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# Multiplicities Associated to Demazure Flags of sl2[t]-Modules

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by

Lisa Marie Schneider

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To my parents for their unwavering love and support.

ABSTRACT OF THE DISSERTATION

Multiplicities Associated to Demazure Flags of sl2[t]-Modules

by

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Doctor of Philosophy, Graduate Program in Mathematics
University of California, Riverside, August 2015
Professor Vyjayanthi Chari, Chairperson

In this paper, we study the multiplicities of a level $\ell + 1$–Demazure flag of a level $\ell$–Demazure module for the current algebra $\mathfrak{sl}_2[t]$. We establish a recursion of the graded multiplicities and explicitly calculate these multiplicities when $\ell = 1, 2$. Taking the specialization $q = 1$ of the graded multiplicity (i.e. numerical multiplicity), we establish a new recursion of polynomials related to the numerical multiplicities. We give a solution for these polynomials in two ways: as solutions of matrix equations and as coefficients in the series expansion of rational functions.

# Contents

# Introduction

In studying the representations of affine Lie algebras, we begin with understanding the highest weight irreducible integrable representations particularly for the affine Lie algebra associated to $\mathfrak{sl}_2$. Since these modules are infinite dimensional, we focus on a family of Demazure modules which occur in a highest weight integrable representation of the affine Lie algebra associated to $\mathfrak{sl}_2$. These Demazure modules are stable under the action of $\mathfrak{sl}_2$ and thus we study their structure as modules for the current algebra $\mathfrak{sl}_2[t]$ which is defined to be the Lie algebra of polynomial maps from $\mathbb{C}$ to $\mathfrak{sl}_2$. Alternatively, the current algebra is a maximal parabolic subalgebra of the affine Lie algebra. The action of the element $d$ of the affine Lie algebra defines an integer grading on the current algebra and also a compatible grading on the $\mathfrak{sl}_2$-stable Demazure modules. This work only reflects the work on $\mathfrak{sl}_2$-stable Demazure modules, thus we refer to these modules by simply using the term *Demazure module*.

The Demazure modules are indexed by triples $(\ell, n, r)$ where $n \in \mathbb{Z}_+$, $r \in \mathbb{Z}$ and $\ell \in \mathbb{N}$ and are denoted as $\tau_r^* D(\ell, n)$. The integer $\ell$ is called the level of the Demazure module and is given the action of the canonical central element of the affine algebra and $r \in \mathbb{Z}$ is minimal so that the corresponding graded component is non–zero. A key result due to Naoi [9] states that if $m \geq \ell \geq 1$ then Demazure module $D(\ell, n)$ admits a filtration such that the successive quotients are isomorphic to level $m$ Demazure modules. In fact Naoi proves this result for an affine Lie algebra associated to a simply–laced simple Lie algebra. His proof is indirect using results of [6] and[7]. Moreover, Naoi proved that the local Weyl module has a filtration of level 1-Demazure modules for an affine Lie algebra associated to a non-simply laced simple Lie algebra.

A direct and constructive proof of Naoi's result was obtained in [4] for $\mathfrak{sl}_2$. The methods of this paper also showed the existence of a level $m$ Demazure flag in a much wider class of modules for $\mathfrak{sl}_2[t]$. As a result, explicit recurrence relations were given for the multiplicity of a level $(\ell+1)$–Demazure module ocurring in a filtration of $\tau_r^* D(\ell, n)$. A closed form solution of these recurrences was, however, only obtained in some special cases: the numerical multiplicities (the $q = 1$ case) were computed for $\ell = 2, m = 3$, and the $q$-multiplicities for $\ell = 1, m = 2$.

In this thesis, we give the closed form solution of the recurrences for $\ell = 1, 2, m = 3$. When $\ell = 1, m = 3$, these closed forms give multiplicities of the filtration of a local Weyl module by level 1-Demazure modules for type $G_2$. Additionally, we show that the generating series can be written in terms of partial theta functions. We also establish a simple matrix solution to find the multiplicities with specialization $q = 1$ of a level $\ell+1$–Demazure flag of a level $\ell$-Demazure module. Lastly, we establish addional relationships between the numerical multiplicities for different values of $\ell$.

# Part I

# Demazure Flags and Multiplicity

# Chapter 1

# Demazure Modules

In this chapter, we introduce the necessary notation and preliminary results for simple and affine Lie algebras in the case of $\mathfrak{g} = \mathfrak{sl}_2$. Then, we recall the definition of a Demazure module occurring in a highest weight integrable irreducible representation of the affine Lie algebra $\widehat{\mathfrak{sl}_2}$. We restrict our attention to stable Demazure modules as representations of the current algebra and state several results from [5] about these modules.

## 1.1 Simple Lie algebra $\mathfrak{sl}_2$

We first fix some notation. We let $\mathbb{C}(\mathbb{Z}, \mathbb{Z}_+, \mathbb{N})$ represent the complex numbers (integers, non-negative integers, positive integers respectively).

Recall that $\mathfrak{sl}_2$ is the complex simple Lie algebra of two by two matrices of trace zero and that $\{x, h, y\}$ is the standard basis of $\mathfrak{sl}_2$, with $[h, x] = 2x$, $[h, y] = -2y$ and $[x, y] = h$ and the Cartan subalgebra is $\mathfrak{h} = \mathbb{C}h$. For $n \in \mathbb{Z}_+$, there is a unique finite dimensional irreducible $\mathfrak{sl}_2$-module, $V(n)$ generated by a nonzero vector $v$ with the relations:

$$x.v = 0, \qquad h.v = nv, \qquad y^{n+1}.v = 0.$$

Moreover, $\dim V(n) = n + 1$.

For any finite dimensional $\mathfrak{sl}_2$-module, $V$, we can define the character of $V$ as the Laurent polynomial in one variable $x$ given by

$$\operatorname{ch} V = \sum_{m \in \mathbb{Z}} \dim V_m x^m,$$

where $V_m = \{v \in V | h.v = mv\}$.

## 1.2 Associated Affine Lie algebra $\widehat{\mathfrak{sl}_2}$

The associated affine Lie algebra $\widehat{\mathfrak{sl}_2}$ with canonical central element $c$ and scaling operator $d$ can be realized as follows: as vector spaces we have

$$\widehat{\mathfrak{sl}_2} = \mathfrak{sl}_2 \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c \oplus \mathbb{C}d,$$

where $\mathbb{C}[t, t^{-1}]$ is the Laurent polynomial ring in an indeterminate $t$, and the commutator is given by

$$[a \otimes t^r, b \otimes t^s] = [a, b] \otimes t^{r+s} + (a, b) r \delta_{r,-s} c, \qquad [d, a \otimes f] = a \otimes t df/dt, \qquad [c, \widehat{\mathfrak{sl}_2}] = 0 = [d, d],$$

where $(\ ,\ )$ is the Killing form on $\mathfrak{sl}_2$. The action of $d$ can also be regarded as defining a $\mathbb{Z}$–grading on $\widehat{\mathfrak{sl}_2}$ where we declare the grade of $d$ and $c$ to be zero and the grade of $a \otimes t^r$ to be $r$ for $a \in \mathfrak{sl}_2$.

Let $\widehat{\mathfrak{h}} = \mathbb{C}h \oplus \mathbb{C}c \oplus \mathbb{C}d$ be the Cartan subalgebra and define the Borel and the standard maximal parabolic subalgebras by

$$\widehat{\mathfrak{b}} = \mathfrak{sl}_2 \otimes t\mathbb{C}[t] \oplus \mathbb{C}x \oplus \widehat{\mathfrak{h}}, \qquad \widehat{\mathfrak{p}} = \widehat{\mathfrak{b}} \oplus \mathbb{C}y = \mathfrak{sl}_2 \otimes \mathbb{C}[t] \oplus \mathbb{C}c \oplus \mathbb{C}d.$$

Notice that $\widehat{\mathfrak{b}}$ and $\widehat{\mathfrak{p}}$ are $\mathbb{Z}_+$–graded subalgebras of $\widehat{\mathfrak{g}}$. We identify $\mathfrak{sl}_2$ with the grade zero subalgebra $\mathfrak{sl}_2 \otimes 1$ of $\mathfrak{sl}_2 \otimes \mathbb{C}[t]$. Define $\delta \in \widehat{\mathfrak{h}}^*$ by: $\delta(d) = 1, \delta(\mathfrak{h} \oplus \mathbb{C}c) = 0$. Let $\widehat{W}$ be the affine Weyl group associated to $\widehat{\mathfrak{g}}$ and recall that it acts on $\widehat{\mathfrak{h}}$ and $\widehat{\mathfrak{h}}^*$ and leaves $c$ and $\delta$ fixed.

### 1.2.1 Demazure Module associated to Integrable Highest Weight Module

Suppose that $\Lambda \in \widehat{\mathfrak{h}}^*$ is dominant integral: i.e., $\Lambda(h), \Lambda(c - h) \in \mathbb{Z}_+$ and $\Lambda(d) \in \mathbb{Z}$. Let $V(\Lambda)$ be the irreducible integrable highest weight $\widehat{\mathfrak{g}}$–module generated by a highest weight vector $v_\Lambda$. The action of $\widehat{\mathfrak{h}}$ on $V(\Lambda)$ is diagonalizable and the central element $c$ acts via the scalar $\Lambda(c)$ on $V(\Lambda)$. The non–negative integer $\Lambda(c)$ is called the level of $V(\Lambda)$. For all $w \in \widehat{W}$ the element $w\Lambda$ is also an eigenvalue for the action of $\widehat{\mathfrak{h}}$ on $V(\Lambda)$ with corresponding eigenspace $V(\Lambda)_{w\Lambda}$. The Demazure module associated to $w$ and $\Lambda$ is defined to be

$$V_w(\Lambda) = \mathbf{U}(\widehat{\mathfrak{b}}) V(\Lambda)_{w\Lambda}.$$

The Demazure modules are finite–dimensional and if $w\Lambda(h) \leq 0$, then $V_w(\Lambda)$ is a module for $\widehat{\mathfrak{p}}$. *From now on, we shall only be interested in such Demazure modules.* Notice that these Demazure modules are indexed by the integers

$$-s = w\Lambda(h) \leq 0, \quad \ell = \Lambda(c), \quad p = w\Lambda(d),$$

The action of $d$ on the Demazure modules defines a $\mathbb{Z}$–grading on them compatible with $\mathbb{Z}_+$–grading on $\mathfrak{sl}_2[t]$. Moreover, since $w(\Lambda + p\delta) = w\Lambda + p\delta$ and $(\Lambda + p\delta)(\mathfrak{h} \oplus \mathbb{C}cc) = \Lambda(\mathfrak{h} \oplus \mathbb{C}c)$, it follows that for a fixed $\ell$ and $s$ the modules are just grade shifts. If $s = 0$ then $D(\ell, 0)$ is the trivial $\mathfrak{sl}_2[t]$–module.

## 1.3 Category of Finite-dimensional $\mathbb{Z}$-graded $\mathfrak{sl}_2[t]$-modules

As the discussion in Section 1.2.1 shows, the proper setting for our study is the category of finite–dimensional $\mathbb{Z}$–graded $\mathfrak{sl}_2[t]$–modules. We recall briefly some of the elementary definitions and properties of this category. A finite–dimensional $\mathbb{Z}$–graded $\mathfrak{sl}_2[t]$–module is a $\mathbb{Z}$–graded vector space space admitting a compatible graded action of $\mathfrak{sl}_2[t]$:

$$V = \bigoplus_{k \in \mathbb{Z}} V[k], \qquad (a \otimes t^r)V[k] \subset V[k+r] \quad a \in \mathfrak{sl}_2, \quad r \in \mathbb{Z}_+.$$

In particular, $V[r]$ is a module for the subalgebra $\mathfrak{sl}_2$ of $\mathfrak{sl}_2[t]$ and hence the action of $\mathfrak{h}$ on $V[r]$ is semisimple,i.e.,

$$V[r] = \bigoplus_{m \in \mathbb{Z}} V[r]_m, \quad V[r]_m = \{v \in V[r] : hv = mv\}.$$

The *graded character* of $V$ is the Laurent polynomial in two variables $x, q$ given by

$$\operatorname{ch}_{\mathrm{gr}} V = \sum_{r \in \mathbb{Z}} \operatorname{ch} V[r] q^r = \sum_{m, r \in \mathbb{Z}} \dim V[r]_m x^m q^r.$$

A map of graded $\mathfrak{sl}_2[t]$–modules is a degree zero map of $\mathfrak{sl}_2[t]$–modules (i.e. $f : V \to W$ such that $f(V[r]) \subset W[r]\ \forall r \in \mathbb{Z}$. If $V_1$ and $V_2$ are graded $\mathfrak{sl}_2[t]$–modules, then the direct sum and tensor product are again graded $\mathfrak{sl}_2[t]$–modules, with grading,

$$(V_1 \oplus V_2)[k] = V_1[k] \oplus V_2[k], \qquad (V_1 \otimes V_2)[k] = \bigoplus_{s \in \mathbb{Z}} (V_1[s] \otimes V_2[k-s]).$$

The graded character is additive on short exact sequences and multiplicative on tensor products.

Given a $\mathbb{Z}$–graded vector space $V$, we let $\tau_p^* V$ be the graded vector space whose $r$–th graded piece is $V[r+p]$. Clearly, a graded action of $\mathfrak{sl}_2[t]$ on $V$ also makes $\tau_p^* V$ into a graded

$\mathfrak{sl}_2[t]$–module. It is now easy to prove (see [3] for instance) that an irreducible object of this category must be of the form $\tau_p^* V(n)$. It follows that if $V$ is an arbitrary finite–dimensional graded $\mathfrak{sl}_2[t]$–module, then $\mathrm{ch}_{\mathrm{gr}}\, V$ can be written uniquely as a non–negative integer linear combination of $q^p\, \mathrm{ch}_{\mathrm{gr}}\, \tau_0^* V(n)$, $p \in \mathbb{Z}$, $n \in \mathbb{Z}_+$.

## 1.4 Demazure Module for the Current Algebra

We also give the definition of $\tau_r^* D(\ell, s)$ as an $\mathfrak{sl}_2[t]$ module. Let $\ell, s \in \mathbb{Z}_+$ and write $s = \ell s_1 + s_0$ with $s_1 \geq -1$ and $s_0 \in \mathbb{N}$ with $s_0 \leq \ell$. Then $D(\ell, s)$ is generated by an element $v_s$ and defining relations:

$$(x \otimes \mathbb{C}[t])v_s = 0, \quad (h \otimes f)v_s = sf(0)v_s, \qquad (y \otimes 1)^{s+1} v_s = 0, \tag{1.4.1}$$

$$(y \otimes t^{s_1+1})v_s = 0, \qquad (y \otimes t^{s_1})^{s_0+1} v_s = 0, \quad \text{if} \quad s_0 < \ell. \tag{1.4.2}$$

Let $\tau_r^* D(\ell, s)$ be the graded $\mathfrak{sl}_2[t]$– module obtained by defining the grade of the element $v_s$ to be $r$. The following result is a special case of a result established in [5, Theorem 2, Proposition 6.7 ] for $s > 0$.

**Proposition 1.** *Let $\Lambda$ be a dominant integral weight for $\widehat{\mathfrak{h}}$ and let $w \in \widehat{W}$ be such that*

$$\Lambda(c) = \ell, \quad w\Lambda(h) = -s, \quad w\Lambda(d) = r.$$

*We have an isomorphism of graded $\mathfrak{sl}_2[t]$–modules*

$$V_w(\Lambda) \cong \tau_r^* D(\ell, s).$$

□

**Remark:** A few remarks are in order here. In the case when $s_0 = \ell, \ell - 1$ the second relation in equation (1.4.2) is a consequence of the other relations. A presentation of all Demazure modules was given in [6], [8] in the case of simple and Kac–Moody algebras respectively and includes infinitely many relations of the form $(y \otimes t^a)V_w(\Lambda) = 0$. However, it was shown in [5, Theorem 2] that in the case of the $\mathfrak{sl}_2$–stable Demazure modules these relations are all consequences of the ones stated in the proposition.

### 1.4.1 Properties of $D(\ell, s)$

We isolate further results from [5, Section 6] that will be needed for our study.

**Proposition 2.** *Let $\ell, s \in \mathbb{Z}_+$ and write $s = \ell s_1 + s_0$ with $s_1 \geq -1$ and $s_0 \in \mathbb{N}$ with $s_0 \leq \ell$.*

*(i) For $0 \leq s \leq \ell$ we have*

$$D(\ell, s) \cong \tau_0^* V(s), \quad \text{i.e.} \quad , \quad (\mathfrak{sl}_2 \otimes t\mathbb{C}[t])\, D(m, s) = 0.$$

*(ii) For $s > 0$, we have $\dim D(\ell, s) = (\ell + 1)^{s_1}(s_0 + 1)$.*

*(iii) The $\mathfrak{sl}_2[t]$–submodule of $D(\ell, s)$ generated by the element $(y \otimes t^{s_1})^{s_0} v_s$ is isomorphic to $\tau_{s_1 s_0}^* D(\ell, s - 2s_0)$. In particular, the quotient $D(\ell, s)/\tau_{s_1 s_0}^* D(\ell, s - 2s_0)$ is generated by an element $\bar{v}_s$ with defining relations, (1.4.1) and,*

$$(y \otimes t^{s_1+1})\bar{v}_s = 0, \qquad (y \otimes t^{s_1})^{s_0}\bar{v}_s = 0. \tag{1.4.3}$$

□

# Chapter 2

# Demazure Flags

In this chapter, we define the Demazure flag for a $\mathfrak{sl}_2[t]$–module. Using the independence of choice of flag, we define the graded multiplicities for a level $\ell$–Demazure flag of a $\mathfrak{sl}_2[t]$–module. Then, we then give results for when a module has a level $\ell$–Demazure flag and a recursion for computing the graded multiplicities.

## 2.1 Jordan–Holder constituents for a Demazure modules

The following is a straightforward application of the Poincare–Birkhoff–Witt theorem.

**Lemma 1.** *Let $\ell \in \mathbb{N}$ and $s \in \mathbb{Z}_+$. The module $\tau_0^* V(s)$ is the unique irreducible quotient of $D(\ell, s)$ and occurs with multiplicity one in the Jordan–Holder series of $D(\ell, s)$. Moreover, if $\tau_p^* V(m)$, $m \neq s$ is a Jordan–Holder constituent of $D(\ell, s)$ then $p \in \mathbb{N}$ and $s - m \in 2\mathbb{N}$.*$\square$

Let $\ell \in \mathbb{N}$. It follows from the Lemma that if $V$ is a graded finite–dimensional module for $\mathfrak{sl}_2[t]$, then $\operatorname{ch}_{\mathrm{gr}} V$ can be written uniquely as a $\mathbb{Z}[q, q^{-1}]$ linear combination of $\operatorname{ch}_{\mathrm{gr}} D(\ell, s)$, $s \in \mathbb{Z}_+$.

## 2.2 Definition of a Demazure Flag

Let $V$ be a finite–dimensional graded $\mathfrak{sl}_2[t]$–module. We say that a decreasing sequence

$$\mathcal{F}(V) = \{V = V_0 \supsetneq V_1 \supsetneq \cdots V_k \supsetneq V_{k+1} = 0\}$$

of graded $\mathfrak{sl}_2[t]$-submodules of $V$ is a Demazure flag of level $m$, if

$$V_i/V_{i+1} \cong \tau_{p_i}^* D(m, n_i), \quad (n_i, p_i) \in \mathbb{Z}_+ \times \mathbb{Z}, \quad 0 \le i \le k.$$

Given a flag $\mathcal{F}(V)$ we say that the multiplicity of $\tau_p^* D(m,n)$ in $\mathcal{F}(V)$ is the cardinality of the set $\{j : V_j/V_{j+1} \cong \tau_p^* D(m,n)\}$. It is not hard to show that the cardinality of this set is independent of the choice of the Demazure flag (see for instance [4, Lemma 2.1]) of $V$ and we denote this number by $[V : \tau_p^* D(m,n)]$. Define

$$[V : D(m,n)]_q = \sum_{p\in\mathbb{Z}} [V : \tau_p^* D(m,n)] q^p, \quad n \ge 0, \quad [V : D(m,n)]_q = 0, \quad n < 0.$$

It is known that

$$[D(\ell, s) : D(m,s)]_q = 1, \qquad [D(\ell,s) : D(m,n)]_q = 0 \quad s - n \notin 2\mathbb{Z}_+.$$

Moreover, for $m \ge \ell' \ge \ell$ we have

$$[D(\ell, s) : D(m,n)]_q = \sum_{p\in\mathbb{Z}_{\ge 0}} [D(\ell,s) : D(\ell',p)]_q \, [D(\ell',p) : D(m,n)]_q. \tag{2.2.1}$$

It follows from the discussion in Section 1.3 and Section 2.1 that if $V$ admits a Demazure flag of level $m$, then

$$\mathrm{ch}_{\mathrm{gr}} V = \sum_{s\in\mathbb{Z}} [V : D(m,s)]_q \, \mathrm{ch}_{\mathrm{gr}} D(m,s). \tag{2.2.2}$$

The following result was first proved in [9] for Demazure modules for arbitrary simply–laced simple algebras using the theory of canonical basis. An alternate more constructive and self contained proof was given in [4] for $\mathfrak{sl}_2[t]$.

**Proposition 3.** *Let $\ell$ be a positive integer. For all non–negative integers $s$ and $m$ with $m \ge \ell$, the module $D(\ell, s)$ has a Demazure flag of level $m$.* □

## 2.3 Demazure Flag of a Quotient of Demazure Modules

Theorem 3.3 of [4] shows that there is a very large class of modules admitting a Demazure flag of level $m$. We do not state that result in full generality since it requires introducing a significant amount of notation which is not needed in this paper. In the special case we need, Theorem 3.3 and Lemma 3.8 of [4] give the first and second parts of the next proposition.

**Proposition 4.** *Let $\ell \in \mathbb{N}$ and $s = \ell s_1 + s_0$ with $s_1, s_0 \in \mathbb{Z}_+$ and $0 < s_0 \leq \ell$.*

*(1) Consider the embedding $\tau^*_{s_1 s_0} D(\ell, s - 2s_0) \hookrightarrow D(\ell, s)$. The corresponding quotient admits a Demazure flag of level $m$ for all $m > \ell$.*

*(2) We have*

$$[D(\ell, s)/\tau^*_{s_1 s_0} D(\ell, s-2s_0) : D(\ell+1, n)]_q = q^{(s-n)/2}[D(\ell, s-\ell-1)) : D(\ell+1, n-\ell-1)]_q.$$

□

The following corollary is immediate.

**Corollary 1.** *Keep the notation of the proposition. We have*

$$[D(\ell, s) : D(\ell + 1, n)]_q = q^{s_1 s_0}[D(\ell, s - 2s_0) : D(\ell + 1, n)]_q$$
$$+q^{(s-n)/2}[D(\ell, s - \ell - 1)) : D(\ell + 1, n - \ell - 1)]_q.$$

## 2.4 Recursion of Graded Multliplicities

We can now prove the following proposition.

**Proposition 5.** *Let $\ell$ be a positive integer.*

*(i) for $0 \leq n, k \leq \ell$, we have $[D(\ell, k) : D(\ell + 1, n)]_q = \delta_{k,n}$ and*

$$[D(\ell, 2\ell j \pm k) : D(\ell + 1, n)]_q = \delta_{k,n}\, q^{j(\ell j \pm n)}, \quad j \in \mathbb{N}.$$

*(ii) if $n \geq \ell + 1$ and $s_0 \in \mathbb{N}$ with $s_0 \leq \ell$ and $s_1 \in \mathbb{Z}_+$, we have*

$$[D(\ell, \ell s_1 + s_0) : D(\ell + 1, n)]_q = q^{s_0 s_1}[D(\ell, \ell(s_1 - 1) + (\ell - s_0)) : D(\ell + 1, n)]_q$$
$$+ q^{(\ell s_1 + s_0 - n)/2}[D(\ell, \ell(s_1 - 1) + (s_0 - 1)) : D(\ell + 1, n - (\ell + 1))]_q.$$

*Proof.* To prove part (i) of the proposition we proceed by induction on $j$. Since $0 \leq n \leq \ell$ we have by Proposition 2(i) that

$$D(\ell, n) \cong D(\ell + 1, n) \cong \tau^*_0 V(n),$$

and so, if $0 \le k \le \ell$, we get $[D(\ell, k) : D(\ell+1, n)]_q = \delta_{k,n}$. This shows that induction begins and for the inductive step we assume that

$$[D(\ell, 2j'\ell + k) : D(\ell + 1, n)]_q = \delta_{k,n} q^{j'(j'\ell+n)},$$

holds for all $0 \le j' < j$ and all $0 \le k, n \le \ell$. Using Corollary 4 and noting that the second term on the right hand side is zero since $n \le \ell$, and using the inductive hypothesis, we get

$$\begin{aligned}
[D(\ell, 2j\ell + k)) : D(\ell + 1, n)]_q &= q^{2kj}[D(\ell, 2j\ell - k) : D(\ell + 1, n)]_q \\
&= q^{2kj} q^{(2j-1)(\ell-k)}[D(\ell, 2(j-1)\ell + k) : D(\ell + 1, n)]_q \\
&= \delta_{k,n} q^{2nj+(2j-1)(\ell-n)+(j-1)(\ell(j-1)+n)} = \delta_{k,n} q^{j(\ell j+n)}.
\end{aligned}$$

This proves the inductive step. It also proves that if $j \ge 1$, then

$$\begin{aligned}
[D(\ell, 2j\ell - k) : D(\ell + 1, n)_q &= q^{(\ell-k)(2j-1)}[D(\ell, 2(j-1)\ell + k) : D(\ell + 1, n)]_q \\
&= \delta_{k,n} q^{(\ell-n)(2j-1)} q^{(j-1)((j-1)\ell+n)} = \delta_{k,n} q^{j(j\ell-n)}.
\end{aligned}$$

This completes the proof of part (i). Part (ii) is precisely the statement of Corollary 4. □

## 2.5 Example of a Demazure Flag

In this section, we give an example of how a flag is constructed using Proposition 4 and identify the graded multiplicities associated with this flag.

Let us consider $D(3, 7)$ and look at the Demazure flag of level 4 of this module. Let $v$ be the nonzero generator of the $D(3, 7)$ with the relations given in Proposition 2(iii). We have the following increasing sequence of graded $\mathfrak{sl}_2[t]$-submodules

$$0 \subsetneq \mathbf{U}(\mathfrak{sl}_2[t])((y \otimes t)^2(y \otimes t^2)v) \subsetneq \mathbf{U}(\mathfrak{sl}_2[t])((y \otimes t^2)v) \subsetneq \mathbf{U}(\mathfrak{sl}_2[t])v = D(3, 7).$$

It can easily be shown that we have the following surjective maps of graded $\mathfrak{sl}_2[t]$ modules:

$$\tau_4 D(4, 1) \to \mathbf{U}(\mathfrak{sl}_2[t])((y \otimes t)^2(y \otimes t^2)v),$$

$$\tau_2 D(4, 5) \to \mathbf{U}(\mathfrak{sl}_2[t])((y \otimes t^2)v)/\mathbf{U}(\mathfrak{sl}_2[t])((y \otimes t)^2(y \otimes t^2)v),$$

$$D(4, 7) \to \mathbf{U}(\mathfrak{sl}_2[t])v/\mathbf{U}(\mathfrak{sl}_2[t])((y \otimes t^2)v).$$

Using a dimension argument along with Proposition 2(ii) gives up that these surjections are actually isomorphisms. Therefore, we can explicitly write out the graded multiplicities as

$$[D(3,7) : D(4,n)]_q = \begin{cases} q^4, & n = 1 \\ q^2, & n = 5 \\ 1, & n = 7 \\ 0, & \text{otherwise.} \end{cases}$$

# Chapter 3

# Graded Multiplicities

In this chapter, we study the graded multiplicities of level $\ell$–Demazure flags of level $m$–Demazure flags for $\ell = 2, 3$ with $m = 1, 2$ respectively. We use the recursion from Chapter 2 along with generating functions to determine closed forms for these graded multiplicities. Moreover, we relate these generating functions to partial theta series. Lastly, we connect these multiplicities to the multiplicities for the level 2-Demazure flag of the local Weyl module for type $G_2$ using the results of Naoi.

## 3.1 Notation

To begin this chapter, we introduce notation for $q$-binomial (or Gaussian) coefficients. Given $n \in \mathbb{Z}_+$ and $m \in \mathbb{Z}$, set

$$\begin{bmatrix} n \\ m \end{bmatrix}_q = \frac{(1-q^n)...(1-q^{n-m+1})}{(1-q)...(1-q^m)}, \quad m > 0,$$

$$\begin{bmatrix} n \\ 0 \end{bmatrix}_q = 1, \quad \begin{bmatrix} n \\ m \end{bmatrix}_q = 0, \quad m < 0.$$

Notice that $\begin{bmatrix} n \\ n-m \end{bmatrix}_q = \begin{bmatrix} n \\ m \end{bmatrix}_q$.

To understand the polynomials $[D(\ell, s) : D(m, n)]_q$, we consider the associated generating series: given $\ell, m \in \mathbb{N}$ with $m \geq \ell$, set

$$A_n^{\ell \to m}(x, q) = \sum_{k \geq 0} [D(\ell, n+2k) : D(m, n)]_q \, x^k, \quad n \geq 0.$$

It will be convenient to set $A_{-1}^{1 \to m}(x, 1) = 1$.

## 3.2 $q$-identities

In this section, we state some $q$-identities with proofs that will be used to verfiy the results in the remainder of this chapter.

**Lemma 3.2.1** (Analogs of Pascal's Identity)**.** *For $n, m \in \mathbb{Z}$, we have the following identities:*

*(i)*
$$\begin{bmatrix} n \\ m \end{bmatrix}_q = q^m \begin{bmatrix} n-1 \\ m \end{bmatrix}_q + \begin{bmatrix} n-1 \\ m-1 \end{bmatrix}_q,$$

*(ii)*
$$\begin{bmatrix} n \\ m \end{bmatrix}_q = \begin{bmatrix} n-1 \\ m \end{bmatrix}_q + q^{n-m} \begin{bmatrix} n-1 \\ m-1 \end{bmatrix}_q.$$

*Proof.* For a simple proof, use the definition of $\left[\begin{smallmatrix} n-1 \\ m \end{smallmatrix}\right]_q$ and $\left[\begin{smallmatrix} n-1 \\ m-1 \end{smallmatrix}\right]_q$ to get

$$\begin{aligned} q^m \begin{bmatrix} n-1 \\ m \end{bmatrix}_q + \begin{bmatrix} n-1 \\ m-1 \end{bmatrix}_q &= \frac{(1-q^{n-1})\cdots(1-q^{n-m+1})}{(1-q)\cdots(1-q^m)} \big(q^m(1-q^{n-m}) + (1-q^m)\big) \\ &= \frac{(1-q^{n-1})\cdots(1-q^{n-m+1})}{(1-q)\cdots(1-q^m)} \big(1-q^n\big) = \begin{bmatrix} n \\ m \end{bmatrix}_q. \end{aligned}$$

The proof for the other identity is similar. □

We refer the reader to [11] for a combinatorial proof using the fact that $\left[\begin{smallmatrix} n \\ m \end{smallmatrix}\right]_q$ is the number of $m$-dimensional subspaces of a $n$-dimensional vector space over $\mathbb{F}_q$.

**Theorem 3.2.2** ($q$-Binomial Theorem)**.** *For $n \geq 0$, we have*

$$\sum_{p=0}^{n} q^{p(p-1)/2} \begin{bmatrix} n \\ p \end{bmatrix}_q x^p = (-x; q)_n.$$

*Proof.* We prove this identity by induction on $n$. For $n = 0$, this equality trivially holds.

Now, assume the equality holds for $n = k - 1$. For $n = k$, we have

$$\begin{aligned}
(-x;q)_k &= (1 + xq^{k-1})(-x;q)_{k-1} \\
&= (1 + xq^{k-1}) \sum_{p=0}^{k-1} q^{p(p-1)/2} \begin{bmatrix} k-1 \\ p \end{bmatrix}_q x^p \\
&= \sum_{p=0}^{k-1} q^{p(p-1)/2} \begin{bmatrix} k-1 \\ p \end{bmatrix}_q x^p + \sum_{p=0}^{k-1} q^{k-1+p(p-1)/2} \begin{bmatrix} k-1 \\ p \end{bmatrix}_q x^{p+1} \\
&= \sum_{p=0}^{k-1} q^{p(p-1)/2} \begin{bmatrix} k-1 \\ p \end{bmatrix}_q x^p + \sum_{p=1}^{k} q^{k-1+(p-1)(p-2)/2} \begin{bmatrix} k-1 \\ p-1 \end{bmatrix}_q x^p \\
&= \sum_{p=0}^{k-1} q^{p(p-1)/2} \begin{bmatrix} k-1 \\ p \end{bmatrix}_q x^p + \sum_{p=1}^{k} q^{k-1+(p-1)(p-2)/2} \begin{bmatrix} k-1 \\ p-1 \end{bmatrix}_q x^p \\
&= \sum_{p=0}^{k} x^p q^{p(p-1)/2} \left( \begin{bmatrix} k-1 \\ p \end{bmatrix}_q + q^{k-p} \begin{bmatrix} k-1 \\ p-1 \end{bmatrix}_q \right).
\end{aligned}$$

We can use Lemma 3.2.1 to get the identity for $n = k$ thus completing the proof. □

In the following sections, we give closed formulae for special cases of $\ell$ and $m$.

## 3.3 Level 2–Demazure flag of level 1-Demazure modules

In certain special cases, it is possible to write down the polynomials $[D(\ell, s) : D(m, n)]_q$ explicitly as sums of products of $q$–binomials, i.e., by fermionic formulae. If $\ell = 1$ and $m = 2$, it was shown in [4], that for all $k, n \in \mathbb{Z}_+$, we have

$$[D(1, n + 2k) : D(2, n)]_q = q^{k\lceil (n+2k)/2 \rceil} \begin{bmatrix} \lfloor (n + 2k)/2 \rfloor \\ k \end{bmatrix}_q . \tag{3.3.1}$$

## 3.4 Level 3–Demazure flag of level 2-Demazure modules

**Proposition 6.** *For $r \in \{0, 1, 2\}$ and $s \in \mathbb{Z}_+$ , set*

$$\overline{r} = \begin{cases} 1 & r = 1 \\ 0 & r = 0,\ 2 \end{cases} , \qquad s' = \left\lfloor \frac{s + 1 + \overline{r}}{2} \right\rfloor .$$

*For all $p \in \mathbb{Z}_+$, we have*

$$[D(2, 3s + r + 2p) : D(3, 3s + r)]_q = q^{\frac{1}{2}(p^2 + p(2s+r))} \sum_{\substack{j=0 \\ j \equiv p \,(\mathrm{mod}\ 2)}}^{s'} q^{j(j-\overline{r})/2} \begin{bmatrix} \frac{p-j}{2} + s \\ s \end{bmatrix}_q \begin{bmatrix} s' \\ j \end{bmatrix}_q .$$

Preliminary work using [10] assisted in the identification of the closed formulae in the proposition.

To prove this proposition, we will first create a recursion of generating functions using 5. Then, we will verify the graded multiplicities.

## 3.5 Recursion of Generating Functions $A_n^{2\to 3}(x, q)$

We first use Proposition 5 to give closed formulae for $A_n^{2\to 3}(x, q)$. In terms of generating series, Proposition 5(i) gives,

$$A_0^{2\to 3}(x, q) = \sum_{s\geq 0} q^{2s^2} x^{2s}, \quad A_1^{2\to 3}(x, q) = \sum_{s\geq 0} q^{\frac{s(s+1)}{2}} x^s, \quad A_2^{2\to 3}(x, q) = \sum_{s\geq 0} q^{2s(s+1)} x^{2s}. \tag{3.5.1}$$

and Proposition 5(ii) gives for $k \geq 3$,

$$A_{k-3}^{2\to 3}(xq, q) = \begin{cases} A_k^{2\to 3}(x, q) - xq^{\frac{k+1}{2}} A_k^{2\to 3}(xq, q) & \text{if } k \text{ is odd.} \\ \\ A_k^{2\to 3}(x, q) - x^2 q^{k+2} A_k^{2\to 3}(xq^2, q) & \text{if } k \text{ is even.} \end{cases} \tag{3.5.2}$$

We have the following result which solves this recurrence explicitly.

**Proposition 3.5.1.** *Let $r \in \{0, 1, 2\}$ and $s \geq 0$, and set*

$$\overline{r} = \begin{cases} 1, & r = 1, \\ 0, & r = 0,\ 2. \end{cases}$$

*Then, we have*

$$A_{3s+r}^{2\to 3}(x, q) = \sum_{p=0}^{\infty} x^p \, q^{\frac{1}{2}(p^2+p(2s+r))} \sum_{\substack{j=0 \\ j\equiv p \,(\mathrm{mod}\ 2)}}^{p} q^{j(j-\overline{r})/2} \begin{bmatrix} \frac{p-j}{2} + s \\ s \end{bmatrix}_q \begin{bmatrix} \left\lfloor \frac{s+1+\overline{r}}{2} \right\rfloor \\ j \end{bmatrix}_q . \tag{3.5.3}$$

## 3.6 Proof of Proposition 3.5.1

We check the initial conditions first. Let $s = 0$; in this case, the inner sum in equation (3.5.3) equals 1 if either (i) $\overline{r} = 1$, or (ii) $\overline{r} = 0$ and $p \equiv 0 \pmod 2$, and is zero otherwise. From this, it is clear that (3.5.3) reduces to equations (3.5.1) when $s = 0$.

Next, for $s \geq 1$, we verify that the recurrence relation (3.5.2) holds. We set $\alpha(s) = \left\lfloor \frac{s+\overline{r}+1}{2} \right\rfloor$ and $\beta(j,p,s) = \frac{p-j}{2} + s$.

First, suppose $k = 3s + r$ is even. Then, $s + \overline{r}$ is even, and the recurrence in Equation (3.5.2) is equivalent to the statement that the following sum vanishes for all $p \geq 0$:

$$\sum_{\substack{j=0 \\ j \equiv p \,(\mathrm{mod}\ 2)}}^{p} q^{j(j-\overline{r})/2} \begin{bmatrix} \alpha(s) \\ j \end{bmatrix}_q \left( \begin{bmatrix} \beta(j,p,s) \\ s \end{bmatrix}_q - \begin{bmatrix} \beta(j,p,s-1) \\ s-1 \end{bmatrix}_q - q^s \begin{bmatrix} \beta(j,p-1,s) \\ s \end{bmatrix}_q \right). \quad (3.6.1)$$

Notice that $\beta(j,p,s-1) = \beta(j,p-1,s) = \beta(j,p,s) - 1$. But, from the $q$-binomial identity 3.2.1(ii) with $n = \beta(j,p,s)$, $m = \beta(j,p,s) - s$, we see that each summand in (3.6.1) is in fact zero. This proves the recurrence relation for $k$ even.

Next, let $k = 3s + r$ be odd. In this case, the recurrence relation of Equation (3.5.2) is equivalent to the statement that the following sum vanishes for all $p \geq 0$:

$$\sum_{\substack{j=0 \\ j \equiv p (\mathrm{mod}\ 2)}}^{p} q^{\frac{j(j-\overline{r})}{2}} \left( \begin{bmatrix} \beta(j,p,s) \\ s \end{bmatrix}_q \begin{bmatrix} \alpha(s) \\ j \end{bmatrix}_q - q^{\alpha(s)-j} \begin{bmatrix} \beta(j,p,s) \\ s \end{bmatrix}_q \begin{bmatrix} \alpha(s) \\ j-1 \end{bmatrix}_q - \begin{bmatrix} \beta(j,p,s-1) \\ s-1 \end{bmatrix}_q \begin{bmatrix} \alpha(s-1) \\ j \end{bmatrix}_q \right). \quad (3.6.2)$$

Notice that $\beta(j,p,s-1) = \beta(j,p,s) - 1$ and $\alpha(s-1) = \alpha(s) - 1$ since $s + \overline{r}$ is odd. Using the identity 3.2.1(ii) twice in succession, first with $(n,m) = (\alpha(s), j)$ and then with $(n,m) = (\alpha(s), j-1)$, we obtain:

$$\begin{bmatrix} \alpha(s) \\ j \end{bmatrix}_q = \begin{bmatrix} \alpha(s) - 1 \\ j \end{bmatrix}_q + q^{\alpha(s)-j} \begin{bmatrix} \alpha(s) \\ j-1 \end{bmatrix}_q - q^{2(\alpha(s)-j)+1} \begin{bmatrix} \alpha(s) - 1 \\ j-2 \end{bmatrix}_q. \quad (3.6.3)$$

Similarly, choosing $n = \beta(j,p,s)$ and $m = \beta(j,p,s) - s$ in Lemma3.2.1(ii) gives:

$$\begin{bmatrix} \beta(j,p,s) - 1 \\ s-1 \end{bmatrix}_q = \begin{bmatrix} \beta(j,p,s) \\ s \end{bmatrix}_q - q^s \begin{bmatrix} \beta(j,p,s) - 1 \\ s \end{bmatrix}_q. \quad (3.6.4)$$

Substituting Equations (3.6.3), (3.6.4) into the first and third terms of (3.6.2) respectively, and simplifying, the expression in (3.6.2) becomes

$$\sum_{\substack{j=0 \\ j \equiv p \,(\mathrm{mod}\ 2)}}^{p} q^{\frac{j(j-\overline{r})}{2}} \begin{bmatrix} \beta(j,p,s) - 1 \\ s \end{bmatrix}_q \begin{bmatrix} \alpha(s) - 1 \\ j \end{bmatrix}_q - \sum_{\substack{j=0 \\ j \equiv p \,(\mathrm{mod}\ 2)}}^{p} q^{\frac{(j-2)(j-\overline{r}-2)}{2}} \begin{bmatrix} \beta(j,p,s) \\ s \end{bmatrix}_q \begin{bmatrix} \alpha(s) - 1 \\ j-2 \end{bmatrix}_q.$$

Re-indexing the second sum with $j' = j - 2$ proves this expression is zero. This completes the proof.

### 3.6.1 Simplification of Closed Form

We are now able to deduce Proposition 6. We define $s' = \lfloor \frac{s+1+\overline{r}}{2} \rfloor$. We first note that for $j > \min(p, s')$,

$$\begin{bmatrix} \frac{p-j}{2} + s \\ s \end{bmatrix}_q \begin{bmatrix} s' \\ j \end{bmatrix}_q = 0,$$

and thus we can take the inner summation in (3.5.3) from $j = 0$ to $j = s'$ with $j \equiv p \pmod 2$. Extracting the coefficient of $x^p$ in equation (3.5.3), we obtain the explicit polynomial for $[D(2, 3s + r + 2p) : D(3, 3s + r)]_q$ in Proposition 6.

## 3.7 Level 3–Demazure flag of a level 1–Demazure module

Using equation (2.2.1) with $\ell = 1$, $\ell' = 2$ and $m = 3$ and the formulae in (3.3.1) and Proposition 6 we get:

$$A^{1\to 3}_{3s+r}(x, q) = \sum_{n=0}^{\infty} \sum_{p=0}^{n} \sum_{\substack{j=0 \\ j\equiv p \\ (\mathrm{mod}\ 2)}}^{p} x^n\, q^{\frac{1}{2}\gamma(n,p,j)} \begin{bmatrix} n + \lfloor \frac{3s+r}{2} \rfloor \\ n-p \end{bmatrix}_q \begin{bmatrix} \frac{p-j}{2} + s \\ s \end{bmatrix}_q \begin{bmatrix} s' \\ j \end{bmatrix}_q \tag{3.7.1}$$

where $\gamma(n, p, j) = \left(n^2 + (n - p)^2 + j^2\right) + n\,(2s + r) + (n - p)\left(2\lceil \frac{s-r}{2} \rceil + r\right) + j\,\left(-2\lceil \frac{r}{2} \rceil + r\right)$.

We isolate the formulae for $A^{1\to 3}_{3s+r}(x, q)$ for $3s + r = 0, 1, 2$. In equation (3.7.1), when $3s + r = 0$, we have $s = r = \overline{r} = s' = 0$ and hence $\begin{bmatrix} s' \\ j \end{bmatrix}_q = 0$ unless $j = 0$. We also have $\gamma(n, p, 0) = n^2 + (n - p)^2$. Reindexing by $p \mapsto n - p$, we have

$$A^{1\to 3}_{0}(x, q) = \sum_{n=0}^{\infty} x^n\, q^{n^2/2} \sum_{\substack{p=0 \\ p\equiv n \\ (\mathrm{mod}\ 2)}}^{n} q^{p^2/2} \begin{bmatrix} n \\ p \end{bmatrix}_q. \tag{3.7.2}$$

For $3s + r = 2$, we have $s = \overline{r} = s' = 0$, $r = 2$ and $\gamma(n, p, 0) = n^2 + (n - p)^2 + 2n - p$. Using the same reasoning in the previous case, we obtain

$$A^{1\to 3}_{2}(x, q) = \sum_{n=0}^{\infty} x^n\, q^{\frac{n^2+2n}{2}} \sum_{\substack{p=0 \\ p\equiv n \\ (\mathrm{mod}\ 2)}}^{n} q^{\frac{p^2}{2}} \begin{bmatrix} n+1 \\ p \end{bmatrix}_q = (xq^{\frac{1}{2}})^{-1} \sum_{n=1}^{\infty} x^n\, q^{\frac{n^2}{2}} \sum_{\substack{p=0 \\ p\not\equiv n \\ (\mathrm{mod}\ 2)}}^{n} q^{\frac{p^2}{2}} \begin{bmatrix} n \\ p \end{bmatrix}_q. \tag{3.7.3}$$

Lastly, for $3s+r=1$, we have $s=0$ and $r=\overline{r}=s'=1$ thus $\left[{s' \atop j}\right]_q=0$ unless $j=0,1$. Since $\gamma(n,p,0)=n^2+(n-p)^2+2n-p=\gamma(n,p,1)$, we have

$$A_1^{1\to 3}(x,q)=\sum_{n=0}^{\infty}x^n\,q^{\frac{n(n+1)}{2}}\sum_{p=0}^{n}q^{\frac{p(p+1)}{2}}\begin{bmatrix}n\\p\end{bmatrix}_q. \tag{3.7.4}$$

**Remark**: In [2], we showed that certain specializations of $A_n^{1\to 3}(x,q)$ reduce to expressions involving the fifth order mock theta functions of Ramanujan.

## 3.8 Multiplicities of level 1- Demazure flag of a local Weyl module for type $G_2$

Using the work of Naoi (see [9]), we are able to determine the graded multiplicities associated to the level 1-Demazure flags of local Weyl modules for $\mathfrak{g}$ of type $G_2$. First, we give the definition of the local Weyl module and the Demazure module as modules for the current algebra.

### 3.8.1 Structure of $\mathfrak{g}$ of type $G_2$

Fix $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$ and let $R$ be the set of roots of $\mathfrak{g}$ with respect to $\mathfrak{h}$. The restriction of the Killing form of $\mathfrak{g}$ to $\mathfrak{h}$ induces an isomorphism between $\mathfrak{h}$ and $\mathfrak{h}^*$ and hence also a symmetric non–degenerate form $(\ ,\ )$ on $\mathfrak{h}^*$. We shall assume that this form on $\mathfrak{h}^*$ is normalized so that the square length of a long root is two and for $\alpha\in R$ set $d_\alpha=\frac{2}{(\alpha,\alpha)}$. Fix a set $\{\alpha_i : i\in\{1,2\}\}$ of simple roots for $R$ and let $\{\omega_i : i\in\{1,2\}\}\subset\mathfrak{h}^*$ be the set of fundamental weights. Let $Q$ (resp. $Q^+$) be the $\mathbb{Z}$ span (resp. the $\mathbb{Z}_+$ span) of $\{\alpha_i : i\in\{1,2\}\}$ and similarly define $P$ (resp. $P^+$) to be the $\mathbb{Z}$ (resp. $\mathbb{Z}_+$) span of $\{\omega_i : i\in\{1,2\}\}$ and set $R^+=R\cap Q^+$. Define a partial order on $P^+$ by $\lambda\geq\mu$ if and only if $\lambda-\mu\in Q^+$.

Finally, let $\{x_\alpha^\pm, h_i : \alpha\in R^+, i\in\{1,2\}\}$ be a Chevalley basis for $\mathfrak{g}$ and set $x_i^\pm=x_{\alpha_i}^\pm$ for $i\in\{1,2\}$.

### 3.8.2 Local Weyl Module

We can now define the local Weyl module of highest weight $\lambda\in P^+$, denoted $W_{\mathrm{loc}}(\lambda)$, as the $\mathfrak{g}[t]$–module generated by an element $w_\lambda$ with defining relations: for $i\in\{1,2\}$ and

$s \in \mathbb{Z}_+$,

$$(x_i^+ \otimes \mathbb{C}[t])w_\lambda = 0, \quad (h_i \otimes t^s)w_\lambda = \lambda(h_i)\delta_{s,0}w_\lambda, \tag{3.8.1}$$

$$(x_i^- \otimes 1)^{\lambda(h_i)+1}w_\lambda = 0. \tag{3.8.2}$$

Moreover, declaring the grade of $w_\lambda$ to be zero makes the local Weyl module a graded $\mathfrak{g}[t]$-module.

### 3.8.3 Demazure Module

Then, the Demazure module of level $\ell$ and weight $\lambda$, denoted $D(\ell, \lambda)$, is the graded quotient of $W_{\text{loc}}(\lambda)$ by the submodule generated by the elements

$$\{(x_\alpha^- \otimes t^p)^{r+1}w_\lambda : \quad p \in \mathbb{Z}_+,\ r \geq \max\{0, \lambda(h_\alpha - d_\alpha \ell p, \quad \text{for } \alpha \in R^+\}.$$

### 3.8.4 Reformulation of Result of Naoi

The next result is a reformulation of the result proved in [9, Section 4, Section 9] of which the special case will be of interest to us.

**Proposition 7.** *Suppose that $\mathfrak{g}$ is of type $G_2$. Let $\lambda \in P^+$ be such that $\lambda(h_2) = r \in \mathbb{Z}_+$ where $\alpha_2$ is the unique short root. Then, $W_{\text{loc}}(\lambda)$ has a Demazure flag of level one and for $\mu \in P^+$,*

$$[W_{\text{loc}}(\lambda)) : D(1, \mu)]_q = \begin{cases} [D^{\mathfrak{sl}_2}(1, \lambda(h_n)) : D^{\mathfrak{sl}_2}(3, \mu(h_n))]_q, & (\lambda - \mu) \in \mathbb{Z}_+\alpha_2, \\ \\ 0, \quad \text{otherwise}. \end{cases} \tag{3.8.3}$$

*Here $D^{\mathfrak{sl}_2}(\ell, s)$ is the Demazure module for $\mathfrak{sl}_2$ of weight $s$.* □

Then combining the multiplicities we found in the previous section with this Proposition, we obtain the following.

**Corollary 2.** *Let $\mathfrak{g}$ be of type $G_2$. The graded character of the local Weyl module of highest weight $\lambda \in P^+$ is a $\mathbb{Z}[q]$-linear combination of the graded characters of level one Demazure* *modules. Explicitly, we have*

$$\mathrm{ch}_{\mathrm{gr}}\, W_{\mathrm{loc}}(\lambda)$$

$$= \sum_{\substack{\mu\in P^+,\\ (\lambda-\mu)\in\mathbb{Z}_+\alpha_2}} \sum_{p=0}^{\frac{\lambda(h_2)-\mu(h_2)}{2}} \sum_{\substack{j=0\\ j\equiv p\\ (\mathrm{mod}\ 2)}}^{p} q^{\frac{1}{2}\gamma((\lambda(h_2)-\mu(h_2)/2),p,j)} \begin{bmatrix} \lfloor\frac{\lambda(h_2)}{2}\rfloor \\ \frac{\lambda(h_2)-\mu(h_2)-2p}{2}\end{bmatrix}_q \begin{bmatrix} \frac{p-j}{2}+\lfloor\frac{\mu(h_2)}{3}\rfloor \\ \lfloor\frac{\mu(h_2)}{3}\rfloor\end{bmatrix}_q \begin{bmatrix} s' \\ j\end{bmatrix}_q \mathrm{ch}_{\mathrm{gr}}\, D(1,\mu),$$

*where* $\gamma(n,p,j) = \left(n^2+(n-p)^2+j^2\right)+n\,(2s+r)+(n-p)\left(2\lceil\frac{s-r}{2}\rceil+r\right)+j\,\left(-2\lceil\frac{r}{2}\rceil+r\right)$ *and*

$$s' = \begin{cases} \lfloor\frac{\lfloor\frac{\mu(h_2)}{3}\rfloor+1}{2}\rfloor, & \mu(h_2)\not\equiv 1 \mod 3 \\ \lfloor\frac{\lfloor\frac{\mu(h_2)}{3}\rfloor+2}{2}\rfloor, & \mu(h_2)\equiv 1 \mod 3 \end{cases}$$

## 3.9 Partial Theta Series

Recall that the partial theta function and the $q$-Pochammer symbol $(a;\,q)_n$ are given by,

$$\Theta(q,z) = \sum_{k=0}^{\infty} q^{k^2} z^k, \qquad (a;\,q)_n = \prod_{i=1}^{n}(1-aq^{i-1}),\ n>0,\ \ (a;\,q)_0 = 1.$$

We refer the reader to [1] for more details regarding partial theta functions. We now use the fermionic formulae to prove,

**Theorem 1.** *Let* $s \geq 0$*.*

*(i) For* $r\in\{0,1\}$*, we have*

$$A^{1\to 2}_{2s+r}(x,q) = \frac{1}{(q;\,q)_s}\sum_{i=0}^{s}(-1)^i q^{\frac{i(i+1)}{2}}\begin{bmatrix} s\\ i\end{bmatrix}_q \Theta\left(q,\ xq^{i+s+r}\right). \tag{3.9.1}$$

*(ii) For* $r\in\{0,1,2\}$*, we have*

$$A^{2\to 3}_{3s+r}(x,q) = \frac{1}{(q;\,q)_s}\sum_{i=0}^{s}\sum_{j=0}^{s'}(-1)^i\, x^j\, q^{\beta(i,j)}\begin{bmatrix} s\\ i\end{bmatrix}_q \begin{bmatrix} s'\\ j\end{bmatrix}_q \Theta\left(q^2,\ x^2q^{\alpha(i,j)}\right), \tag{3.9.2}$$

*where*

$$\beta(i,j) = \frac{i(i+1)}{2}+j^2+j\left(s+\frac{r-\overline{r}}{2}\right),$$

$$\alpha(i,j) = i+2j+2s+r.$$

*Proof.* Using the the $q$-binomial theorem 3.2.2 we get

$$\begin{bmatrix} k+s \\ s \end{bmatrix}_q = \frac{(q^{k+1};\, q)_s}{(q;\, q)_s} = \frac{1}{(q;\, q)_s} \sum_{i=0}^{s} (-q^k)^i \begin{bmatrix} s \\ i \end{bmatrix}_q q^{i(i+1)/2}. \tag{3.9.3}$$

Equation (3.3.1) gives

$$A^{1\to 2}_{2s+r}(x,q) = \sum_{k=0}^{\infty} x^k q^{k(k+s+r)} \begin{bmatrix} k+s \\ s \end{bmatrix}_q, \tag{3.9.4}$$

for $s \geq 0$, $r \in \{0, 1\}$ and using (3.9.3) gives part (i).

Similarly, for part (ii), we begin with (3.5.3) and use the change in the inner summation as discussed in section 3.6.1. Then, we interchange the order of summation and let $p = j + 2k$ with $k \geq 0$ to get

$$A^{2\to 3}_{3s+r}(x,q) = \sum_{j=0}^{s'} x^j q^{j^2 + j(s + \frac{r - \overline{r}}{2})} \begin{bmatrix} s' \\ j \end{bmatrix}_q \sum_{k=0}^{\infty} x^{2k} q^{2k^2 + 2jk + k(2s+r)} \begin{bmatrix} k+s \\ s \end{bmatrix}_q.$$

Now, we use (3.9.3) to finish the proof of part (ii). □

## 3.10 Graded Multiplicities for arbitrary $\ell$

We finish this chapter with a discussion of the graded multiplicities of a level $\ell + 1$-Demazure flag of $D(\ell, s)$ for arbitrary $\ell$. Using Proposition 5(i), we can compute $A^{\ell\to\ell+1}_n(x,q)$ for $0 \leq n \leq \ell$ and write these generating series as (sums of) partial theta functions:

$$A^{\ell\to\ell+1}_0(x,q) = \Theta(q^\ell, x^\ell), \qquad A^{\ell\to\ell+1}_\ell(x,q) = \Theta(q^\ell, x^\ell q^\ell),$$
$$A^{\ell\to\ell+1}_k(x,q) = \Theta(q^\ell, x^\ell q^k) + (xq)^{\ell-k}\Theta(q^\ell, x^\ell q^{2\ell-k}),$$

where $1 \leq k \leq \ell - 1$. Beyond the initial conditions, it is very difficult to obtain closed forms for the graded multiplicities. Even forming a recursion of generating functions $A^{\ell\to\ell+1}_n(x,q)$ is difficult. Recall in Proposition 5(ii) we have the term

$$q^{s_1 s_0}[D(\ell, \ell(s_1 - 1) + (\ell - s_0)) : D(\ell + 1, n)]_q.$$

For $\ell = 1, 2$, rewriting the weight $\ell(s_1 - 1) + (\ell - s_0)$ in the form $\ell\widetilde{s_1} + \widetilde{s_0}$ we have $s_0 = \widetilde{s_0}$. For arbitrary $\ell$, this equality does not hold. When we consider the generating series

$$A^{\ell\to\ell+1}_n(x,q) = \sum_{k\geq 0} [D(\ell, n + 2k) : D(\ell + 1, n)]_q x^k,$$

the recursions of the multiplicities (Proposition 5(ii)) depend upon the value of $k \mod \ell$.

# Chapter 4

# Numerical Multiplicities

In the previous chapter, we gave closed formulae in small cases of $\ell$. In general, the graded multiplicities are difficult to calculate. Now, we turn our attention to numerical multiplicities i.e. formally taking the specialization $q = 1$. Using Proposition 5 with $q = 1$, we create a recursion of generating functions. We can in turn create a recursion of polynomials and solve this recursion as a system of equations. Lastly, we relate these polynomials and find further recursions relating different level flags of Demazure modules.

## 4.1 Recursion for $A_n^{\ell \to \ell+1}(x, 1)$

We turn our attention to the study of $A_n^{\ell \to \ell+1}(x, 1)$ for $\ell \geq 1$. We prove,

**Theorem 2.** *For $\ell \geq 1$ and $n \geq 0$, write $n = (\ell+1)p_n - r_n$ where $p_n \in \mathbb{Z}_+$ and $0 \leq r_n \leq \ell$. Then,*

$$A_n^{\ell \to \ell+1}(x, 1) = \begin{cases} A_{n+\ell}^{\ell \to \ell+1}(x, 1) - x^{r_n} A_{n+2r_n}^{\ell \to \ell+1}(x, 1) & \text{if } \ell + 1 \nmid n, \\ \\ A_{n+\ell}^{\ell \to \ell+1}(x, 1) & \text{if } \ell + 1 \mid n. \end{cases} \tag{4.1.1}$$

### 4.1.1 Polynomials $d_n^\ell$

We shall use Theorem 2 to establish the following result, which in particular shows that the functions $A_n^{\ell \to \ell+1}(x, 1)$ are rational. For this we define polynomials $d_n^\ell$, $n \geq 0, \ell \geq 1$ with non–negative integer coefficients as follows. Set

$$K_1 = \begin{bmatrix} 0 & 1 & & & & \\ & 0 & 1 & & & \\ & & 0 & 1 & & \\ & & & 0 & \ddots & \\ & & & & \ddots & 1 \\ & & & & & 0 \end{bmatrix}, \quad K_2 = \begin{bmatrix} & & & & & 0 \\ & & & & \iddots & x^{\ell-1} \\ & & & \iddots & \iddots & \\ & & 0 & x^2 & & \\ & 0 & x & & & \\ 0 & 1 & & & & \end{bmatrix} \text{ and } K = K_1 + K_2.$$

The polynomials $d_n^\ell$ are defined by requiring that the following equality hold for all $p \geq 0$:

$$\begin{bmatrix} d^\ell_{(\ell+1)p} & d^\ell_{(\ell+1)p+1} & \cdots & d^\ell_{(\ell+1)p+\ell} \end{bmatrix}^T = K^{p+1} \begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix}^T. \tag{4.1.2}$$

**Proposition 8.** *Let $\ell \geq 1$. Then, for all $n \geq 0$, we have*

$$A_n^{\ell \to \ell+1}(x, 1) = \frac{d_n^\ell}{(1 - x^\ell)^{\lfloor \frac{n}{\ell+1} \rfloor + 1}}.$$

### 4.1.2 Proof of Theorem 2

To simplify notation, we fix $\ell \geq 1$, and for $s, n \in \mathbb{Z}_+$, set

$$\nu(s, n) := [D(\ell, s) : D(\ell + 1, n)]_{q=1}.$$

Recall that $\nu(s, n) = 0$ if $s < n$. The theorem follows if we prove that for all $s, n \geq 0$, we have

$$\nu(s, n) = \begin{cases} \nu(s + \ell, n + \ell) - \nu(s, n + 2r_n) & \text{if } (\ell + 1) \nmid n. \\ \nu(s + \ell, n + \ell) & \text{if } (\ell + 1) \mid n. \end{cases} \tag{4.1.3}$$

Notice that this equality holds whenever $s < n$, both sides being zero. Hence we have to prove it only in the case when $s \geq n$.

Observe that taking $q = 1$ in Proposition 5(i), gives

$$s \geq 0,\ 0 \leq n \leq \ell \implies \nu(s, n) = \begin{cases} 1 & \text{if } s + n \text{ or } s - n \text{ is a multiple of } 2\ell. \\ 0 & \text{otherwise.} \end{cases} \tag{4.1.4}$$

and taking $q = 1$ in Proposition 5(ii) with $0 < s_0 \leq \ell$, $s_1 > 0$ and $s = \ell s_1 + s_0$ gives

$$s \geq \ell + 1,\ n \geq \ell + 1 \implies \nu(s, n) = \nu(s - \ell - 1, n - \ell - 1) + \nu(s - 2s_0, n). \tag{4.1.5}$$

Observe in the last equation that $s - 2s_0 = \ell s_1 - s_0 \geq 0$.

We now proceed to prove (4.1.3) by induction on $n$. To see that induction begins, we first prove that this assertion holds when $n = 0$ for all $s \geq 0$. Using equation (4.1.4) it follows trivially that $\nu(s, 0) = \nu(s + \ell, \ell)$ as required.

Now let $n > 0$. Assume that we have proved that $\nu(s, n')$ satisfies (4.1.3) for all $0 \leq n' < n$ and for all $s \in \mathbb{Z}_+$. We proceed by induction on $s$ to prove that $\nu(s, n)$ satisfies (4.1.3) for all $s \in \mathbb{Z}_+$. Notice that this induction begins at $s = 0$ since both sides of (4.1.3) are then zero. Further, as remarked earlier, this equality holds for $s < n$; so we can further assume that $s \geq n$. Now, assume that we have proved the result for all $s'$ with $0 \leq s' < s$. We have to consider two cases.

*Case 1:* Suppose $0 < n \leq \ell$, and $s \geq n$. In this case we have $n = \ell + 1 - r_n$ and we have to prove that

$$\nu(s, n) = \nu(s + \ell, n + \ell) - \nu(s, 2\ell + 2 - n).$$

*Case 1(a):* Suppose $s \geq \ell + 1$. Then (4.1.5) can be used for both terms of the right hand side and we get

$$\begin{aligned} \nu(s + \ell, n + \ell) &= \nu(s - 1, n - 1) + \nu(s + \ell - 2s_0, n + \ell), \\ \nu(s, 2\ell + 2 - n) &= \nu(s - \ell - 1, \ell + 1 - n) + \nu(s - 2s_0, 2\ell + 2 - n). \end{aligned}$$

Set

$$T_1 = \nu(s - 1, n - 1) - \nu(s - \ell - 1, \ell + 1 - n)$$

and

$$T_2 = \nu(s + \ell - 2s_0, n + \ell) - \nu(s - 2s_0, 2\ell + 2 - n).$$

Equation (4.1.4) applies to both the terms in $T_1$. Now observing that:

$$\begin{aligned} (s - 1) - (n - 1) &= (s - \ell - 1) + (\ell + 1 - n) \\ (s - 1) + (n - 1) &\equiv (s - \ell - 1) - (\ell + 1 - n) \pmod{2\ell}, \end{aligned}$$

we deduce that $T_1 = 0$. Further, since $s - 2s_0 < s$, the inductive hypothesis gives $T_2 = \nu(s - 2s_0, n)$. We must thus prove that $\nu(s, n) = \nu(s - 2s_0, n)$. Since $s \equiv s_0 \pmod{\ell}$, we obtain $s - 2s_0 \equiv -s \pmod{2\ell}$; hence $s \pm n \equiv (s - 2s_0) \mp n \pmod{2\ell}$; applying (4.1.4) completes the proof.

*Case 1(b):* Suppose $s \leq \ell$. Then since $2\ell + 2 - n > \ell$, we have $\nu(s, 2\ell + 2 - n) = 0$. We thus need to show that $\nu(s, n) = \nu(s + \ell, n + \ell)$. Applying equation (4.1.5) again:

$$\nu(s + \ell, n + \ell) = \nu(s - 1, n - 1) + \nu(s + \ell - 2s_0, n + \ell).$$

But since $0 < s \leq \ell$, we have $s = s_0$, and hence $s + \ell - 2s_0 < \ell < n + \ell$. Thus the second term vanishes. We need to now show that $\nu(s - 1, n - 1) = \nu(s, n)$. But from (4.1.4), it is clear that for $1 \leq s, n \leq \ell$, $\nu(s - 1, n - 1) = \nu(s, n) = \delta_{s,n}$ . This completes Case 1 of the inductive step.

*Case 2:* Suppose $n \geq \ell + 1$ and $s \geq n$. Suppose first that $\ell + 1 \nmid n$. Consider

$$S = \nu(s + \ell, n + \ell) - \nu(s, n + 2r_n) - \nu(s, n).$$

By applying (4.1.5) to each of these terms, we have

$$\begin{aligned} S = {} & \nu(s - 1, n - 1) + \nu(s + \ell - 2s_0, n + \ell) - \nu(s - \ell - 1, n + 2r_n - \ell - 1) - \nu(s - 2s_0, n + 2r_n) \\ & - \nu(s - \ell - 1, n - \ell - 1) - \nu(s - 2s_0, n). \end{aligned}$$

Since $n - \ell - 1 < n$ and $s - 2s_0 < s$, the inductive hypothesis gives

$$\begin{aligned} \nu(s - \ell - 1, n - \ell - 1) &= \nu(s - 1, n - 1) - \nu(s - \ell - 1, n + 2r_n - \ell - 1), \\ \nu(s - 2s_0, n) &= \nu(s + \ell - 2s_0, n + \ell) - \nu(s - 2s_0, n + 2r_n). \end{aligned}$$

Using these equations to replace $\nu(s - \ell - 1, n - \ell - 1)$ and $\nu(s - 2s_0, n)$ in our equation for $S$, we obtain $S = 0$ as required.

Now, suppose $\ell + 1 \mid n$. Consider $S' = \nu(s + \ell, n + \ell) - \nu(s, n)$. As in the case for $\ell + 1 \nmid n$, apply (4.1.5) to each term to get

$$S' = \nu(s - 1, n - 1) + \nu(s + \ell - 2s_0, n + \ell) - \nu(s - \ell - 1, n - \ell - 1) - \nu(s - 2s_0, n).$$

Since $n - \ell - 1 < n$, $s - 2s_0 < s$ and $\ell + 1 \mid (n - \ell - 1)$, the inductive hypothesis gives

$$\nu(s - \ell - 1, n - \ell - 1) = \nu(s - 1, n - 1), \qquad \nu(s - 2s_0, n) = \nu(s + \ell - 2s_0, n + \ell).$$

This gives us $S' = 0$ as required. $\square$

### 4.1.3 Proof of Proposition 8

*Proof.* Let $n \geq 0$, with $n = (\ell+1)p_n - r_n$ and $0 \leq r_n \leq \ell$. We consider three cases in equation (4.1.1):

(i) $1 \leq r_n \leq \ell - 1$. In this case, define

$$n' = n + 2r_n - \ell = (\ell+1)p_n - (\ell - r_n),$$

and consider $A_n^{\ell\to\ell+1}(x,1)$ and $A_{n'}^{\ell\to\ell+1}(x,1)$. Equation (4.1.1) gives us the system of equations:

$$A_n^{\ell\to\ell+1}(x,1) = A_{n+\ell}^{\ell\to\ell+1}(x,1) - x^{r_n}\, A_{n'+\ell}^{\ell\to\ell+1}(x,1)$$
$$A_{n'}^{\ell\to\ell+1}(x,1) = A_{n'+\ell}^{\ell\to\ell+1}(x,1) - x^{\ell-r_n}\, A_{n+\ell}^{\ell\to\ell+1}(x,1)$$

(this becomes a single equation if $r_n = \ell - r_n$, i.e., if $n = n'$). Solving, we obtain:

$$A_{n+\ell}^{\ell\to\ell+1}(x,1) = \frac{1}{1-x^\ell}\left(A_n^{\ell\to\ell+1}(x,1) + x^{r_n}\, A_{n'}^{\ell\to\ell+1}(x,1)\right) \tag{4.1.6}$$

(ii) $r_n = 0$, i.e., $n = (\ell+1)p_n$. Here we obtain $A_{n+\ell}^{\ell\to\ell+1}(x,1) = A_n^{\ell\to\ell+1}(x,1)$

(iii) $r_n = \ell$, i.e., $n = (\ell+1)p_n - \ell$. Then, $A_n^{\ell\to\ell+1}(x,1) = A_{n+\ell}^{\ell\to\ell+1}(x,1) - x^\ell\, A_{(\ell+1)p_n+\ell}^{\ell\to\ell+1}(x,1)$. Using case (ii) above, we obtain $A_n^{\ell\to\ell+1}(x,1) = (1-x^\ell)A_{n+\ell}^{\ell\to\ell+1}(x,1)$.

Now, for $n, p \geq 0$, define

$$d_n^\ell := A_n^{\ell\to\ell+1}(x,1)\cdot(1-x^\ell)^{\lfloor \frac{n}{\ell+1}\rfloor + 1},$$

and

$$\zeta_p := \begin{bmatrix} d_{(\ell+1)p} & d_{(\ell+1)p+1} & \cdots & d_{(\ell+1)p+\ell}\end{bmatrix}^T.$$

We will prove by induction that

$$\zeta_p = K^{p+1}\begin{bmatrix}1 & 1 & \cdots & 1\end{bmatrix}^T$$

for $p \geq 0$. When $p = 0$, we use equation (4.1.4) to get

$$d_n^\ell = (1-x^\ell)A_n^{\ell\to\ell+1}(x,1) = \begin{cases}(1-x^\ell)\sum_{k\geq 0} x^{\ell k}, & n = 0, \ell \\ (1-x^\ell)(\sum_{k\geq 0} x^{\ell k} + \sum_{k\geq 1} x^{\ell k - n}), & 0 < n < \ell.\end{cases}$$

Thus, we have

$$d_n^\ell = \begin{cases} 1, & n = 0, \ell \\ 1 + x^{\ell - n}, & 0 < n < \ell.\end{cases} \tag{4.1.7}$$

These polynomials are the entries in $\zeta_0$ and satisfy $\zeta_0 = K \begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix}^T$. Now, let $p \geq 1$ and assume

$$\zeta_{p-1} = K^p \begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix}^T.$$

We now have $K\zeta_{p-1} = K_1\zeta_{p-1} + K_2\zeta_{p-1}$, where

$$K_1\zeta_{p-1} = \begin{bmatrix} d^{\ell}_{(\ell+1)(p-1)+1} & \cdots & d^{\ell}_{(\ell+1)(p-1)+\ell} & 0 \end{bmatrix}^T,$$

and

$$K_2\,\zeta_{p-1} = \begin{bmatrix} 0 & x^{\ell-1}d^{\ell}_{(\ell+1)(p-1)+\ell} & x^{\ell-2}d^{\ell}_{(\ell+1)(p-1)+\ell-1} & \cdots & d^{\ell}_{(\ell+1)(p-1)+1} \end{bmatrix}^T.$$

Dividing these vectors by $(1-x^{\ell})^{p+1}$, the equations (4.1.6) for $0 < r < \ell$ and the cases for $r = 0, \ell$ give us that $(K_1 + K_2)\zeta_{p-1} = \begin{bmatrix} d^{\ell}_{(\ell+1)p} & d^{\ell}_{(\ell+1)p+1} & \cdots & d^{\ell}_{(\ell+1)p+\ell} \end{bmatrix} = \zeta_p$. Then, by the inductive hypothesis, we have

$$\zeta_p = K\zeta_{p-1} = KK^p \begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix}^T$$

as desired.

$\square$

## 4.2 Generating Series Associated to $d^{\ell}_{(\ell+1)p}$

In the remaining sections of this chapter, we work towards giving a closed form for the generating series $h_{\ell}(t,x) = \sum_{n\geq 0} d^{\ell}_{(\ell+1)n}t^n$. Through this process, we also relate the numerical multiplicities between Demazure modules of different levels. Denote $D(n,\ell) := d^{\ell}_{(\ell+1)n}$. We can find a rational expression for $h_{\ell}(t,x)$.

**Proposition 9.** *Let $h_{\ell}(t,x) = \sum_{n=0}^{\infty} D(n,\ell)t^n$ and $i \in \mathbb{Z}_+$. We can write $h_{\ell}(t,x)$ as a rational expression.*

*(i) For $\ell = 2i+1$, We write $h_{\ell}(t,x)$ as*

$$\left(x^{\frac{\ell+1}{2}} - tx^{\ell} - \sum_{k=1}^{\frac{\ell-1}{2}} x^{\frac{3\ell+1}{2}-k}t^{2k}P_k^{\frac{\ell+1}{2}} - \sum_{k=1}^{\frac{\ell-1}{2}}(1-x^{\ell})x^{\ell-k}t^{2k+1}P_k^{\frac{\ell+1}{2}} + \sum_{m=1}^{\frac{\ell-3}{2}}\sum_{k=0}^{m-1} x^{\ell-k}t^{2k+3}P_k^{\ell-(m-1)}\right)^{-1}$$

$$\times\left(x^{\frac{\ell+1}{2}}\sum_{j=0}^{\ell} t^jD(j,\ell) - x^{\ell}t\sum_{j=0}^{\ell-1} t^jD(j,\ell) - \sum_{k=1}^{\frac{\ell-1}{2}} x^{\frac{3\ell+1}{2}-k}t^{2k}P_k^{\frac{\ell+1}{2}}\sum_{j=0}^{\ell-2k} t^jD(j,\ell)\right.$$

$$\left. - \sum_{k=1}^{\frac{\ell-1}{2}}(1-x^{\ell})x^{\ell-k}t^{2k+1}P_k^{\frac{\ell+1}{2}}\sum_{j=0}^{\ell-2k-1}t^jD(j,\ell) + \sum_{m=1}^{\frac{\ell-3}{2}}\sum_{k=0}^{m-1} x^{\ell-k}t^{2k+3}P_k^{\ell-(m-1)}\sum_{j=0}^{\ell-2k-3}t^jD(j,\ell)\right).$$

*(ii) For $\ell = 2i+2$, we write $h_\ell(t,x)$ as*

$$\left(1-(1+x^{\ell/2})\sum_{k=0}^{\frac{\ell}{2}-1}x^{\frac{\ell}{2}-1-k}t^{2k+2}P_k^{\ell-(\frac{\ell}{2}-1)}-\sum_{m=0}^{\frac{\ell}{2}-1}\sum_{k=0}^{m}x^{\ell-k}t^{2k+2}P_k^{\ell-m}\right)^{-1}$$
$$\times\left(\sum_{j=0}^{\ell-1}t^jD(j,\ell)-(1+x^{\ell/2})\sum_{k=0}^{\frac{\ell}{2}-1}x^{\frac{\ell}{2}-1-k}t^{2k+2}P_k^{\ell-(\frac{\ell}{2}-1)}\sum_{j=0}^{\ell-1-2k}t^jD(j,\ell)\right.$$
$$\left.-\sum_{m=0}^{\frac{\ell}{2}-1}\sum_{k=0}^{m}x^{\ell-k}t^{2k+2}P_k^{\ell-m}\sum_{j=0}^{\ell-1-2k}t^jD(j,\ell)\right).$$

### 4.2.1 Useful Lemma

To prove the proposition, we first need to write the polynomials $d^\ell_{(\ell+1)n+j}$ for $0<j<\ell$ as $\mathbb{Z}[x]$-linear combinations of $D(p,\ell)$. We have the following result.

**Lemma 4.2.1.** *Let $D(n,\ell)$ denote $d^\ell_{(\ell+1)n}$ and let $m,k\in\mathbb{N}$ and $p\in\mathbb{Z}$. Then for $1\le m<\frac{\ell-1}{2}, p\ge m-1$ in the first equality and $0\le m\le\frac{\ell+1}{2}, p\ge m-1$ in the second, we have*

$$d^\ell_{(\ell+1)p+\ell-m}=\sum_{k=0}^{m}x^{m-k}P_k^{\ell-m}D(p+m-2k,\ell)$$
$$d^\ell_{(\ell+1)p+m}=D_{p+m}-\sum_{k=1}^{m-1}x^{\ell-k}P_k^{m}D(p+m-2k,\ell)$$

*where*

$$P_k^{\ell-m}=\begin{cases}\sum_{i=0}^{k}(-1)^i\binom{m-k+i}{i}\binom{k}{i}x^{i\ell}, & 0\le 2k\le m\\ \\ (1-x^\ell)^{-(m-2k)}P^{\ell-m}_{m-k}, & m<2k\end{cases}$$

*and*

$$P_k^{m}=\begin{cases}\sum_{i=0}^{k-1}(-1)^i\binom{m-k+i}{i+1}\binom{k-1}{i}x^{i\ell}, & 0\le 2k\le m\\ \\ \frac{m-k}{k}(1-x^\ell)^{-(m-2k)}P^{m}_{m-k}, & m<2k.\end{cases}$$

*Proof.* For $p\ge -1$, we have $d^\ell_{(\ell+1)p+\ell}=D(p,\ell)$ and $d^\ell_{(\ell+1)p+1}=D(p+1,\ell)$ using (4.1.2) where $d_{-k}=1$ for $1\le k\le\ell$. Then, we check the recursions for $d^\ell_{(\ell+1)p+m+1}$ with $1\le m<\frac{\ell-1}{2}, p\ge m+2$. We plug our solutions into the equation

$$d^\ell_{(\ell+1)p+m}=d^\ell_{(\ell+1)(p-1)+m+1}+x^{\ell-m}d^\ell_{(\ell+1)(p-1)+\ell-(m-1)}. \tag{4.2.1}$$

Using our solutions on the right hand side, we have

$$\begin{aligned}
&d^{\ell}_{(\ell+1)(p-1)+m+1} + x^{\ell-m} d^{\ell}_{(\ell+1)(p-1)+\ell-(m-1)} \\
&\quad = D(p+m,\ell) - \sum_{k=1}^{m} x^{\ell-k} P_k^{m+1} D(p+m-2k,\ell) \\
&\qquad + x^{\ell-m} \sum_{k=0}^{m-1} x^{m-1-k} P_k^{\ell-(m-1)} D(p+m-2-2k,\ell) \\
&\quad = D(p+m,\ell) + \sum_{k=1}^{m} x^{\ell-k} (P_{k-1}^{\ell-(m-1)} - P_k^{m+1}) D(p+m-2k,\ell).
\end{aligned}$$

Now, due to the piecewise definition of $P_k^n$, we must split our verification into cases.

*Case 1:* Take $k=0$. The coefficient of $D(p+m,\ell)$ is 1 on both sides of (4.2.1).

*Case 2:* Take $1 \le k \le \frac{m}{2}$. The coefficient of $D(p+m-2k,\ell)$ on the left side of (4.2.1) is

$$-x^{\ell-k} P_k^m = -x^{\ell-k} \sum_{i=0}^{k} (-1)^i \binom{m-k+i}{i} \binom{k}{i} x^{i\ell}.$$

The coefficient of $D(p+m-2k,\ell)$ on the right hand side is

$$\begin{aligned}
&x^{\ell-k} \sum_{i=0}^{k-1} (-1)^i \binom{m-k+i}{i} \binom{k-1}{i} x^{i\ell} - (-1)^i \binom{m+1-k+i}{i+1} \binom{k-1}{i} x^{i\ell} \\
&\quad = x^{\ell-k} \sum_{i=0}^{k-1} (-1)^i x^{i\ell} \binom{k-1}{i} \left( \binom{m-k+i}{i} - \binom{m+1-k+i}{i+1} \right) \\
&\quad = -x^{\ell-k} \sum_{i=0}^{k-1} (-1)^i x^{i\ell} \binom{k-1}{i} \binom{m-k+i}{i+1} \\
&\quad = -x^{\ell-k} P_k^m,
\end{aligned}$$

where we used the identity 3.2.1 with $q=1$.

*Case 3:* Take $k = \frac{m+1}{2}$ for $m$ odd. For the left side of (4.2.1), the condition $2k > m$ gives us the coefficient of $D(p+m-2k,\ell)$ is

$$\begin{aligned}
-x^{\ell-k} P_k^m &= -\frac{\frac{m-1}{2}}{\frac{m+1}{2}} x^{\ell-\frac{m+1}{2}} (1-x^{\ell}) P^m_{\frac{m-1}{2}} \\
&= -\frac{\frac{m-1}{2}}{\frac{m+1}{2}} x^{\ell-\frac{m+1}{2}} (1-x^{\ell}) \sum_{i=0}^{(m-3)/2} (-1)^i \binom{\frac{m+1}{2}+i}{i+1} \binom{\frac{m-3}{2}}{i} x^{i\ell} \\
&= -x^{\ell-\frac{m+1}{2}} \sum_{i=0}^{(m-1)/2} (-1)^i x^{i\ell} \binom{\frac{m-1}{2}}{i} \binom{\frac{m-1}{2}+i}{i+1},
\end{aligned}$$

which is equal to the coefficient of the same term on the right hand side of (4.2.1) using the calculation from case 2.

*Case 4:* Take $k > \frac{m+1}{2}$. The coefficient of $D(p+m-2k,\ell)$ on the left hand side of (4.2.1) is

$$\begin{aligned}
-x^{\ell-k}P_k^m &= -\frac{m-k}{k}x^{\ell-k}(1-x^\ell)^{2k-m}P_{m-k}^m \\
&= \frac{k-m}{k}x^{\ell-k}(1-x^\ell)^{2k-m}\sum_{i=0}^{m-k-1}(-1)^i\binom{k+i}{i+1}\binom{m-k-1}{i}x^{i\ell} \\
&= \frac{k-m}{k}x^{\ell-k}(1-x^\ell)^{2k-m-1}\sum_{i=0}^{m-k-1}(-1)^i\binom{k+i}{i+1}\binom{m-k-1}{i}x^{i\ell} \\
&\quad+\sum_{i=1}^{m-k}\binom{k+i-1}{i}\binom{m-k-1}{i-1}x^{i\ell}.
\end{aligned}$$

Similarly for the coefficient on the right hand side of (4.2.1), we have

$$\begin{aligned}
&x^{\ell-k}(P_{k-1}^{\ell-(m-1)}-P_k^{m+1}) \\
&= x^{\ell-k}((1-x^\ell)^{2k-m-1}P_{m-k}^{\ell-(m-1)}-\frac{m+1-k}{k}(1-x^\ell)^{2k-m-1}P_{m+1-k}^{m+1}) \\
&= x^{\ell-k}(1-x^\ell)^{2k-m-1}\sum_{i=0}^{m-k}(-1)^i x^{i\ell}\binom{m-k}{i}\left(\binom{k+i-1}{i}-(m+1-k)\binom{k+i}{i+1}\right)
\end{aligned}$$

To prove these coefficients are equivalent, it is enough to show for all $i=0,\cdots m-k$,

$$\begin{aligned}
(k-m)\left(\binom{k+i}{i+1}\binom{m-k-1}{i}+\binom{k+i-1}{i}\binom{m-k-1}{i-1}\right) \\
=\binom{m-k}{i}\left(\binom{k+i-1}{i}-(m+1-k)\binom{k+i}{i+1}\right).
\end{aligned}$$

This can be proven rewriting the binomial coefficients using factorials.

The proof for verifying the expression for $d_{(\ell+1)p+\ell-m}$ for $1\le m<\frac{\ell-1}{2}$ is similar. □

### 4.2.2 Proof of Proposition 9

Now, we are able to prove Proposition 9.

First, assume $\ell$ is odd. We note that

$$\begin{aligned}
d^\ell_{(\ell+1)p+\frac{\ell-1}{2}} &= d^\ell_{(\ell+1)(p-1)+\frac{\ell+1}{2}}+x^{\frac{\ell+1}{2}}d^\ell_{(\ell+1)(p-1)+\frac{\ell+3}{2}} \\
&= x^{\frac{\ell+1}{2}}d^\ell_{(\ell+1)(p)+\frac{\ell+1}{2}}+(1-x^\ell)d^\ell_{(\ell+1)(p-1)+\frac{\ell+1}{2}}
\end{aligned}$$

and we can write $d_{(\ell+1)(p-1)+\frac{\ell+1}{2}}$ as a combination of $d_n^\ell$ using Lemma 4.2.1. For $p \geq 0$, we have

$$\begin{aligned}
D(p+\ell,\ell) &= d^\ell_{(\ell+1)(p+\ell)} \\
&= d^\ell_{(\ell+1)(p+\ell-\frac{\ell-1}{2})+\frac{\ell-1}{2}} + \sum_{m=1}^{\frac{\ell-3}{2}} x^{\ell-m} d^\ell_{(\ell+1)(p+\ell-m-1)+\ell-(m-1)} \\
&= x^{\frac{\ell+1}{2}} d^\ell_{(\ell+1)(p+\ell-\frac{\ell-1}{2})+\frac{\ell+1}{2}} + (1-x^\ell) d^\ell_{(\ell+1)(p+\ell-\frac{\ell+1}{2})+\frac{\ell+1}{2}} \\
&\quad + \sum_{m=1}^{\frac{\ell-3}{2}} x^{\ell-m} d^\ell_{(\ell+1)(p+\ell-m-1)+\ell-(m-1)} \\
&= x^{\frac{\ell+1}{2}} (D(p+\ell+1,\ell) - \sum_{k=1}^{\frac{\ell-1}{2}} x^{\ell-k} P_k^{\frac{\ell+1}{2}} D(p+\ell+1-2k,\ell)) \\
&\quad + (1-x^\ell)(D(p+\ell,\ell) - \sum_{k=1}^{\frac{\ell-1}{2}} x^{\ell-k} P_k^{\frac{\ell+1}{2}} D(p+\ell-2k,\ell)) \\
&\quad + \sum_{m=1}^{\frac{\ell-3}{2}} x^{\ell-m} (\sum_{k=0}^{m-1} x^{m-1-k} P_k^{\ell-(m-1)} D(p+\ell-2-2k,\ell)).
\end{aligned}$$

Rearranging some terms, we have

$$\begin{aligned}
x^{\frac{\ell+1}{2}} D(p+\ell+1,\ell) = x^\ell D(p+\ell,\ell) &+ x^{\frac{\ell+1}{2}} \sum_{k=1}^{\frac{\ell-1}{2}} x^{\ell-k} P_k^{\frac{\ell+1}{2}} D(p+\ell+1-2k,\ell) \\
&+ (1-x^\ell) \sum_{k=1}^{\frac{\ell-1}{2}} x^{\ell-k} P_k^{\frac{\ell+1}{2}} D(p+\ell-2k,\ell) \\
&- \sum_{m=1}^{\frac{\ell-3}{2}} \sum_{k=0}^{m-1} x^{\ell-1-k} P_k^{\ell-(m-1)} D(p+\ell-2-2k,\ell).
\end{aligned}$$

Then we multiply by $t^p$ and sum over $p \geq 0$ to get

$$\begin{aligned}
\frac{x^{\frac{\ell+1}{2}}}{t^{\ell+1}}\Big(h_\ell(t,x) - \sum_{j=0}^{\ell} t^j D(j,\ell)\Big) &= \frac{x^\ell}{t^\ell}\Big(h_\ell(t,x) - \sum_{j=0}^{\ell-1} t^j D(j,\ell)\Big) \\
&+ \sum_{k=1}^{\frac{\ell-1}{2}} \frac{x^{\frac{3\ell+1}{2}-k}}{t^{\ell+1-2k}} P_k^{\frac{\ell+1}{2}}\Big(h_\ell(t,x) - \sum_{j=0}^{\ell-2k} t^j D(j,\ell)\Big) \\
&+ \sum_{k=1}^{\frac{\ell-1}{2}} \frac{(1-x^\ell)x^{\ell-k}}{t^{\ell-2k}} P_k^{\frac{\ell+1}{2}}\Big(h_\ell(t,x) - \sum_{j=0}^{\ell-2k-1} t^j D(j,\ell)\Big) \\
&- \sum_{m=1}^{\frac{\ell-3}{2}} \sum_{k=0}^{m-1} \frac{x^{\ell-1-k}}{t^{\ell-2-2k}} P_k^{\ell-(m-1)}\Big(h_\ell(t,x) - \sum_{j=0}^{\ell-2k-3} t^j D(j,\ell)\Big).
\end{aligned}$$

Solving for $h_\ell(t,x)$ completes the proof of part (i).

For part (ii) of the lemma, we assume $\ell$ is even. Notice, that we have

$$d^\ell_{(\ell+1)p+\frac{\ell}{2}} = (1+x^{\frac{\ell}{2}})d^\ell_{(\ell+1)(p-1)+\frac{\ell}{2}+1}.$$

For $p \geq 0$, we have

$$\begin{aligned}
D(p+\ell,\ell) &= d^\ell_{(\ell+1)(p+\ell)} \\
&= d^\ell_{(\ell+1)(p+\ell-\frac{\ell}{2})+\frac{\ell}{2}} + \sum_{m=1}^{\frac{\ell}{2}-1} x^{\ell-m} d^\ell_{(\ell+1)(p+\ell-m-1)+\ell-(m-1)} \\
&= (1+x^{\frac{\ell}{2}})d^\ell_{(\ell+1)(p+\ell-\frac{\ell}{2}-1)+\frac{\ell}{2}+1} + \sum_{m=1}^{\frac{\ell}{2}-1} x^{\ell-m} d^\ell_{(\ell+1)(p+\ell-m-1)+\ell-(m-1)} \\
&= (1+x^{\frac{\ell}{2}}) \sum_{k=0}^{\frac{\ell}{2}-1} x^{\frac{\ell}{2}-1-k} P_k^{\ell-(\frac{\ell}{2}-1)} D(p+\ell-2-2k,\ell) \\
&\quad + \sum_{m=1}^{\frac{\ell}{2}-1} x^{\ell-m}(\sum_{k=0}^{m} x^{m-k} P_k^{\ell-m} D(p+\ell-2-2k,\ell)).
\end{aligned}$$

Now, we repeat the steps as in the proof of (i) to get the following equality with generating functions

$$\begin{aligned}
h_\ell(t,x) - \sum_{j=0}^{\ell-1} t^j D(j,\ell) &= (1+x^{\ell/2}) \sum_{k=0}^{\frac{\ell}{2}-1} x^{\frac{\ell}{2}-1-k} t^{2k+2} P_k^{\ell-(\frac{\ell}{2}-1)}\Big(h_\ell(t,x) - \sum_{j=0}^{\ell-1-2k} t^j D(j,\ell)\Big) \\
&+ \sum_{m=0}^{\frac{\ell}{2}-1} \sum_{k=0}^{m} x^{\ell-k} t^{2k+2} P_k^{\ell-m}\Big(h_\ell(t,x) - \sum_{j=0}^{\ell-1-2k} t^j D(j,\ell)\Big).
\end{aligned}$$

We have completed part (ii) of the lemma and thus completed proving the two expressions for $h_\ell(t,x)$ as rational functions of $t$ and $x$.

This completes the proof of Proposition 9.

## 4.3 Recursion for $d^{\ell}_{(\ell+1)p}$

We now restrict our attention to obtaining $D(n,\ell)$ for $0 \le n \le \ell$. Identifying these polynomials would complete an explicit form for $h_\ell(t,x)$.

First, we define the function $\phi_{p,\ell} : \mathbb{Z} \to \mathbb{Z}$ as $\phi_{p,\ell}(m) = m + \lceil \frac{m}{p} \rceil (\ell - p)$. Then, we can define the map $\Phi_{p,\ell} : \mathbb{Z}[x] \to \mathbb{Z}[x]$ as $\Phi_{p,\ell}(x^m) = x^{\phi_{p,\ell}(m)}$. Note that this map is linear, but not always preserved under multiplication.

We have the following relationship between the polynomials $d^{\ell}_{n}$.

**Lemma 4.3.1.** *For $p, r \ge 0$, we have an equality of polynomials*

$$d^{p+2+r}_{(p+3+r)p} = \Phi_{p+r+1,p+r+2}(d^{p+1+r}_{(p+2+r)p}).$$

In this lemma, we take $\ell = p+2+r$. Then, we have

$$d^{\ell}_{(\ell+1)p} = \Phi_{\ell-1,\ell}(d^{\ell-1}_{\ell p}).$$

for $\ell \ge p+2$.

We observe that $\Phi_{r,s} \circ \Phi_{p,r} = \Phi_{p,s}$ for $p \le r \le s$. Thus, we have

$$d^{\ell}_{(\ell+1)p} = \Phi_{p+1,\ell}(d^{p+1}_{(p+2)p}).$$

Thus, in terms of $h_\ell(t,x)$ for any $\ell$, we can calculate $d^{p+1}_{(p+2)p}$ for $0 \le p \le \ell - 2$ and apply the function $\Phi_{p+1,\ell}$ to obtain the necessary $d^{\ell}_{(\ell+1)p}$ for $0 \le p \le \ell - 2$.

We finish this section with a proof of Lemma 4.3.1

### 4.3.1 Proof of Lemma 4.3.1

Applying the map $\Phi_{p+r+1,p+r+2}$, we get

$$\begin{aligned} &\Phi_{p+r+1,p+r+2}(d_{(p+2+r)p,p+1+r}) \\ &= (1 - x^{p+2+r})^{p+1} \sum_{m \ge 0} [D(p+1+r, (p+2+r)p + 2m) : D(p+2+r)p] x^{m + \lceil \frac{m}{p+1+r} \rceil}. \end{aligned}$$

Comparing coefficients in $\Phi_{p+r+1,p+r+2}(d_{(p+2+r)p,p+1+r})$ and $d_{(p+3+r)p,p+2+r}$, it is enough to prove the following equalities of multiplicities. For $1 \le k \le p+1+r$, we have

$$\begin{aligned}[D(p+2+r,(p+3+r)p+2(p+2+r)j+2k+2) : D(p+3+r,(p+3+r)p)]_{q=1} \\ = [D(p+1+r,(p+2+r)p+2(p+1+r)j+2k) : D(p+2+r,(p+2+r)p)]_{q=1}\end{aligned} \tag{4.3.1}$$

and

$$[D(p+2+r,(p+3+r)p+2(p+2+r)j+2) : D(p+3+r,(p+3+r)p)]_{q=1} = 0 \tag{4.3.2}$$

for all $p, r, j \ge 0$.

Denote $\nu(s,n,\ell) := [D(\ell,s) : D(\ell+1,n)]_{q=1}$. To prove these equalities, we will first proceed by induction on $p$. We will prove the equality when $p = 0$ for all $r, j \ge 0$. Then to prove the inductive step, we will use induction on $j$. The induction on $j$ will require different cases based on the value of $k$.

We begin with induction on $p \ge 0$.

*Base case $p = 0, \forall r, j \ge 0$:* First, we comment on the values of $k$. We are allowing $1 \le k \le r+1$, then we have $4 \le 2k+2 \le 2r+4$. Using Corollary 5(i) with $q = 1$, we calculate the left side of (4.3.1)

$$\nu(2(2+r)j+2k+2, 0, r+2) = \delta_{2k+2,2r+4} = \delta_{k,r+1},$$

for all $r, j \ge 0$. Similarly, calculating the right side of (4.3.1) we have

$$\nu(2(1+r)j+2k, 0, r+1) = \delta_{k,r+1},$$

for all $j, r \ge 0$. Moreover, $\nu(2(2+r)j+2k+2, 0, r+2) = \delta_{k,r+1}$ with $k = 0$ proves $p = 0$ case for (4.3.2) for all $r, j \ge 0$. This completes the base case for the induction on $p$.

*Inductive step on $p, \forall r, j \ge 0$:* Now, assume the equalities hold for all $p \le i$ and $r, j \ge 0$. Then we must show (4.3.1) and (4.3.2) hold for $p = i+1, \forall r, j \ge 0$. We will prove this by induction on $j \ge 0$.

*Base Case* $j = 0, p = i + 1, \forall r \geq 0$: First, assume $j = 0, p = i + 1$. We discuss these results in cases. First, suppose $2 \leq 2k \leq r$. Then, we have the following

$$\begin{aligned}
&\nu((i+4+r)(i+1)+2k+2,(i+4+r)(i+1),(i+3+r)) \\
&\quad = \nu((i+4+r)i+2k+2,(i+4+r)i,(i+3+r)) \\
&\quad = \nu((i+3+r)i+2k,(i+3+r)i,(i+2+r)),
\end{aligned}$$

using Proposition 5(ii) with $\nu((i+4+r)i+i+4+r-2k-2,(i+4+r)(i+1),(i+4+r)) = 0$ and the inductive hypothesis with $p = i$. Applying Proposition 5(ii) to $\nu((i+3+r)(i+1)+2k,(i+3+r)(i+1),(i+3+r))$ gives the same result.

Consider the case when $r+1 \leq 2k \leq 2r+1$. We have

$$\begin{aligned}
&\nu(((i+4+r)(i+1)+2k+2),(i+4+r)(i+1),i+3+r) \\
&\quad = \nu((i+4+r)(i+1)+2+2r-2k,(i+4+r)(i+1),i+3+r) \\
&\qquad + \nu((i+4+r)i+2k+2,(i+4+r)i,i+3+r) \\
&\quad = \nu((i+4+r)(i+1)+2+2r-2k,(i+4+r)(i+1),i+3+r) \\
&\qquad + \nu((i+3+r)i+2k,(i+3+r)i,i+2+r).
\end{aligned}$$

Moreover, if we apply Proposition 5(ii) to the first term $i+1$ times, we obtain

$$\nu((i+4+r)(i+1)+2+2r-2k,(i+4+r)(i+1),i+3+r) = \nu(2+2r-2k,0,i+3+r) = 0,$$

since $0 < 2+2r-2k \leq r+1 < i+3+r$ and using the fact that the multplicity term from inclusion map of Proposition4 is 0 since $\nu(s,n,\ell) = 0$ for $s < n, \forall \ell \in \mathbb{Z}_+$. Similarly, we have for $r+1 < 2k \leq 2r+1$,

$$\begin{aligned}
&\nu(((i+3+r)(i+1)+2k),(i+3+r)(i+1),i+2+r) \\
&\quad = \nu((i+3+r)(i+1)+2+2r-2k,(i+3+r)(i+1),i+2+r) \\
&\qquad + \nu((i+3+r)i+2k,(i+3+r)i,i+2+r),
\end{aligned}$$

where again obtain

$$\nu((i+3+r)(i+1)+2+2r-2k,(i+3+r)(i+1),i+2+r) = \nu(2+2r-2k,0,i+2+r) = 0,$$

by applying Proposition 5(ii) to the first term $i+1$ times. For the case $2k = r+1$, we have

$$\begin{aligned}
&\nu(((i+3+r)(i+1)+2k),(i+3+r)(i+1),i+2+r)\\
&=\nu((i+2+r)i,(i+3+r)(i+1),i+2+r)\\
&\quad+\nu((i+3+r)i+2k,(i+3+r)i,i+2+r)\\
&=\nu((i+3+r)i+2k,(i+3+r)i,i+2+r).
\end{aligned}$$

This completes the base case when $r+1 \leq 2k \leq 2r+1$.

Now, consider the case when $2r+2 \leq 2k \leq 2r+i+3$.We have

$$\begin{aligned}
&\nu(((i+4+r)(i+1)+2k+2),(i+4+r)(i+1),i+3+r)\\
&=\nu((i+3+r)(i+2)+2+2r-2k,(i+4+r)(i+1),i+3+r)\\
&\quad+\nu((i+4+r)i+2k+2,(i+4+r)i,i+3+r)\\
&=\delta_{k,r+1}+\nu((i+3+r)i+2k,(i+3+r)i,i+2+r).
\end{aligned}$$

using Proposition 5(ii), the inductive hypthesis on the second term and the observation that $(i+3+r)(i+2)+2+2r-2k \leq (i+4+r)(i+1)$ with equality holding only for $2k = 2r+2$ implies $\nu(i+3+r)(i+2)+2+2r-2k,(i+4+r)(i+1),i+3+r) = \delta_{k,r+1}$ for calculating the first term. Similarly, using Proposition 5(ii) and the same observation, we get

$$\begin{aligned}
&\nu(((i+3+r)(i+1)+2k),(i+3+r)(i+1),i+2+r)\\
&=\delta_{k,r+1}+\nu(((i+3+r)i+2k),(i+3+r)i,i+2+r),
\end{aligned}$$

thus completing the proof in this case.

Lastly, consider the case: $2r+i+4 \leq 2k \leq 2r+2i+4$. We have

$$\begin{aligned}
&\nu((i+4+r)(i+1)+2k+2,(i+4+r)(i+1),i+3+r)\\
&=\nu((i+4+r)(i+1)+2i+4r+8-2k,(i+4+r)(i+1),i+3+r)\\
&\quad+\nu((i+4+r)i+2k+2,(i+4+r)i,i+3+r)\\
&=\nu((i+3+r)(i+1)+2k-2r-i-3,(i+4+r)(i+1),i+3+r)\\
&\quad+\nu((i+4+r)i+2i+4r+8-2k,(i+4+r)i,i+3+r)\\
&\quad+\nu((i+4+r)i+2k+2,(i+4+r)i,i+3+r)\\
&=\delta_{k,r+i+2}+\nu((i+3+r)i+2k',(i+3+r)i,i+2+r)\\
&\quad+\nu((i+3+r)i+2k,(i+3+r)i,i+2+r),
\end{aligned}$$

where we set $k' = 2r+3+i-k$, use the inductive step on $p = i$ to obtain the second and third terms and compute $\nu((i+3+r)(i+1)+2k-2r-i-3, (i+4+r)(i+1), i+3+r)$ since $(i+3+r)(i+1)+2k-2r-i-3 < (i+4+r)(i+1)$ unless $k = r+i+2$. We also have

$$\begin{aligned}
&\nu((i+3+r)(i+1)+2k, (i+3+r)(i+1), i+2+r) \\
&\quad = \nu((i+2+r)(i+1)+2i+4r+5-2k, (i+3+r)(i+1), i+2+r) \\
&\quad\quad + \nu((i+3+r)i+2k, (i+3+r)i, i+2+r) \\
&\quad = \nu((i+2+r)(i+1)+2k-2r-i-3, (i+3+r)(i+1), i+2+r) \\
&\quad\quad + \nu((i+3+r)i+2i+4r+6-2k, (i+3+r)i, i+2+r) \\
&\quad\quad + \nu((i+3+r)i+2k, (i+3+r)i, i+2+r) \\
&\quad = \delta_{k,r+i+2} + \nu((i+3+r)i+2k', (i+3+r)i, i+2+r) \\
&\quad\quad + \nu((i+3+r)i+2k, (i+3+r)i, i+2+r).
\end{aligned}$$

This completes the this case and also completes the base case for the induction on $j$ for (4.3.1).

Now, to prove the base case for (4.3.2), we have

$$\begin{aligned}
&\nu((i+4+r)(i+1)+2, (i+4+r)(i+1), i+3+r) \\
&\quad = \nu((i+3+r)i+r, (i+4+r)(i+1), i+3+r) + \nu((i+4+r)i+2, (i+4+r)i, i+3+r) = 0,
\end{aligned}$$

using the inductive hypothesis for the second term and the fact that

$$(i+3+r)i < (i+4+r)(i+1)$$

to get the first term is zero. The base case with $j = 0$ is complete.

*Inductive step on $j$, $\forall r \geq 0, p = i+1$:* Now assume equality holds for $j \leq n$ for $p = i+1$ and for all $r \geq 0$. Again, we break the proof for (4.3.1) into cases. First, consider $2 \leq 2k < r$. We have

$$\begin{aligned}
&\nu((i+4+r)(i+1)+2(n+1)(i+3+r)+2k+2, (i+4+r)(i+1), i+3+r) \\
&\quad = \nu((i+4+r)i+2(n+1)(i+3+r)+2k+2, (i+4+r)i, i+3+r) \\
&\quad\quad + \nu((i+4+r)(i+1)+2n(i+3+r)+2(r-k)+2, (i+4+r)(i+1), i+3+r) \\
&\quad = \nu((i+3+r)i+2(n+1)(i+2+r)+2k, (i+3+r)i, i+2+r) \\
&\quad\quad + \nu((i+4+r)(i+1)+2n(i+3+r)+2(r-k)+2, (i+4+r)(i+1), i+3+r),
\end{aligned}$$

where we used Proposition 5(ii) and the inductive hypothesis on $p$ on the first term. Calculating the other multiplicity using Proposition 5, we obtain

$$\begin{aligned}
&\nu((i+3+r)(i+1)+2(n+1)(i+2+r)+2k,(i+3+r)(i+1),i+2+r)\\
&\quad=\nu((i+3+r)i+2(n+1)(i+2+r)+2k,(i+3+r)i,i+2+r)\\
&\qquad+\nu((i+3+r)(i+1)+2n(i+2+r)+2(r-k)+2,(i+3+r)(i+1),i+2+r).
\end{aligned}$$

We need to show

$$\begin{aligned}
&\nu((i+4+r)(i+1)+2n(i+3+r)+2(r-k)+2,(i+4+r)(i+1),i+3+r)\\
&\quad=\nu((i+3+r)(i+1)+2n(i+2+r)+2(r-k)+2,(i+3+r)(i+1),i+2+r).
\end{aligned}$$

We can use the inductive hypothesis on $j$ $(r-k)$-times to get

$$\begin{aligned}
&\nu((i+4+r)(i+1)+2n(i+3+r)+2(r-k)+2,(i+4+r)(i+1),i+3+r)\\
&\quad=\nu((i+4+k)(i+1)+2n(i+3+k)+2,(i+4+k)(i+1),i+3+k)\\
&\quad=0,
\end{aligned}$$

and similarly

$$\begin{aligned}
&\nu((i+3+r)(i+1)+2n(i+2+r)+2(r-k)+2,(i+3+r)(i+1),i+2+r)\\
&\quad=\nu((i+3+k)(i+1)+2n(i+2+k)+2,(i+3+k)(i+1),i+2+k)\\
&\quad=0.
\end{aligned}$$

Thus, the multiplicities are equal completing the case when $2\le 2k<r$. Calculating the other multiplicity using Proposition 5, we obtain

$$\begin{aligned}
&\nu((i+3+r)(i+1)+2(n+1)(i+2+r)+2k,(i+3+r)(i+1),i+2+r)\\
&\quad=\nu((i+3+r)i+2(n+1)(i+2+r)+2k,(i+3+r)i,i+2+r)\\
&\qquad+\nu((i+3+r)(i+1)+2n(i+2+r)+2(r-k)+2,(i+3+r)(i+1),i+2+r)\\
&\quad=\nu((i+3+r)i+2(n+1)(i+2+r)+2k,(i+3+r)i,i+2+r)\\
&\qquad+\nu((i+3+r)(i+1)+2n(i+2+r)+2k,(i+3+r)(i+1),i+2+r)\\
&\qquad+\nu((i+3+r)i+2n(i+2+r)+2(r-k)+2,(i+3+r)i,i+2+r).
\end{aligned}$$

As in the previous case, we can use the inductive hypothesis on $j$ $(r-k)$-times to obtain

$$
\begin{aligned}
&\nu((i+3+r)(i+1)+2n(i+2+r)+2(r-k)+2,(i+3+r)(i+1),i+2+r)\\
&\quad=\nu((i+3+k)(i+1)+2n(i+2+k)+2,(i+3+k)(i+1),i+2+k)\\
&\quad=0.
\end{aligned}
$$

We consider the case $2k=r,r+1$ separately. For $2k=r$, we have

$$
\begin{aligned}
&\nu((i+4+r)(i+1)+2(n+1)(i+3+r)+2k+2,(i+4+r)(i+1),i+3+r)\\
&\quad=\nu((i+4+r)i+2(n+1)(i+3+r)+2k+2,(i+4+r)i,i+3+r)\\
&\qquad+\nu((i+4+r)(i+1)+2n(i+3+r)+2k+2,(i+4+r)(i+1),i+3+r)\\
&\quad=\nu((i+3+r)i+2(n+1)(i+2+r)+2k,(i+3+r)i,i+2+r)\\
&\qquad+\nu((i+4+r)(i+1)+2n(i+3+r)+2k,(i+4+r)(i+1),i+3+r),
\end{aligned}
$$

where we used Proposition 5(ii) and the inductive hypothesis on $p$ and $j$ on the first and second terms respectively. Similarly, when $2k=r+1$, we have

$$
\begin{aligned}
&\nu(((i+4+r)(i+1)+2(n+1)(i+3+r)+2k+2,(i+4+r)(i+1),i+3+r)\\
&\quad=\nu((i+4+r)(i+1)+2(n+1)(i+3+r)+2r-2k+2,(i+4+r)(i+1),i+3+r)\\
&\qquad+\nu(((i+4+r)i+2(n+1)(i+3+r)+2k+2,(i+4+r)i,i+3+r)\\
&\quad=\nu((i+4+r)(i+1)+2n(i+3+r)+2k+2,(i+4+r)(i+1),i+3+r)\\
&\qquad+\nu((i+4+r)i+2(n+1)(i+3+r)+2r-2k+2,(i+4+r)i,i+3+r)\\
&\qquad+\nu(((i+4+r)i+2(n+1)(i+3+r)+2k+2,(i+4+r)i,i+3+r)\\
&\quad=\nu((i+3+r)(i+1)+2n(i+2+r)+2k,(i+3+r)(i+1),i+2+r)\\
&\qquad+\nu((i+4+r)i+2(n+1)(i+3+r)+2r-2k+2,(i+4+r)i,i+3+r)\\
&\qquad+\nu(((i+3+r)i+2(n+1)(i+2+r)+2k,(i+3+r)i,i+2+r),
\end{aligned}
$$

where we use the inductive hypothesis on $j$ and $p$ on the first and third terms, respectively. Moreover, applying the inductive hypothesis on $p$ $(r-k)$-times as in the previous case, we get that the second term is 0 i.e.,

$$\nu((i+4+r)i+2(n+1)(i+3+r)+2r-2k+2,(i+4+r)i,i+3+r)=0.$$

Calculating the other multiplicity, we have

$$\begin{aligned}&\nu(((i+3+r)(i+1)+2(n+1)(i+2+r)+2k,(i+3+r)(i+1),i+2+r)\\&\quad=\nu((i+3+r)(i+1)+2n(i+2+r)+2k+2,(i+3+r)(i+1),i+2+r)\\&\qquad+\nu(((i+3+r)i+2(n+1)(i+2+r)+2k,(i+3+r)i,i+2+r),\end{aligned}$$

which completes the cases for $2k=r,r+1$.

Now, consider $r+1<2k\leq 2r+1$. We have

$$\begin{aligned}&\nu(((i+4+r)(i+1)+2(n+1)(i+3+r)+2k+2,(i+4+r)(i+1),i+3+r)\\&\quad=\nu((i+4+r)(i+1)+2(n+1)(i+3+r)+2r-2k+2,(i+4+r)(i+1),i+3+r)\\&\qquad+\nu(((i+4+r)i+2(n+1)(i+3+r)+2k+2,(i+4+r)i,i+3+r)\\&\quad=\nu((i+4+r)(i+1)+2n(i+3+r)+2k+2,(i+4+r)(i+1),i+3+r)\\&\qquad+\nu((i+4+r)i+2(n+1)(i+3+r)+2r-2k+2,(i+4+r)i,i+3+r)\\&\qquad+\nu(((i+4+r)i+2(n+1)(i+3+r)+2k+2,(i+4+r)i,i+3+r)\\&\quad=\nu((i+3+r)(i+1)+2n(i+2+r)+2k,(i+3+r)(i+1),i+2+r)\\&\qquad+\nu((i+4+r)i+2(n+1)(i+3+r)+2r-2k+2,(i+4+r)i,i+3+r)\\&\qquad+\nu(((i+3+r)i+2(n+1)(i+2+r)+2k,(i+3+r)i,i+2+r),\end{aligned}$$

where we use the inductive hypothesis on $j$ and $p$ on the first and third terms, respectively. Moreover, applying the inductive hypothesis on $p$ $(r-k)$-times as in the previous case, we get that the second term is 0 i.e.,

$$\nu((i+4+r)i+2(n+1)(i+3+r)+2r-2k+2,(i+4+r)i,i+3+r)=0.$$

Similarly, we have

$$\begin{aligned}&\nu(((i+3+r)(i+1)+2(n+1)(i+2+r)+2k,(i+3+r)(i+1),i+2+r)\\&\quad=\nu((i+3+r)(i+1)+2(n+1)(i+2+r)+2r-2k+2,(i+3+r)(i+1),i+2+r)\\&\qquad+\nu(((i+3+r)i+2(n+1)(i+2+r)+2k,(i+3+r)i,i+2+r)\\&\quad=\nu((i+3+r)(i+1)+2n(i+2+r)+2k,(i+3+r)(i+1),i+2+r)\\&\qquad+\nu((i+3+r)i+2(n+1)(i+2+r)+2r-2k+2,(i+3+r)i,i+2+r)\\&\qquad+\nu(((i+3+r)i+2(n+1)(i+2+r)+2k,(i+3+r)i,i+2+r)\end{aligned}$$

where the second term is zero by applying the inductive hypothesis on $p$ $(r-k)$-times i.e.

$$\nu((i+3+r)i+2(n+1)(i+2+r)+2r-2k+2,(i+3+r)i,i+2+r)=0.$$

This completes the case when $r+1 \leq 2k \leq 2r+1$.

Now, we consider $2r+2 \leq k \leq 2r+i+3$. We have

$$\begin{aligned}
&\nu(((i+4+r)(i+1)+2(n+1)(i+3+r)+2k+2,(i+4+r)(i+1),i+3+r)\\
&\quad=\nu((i+4+r)(i+1)+2(n+1)(i+3+r)+2r-2k+2,(i+4+r)(i+1),i+3+r)\\
&\quad\ +\nu(((i+4+r)i+2(n+1)(i+3+r)+2k+2,(i+4+r)i,i+3+r)\\
&\quad=\nu((i+4+r)(i+1)+2n(i+3+r)+2k+2,(i+4+r)(i+1),i+3+r)\\
&\quad\ +\nu((i+4+r)i+2(n+1)(i+3+r)+2r-2k+2,(i+4+r)i,i+3+r)\\
&\quad\ +\nu(((i+4+r)i+2(n+1)(i+3+r)+2k+2,(i+4+r)i,i+3+r)\\
&\quad=\nu((i+3+r)(i+1)+2n(i+2+r)+2k,(i+3+r)(i+1),i+2+r)\\
&\quad\ +\nu((i+4+r)i+2(n+1)(i+3+r)+2r-2k+2,(i+4+r)i,i+3+r)\\
&\quad\ +\nu(((i+3+r)i+2(n+1)(i+2+r)+2k,(i+3+r)i,i+2+r),
\end{aligned}$$

where we use the inductive hypothesis on $j$ and $p$ on the first and third terms, respectively. For the second term, since $2r-2k+2 \leq 0$, we have

$$\begin{aligned}
&\nu((i+4+r)i+2(n+1)(i+3+r)+2r-2k+2,(i+4+r)i,i+3+r)\\
&\quad=\nu((i+4+r)i+2n(i+3+r)+2k'+2,(i+4+r)i,i+3+r)\\
&\quad=\nu((i+3+r)i+2n(i+2+r)+2k',(i+3+r)i,i+2+r),
\end{aligned}$$

where we use the inductive hypothesis on $j$ and $k'=i+3+2r-k>0$.In the same fashion, we have

$$
\begin{aligned}
&\nu(((i+3+r)(i+1)+2(n+1)(i+2+r)+2k,(i+3+r)(i+1),i+2+r)\\
&\quad=\nu((i+3+r)(i+1)+2(n+1)(i+2+r)+2r-2k+2,(i+3+r)(i+1),i+2+r)\\
&\quad\quad+\nu(((i+3+r)i+2(n+1)(i+2+r)+2k,(i+3+r)i,i+2+r)\\
&\quad=\nu((i+3+r)(i+1)+2n(i+2+r)+2k,(i+3+r)(i+1),i+2+r)\\
&\quad\quad+\nu((i+3+r)i+2(n+1)(i+2+r)+2r-2k+2,(i+3+r)i,i+2+r)\\
&\quad\quad+\nu((i+3+r)i+2(n+1)(i+2+r)+2k,(i+3+r)i,i+2+r)\\
&\quad=\nu((i+3+r)(i+1)+2n(i+2+r)+2k,(i+3+r)(i+1),i+2+r)\\
&\quad\quad+\nu((i+3+r)i+2n(i+2+r)+2k',(i+3+r)i,i+2+r)\\
&\quad\quad+\nu(((i+3+r)i+2(n+1)(i+2+r)+2k,(i+3+r)i,i+2+r)
\end{aligned}
$$

using the same notation for $k'$ as above. This completes the case for $2r+2 \leq k \leq 2r+i+3$

Lastly, consider $2r+i+4 \leq 2k \leq 2r+2i+4$. Then, we have

$$
\begin{aligned}
&\nu((i+4+r)(i+1)+2(n+1)(i+3+r)+2k+2,(i+4+r)(i+1),i+3+r)\\
&\quad=\nu((i+4+r)(i+1)+2(n+1)(i+3+r)+2i+8+4r-2k,(i+4+r)(i+1),i+3+r)\\
&\quad\quad+\nu((i+4+r)i+2(n+1)(i+3+r)+2k+2,(i+4+r)i,i+3+r)\\
&\quad=\nu((i+4+r)(i+1)+2n(i+3+r)+2k+2,(i+4+r)i,i+3+r)\\
&\quad\quad+\nu((i+4+r)i+2(n+1)(i+3+r)+2i+8+4r-2k,(i+4+r)(i+1),i+3+r)\\
&\quad\quad+\nu((i+4+r)i+2(n+1)(i+3+r)+2k+2,(i+4+r)i,i+3+r)\\
&\quad=\nu((i+3+r)(i+1)+2n(i+2+r)+2k,(i+3+r)i,i+2+r)\\
&\quad\quad+\nu((i+3+r)i+2(n+1)(i+2+r)+2(i+3+2r-k),(i+3+r)(i+1),i+2+r)\\
&\quad\quad+\nu((i+3+r)i+2(n+1)(i+2+r)+2k,(i+3+r)i,i+2+r),
\end{aligned}
$$

using the inductive hypothesis on $p$ for the first and second terms and the inductive hypothesis on $j$ for the third term. We also have

$$\begin{aligned}
&\nu((i+3+r)(i+1)+2(n+1)(i+2+r)+2k,(i+3+r)(i+1),i+2+r)\\
&\quad=\nu((i+3+r)(i+1)+2(n+1)(i+2+r)+2i+6+4r-2k,(i+3+r)(i+1),i+2+r)\\
&\qquad+\nu((i+3+r)i+2(n+1)(i+2+r)+2k,(i+3+r)i,i+2+r)\\
&\quad=\nu((i+3+r)(i+1)+2n(i+2+r)+2k,(i+3+r)i,i+2+r)\\
&\qquad+\nu((i+3+r)i+2(n+1)(i+2+r)+2(i+3+2r-k),(i+3+r)(i+1),i+2+r)\\
&\qquad+\nu((i+3+r)i+2(n+1)(i+2+r)+2k,(i+3+r)i,i+2+r),
\end{aligned}$$

thus equality in this case. We have completed the inductive step for $j$ for (4.3.2). This also completes the inductive step for $p$ for (4.3.2). Hence, we have established (4.3.2).

Now, to prove

$$\nu((i+4+r)(i+1)+2(n+1)(i+3+r)+2,(i+4+r)(i+1),i+3+r)=0.$$

Recall, we assume this equality holds for $j \leq n$ for $p = i+1$ and for all $r \geq 0$. We have

$$\begin{aligned}
&\nu((i+4+r)(i+1)+2(n+1)(i+3+r)+2,(i+4+r)(i+1),i+3+r)\\
&\quad=\nu((i+4+r)(i+1)+2n(i+3+r)+2r+2,(i+4+r)(i+1),i+3+r)\\
&\qquad+\nu((i+4+r)i+2(n+1)(i+3+r)+2,(i+4+r)i,i+3+r)\\
&\quad=\nu((i+4+r)(i+1)+2n(i+3+r)+2r+2,(i+4+r)(i+1),i+3+r),
\end{aligned}$$

by induction on $p$ for the second multiplicity. By $r$-applications of (4.3.1), we have

$$\nu((i+4+r)(i+1)+2n(i+3+r)+2r+2,(i+4+r)(i+1),i+3+r)=0.$$

This completes the inductive step on $j$ for (4.3.2). Moreover, this completes the inductive step on $p$ and the proof of the lemma.

# Conclusion

In this paper, we used the constructive proof of level $(\ell+1)$–Demazure flags of a level $\ell$–Demazure module in [4] to give a well-defined recursion for the graded multiplicities. For $\ell = 1, 2$, we gave closed formulae for the graded multiplicities. Moreover, the proper combination of these multiplicities with Naoi's result in [9] gave exact multiplicities for the flag of the local Weyl module for type $G_2$ by level 1-Demazure modules. We then related the results to partial theta functions. As it is difficult to find a closed form for arbitrary $\ell \in \mathbb{N}$, we turned our focus to numerical multiplicities. We proved recursions of generating functions for the specialization $q = 1$. This gave us two ways to determine the numerical multiplicities. Moreover, we also found relationships between the numerical multiplicities for different values of the level $\ell$.

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