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ON MULTILINEAR OSCILLATORY INTEGRALS, NONSINGULAR AND SINGULAR

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ABSTRACT. Basic questions concerning nonsingular multilinear operators with oscillatory factors are posed and partially answered. L^p norm inequalities are established for multilinear integral operators of Calderón-Zygmund type which incorporate oscillatory factors e^{iP}, where P is a real-valued polynomial.

1. INTRODUCTION

Consider multilinear functionals

$$(1.1) \qquad \Lambda_\lambda(f_1, f_2, \cdots, f_n) = \int_{\mathbb{R}^m} e^{i\lambda P(x)} \prod_{j=1}^n f_j(\pi_j(x)) \eta(x)\, dx$$

where $\lambda \in \mathbb{R}$ is a parameter, $P : \mathbb{R}^m \to \mathbb{R}$ is a measurable real-valued function, $m \geq 2$, and $\eta \in C_0^1(\mathbb{R}^m)$ is compactly supported. Each π_j denotes the orthogonal projection from $\mathbb{R}^m$ to a linear subspace $V_j \subset \mathbb{R}^m$ of any dimension $\kappa < m$, and $f_j : V_j \to \mathbb{C}$ is always assumed to be locally integrable with respect to Lebesgue measure on V_j. We assume for simplicity that κ is independent of j.

The integral Λ_λ is well-defined if all f_j belong to L^∞, and satisfies $|\Lambda_\lambda(f_1, \cdots, f_n)| \leq C \prod_j \|f_j\|_{L^\infty}$.

Definition 1.1. A measurable real-valued function P is said to have the power decay property, relative to a collection of subspaces $\{V_j\}$ of $\mathbb{R}^m$, on an open set $U \subset \mathbb{R}^m$ if for any $\eta \in C_0^1(U)$ there exist $\varepsilon > 0$ and $C < \infty$ such that

$$(1.2) \qquad |\Lambda_\lambda(f_1, \cdots, f_n)| \leq C(1 + |\lambda|)^{-\epsilon} \prod_{j=1}^n \|f_j\|_{L^\infty} \quad \text{for all } f_j \in L^\infty(V_j) \text{ and all } \lambda \in \mathbb{R}.$$

Moreover, for η supported in any fixed compact set, ε may be taken to be independent of η, and $C = O(\|\eta\|_{C^1})$.

Our goal is to characterize those data $(P, \{V_j\})$ for which the power decay property holds. This problem has been well studied in the bilinear case where $n = 2$, and to a lesser but still appreciable extent when $\kappa = 1$ and $n = m$, but in the general case basic issues seem not to have been thoroughly explored.

A necessary condition for power decay is that P can not be expressed as a linear combination of measurable functions $p_j \circ \pi_j$. Indeed, if $P = \sum_j p_j \circ \pi_j$ then for $f_j = e^{-i\lambda p_j} \in L^\infty$ one has $e^{i\lambda P(x)} \prod_j f_j(\pi_j(x)) \equiv 1$, and there is consequently no decay.

Definition 1.2. A polynomial P is said to be degenerate (relative to $\{V_j\}$) if there exist polynomials $p_j : V_j \to \mathbb{R}$ such that $P = \sum_{j=1}^{n} p_j \circ \pi_j$. Otherwise P is nondegenerate.

In the case $n = 0$, where the collection of subspaces $\{V_j\}$ is empty, P is considered to be nondegenerate if and only if it is nonconstant.

We will show in Lemma 3.4 that degeneracy is also equivalent to the existence, in some nonempty open set, of a representation of the more general form $P = \sum_{j=1}^{n} h_j \circ \pi_j$ where the h_j are arbitrary distributions rather than polynomials.

More generally, a (measurable) function P is said to be degenerate if it can be expressed as $\sum_j p_j \circ \pi_j$ for some (measurable) functions p_j, or distributions. But we will restrict attention to polynomials henceforth.

Question. Is the power decay property equivalent to nondegeneracy, for real-valued polynomials?

We were led to this question by a problem concerning multilinear singular integral operators; see Theorems 2.5 and 2.6 below. We have been able to answer it, always in the affirmative, only in special cases.

Numerous variants can be formulated. One can ask when (1.2) holds with the right-hand side replaced by $\Theta(\lambda) \prod_{j=1}^{n} \|f_j\|_{L^\infty}$ where $\Theta(\lambda)$ is a specific function of $(1+|\lambda|)$, or whether there merely exists some function Θ tending to zero as $|\lambda| \to \infty$ for which it holds. When (1.2) does hold, one can ask what is the optimal power ε in (1.2). We will focus on the first formulation, which is the one most directly relevant to our applications to multilinear singular integral operators, and is possibly the most fundamental.

The extreme formulation in which all f_j are measured in the strongest Lebesgue norm L^∞ is the essence of the matter. Since $|\Lambda_\lambda(f_1, \cdots, f_n)| \leq C \|f_k\|_{L^1} \prod_{j \neq k} \|f_j\|_{L^\infty}$ uniformly in λ for any k, a simple interpolation argument shows that if (1.2) does hold, then a decay estimate of the same type holds with $\prod_j \|f_j\|_\infty$ replaced by $\prod_j \|f_j\|_{p_j}$ for various n-tuples of indices $p_j \in (1, \infty]$. More precisely, if the integral converges absolutely whenever each $f_j \in L^{p_j}$, then $|\Lambda_\lambda(f_1, \cdots, f_n)| \leq C(1 + |\lambda|)^{-\varepsilon} \prod_j \|f\|_{q_j}$ whenever each $q_j > p_j$, where $\varepsilon > 0$ depends on $\{p_j\}, \{q_j\}$, and other n-tuples q can in general be obtained via further interpolations. On the other hand, the combinations of exponents p_j for which the integral is guaranteed to converge absolutely depend strongly on $\{V_j\}$.

When $n = 0$, that is, when $\{V_j\}$ is empty, one is dealing with oscillatory integrals $\Lambda_\lambda = \int e^{i\lambda P(x)} \eta(x)\, dx$, which are of the first type in the terminology of Stein [10]. The general case can also be regarded as concerning oscillatory integrals of the first type. Indeed, (1.2) with L^∞ bounds on the functions f_j is equivalent to $|\int e^{i\lambda\phi(x)}\eta(x)\,dx| \leq C|\lambda|^{-\epsilon}$ *uniformly* for all phase functions of the form $\phi = P - \sum_j h_j \circ \pi_j$, where the h_j are arbitrary real-valued measurable functions; one implication is tautologous, while the converse is nearly trivial and is proved in [1].

The regularity condition $\eta \in C^1$ is rather arbitrary. If η is merely Hölder continuous then for any $s < \infty$, η may be decomposed as a smooth function whose C^s norm is $O(|\lambda|^{C\delta})$ plus a remainder which is $O(|\lambda|^{-\delta})$ in supremum norm. If (1.2) holds for all $\eta \in C_0^s$ with a constant C which is $O(\|\eta\|_{C^s})$ then it follows from this decomposition, with $\delta = \varepsilon/2C$, that it continues to hold for all Hölder continuous η.

$\prod_{j=1}^{n} f_j \circ \pi_j$ could be replaced by $\prod_{j=1}^{n} g_j \circ \ell_j$ in (1.1), where each $\ell : \mathbb{R}^m \to \mathbb{R}^\kappa$ is linear and has full rank, without any increase in generality. For any function $g \circ \ell$ may be expressed as $f \circ \pi$, where π is the orthogonal projection of $\mathbb{R}^m$ onto the orthocomplement V of the nullspace of ℓ; $f \in L^p(V)$ if and only if $g \in L^p(\mathbb{R}^\kappa)$ with comparable norms.

The linear prototype for power decay is the inequality (see Stein [10], p. 416 and Phong and Stein [7])

$$(1.3) \qquad \Big| \int e^{i\lambda P(x,y)} f(x) g(y) \eta(x,y) \, dx \, dy \Big| = O\big(|\lambda|^{-\varepsilon} \|f\|_{L^2} \|g\|_{L^2} \big),$$

with suitable uniformity, whenever P is a real-valued polynomial, of bounded degree, which is nondegenerate in the sense that some mixed partial derivative $\partial^{\alpha+\beta} P / \partial x^\alpha \partial y^\beta$, with both $\alpha, \beta \neq 0$, does not vanish identically; this is equivalent to the impossibility of decomposing $P(x,y)$ as $p(x) + q(y)$.

A related result is the lemma of van der Corput, which asserts for instance that $\int_a^b e^{i\lambda\varphi(t)} \, dt = O(1 + |\lambda|)^{-\varepsilon}$ provided that φ is real-valued and that some derivative of φ is bounded away from zero (in the case of the first derivative, it is also assumed to be monotone). Here the upper bound for the integral depends only on a lower bound for the derivative; no upper bound on φ or its derivatives, hence in some sense no *a priori* smoothness condition, are imposed. Higher-dimensional versions of van der Corput's lemma, likewise without assumptions of upper bounds or extra smoothness, were established in [1].

Our nondegeneracy condition replaces the hypothesis of a nonvanishing derivative. (1.2), like (1.3), is invariant under the replacement of P by $P - \sum_j p_j \circ \pi_j$, for arbitrary real-valued measurable functions p_j, and there is no *a priori* smoothness condition or upper bound on $P - \sum_j p_j \circ \pi_j$. Formally the power decay property more closely resembles (1.3), but our analysis is closely related to [1], and combinatorial issues lurk here as they did there.

2. Results

2.1. **Further definitions.** Given a degree d, fix a norm $\| \cdot \|_{\mathcal{P}_d}$ on the finite-dimensional vector space $\mathcal{P}_d$ of all polynomials in $\mathbb{R}^m$ of degree $\leq d$. Given d, the norm of P relative to a finite collection of linear subspaces $\{V_j\}$ is defined to be $\inf \|P - \sum_j p_j \circ \pi_j\|_{\mathcal{P}_d}$, where the infimum is taken over all real-valued polynomials p_j of degrees $\leq d$. Polynomials P_α are said to be uniformly nondegenerate if they are all of degrees $\leq d$ for some finite d, and if there exists $c > 0$ such that the norm of P_α relative to $\{V_j\}$ (and relative to this degree d) is $\geq c$ for all α.

More generally, if we are given given a finite collection of subspaces $\{V_j^\alpha : \alpha \in A\}$ and a polynomial P_α of uniformly bounded degree, we may still define uniform nondegeneracy by requiring that $\inf_\alpha \inf_{\{p_j\}} \|P_\alpha - \sum_j p_j \circ \pi_j^\alpha\|_{\mathcal{P}_d} \geq c > 0$, where π_j^α is the orthogonal projection onto V_j^α.

These definitions are independent of d provided only that each P has degree $\leq d$, a simple consequence of the equivalence of any two norms on a finite-dimensional vector space, and the fact that if $P = \sum_j p_j \circ \pi_j$ then $P = \sum_j \tilde{p}_j \circ \pi_j$ where $\tilde{p}_j$ is the sum of all terms of degrees $\leq \deg(P)$ in the decomposition of π_j as a linear combination of monomials.

Because the family of polynomials of degree $\leq d$ has finite dimension, it is easily verified that the infimum defining the relative norm is actually assumed for some polynomials p_j. Thus either P is degenerate, or the infimum is strictly positive.

Definition 2.1. A finite collection of subspaces $\{V_j\}$ is said to have the power decay property if every real-valued polynomial P which is nondegenerate relative to $\{V_j\}$ has the power decay property (1.2), in every open set U.

A collection of subspaces $\{V_j\}$ is said to have the uniform power decay property if (1.2) holds, with uniform constants C, ε, for any family of real-valued polynomials of bounded degrees which are uniformly nondegenerate relative to $\{V_j\}$.

A related concept turns out to be somewhat easier to analyze. To any distribution $h \in \mathcal{D}'(V_j)$ is naturally associated a distribution $h \circ \pi_j \in \mathcal{D}'(\mathbb{R}^m)$. Denote by $V_j^{\perp}$ the orthogonal complement of V_j. Then $h \circ \pi_j$ is annihilated by any first-order constant-coefficient differential operators of the form $w \cdot \nabla$, where $w \in V_j^{\perp}$.

Definition 2.2. A polynomial P is said to be simply nondegenerate relative to $\{V_j : 1 \leq j \leq n\}$ if there exists a differential operator L of the form $L = \prod_{j=1}^{n}(w_j \circ \nabla)$, with each $w_j \in V_j^{\perp}$, such that $L(P)$ does not vanish identically.

Simple nondegeneracy implies nondegeneracy. Indeed, for any distribution f defined on some V_j, the j-th factor of L annihilates $f \circ \pi_j$, and hence L does so since all factors commute. Thus $L(P - \sum_j f_j \circ \pi_j) \equiv L(P)$.

The converse is not true in general. Consider the case $\kappa = 1$ where each subspace V_j has dimension one. Let $\mathcal{L}$ be any homogeneous constant-coefficient linear partial differential operator with real coefficients, and let σ be its symbol. Identify the dual space of $\mathbb{R}^m$ with $\mathbb{R}^m$ via the inner product. Let e be any unit vector and let $\pi(x) = \langle x, e \rangle e$. Then $\mathcal{L}$ annihilates every function $f \circ \pi$ if and only if $\sigma(e) = 0$. To create examples of nondegenerate polynomials, fix any such nonelliptic $\mathcal{L}$, choose $\{e_j\}$ to be any finite collection of distinct unit vectors satisfying $\sigma(e_j) = 0$, and choose any real-valued polynomial P such that $\mathcal{L}(P)$ does not vanish identically. This forces P to be nondegenerate relative to $\{V_j\}$, where V_j is the span of e_j. If $m \geq 3$ then $P, \mathcal{L}, \{V_j\}$ can be chosen so that P has degree two, yet $\{V_j\}$ has arbitrarily large finite cardinality.

2.2. **Decay for nonsingular oscillatory multilinear functionals.** We now state several theorems asserting that nondegeneracy implies the power decay property, under various auxiliary hypotheses. To formulate the first of these, let $\{V_j : 1 \leq j \leq n\}$ be a collection of subspaces of $\mathbb{R}^m$ of dimension κ. We say that they are in general position if any subcollection of cardinality $k \geq 1$ spans a subspace of dimension $\min(k\kappa, m)$. It is elementary that for subspaces in general position, if each f_j belongs to $L^2(\mathbb{R})$, then the product of any k functions $f_j \circ \pi_j$ belongs to $L^2(\mathbb{R}^m)$ provided that $k\kappa \leq m$. When $k\kappa \leq 2m$ the product belongs to L^1, by Cauchy-Schwarz. Therefore the integral defining Λ_λ converges absolutely, and $|\Lambda_\lambda(f_1, \cdots, f_n)| \leq C \prod_j \|f_j\|_{L^2}$, uniformly in $\lambda \in \mathbb{R}$.

Theorem 2.1. *Suppose that $n < 2m$. Then any family $\{V_j : 1 \leq j \leq n\}$ of one-dimensional subspaces of $\mathbb{R}^m$ which lie in general position has the uniform power decay property. Moreover under these hypotheses*

$$(2.1) \qquad |\Lambda_\lambda(f_1, \cdots, f_n)| \leq C(1 + |\lambda|)^{-\varepsilon} \prod_{j=1}^{n} \|f_j\|_{L^2}$$

for all polynomials P of bounded degree which are uniformly nondegenerate with respect to $\{V_j\}$, for all functions $f_j \in L^2(\mathbb{R}^1)$, with uniform constants $C, \varepsilon \in \mathbb{R}^+$.

The case $n = m$ is known; Phong, Stein, and Sturm [8] have obtained much more precise results on the exponent in the power decay estimate, phrased in terms of the reduced Newton polyhedron of P. The case $n < m$ follows from $n = m$. Indeed, choose coordinates $(x', x'') \in \mathbb{R}^n \times \mathbb{R}^{m-n}$, in such a way that the first factor $\mathbb{R}^n$ equals the span of the subspaces V_j. If $\partial P / \partial x''$ does not vanish identically then (1.2) does hold, as one sees by integrating with respect to x'' for generic fixed x' and using well-known results for oscillatory integrals of the first type in the sense of [10]. If P is independent of x'' then matters have been reduced to the case where the ambient dimension is n.

As will be clear from its proof, (2.1) continues to hold with uniform constants C, ε if the collection $\{V_j\}$ varies over a compact subset of the open subset of the relevant Grassmannian manifold consisting of all such collections in general position, and the polynomials P are uniformly nondegenerate relative to $\{V_j\}$.

The condition $n < 2m$ is necessary for (2.1), with $\|f_j\|_{L^2}$ on the right-hand side rather than $\|f_j\|_{L^\infty}$ as in Definition 1.1. To see this in the main case where $\{V_j\}$ span $\mathbb{R}^m$, consider without loss of generality the case $n = 2m$, and let each f_j equal the characteristic function of $[-\delta, \delta]$. If δ is chosen to be a small fixed constant times $|\lambda|^{-1}$, then $|\Lambda_\lambda(f_1, \cdots, f_n)| \geq c\delta^m$. On the other hand $\prod_j \|f_j\|_{L^2} = c'\delta^{n/2}$. It follows from the same construction that if $n > 2m$ then the integral defining $\Lambda_\lambda(f_1, \cdots, f_n)$ will in general not even be absolutely convergent for general $f_j \in L^2$.

Theorem 2.2. *Any collection of subspaces $\{V_j\}$ of codimension one has the uniform power decay property.*

In particular, when $m = 2$ then $\kappa = 1 = m - 1$ and hence nondegeneracy of P implies (1.2), no matter how large the collection of subspaces V_j may be.

Theorem 2.3. *Let m, κ be arbitrary. Then any simply nondegenerate polynomial has the power decay property (1.2) in every open set.*

More precisely, let $d \in \mathbb{N}$ and $c > 0$. Let $L = \prod_{j=1}^n (w_j \cdot \nabla)$ where each $w_j \in V_j^\perp$ is a unit vector. Then there exist $C < \infty$ and $\varepsilon > 0$ such that for any real-valued polynomial P of degree $\leq d$ such that $\max_{|x| \leq 1} |L(P)(x)| \geq c$,

$$|\Lambda_\lambda(f_1, \cdots, f_n)| \leq C(1 + |\lambda|)^{-\varepsilon} \prod_j \|f_j\|_\infty$$

for all functions $f_j \in L^\infty$ and all $\lambda \in \mathbb{R}$.

In each of these theorems, the regularity hypothesis on η can be relaxed to Hölder continuity, as discussed above.

Theorem 2.2 directly implies Theorem 2.3. Indeed, if w_j are as in the latter theorem, define $\tilde{V}_j = w_j^\perp$, thus obtaining codimension one subspaces $\tilde{V}_j \supset V_j$. Each $f_j \circ \pi_j$ can be rewritten as $\tilde{f}_j \circ \tilde{\pi}_j$ where $\tilde{\pi}_j$ is the orthogonal projection onto $\tilde{V}_j$ and $\tilde{f}_j(y) = f_j \circ \pi_j(y)$ for $y \in \tilde{V}_j$. The hypothesis $L(P) \neq 0$ guarantees that P is nondegenerate relative to $\{\tilde{V}_j\}$, with appropriate uniformity. On the other hand, we will show in Proposition 3.1 that for $\kappa = m - 1$, nondegeneracy is equivalent to simple nondegeneracy, so that Theorem 2.2 conversely implies Theorem 2.3.

The next result concerns the case of small codimension, but is less satisfactory than the definitive statement of Theorem 2.2 for codimension one.

Theorem 2.4. *Let $\{V_j : 1 \leq j \leq n\}$ be a finite collection of subspaces of $\mathbb{R}^m$, each having dimension κ and lying in general position. If*

$$n \leq \frac{m}{m - \kappa}$$

then $\{V_j\}$ has the uniform decay property.

In this result, general position of $\{V_j\}$ means instead that the orthocomplements $\{V_j^\perp\}$ lie in general position in the sense defined above, that is, their joint span has dimension $\leq (m - \kappa)n$. Lemma 3.5 will assert that nondegeneracy is equivalent to simple nondegeneracy when $n \leq \frac{m}{m - \kappa}$, so that Theorem 2.4 will be a corollary of Theorem 2.3.

2.3. **Singular integrals.** For any real valued polynomial $P(x,t)$ of degree d, consider the operator

$$(2.2) \qquad T(f,g)(x) = \int_{-\infty}^{\infty} e^{iP(x,t)} f(x-t)g(x+t)\, t^{-1}\, dt$$

where it is initially assumed that $f, g \in C_0^1$, the class of all continuously differentiable functions having compact supports, and the integral is taken in the principal-value sense. One of the purposes of this note is to establish the following L^p bounds for these operators.

Theorem 2.5. *For any exponents $p_1, p_2, q \in (0, \infty)$ such that $q^{-1} = p_1^{-1} + p_2^{-1}$, $p_1, p_2 > 1$ and $q > 2/3$, and any degree $d \geq 1$, there exists $C < \infty$ such that $\|T(f,g)\|_q \leq C \|f\|_{p_1} \|g\|_{p_2}$ for all $f, g \in C_0^1$, uniformly for all real-valued polynomials P of degrees $\leq d$.*

This answers a question raised by Lacey and Thiele. The cases $d = 0, 1, 2$ were previously known. Indeed, the case $d = 0$ is a celebrated theorem of Lacey and Thiele [4]. The case $d = 1$, that is $P(x,t) = a_0 x + a_1 t$, can be reduced to $d = 0$ by replacing f by $\tilde{f}(x) = e^{ia_0 x/2 - ia_1 x/2} f(x)$ and g by $\tilde{g}(x) = e^{ia_0 x/2 + ia_1 x/2} f(x)$. The case $d = 2$, that is, $P(x,t) = a_0 x^2 + a_1 xt + a_2 t^2 + b_1 x + b_2 t$, is likewise reducible to $d = 0$ by the substitution $\tilde{f}(x) = e^{-ia_1 x^2/4 + ia_2 x^2/2} f$, $\tilde{g}(x) = e^{ia_1 x^2/4 + ia_2 x^2/2} g$. But for $d \geq 3$ no such simple reduction exists.

Theorem 2.5 is a bilinear analogue of a theorem of Ricci and Stein [9], who proved L^p estimates for linear operators $f \mapsto \int e^{iP(x,t)} f(x-t) K(t)\, dt$, for arbitrary real-valued polynomials P and Calderón-Zygmund kernels K. It is a special case of the following more general result. Let $n \geq 1$, $\Gamma = \{\xi \in \mathbb{R}^{n+1} : \xi_1 + \xi_2 + \cdots + \xi_{n+1} = 0\}$, and Γ' be the orthogonal complement of a subspace of Γ such that the dimension of $\Gamma' \cap \Gamma$ is k. Let K be a k-dimensional Calderón-Zygmund kernel on $\Gamma' \cap \Gamma$, that is, K is Lipschitz continuous except at the origin, $K(r\gamma) \equiv r^{-k} K(\gamma)$ for all $r > 0$ and $\gamma \neq 0$, and $\int_{S^{k-1}} K\, d\sigma = 0$ where σ denotes surface measure on the unit sphere S^{k-1} in $\Gamma' \cap \Gamma$. We define the n-linear operator T by

$$(2.3) \qquad T(f_1, f_2, \cdots, f_n)(x) = \int_{\Gamma' \cap \Gamma} f_1(x + \gamma_1) f_2(x + \gamma_2) \cdots f_n(x + \gamma_n) K(\gamma) d\gamma\,,$$

for $x \in \mathbb{R}^1$, where $d\gamma$ is Lebesgue measure on $\Gamma' \cap \Gamma$, and $\gamma_i \in \mathbb{R}^1$ is the i-th coordinate of $\gamma \in \mathbb{R} \times \mathbb{R} \cdots \times \mathbb{R}$ as an element of $\mathbb{R}^{n+1}$. The integral is interpreted in the principal-value sense, so that $T(f_1, \cdots, f_n)(x)$ is well-defined provided that each $f_j \in C^1$ has compact support. We always assume that any $k+1$ variables in $\{x + \gamma_1, \cdots, x + \gamma_n, x\}$ are linearly independent.

To such an operator T and to any real-valued polynomial $P(x, \gamma_1, \cdots, \gamma_n)$ we associate the multilinear operator

$$(2.4) \quad T_P(f_1, f_2, \cdots, f_n)(x) = \int_{\Gamma' \cap \Gamma} e^{iP(x, \gamma_1, \cdots, \gamma_n)} f_1(x + \gamma_1) f_2(x + \gamma_2) \cdots f_n(x + \gamma_n) K(\gamma) d\gamma\,,$$

which is again well-defined when each $f_j \in C_0^1$.

Write $p_0 = q'$.

Theorem 2.6. *Suppose that $n \leq 2k$. Then for any real-valued polynomial P, T_P maps $\otimes_{j=1}^n L^{p_j}$ boundedly to L_0^p whenever $p_0 > n^{-1}$, $1 < p_j \leq \infty$ and $p_0^{-1} = \sum_j p_j^{-1}$, provided that*

$$(2.5) \qquad \frac{1}{p_{i_1}} + \frac{1}{p_{i_2}} + \cdots + \frac{1}{p_{i_r}} < \frac{2k + r + 1 - n}{2}$$

for all $0 \le i_1 < i_2 < \cdots < i_r \le n$ and all $1 \le r \le n+1$. This conclusion holds uniformly for all polynomials of degrees $\le D$, for any $D < \infty$.

Under these hypotheses, the nonoscillatory case $P \equiv 0$ was treated in [6]. Theorems 2.5 and 2.6 will be proved by combining previously known results for nonoscillatory multilinear singular integral operators with our new results for nonsingular oscillatory integrals.

The remainder of the paper is devoted to proofs of the theorems. Certain algebraic aspects of nondegeneracy come into play in the analysis and are developed in §3; this material is used throughout the paper. The simply nondegenerate case is treated in §4. The analysis when the dimension κ equals 1 involves an additional ingredient and is presented in §5. Applications to multilinear operators with Calderón-Zygmund singularities and oscillatory factors are carried out in §6. §7 is devoted to an extension of Theorem 2.2 and, as an application, a simpler reorganization of the proof of Theorem 2.1.

Our results are certainly not definitive. One known inequality not covered by our theorems is the twisted convolution inequality

$$\Big| \iint_{\mathbb{C}^{n+n}} f_1(z) f_2(z-w) f_3(w) e^{i\lambda \operatorname{Im}(z \cdot \bar{w})} \Big| \le C|\lambda|^{-n/2} \prod_{j=1}^{3} \|f_j\|_{L^2(\mathbb{C}^n)},$$

where the integration is with respect to Lebesgue measure on $\mathbb{C}^{n+n}$. See Proposition 3, page 556 of [10].

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3. Algebraic aspects of nondegeneracy

This section develops various general properties, largely algebraic in nature, of nondegeneracy. The first observation is that for given dimensions m, κ, degree of multilinearity n, and subspaces V_j, there exist nondegenerate homogeneous polynomials P of all sufficiently high degrees. Indeed, the dimension of the vector space of all $\sum_j q_j \circ \pi_j$, where q_j is an arbitrary homogeneous polynomial of degree D on $\mathbb{R}^\kappa$, is $O(nD^\kappa)$, while the space of all homogeneous polynomials $P : \mathbb{R}^m \to \mathbb{R}$ is $\sim cD^m \gg D^\kappa$ as $D \to \infty$.

As noted above, simple nondegeneracy implies nondegeneracy. In the codimension one case they are equivalent:

Proposition 3.1. *If each subspace V_j has codimension one, that is if $\kappa = m - 1$, then any nondegenerate polynomial is simply nondegenerate.*

Proof. Let a polynomial P and distinct subspaces V_j of codimension 1 be given. Let w_j be unit vectors orthogonal to V_j, let $L = \prod_{j=1}^{n}(w_j \cdot \nabla)$, and let $L' = \prod_{j=1}^{n-1}(w_j \cdot \nabla)$. We must show that if $LP \equiv 0$, then P is degenerate.

Proceed by induction on n, which is the total number of subspaces V_j. Then since $L'(w_n \cdot \nabla)P \equiv 0$, $(w_n \cdot \nabla)P = \sum_{j=1}^{n-1} q_j \circ \pi_j$ for some polynomials q_j. By a rotation of the coordinate system it may be arranged that $w_n \cdot \nabla = \partial_{x_m}$.

$(w_n \cdot \nabla)(f \circ \pi_j) = (D_j f) \circ \pi_j$ for some nonvanishing constant-coefficient real vector fields D_j, for w_n cannot be orthogonal to V_j for $j < n$ since it is orthogonal to V_n, and these codimension one subspaces are distinct. Consequently there exist polynomials of the form $p_j \circ \pi_j$ for $1 \le j \le n-1$ such that $(w_n \cdot \nabla)(p_j \circ \pi_j) = q_j \circ \pi_j$.

Thus $(w_n \cdot \nabla)(P - \sum_{j<n} p_j \circ \pi_j) \equiv 0$, so the polynomial $(P - \sum_{j<n} p_j \circ \pi_j)(x)$ is a function of $x_m = \pi_m(x)$ alone, and hence is of the form $p_n \circ \pi_n$. This proves that P is degenerate. $\square$

Lemma 3.2. *Let P be a homogeneous polynomial of some degree d. Then P is nondegenerate relative to a finite collection of subspaces $\{V_j\}$, of any dimension, if and only if there exists a constant-coefficient partial differential operator $\mathcal{L}$, homogeneous of degree d, such that $\mathcal{L}(P) \neq 0$ but $\mathcal{L}(p_j \circ \pi_j) \equiv 0$ for every polynomial $p_j : V_j \to \mathbb{C}$ of degree d.*

Proof. If such an operator $\mathcal{L}$ exists then P is obviously nondegenerate. To prove the converse, fix d and denote by $\mathcal{P}_d$ the vector space of all homogeneous polynomials of degree d. The pairing $\langle \mathcal{L}, P \rangle = \mathcal{L}(P)$ between homogeneous constant-coefficient differential operators of the same degree d and elements of $\mathcal{P}_d$ is nondegenerate. Thus the dual space of $\mathcal{P}_d$ may be canonically identified with the vector space of all such differential operators.

If P is nondegenerate then it does not belong to the subspace of all degenerate homogeneous polynomials, so there exists a linear functional which annihilates that subspace, but not P. This functional may be realized as $Q \mapsto \mathcal{L}(Q)$ for some homogeneous constant-coefficient linear partial differential operator $\mathcal{L}$, completing the proof in the converse direction. $\square$

From this it is apparent that if D and $\{\pi_j\}$ are held fixed, then the collection of all nondegenerate polynomials P of degrees $\leq D$ is an open subset of the vector space of all polynomials of degrees $\leq D$. On the other hand, nondegeneracy is not an open condition on $\{\pi_j : 1 \leq j \leq n\}$; if n is large relative to m then the condition that there exist nontrivial differential operators annihilating all functions $q_j \circ \pi_j$ is unstable under small perturbations.

Any polynomial may be expressed in a unique way as a sum of homogeneous polynomials of distinct degrees.

Lemma 3.3. *A polynomial P is nondegenerate relative to $\{V_j\}$ if and only if at least one of its homogeneous summands is nondegenerate. Moreover, a homogeneous polynomial is degenerate if and only if it may be expressed as $\sum_j p_j \circ \pi_j$ where each p_j is a homogeneous polynomial of the same degree.*

Proof. Both of these assertions follow from the simple fact that a polynomial $p_j : \mathbb{R}^\kappa \to \mathbb{R}$ is homogeneous of some degree if and only if $p_j \circ \pi_j : \mathbb{R}^k \to \mathbb{R}$ is homogeneous, of the same degree. $\square$

If P can be decomposed as $\sum_j f_j \circ \pi_j$ for some functions h_j which are not necessarily polynomials, then the decay bound (1.2) certainly fails, for the same reason noted earlier (set $f_j = e^{-i\lambda h_j}$). The next lemma says that it makes no difference whether the h_j are taken to be polynomials, or arbitrary functions, in the definition of nondegeneracy.

Lemma 3.4. *A polynomial P is degenerate with respect to a collection of subspaces $\{V_j\}$ if and only if there exist distributions h_j in $\mathbb{R}^\kappa$ such that $P = \sum_j h_j \circ \pi_j$ in some open set.*

Proof. Let $\{V_j\}$ be given, and let P be any nondegenerate polynomial of some degree d. We need only show that P can't be decomposed locally as $\sum_j h_j \circ \pi_j$, since the converse is a tautology. It is no loss of generality to suppose that the homogeneous summand of P of degree d is nondegenerate. For otherwise we may express that summand as $\sum_j p_j \circ \pi_j$ where the p_j are homogeneous polynomials of degree d, then replace P by $P - \sum_j p_j \circ \pi_j$ to reduce the degree.

By Lemma 3.2, there exists a constant-coefficient linear partial differential operator $\mathcal{L}$, homogeneous of degree d, such that $\mathcal{L}(P) \neq 0$, yet $\mathcal{L}(p_j \circ \pi_j) = 0$ for any polynomial p_j homogeneous of degree d, for any index j.

$\mathcal{L}(Q)(0) = 0$ for any polynomial $Q : \mathbb{R}^m \to \mathbb{R}$ which is homogeneous of any degree except d. By combining this with the hypothesis we find that $\mathcal{L}(p \circ \pi_j)(0) = 0$ for any polynomial $p : \mathbb{R}^\kappa \to \mathbb{R}$ and any index j. Consequently $\mathcal{L}(p \circ \pi_j)(x) = 0$ for all $x \in \mathbb{R}^m$. Since any distribution is a limit of polynomials in the distribution topology, $\mathcal{L}(h_j \circ \pi_j)$ likewise vanishes identically, in the sense of distributions. Yet $\mathcal{L}$ does not annihilate P, so P cannot be decomposed as $\sum_j h_j \circ \pi_j$. $\qquad\square$

In Lemmas 3.5 and 3.6 we say that a polynomial is a *monomial* if it can be expressed as a product of linear factors. Thus $(x_1 + x_2)x_1^2 x_3$ is a monomial. By a *differential monomial* we mean a linear partial differential operator which can be expressed as a product of finitely many real vector fields, each with constant coefficients.

Lemma 3.5. *Let $\{V_j : 1 \leq j \leq n\}$ be a finite collection of subspaces of $\mathbb{R}^m$, each having dimension κ and lying in general position. If $(m - \kappa)n \leq m$ then any polynomial which is nondegenerate relative to $\{V_j\}$ is simply nondegenerate relative to $\{V_j\}$.*

In Lemmas 3.5 and 3.6, general position means that the orthocomplements $V_j^\perp$ are linearly independent, that is, together span a space of dimension $(m - \kappa)n$. This only makes sense under the hypothesis that $(m - \kappa)n \leq m$.

Proof. Consider the variety $\mathcal{V} = \cup_j V_j$. Let $\mathcal{I}$ be the ideal in $\mathbb{C}[x]$ consisting of all real polynomials which vanish on $\mathcal{V}$. Then $\mathcal{I}$ is of course finite-dimensional as a $\mathbb{C}[x]$-module. According to the next lemma, $\mathcal{I}$ is generated by some finite set of monomials.

Now consider the ring $\mathcal{L}$ of all constant-coefficient linear partial differential operators in $\mathbb{R}^m$, and in it the ideal $\mathcal{L}_{\{V_j\}}$ consisting of all elements which annihilate all polynomials $\sum_j p_j \circ \pi_j$. The map $L \mapsto \sigma(L)$ from L to its symbol is an isomorphism of $\mathcal{L}$ with $\mathbb{C}[x]$, which maps $\mathcal{L}_{\{V_j\}}$ to $\mathcal{I}$. Thus any element $L \in \mathcal{L}_{\{V_j\}}$ may expressed as a finite sum $\sum_\alpha \ell_\alpha \circ \mathcal{M}_\alpha$ where each $\mathcal{M}_\alpha$ is a differential monomial which likewise annihilates all $p_j \circ \pi_j$, and each ℓ_α is some constant-coefficient linear partial differential operator.

If a polynomial P is nondegenerate relative to $\{V_j\}$, then there exists $L \in \mathcal{L}_{\{V_j\}}$ such that $L(P)$ does not vanish identically. Therefore there exists a differential monomial $\mathcal{M}_\alpha$ such that $\mathcal{M}_\alpha(P)$ does not vanish identically, so that P is simply nondegenerate. $\qquad\square$

Lemma 3.6. *Let $\mathcal{V} = \cup_{j=1}^n \mathcal{V}_j$ where each $\mathcal{V}_j$ is a linear subspace of $\mathbb{R}^m$ of dimension κ. Suppose that $(m - \kappa)n \leq m$, and that the subspaces $\mathcal{V}_j$ lie in general position. Then the ideal $\mathcal{I} \subset \mathbb{R}[x]$ of all polynomials vanishing identically on $\mathcal{V}$ is generated by monomials.*

If $\{\ell_i : 1 \leq i \leq m\}$ is some collection of m linearly independent linear functionals from $\mathbb{R}^m$ to $\mathbb{R}$, then we may regard the functions $x \mapsto \ell_i(x)$ as coordinates on $\mathbb{R}^m$. Any polynomial $Q : \mathbb{R}^m \to \mathbb{R}$ can be expressed in a unique way as a finite constant-coefficient linear combination $\sum_\alpha c_\alpha \ell^\alpha$ where $\ell^\alpha = \prod_i \ell_i^{\alpha_i}$ and $\alpha = (\alpha_1, \cdots, \alpha_m)$. We say that Q is independent of some coordinate ℓ_j if $c_\alpha = 0$ whenever $\alpha_j \neq 0$.

Proof. Fix linear functions $y_{j,k}(x)$, for $1 \leq j \leq n$ and $1 \leq k \leq m - \kappa$, such that $\mathcal{V}_j = \{x : y_{j,k}(x) = 0 \text{ for all } 1 \leq k \leq m - \kappa\}$. The total number of such functions is $(m - \kappa)n \leq m$, and they are linearly independent by the general position hypothesis. If $m = (m - \kappa)n$, we regard these as coordinates on $\mathbb{R}^m$; if $m > (m - \kappa)n$ then we take these as being among the coordinates and choose some $m - (m - \kappa)n$ additional linear functions to obtain a full

coordinate system. In either case this gives a collection of linear functionals ℓ_i as in the preceding paragraph.

Each of the $(m-\kappa)^n$ monomials of the form $\prod_{j=1}^n y_{j,k(j)}$ belongs to $\mathcal{I}$. We claim conversely that $\mathcal{I}$ is generated by these monomials.

Let $P \in \mathcal{I}$. Since $P \equiv 0$ on $\mathcal{V}_1$, it may be expressed as $P = \sum_{k=1}^{m-\kappa} y_{1,k} r_k$ for certain polynomials r_k, which have the additional property that r_k is independent of $y_{1,i}$ for all $i < k$, in the sense discussed above. Consider the restriction of P to the subspace $U = \{x : y_{1,k}(x) = 0$ for all $k < m - \kappa\}$. Since P vanishes on $\mathcal{V}_2$ and thence on $U \cap \mathcal{V}_2$, $y_{1,m-\kappa} r_{m-\kappa}$ must vanish on $U \cap \mathcal{V}_2$, and hence, by the general position hypothesis, so must $r_{m-\kappa}$. Since $r_{m-\kappa}$ is independent of $y_{1,i}$ for all $i < k$, this means that $r_{m-\kappa}$ vanishes identically on $\mathcal{V}_2$.

Next consider the restriction of P to the subspace $\{x : y_{1,k}(x) = 0$ for all $k < m - \kappa - 1\}$. Since $r_{m-\kappa} \equiv 0$ on $\mathcal{V}_2$, a repetition of the reasoning of the preceding paragraph demonstrates that $r_{m-\kappa-1} \equiv 0$ on $\mathcal{V}_2$. By descending induction on k we eventually conclude that every r_k likewise vanishes identically there.

Of course the same applies to $\mathcal{V}_j$ for all $j \geq 3$. Thus $P = \sum_{k=1}^{m-\kappa} y_{1,k} r_k$ where each r_k belongs to the ideal $\tilde{\mathcal{I}}$ of all polynomials vanishing on $\cup_{j=2}^n \mathcal{V}_j$. By induction on the number n of subspaces $\mathcal{V}_j$, we conclude that each r_k is generated by monomials belong to $\tilde{\mathcal{I}}$, hence that P is generated by monomial elements of $\mathcal{I}$. $\square$

Example. In $\mathbb{R}^4$ let $\mathcal{V}_1 = \{x : x_1 = x_2 = 0\}$, $\mathcal{V}_2 = \{x : x_3 = x_4 = 0\}$, $\mathcal{V}_3 = \{x : x_1 - x_3 = x_2 - x_4 = 0\}$. Thus $m = 4$, $\kappa = 2$, and $n = 3 > m/(m - \kappa)$; the dimensional hypotheses of the preceding two lemmas are not satisfied. The polynomial $p(x) = x_1 x_4 - x_2 x_3$ belongs to the associated ideal $\mathcal{I}$. On the other hand, any monomial $\mathcal{M} \in \mathcal{I}$ must have degree at least three. Indeed, if $\mathcal{M}$ has degree two then in order to vanish on $\mathcal{V}_1 \cup \mathcal{V}_2$, it must take the form $\mathcal{M}(x) = (ax_1 + bx_2)(cx_3 + dx_4)$. Then $\mathcal{M}(x_1, x_2, x_1, x_2)$ plainly cannot vanish identically on $\mathcal{V}_3$ unless $\mathcal{M}$ vanishes identically on $\mathbb{R}^4$.

The reasoning underlying Lemma 3.6 and therefore Lemma 3.5 applies more generally. If $\{V_j\}$ is a finite collection of linear subspaces having arbitrary dimensions κ_j, then the conclusion of Lemma 3.5 continues to hold whenever $\sum_j (m - \kappa_j) \leq m$. Moreover, if $N < n$ and if $\sum_{j=1}^N (m - \kappa_j) \leq m$, then for any homogeneous polynomial P which is nondegenerate relative to $\{V_j : 1 \leq j \leq n\}$, there exists a homogeneous constant-coefficient partial differential operator $\mathcal{L}$ which annihilates $h_j \circ \pi_j$ for all $j \leq n$ and all distributions h_j, but which does not annihilate P, and which takes the special form $\sum_\alpha M_\alpha \circ L_\alpha$ where M_α, L_α are likewise homogeneous constant-coefficient partial differential operators, and L_α is a monomial $\prod_{j=1}^N (w_{j,\alpha} \cdot \nabla)$ with each vector $w_{j,\alpha}$ orthogonal to V_j.

4. The simply nondegenerate case

In proving (1.2), each function f_j may be assumed to be supported in a fixed compact set, the image under π_j of the support of the cutoff function η. We will assume this throughout the proofs of Theorems 2.1 and 2.3.

Proof of Theorem 2.3. The proof proceeds by induction on the degree n of multilinearity of Λ_λ, and is an adaptation of an argument in [1]. Adopt coordinates in which $V_n = \{(y, z) \in \mathbb{R}^{m-1} \times \mathbb{R}^1 : z = 0\}$ and hence $\pi_n(y, z) = y$. Write

$$\Lambda_\lambda(f_1, \cdots, f_n) = \int f_n(y) \Big(\int e^{i\lambda P(y,z)} \prod_{j=1}^{n-1} f_j(\pi_j(y, z)) \, \eta(y, z) \, dz \Big) dy.$$

This equals $\langle T_\lambda(f_1, \cdots, f_{n-1}), \overline{f_n} \rangle$ for a certain linear operator T_λ, whence

$$|\Lambda_\lambda(f_1, \cdots, f_n)| \leq \|f_n\|_2 \|T_\lambda(f_1, \cdots, f_{n-1})\|_2 \leq C\|f_n\|_\infty \|T_\lambda(f_1, \cdots, f_{n-1})\|_2.$$

Thus it suffices to bound T_λ as an operator from $L^\infty \times \cdots \times L^\infty$ to L^2.

Now $\int |T_\lambda(f_1, \cdots, f_{n-1})(y)|^2 \, dy$ equals

$$\int_{\mathbb{R}^{m-1}} \iint_{\mathbb{R}^2} e^{i\lambda(P(y,z) - P(y,z'))} \prod_{j<n} f_j(\pi_j(y,z)) \overline{f_j}(\pi_j(y,z')) \eta(y,z) \overline{\eta}(y,z') \, dz \, dz' \, dy.$$

Define $Q_\zeta(y,z) = P(y,z) - P(y, z+\zeta)$, and $\tilde{\eta}_\zeta(y,z) = \eta(y,z)\overline{\eta}(y, z+\zeta)$. Likewise define $F_j^\zeta : V_j \to \mathbb{C}$ so that $F_j^\zeta \circ \pi_j(y,z) = f_j(\pi_j(y,z))\overline{f_j}(\pi_j(y, z+\zeta))$; the right-hand side is a function of $(\pi_j(y,z), \zeta)$ alone because of the linearity of π_j. Of course $\|F_j^\zeta\|_\infty \leq \|f_j\|_\infty^2$. With these definitions and the substitutions $x = (y,z)$, $z' = z + \zeta$,

$$\int |T_\lambda(f_1, \cdots, f_{n-1})(y)|^2 \, dy = \int \Lambda_\lambda^\zeta(F_1^\zeta, \cdots, F_{n-1}^\zeta) \, d\zeta$$

where

$$\Lambda_\lambda^\zeta(F_1^\zeta, \cdots, F_{n-1}^\zeta) = \int_{\mathbb{R}^m} e^{i\lambda Q_\zeta(x)} \prod_{j=1}^{n-1} F_j^\zeta(\pi_j(x)) \tilde{\eta}_\zeta(x) \, dx.$$

The outer integral may be taken over a bounded subset of $\mathbb{R}^{m-\kappa}$, since $\tilde{\eta}_\zeta \equiv 0$ when $|\zeta|$ is sufficiently large. Henceforth we assume ζ to be restricted to such a bounded set.

For each ζ, consider the polynomial phase Q_ζ. We may assume that $|\lambda| \geq 1$, since (1.2) holds trivially otherwise. It is given as a hypothesis that there exists a differential operator of the form $L = \prod_{j=1}^n (w_j \cdot \nabla)$, with each $w_j \in V_j^\perp$, such that $\sup_{|x| \leq 1} |L(P)(x)| \geq c > 0$. Let $L' = \prod_{j<n}(w_j \cdot \nabla)$. For any $\rho \in (0,1)$ define

$$E_\rho = \{\zeta : \max_{|x| \leq 1} |L'Q_\zeta(x)| \leq \rho\}.$$

For any $\zeta \notin E_\rho$, write $\lambda Q_\zeta = (\lambda\rho)\tilde{Q}$ where $\tilde{Q} = \rho^{-1}Q_\zeta$. Then by the induction hypothesis, there exist $C, \varepsilon' \in \mathbb{R}^+$ such that for all $\zeta \notin E_\rho$,

$$|\Lambda_\lambda^\zeta(F_1^\zeta, \cdots, F_{n-1}^\zeta)| \leq C(1 + |\lambda|\rho)^{-\varepsilon'} \prod_{j<n} \|F_j^\zeta\|_\infty \leq C(1 + |\lambda|\rho)^{-\varepsilon'} \prod_{j<n} \|f_j\|_\infty^2.$$

The same bound holds for the integral over all $\zeta \notin E_\rho$, since ζ is confined to a bounded set.

For $\zeta \in E_\rho$ there is the trivial estimate

$$|\Lambda_\lambda^\zeta(F_1^\zeta, \cdots, F_{n-1}^\zeta)| \leq C \prod_{j<n} \|F_j^\zeta\|_\infty,$$

so

$$\int_{E_\rho} \left| \Lambda_\lambda^\zeta(F_1^\zeta, \cdots, F_{n-1}^\zeta) \right| d\zeta \leq C|E_\rho| \prod_{j<n} \|f_j\|_{L^\infty}^2.$$

Now

$$(4.1) \qquad\qquad |E_\rho| \leq C\rho^\delta \quad \text{for some } \delta > 0 \text{ and } C < \infty.$$

Indeed, by hypothesis $\sup_{(x,\zeta)} |\partial_\zeta(L'Q_\zeta(x))| \geq c > 0$, and as is well known, this implies that if (x, ζ) is restricted to any fixed bounded set, then

$$(4.2) \qquad\qquad |\{(x,\zeta) : |L'Q_\zeta(x)| \leq r\}| \leq Cr^a$$

for some $a > 0$ where C, a depend only on c and on an upper bound for the degree of Q as a polynomial in (x, ζ). In particular, so long as x, ζ are restricted to lie in any fixed bounded set,

$$|\{\zeta : \max_x |L'Q_\zeta(x)| \leq r\}| \leq Cr^a.$$

Thus in all

$$\int \left| \Lambda_\lambda^\zeta(F_1^\zeta, \cdots, F_{n-1}^\zeta) \right| d\zeta \leq C[(|\lambda|\rho)^{-\varepsilon'} + \rho^\delta] \prod_{j<n} \|f_j\|_\infty^2.$$

Choosing $\rho = |\lambda|^{-c}$ for any fixed $c \in (0, 1)$ yields the desired bound.   □

5. The power decay property for $\kappa = 1$

This section is devoted to the proof of Theorem 2.1. The proof turns on a concept related to a notion of uniformity employed by Gowers [2]. Let $d \geq 1$, and fix a bounded ball $B \subset \mathbb{R}^m$. Let $\tau > 0$ be a small quantity to be chosen below, let $|\lambda| \geq 1$, and consider any function $f \in L^2(\mathbb{R}^m)$ supported in B.

Definition 5.1. f is λ-nonuniform if there exist a polynomial q of degree $\leq d$ and a scalar c such that

(5.1) $$\|f - ce^{iq}\|_{L^2(B)} \leq (1 - |\lambda|^{-\tau})\|f\|_{L^2}.$$

Otherwise f is said to be λ-uniform.

This notion depends on the parameters d, τ. So long as they remain fixed there exists $C < \infty$, depending also on B, such that any λ-uniform function $f \in L^2(B)$ satisfies favorable bounds for generalized Fourier coefficients:

(5.2) $$\left| \int f(t)e^{-iq(t)} dt \right| \leq C|\lambda|^{-\tau/2}\|f\|_{L^2(B)}$$

uniformly for all real-valued polynomials q of degree $\leq d$. Indeed, f could otherwise be decomposed in $L^2(B)$ into its projection onto e^{iq} plus an orthogonal vector, implying (5.1).

The proof of Theorem 2.1 will proceed by induction on n, the number of subspaces $\{V_j\}$. The inductive step enters in the following way: If $f_1 = e^{ip}$ for some polynomial p, then $\Lambda_\lambda(e^{ip}, f_2, \cdots, f_n) = \int e^{i\tilde{P}(x)} \prod_{j=2}^n f_j(\pi_j(x))\, \eta(x)\, dx$ where $\tilde{P} = P + p \circ \pi_1$. With p held fixed, this may be regarded as an $(n-1)$-multilinear operator acting on $(f_2, \cdots, f_n)$, with the new phase $\tilde{P}$ and the smaller collection of subspaces $\{V_j : 2 \leq j \leq n\}$. $\tilde{P}$ is nondegenerate relative to this subcollection; moreover, this nondegeneracy is uniform as p varies over all polynomials of uniformly bounded degrees. Thus it is a consequence of the inductive hypothesis that

(5.3) $$|\Lambda_\lambda(e^{ip}, f_2, \cdots, f_n)| \leq C|\lambda|^{-\varepsilon}$$

provided that $\|f_j\|_2 \leq 1$ for all $2 \leq j \leq n$, uniformly for all polynomials p of uniformly bounded degrees.

Proof of Theorem 2.1. Given the cutoff function η, there exist intervals B_j of finite lengths in $\mathbb{R}^1$ such that $\Lambda_\lambda(f_1, \cdots, f_n)$ depends only on the restriction of each f_j to B_j, so henceforth we assume f_j to be supported in B_j. Thus $\|f_j\|_{L^2}$ equals $\|f_j\|_{L^2(B_j)}$.

We may assume henceforth[1] that n is strictly larger than m, since the theorem is already known in a more precise form [8],[9],[10] for the case $n = m$. Let e_1 be a unit vector

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[1]Our proof can be adapted to the simpler case $n \leq m$ as well.

orthogonal to the span of $\{V_j : 2 \leq j \leq m\}$. Since the subspaces $\{V_j : 2 \leq j \leq m\}$ span a space of codimension one, e_1 is uniquely determined modulo multiplication by -1, and it cannot be orthogonal to V_1 because of the general position hypothesis.

Likewise choose some unit vector e_2 orthogonal to $\mathrm{span}\,(\{V_j : j > m\})$, and not orthogonal to V_1. $\cup_{j>m} V_j$ spans a space of dimension $n - m < m$ by the assumption that $n < 2m$ and the general position hypothesis, and the only way that all unit vectors in its orthocomplement could be forced to be orthogonal to V_1 is if V_1 were to be contained in $\mathrm{span}\,(\{V_j : j > m\})$. But this cannot happen, again by general position and the restriction $n - m < m$. Thus there exists at least one vector e_2 with the required properties.

e_2 cannot be orthogonal to $\mathrm{span}\,(\{V_j : 2 \leq j \leq m\})$, since $\cup_{j=2}^{n} V_j$ spans $\mathbb{R}^m$ by the general position assumption $(n > m)$. e_1 cannot be orthogonal to $\mathrm{span}\,(\{V_j : j > m\})$, since then it would be orthogonal to $\cup_{j \geq 2} V_j$, which we have just seen to be impossible. Thus e_2 is automatically linearly independent of e_1.

Given $\{V_j\}$, P, a cutoff function η, and λ, define $A(\lambda)$ to be the best constant (that is, the infimum of all admissible constants) in the inequality

$$(5.4) \qquad |\Lambda_\lambda(f_1, \cdots, f_n)| \leq A(\lambda) \prod_j \|f_j\|_{L^2}.$$

As noted before the statement of Theorem 2.1, the hypotheses of general position and $n \leq 2m$ ensure that $A(\lambda)$ is finite for all λ. We will assume that $\|f_j\|_{L^2} \leq 1$ for all j, and that $|\lambda|$ exceeds some sufficiently large constant. To prove the theorem it suffices to obtain a bound of $C|\lambda|^{-\varepsilon}$ under these additional hypotheses.

The analysis of $\Lambda_\lambda(f_1, \cdots, f_n)$ is divided into two cases, depending on whether or not f_1 is λ-uniform. If it is not, let $f = f_1, c, q$ satisfy (5.1). Then

$$|\Lambda_\lambda(f_1 - ce^{iq}, f_2, \cdots, f_n)| \leq A(\lambda)(1 - |\lambda|^{-\tau}).$$

Moreover $|c|$ is majorized by an absolute constant since $\|f_1\|_{L^2} = 1$ and e^{iq} is unimodular, so by the inductive hypothesis (5.3)

$$|\Lambda_\lambda(ce^{iq}, f_2, \cdots, f_n)| \leq C|\Lambda_\lambda(e^{iq}, f_2, \cdots, f_n)| \leq C|\lambda|^{-\sigma}$$

for certain $C, \sigma \in (0, \infty)$; this bound holds uniformly provided that P varies over a set of uniformly nondegenerate polynomials. Combining the two terms yields

$$(5.5) \qquad |\Lambda_\lambda(f_1, \cdots, f_n)| \leq A(\lambda)(1 - |\lambda|^{-\tau}) + |\lambda|^{-\sigma}$$

provided that $\|f_j\|_{L^2} \leq 1$ for all j, and that f_1 is λ-nonuniform.

Suppose next that f_1 is λ-uniform. Adopt coordinates $(t, y) \in \mathbb{R}^2 \times \mathbb{R}^{m-2}$ where $\mathbb{R}^2$ is the span of e_1, e_2 and $\mathbb{R}^{m-2}$ is its orthocomplement, by writing $x = t_1 e_1 + t_2 e_2 + y$. Let $P^y(t) = P(t, y)$. Define

$$F_1^y(t_2) = \prod_{j=2}^{m} f_j \circ \pi_j(t, y);$$

the right-hand side is independent of t_1 since e_1 is orthogonal to V_j for all $2 \leq j \leq m$. Likewise define

$$F_2^y(t_1) = \prod_{j=m+1}^{n} f_j \circ \pi_j(t, y),$$

which is independent of t_2. In the same way, since neither e_1 nor e_2 is orthogonal to V_1, $f_1 \circ \pi_1(t, y)$ depends for each y only on a certain projection $\pi(t)$ of $t \in \mathbb{R}^2$ onto $\mathbb{R}^1$, so we

may write

$$G^y(\pi(t)) = f_1 \circ \pi_1(t, y).$$

The general position hypothesis, the hypothesis $n \le 2m$, and Fubini's theorem together imply that

$$\int \|F_1^y\|_{L^2}^2 \|G^y\|_{L^2}^2 \, dy = C \prod_{j=1}^{m} \|f_j\|_{L^2}^2$$

$$\int \|F_2^y\|_{L^2}^2 \, dy = C \prod_{j=m+1}^{n} \|f_j\|_{L^2}^2$$

for a certain positive constant C, and consequently

(5.6) $$\int \|F_1^y\|_{L^2} \|F_2^y\|_{L^2} \|G^y\|_{L^2} \, dy \le C \prod_{j=1}^{n} \|f_j\|_{L^2}$$

by Cauchy-Schwarz.

For each $y \in \mathbb{R}^{m-2}$ consider

$$\Lambda_\lambda^y = \int_{\mathbb{R}^2} e^{i\lambda P^y(t)} F_1^y(t_2) F_2^y(t_1) G^y(\pi(t)) \eta(t, y) \, dt.$$

Define the set $\mathcal{B} \subset \mathbb{R}^{m-2}$ of bad parameters to be the set of all y for which P^y may be decomposed as

$$P^y(t) = Q_1(t_1) + Q_2(t_2) + Q_3(\pi(t)) + R(t)$$

where Q_j are real-valued polynomials of degrees $\le d$ on $\mathbb{R}^1$ for $j = 1, 2$, t_j is a certain linear function of t, $R : \mathbb{R}^2 \to \mathbb{R}$ is likewise a real-valued polynomial of degree $\le d$, and

$$\|R\| \le |\lambda|^{-\rho};$$

that is, P^y has small norm in the quotient space of polynomials modulo degenerate polynomials, relative to the three projections $t \mapsto t_1, t_2, \pi(t)$. Here $\|\cdot\|$ denotes any fixed norm on the vector space of all polynomials of degree $\le d$, and $\rho \in (0, 1)$ is another parameter to be specified. If $y \notin \mathcal{B}$ then $|\lambda|^\rho P^y$ is at least a fixed positive distance from the span of all polynomials $Q_1(t_1) + Q_2(t_2) + Q_3(\pi(t))$, so we may apply Theorem 2.3 with $n = 3$, $m = 2$, and the phase $|\lambda|^\rho P^y$, to obtain

(5.7) $$|\Lambda_\lambda^y| \le C(|\lambda|^{1-\rho})^{-\tilde{\rho}} \|F_1^y\|_{L^2} \|F_2^y\|_{L^2} \|G^y\|_{L^2} \text{ for some } C, \tilde{\rho} \in \mathbb{R}^+.$$

This together with (5.6) implies that

$$\int_{y \notin \mathcal{B}} |\Lambda_\lambda^y| \, dy \le C|\lambda|^{-(1-\rho)\tilde{\rho}},$$

as desired.

Bad parameters cannot be handled in this way. Nor is it true that the set of bad parameters has measure $O(|\lambda|^{-\delta})$ for some $\delta > 0$; it could happen that every parameter is bad. Nonetheless we claim that if the exponent ρ appearing in the definition of bad parameters is chosen to be sufficiently small, then there exists $\varepsilon > 0$ such that uniformly for all $y \in \mathcal{B}$,

(5.8) $$|\Lambda_\lambda^y| \le C|\lambda|^{-\varepsilon} \|F_1^y\|_{L^2} \|F_2^y\|_{L^2} \|G^y\|_{L^2},$$

which by (5.6) again implies the desired bound. To verify this fix $y \in \mathcal{B}$, and let $P^y(t) = Q_1(t_1) + Q_2(t_2) + Q_3(\pi(t)) + R(t)$ as above. Write $\pi(t) = c_1 t_1 + c_2 t_2$ where c_1, c_2 are both nonzero. Introduce the functions

$$\tilde{F}_1(t_2) = F_1^y(t_2) e^{i\lambda Q_2(t_2)},$$
$$\tilde{F}_2(t_1) = F_2^y(t_1) e^{i\lambda Q_1(t_1)},$$
$$\tilde{G}(s) = G^y(s) e^{i\lambda Q_3(s)},$$
$$\zeta(t) = \eta(t, y) e^{i\lambda R(t)}.$$

Since λR is a polynomial of bounded degree which is $O(|\lambda|^{1-\rho})$ on the support of ζ, the same holds for all its derivatives and therefore

(5.9) $$|\widehat{\zeta}(\xi)| \leq C_N |\lambda|^{(1-\rho)N} (1 + |\xi|)^{-N} \text{ for all } \xi \text{ and all } N.$$

By Fourier inversion,

$$\Lambda_\lambda^y = c \int_{\mathbb{R}^3} \widehat{\tilde{G}}(\xi_0) \widehat{\tilde{F}_1}(-c_2 \xi_0 - \xi_2) \widehat{\tilde{F}_2}(-c_1 \xi_0 - \xi_1) \widehat{\zeta}(\xi_1, \xi_2) \, d\xi_0 d\xi_1 d\xi_2.$$

From this representation, the bound (5.9) with say $N = 4$, and Plancherel's theorem, it follows that

$$|\Lambda_\lambda^y| \leq C|\lambda|^{4(1-\rho)} \|\widehat{\tilde{G}}\|_{L^\infty} \|\tilde{F}_1\|_{L^2} \|\tilde{F}_2\|_{L^2}.$$

Here $\|\tilde{F}_i\|_{L^2} = \|F_i^y\|_{L^2}$. $\tilde{G}$ is obtained from f_1 by composing with a fixed linear mapping, translating by an arbitrary amount, and multiplying with e^{iq} for some real-valued polynomial q of degree $\leq d$. Therefore the uniformity condition (5.2) for f_1 implies that

$$\|\widehat{\tilde{G}}\|_{L^\infty} \leq C|\lambda|^{-\tau}$$

for some $C, \tau \in \mathbb{R}^+$ independent of y. Thus if f_1 is λ-uniform then

$$|\Lambda_\lambda^y| \leq C|\lambda|^{4(1-\rho)-\tau} \|F_1^y\|_{L^2} \|F_2^y\|_{L^2} \leq C|\lambda|^{4(1-\rho)-\tau}.$$

The parameter $\rho \in (0, 1)$ is at our disposal; we choose $\rho < 1$ sufficiently close to 1 such that $4(1-\rho) - \tau = -\varepsilon$ is strictly negative. This completes the analysis of the case where f_1 is λ-uniform.

Combined with (5.5) and (5.7), this proves that

(5.10) $$A(\lambda) \leq C \max \left(|\lambda|^{-\varepsilon}, |\lambda|^{-(1-\rho)\tilde{\rho}}, A(\lambda)(1 - |\lambda|^{-\tau}) + |\lambda|^{-\sigma} \right).$$

Since $A(\lambda)$ is certainly finite, this implies that it is majorized by a constant times some negative power of $|\lambda|$, as was to be proved. $\qquad\square$

6. Oscillatory singular integral operators

In this section we combine the results on nonsingular oscillatory integral operators proved above with the currently existing theory of multilinear Calderón-Zygmund singular integral operators to prove Theorems 2.5 and 2.6. Let $d, k, n \geq 1$, and let K be a Calderón-Zygmund distribution in $\mathbb{R}^k$. Thus K is a tempered distribution which agrees with a Lipschitz function on $\mathbb{R}^k \setminus \{0\}$, and $K(ry) \equiv r^{-k} y$ in the sense of distributions. For $1 \leq j \leq n$ let $\ell_j : \mathbb{R}^k \to \mathbb{R}^d$ be a linear mapping, and suppose that the intersection of the nullspaces of all the ℓ_j is $\{0\}$. Consider the multilinear operators

(6.1) $$T(f_1, \cdots, f_n)(x) = \left\langle K, \prod_{j=1}^n f_j(x + \ell_j(\cdot)) \right\rangle$$

where $x \in \mathbb{R}^d$, $f_j \in C_0^\infty(\mathbb{R}^d)$, and the pairing is that of the distribution K with the test function $y \mapsto \prod_{j=1}^n f_j(x + \ell_j(y))$. To any real-valued polynomial $P(x, y)$ defined on $\mathbb{R}^{d+k}$ is associated the operator

$$(6.2) \qquad T_P(f_1, \cdots, f_n)(x) = \big\langle e^{iP(x,\cdot)} K, \ \prod_{j=1}^n f_j(x + \ell_j(\cdot)) \big\rangle,$$

which is again well-defined when each $f_j \in C_0^\infty(\mathbb{R}^d)$.

If $q^{-1} = \sum_j p_j^{-1}$ then a necessary condition for T_P to map $\otimes_j L^{p_j}$ boundedly to L^q is that T should do so. It is not known under what conditions T acts boundedly on Lebesgue spaces, but we can assert a conditional result, generalizing Theorem 2.6. Set $\ell_0(y) \equiv 0$ so that $x + \ell_0(y) \equiv x$, and define $\pi_0(x, y) = x$. Define $V_j \subset \mathbb{R}^{d+k}$ for each $j \geq 1$ to be the orthocomplement of the nullspace of the mapping $(x, y) \mapsto x + \ell_j(y)$, and let π_j be the orthogonal projection onto V_j. Then $f_j(x + \ell_j(y)) = \tilde{f}_j(\pi_j(x, y))$ where $\|\tilde{f}_j\|_{p_j}$ is a constant multiple of $\|f_j\|_{p_j}$.

Theorem 6.1. *Let $d, n, K, T, \{\ell_j\}$ be as above. Suppose that T maps $\otimes_{j=1}^n L^{p_j}(\mathbb{R}^d)$ boundedly to $L^q(\mathbb{R}^d)$ for some exponents $1 < p_j < \infty$ and $0 < q < \infty$ satisfying $q^{-1} = \sum_j p_j^{-1}$. Suppose furthermore that the collection $\{V_j : 0 \leq j \leq n\}$ of orthocomplements of the nullspaces of ℓ_j has the uniform decay property. Then for any real-valued polynomial P, T_P maps $\otimes_{j=1}^n L^{p_j}$ boundedly to L^q. Moreover for any D there exists $C_D < \infty$ such that $\|T_P(f_1, \cdots, f_n)\|_q \leq C_D B \prod_j \|f_j\|_{p_j}$ uniformly for all real-valued polynomials P of degrees $\leq D$, where B is the operator norm of T from $\otimes_j L^{p_j}$ to L^q.*

If $\{V_j\}$ merely has the uniform decay property for all polynomials of degrees $\leq D$, then the conclusion of the theorem holds for polynomials of degrees $\leq D$.

A sufficient condition for T to be bounded was established in [6]. Combining it with Theorem 6.1 gives Theorem 2.6, and its special case Theorem 2.5.

The proof of Theorem 6.1 is little different from its linear prototype [9]. One step in the proof requires passing to associated truncated operators. This can be done quite generally by means of the following lemma, versions of which have appeared elsewhere in the literature.

Lemma 6.2. *Let $k \geq \kappa \geq 1$ and $n \geq 0$ be integers. For $1 \leq j \leq n$ let $\ell_j : \mathbb{R}^k \to \mathbb{R}^\kappa$ be linear mappings, the intersection of all of whose nullspaces is $\{0\}$. Let $p_j \in [1, \infty]$ and let $q^{-1} = \sum_j p_j^{-1}$.*

Let $\eta \in C_0^\infty(\mathbb{R}^m)$ have compact support. There exists $A < \infty$, depending only on a sufficiently high order C^M norm of η and the support of η, with the following property. Let $\mathcal{K}$ be any tempered distribution in $\mathbb{R}^k$, and suppose there exists $C < \infty$ such that

$$(6.3) \qquad \|\langle \mathcal{K}, \prod_{j=1}^n f_j(x + \ell_j(\cdot)) \rangle\|_{L^q} \leq C \prod_j \|f_j\|_{L^{p_j}} \quad \text{for all } f_j \in C_0^\infty(\mathbb{R}^\kappa).$$

Then

$$(6.4) \qquad \|\langle \eta \mathcal{K}, \prod_{j=1}^n f_j(x + \ell_j(\cdot)) \rangle\|_{L^q} \leq AC \prod_j \|f_j\|_{L^{p_j}} \quad \text{for all } f_j \in C_0^\infty(\mathbb{R}^\kappa).$$

As in Theorem 6.1, q is permitted to be strictly less than one. Here $\langle \mathcal{K}, \cdot \rangle$ denotes the pairing of distribution with test function; the hypothesis that the intersection of the nullspaces is trivial guarantees that $\prod_{j=1}^n f_j \circ \ell_j$ is compactly supported.

Proof. Write $\mathcal{T}_\mathcal{K}(f_1, \cdots, f_n)(x) = \langle \mathcal{K}, \prod_j f_j(x + \ell_j(\cdot)) \rangle$. Let $h \in C^\infty(\mathbb{R}^\kappa)$ satisfy $\sum_{\nu \in \mathbb{Z}^\kappa} h(x - \nu) = 1$ for all $x \in \mathbb{R}^\kappa$. Then

$$\mathcal{T}_{\eta\mathcal{K}}(f_1, \cdots, f_n)(x) = \sum_{(\nu_0, \cdots, \nu_n) \in \mathbb{Z}^{(n+1)\kappa}} h(x - \nu_0) \langle \mathcal{K}\eta, \prod_{j=1}^n f_j(x + \ell_j(\cdot)) h(x + \ell_j(\cdot) - \nu_j) \rangle.$$

Because η has compact support, there exists $C_0 < \infty$ such that $h(x - \nu_0)\langle \mathcal{K}\eta, \prod_{j=1}^n f_j(x + \ell_j(\cdot)) h(x + \ell_j(\cdot) - \nu_j) \rangle \equiv 0$ unless $|\nu_j - \nu_0| \leq C_0$ for all $1 \leq j \leq n$. We write $\sum_\nu^*$ to denote the summation over all $\nu = (\nu_0, \cdots, \nu_n) \in \mathbb{Z}^{(n+1)\kappa}$ satisfying this restriction. Thus

$$\mathcal{T}_{\eta\mathcal{K}}(f_1, \cdots, f_n)(x) = \sum_\nu^* h(x - \nu_0) \langle \mathcal{K}\eta, \prod_{j=1}^n f_j(x + \ell_j(\cdot)) h(x + \ell_j(\cdot) - \nu_j) \rangle.$$

There exists $C_1 < \infty$ such that for any $\nu \in \mathbb{Z}^{(n+1)\kappa}$ satisfying $|\nu_j - \nu_0| \leq C_0$ for all $j \geq 1$, and for any point $(x, y) \in \mathbb{R}^{\kappa+m}$ such that $h(x - \nu_0) \neq 0$ and $h(x + \ell_j(y) - \nu_j) \neq 0$ for all $j \geq 1$, necessarily $|y| \leq C_1$. This follows directly from the assumption that the intersection of the nullspaces of all the linear mappings ℓ_j equals $\{0\}$. Define $\tilde{\eta} : \mathbb{R}^k \to \mathbb{R}$ to be the unique function that is periodic with periodic $2C_1$ with respect to each coordinate in $\mathbb{R}^m$, and that satisfies $\tilde{\eta}(x) = \eta(x)$ whenever $x = (x_1, \cdots, x_k)$ with every $|x_j| \leq C_1$. Then $\tilde{\eta} \in C^\infty$. We may replace η by $\tilde{\eta}$:

$$\mathcal{T}_{\eta\mathcal{K}}(f_1, \cdots, f_n)(x) = \sum_\nu^* h(x - \nu_0) \langle \mathcal{K}\tilde{\eta}, \prod_{j=1}^n f_j(x + \ell_j(\cdot)) h(x + \ell_j(\cdot) - \nu_j) \rangle.$$

By dilating all variables by π/C_1 we may assume that $C_1 = \pi$, so that $\tilde{\eta}$ is 2π-periodic with respect to each coordinate and hence $\tilde{\eta}(z) = \sum_{\mu \in \mathbb{Z}^k} a_\mu e^{i\mu \cdot z}$ where $a_\mu = O(|\mu|^{-N})$ as $|\mu| \to \infty$, for all finite N. Inserting this into the preceding expression gives

$$\mathcal{T}_{\eta\mathcal{K}}(f_1, \cdots, f_n)(x) = \sum_{\mu \in \mathbb{Z}^k} \sum_\nu^* h(x - \nu_0) \langle \mathcal{K}, e^{i\mu \cdot y} \prod_{j=1}^n f_j(x + \ell_j(y)) h(x + \ell_j(y) - \nu_j) \rangle_y,$$

where this notation indicates that the pairing is between the distribution $\mathcal{K}$ and the function $\mathbb{R}^k \ni y \mapsto e^{i\mu \cdot y} \prod_{j=1}^n f_j(x + \ell_j(y)) h(x + \ell_j(y) - \nu_j)$.

Since the intersection of the nullspaces of $\{\ell_j\}$ is trivial, there exist linear mappings $L_j : \mathbb{R}^\kappa \to \mathbb{R}^k$ such that $y \equiv \sum_{j=1}^n L_j \circ \ell_j(y)$ for all $y \in \mathbb{R}^k$. Therefore

$$e^{i\mu \cdot y} = e^{-i\mu \cdot \sum_j L_j(x)} \prod_{j=1}^n e^{i\mu \cdot L_j(x + \ell_j(y))}.$$

Inserting this into the preceding expression gives

$$\mathcal{T}_{\eta\mathcal{K}}(f_1, \cdots, f_n)(x) = \sum_{\mu \in \mathbb{Z}^k} a_\mu \sum_\nu^* e^{-i\mu \cdot \sum_j L_j(x)} h(x - \nu_0) \langle \mathcal{K}, \prod_{j=1}^n \tilde{f}_{j,\mu,\nu_j}(x + \ell_j(\cdot)) \rangle$$

where $\tilde{f}_{j,\mu,\nu_j}(z) = f(z) e^{i\mu \cdot L_j(z)} h(z - \nu_j)$.

By hypothesis,

$$\|\langle \mathcal{K}, \prod_{j=1}^n \tilde{f}_{j,\mu,\nu_j}(x + \ell_j(\cdot)) \rangle \|_q \leq C \prod_{j=1}^n \|\tilde{f}_{j,\mu,\nu_j}\|_{p_j} = C \prod_{j=1}^n \|f_j h(\cdot - \nu_j)\|_{p_j}.$$

If $q \geq 1$ then by the triangle inequality, the fact that no point belongs to the support of $h(\cdot - \nu_0)$ for more than a uniformly bounded number of indices ν_0, and the finiteness of $\sum_\mu |a_\mu|$,

$$\|\mathcal{T}_{\eta\mathcal{K}}(f_1, \cdots, f_n)(x)\|_q^q \leq C \sum_\nu^* \prod_{j=1}^n \|f_j h(\cdot - \nu_j)\|_{p_j}^q.$$

Fix any $(v_1, \cdots, v_n) \in (\mathbb{Z}^\kappa)^n$. Then

$$\sum_{\nu_0 \in \mathbb{Z}^\kappa} \prod_{j=1}^n \|f_j h(\cdot - \nu_0 - v_j)\|_{L^{p_j}}^q \leq \prod_{j=1}^n \Big(\sum_{\nu_0 \in \mathbb{Z}^\kappa} \|f_j h(\cdot - \nu_0 - v_j)\|_{L^{p_j}}^{p_j} \Big)^{q/p_j}$$

by Hölder's inequality with exponents $p_1/q, \cdots, p_n/q$. The right-hand side does not exceed $C \prod_{j=1}^n \|f_j\|_{p_j}^{q/p_j}$. Now because the sum $\sum_\nu^*$ is subject to the restriction that $|\nu_j - \nu_0| \leq C_0$ for all $j \geq 1$, summming over an appropriate finite set of values of $(v_1, \cdots, v_n)$ gives a majorization for the total. This completes the proof for $q \geq 1$.

For $0 < q < 1$ we use instead the modified triangle inequality $|\sum_i b_i|^q \leq \sum_i |b_i|^q$ together with the fact that $|a_\mu| = O(|\mu|^{-N})$ for all N as $|\mu| \to \infty$. If N is taken to be sufficiently large, depending on q and on k, then this bound permits a repetition of the above reasoning. $\square$

Proof of Theorem 6.1. Let subspaces V_j, a degree $D \geq 1$, and a polynomial P of degree $\leq D$ be given. We proceed by induction on D, the result for $D = 0$ being given as a hypothesis.

Decompose $P = P_D + R$ where R has degree $< D$ and P_D is homogeneous of degree D. If P_D is degenerate then by decomposing $P_D = \sum_j q_j(x + \ell_j(y))$ for certain real-valued polynomials q_j, we may rewrite $\mathcal{T}_P(f_0, \cdots, f_n) = \mathcal{T}_R(e^{iq_0} f_0, \cdots, e^{iq_n} f_n)$. By induction on the degree, $\mathcal{T}_R$ satisfies the desired estimate, which is equivalent to the desired estimate for $\mathcal{T}_P$ since $\|e^{iq_j} f_j\|_{p_j} = \|f_j\|_{p_j}$.

If P_D is nondegenerate, there exists a smallest integer N such that $P_D(2^N x, 2^N y)$ has norm ≥ 1 in the quotient space of homogeneous polynomials of degree D modulo degenerate homogeneous polynomials of that degree, with respect to some fixed choice of norm on that finite-dimensional space. This norm is then also $\leq 2^D$.

By rescaling the variables $(x, y) \in \mathbb{R}^{\kappa+m}$ by a factor of 2^N we may reduce henceforth to the case $N = 0$. Because K is homogeneous of the critical degree and $\sum_{j=1}^n p_j^{-1} = q^{-1}$, such a rescaling does not affect the inequality to be proved.

By replacing P_D by $P_D - \sum_j S_j(x + \ell_j(y))$ for appropriate homogeneous real-valued polynomials S_j of degree D, we may assume henceforth that P_D has norm $\lesssim 1$ in the space of all homogeneous polynomials of degree D, rather than merely in the quotient space.

Fix a cutoff function $\zeta \in C_0^\infty(\mathbb{R}^k)$ which is $\equiv 1$ in some neighborhood of the origin, and consider the truncated multilinear functional $T'_P(f_1, \cdots, f_n)(x) = \langle \zeta(\cdot) K e^{iP(x,\cdot)}, \prod_{j=1}^n f_j(x + \ell_j(\cdot)) \rangle$. Introduce also an auxiliary function $h \in C_0^\infty(\mathbb{R}^d)$ such that $\sum_{\nu \in \mathbb{Z}^d} h(x - \nu) \equiv 1$ and decompose

$$T'_P(f_1, \cdots, f_n)(x) = \sum_{\nu = (\nu_0, \cdots, \nu_n) \in \mathbb{Z}^{(n+1)d}} \langle \zeta(\cdot) K e^{iP(x,\cdot)}, \prod_{j=0}^n f_j(x + \ell_j(\cdot)) h(x + \ell_j(\cdot) - \nu_j) \rangle.$$

Because of the presence of the compactly supported factor $\zeta(y)$, there exists $C_0 < \infty$ such that any term with parameter ν vanishes unless $\max_j |\nu_j - \nu_0| \leq C_0$. Therefore by the triangle inequality (modified as above when $q < 1$), Hölder's inequality, and the hypothesis

$\sum_j p_j^{-1} = q^{-1}$, it suffices to prove the desired bound for fixed ν, with $f_j(\cdot)$ replaced by $\tilde{f}_j(\cdot) = f_j(\cdot)h(\cdot - \nu_j)$, so long as a majorization uniform in ν is obtained. Henceforth we fix ν_0, and assume that each function f_j is supported within a fixed distance of ν_0.

Make the change of variables $x = z + \nu_0$. Then $P(x,y) = P_D(x,y) + R(x,y) = P_D(z,y) + \tilde{R}_{\nu_0}(z,y)$ where $\tilde{R}_{\nu_0}$ has degree $< D$. Moreover, if another cutoff function $\tilde{\zeta} \in C_0^\infty$ is chosen to be $\equiv 1$ in a sufficiently large neighborhood of the origin, then as a function of z,

$$T_P'(\tilde{f}_1, \cdots, \tilde{f}_n)(x) = \langle K(\cdot) e^{i\tilde{R}_{\nu_0}(z,\cdot)} \big(e^{iP_D(z,\cdot)} \zeta(\cdot)\tilde{\zeta}(z,\cdot)\big), \prod_{j=0}^n \tilde{f}_j(z + \ell_j(\cdot)) \rangle.$$

Now the factor $\zeta(y) e^{iP_D(z,y)} \tilde{\zeta}(z,y)$ is smooth and compactly supported as a function of (z,y), and is bounded above in any C^s norm by a constant depending only on s, D and the choice of $\tilde{\zeta}$. By induction on D,

$$\|\langle Ke^{i\tilde{R}_{\nu_0}(z,\cdot)}, \prod_{j=0}^n \tilde{f}_j(z + \ell_j(\cdot)) \rangle\|_{L^q(dz)} \le C \prod_{j=1}^n \|\tilde{f}_j\|_{L^{p_j}},$$

uniformly for all polynomials $\tilde{R}_{\nu_0}$ of degree $< D$.

Therefore we may invoke Lemma 6.2 to conclude that

$$\Big\|\langle Ke^{i\tilde{R}_{\nu_0}(z,\cdot)} \big(e^{iP_D(z,\cdot)} \zeta(\cdot)\tilde{\zeta}(z,\cdot)\big), \prod_{j=0}^n \tilde{f}_j(z + \ell_j(\cdot)) \rangle\Big\|_{L^q(dz)} \le C \prod_j \|\tilde{f}_j\|_{L^{p_j}},$$

uniformly in ν.

It remains to analyze

$$\langle Ke^{iP}(1 - \zeta(\cdot)), \prod_j f_j(x + \ell_j(\cdot)) \rangle = \sum_{r=0}^\infty \langle Ke^{iP}\eta(2^{-r}\cdot), \prod_j f_j(x + \ell_j(\cdot)) \rangle$$

where $\eta(y) = \zeta(y/2) - \zeta(y)$. We claim that

(6.5) $$\|\langle Ke^{iP}\eta(2^{-r}\cdot), \prod_j f_j(x + \ell_j(\cdot)) \rangle\|_q \le C2^{-\varepsilon r} \prod_j \|f\|_{p_j}$$

for some $\varepsilon > 0$ and $C < \infty$ depending only on $D, \{V_j\}$. Once (6.5) is proved, the proof may immediately be completed if $q \ge 1$, by invoking the triangle inequality. When $0 < q < 1$, the rapidly decaying factor $C2^{-\varepsilon r}$, together with the modified triangle inequality $|\sum_i b_i|^q \le \sum_i |b_i|^q$, likewise suffice to complete the proof.

Because of the homogeneity of K and the condition $\sum_j p_j^{-1} = q^{-1}$, (6.5) is equivalent to

(6.6) $$\|\langle K(\cdot)\eta(\cdot)e^{iP_r}, \prod_j f_j(x + \ell_j(\cdot)) \rangle\|_{L^q(dx)} \le C2^{-\varepsilon r} \prod_j \|f\|_{L^{p_j}},$$

via the substitution $(x,y) = (2^r x', 2^r y')$, where $P_r(x,y) = P(2^r x, 2^r y) = 2^{Dr}P_D(x,y) + \tilde{R}_r(x,y)$, and the remainder term $\tilde{R}_r$ has degree strictly less than D and is real-valued.

As above, by introducing a partition of unity we may reduce matters to the case where all f_j are supported in a fixed bounded set, by introducing a partition of unity. $\tilde{R}_r$ is thereby further modified and depends also on the index ν_0; it satisfies no useful upper or lower bounds but still has degree strictly $< D$.

Any such polynomial $2^{Dr}P_D + \tilde{R}$, where $\tilde{R}$ has degree $< D$, has norm $\ge c2^{Dr}$ in the quotient space of polynomials of degree $\le D$ modulo degenerate polynomials of degreee $\le D$, for some fixed constant $c > 0$, depending only on D. The function $K\eta$ is Lipschitz,

since K is a Calderón-Zygmund kernel and η vanishes identically in some neighborhood of the origin. Therefore (6.6) follows from the assumption that $\{V_j\}$ has the uniform decay property. $\qquad\square$

7. Further results

Theorem 2.4 is a direct consequence of Lemma 3.5 together with Theorem 2.3. The next result extends Theorem 2.2.

Theorem 7.1. *Let $\{V_j\}$ be any finite collection of subspaces of $\mathbb{R}^m$, and let $\{W_i\}$ be any finite collection of codimension one subspaces of $\mathbb{R}^m$. If $\{V_j\}$ has the uniform decay property, then so does $\{V_j\} \cup \{W_i\}$.*

Proof. It suffices to prove this in the case where a single codimension one subspace W is given. Choose coordinates with respect to which $W = \{(x', x_m) : x_m = 0\}$. Let P be any nondegenerate polynomial, relative to the augmented collection $\{V_j\} \cup \{W\}$. It is no loss of generality to assume that no subspaces V_j are contained in W, for any such spaces may be deleted from $\{V_j\}$ without affecting the nondegeneracy of P.

$\partial P/\partial x_m$ is nondegenerate relative to $\{V_j\}$. For if not, then there exists a polynomial decomposition $\partial P/\partial x_m = \sum_j q_j \circ \pi_j$. Since no space V_j is contained in W, there exist nonzero vectors $v_j \in V_j$ such that $\partial(f \circ \pi_j)/\partial x_m = (v_j \cdot \nabla f) \circ \pi_j$ for all functions $f : V_j \to \mathbb{R}$. Since $0 \neq v_j \in V_j$, there exist polynomials Q_j such that $v_j \cdot \nabla Q_j \equiv q_j$. Consequently $\partial(Q_j \circ \pi_j)/\partial x_m = q_j \circ \pi_j$, and hence $\tilde{P} = \sum_j Q_j \circ \pi_j$ satisfies $\partial(P - \tilde{P})/\partial x_m \equiv 0$. Thus $P - \sum_j Q_j \circ \pi_j$ is a function of x_m alone, contradicting the hypothesized nondegeneracy of P.

It follows that there exists $z \in \mathbb{R}^m$ for which $P_z(x) = P(x', x_m) - P(x', x_m + z)$ is nondegenerate relative to $\{V_j\}$. For if not, since $\partial P/\partial x_m$ can be reconstructed from $\{P_{z_i}\}$ for a suitable finite collection of points z_i by the Lagrange interpolation formula, degeneracy of each P_{z_i} implies degeneracy of $\partial P/\partial x_m$.

Consider the quotient space $\mathcal{P}$ of all polynomials of degrees not exceeding the degree of P modulo the subspace of all such polynomials which are degenerate relative to $\{V_j\}$, and equip it some inner product structure. Then $\|P_z\|_{\mathcal{P}}^2$ is a polynomial in z which does not vanish identically. Hence there exist $C, \delta \in \mathbb{R}^+$ such that for any ball B of fixed finite radius, for any $\varepsilon > 0$,

$$(7.1) \qquad\qquad |\{z \in B : \|P_z\|_{\mathcal{P}}^2 < \varepsilon\}| \leq C\varepsilon^\delta.$$

We may now argue as in the proof of Theorem 2.1 to conclude the proof that P has the decay property.

To establish the uniform decay property, fix $c \in (1, \infty$ and a degree d. Let P be any polynomial of degree $\leq d$ whose norm, in the quotient space of polynomials of degrees $\leq d$ modulo polynomials of degrees $\leq d$ that are degenerate relative to $\{V_j\} \cup \{W\}$, lies in $[c^{-1}, c]$. Since this quotient space is a finite-dimensional vector space, P thus belongs to a compact subset. Together with the above reasoning, this implies that (7.1) holds with a constant depending on d, c but not on P. The proof of Theorem 2.1 then yields a decay estimate independent of P. $\qquad\square$

Corollary 7.2. *Let a real-valued polynomial P be nondegenerate relative to a finite collection $\{V_j\}$ of subspaces of $\mathbb{R}^m$, and suppose that V_j has codimension one for all $j > 1$. Then*

$\{V_j\}$ *has the uniform decay property. Moreover*

(7.2) $$|\Lambda_\lambda(f_1, \cdots, f_n)| \leq C|\lambda|^{-\varepsilon} \|f_1\|_2 \prod_{j>1} \|f_j\|_\infty,$$

with uniform bounds if P belongs to a family of uniformly nondegenerate polynomials.

A decay estimate with $\prod_j \|f_j\|_\infty$ on the right-hand side of the inequality follows by repeated applications of the preceding theorem. (7.2) then follows by interpolation with the trivial bound $|\Lambda_\lambda(f_1, \cdots, f_n)| \lesssim \|f_1\|_1 \prod_{j>1} \|f_j\|_\infty$.

With Corollary 7.2 in hand, the proof of Theorem 2.1 can be reorganized in a simpler way, though without altering the main idea, as follows. We are given a collection of one-dimensional subspaces $\{V_j : 1 \leq j \leq n\}$ of $\mathbb{R}^m$, lying in general position, and it is assumed that $n < 2m$. Again we discuss only the most extreme case where $n = 2m - 1$. Let P be some polynomial which is nondegenerate relative to $\{V_j\}$.

We will prove power decay in the form (7.2). Let W be the span of those V_j with $2 \leq j \leq m$, and likewise let $\tilde{W}$ be the span of those with $m < j \leq n = 2m - 1$. By the general position hypothesis, W and $\tilde{W}$ are distinct subspaces of codimension one, neither of which contains V_1.

Let functions $f_1 \in L^2(V_1)$, and $f_j \in L^\infty(V_j)$ for $j > 1$, be given. As usual, we may assume each f_j to be supported in a fixed bounded subset of V_j.

The proof is divided into two main cases, depending on whether or not f_1 is λ-uniform, as defined in Definition 5.1; the parameter d appearing in that definition is here taken to be the degree of P. The case where f_1 is not λ-uniform is handled as already discussed in the proof of Theorem 2.1.

Let $A(\lambda)$ be the best constant in the inequality (7.2). Define functions $F, \tilde{F}$ on $W, \tilde{W}$ respectively by the relations $F \circ \pi = \prod_{j=2}^m f_j \circ \pi_j$, $\tilde{F} \circ \tilde{\pi} = \prod_{j>m} f_j \circ \pi_j$ where $\pi, \tilde{\pi}$ denote the orthogonal projections from $\mathbb{R}^m$ onto $W, \tilde{W}$ respectively. Then $\|F\|_2 \lesssim \prod_{j=2}^m \|f_j\|_\infty$ and $\|\tilde{F}\|_2 \lesssim \prod_{j>m} \|f_j\|_\infty$. Our multilinear oscillatory integral is

$$\Lambda_\lambda(f_1, \cdots, f_n) = \int_{\mathbb{R}^m} e^{i\lambda P(x)}(F \circ \pi)(\tilde{F} \circ \tilde{\pi})(f_1 \circ \pi_1)\eta \, dx.$$

If P happens to be nondegenerate relative to the collection of three subspaces $\{V_1, W, \tilde{W}\}$, then the proof is complete by virtue of Corollary 7.2, in its more precise version (7.2). If not, then $P = p \circ \pi + \tilde{p} \circ \tilde{\pi} + q \circ \pi_1$ for certain polynomials $p, \tilde{p}, q$, whose degrees do not exceed the degree of P.

Suppose without loss of generality that $\|f_1\|_{L^2} = 1$, and that f_1 is λ-uniform. Set $G = e^{i\lambda p}F$, $H = e^{i\lambda \tilde{p}}\tilde{F}$, and $g = e^{i\lambda q}f_1$. Our integral becomes

$$\Lambda_\lambda = \int (G \circ \pi)(H \circ \tilde{\pi})(g \circ \pi_1)\eta = c \iiint \widehat{G}(\xi)\widehat{H}(\tilde{\xi})\widehat{g}(\zeta)\widehat{\eta}(\pi^*\xi + \tilde{\pi}^*\tilde{\xi} + \pi_1^*\zeta) \, d\xi \, d\tilde{\xi} \, d\zeta$$

where π_1^* is the adjoint of $\pi_1 : \mathbb{R}^m \to V_1$, and analogously for $\pi^*, \tilde{\pi}^*$; here $\zeta \in \mathbb{R}^1$, $\xi \in \mathbb{R}^m$, and $\tilde{\xi} \in \mathbb{R}^m$. The λ-uniformity condition gives a bound $\|\widehat{g}\|_{L^\infty} = O(|\lambda|^{-\tau})$ for some fixed $\tau > 0$. Therefore for $|\lambda| \geq 1$

$$|\Lambda_\lambda| \lesssim |\lambda|^{-\tau} \iiint |\widehat{G}(\xi)\widehat{H}(\tilde{\xi})| \cdot |\widehat{\eta}(\pi^*\xi + \tilde{\pi}^*\tilde{\xi} + \pi_1^*\zeta)| \, d\xi \, d\tilde{\xi} \, d\zeta.$$

It therefore suffices to have

$$(7.3) \qquad \sup_{\tilde{\xi}} \iint |\widehat{\eta}(\pi^*\xi + \tilde{\pi}^*\tilde{\xi} + \pi_1^*\zeta)| \, d\xi \, d\zeta < \infty$$

and the same with the roles of $\tilde{\xi}, \xi$ interchanged. (7.3) holds because η may be taken to belong to C_0^{m+1}, so that $\widehat{\eta}$ decays rapidly, and the mapping $\mathbb{R} \times \mathbb{R}^{m-1} \ni (\zeta, \xi) \to \pi^*\xi + \pi_1^*\zeta \in \mathbb{R}^m$ is linear and injective. $\qquad \square$

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