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Publication Date
1983-03-01

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REPORT NO.
UCB/SESM-83/04

STRUCTURAL ENGINEERING AND
STRUCTURAL MECHANICS

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MARCH 1983

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MIXED FINITE ELEMENT APPROXIMATIONS FOR NON-LINEAR PLATES

by

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1.- INTRODUCTION

Rod plate and shell theories are characterized by the use of stress resultants over the thickness of the body in the formulation of the governing field equations. In the process of averaging stress distributions over the thickness, loss of information with respect to local phenomena inevitably occurs. In particular this is significant in the range of inelastic response where elastic-plastic interfaces across the thickness can not be obtained. Such theories, however, provide a realistic alternative to fully two or three dimensional non-linear analyses.

Although fully (geometrically) non-linear plate theories are available [1,2,3], in most engineering applications the use of second order theories often suffices. The von-Karman plate model is perhaps the best known example of such an approximate theory. The major shortcoming of this widely used model lies in its inability to account for transverse shear deformation of the plate (due to the Kirchhoff kinematic assumption which is characteristic of this model). In the context of a second order theory, a plate model was presented in [4,5] which is capable of including transverse shear effects. In addition, the warping effect which appears as a result of shear deformation, and the effect of a non-vanishing transverse normal stress, are also accounted for in the formulation proposed in [4,5]

In this paper, we examine mixed finite element formulations of a second order theory capable of accounting for the effect of transverse shear. The plate model employed is summarized in Section 2., and may be viewed as an extension to the non-linear range of the linear plate theory due to Mindlin. Such a non-linear model results from that presented in [4,5] if the

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warping effect and the effect of the transverse normal stress are neglected.

Our mixed finite element formulation is obtain by constructing weak forms of the local equilibrium and constitutive equations of the plate, and employs a consistent linearization process leading naturally to an iterative solution procedure. For simplicity we confine our presentation of the theory essentially to elastic behavior, although the formulation may be extended to elasto-viscoplastic material response as indicated in Section 4. Indeed, our numerical implementation includes such capabilities.

Numerical examples are presented in Section 4 which illustrate the formulation discussed in this paper.

2.- SUMMARY OF THE NON-LINEAR PLATE MODEL

We consider a plate of thickness h whose mid-plane spans a domain $\Omega \subset \mathbb{R}^2$, a bounded open set with smooth boundary $\partial\Omega$. We choose the undeformed configuration $B \equiv \Omega \times (-h/2,\ h/2)$ as the reference configuration. The material frame, designated by $\{\hat{\mathbf{E}}_i\}$, is taken as the standard basis in $\mathbb{R}^3$, $\hat{\mathbf{E}}_i \equiv \hat{\mathbf{E}}^i$, with $\{\hat{\mathbf{E}}_1,\ \hat{\mathbf{E}}_2\}$ spanning the mid-plane Ω and $\hat{\mathbf{E}}_3$ normal to the mid-plane. Points in B are designated by X with coordinates $X_i(X) \equiv X^i(X)$ with respect to the standard basis. We shall designate by $\Gamma^{\pm}$ the two faces of the plate which are defined by $\Gamma^+ \equiv \Omega \times \{h/2\}$ and $\Gamma^- \equiv \Omega \times \{-h/2\}$, respectively.

The deformation map is denoted by $\boldsymbol{\phi} : B \rightarrow \mathbb{R}^3$. We choose the *fixed* spatial frame as coincident with the material frame $\{\hat{\mathbf{E}}_i\}$, and denote the coordinates of points $x = \boldsymbol{\phi}(X)$ in the deformed configuration by $x_i(x) = x^i(x)$. By slight abuse of notation points $X \in B$ and $x \in \boldsymbol{\phi}(B)$ will be often identified with their position vectors $\mathbf{X}$ and $\mathbf{x}$, respectively. In addition, we employ the convention

$$\mathbf{X} = \mathbf{X}^o + X_3\, \hat{\mathbf{E}}_3\,, \qquad \mathbf{X}^o \in \Omega\ ,\ \ X_3 \in (-h/2,\ h/2)$$

Following standard practice, Latin indices range from $\{1, 2, 3\}$, Greek indices from $\{1, 2\}$, and the summation convention on repeated indices is assumed unless otherwise stated.

A systematic procedure of developing rod and plate theories from the three dimensional theory, is to constrain the form of the deformation map by introducing a kinematic assumption [1,2,3,4,5,6]. This approach will be employed here.

2.1.-Kinematic Assumption.

We assume the following particular form for the deformation map $\boldsymbol{\phi} : B \equiv \Omega \times (-h/2, h/2) \to \mathbb{R}^3$ of the plate:

$$\mathbf{x} = \boldsymbol{\phi}(\mathbf{X}) \equiv \boldsymbol{\phi}(\mathbf{X}^o) + X_3 \, \boldsymbol{\psi}(\mathbf{X}^o) \,, \quad \mathbf{X}^o \in \Omega, \; X_3 \in (-h/2, h/2) \tag{2.1}$$

where $\boldsymbol{\phi}(\mathbf{X}^o) \equiv \boldsymbol{\phi}(\mathbf{X})|_{X_3=0}$ and the vector field $\boldsymbol{\psi} : \Omega \to \mathbb{R}^3$ are given by:

$$\boldsymbol{\phi}(\mathbf{X}^o) = [X_\alpha + u^o_\alpha(\mathbf{X}^o)] \, \hat{\mathbf{E}}_\alpha + w(\mathbf{X}^o) \, \hat{\mathbf{E}}_3$$

$$\boldsymbol{\psi}(\mathbf{X}^o) . \boldsymbol{\psi}(\mathbf{X}^o) = ||\boldsymbol{\psi}(\mathbf{X}^o)||^2 \equiv 1 \tag{2.2}$$

The vector fields $\mathbf{u}^o(\mathbf{X}^o) = u^o_\alpha(\mathbf{X}^o)\hat{\mathbf{E}}_\alpha$ and $w(\mathbf{X}^o)\hat{\mathbf{E}}_3$ physically represent the in-plane displacement and vertical deflection of the mid-plane of the plate, respectively, whereas $\boldsymbol{\psi}(\mathbf{X}^o)$ may be thought of as a unit vector defining the orientation of lines initially normal to the mid-plane in the undeformed configuration $B \equiv \Omega \times (-h/2, h/2)$ of the plate. The kinematic assumption (2.1)-(2.2) has been employed in the development of fully non-linear rod and plate models [2,3]. In this paper, however, we shall focus our attention on a plate model restricted to "moderate deflections and rotations." Following Simo *et al* [5] and based upon early work of Ciarlet [14], the notion of "moderate" rotations and deflections may be precisely stated as follows. Let $L = dia(\Omega)$ be the diameter of the set $\Omega \subset \mathbb{R}^2$ [†]; let $X_\alpha = L \, \xi_\alpha$ and define $\boldsymbol{\xi} = \xi_\alpha \hat{\mathbf{E}}_\alpha$. In addition, let $X_\alpha = \xi_3 \hat{\mathbf{E}}_3$ and consider the map:

$$\mathbf{X} \equiv \mathbf{X}^o + X_3 \hat{\mathbf{E}}_3 \in \Omega \times (-h/2, h/2) \to \boldsymbol{\xi} \equiv \boldsymbol{\xi}^o + \xi_3 \hat{\mathbf{E}}_3 \in \overline{\Omega} \times (-1/2, 1/2) \tag{2.3}$$

which scales the reference configurations $\Omega \times (-h/2, h/2)$ of the plate to a *non–dimensional* configuration $\overline{B} = \overline{\Omega} \times (-1/2, 1/2)$. The order of magnitude of the displacements and rotations of interest may then be precisely stated by postulating the manner in which they transform under the scaling map (2.3). If (non-dimensional) fields defined on $\overline{B}$ are designated by a superposed "-", we restrict our attention to a situation for which the components of the variables entering in (2.1)-(2.2) transform according to

$$u^o_\alpha(\mathbf{X}^o) = \frac{h^2}{L} \, \overline{u}^o_\alpha(\boldsymbol{\xi}^o) \,, \qquad w(\mathbf{X}^o) = h \, \overline{w}(\boldsymbol{\xi}^o)$$

$$\psi_\alpha(\mathbf{X}^o) = \frac{h}{L} \, \overline{\psi}_\alpha(\boldsymbol{\xi}^o) \,, \qquad \psi_3(\mathbf{X}^o) = \frac{h^2}{L^2} \, \overline{\psi}_3(\boldsymbol{\xi}^o) \tag{2.4}$$

[†] i.e; the diameter of the smallest open ball in $\mathbb{R}^2$ which contains $\Omega \subset \mathbb{R}^2$.

The requirement of conditions (2.4) amounts physically to restrict the deformation of the plate to deflections $w(\mathbf{X}^o)$ of the order of the plate thickness h, in-plane displacements $u_\alpha^o(\mathbf{X}^o)$ of the order h^2 and "out of plane" rotations $\psi_\alpha(\mathbf{X}^o)$ of the order h/L. The coordinate expression for the displacement field $\mathbf{u}(\mathbf{X}) = \boldsymbol{\phi}(\mathbf{X}) - \mathbf{X}$ then takes the form:

$$
\begin{aligned}
u_\alpha(X_i) &= u_\alpha^o(X_\alpha) - X_3\,\psi_\alpha(X_\alpha) + O(h^2) \\
u_3(X_i) &= w(X_\alpha) \qquad\qquad\quad + O(h)
\end{aligned}
\tag{2.5}
$$

where we have set $\boldsymbol{\psi} = -\psi_\alpha\,\hat{\mathbf{E}}_\alpha + O(h^2)$. If attention is focussed on the linear theory, (2.5) corresponds to the kinematic assumption characterizing the plate model due to Mindlin [7] which neglects the warping effect of transverse sections of the plate appearing as a result of transverse shear deformation. Within the framework of projection methods, a more general kinematic expression was proposed in [4,5] which accounts for warping effect and contains (2.5) as a particular case. Remarkably, the linear theory resulting form this more general kinematic assumption is equivalent to that first derived by Reissner employing a variational procedure [8,9]. It is emphasized that, as opposed to Mindlin's Theory, the Reissner plate model accounts for both the warping effect due to transverse shear deformation and the effect of transverse normal stress (see [5]).

Next, we turn our attention to the development of the non-linear plate model considered in this paper. Our presentation is restricted to an outline of the key results necessary for the discussion of the mixed finite element implementation. A comprehensive account of the theory in a more general context, which includes the warping effect of transverse sections of the plate as well as the effect of transverse normal stresses, can be found in [4,5].

2.2.- Second Order Approximation to the Equilibrium Equation.

(a) Local Form. The key idea to develop appropriate equilibrium equations is to make use of the method of successive approximations [10] in terms of the -presumably small- parameter $\epsilon \equiv h/L$. Let us denote by $\mathbf{P}(\mathbf{X})$ the first Piola-Kirchhoff (two-point) stress tensor, and by $q_\pm(\mathbf{X}^o)\,\hat{\mathbf{E}}_3 = \pm\mathbf{P}\big|_{\Gamma^\pm}\hat{\mathbf{E}}_3$ the applied forces on the major surfaces $\Gamma^\pm$ of the plate. Consistent with (2.4) and (2.5) the components of $\mathbf{P}(\mathbf{X})$ and $q_\pm(\mathbf{X}^o)$ transform under the scaling map (2.3) according to [4,5].

$$
P_{\alpha\beta}(\mathbf{X}) = \epsilon^2\,\bar{P}_{\alpha\beta}(\xi)\,, \qquad P_{3\beta}(\mathbf{X}) = \epsilon^3\,\bar{P}_{3\beta}(\xi)\,, \qquad q_\pm(\mathbf{X}^o) = \epsilon^4\,\bar{q}_\pm(\xi^o) \tag{2.6}
$$

When use is made of the method of successive approximations and a convenient choice of spatial frame is introduced [4], it can be shown [4,5] that integration over the thickness of the plate of the three dimensional equilibrium equations and use of order of magnitude arguments based upon (2.4) and (2.6); leads to the system of equilibrium equations.

$$DIV\ [\mathbf{N} - \boldsymbol{\psi} \otimes \mathbf{V}] = 0$$

$$DIV\ [\mathbf{V} + \boldsymbol{\psi}.\mathbf{N}] + q = 0 \tag{2.7}$$

$$DIV\ \mathbf{M} + (\mathbf{I} + \nabla \mathbf{u}^o)\,\mathbf{V} - \mathbf{N}.(\nabla w - \boldsymbol{\psi}) = 0$$

where $\mathbf{N} = [N_{\alpha\beta}(\mathbf{X}^o)]$ represents the in-plane forces and $\mathbf{V} = \{V_\beta(\mathbf{X}^o)\}$ the transverse shear forces which may be related to the components of the first Piola-Kirchhoff tensor through the transformation [4,5]

$$\int_{-h/2}^{h/2} P_{\alpha\beta}(\mathbf{X})\ dX_3 = N_{\alpha\beta}(\mathbf{X}^o) - \psi_\alpha(\mathbf{X}^o)\ V_\beta(\mathbf{X}^o)$$

$$\int_{-h/2}^{h/2} P_{3\beta}(\mathbf{X})\ dX_3 = V_\beta(\mathbf{X}^o) + \psi_\alpha(\mathbf{X}^o)\ N_{\alpha\beta}(\mathbf{X}^o) \tag{2.8}$$

$\mathbf{M} = [M_{\alpha\beta}(\mathbf{X}^o)]$ represent the bending moments acting on the plate, $\mathbf{I} = [\delta_{\alpha\beta}]$ is the unit matrix and $q = q_+(\mathbf{X}^o) - q_-(\mathbf{X}^o)$. The operators in (2.7) have the usual meaning, i.e;

$$\nabla\,(.) = [\frac{\partial}{\partial X_1}(.) \quad \frac{\partial}{\partial X_2}(.)]^T\ , \qquad DIV\,(.) = tr[\nabla\,(.)] = \sum_{\alpha=1}^{2} \frac{\partial}{\partial X_\alpha}(.) \tag{2.9}$$

It can be shown (see [5]) that due to the restrictions (2.4)-(2.6), we have the symmetry conditions $\mathbf{N} = \mathbf{N}^T$ and $\mathbf{M} = \mathbf{M}^T$.

(b) Boundary Conditions. Let the boundary $\partial\Omega$ of $\Omega \subset \mathbb{R}^2$, with unit normal $\hat{\mathrm{N}}(\mathbf{X}^o)$, be divided into parts $\Omega_\mathbf{u}$ and $\Omega_\mathbf{t}$ such that $\Omega_\mathbf{u} \cap \Omega_\mathbf{t} = \varnothing$. The stress vector field on $\partial\Omega$ is given by $\mathbf{T}^{(\hat{\mathrm{N}})} = \mathbf{P}(\mathbf{X})|_{\partial\Omega}\ \hat{\mathrm{N}}(\mathbf{X}^o)$. On $\Omega_\mathbf{t}$ we assume that only the resultant force and resultant moment of the stress distribution are prescribed; i.e.,

$$\frac{1}{h}\int_{-h/2}^{h/2} \mathbf{T}^{(\hat{\mathrm{N}})}\ dX_3 = \tilde{\mathbf{t}}\ , \quad -\frac{1}{h}\int_{-h/2}^{h/2} [-X_3\ T_\alpha^{(\hat{\mathrm{N}})} + X_\alpha\ T_3^{(\hat{\mathrm{N}})}]\ dX_3 = \tilde{m}_\alpha, \qquad on\ \ \Omega_\mathbf{t} \tag{2.10}$$

whereas on $\Omega_\mathbf{u}$ the displacement of the mid-plane and the out-of- plane rotations are prescribed as:

$$\psi_\alpha(\mathbf{X}^o) = \bar{\psi}_\alpha\ , \qquad u_\alpha(\mathbf{X}^o) = \bar{u}_\alpha\ , \qquad w(\mathbf{X}^o) = \bar{w}\ , \qquad on\ \ \Omega_\mathbf{u} \tag{2.11}$$

Next, we proceed to construct a weak form of the equilibrium equations (2.7) with boundary conditions (2.10) - (2.11).

(c) Weak Form. We introduce linear spaces of kinematically admissible variations formally defined as:

$$S_1 = \{\ \delta\mathbf{u} = \delta\mathbf{u}^o + \delta w\,\hat{\mathbf{E}}_3 : \Omega \rightarrow \mathbb{R}^3 \ \mid \delta\mathbf{u}|_{\Omega_u} = 0\ \}$$

$$S_2 = \{\ \delta\boldsymbol{\psi} = \delta\psi_\alpha \hat{\mathbf{E}}_\alpha : \Omega \rightarrow \mathbb{R}^3 \ \mid \delta\boldsymbol{\psi}|_{\Omega_u} = 0\ \} \tag{2.12}$$

In addition, if $f(\mathbf{x},\mathbf{y})$ is a vector valued function of its arguments, the (Frechet) derivative in the direction $(\mathbf{h},\mathbf{k})$ will be denoted by $Df(\mathbf{x},\mathbf{y}).(\mathbf{h},\mathbf{k})$ and may be computed as:

$$Df(\mathbf{x},\mathbf{y}).(\mathbf{h},\mathbf{k}) = \frac{d}{d\epsilon}\Big[f(\mathbf{x}+\epsilon\mathbf{h},\mathbf{y}+\epsilon\mathbf{k})\Big]_{\epsilon=0} \tag{2.13}$$

With this convention, a formal computation employing Green's formula leads to the weak form of the equilibrium equations (2.7) expressed as:

$$G = \int_\Omega \Big[\mathbf{N}:D\boldsymbol{\lambda}(\mathbf{u},\boldsymbol{\psi}).(\delta\mathbf{u},\delta\boldsymbol{\psi}) + \mathbf{M}:D\boldsymbol{\theta}(\boldsymbol{\psi}).\delta\boldsymbol{\psi} + \mathbf{V}:D\boldsymbol{\gamma}(\mathbf{u},\boldsymbol{\psi}).(\delta\mathbf{u},\delta\boldsymbol{\psi})\Big]\, d\Omega$$

$$- \int_\Omega q\ \delta w\ d\Omega - \int_{\Omega_t} [\bar{\mathbf{t}}.\delta\mathbf{u} + \bar{\mathbf{m}}.\delta\boldsymbol{\psi}]\ d\Gamma \quad \text{for } any \quad (\delta\mathbf{u},\delta\boldsymbol{\psi}) \in S \equiv S_1 \times S_2 \tag{2.14}$$

where $\boldsymbol{\lambda}(\mathbf{u},\boldsymbol{\psi})$, $\boldsymbol{\theta}(\boldsymbol{\psi})$ and $\boldsymbol{\gamma}(\mathbf{u},\boldsymbol{\psi})$ are defined by

$$\boldsymbol{\lambda}(\mathbf{u},\boldsymbol{\psi}) = \tfrac{1}{2}[\nabla^T\mathbf{u}^o + \nabla\mathbf{u}^o] + \tfrac{1}{2}\nabla w \otimes \nabla w - \tfrac{1}{2}(\nabla w - \boldsymbol{\psi})\otimes(\nabla w - \boldsymbol{\psi})$$

$$\boldsymbol{\gamma}(\mathbf{u},\boldsymbol{\psi}) = \nabla w - (\mathbf{I} + \nabla\mathbf{u}^o).\boldsymbol{\psi} \tag{2.15}$$

$$\boldsymbol{\theta}(\boldsymbol{\psi}) = \tfrac{1}{2}[\nabla^T\boldsymbol{\psi} + \nabla\boldsymbol{\psi}]$$

and represent the deformation measures conjugate to the resultant forces **N**, **V** and **M**, respectively.

We now turn our attention to the formulation of constitutive equations for elasticity. Only the pure mechanical theory is considered.

2.3.- Constitutive Equations. Elasticity.

Since the kinetic variables $\{\mathbf{N},\mathbf{V},\mathbf{M}\}$ are conjugated to the kinematic variables $\{\boldsymbol{\lambda},\boldsymbol{\gamma},\boldsymbol{\theta}\}$ we postulate the simplest constitutive model connecting these variables, which is clearly inspired in the linear theory [5], and is given by:

$$\boldsymbol{\lambda} = \frac{1+\nu}{Eh}\,\mathbf{N} - \frac{\nu}{Eh}\,tr(\mathbf{N})\,\mathbf{I} \equiv \mathbf{C}_N\mathbf{N}$$

$$\boldsymbol{\theta} = \frac{1+\nu}{D}\,\mathbf{M} - \frac{\nu}{D}\,tr(\mathbf{M})\,\mathbf{I} \equiv \mathbf{C}_M\mathbf{M} \tag{2.16}$$

$$\gamma = \frac{1}{Gh\kappa} \mathbf{V} \qquad (\kappa = 5/6)$$

The assumption of a constitutive model such as (2.16) avoids the explicit formulation of a three dimensional model, and is motivated by reasons examined in [5]. Note that in the context of a direct approach which regards the plate as a Cosserat surface with one director [12], assumption (2.16) is perfectly legitimate.

If we assume that constitutive equations (2.16) hold point-wise in $\Omega \subset \mathbb{R}^2$, their substitution into the weak form (2.15) leads to a functional $G[(\mathbf{u},\boldsymbol{\psi});(\delta\mathbf{u},\delta\boldsymbol{\psi})]$ which satisfies, for any $(\delta\mathbf{u},\delta\boldsymbol{\psi})$ and $(\overline{du},\delta\overline{\boldsymbol{\psi}})$ in $S = S_1 \times S_2$, the symmetry condition

$$D_1\, G[(\mathbf{u},\boldsymbol{\psi});(\delta\mathbf{u},\delta\boldsymbol{\psi})].(\delta\bar{\mathbf{u}},\delta\bar{\boldsymbol{\psi}}) = D_1\, G[(\mathbf{u},\boldsymbol{\psi});(\delta\bar{\mathbf{u}},\delta\bar{\boldsymbol{\psi}})].(\delta\mathbf{u},\delta\boldsymbol{\psi}) \tag{2.17}$$

Hence, assuming appropriate smoothness [13], there exists a potential $(\mathbf{u},\boldsymbol{\psi}) \rightarrow J(\mathbf{u},\boldsymbol{\psi})$, which physically represents the total potential energy of the plate, and with explicit expression given by Vainberg's formula ([13] pp.112) in terms of the weak form (2.14). Making use of this result, from (2.14) and (2.16) we arrive at the expression

$$J(\mathbf{u},\boldsymbol{\psi}) = \tfrac{1}{2}\int_{\Omega} \left[\boldsymbol{\lambda}:(\mathbf{C}_N^{-1}\boldsymbol{\lambda}) + \boldsymbol{\theta}:(\mathbf{C}_M^{-1}\boldsymbol{\theta}) + Gh\kappa\,\boldsymbol{\gamma}.\boldsymbol{\gamma}\right] d\Omega$$

$$- \int_{\Omega} q\, w\, d\Omega - \int_{\Omega_t} [\tilde{\mathbf{t}}.\mathbf{u} + \tilde{\mathbf{m}}.\boldsymbol{\psi}]\, d\Gamma \tag{2.18}$$

The total potential energy functional (2.18) can be used as the starting point for the development of *displacement* finite element formulations (see [6] for an application to beam problems). Since our interest focuses on *mixed* finite element models, instead of considering (2.18) we construct a weak form of constitutive equations (2.16) in the usual manner. The result may be written as:

$$H = \int_{\Omega} \left[\delta\mathbf{N}:\boldsymbol{\lambda} + \delta\mathbf{M}:\boldsymbol{\theta} + \delta\mathbf{V}.\boldsymbol{\gamma} - \delta\mathbf{N}:(\mathbf{C}_N\mathbf{N}) - \delta\mathbf{M}:(\mathbf{C}_M\mathbf{M}) - \frac{1}{Gh\kappa}\delta\mathbf{V}.\mathbf{V}\right] d\Omega = 0 \tag{2.19}$$

for any $(\delta\mathbf{N},\delta\mathbf{V},\delta\mathbf{M}) \in W$, where W may be taken as $W = [L(\Omega)^2]^4 \times [L(\Omega)^2]^2 \times [L(\Omega)^2]^4$ and such that $DIV(\delta\mathbf{N}) \in [L^2(\Omega)]^2$, $DIV(\delta\mathbf{V}) \in L^2(\Omega)$ and $DIV(\delta\mathbf{M}) \in [L^2(\Omega)]^2$. The pair of variational equations (2.14) (2.19) comprises the formulation suited for a mixed finite element approximation to the non-linear plate theory governed by equilibrium equations (2.7), and constitutive equations (2.16).

3.- SOLUTION PROCEDURE. FINITE ELEMENT FORMULATION.

Due to the non-linear nature of the strain measures $\boldsymbol{\lambda}(\mathbf{u},\boldsymbol{\psi})$, $\boldsymbol{\gamma}(\mathbf{u},\boldsymbol{\psi})$ and $\boldsymbol{\theta}(\boldsymbol{\psi})$, appearing in the energy functional (2.18) or in the variational equations (2.14) and (2.19); an iterative solution procedure is necessary. The linearization of (2.14) and (2.19) about an intermediate configuration then plays a key note in the solution scheme. A complete account of linearization techniques in the general context of infinite dimensional manifolds can be found in [13]. For the problem at hand, our approach essentially follows that employed in [4,6].

3.1.- The Linearized Variational Problem.

For simplicity in the presentation, we introduce the following matrix notation

$$\begin{aligned} \mathbf{r}(\mathbf{X}^o) &\equiv (\mathbf{u}^o, w, \boldsymbol{\psi}) = [u_1^o \quad u_2^o \quad w \quad \psi_1 \quad \psi_1]^T \\ \mathbf{f}(\mathbf{X}^o) &\equiv (\mathbf{N},\mathbf{V},\mathbf{M}) = [N_{11} \quad N_{22} \quad N_{12} \quad V_1 \quad V_2 \quad M_{11} \quad M_{22} \quad M_{12}]^T \\ \boldsymbol{\Lambda}(\mathbf{X}^o) &\equiv (\boldsymbol{\lambda},\boldsymbol{\gamma},\boldsymbol{\theta}) \equiv [\lambda_{11} \quad \lambda_{22} \quad \lambda_{12} \quad \gamma_1 \quad \gamma_2 \quad \cdots]^T \end{aligned} \tag{3.1}$$

In addition, we define a linear operator $\mathbf{r} \rightarrow \mathbb{B}(\mathbf{r})$ by

$$\mathbb{B}(\mathbf{r}) = (\nabla \mathbf{u}^o, \nabla w, \boldsymbol{\psi}, \nabla \boldsymbol{\psi}) \equiv [u_{1,1}^o \quad u_{2,1}^o \quad u_{1,2}^o \quad \cdots]^T \tag{3.2}$$

Consider an intermediate configuration of the plate characterized by $\bar{\mathbf{r}}(\mathbf{X})$ and acted on by forces $\bar{\mathbf{f}}(\mathbf{X}^o)$. Let $\Delta\mathbf{r}(\mathbf{X}^o) : \Omega \rightarrow \boldsymbol{R}^5$ be a superposed infinitesimal deformation; that is, a vector field covering $\bar{\mathbf{r}}(\mathbf{X}^o) : \Omega \rightarrow \boldsymbol{R}^5$, which gives rise to an incremental configuration characterized by $\mathbf{r}(\mathbf{X}^o) + \Delta\mathbf{r}(\mathbf{X}^o)$, and acted upon by forces $(\bar{\mathbf{f}} + \Delta\mathbf{f})(\mathbf{X})$. It follows from definition (2.15) that the (Frechet) derivative of the of the strain measures $\boldsymbol{\Lambda}(\mathbf{r})$ at $\bar{\mathbf{r}}(\mathbf{X}^o)$ in the direction $\Delta\mathbf{r}(\mathbf{X}^o)$ is given, in matrix notation, by

$$D\,\boldsymbol{\Lambda}(\bar{\mathbf{r}}).\Delta\mathbf{r} = \boldsymbol{\Xi}^T(\bar{\mathbf{r}})\,\mathbb{B}(\Delta\mathbf{r}) \tag{3.3}$$

where $\boldsymbol{\Xi}(\bar{\mathbf{r}})$ is a (10 x 10) matrix. By a procedure similar to that employed in [4,6], the linearization of the weak form of equilibrium equations about the configuration $\bar{\mathbf{r}}(\mathbf{X}^o)$ may be computed from its expression (2.14) and (3.3). Setting the result equal to zero, we arrive at a linearized variational problem which may be written as:

$$L\,[G]_{\bar{\mathbf{r}}} = \int_\Omega \mathbb{B}^T(\delta\mathbf{r}).\Big[\mathbf{A}_G(\bar{\mathbf{f}})\,\mathbb{B}(\Delta\mathbf{r}) + \boldsymbol{\Xi}(\bar{\mathbf{r}})\,\Delta\mathbf{f}\Big]\,d\Omega + G(\bar{\mathbf{r}},\bar{\mathbf{f}},\delta\mathbf{r}) = 0 \tag{3.4a}$$

for any $\delta\mathbf{r}(\mathbf{X}^o) \in S$. $G(\bar{\mathbf{r}},\bar{\mathbf{f}},\delta\mathbf{r})$ gives the "out-of-balance force" at the configuration $\bar{\mathbf{r}}(\mathbf{X}^o)$ and has the explicit expression

$$-G(\bar{\mathbf{r}},\bar{\mathbf{f}},\delta\mathbf{r}) = \int_{\Omega_t} [\delta\mathbf{u}.\bar{\mathbf{t}} + \delta\boldsymbol{\psi}.\bar{\mathbf{m}}]\, d\Gamma - \int_{\Omega} \mathbb{B}^T(\delta\mathbf{r})\, \Xi(\mathbf{r})\, \bar{\mathbf{f}}\, d\Omega \tag{3.4b}$$

A similar computation for the weak form (2.19) of the constitutive equation leads to the linearized variational problem

$$L[H]_{\bar{\mathbf{r}}} = \int_{\Omega} \delta\mathbf{f}^T \Big[\Xi_T(\mathbf{r})\, \mathbb{B}(\Delta\mathbf{r}) - \mathbf{C}\,\Delta\mathbf{f}\Big] d\Omega + \int_{\Omega} \delta\mathbf{f}^T.\Lambda(\bar{\mathbf{r}})\, d\Omega = 0\,, \quad \text{for } \textit{any } \delta\mathbf{f} \in W \tag{3.5}$$

Equations (3.4) and (3.5) define at each intermediate configuration $\bar{\mathbf{r}}(\mathbf{X}^o) : \Omega \to R^3$ a coupled system of variational equations from which a subsequent configuration $\bar{\mathbf{r}}(\mathbf{X}^o) + \Delta\mathbf{r}(\mathbf{X}^o)$ may be obtained, provided (3.4) and (3.5) are well posed at $\bar{\mathbf{r}}(\mathbf{X}^o)$. That is, provided nothing catastrophic such as bifurcation phenomena occur.

3.2.- Mixed Finite Element Approximation.

Finite element formulations of the variational problem defined by equations (3.4) and (3.5) involve the approximation of the spaces S and W by finite dimensional subspaces $S_h \subset S$ and $W_h \subset W$. The domain Ω is partitioned into a collection of disjoint finite elements $\{\Omega_e\}_{e=1}^{E}$ such that and $\bigcup_{e=1}^{E} \Omega_e \approx \Omega$. Global interpolating functions are constructed as usual by patching together local shape functions defined over each finite element. Over a typical element Ω_e, the incremental displacement vector is interpolated as

$$\Delta\mathbf{r}(\mathbf{X}^o)|_{\Omega_e} = \sum_{i=1}^{N_e} \mathbf{s}_i^e(\mathbf{X}^o)\, \Delta\mathbf{R}_i^e \quad => \quad \mathbb{B}|_{\Omega_e}(\Delta\mathbf{r}) = \sum_{i=1}^{N_e} \mathbf{B}_i^e(\mathbf{X}^o)\, \Delta\mathbf{R}_i^e \tag{3.6}$$

where the shape functions $\{\mathbf{s}_i^e\}_{i=1}^{N_e}$ satisfy the usual relations $\mathbf{s}_i^e(\mathbf{X}^o{}_k) = \delta_{ik}$ at nodal points $\{\mathbf{X}^o{}_k\}_{k=1}^{N_e}$. Similarly $\Delta\mathbf{f}$ is interpolated over any Ω_e by

$$\Delta\mathbf{f}(\mathbf{X}^o)|_{\Omega_e} = \sum_{m=1}^{M_e} \mathbf{h}_m^e(\mathbf{X}^o)\, \Delta\mathbf{F}_m^e \tag{3.7}$$

At this stage *the selection of shape functions* $\{\mathbf{h}_m^e\}_{m=1}^{M_e}$ and $\{\mathbf{s}_i^e\}_{i=1}^{N_e}$ as well as the location and meaning of the corresponding nodal quantities, is independent. For each element Ω_e, we define the matrices

$$\mathbf{K}_G^e = [\mathbf{K}_{G\,ij}^e]\,, \qquad \mathbf{K}_{G\,ij}^e = \int_{\Omega_e} \mathbf{B}_i^{eT}\, \mathbf{A}_G(\bar{\mathbf{f}})\, \mathbf{B}^{j^e}\, d\Omega$$

$$\mathbf{T}^e = [\mathbf{T}^e{}_{im}]\,, \qquad \mathbf{T}^e{}_{im} = \int_{\Omega_e} \Xi(\bar{\mathbf{r}})^T\, \mathbf{B}_i^e\, \mathbf{h}_m^e\, d\Omega \tag{3.8}$$

$$\mathbf{C}^e = [\mathbf{C}^e{}_{mn}]\ , \qquad \mathbf{C}^e{}_{mn} = \int_{\Omega_e} \mathbf{h}_m^{e\,T}\,\mathbf{C}\,\mathbf{h}_n^e\,d\Omega$$

$$\mathbf{\Gamma}^e = \{\mathbf{\Gamma}^e{}_m\}\ , \qquad \mathbf{\Gamma}^e{}_m = \int_{\Omega_e} \mathbf{h}_m^{e\,T}\,\Lambda(\bar{\mathbf{r}})\,d\Omega$$

The restriction of the variational equations (3.4) and (3.5) to a typical element Ω_e may be written as:

$$\delta\mathbf{R}^{e\,T}\Big[\mathbf{K}_G^e\,\Delta\mathbf{R}^e + \mathbf{T}^{e\,T}\,\Delta\mathbf{F}^e - (\boldsymbol{\sigma}^e - \mathbf{T}^{e\,T}\,\mathbf{F}^e)\Big] = \mathbf{O} \tag{3.9}$$

$$\delta\mathbf{F}^{e\,T}\Big[\mathbf{T}^e\,\Delta\mathbf{R}^e - \mathbf{C}^e\,\Delta\mathbf{F}^e + \mathbf{\Gamma}^e\Big] = \mathbf{O} \tag{3.10}$$

where $\boldsymbol{\sigma}^e$ gives the contribution of the force boundary conditions on Ω_t to $\partial\Omega_t^e$; and evaluation of the matrices entering in (3.9) and (3.10) at the configuration $\bar{\mathbf{r}}(\mathbf{X}^o) : \Omega \to \mathbb{R}^5$ is understood. $\mathbf{K}_G^e$ is often referred to as the geometric stiffness, $\mathbf{C}^e$ as the "weighted" compliance matrix and $\mathbf{\Gamma}^e$ gives the "constitutive residual" at configuration $\mathbf{r}(\mathbf{X}^o)$. Global forms associated with (3.9) and (3.10) can be constructed upon noting that

(i) The entries in the vector $\mathbf{r}(\mathbf{X}^o)$ must lie at least in $H^1(\Omega)$. Hence, C^0 interelement continuity is required when patching together the local shape functions $(3.6)_1$ to construct global interpolation functions.

(ii) The components of the force vector $\mathbf{f}$ can be taken in $L^2(\Omega)$, therefore, no *no interelement continuity is required* for the assembly of shape functions (3.7).

As a consequence of (ii), equation (3.10) may be *solved at the element level* for the nodal forces $\delta\mathbf{F}^e$ and the result substituted in (3.9) leading to the *generalized displacement model*

$$\delta\mathbf{R}^{e\,T}\Big[(\mathbf{K}_G^e + \mathbf{T}^{e\,T}\mathbf{C}^{e^{-1}}\mathbf{T}^e)\,\Delta\mathbf{R}^e - (\boldsymbol{\sigma}^e - \mathbf{T}^{e\,T}\mathbf{F}^e - \mathbf{T}^{e\,T}\mathbf{C}^{e^{-1}}\mathbf{\Gamma}^e)\Big] = \mathbf{O} \tag{3.11}$$

Note that (3.10) may be numerically solved "exactly" by *local iteration* to within a prescribed norm of the constitutive residual $\mathbf{\Gamma}^e$. As recently shown in [6], such a procedure leads to a particularly convenient algorithm for finite deformation inelasticity.

The success of the mixed formulation depends critically on the selection of the basis spanning the subspace W_h, i.e; on the choice of shape functions $\{\mathbf{h}_m^e\}_{m=1}^{M_e}$ which interpolate the force vector $\mathbf{f} = (\mathbf{N},\mathbf{V},\mathbf{M})$ within an element Ω_e. This choice affects critically the stability of the method which in turn is closely related to a discrete form of the Babuska-Brezzi condition [15,16].

The rest of this paper is devoted to the analysis of this critical step. Our presentation is based upon previous work in [18] and emphasizes the physical aspects involved in the selection of the local basis $\{\mathbf{h}_m^e\}_{m=1}^{M_e}$. Related mathematical aspects have been considered in [15,16] and references therein.

3.3.- Choice of Shape Functions

In this paper we use standard 4-node, isoparametric shape functions for all components of displacements, and displacement increments. The shape functions for the components of the force vector are chosen as polynomials of sufficient order according to (ii). In each element it is possible to select polynomial expressions either in terms of global or local cartesian coordinates, or in terms of the natural coordinates used to construct the displacement shape functions.

Although the problem considered here is nonlinear, the incremental problem obtained by linearization about a stable configuration is linear. Accordingly, the solution of a linear problem by mixed methods provides the necessary insight to solve the nonlinear problem. Error estimates for the strain increments computed from the approximating displacement field may be used as a convenient guide to limit the number of terms used for each force approximating function. For example, use of the bilinear shape functions for displacement quantities implies that only a complete set of constant strain increment polynomials are recovered in each element. While additional terms exist in the strain increments, they are not a complete order of linear polynomials. Thus, in a mixed method it is not possible to obtain force approximations which are of higher order accuracy than a complete constant field. The use of constants for each force component, however, will lead to an element tangent matrix for the generalized displacement model which is rank deficient (i.e., has zero eigenvalues for displacement increments which are not rigid body modes). Consequently, in addition to the terms corresponding to the complete order of polynomials in the strain increments additional terms are required to avoid rank deficiency of the generalized tangent matrix.

If force approximations are employed with all the terms which result from a pure displacement formulation, the mixed model will give identical results with a displacement formulation. This is often called the "limitation principle" [19]. Indeed, if more terms are added then the mixed principle merely discards them (i.e., produces zero coefficients to these terms). Thus, the proper number of terms for each force component to be used in a mixed model is limited on one side by accuracy assessments and on the other by the limitation principle. As additional terms are added to the force terms of a mixed formulation convergence to the solution of the displacement formulation results. One may inquire then about the merits of pursuing mixed formulations. In general, no increase in accuracy of error estimates for a mixed formulation over those of a displacement formulation results. However, for problems with constraints or other complicating factors resulting from the constitutive model employed, the asymptotic rate of convergence may be achieved much earlier if a judicious choice of approximating functions for the force components is made.

In the work reported here we have sought the minimum number of force parameters to give proper rank of the generalized tangent matrix. In addition, we require our element to produce results which are independent of the user (e.g., the numbering sequence of the nodes or orientation of the mesh with respect to the coordinate axes). In a previous study the use of cartesian coordinate approximations has been shown to lead to better accuracy when elements are distorted than use of polynomials in the natural coordinates of the displacement shape functions. Consequently, the present study considers only polynomials of cartesian coordinates with respect to a uniquely defined local set of axes in each element, as shown in Figure 1.

For a 4-node plate element with 5-displacement parameters per node, the rank of the generalized tangent matrix should be 14. Consequently, at least 14 independent terms must be used to approximate the element force vector. The terms selected for the 4-node element considered here use M_e equal to 2 with $\mathbf{h}_1^e$ taken as the identity matrix and $\mathbf{h}_2^e$ defined as:

$$\mathbf{h}_2^e = \begin{bmatrix} \mathbf{A}_1 & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{A}_2 & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{A}_1 \end{bmatrix} \qquad (3.11)$$

where

$$\mathbf{A}_1 = \begin{bmatrix} Y & 0 \\ 0 & X \\ 0 & 0 \end{bmatrix} \qquad \mathbf{A}_2 = \begin{bmatrix} Y & 0 \\ 0 & X \end{bmatrix} \qquad (3.12)$$

The choice of these approximating force functions is largely motivated by intuition. However, the above choice satisfies our initial objectives relative to accuracy, rank of the tangent matrix, and invariance with respect to user inputs. The choice of the specific functions for the in-plane forces is motivated by a desire to decouple the "bending" response from the "shearing" response. (at least with respect to the local coordinate frame). Numerical experiments for a linear elastic problem indicate that a mixed element using the above approximation is significantly more accurate than a displacement model when the primary mode of response is in-plane, pure bending. Moreover, the mixed element results were less sensitive to in-plane mesh distortions. The moments **M** are approximated with the same functions as the in-plane forces **N**. The approximations for in-plane and bending forces consist of 10-terms, consequently, to produce correct rank for the element generalized tangent matrix, the approximations for the transverse shearing forces **V** must contain 4-terms. While this approximation leads to correct rank of the element generalized tangent matrix, and correct asymptotic rate of convergence, issues related to *locking* for thin plate applications should be considered also. Indeed, when thin plate problems were solved using meshes of distorted elements, it was found necessary to introduce a modification for the transverse shear strain-displacement relations to avoid locking. A method analogous to that employed in [20] was adopted to avoid locking for the thin plate limit while retaining proper rank for thick plate applications.

4.- NUMERICAL EXAMPLES

Two example problems are selected to illustrate the performance of the plate theory and mixed model element. The first example is the cylindrical bending of a plate with simply supported ends and uniform vertical loading. Constitutive equations for this example are taken as elastic-perfectly plastic in terms of force resultants. The solution is achieved using a penalty formulation for an elastic-viscoplastic model. The second example is the solution of a simply supported, square plate subjected to uniform vertical loading. Solutions are obtained for elastic and elastic-plastic response.

4.1.- Cylindrical Bending of a Uniformly Loaded Plate

The first example considers the cylindrical bending of a plate subjected to uniform vertical loading. The plate considered has a 20 inch span and a 1 inch thickness. The material is assumed to be elastic-perfectly plastic with the following values for constants:

$E = 30000$ ksi.

$\nu = 0.3$

$\sigma_y = 36$ ksi

where ν is Poisson's ratio and σ_y is the uniaxial yield stress. In all problems considered in this paper the constitutive equations are given in terms of force resultants (e.g., see (2.16)). The elastic-plastic relations are developed as a penalty form of an elastic-viscoplastic model in terms of appropriate objective rates [6]. The viscoplastic flow potential is taken as

$$f = Q_N + Q_M + \frac{1}{\sqrt{3}} |Q_{NM}| + Q_V - 1$$

where

$$Q_N = n_{11}^2 - n_{11} n_{22} + n_{22}^2 + 3\, n_{12}^2$$

$$Q_M = m_{11}^2 - m_{11} m_{22} + m_{22}^2 + 3\, m_{12}^2$$

$$Q_{NM} = n_{11} m_{11} - \tfrac{1}{2}(n_{11} m_{22} + n_{22} m_{11}) + n_{22} m_{22} + 3\, n_{12} m_{12}$$

$$Q_V = v_1^2 + v_2^2$$

and the resultant variables appearing are normalized according to

$$n_{ab} = \frac{N_{ab}}{\sigma_y\, h} \; ; \qquad m_{ab} = \frac{4\, M_{ab}}{\sigma_y\, h^2} \; ; \qquad v_a = \frac{\sqrt{3}\, V_a}{\sigma_y\, h}$$

This potential was suggested by Shapiro [21]. Comparisons between resultant form of the constitutive equations and forms in terms of the stress tensor directly have been performed for beams [6]. In general, excellent agreements result. The cost of analysis is reduced considerably by using the resultant forms since no numerical integration through the plate thickness is required.

The inclusion of the transverse shearing forces in the yield function plays a key role in the prediction of plastic collapse of the plate. Essentially, the plate initially yields at the center and gradually through increased rotation transfers the loading to the membrane forces. Near the supports, however, an interaction of the shearing and in-plane loadings occurs which eventually leads to a fully developed plastic zone and a resultant collapse mechanism. The plate model developed here predicts an ultimate load as shown in Figure 2. It should be noted that the predicted collapse load is significantly higher than that predicted from the small displacement theory. In the small displacement theory the ultimate load occurs when the plastic moment at the center of the plate is exceeded. In an attempt to evaluate the predicted collapse load, the cylindrical bending of the plate was modeled as a large deformation, plane strain problem using the computer program NIKE2D [22]. The constitutive equation is assumed to be elastic-perfectly plastic with a Mises potential function defining the yield condition. The results for the transverse center displacement from the NIKE2D analysis is also plotted in Figure 2. The correlation between the two analyses is quite good for displacements up to twice the plate thickness (i.e., 2 inches). Above this value however the two solutions differ, essentially due to the collapse mechanisms occurring. These results are consistent with the *apriori* estimates for the range of validity of the second order theory given by (2.4). A plot of the deformed plate at collapse, as predicted by NIKE2D, is shown in Figure 3. It can be observed that the plate *necks* in the vicinity of the support, clearly a phenomenon which the plate kinematics given in (2.1), (2.2) do not include.

4.2.- Simply Supported Square Plate, Uniformly Loaded

The second example considered is a simply supported, square plate subjected to uniform vertical loading. The material model and properties are the same as given for the first example. The plate has side lengths of 20 inches and the thickness is again 1 inch. Bounds for the small deflection collapse load for a perfectly plastic plate have been given in [23]. If the non-linear terms in the strain-displacement equations are suppressed the plate model described here produces answers within the bounds for a 16 element (4x4) mesh of elements [24]. If the yield stress is set artificially large (e.g., equal to the elastic modulus), the plate model gives answers corresponding to the Levy solution [25]. If the transverse shear terms in the potential defining yielding are suppressed, then after initial yielding the loading is transformed to the in-plane force resultants and the plate behaves like a membrane with tension equal to the limiting in-plane yield loads. Finally, when all terms are included in the kinematics and the yield function the plate initially yields due to bending, gradually transfers the loads into the in-plane force resultants, and, finally, collapses under a combined in-plane and transverse shear mechanism. For all the cases described above, the results for the displacement at the center of the plate are shown in Figure 4.

The two examples described above illustrate some of the aspects of the behavior for the plate theory and a mixed-model finite element implementation described in this paper. In particular, the inclusion of transverse shear deformation permits inclusion of yield effects related to transverse shear force resultants. This effect, combined with the kinematic approximations for the plate, permits consideration of collapse mechanisms due to combined in-plane, bending, and shear effects. For metals, a comparison with results obtained using the finite deformation plasticity model in NIKE2D indicates that the predicted collapse load from the plate model is unconservative. This is primarily due to the fact that NIKE2D predicts a "necking" phenomena near the edge which is not included in the kinematic approximations of the plate theory.

The kinematic approximations employed in the plate theory developed here are more appropriate for applications where the plate is thick or where the transverse shear effects are large compared to in-plane and bending effects -- for example, in sandwich plates. An example of the discrepancy of results produced using commonly employed kinematic approximations (i.e., von Karman plate theory) and those described here is shown for a beam in [4,6]. Similar results will occur for applications to typical plate problems with these properties.

Acknowledgements

Partial support for the work reported here was provided by the Lawrence Livermore National Laboratory. We acknowledge especially the interest and encouragement of Dr. G. L. Goudreau of LLNL. The third author expresses appreciation to the Fannie and John Hertz Foundation who provided support during his graduate study at the University of California.

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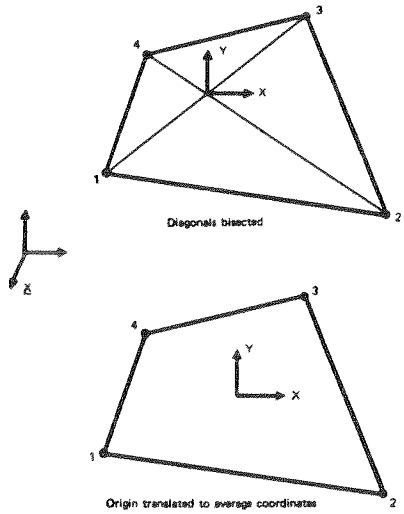

Figure 1. Local Coordinate System for Element

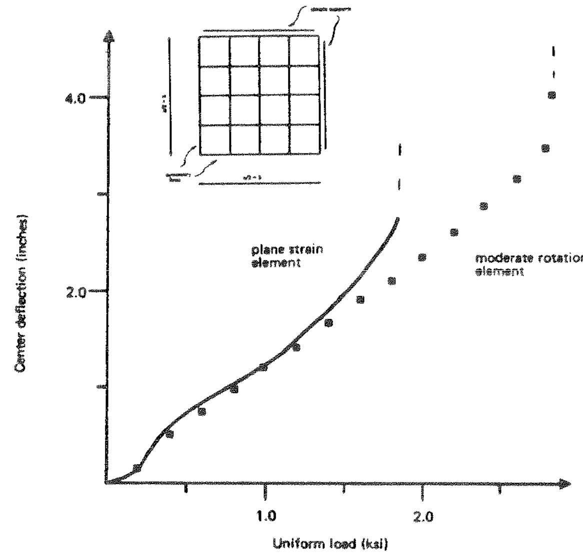

Figure 2. Displacement at the Center of a Uniformly Loaded, Simply Supported Plate for Cylindrical Bending.

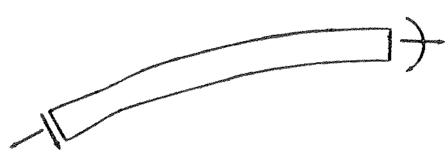

Figure 3. Collapse Mechanism for Cylindrical Bending of a Plate. NIKE2D Solution.

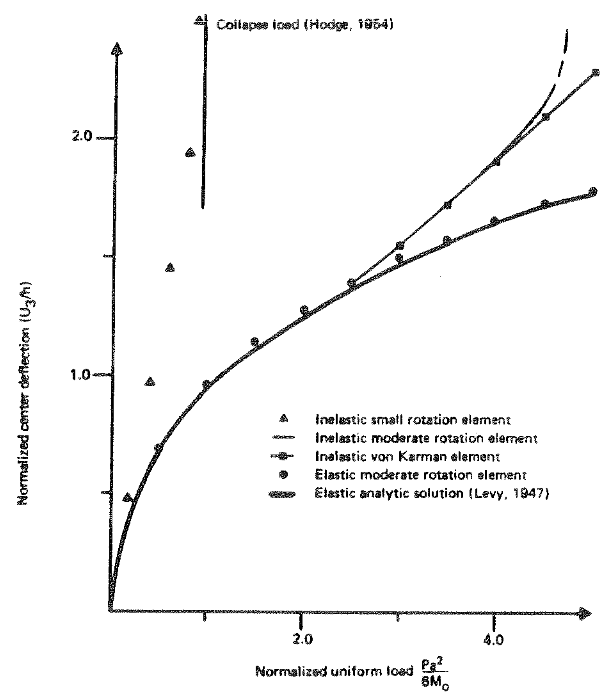

Figure 4. Displacement at the Center of a Uniformly Loaded, Simply Supported Square Plate for Various Models.