gamma\mu^*}$$

In particular, for the normal derivatives on the boundaries:

$$v_x(x_1,y_j) = w_{1+j\mu^*}$$

$$v_x(x_m,y_j) = w_{1+\mu+j\mu^*}$$

$$v_y(x_i,y_1) = w_{1+i}$$

$$v_y(x_i,y_n) = w_{1+i+\gamma\mu^*}$$

Replacing u and its derivatives in Equations (1) to (5) by v and its derivatives as expressed above in terms of w and including the dummy variables as defined we obtain a linear system consisting of $mn + 2n + 2m + 4$ equations in a like number of unknowns, w_λ. Details are shown in Appendix I.

It should be noted that either inconsistency or redundancy in the above system occurs if the two boundary conditions applied at a vertex of R do not involve normal derivatives. For example, at (x_1,y_1), if

$$\underline{p}(x_1) = 0 \quad \text{and} \quad \underline{q}(y_1) = 0$$

then

$$\underline{f}(x_1)v(x_1,y_1) = \underline{\Psi}(x_1) \quad \text{and} \quad \underline{g}(y_1)v(x_1,y_1) = \Phi(y_1)$$

and, if

$$\underline{\psi}(x_1)/\underline{f}(x_1) \neq \underline{\phi}(y_1)/\underline{g}(y_1)$$

then two distinct values for $v(x_1,y_1)$ are prescribed and the system is inconsistent. On the other hand if

$$\underline{\psi}(x_1)/\underline{f}(x_1) = \underline{\phi}(y_1)/\underline{g}(y_1)$$

the two equations resulting from the boundary conditions are linearly dependent.

Nothing can be done about inconsistency, the problem is not solvable. The second situation (redundancy) can be relieved by retaining one of the boundary condition equations and replacing the other by an equation approximating u_x by the assumption that u is very nearly cubic in x for any fixed y. (See Appendix II for details.)

A similar situation occurring at any or all of the other three vertices may be treated in an analogous manner.

RECOVERY AND REFINEMENT OF APPROXIMATE SOLUTION

After the linear system has been solved for

$$w_\lambda \quad \text{for} \quad \lambda = 1 \text{ to } mn + 2n + 2m + 4$$

we obtain v and its normal derivatives by:

$$v(x_i,y_j) = w_{1+i+j\mu^*} \quad \text{for } i = 1 \text{ to } m \text{ and } j = 1 \text{ to } n.$$

$$v_x(x_1,y_j) = w_{1+j\mu^*} \quad \text{for } j = 1 \text{ to } n$$

$$v_x(x_m,y_j) = w_{1+\mu+j\mu^*} \quad \text{for } j = 1 \text{ to } n$$

$$v_y(x_i,y_1) = w_{1+i} \quad \text{for } i = 1 \text{ to } m$$

$$v_y(x_i,y_n) = w_{1+i+\gamma\mu^*} \quad \text{for } i = 1 \text{ to } m$$

Observing that we know nothing about v other than as indicated above, and noting use of large m and/or n is precluded by the size of the resulting linear system, it seems desirable to replace our approximation v by a function $z(x,y)$ which is defined for any point (x,y) in R. We do this by constructing z as the optimal bicubic spline[2] on R with knots at the (x_i,y_j) and assuming for z the prescribed values of v and its normal derivatives.

Now approximate values for u on a much finer mesh over R can be obtained by computing the values of z at the desired points.

NUMERICAL EXAMPLES

We consider a few simple examples to illustrate the method.

EXAMPLE 1

$$u_{xx} + u_{yy} = 6x + 6y \qquad \text{on } [0,1] \times [0,1]$$

$$u(0,y) + u_x(0,y) = 1 + y^3 \qquad \text{along } x = 0$$

$$u(1,y) + u_x(1,y) = 5 + y^3 \qquad \text{along } x = 1$$

$$u(x,0) + u_y(x,0) = 1 + x^3 \qquad \text{along } y = 0$$

$$u(x,1) + u_y(x,1) = 5 + x^3 \qquad \text{along } y = 1$$

in terms of the notation of Equations (1) to (5)

$a(x,y) = 1 \qquad b(x,y) = 1 \qquad c(x,y) = 0 \qquad d(x,y) = 0 \qquad e(x,y) = 0$

$\underline{g}(y) = 1 \qquad \underline{q}(y) = 1 \qquad \underline{\phi}(y) = 1 + y^3 \qquad r(x,y) = 6x+6y$

$\bar{g}(y) = 1 \qquad \bar{q}(y) = 1 \qquad \bar{\phi}(y) = 5 + y^3$

$\underline{f}(x) = 1 \qquad \underline{p}(x) = 1 \qquad \underline{\psi}(x) = 1 + x^3$

$\bar{f}(x) = 1 \qquad \bar{p}(x) = 1 \qquad \bar{\psi}(x) = 5 + x^3$

Analytic solution. It may readily be shown that

$$u = x^3 + y^3 + 1$$

satisfies all the conditions.

Approximate solution. Using $m = 6$ and $n = 6$ we obtain for

$$v(x,y)$$

y \ x	0.	0.2	0.4	0.6	0.8	1.0
1.0	2.000	2.008	2.064	2.216	2.512	3.000
0.8	1.512	1.520	1.576	1.728	2.024	2.512
0.6	1.216	1.224	1.280	1.432	1.728	2.216
0.4	1.064	1.072	1.128	1.280	1.576	2.064
0.2	1.008	1.016	1.072	1.224	1.520	2.008
0.0	1.000	1.008	1.064	1.216	1.512	2.000

and for normal derivatives

v_y

y \ x	0.	0.2	0.4	0.6	0.8	1.0
1.0	3.0	3.0	3.0	3.0	3.0	3.0
0.0	0.0	0.0	0.0	0.0	0.0	0.0

and v_x

y \ x	0.0	1.0
1.0	0.0	3.0
0.8	0.0	3.0
0.6	0.0	3.0
0.4	0.0	3.0
0.2	0.0	3.0
0.0	0.0	3.0

Comparison of values of u and its normal derivatives computed from the analytic solution reveals exact agreement with the values for v given above. This is to be expected since u is cubic in x and y. Construction of the optimal cubic spline z, to fit v and computation of its values in a finer mesh, likewise reveals full agreement with u.

EXAMPLE 2

$$u_{xx} + u_{yy} + u = x^3y^3 + 6x^3y + 6xy^3 + x + y \text{ on } [0,1] \times [0,1]$$

$$u(0,u) + u_x(0,y) = 1 + y$$

$$u(1,y) - u_x(1,y) = -2y^3 + y$$

$$u(x,0) + u_y(x,0) = 1 + x$$

$$u(x,1) - u_y(x,1) = -2x^3 + x$$

Analytic Solution. It may be readily verified that

$$u = x^3y^3 + x + y$$

satisfies all the conditions.

Approximate Solution. With m = 6 and n = 6 we obtain

v(x,y)

y \ x	0.0	0.2	0.4	0.6	0.8	1.0
1.0	1.000	1.208	1.464	1.816	2.312	3.000
0.8	0.800	1.004	1.233	1.511	1.862	2.312
0.6	0.600	0.802	1.014	1.247	1.511	1.816
0.4	0.400	0.600	0.804	1.014	1.233	1.464
0.2	0.200	0.400	0.601	0.802	1.004	1.208
0.0	0.000	0.200	0.400	0.600	0.800	1.000

and normal derivatives

v_x and v_y

y \ x	0.0	1.0
1.0	1.000	4.000
0.8	1.000	2.536
0.6	1.000	1.648
0.4	1.000	1.192
0.2	1.000	1.024
0.0	1.000	1.000

y \ x	0.0	0.2	0.4	0.6	0.8	1.0
1.0	1.000	1.024	1.192	1.648	2.536	4.000
0.0	1.000	1.000	1.000	1.000	1.000	1.000

The values for v are seen to agree with those for u. Further values for w on a finer mesh display the same agreement. This is to be expected since u is a bicubic in x and y.

EXAMPLE 3

$$u_{xx} + u_{yy} + u_x + u_y + 2u = 2. + \cos(x+y) \text{ on } [0,1] \times [0,1]$$

$$u(0,y) = 1.$$
$$u(1,y) = 1. + \sin(1)\cos(y)$$
$$u(x,0) = 1. + \sin(x)$$
$$u(x,1) = 1. + \sin(x)\cos(1)$$

Analytic Solution. The above conditions are satisfied by

$$u = 1 + \sin(x)\cos(y)$$

Approximate Solution. With $m = 6$ and $n = 6$ we obtain

$v(x,y)$

y \ x	0.0	0.2	0.4	0.6	0.8	1.0
1.0	1.000	1.107	1.210	1.305	1.388	1.455
0.8	1.000	1.138	1.271	1.393	1.500	1.586
0.6	1.000	1.164	1.321	1.466	1.592	1.694
0.4	1.000	1.183	1.358	1.520	1.661	1.775
0.2	1.000	1.195	1.381	1.553	1.703	1.825
0.0	1.000	1.199	1.389	1.565	1.717	1.841

v_x

y \ x	0.0	1.0
1.0	0.5402	0.2923
0.8	0.6983	0.3777
0.6	0.8247	0.4449
0.4	0.9203	0.5072
0.2	0.9798	0.4998
0.0	0.9991	0.6557

v_y

y \ x	0.0	0.2	0.4	.06	0.8	1.0
1.0	-.0001	-.1668	-.3270	-.4744	-.6027	-.7086
0.0	0.0005	-.0002	-.0036	0.0079	-.0359	0.1292

For the analytic solution we have:

$u(x,y)$

y \ x	0.0	0.2	0.4	0.6	0.8	1.0
1.0	1.000	1.107	1.210	1.305	1.388	1.455
0.8	1.000	1.138	1.271	1.393	1.500	1.586
0.6	1.000	1.164	1.321	1.466	1.592	1.694
0.4	1.000	1.183	1.359	1.520	1.661	1.775
0.2	1.000	1.195	1.381	1.553	1.703	1.825
0.0	1.000	1.199	1.389	1.565	1.717	1.841

u_x

y \ x	0.0	1.0
1.0	0.5403	0.2919
0.8	0.6967	0.3764
0.6	0.8253	0.4459
0.4	0.9211	0.4977
0.2	0.9801	0.5295
0.0	1.0000	0.5403

u_y

y \ x	0.0	0.2	0.4	0.6	0.8	1.0
1.0	0.0000	-.1672	-.3277	-.4751	-.6037	-.7081
0.0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

As seen from the tables above, values for v are in agreement with those for u to three decimal places, however values for the normal derivatives, particularly along $x = 1$ and $y = 0$ depart considerably from those of u. We should anticipate that interpolation on v to obtain w values would not give very good agreement with u. In fact we find near $(1,0)$ we have:

x	y	z	u
0.92	0.00	1.792	1.796
1.00	0.12	1.839	1.835

In general we have only two decimal place agreement in values for z and u.

Repeating the problem with $m = 11$ and $n = 11$ and interpolating we obtain

$$z(0.92,0.00) = 1.795$$

$$z(1.00,0.12) = 1.835$$

and, essentially, three decimal place agreement between interpolated values of z and actual values of u .

COMPUTER CODES

Two computer subroutines have been written in FORTRAN for the CDC 6600. The first of these, PDYBCS, computes values for the approximate solution, v, at grid-points and for its normal derivatives at boundary grid-points. The user must supply a "main" program to specify m, n, $\underline{x}$, $\overline{x}$, $\underline{y}$ and $\overline{y}$ and provide for output of results. Suitable dimension for various arrays must also be set in the main (calling) program. The user must also supply subroutines which define the differential equation and boundary conditions.

The second subroutine uses the results obtained above to construct the optimal bicubic spline, z, and to interpolate for values of z on a finer mesh. The user must supply a main program to read in the values of v, v_x and v_y as appropriate.

Listings and fuller description of the use of these subroutines may be obtained from the author.

CONCLUSION

We have confirmed that a bicubic spline approximation can be made to a wide variety of boundary-conditioned partial differential equations in two variables on a rectangular domain. In the examples given and in many others tried, a reasonable approximation was attained to a known solution. In all cases the approximation (if not already exact as in Examples 1 and 2) was significantly improved by increasing the number of grid-points. The size of linear system to be solved places a practical limit on this increase.

In actual practice, there would be no point to approximating the solution if it could be obtained analytically. In any case, the bicubic spline approximation does satisfy the differential equation at all grid-points and the boundary conditions at boundary grid-points. The accuracy of the approximation (irrespective of whether the solution u is known or not) is certainly indicated by how well the approximation, w, satisfies the boundary conditions and differential equation at points (other than grid-points) in R. In particular one might compute z, z_x or z_y as appropriate at mid-points of subintervals, $[x_i, x_{i+1}]$ for $y=\underline{y}$ and $y=\bar{y}$ or $[y_j, y_{j+1}]$ for $x=\underline{x}$ and $x=\bar{x}$ and compute errors in meeting boundary conditions. Then at each midpoint of a subrectangle compute z, z_{xx}, z_{yy}, z_x, z_y and compute the errors in satisfying the differential equation at these points.

ACKNOWLEDGEMENT

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REFERENCES

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APPENDIX I

DISCRETIZATION OF DIFFERENTIAL EQUATIONS AND BOUNDARY CONDITIONS

Applying Equation (1) at any grid-point, (x_i, y_j) for $i = 1$ to m and $j = 1$ to n we obtain

$$a(x_i,y_j)u_{xx}(x_i,y_j) + b(x_i,y_j)u_{yy}(x_i,y_j) + c(x_i,y_j)u_x(x_i,y_j) +$$

$$+ d(x_i,y_j)u_y(x_i,y_j) + e(x_i,y_j)u(x_i,y_j) = r(x_i,y_j)$$

Replacing u by v with appropriate subscripts as obtained from Equation (8), we have

$$a_{ij}\sum_0^{\mu} s''_\ell(x_i)v_{\ell j} + b_{ij}\sum_0^{\gamma} t''_k(y_j)v_{ik} +$$

$$+ c_{ij}\sum_0^{\mu} s'_\ell(x_i)v_{\ell j} + d_{ij}\sum_0^{\gamma} t'_k(y_j)v_{ik} + e_{ij}v_{ij} = r_{ij}$$

or in terms of w

$$\sum_{\ell=0}^{\mu}[a_{ij}s''_\ell(x_i)+c_{ij}s'_\ell(x_i)]w_{1+\ell+j\mu*} + \sum_{k=0}^{\gamma}[b_{ij}t''_k(y_j)+d_{ij}t'_k(y_j)]w_{1+i+k\mu*}$$

$$+ e_{ij}w_{1+i+j\mu*} = r_{ij} \qquad \text{for } i = 1 \text{ to } m \text{ and } j = 1 \text{ to } n. \quad (1')$$

Similarly, applying Equations (2) to (4) at appropriate boundary grid-points we have.

$$\underline{g}_j \, w_{2+j\mu*} + \underline{q}_j \, w_{1+j\mu*} = \phi_j \qquad \text{for } j = 1 \text{ to } n \ (i=1) \quad (2')$$

$$\bar{g}_j \, w_{1+m+j\mu*} + \bar{q}_j \, w_{1+\mu+j\mu*} = \bar{\Phi}j \qquad \text{for } j = 1 \text{ to } n \ (i=m) \quad (3')$$

$$\underline{f}_i \, w_{1+i+\mu*} + \underline{p}_i \, w_{1+i} = \underline{\psi}_i \qquad \text{for } i = 1 \text{ to } m \ (j=1) \quad (4')$$

$$\bar{f}_i \, w_{1+i+n\mu*} + \bar{p}_i \, w_{1+i+\Upsilon\mu*} = \bar{\Psi}_i \qquad \text{for } i = 1 \text{ to } m \ (j=n) \quad (5')$$

APPENDIX II

SPECIAL CORNER CASES

At (x_1, y_1) if $\underline{p}(x_1)$ and $\underline{q}(y_1)$ both are zero, from Appendix I, we retain Equation (4) as

$$\underline{f}_1 w_{2+\mu^*} = \underline{\psi}_1$$

but we replace Equation (2) by expressing $v_x(x_1, y_1)$ in terms of a cubic approximation of $v(x,y_1)$:

$$v_x(x_1,y_1) = \frac{11}{6\delta} v(x_1,y_1) - \frac{3}{\delta} v(x_2,y_1) + \frac{3}{2\delta} v(x_3,y_1) - \frac{1}{3\delta} v(x_4,y_1)$$

and using

$$w_{1+\mu^*} - \frac{11}{6\delta} w_{2+\mu^*} + \frac{3}{\delta} w_{3+\mu^*} - \frac{3}{2\delta} w_{4+\mu^*} + \frac{1}{3\delta} w_{5+\mu^*} = 0 .$$

By a similar procedure at (x_m,y_1) if $\underline{p}(x_m)$ and $\bar{q}(y_1)$ are both zero we obtain

$$\frac{1}{3\delta} w_{2m} - \frac{3}{2\delta} w_{2m+1} + \frac{3}{\delta} w_{2\mu} - \frac{11}{6\delta} w_{2\mu+1} - w_{2\mu^*} = 0 .$$

Redundancy at (x_1,y_n) or (x_m,y_n) is treated in an analogous manner.

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