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by

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Professor Robert M. Anderson

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Abstract

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The semidefinite linear complementarity problem (SDLCP) is a generalization of the linear complementarity problem (LCP) in which linear transformations replace matrices and the cone of positive semidefinite matrices replaces the nonnegative orthant. We study a number of linear transformation classes (some of which are introduced for the first time) and extend several known results in LCP theory to the SDLCPs, and in particular, results which are related to the key properties of uniqueness, feasibility and convexity. Finally, we introduce some new characterizations related to the class of matrices $\mathbf{E^*}$ and the uniqueness of the LCPs.

To my parents,
Yukai Wang and Jine Feng,
who sacrifice their whole lives for me.

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Chapter 1

Introduction

1.1 Introduction

The goal of the linear complementarity problem (LCP) is to find a real vector of finite dimension that satisfies a system of linear inequalities with an additional quadratic constraint. Specifically, given a vector $q \in \boldsymbol{R^n}$ and a square matrix $M \in \boldsymbol{R^{n \times n}}$, it is to find a vector $z \in \boldsymbol{R^n}$ such that

$$\begin{aligned} z &\geq 0, && (1.1) \\ q + Mz &\geq 0, && (1.2) \\ z^T(q + Mz) &= 0, && (1.3) \end{aligned}$$

or to show no such vector z exists.

Extensive coverage of linear complementarity problem theory is available in the monographs (Cottle et al., 1992; Murty, 1988) and the research articles in their reference lists. One of the interesting aspects of LCPs is its range of applications, from well understood and relatively easy problems such as linear and convex quadratic programming problems, to NP-hard problems.

Besides covering several important classes of mathematical programming problem such as linear programming (Cottle and Dantzig, 1968a), convex quadratic programming (Hildreth, 1954, 1957; Frank and Wolfe, 1956b), Nash equilibrium points for nonzero sum games (Lemke and Howson, 1964; Hansen and Scarf, 1974), several economic equilibrium problems (Manne, 1985) and the knapsack problem (Chung, 1989), LCP is also used to model many applications such as the contact problem (Conry and Seirge, 1971), which studies the situation in which one deformable body comes in contact with another, and how loads are delivered to the structure and how the structure are supported to sustain loads; the porous flow problem (Oden and Kikuchi, 1986), which analyzes the seepage through porous media in the presence of a free surface; the obstacle problem (Cottle and Coheen, 1978; Moré and Toraldo, 1991), which finds the equilibrium position of an elastic membrane whose boundary is held fixed, and which is constrained to lie above a given obstacle; the journal bearing problem (Goenka, 1984), which analyzes dynamically loaded journal bearings; and many other free boundary problems (Cryer, 1977).

A great deal of research effort was devoted to classifying LCP problems according to the type of matrices that appear in the formulation. Typically, these classifications were investigated in two major directions: The first one was related to the intrinstic properties of the matrix itself (e.g. positive semidefinite matrices); the second one was related to properties of the corresponding LCP solution set (e.g. the matrices that yield convex solution sets). The ultimate goal was to find the connections between the two sets of characterizations, that is, to find the intrinstic properties of the classes of matrices that leads to a specific properties of the solution set.

Many classes were identified over the years with some major results related to uniqueness, feasibility and convexity. (Some of these results are discussed throughout this dissertation.) These studies provided crucial information and insight for the development and analysis of specific algorithms.

Motivated by the generalization of linear programming to semideinite programming that was introduced by Bellman and Fan (Bellman and Fan, 1963), similiar generalization of the LCP was first introduced by Kojima, Shindoh, and Hara (Kojima et al., 1997). In the following we present an equivalent, but slightly different, form that was introduced by (Gowda and Song, 2000), and is the focus of this dissertation.

- In (1.1), we replace $z \in \boldsymbol{R^n}$ with $X \in \boldsymbol{S^n}$, where $\boldsymbol{S^n}$ denotes the set of the all real symmetric $n \times n$ matrices. Then we replace $z \geq 0$ with

$$X \succeq 0, \tag{1.4}$$

  where $X \succeq 0$ means that X is positive semidefinite (that is, $y^T X y \geq 0$ for every real n-vector y).

- In (1.2), we replace Mz with a linear transformation $L(X)$ from $\boldsymbol{S^n}$ to $\boldsymbol{S^n}$, that is, for any $X,\ Y \in \boldsymbol{S^n}$ and $\alpha,\ \beta \in \boldsymbol{R}$, L satisfies $L(\alpha X + \beta Y) = \alpha L(X) + \beta L(Y)$. Similarly to (1.4), we replace $Mz + q \geq 0$ with

$$L(X) + Q \succeq 0, \tag{1.5}$$

  where Q is a given matrix in $\boldsymbol{S^n}$. Notice that the linear transformation $L(X)$ is very general and does not have simple form of expression. For example, the Lyapunov transformation $L_A(X) := AX + XA^T$ (see e.g. Horn and Johnson, 1985), only represent an explicit kind of linear transformation.

- In (1.6), we replace $z^T(Mz + q) = 0$ with

$$X(L(X) + Q) = 0. \tag{1.6}$$

  Note that since the operation of product of matrices is not necessarily commutative, it is not clear in (1.6) whether to require $X(L(X) + Q) = 0$ or $(L(X) + Q)X = 0$. However, given (1.4) and (1.5), it can be easily shown (see e.g. Horn and Johnson, 1985) that $X(L(X)+Q) = 0$ iff $(L(X)+Q)X = 0$. Moreover, in this case $X(L(X)+Q) = 0$ iff $\text{trace}(X(L(X) + Q)) = 0$[1], thus it is customary to write (1.6) as

$$\langle X, L(X) + Q \rangle = 0, \tag{1.7}$$

  where for $A,\ B \in \boldsymbol{R^{n\times n}}$, $\langle A, B \rangle$ denotes $\text{trace}(A^T B)$.

[1]For any $A \in \boldsymbol{R^{n\times n}}$, $\text{trace}(A) = \sum_{i=1}^{n} A_{ii}$, where A_{ij} is (i, j) element of matrices A. Notice that for any $X, Y \in \boldsymbol{R^{n\times n}}$, $\text{trace}(XY) = \text{trace}(YX)$.

To summarize, this problem, which is called the *Semidefinite Linear Complementarity Problem* (SDLCP), is to find $X \in \boldsymbol{S^n}$ such that

$$\begin{aligned} X &\succeq 0, & (1.8)\\ L(X)+Q &\succeq 0, & (1.9)\\ \langle X, L(X)+Q\rangle &= 0, & (1.10)\end{aligned}$$

or to show that no such matrix X exists, where $L : \boldsymbol{S^n} \to \boldsymbol{S^n}$ is a given linear transformation, and Q is a given real symmetric matrix (i.e., $Q \in \boldsymbol{S^n}$).

It is well known (see e.g. Ferris and Pang, 1997) that LCP is a special case of the semidefinite linear complementarity problem. Specifically, when the matrix Q is a real diagonal matrix, and the linear transformation L is a mapping from the set of all real diagonal matrices to itself, the semidefinite linear complementarity problem becomes a linear complementarity problem.

It should also be noted that the semidefinite linear complementarity problem is a special case of the cone complementarity problem (Isac, 1992) and also a special case of the variational inequality (VI) problem (Harker and Pang, 1990). Though general results on the cone complementarity problem and the variational inequality problem are applicable to SDLCPs, with the additional structural conditions as of the SDLCP it is possible to get stronger results.

Given the many results concerning the classifications of LCPs through properties related to their matrices M, it is natural to investigate the possibility of generalizing these classifications to SDLCPs. The goal of this dissertation is to investigate some of these possibilities with respect to classes of linear transformations. We also look into the relationships among different classes of linear transformations in SDLCPs.

The materials in this dissertation are organized as follows: After introducing notations and preliminary results in the next section, we study in Chapter 2 conditions for uniqueness of the solution to the SDLCP. In particular, we first introduce the classes of matrices that are related to the uniqueness of the solution of the LCP in general as well as for specific sets of q ($q \geq 0$ and $q > 0$). In addition, we consider the class of matrices which yield unique solution for LCPs with $0 \neq q \geq 0$ and develop several new results with respect to it. We then review some known results on several classes of linear transformations related to the uniqueness of the solution of SDLCPs. Finally, we characterize linear transformations that lead to unique solutions to SDLCPs with $Q \succeq 0$, $Q \succ 0$ and $0 \neq Q \succeq 0$. In Chapter 3, we study issues related to the convexity of the solution set of the SDLCP. In particular, following the introduction of some known results related to LCPs, we generalize three known classes of matrices in LCPs which are related to convexity to corresponding three classes of linear transformations for SDLCPs, and present necessary and sufficient condition for linear transformations to be of those classes. Finally in Chapter 4, we review some known results regarding the relationships among property classes of linear transformations in the context of the SDLCP and then establish several more relationships.

1.2 Preliminaries

We denote the LCP (1.1)-(1.3) with parameters q and M by LCP(q, M). A vector z is called *feasible* if it satisfies (1.1) and (1.2). The set of all feasible vectors z is the *feasible region* of LCP(q, M) and is denoted by FEA(q, M). LCP(q, M) is said to be *feasible* if it has a nonempty feasible region. A feasible vector z which also satisfies (1.3) is called a *solution* to LCP(q, M). The set of all solutions to LCP(q, M) is called the *solution set* of LCP(q, M) and is denoted by SOL(q, M). We say that LCP(q, M) is *solvable* if its solution set is nonempty.

For the SDLCP, we adopt a similar terminology. We denote the SDLCP (1.8)-(1.10) with parameters Q and L by SDLCP(L, Q). A matrix X is *feasible* if it satisfies (1.8) and (1.9). The set of all feasible matrices X is called the *feasible region* of the SDLCP(L, Q) and is denoted by FEA(L, Q). SDLCP(L, Q) is said to be *feasible* if it has a nonempty feasible region. A feasible matrix X which also satisfies (1.10) is called a *solution* to SDLCP(L, Q). The set of all solutions to SDLCP(L, Q) is the *solution set* of SDLCP(L, Q), and is denoted by SOL(L, Q). We say that SDLCP(L, Q) is *solvable* if its solution set is nonempty.

The following notations are used throughout this dissertation:

- I denotes the identity matrix.
- Given an index set $\alpha \subset \{1, \cdots, n\}$, we denote the principal submatrix $M_{\alpha\alpha} := (M_{ij})_{i,j\in\alpha}$.
- $X \succeq 0$ means $X \in \boldsymbol{S^n}$ is positive semidefinite, (that is, $y^T X y \geq 0$ for any read n-vector y).
- $X \succ 0$ means $X \in \boldsymbol{S^n}$ is positive definite, (that is, $y^T X y > 0$ for any nonzero read n-vector y).
- $X \preceq 0$ means $-X \succeq 0$, (similarly, $X \prec 0$ means $-X \succ 0$).
- For $a \in \boldsymbol{R}$, we denote $a^+ := \max\{a, 0\}$ and $a^- := a^+ - a$.
- $D = \mathrm{diag}(d_1, d_2, \ldots, d_n)$ denotes a diagonal matrix in $\boldsymbol{R^{n\times n}}$ whose diagonal elements are $d_1, d_2, \ldots, d_n$.
- For $D = \mathrm{diag}(d_1, d_2, \ldots, d_n)$, we denote $D^+ := \mathrm{diag}(d_1^+, d_2^+, \ldots, d_n^+)$ and $D^- := \mathrm{diag}(d_1^-, d_2^-, \ldots, d_n^-)$.
- For $x \in \boldsymbol{R^n}$, supp(x) denotes the index set of all the nonzero elements in x.

The following matrix theoretic properties are used in this dissertation (see e.g. Horn and Johnson, 1985):

- $X \in \boldsymbol{S^n}$, $X \succeq 0 \;\Rightarrow\; UXU^T \succeq 0$ for any orthogonal matrix[2] $U \in \boldsymbol{R^{n\times n}}$.
- $X,\ Y \in \boldsymbol{S^n}$, $X \succeq 0,\ Y \succeq 0 \;\Rightarrow\; \langle X, Y\rangle \geq 0$.
- $X,\ Y \in \boldsymbol{S^n}$, $X \succeq 0,\ Y \succeq 0,\ \langle X, Y\rangle = 0 \;\Rightarrow XY = YX = 0$.
- $X \in \boldsymbol{S^n},\ \langle X, Y\rangle \geq 0,\ \forall\, Y \succeq 0 \;\Rightarrow X \succeq 0$. (*i.e.*, the cone $\boldsymbol{S^n_+} = \{X \in \boldsymbol{S^n} \mid X \succeq 0\}$ is self dual.)

[2]A matrix $U \in \boldsymbol{R^{n\times n}}$ is orthogonal if $U^T U = UU^T = I$.

- For $X \in \boldsymbol{S^n}$, the diagonal decomposition of X is denoted by $X = UDU^T$ with an orthogonal matrix U and a diagonal matrix $D := \text{diag}(X(1), X(2), \ldots, X(n))$, where D is uniquely determined up to ordering of the diagonal entries. We denote $X^+ := UD^+U^T$ and $X^- := UD^-U^T$, and call U the decomposition matrix of X. In addition, if $X, Y \in \boldsymbol{S^n}$ with $XY = YX$, there exists a decomposittion matrix U, and diagonal matrices D, E such that $X = UDU^T$ and $Y = UEU^T$.

Unlike the operation of Hadamard product[3] which is closed on the n-dimensional vector space (e.g. for any $x \in \boldsymbol{R^n}$, $y \in \boldsymbol{R^n}$, $x*y \in \boldsymbol{R^n}$), the operation of matrix product is not closed on the real symmetric matrix space (e.g. for arbitrary $X,\ Y \in \boldsymbol{S^n}$, XY is not necessarily in $\boldsymbol{S^n}$). Since in SDLCPs, the variables are symmetric real matrix, the commutativity of matrix products is often required in order to get meaningful results. In particular, we shall frequently use a particular commutativity assumption called the cross commutativity, which is defined below.

Definition 1.2.1. A linear transformation $L : \boldsymbol{S}^n \to \boldsymbol{S}^n$ is said to have the *cross commutative property* if for every $Q \in \boldsymbol{S^n}$, any two solutions X_1 and X_2 of SDLCP(L, Q) satisfy $X_1Y_2 = Y_2X_1$ and $X_2Y_1 = Y_1X_2$, where $Y_i = L(X_i) + Q,\ i = 1, 2$.

In LCP theory, matrix transpose is frequently used. To generalize matrix transpose to linear transformation, we define the transpose of a linear transformation as follows:

Definition 1.2.2. Given a linear transformation $L : \boldsymbol{S}^n \to \boldsymbol{S}^n$, its *transpose* $L^T : \boldsymbol{S}^n \to \boldsymbol{S}^n$ is defined by

$$\langle L^T(Y), X\rangle = \langle Y, L(X)\rangle,\ \forall\ X \in \boldsymbol{S}^n.$$

Since the transpose of a linear transformation is not explicitly given in the above definition, we need to show that it is properly defined as a linear transformation. This is exhibited in the following theorem.

Theorem 1.2.1. The transpose L^T defined in Definition 1.2.2 is uniquely defined and is a linear transformation.

Proof We first prove that L^T is a function. Suppose that for $Y \in \boldsymbol{S}^n$, $\langle L(X), Y\rangle = \langle X, Z_1\rangle = \langle X, Z_2\rangle$, where $Z_1,\ Z_2 \in \boldsymbol{S}^n$. Then $\langle X, Z_1 - Z_2\rangle = 0$, for every $X \in \boldsymbol{S}^n$. Thus, $Z_1 - Z_2 = 0$, i.e., for each $Y \in \boldsymbol{S}^n$, there is only one $L^T(Y)$ corresponding it.

Next, we prove that L^T is linear. For any $\alpha,\ \beta \in \boldsymbol{R}^n$ and $Y_1,\ Y_2 \in \boldsymbol{S}^n$, we have

$$\langle X, L^T(\alpha Y_1 + \beta Y_2)\rangle = \langle L(X), \alpha Y_1 + \beta Y_2\rangle = \alpha\langle L(X), Y_1\rangle + \beta\langle L(X), Y_2\rangle$$

$$= \alpha\langle X, L^T(Y_1)\rangle + \beta\langle X, L^T(Y_2)\rangle = \langle X, \alpha L^T(Y_1) + \beta L^T(Y_2)\rangle.$$

Thus, $L^T(\alpha Y_1 + \beta Y_2) = \alpha L^T(Y_1) + \beta L^T(Y_2)$, i.e., L^T is linear. These complete the proof. □

[3] The Hadamard product which is denoted by $*$ is defined as follows: for $x, y \in \boldsymbol{R^n}$, $z = x * y$ means $z_i = x_iy_i,\ i = 1, \cdots, n$.

Chapter 2

Uniqueness of the Solution

In this chapter, we study issues regarding the uniqueness of the solution to both the linear complementarity problem and the semidefinite linear complementarity problem. In Section 2.1, We survey some of the known results in the linear complementarity problems (Cottle, 2009; Cottle et al., 1992). In addition, in Section 2.1.3, we introduce some new results regarding the $\mathbf{E}^*$ matrix class as related to uniqueness of the solution to the LCP. In Section 2.2, we discuss the generalization of these LCP results to the SDLCP.

2.1 The Linear Complementarity Problem

2.1.1 The Class of P Matrices

One of the earliest results about the linear complementarity problem is related to the class of **P**-matrices which includes the positive definite matrices as a special case.

A matrix $M \in \boldsymbol{R}^{n \times n}$ is said to be a **P**-*matrix* if all its principal minors are positive. The class of such matrices is denoted by **P**. Obviously, if M is a **P**-matrix, so are all of its principal submatrices as well as its transpose. A real symmetric matrix is positive definite if and only if it belongs to **P**, yet the class of **P**-matries is not equivalent to the class of positive definite matrices when the assumption of symmetry is dropped (see e.g. Cottle et al., 1992, p.147). There are other characterizations for the class of **P**-matrices, which are summarized in the following proposition (see e.g. Cottle et al., 1992, Theorem 3.3.4).

Proposition 2.1.1. Given $M \in \boldsymbol{R}^{n \times n}$. The following statements are equivalent:

1. $M \in \mathbf{P}$.

2. M reverses the sign of no nonzero vectors, *i.e.*,

$$[z_i(Mz)_i \leq 0 \text{ for all } i] \Rightarrow [z = 0].$$

3. All real eigenvalues of M and its principal submatrices are positive.

The importance of the class of P-matrices to LCP theory is expressed in the following proposition (see e.g. Cottle et al., 1992, Theorem 3.3.7).

Proposition 2.1.2. A matrix $M \in \boldsymbol{R}^{n \times n}$ is a **P**-matrix if and only if $\mathrm{LCP}(q, M)$ has a unique solution for all vectors $q \in \boldsymbol{R}^n$.

2.1.2 The Classes of E and E_0 Matrices

Relaxing somewhat the characterization of **P**-matrices as in Proposition 2.1.1 (2), leads to two new classes of matrices, $\mathbf{E_0}$ and **E**, which have a unique solution for any vector q in the nonnegative and positive orthant, respectively.

A matrix $M \in \boldsymbol{R^{n\times n}}$ is said to be *semimonotone* if

$$[0 \neq x \geq 0] \Rightarrow [x_k > 0 \text{ and } (Mx)_k \geq 0 \text{ for some index k}].$$

The class of such matrices is denoted by $\mathbf{E_0}$, and its elements are called $\mathbf{E_0}$-matrices.

Note that, as follows directly from the preceding definition, every principal submatrix of an $\mathbf{E_0}$-matrix is also an $\mathbf{E_0}$-matrix.

There are several ways to characterize further the class of $\mathbf{E_0}$-matrices, one of which is through the uniqueness of the solution to the corresponding LCPs with positive vector q (see e.g. Cottle et al., 1992, Theorem 3.9.3).

Proposition 2.1.3. Given $M \in \boldsymbol{R^{n\times n}}$, the following statements are equivalent:

1. $M \in \mathbf{E_0}$.

2. LCP(q, M) has a unique solution for every $q > 0$.

3. For every index set $\alpha \subseteq \{1, \cdots, n\}$, the system

   $$M_{\alpha\alpha}x_\alpha < 0, \;\; x_\alpha \geq 0$$

   has no solution.

A natural subclass of $\mathbf{E_0}$ is the following: A matrix $M \in \boldsymbol{R^{n\times n}}$ is said to be *strictly semimonotone* if

$$[0 \neq x \geq 0] \Rightarrow [x_k > 0 \text{ and } (Mx)_k > 0 \text{ for some index } k].$$

The class of such matrices is denoted by **E**, and its elements are called **E**-matrices.

As it is shown in the next proposition, the results of Proposition 2.1.3 are extended to the class of **E**-matrices . In particular, the proposition states that a matrix is an **E**-matrix if and only if there is a unique solution for all LCPs with nonnegative q (see e.g. Cottle et al., 1992, Theorem 3.9.11).

Proposition 2.1.4. Let $M \in \boldsymbol{R^{n\times n}}$. The following statements are equivalent:

1. $M \in \mathbf{E}$.

2. LCP(q, M) has a unique solution for every $q \geq 0$.

3. For every index set $\alpha \subseteq \{1, \cdots, n\}$, the system

   $$M_{\alpha\alpha}x_\alpha \leq 0, \;\; 0 \neq x_\alpha \geq 0$$

   has no solution.

2.1.3 The Class of $\mathbf{E}^*$ Matrices

A proper subclass of $\mathbf{E_0}$ which properly contains $\mathbf{E}$ is the class of $\mathbf{E}^*$-matrices which is defined as follows:

A matrix $M \in \boldsymbol{R}^{n\times n}$ is said to be an $\mathbf{E}^*$-*matrix* if LCP(q, M) has a unique solution for every $0 \neq q \geq 0$. The class of such matrices is denoted by $\mathbf{E}^*$. Note that this definition implies that for $M \in \mathbf{E}^*$ and $0 \neq q \geq 0$, SOL$(q, M) = \{0\}$.

This class of matrices was presented and discussed in (Danao, 1993, 1997), but no alternative characterizations in term of intrinsic properties of the matrices was ever introduced. Such characterizations (in the spirit of Proposition 2.1.3 and 2.1.4) are offered in the next theorem.

Theorem 2.1.1. Given $M \in \boldsymbol{R}^{n\times n}$. The following three statements are equivalent:

1. $M \in \mathbf{E}^*$.

2. (a) For every index set $\alpha \subset \{1, \cdots, n\}$, $M_{\alpha\alpha} \in \mathbf{E}$;
   (b) The linear system
   $$\begin{cases} 0 \neq Mx \leq 0 \\ \quad\; x \geq 0 \end{cases}$$
   has no solution.

3. $[0 \neq x \geq 0,\ x_i(Mx)_i \leq 0,\ \forall\ i] \ \Rightarrow\ [Mx = 0,\ x > 0]$.

Proof of Theorem 2.1.1
$[1 \Rightarrow 2]$ (a) Assume that for some nonempty index set $\alpha \subset \{1, \cdots, n\}$, $M_{\alpha\alpha} \notin \mathbf{E}$, which means that there exists a vector $x \in \boldsymbol{R}^n$, a nonempty index set $\alpha \subset \{1, \cdots, n\}$ and a vector $q_\alpha \in \boldsymbol{R}^{|\alpha|}$ such that LCP(q_α, M_α) has a nonzero solution x_α. Denote $\bar{\alpha} = \{1, \cdots, n\} \setminus \alpha$. Set

$$x = \begin{pmatrix} x_\alpha \\ x_{\bar{\alpha}} \end{pmatrix} \begin{pmatrix} x_\alpha \\ 0 \end{pmatrix} \neq 0$$

and

$$q = \begin{pmatrix} q_\alpha \\ q_{\bar{\alpha}} \end{pmatrix},$$

where

$$q_\alpha = -M_{\alpha\alpha}x_\alpha \geq 0,$$
$$q_i = \max\{0, -(M_{\bar{\alpha}\alpha}x_\alpha)_i\} + 1 > 0, \text{ for } i \in \bar{\alpha}.$$

Then, $0 \neq q \geq 0$, and x is a nonzero solution for LCP(q, M). Since LCP(q, M) also has the zero vector as another solution, this contradicts the fact that $M \in \mathbf{E}^*$ by its definition.
(b) Now assume that the linear system

$$\begin{cases} 0 \neq Mx \leq 0 \\ \quad\; x \geq 0 \end{cases}$$

has a solution $\bar{x}$. Then,

$$M\bar{x} \neq 0 \Rightarrow \bar{x} \neq 0.$$

Let $\bar{q} = -M\bar{x}$. Then, $0 \neq \bar{q} \geq 0$. It is easy to see that $\bar{x}$ is a nonzero solution to LCP$(\bar{q}, M)$, and again since LCP$(\bar{q}, M)$ also the zero vector as its solution, this contradicts the fact that $M \in \mathbf{E}^*$.
[2 $\Rightarrow$ 3] We prove it by contradiction. Assume that $0 \neq x \geq 0$ and $x_i(Mx)_i \leq 0, \ \forall\ i$. Assume that either $Mx \neq 0$ or $x \ngtr 0$. By part (b) of 2, $Mx = 0$. Assume $x \ngtr 0$. Let $\alpha = \text{supp}\{x\}$, $\bar{\alpha} = \{1, \cdots, n\} \setminus \alpha$. Then

$$x \ngtr 0 \Rightarrow \bar{\alpha} \neq \emptyset.$$

Moreover, we have,

$$x_\alpha > 0,$$

$$(x_\alpha)_i(M_{\alpha\alpha}x_\alpha)_i \leq 0, \ \forall\ i \in \alpha,$$

which contradicts the fact that $M_{\alpha\alpha} \in \mathbf{E}$.
[3 $\Rightarrow$ 1] It is obvious that the zero vector is a solution to LCP(q, M), $\forall\ 0 \neq q \geq 0$. Now assume there is also a nonzero solution x to LCP(q, M). Then, $0 \neq x \geq 0$, $Mx + q \geq 0$ and $x^T(Mx + q) = 0$. Since

$$0 \neq x \geq 0, \ x_i q_i \geq 0, \ \text{and } x_i(Mx)_i \geq 0, \ \forall\ i,$$

we have

$$Mx = 0 \text{ and } x > 0,$$

then by $0 \neq q \geq 0$, $0 \neq Mx + q \geq 0$, we get the conclusion that $x^T(Mx + q) \neq 0$, a contradiction. Thus, LCP$(q.M)$ has no nonzero solutions. Therefore, LCP(q, M) has a unique solution, $\forall\ 0 \neq q \geq 0$. □

To prove the next result, we will need the following well-known version of the so-called Theorem of the Alternatives (see e.g. Cottle et al., 1992, p.109-111):

Lemma 2.1.1. (Theorem of the Alternatives)

1. The system $\begin{cases} 0 \neq x \geq 0 \\ Mx \leq 0 \end{cases}$ has no solution. $\Leftrightarrow$ The system $\begin{cases} M^T y > 0 \\ y > 0 \end{cases}$ has a solution.

2. The system $\begin{cases} x \geq 0 \\ 0 \neq Mx \leq 0 \end{cases}$ has no solution. $\Leftrightarrow$ The system $\begin{cases} M^T y \geq 0 \\ y > 0 \end{cases}$ has a solution.

Corollary 2.1.1. Given $M \in \boldsymbol{R}^{n \times n}$. The following two statements are equivalent:

1 $M \in \mathbf{E}^*$.

2 For every index set $\alpha \subset \{1, \cdots, n\}$, $M_{\alpha\alpha} \in \mathbf{E}$, and the linear system

$$\begin{cases} M^T y \geq 0 \\ y > 0 \end{cases}$$

has a solution.

The proof of the preceding corollary follows directly from the second characterization of $\mathbf{E}^*$-matrices in Theorem 2.1.1 and from Lemma 2.1.1.

In Cottle's recent paper (Cottle, 2009), several structural properties (complete, full, reflective and sign-change invariance) are defined and then their existence (or lack of) are discussed for many matrix classes.

A matrix M belonging to a class $\boldsymbol{Y}$ is said to be *completely-$\boldsymbol{Y}$* if every principal submatrix of M also belongs to $\boldsymbol{Y}$. The class of all completely-$\boldsymbol{Y}$ matrices is denoted by $\boldsymbol{Y}^c$. To say that $\boldsymbol{Y}$ is a *complete class* means that $\boldsymbol{Y} = \boldsymbol{Y}^c$.

The classes of $\mathbf{P}$, $\mathbf{E_0}$ and $\mathbf{E}$ matrices are all complete classes (see e.g. Cottle et al., 1992, Corollary 3.9.7 and 3.9.11).

Consider the specific system of linear equations

$$w = q + Mz, \tag{2.1}$$

where $q \in \boldsymbol{R^n}$ and $M \in \boldsymbol{R^{n\times n}}$. Let α be a subset of the index set $\{1, \cdots, n\}$ and suppose $M_{\alpha\alpha}$ is nonsingular. Let $\bar{\alpha} = \{1, \cdots, n\} \setminus \alpha$. Then,

$$M' = \begin{pmatrix} M'_{\alpha\alpha} & M'_{\alpha\bar{\alpha}} \\ M'_{\bar{\alpha}\alpha} & M'_{\bar{\alpha}\bar{\alpha}} \end{pmatrix} \tag{2.2}$$

is called a *principal pivotal transform* of M with respect to the index set α (and the nonsingular principal submatrix $M_{\alpha\alpha}$), where

$$M'_{\alpha\alpha} = M_{\alpha\alpha}^{-1}, \quad M'_{\alpha\bar{\alpha}} = -M_{\alpha\alpha}^{-1} M_{\alpha\bar{\alpha}}, \tag{2.3}$$

$$M'_{\bar{\alpha}\alpha} = M_{\bar{\alpha}\alpha} M_{\alpha\alpha}^{-1}, \quad M'_{\bar{\alpha}\bar{\alpha}} = M_{\bar{\alpha}\bar{\alpha}} - M_{\bar{\alpha}\alpha} M_{\alpha\alpha}^{-1} M_{\alpha\bar{\alpha}}. \tag{2.4}$$

A matrix M belonging to a class $\boldsymbol{Y}$ is said to be *fully-$\boldsymbol{Y}$* if for every nonsingular principal submatrix of M the associated principal pivotal transform of M also belongs to $\boldsymbol{Y}$. The class of all fully-$\boldsymbol{Y}$ matrices is denoted by $\boldsymbol{Y}^f$. To say that $\boldsymbol{Y}$ is a *full class* means that $\boldsymbol{Y} = \boldsymbol{Y}^f$.

The class of $\mathbf{P}$-matrices is full (see e.g. Cottle et al., 1992, Theorem 6.6.9). The classes of $\mathbf{E_0}$ and $\mathbf{E}$ matrices are not full classes (see e.g. Cottle et al., 1992, Theorem 6.6.6).

A matrix M belonging to a class $\boldsymbol{Y}$ is said to be *reflectively-$\boldsymbol{Y}$* if $M^T \in \boldsymbol{Y}$. The class of all reflectively-$\boldsymbol{Y}$ matrices is denoted by $\boldsymbol{Y}^r$. To say that $\boldsymbol{Y}$ is a *reflective class* means that $\boldsymbol{Y} = \boldsymbol{Y}^r$.

The classes of $\mathbf{P}$, $\mathbf{E_0}$ and $\mathbf{E}$ matrices are all reflective (Cottle, 2009).

A matrix M belonging to a class $\boldsymbol{Y}$ is said to be *sign-change invariant-$\boldsymbol{Y}$* if $SMS \in \boldsymbol{Y}$ for every diagonal matrix S such that $S^2 = I$. The class of all sign-change invariant-$\boldsymbol{Y}$ matrices is denote by $\boldsymbol{Y}^s$. To say that $\boldsymbol{Y}$ is a *sign-change invariant class* means that $\boldsymbol{Y} = \boldsymbol{Y}^s$.

It can be shown that the class of $\mathbf{P}$-matrices is sign-change invariant, while the classes of $\mathbf{E_0}$ and $\mathbf{E}$ matrices are not sign-change invariant (Cottle, 2009).

Here we study the structural properties shown above for the $\mathbf{E}^*$ class as well.

The completeness of the $\mathbf{E}^*$ matrix class can be easily derived as a corollary of Theorem 2.1.1.

Corollary 2.1.2. The $\mathbf{E}^*$ class is complete.

The proof follows directly from the second definition of the $\mathbf{E}^*$ matrix class in Theorem 2.1.1, and the fact that $\mathbf{E} \subseteq \mathbf{E}^*$.

The $\mathbf{E}^*$ matrix class is not full as shown in the following example.

Example 2.1.1. Consider the matrix

$$M = \begin{pmatrix} 1 & -2 & 0 \\ 0 & 1 & 2 \\ 2 & 0 & 1 \end{pmatrix}.$$

It is an **E**-matrix, thus it is also an **E***-matrix. Let $\alpha = \{1, 2\}$. The principal pivotal transform of M with respect to the nonsingular matrix $M_{\alpha\alpha}$ is

$$M' = \begin{pmatrix} 1 & -2 & -4 \\ 0 & 1 & -2 \\ 2 & 4 & -7 \end{pmatrix}.$$

M' is not an **E***-matrix, since its principal submatrix

$$\begin{pmatrix} 1 & -2 \\ 4 & -7 \end{pmatrix}$$

is not an **E**-matrix, thus, M' is not an **E***-matrix. Therefore, the **E*** class is not full. ♢

The following theorem shows that the **E*** class is reflective.

Theorem 2.1.2. $M \in \mathbf{E}^* \Leftrightarrow M^T \in \mathbf{E}^*$

Proof

Since $(M^T)^T = M$, it is enough to prove that $M \in \mathbf{E}^* \Rightarrow M^T \in \mathbf{E}^*$. We will use the second characterization of **E***-matrices in Theorem 2.1.1 to prove the result.

We prove part (a) of the characterization by induction on $|\alpha|$.[1] where α is a nonempty index set of $\{1, \cdots, n\}$. When $|\alpha| = 1$, if the linear system

$$\begin{cases} 0 \neq x_\alpha \geq 0 \\ M_{\alpha\alpha} x_\alpha \leq 0 \end{cases}$$

has no solution, then obviously since $M_{\alpha\alpha} = M^T_{\alpha\alpha}$, the linear system

$$\begin{cases} 0 \neq x_\alpha \geq 0 \\ M^T_{\alpha\alpha} x_\alpha \leq 0 \end{cases}$$

has no solution as well.

Now consider any index set α with $|\alpha| = k$, $1 < k < n$. Suppose that for all the index sets $\tilde{\alpha}$ with $1 \leq |\tilde{\alpha}| \leq k - 1$, the linear system

$$\begin{cases} 0 \neq x_{\tilde{\alpha}} \geq 0 \\ M^T_{\tilde{\alpha}\tilde{\alpha}} x_{\tilde{\alpha}} \leq 0 \end{cases}$$

has no solution.

Assume on the contrary, that the linear system

$$\begin{cases} 0 \neq x_\alpha \geq 0 \\ M^T_{\alpha\alpha} x_\alpha \leq 0 \end{cases}$$

[1] For a set S of finite elements, $|S|$ denotes the number of elements in S.

has a solution x_α. Assume that $x_\alpha \not> 0$, then letting $\beta = \text{supp}(x_\alpha) \subset \alpha$, (so $1 \leq |\beta| \leq k-1$), we have

$$\begin{cases} x_\beta > 0 \\ M^T_{\beta\beta} x_\beta \leq 0, \end{cases}$$

which contradicts the assumption in the induction, so $x_\alpha > 0$.

Since $M \in \mathbf{E^*}$, the linear system

$$\begin{cases} y_\alpha > 0 \\ M^T_{\alpha\alpha} y_\alpha > 0 \end{cases}$$

has a solution.

Then, there exists a scalar $\lambda > 0$ such that $x_\alpha - \lambda y_\alpha \geq 0$, $x_\alpha - \lambda y_\alpha \not> 0$ and $x_\alpha - \lambda y_\alpha \neq 0$. Also $0 \neq M^T_{\alpha\alpha}(x_\alpha - \lambda y_\alpha) \leq 0$.

Let $\gamma = \text{supp}(x_\alpha - \lambda y_\alpha)$, then $\gamma \subset \alpha$ and

$$\begin{cases} (x_\alpha - \lambda y_\alpha)_\gamma > 0 \\ M^T_{\gamma\gamma}(x_\alpha - \lambda y_\alpha)_\gamma \leq 0, \end{cases}$$

this again contradicts the induction assumption.

Therefore, for all index sets α for which $1 \leq |\alpha| \leq n-1$, the linear system

$$\begin{cases} 0 \neq x_\alpha \geq 0 \\ M^T_{\alpha\alpha} x_\alpha \leq 0 \end{cases}$$

has no solution.

Again, we prove part (b) of the characterization by contradiction. Assume that the linear system

$$\begin{cases} x \geq 0 \\ 0 \neq M^T x \leq 0 \end{cases}$$

has a solution x.

Suppose $x \not> 0$. Set $\alpha = \text{supp}\{x\}$ and $\alpha \subset \{1, \cdots, n\}$. Then we have

$$\begin{cases} x_\alpha > 0 \\ M^T_{\alpha\alpha} x_\alpha \leq 0, \end{cases}$$

which contradicts part (a). Thus, $x > 0$.

Since $M \in \mathbf{E^*}$, the linear system

$$\begin{cases} y > 0 \\ M^T y \geq 0 \end{cases}$$

has a solution. Then, there exists a scalar $\lambda > 0$ such that $x - \lambda y \geq 0$, $x - \lambda y \not> 0$ and $x - \lambda y \neq 0$. Set $\beta = \text{supp}\{x - \lambda y\} \subset \{1, \cdots, n\}$. Then

$$\begin{cases} (x - \lambda y)_\beta > 0 \\ M^T_{\beta\beta}(x - \lambda y)_\beta \leq 0, \end{cases}$$

which also contradicts part (a).

Thus, the linear system

$$\begin{cases} x \geq 0 \\ 0 \neq M^T x \leq 0 \end{cases}$$

has no solution.

Combining part (a) and (b) of the characterization, we conclude that $M^T \in \mathbf{E}^*$. $\square$

As we mentioned earlier, neither the $\mathbf{E}$ class nor the $\mathbf{E_0}$ class is sign-changing invariance. We show that the class of $\mathbf{E}^*$-matrices is not sign-changing invariance as well.

Example 2.1.2. Consider the matrix

$$M = \begin{pmatrix} 1 & -2 & 0 \\ 0 & 1 & 2 \\ 2 & 0 & 1 \end{pmatrix},$$

and consider the following sign-changing matrix

$$S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Then

$$SMS = \begin{pmatrix} 1 & -2 & 0 \\ 0 & 1 & -2 \\ -2 & 0 & 1 \end{pmatrix},$$

which is not an $\mathbf{E}^*$-matrix. Therefore, the class of $\mathbf{E}^*$-matrices is not sign-change invariance. $\Diamond$

Next, we turn our attention to the $\boldsymbol{L} = \mathbf{E_0} \cap \mathbf{E_1}$ matrix class (where the $\mathbf{E_1}$ class is defined below) which was introduced in (Eaves, 1971) and is a well-studied class in LCP theory.

A matrix $M \in \boldsymbol{R}^{\boldsymbol{n}\times\boldsymbol{n}}$ is said to be an $\mathbf{E_1}$-*matrix* if for every $0 \neq z \in \mathrm{SOL}(0, M)$, there exist nonnegative diagonal matrices D_1 and D_2 such that $D_2 z \neq 0$ and $(D_1 M + M^T D_2)z = 0$. The class of such matrices is denoted by $\mathbf{E_1}$.

It has been established that $\mathbf{E} \subseteq \mathbf{E_1}$ (see e.g. Cottle et al., 1992, p.192).

We show, through the corollary of the following theorem, that the class of $\mathbf{E}^*$-matrices belongs to the class of $\boldsymbol{L}$-matrices, which in turn (see e.g. Cottle et al., 1992, Corollary 3.9.19) belongs to the class of $\mathbf{Q_0}$-matrices, where $\mathbf{Q_0}$-matrix is a matrix for which the related LCP is solvable whenever feasible. It should be noted, as detailed in Section 3.1.1, that the $\mathbf{Q_0}$ class is a particularly interesting class.

Theorem 2.1.3. $\mathbf{E}^* \subseteq \mathbf{E_1}$.

Proof Let $M \in \mathbf{E}^*$. Suppose that $z \in \mathrm{SOL}(0, M) \setminus \{0\}$, then

$$\begin{aligned} 0 \neq z &\geq 0, \\ Mz &\geq 0, \\ z^T M z &= 0. \end{aligned}$$

Thus, z satisfies

$$0 \neq z \geq 0 \text{ and } z_i (Mz)_i \leq 0, \ \forall\, i = 1, \cdots, n$$

Since $M \in \mathbf{E^*}$, by its definition, $Mz = 0$ and $z > 0 \Rightarrow\ M \notin \mathbf{E}\ \Rightarrow\ M^T \notin \mathbf{E}$. Moreover, by Theorem 2.1.2, $M^T \in \mathbf{E^*}$. So, $M^T \in \mathbf{E^*} \setminus \mathbf{E}$. Thus, there exists $0 \neq x \geq 0$ such that $M^T x = 0$. Set

$$D_2 = \text{diag}(\frac{x_1}{z_1}, \frac{x_2}{z_2}, \cdots, \frac{x_n}{z_n}) \succeq 0,$$

and let $D_1 = 0$, then we have

$$(D_1 M + M^T D_2)z = M^T(D_2 z) = M^T x = 0$$

Therefore, since $D_2 z \neq 0$, by definition, we have that M is a $\mathbf{E_1}$-matrix. □

As a corollary of the above theorem, we have that the class of $\mathbf{E^*}$-matrices belongs to the class of $\boldsymbol{L}$-matrices.

Corollary 2.1.3. $\mathbf{E^*} \subseteq \boldsymbol{L}$.

The proof directly follows from Theorem 2.1.3 and from the definition of $\boldsymbol{L}$ as $\mathbf{E_0} \cap \mathbf{E_1}$, and the fact that $\mathbf{E^*} \subseteq \mathbf{E_0}$.

2.2 The Semidefinite Linear Complementarity Problem

In this section, we extend some of the results regarding the uniqueness of the solution in LCPs as presented in the previous section to semidefinite linear complementarity problems (SDLCPs). Unlike the case of the Hadamard product of vectors in which $x * y = y * x$, for any $x, y \in \boldsymbol{R^n}$; we do not necessarily have commutativity for product of matrices (that is, for any $X, Y \in \boldsymbol{S^n}$, we do not necessarily have that $XY = YX$). This (as will be shown later) necessitates somewhat more restricted results for the SDLCP. In the definitions introduced later, whenever X and $L(X)$ commute, we consider the decompositions of X and $L(X)$ with the same orthogonal matrix U, *i.e.*, $X = U^T \text{diag}(X(1), \cdots, X(n))U$ and $L(X) = U^T \text{diag}(L(X)(1), \cdots, L(X)(n))U$.

2.2.1 The Linear Transformation Class of P

Motivated by the results that relate **P**-matrices and uniqueness of the solution to the corresponding LCPs, Gowda and Song (Gowda and Song, 2000) introduced an analogous property for the linear transformation L of the SDLCP as follows:

Definition 2.2.1. A linear transformation $L : \boldsymbol{S^n} \to \boldsymbol{S^n}$ has the **P**-*property* if

$$[X \neq 0] \ \Rightarrow\ \ [\text{if } X \text{ and } L(X) \text{ commute, } \exists\, i \text{ such that } X(i) \times L(X)(i) > 0].$$

The class of linear transformations having such property is denoted by **P**.

The above definition for the **P**-property is equivalent though slightly different from the one in (Gowda and Song, 2000, Definition 2). We choose this definition to facilitate our proofs. Note that the additional assumption of the commutativity of $L(X)$ and X in the definition of the **P** property is necessary to obtain uniqueness properties for the SDLCP.

This additional commutativity assumption will be part of most of the definitions of classes of linear transformations to be introduced in the rest of this dissertation.

As discussed in the introduction, most of the results regarding extending properties of matrices in LCP theory to linear transformations in SDLCP theory require the additional assumption of cross commutativity (see Definition 1.2.1). To facilitate the presentation, we introduce the following convention:
Let $\boldsymbol{Y}$ denote a (fixed) class of linear transformations $L : \boldsymbol{S^n} \to \boldsymbol{S^n}$. The class of linear transformations that belong to $\boldsymbol{Y}$ and have the cross commutative property is denoted by $\hat{\boldsymbol{Y}}$.

We start by quoting the results in (Gowda and Song, 2000, Theorem 7) regarding the class $\mathbf{P}$ and the uniqueness of solution in the SDLCP.

Proposition 2.2.1. Given a linear transformation $L : \boldsymbol{S^n} \to \boldsymbol{S^n}$, the following three statements are equivalent:

1. For all $Q \in \boldsymbol{S^n}$, SDLCP(L, Q) has at most one solution.
2. $L \in \hat{\mathbf{P}}$.
3. For all $Q \in \boldsymbol{S^n}$, SDLCP(L, Q) has a unique solution.

2.2.2 The Linear Transformation Classes of E and $\mathbf{E_0}$

Motivated by Gowda and Song's result regarding the $\mathbf{P}$ class of linear transformations, we will extend the known LCP uniqueness results related to vectors q in the nonnegative orthant (as presented in Section 2.1.2) to the SDLCP.

Definition 2.2.2. A linear transformation $L : \boldsymbol{S^n} \to \boldsymbol{S^n}$ is said to have the $\mathbf{E_0}$-*property* if

$$[0 \neq X \succeq 0] \;\Rightarrow\; [\text{if } X \text{ and } L(X) \text{ commute, } \exists\, i \text{ such that } X(i) > 0 \text{ and } L(X)(i) \geq 0].$$

The class of linear transformations having such property is denoted by $\mathbf{E_0}$.

Definition 2.2.3. A linear transformation $L : \boldsymbol{S^n} \to \boldsymbol{S^n}$ is said to have the $\mathbf{E}$-*property* if

$$[0 \neq X \succeq 0] \;\Rightarrow\; [\text{if } X \text{ and } L(X) \text{ commute, } \exists\, i \text{ such that } X(i) > 0 \text{ and } L(X)(i) > 0].$$

The class of linear transformations having such property is denoted by $\mathbf{E}$.

Next, we extend the results of Propositions 2.1.3 and 2.1.4 to the linear transformation classes $\mathbf{E_0}$ and $\mathbf{E}$ with respect to the SDLCP.

Theorem 2.2.1. Given a linear transformation $L : \boldsymbol{S}^n \to \boldsymbol{S}^n$, we have:

1. If SDLCP(L, Q) has a unique solution for all $Q \succ 0$, then $L \in \mathbf{E_0}$.
2. If $L \in \hat{\mathbf{E}}_\mathbf{0}$, then SDLCP$(L, Q)$ has a unique solution for all $Q \succ 0$.

Proof We prove these results by contradiction.
1. Suppose on the contrary, that $L \notin \mathbf{E_0}$. Then there exists a matrix $X \in \boldsymbol{S^n}$ such that

$$0 \neq X \succeq 0, \; X \text{ and } L(X) \text{ commute}$$

And we can decompose X and $L(X)$ with the same decomposition matrix U:

$$X := U^T DU \text{ and } L(X) := U^T EU,$$

where $D = \mathrm{diag}(d_1, \cdots, d_n)$ and $E = \mathrm{diag}(e_1, \cdots, e_n)$. Since $L \notin \mathbf{E_0}$, for all the index k such that $d_k > 0$, we have $e_k < 0$.

Now set $Q := U^T GU$, where $G = \mathrm{diag}(g_1, \cdots, g_n)$ with

$$g_i = \begin{cases} 1, & \text{if } e_i \geq 0 \\ -e_i, & \text{if } e_i < 0. \end{cases}$$

Then, $Q \succ 0$ and $X(L(X)+Q) = 0$, and also $L(X)+Q \succeq 0$. Therefore, $X \neq 0$ is a solution to SDLCP(L, Q), with $Q \succ 0$. Since the zero matrix is another solution to SDLCP(L, Q), this contradicts the assumption that SDLCP(L, Q) has a unique solution.

2. Suppose that on the contrary, for some positive definite matrix Q, SDLCP(L, Q) does not have a unique solution. Since it is clear that the zero matrix is a solution to SDLCP(L, Q), there exists an $X \neq 0$ such that $X \in \mathrm{SOL}(L, Q)$. Then, $X \succeq 0$, $L(X) + Q \succeq 0$ and $X(L(X) + Q) = 0$. Therefore, X and $L(X) + Q$ commute. By the cross commutative property of L, and because the zero matrix and X are both solutions to SDLCP(L, Q), we have that X and $L(0) + Q = Q$ commute. Therefore, X and $L(X)$ commute as well. Then, since $Q \succ 0$,

$$X(L(X) + Q) = 0 \Rightarrow XL(X) = -XQ \prec 0.$$

It contradicts the assumption that $L \in \mathbf{E_0}$. Therefore, there exists no nonzero solution to SDLCP(L, Q), for all matrix $Q \succ 0$. *i.e.*, SDLCP(L, Q) has a unique solution, for all $Q \succ 0$. □

Unlike the result for the class $\mathbf{P}$ in Proposition 2.2.1, for the class $\mathbf{E_0}$, the uniqueness of the solution for all the positive definite matrix Q does not lead to the cross commutativity property. This is also true with respect to the class $\mathbf{E}$ (see Theorem 2.2.2).

The following example shows that the cross commutative property is indeed necessary in Theorem 2.2.1 part 2. In the example below, $L \in \mathbf{E_0}$, but SDLCP(L, Q) has no unique solution for a positive definite matrix Q.

Example 2.2.1. Consider the linear transformation

$$L : \begin{pmatrix} a & b \\ b & c \end{pmatrix} \to \begin{pmatrix} b & 0 \\ 0 & 0 \end{pmatrix}.$$

Then for any matrix $X = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \succeq 0$, we have $a \geq 0,\ c \geq 0,\ ac \geq b^2$. Since

$$XL(X) = \begin{pmatrix} ab & 0 \\ b^2 & 0 \end{pmatrix},$$

X and $L(X)$ commute if and only if $b = 0$. Thus, it is easy to see that $L \in \mathbf{E_0}$.

Now set

$$Q = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \succ 0.$$

Then

$$\begin{aligned} X &= \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \succeq 0, \\ L(X) + Q &= \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \succeq 0, \\ \langle X, L(X) + Q \rangle &= 0. \end{aligned}$$

Therefore, $X \neq 0$ is another solution to SDLCP(L, Q), so SDLCP(L, Q) has no unique solution even though $Q \succ 0$. ♢

In addition, it is possible to have a linear transformation L that guarantees uniqueness of the solution to SDLCP(L, Q) for any positive definite matrix Q without having the cross commutative property.

Example 2.2.2. Consider the linear transformation

$$L : \begin{pmatrix} x & y \\ y & z \end{pmatrix} \rightarrow \begin{pmatrix} z & y \\ y & x \end{pmatrix}.$$

Next we show SDLCP(L, Q) has a unique solution for all $Q \succ 0$.
Consider any positive definite matrix

$$Q = \begin{pmatrix} a & b \\ b & c \end{pmatrix},$$

with $a > 0, \ c > 0, \ ac > b^2$. Assume that

$$\bar{X} = \begin{pmatrix} \bar{x} & \bar{y} \\ \bar{y} & \bar{z} \end{pmatrix} \succeq 0$$

is a solution to SDLCP(L, Q). Then,

$$\langle \bar{X}, L(\bar{X}) + Q \rangle = 2\bar{x}\bar{z} + 2\bar{y}^2 + \bar{x}a + \bar{z}c + 2\bar{y}b = 0.$$

But

$$(2\bar{y}b)^2 = 4\bar{y}^2 b^2 \leq 4(\bar{x}\bar{z})(ac) \leq (a\bar{x} + c\bar{z})^2,$$

with both equalities hold only when $\bar{x} = \bar{y} = \bar{z} = 0$. Therefore, $\bar{X} = 0$, *i.e.*, SDLCP(L, Q) has a unique solution, namely the zero matrix, for any positive definite matrix Q.

Now set

$$Q = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}.$$

Then, both

$$X_1 = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

and

$$X_2 = \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{4} \end{pmatrix}$$

are solutions to SDLCP(L, Q). But

$$X_1(L(X_2) + Q) = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \frac{1}{4} & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{pmatrix} = \begin{pmatrix} -\frac{1}{8} & \frac{1}{4} \\ -\frac{1}{8} & \frac{1}{4} \end{pmatrix}$$

is not symmetric, *i.e.*, X_1 and $L(X_2) + Q$ do not commute. Thus, L doesn't have the cross commutative property. ♢

Following similar arguments as in Theorem 2.2.1, the following results can be proved:

Theorem 2.2.2. Given a linear transformation $L : \boldsymbol{S}^n \to \boldsymbol{S}^n$, we have the following implications:

1. If SDLCP(L, Q) has a unique solution, for all $Q \succeq 0$, then $L \in \mathbf{E}$;

2. If $L \in \hat{\mathbf{E}}$, then SDLCP(L, Q) has a unique solution, for all $Q \succeq 0$.

In the next example we show that the requirement of cross commutativity in the second part of the preceding theorem is necessary. In particular, we exhibit a linear transformation L in which $L \in \mathbf{E}$ but $\boldsymbol{L} \notin \hat{\mathbf{E}}$ and shows that for a certain $Q \succeq 0$, the corresponding SDLCP(L, Q) has multiple solutions.

Example 2.2.3. Consider the linear transformation $L(X) = AX + XA^T$, where

$$A = \begin{pmatrix} -1 & 2 \\ 2 & 2 \end{pmatrix}.$$

It can be shown that $L \in \mathbf{E}$ and $L \in \mathbf{E_0}$. (In fact, $L \in \mathbf{P}$ as well).

Set

$$Q = \begin{pmatrix} 2 & 2 \\ 2 & 4 \end{pmatrix} \succeq 0.$$

Then, both

$$X_1 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

and

$$X_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

are solutions to SDLCP(L, Q). Thus, SDLCP(L, Q) has multiple solutions for the positive semidefinite matrix Q. $\diamondsuit$

2.2.3 The Linear Transformation Class of $\mathbf{E}^*$

We also generalize the $\mathbf{E}^*$ matrix class to the corresponding class of linear transformations.

Definition 2.2.4. A linear transformation $L : \boldsymbol{S^n} \to \boldsymbol{S^n}$ is said to have the $\mathbf{E}^*$-*property* if

$$[0 \neq X \succeq 0, \ X \text{ and } L(X) \text{ commute}, \ XL(X) \preceq 0]$$

$$\Rightarrow [L(X) = 0, \ X \succ 0]$$

The class of linear transformations having such property is denoted by $\mathbf{E}^*$.

As in the LCP result in Theorem 2.1.1, the $\mathbf{E}^*$ class of linear transformations is also related to the uniqueness of the solution to the SDLCPs:

Theorem 2.2.3. Given a linear transformation $L : \boldsymbol{S}^n \to \boldsymbol{S}^n$:

1. If SDLCP(L, Q) has a unique solution for all $0 \neq Q \succeq 0$, then $L \in \mathbf{E}^*$;

2. If $L \in \hat{\mathbf{E}}^*$, then SDLCP(L, Q) has a unique solution for all $0 \neq Q \succeq 0$.

Proof

1. Consider any matrix $0 \neq X \succeq 0$, such that X and $L(X)$ commute, and also $XL(X) \preceq 0$. We can decompose X and $L(X)$ with the same decomposition matrix U:

$$\begin{aligned} X &:= U^T D U, \\ L(X) &:= U^T E U, \end{aligned}$$

where $D = \text{diag}(d_1, \cdots, d_n)$, $E = \text{diag}(e_1, \cdots, e_n)$. Note that $XL(X) \preceq 0 \Rightarrow d_i e_i \leq 0$. We now study two cases:

Case 1: $L(X) \neq 0$.

Set

$$Q := U^T F U,$$

where $F = \text{diag}(f_1, \cdots, f_n)$ and

$$f_i = \begin{cases} \max\{-e_i, 1\}, & \text{if } e_i = 0 \\ -e_i & \text{if } e_i > 0. \end{cases}$$

Then, $0 \neq Q \succeq 0$, and $X \neq 0$ is another solution to SDLCP(L, Q), which contradicts the fact that SDLCP(L, Q) has a unique solution for all the nonzero positive semidefinite matrix Q.

Case 2: $L(X) = 0$ but $X \not\succ 0$.

Since $X \succeq 0$, there exists an index j such that $d_j = 0$. Set $Q := U^T F U$, where $F = \text{diag}(f_1, \cdots, f_n)$ and

$$f_i = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{otherwise.} \end{cases}$$

Thus, we have $0 \neq Q \succeq 0$. Thus $X \neq 0$ is another solution to SDLCP(L, Q) (besides $X = 0$), which again contradicts the fact that SDLCP(L, Q) has a unique solution for all the nonzero positive semidefinite matrix Q.

From the two cases above, we conclude that $L(X) = 0$ and $X \succeq 0$.

2. Suppose on the contrary, there exists a matrix $0 \neq Q \succeq 0$ such that SDLCP(L, Q) has more than one solution. Let X be a nonzero solution to SDLCP(L, Q), then

$$\begin{aligned} 0 \neq X &\succeq 0, \\ L(X) + Q &\succeq 0, \\ \langle X, L(X) + Q \rangle &= 0. \end{aligned}$$

Since X and the zero matrix are two distinct solutions to SDLCP(L, Q), by the cross commutative property of the linear transformation L, we can conclude that X and Q commute. Then, since $X \succeq 0$ and $Q \succeq 0.$,

$$X(L(X) + Q) = 0 \Rightarrow XL(X) = -XQ \preceq 0.$$

By the definition of the $\mathbf{E}^*$-property, $L(X) = 0$ and $X \succ 0$. Thus,

$$0 = XL(X) = -XQ \prec 0,$$

a contradiction. Therefore, SDLCP(L, Q) has a unique solution for all $0 \neq Q \succeq 0$. □

Chapter 3

Convexity and the Class of Sufficient Linear Transformations

In this chapter, we discuss the properties of LCPs as well as SDLCPs with respect to convexity. In Section 3.1, we recall and discuss several well-known results regarding three classes of matrices which have properties related to the convexity of the solution sets of LCPs. In Section 3.2, we extend these results to classes of linear transformations related to SDLCPs.

3.1 The Linear Complementarity Problem

In this section, we show several known results on the classes of $\mathbf{Q_0}$, column sufficient and row sufficient matrices (will be defined later in the section), and their properties with respect to the convexity of the solution sets of LCPs.

3.1.1 The Classes of Q and $\mathbf{Q_0}$ Matrices

The solvability of a linear complementarity problem is often the first question asked when analyzing the problem. A matrix $M \in \boldsymbol{R^{n\times n}}$ is called a $\mathbf{Q_0}$-*matrix* if the LCP(q, M) is solvable whenever it is feasible[1]. The class of such matrices is denoted by $\mathbf{Q_0}$.

From its definition, we can see that $\mathbf{Q_0}$ is an important class of matrices: if M is a $\mathbf{Q_0}$-matrix, we can check the solvability of LCP(q, M) by checking the solvability of only the first two linear inequalities (1.1) and (1.2), which greatly simplifies the problem, and in fact can be done in polynomial time (Khachian, 1979). However, there is no known polynomial algorithm to examine whether a matrix belongs to $\mathbf{Q_0}$. The following is a key convexity result that characterizes the class of $\mathbf{Q_0}$-matrices (see e.g. Cottle et al., 1992, Proposition 3.2.1):

Proposition 3.1.1. Given $M \in \boldsymbol{R^{n\times n}}$. The following statements are equivalent:

1. $M \in \mathbf{Q_0}$.
2. $K(M) = \{q \mid \text{SOL}(q, M) \neq \emptyset\}$ is convex.

[1] Recall that a LCP(q, M) is feasible if there exists a vector z such that (1.1) and (1.2) are satisfied and is solvable if (1.3) is satisfied as well.

A natural subclass of $\mathbf{Q_0}$ is $\mathbf{Q}$, which is the class of matrices $M \in \boldsymbol{R^{n\times n}}$ such that LCP(q, M) is solvable for all vector q. The elements in this class are called **Q**-*matrices*.

As in the case of the $\mathbf{Q_0}$ class, there is no known easily checked characterization for the **Q**-matrices. But again, some well studied classes of matrices are proved to belong to **Q**. In particular, the **E** class (that contains the **P** class) belongs to **Q** (see e.g. Cottle, 2009).

3.1.2 The Classes of Column and Row Sufficient Matrices

Convexity plays an important role in analyzing properties of linear complementarity problems. A slight generalization of the class **P** leads to the so-called column sufficient class of matrices which is further characterized as the class of matrices for which the solution set (if not empty) is convex for all possible vectors q.

A matrix $M \in \boldsymbol{R^{n\times n}}$ is said to be *column sufficient* if it satisfies

$$[z_i(Mz)_i \leq 0 \text{ for all } i] \Rightarrow [z_i(Mz)_i = 0 \text{ for all } i].$$

The class of column sufficient matrices is denoted by **CS**, and its elements are called **CS**-matrices. As mentioned above, the class of column sufficient matrices contains all the matrices for which the solution set (if not empty) is convex for all possible vector q (see e.g. Cottle et al., 1992, Theorem 3.5.8).

Proposition 3.1.2. Given $M \in \boldsymbol{R^{n\times n}}$. The following statements are equivalent:

1. $M \in \mathbf{CS}$.
2. For every $q \in \boldsymbol{R^n}$, SOL(q, M) (if not empty) is convex.

As is well known (Cottle, 2009), the class of column sufficient matrices is not reflective. The class of all matrices whose transpose belong to **CS** is (naturally) called row sufficient. In particular, a matrix M is said to be *row sufficient* if its transpose is column sufficient. The class of such matrices is denoted by **RS**, and its elements are called **RS**-matrices. A matrix M is said to be *sufficient* if it is both column and row sufficient. The row sufficient matrices have shown to have an important property with respect to the Karush-Kuhn-Tucker conditions of the following quadratic programming problem derived from LCP(q, M):

$$\begin{aligned} \min \quad & z^T(q + Mz) \\ \text{s.t.} \quad & q + Mz \geq 0 \\ & z \geq 0. \end{aligned} \tag{3.1}$$

Note that the constraints in (3.1) are the same as the first two linear inequalities (1.1) and (1.2) of LCP(q, M). Clearly, LCP(q, M) has a solution if and only if there exists an optimal solution to (3.1) whose objective function value is equal to 0.

We say that (z, u) is a *Karush-Kuhn-Tucker(KKT) pair* associated with (3.1) if it satisfies

$$z \geq 0,\ u \geq 0 \tag{3.2}$$
$$q + Mz \geq 0 \tag{3.3}$$
$$q + (M + M^T)z - M^T u \geq 0 \tag{3.4}$$
$$z^T(q + (M + M^T)z - M^T u) = 0 \tag{3.5}$$
$$u^T(q + Mz) = 0 \tag{3.6}$$

The system (3.2)-(3.6) is called the *Karush-Kuhn-Tucker (KKT) conditions* associated with the quadratic programming problem (3.1).

It is well known (see e.g. Boyd and Vandenberghe, 2004) that if z is a local optimal solution to the quadratic program (3.1), then there exists a vector $u \in \boldsymbol{R^n}$ such that the pair (z, u) satisfies the KKT conditions. In general, a nonlinear minimization program is called *convex* if it has a convex objective function and a convex feasible region. In this case, the KKT conditions are known to be both necessary and sufficient for global optimality. In particular, in LCP(q, M), if M is positive semidefinite, then the KKT conditions are sufficient for global optimality, *i.e.*, if (z, u) satisfies the KKT conditions, then z is a global optimal solution for the quadratic program (3.1).

The following key result holds for the class of **RS**-matrices (see e.g. Cottle et al., 1992, Theorem 3.5.4):

Proposition 3.1.3. Given $M \in \boldsymbol{R^{n\times n}}$. The following statements are equivalent:

1. $M \in$ **RS**.

2. For every $q \in \boldsymbol{R^n}$, if (z, u) is a KKT pair for the quadratic program (3.1), then z solves LCP(q, M).

From the above proposition, we can not conclude that if M is row sufficient, then LCP(q, M) is necessarily solvable for all $q \in \boldsymbol{R^n}$. This is because there is no guarantee that a KKT pair will exist for all LCPs for which $q \in \boldsymbol{R^n}$. However, if the quadratic programming (3.1) is feasible, then by Frank-Wolfe Thoerem (Frank and Wolfe, 1956a) there exists a KKT pair for (3.1), and a solution to LCP(q, M). That is, the row sufficient class belongs to the class of $\mathbf{Q_0}$-matrices (see e.g. Cottle et al., 1992, Corollary 3.5.5).

3.2 The Semidefinite Linear Complementarity Problem

In this section, we extend some of the results in the previous section regarding linear complementarity problems to semidefinite linear complementarity problems. As in the previous chapter, cross commutativity is assumed for most of the results.

3.2.1 The Linear Transformation Classes of Q and $\mathbf{Q_0}$

The **Q** and $\mathbf{Q_0}$ classes of matrices can be straight forwardly generalized to linear transformations for SDLCPs as follows.

Definition 3.2.1. We say that a linear transformation $L : \boldsymbol{S^n} \to \boldsymbol{S^n}$ has the **Q**-*property* if for any matrix $Q \in \boldsymbol{S^n}$, SDLCP(L, Q) has a solution. The class of linear transformations having such property is denoted by **Q**.

Similarly, we define the class $\mathbf{Q_0}$ of linear transformations for SDLCPs.

Definition 3.2.2. A linear transformation $L : \boldsymbol{S^n} \to \boldsymbol{S^n}$ is said to have the $\mathbf{Q_0}$-*property* if SDLCP(L, Q) is solvable whenever it is feasible. The class of linear transformations having such property is denoted by $\mathbf{Q_0}$.

As in the LCP case, $\mathbf{Q_0}$ is characterized as the class of linear transformations L with the following property: the set of symmetric matrices Q for which SDLCP(L, Q) has a solution forms a convex set.

Theorem 3.2.1. Given a linear transformation $L : \boldsymbol{S^n} \to \boldsymbol{S^n}$, the following two statements are equivalent:

1. $L \in \mathbf{Q_0}$;
2. $K(L) = \{Q \in \boldsymbol{S^n} \mid \mathrm{SOL}(L, Q) \neq \emptyset\}$ is convex.

Proof
We can write $K(L)$ as

$$K(L) = \{Y - L(X) \mid Y \succeq 0,\ X \succeq 0,\ XY = 0\}.$$

Denote

$$H(L) = \{Y - L(X) \mid Y \succeq 0,\ X \succeq 0\}.$$

Then, $L \in \mathbf{Q_0} \Leftrightarrow K(L) = H(L)$.
[$1 \Rightarrow 2$] Since the semidefinite matrix cone is convex, we have $H(L)$ is convex. In addition, since $L \in \mathbf{Q_0} \Rightarrow K(L) = H(L)$, we have $K(L)$ convex.
[$2 \Rightarrow 1$] It is easy to see that $K(L) \subseteq H(L)$, and $H(L)$ is convex. In addition, for any $X, Y \succeq 0$, $(X, Y) = \frac{1}{2}(2X, 0) + \frac{1}{2}(0, 2Y)$ and $(2X) \times 0 = 0 \times (2Y) = 0$, thus, $H(L) \subseteq \mathrm{conv}(K(L))$.[2] Moreover, since $K(L)$ is convex, *i.e.*, $K(L) = \mathrm{conv}(K(L))$, therefore, $K(L) = H(L)$, *i.e.*, $L \in \mathbf{Q_0}$. □

3.2.2 The Linear Transformation Classes of Column and Row Sufficient

In this section, we generalize the sufficient classes of matrices in LCP theory to the corresponding classes of linear transformations in SDLCP theory. Again, because of the noncommutativity and nonpolyhedrality of the symmetric positive semidefinite matrix cone, the additional assumption of cross commutativity is necessary.

Definition 3.2.3. A linear transformation $L : \boldsymbol{S}^n \to \boldsymbol{S}^n$ is said to have the *column sufficient property* if

$$\forall\ X \in \boldsymbol{S}^n,\quad X \text{ and } L(X) \text{ commute},\ XL(X) \preceq 0\ \Rightarrow XL(X) = 0.$$

The class of linear transformations having such property is denoted by **CS**.

It was shown by Gowda and Song (Gowda and Song, 2000, Theorem 6) that if a linear transformation $L : \boldsymbol{S}^n \to \boldsymbol{S}^n$ is *monotone*, *i.e.*,

$$X \in \boldsymbol{S}^n \Rightarrow \langle X, L(X)\rangle \geq 0,$$

then for every $Q \in \boldsymbol{S}^n$, the solution set (if not empty) of SDLCP(L, Q) is convex. Next we show that linear transformations L for which SDLCP(L, Q) has a convex solution set consist a larger class of linear transformations. The result is a generalization of Theorem 3.1.2 and it was also recently proved in (Qin et al., 2009).

[2] For any set S, $\mathrm{conv}(S)$ denote the convex hull of S.

Theorem 3.2.2. Given a linear transformation $L : \boldsymbol{S^n} \to \boldsymbol{S^n}$, the following two statements are equivalent:

1. $L \in \widehat{\mathbf{CS}}$.[3]

2. For every $Q \in \boldsymbol{S}^n$, the solution set (possibly empty) of SDLCP(L, Q) is convex.

Proof

[$1 \Rightarrow 2$] Consider any $Q \in \boldsymbol{S^n}$. If SDLCP(L, Q) has only one solution, then its solution set is trivially convex. Thus, assume that SDLCP(L, Q) has more than one solution. Let X_1 and X_2 be any two solutions to SDLCP(L, Q). Denote $Y_1 := L(X_1) + Q$ and $Y_2 := L(X_2) + Q$. Then we have $X_1 \succeq 0,\ X_2 \succeq 0,\ Y_1 \succeq 0,\ Y_2 \succeq 0$ and

$$\langle X_1, Y_1\rangle = 0,\ \langle X_2, Y_2\rangle = 0.$$

Therefore, $X_1Y_1 = 0$ and $X_2Y_2 = 0$. Now, since

$$X_1 \succeq 0,\ X_2 \succeq 0,\ Y_1 \succeq 0,\ Y_2 \succeq 0,$$

and since X_1 and Y_2 commute, X_2 and Y_1 commute, we have

$$X_1Y_2 = Y_2X_1 \succeq 0,\ X_2Y_1 = Y_1X_2 \succeq 0.$$

Therefore,

$$\begin{aligned}(X_1 - X_2)L(X_1 - X_2) &= (X_1 - X_2)(Y_1 - Y_2) \\ &= X_1Y_1 - X_1Y_2 - X_2Y_1 + X_2Y_2 \\ &= -X_1Y_2 - X_2Y_1 \preceq 0.\end{aligned}$$

Thus, by the definition of the column sufficient property, we have $(X_1 - X_2)L(X_1 - X_2) = 0$, which (because $(X_1 - X_2)(Y_1 - Y_2) = 0$) leads to

$$X_1Y_2 = 0,\ X_2Y_1 = 0.$$

Now since the cone of positive semidefinite matrices is convex, we have,

$$\alpha X_1 + (1-\alpha)X_2 \succeq 0$$

$$\alpha Y_1 + (1-\alpha)Y_2 \succeq 0,$$

for any scalar $\alpha \in [0, 1]$. Moreover,

$$(\alpha X_1 + (1-\alpha)X_2)(L(\alpha X_1 + (1-\alpha)X_2) + Q) = (\alpha X_1 + (1-\alpha)X_2)(\alpha Y_1 + (1-\alpha)Y_2) = 0.$$

Therefore, $\alpha X_1 + (1-\alpha)X_2$ is also a solution to SDLCP(L, Q). Since the above holds for all $\alpha \in [0, 1]$, the solution set of SDLCP(L, Q) is convex.

[$2 \Leftarrow 1$] First, we will show that L has the cross commutative property. If SDLCP(L, Q) has only one solution for all $Q \in \boldsymbol{S^n}$, then the cross commutative property obviously satisfies for

[3] Recall (as in Section 2.2.1) that $\widehat{\mathbf{CS}}$ denotes the class of linear transformations having both the column sufficient property and the cross commutative property.

L. Thus consider any $Q \in \boldsymbol{S^n}$ such that SDLCP(L, Q) has more than one solution. Let X_1 and X_2 be any two solutions to SDLCP(L, Q). Denote $Y_1 := L(X_1)+Q$ and $Y_2 := L(X_2)+Q$. Since SOL(L, Q) is convex, we have that for any $\alpha \in [0, 1]$, $\alpha X_1+(1-\alpha)X_2 \in$ SOL(L, Q). By the complementarity condition (1.10), we have $X_1Y_2 + X_2Y_1 = 0$, which by taking the trace of both side, results in $\text{trace}(X_1Y_2) + \text{trace}(X_2Y_1) = 0$. Also since $X_1 \succeq 0,\ X_2 \succeq 0,\ Y_1 \succeq 0,\ Y_2 \succeq 0$, we have that $\text{trace}(X_1Y_2) \geq 0$, $\text{trace}(X_2Y_1) \geq 0$. Therefore, $\text{trace}(X_1Y_2) = 0$, $\text{trace}(X_2Y_1) = 0$. Again, because $X_1 \succeq 0,\ X_2 \succeq 0,\ Y_1 \succeq 0,\ Y_2 \succeq 0$, we have $X_1Y_2 = Y_2X_1 = 0$ and $X_2Y_1 = Y_1X_2 = 0$. *i.e.*, L has the cross commutative property.

To prove $L \in \mathbf{CS}$, we assume on the contrary that there exists an $X \in \boldsymbol{S}^n$, such that

$$X \text{ and } L(X) \text{ commute with } XL(X) \preceq 0 \text{ and } XL(X) \neq 0.$$

Since X and $L(X)$ commute, X and $L(X)$ can be decomposed with the same decomposition matrix U, that is

$$X = UDU^T \text{ and } L(X) = UEU^T,$$

where D and E are diagonal matrices. Denote

$$D := \text{diag}\{d_1, \cdots, d_n\},\ E := \text{diag}\{e_1, \cdots, e_n\}.$$

Also denote

$$L(X)^+ := UE^+U^T,\ L(X)^- := UE^-U^T.$$

Let

$$Q = L(X)^+ - L(X^+).$$

Since $0 \neq XL(X) \preceq 0$, we have that for $\forall\, i$, $d_ie_i \leq 0$ and exists an index j such that $d_je_j < 0$. Therefore,

$$\langle X^+, L(X^+) + Q\rangle = \langle X^+, L(X)^+\rangle = 0.$$

Moreover, since

$$X^+ \succeq 0,\ L(X^+) + Q = L(X)^+ \succeq 0,$$

then, X^+ is a solution to SDLCP(L, Q).

Since $L(X) = L(X)^+-L(X)^- = L(X^+-X^-) = L(X^+)-L(X^-)$, $Q = L(X)^+-L(X^+) = L(X)^- - L(X^-)$. By the similar argument and using $Q = L(X)^- - L(X^-)$, it can be easily shown that X^- is another solution to SDLCP(L, Q).

But either $d_j^+e_j^- > 0$ or $d_j^-e_j^+ > 0$, thus, either $\langle X^+, L(X^-) + Q\rangle$ or $\langle X^-, L(X^+) + Q\rangle$ is not zero, which contradicts the fact that the solution set of SDLCP(L, Q) is convex. Therefore, $L \in \mathbf{CS}$ and this completes the proof. □

In the following example, we show that the cross commutative property of linear transformations is necessary for the preceding theorem. Here, we present a linear transformation $L \in \mathbf{CS}$ which does not have the cross commutative property, but for a symmetric matrix Q, the solution set of SDLCP(L, Q) is not convex.

Example 3.2.1. Consider the linear transformation

$$L : \begin{pmatrix} x & y \\ y & z \end{pmatrix} \rightarrow \begin{pmatrix} -x + 4y & -2x + y + 2z \\ -2x + y + 2z & -4y + 4z \end{pmatrix}.$$

From the definition, it can easily verified that $L \in \mathbf{CS}$.

Set

$$Q = \begin{pmatrix} 2 & 2 \\ 2 & 4 \end{pmatrix},$$

then both

$$X_1 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

and

$$X_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

are solutions to SDLCP(L, Q), but their linear combination

$$\frac{1}{2}X_1 + \frac{1}{2}X_2 = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 0 \end{pmatrix}$$

is not a solution to SDLCP(L, Q). So SOL(L, Q) is not convex. $\Diamond$

To define the row sufficient property for linear transformations in the SDLCPs, we need to consider the transpose of a linear transformation, which is introduced in Definition 1.2.2.

Definition 3.2.4. A linear transformation $L : \boldsymbol{S}^n \to \boldsymbol{S}^n$ is said to have the *row sufficient property* if its transpose L^T has the column sufficient property. The class of linear transformations having such property is denoted by **RS**. We say that a linear transformation L has the *sufficient property* if it has both the column sufficient property and the row sufficient property.

Next, we generalize Proposition 3.1.3 which specifies the relationships between LCPs and the KKT point of the quadratic programming problem (3.1) to SDLCPs. We start by presenting the analog to problem (3.1) as applied to SDLCPs. The quadratic semidefinite programming problem derived from SDLCP(L, Q) is

$$\begin{array}{cl} \min & \langle X, L(X) + Q \rangle \\ \text{subject to} & X \succeq 0 \\ & L(X) + Q \succeq 0. \end{array} \tag{3.7}$$

Obviously, SDLCP(L, M) is solvable if and only if there exists an optimal solution $\bar{X}$ to (3.7) satisfying $\langle \bar{X}, L(\bar{X}) + Q \rangle = 0$.

The Karush-Kuhn-Tucker (KKT) conditions for problem (3.7) are

$$X \succeq 0, U \succeq 0, \tag{3.8}$$
$$L(X) + Q \succeq 0, \tag{3.9}$$
$$Q + L(X) + L^T(X - U) \succeq 0, \tag{3.10}$$
$$\langle X, Q + L(X) + L^T(X - U) \rangle = 0, \tag{3.11}$$
$$\langle U, Q + L(X) \rangle = 0. \tag{3.12}$$

We say that (X, U) is a *Karush-Kuhn-Tucker pair* of the quadratic semidefinite programming problem (3.7) if it satisfies the KKT conditions.

It is well known (Boyd and Vandenberghe, 2004) that if the objective function of (3.7) is convex, *i.e.*, the linear transformation L is monotone, the KKT conditions are both necessary and sufficient for global optimality, thus, if (X, U) is a KKT pair of (3.7), then X solves

SDLCP(L, Q).

The following theorem is a generalization of Proposition 3.1.3 to SDLCPs and it was also recently proved in (Qin et al., 2009).

Theorem 3.2.3. Given a linear transformation $L : \boldsymbol{S}^n \rightarrow \boldsymbol{S}^n$, the following two statements are equivalent:

1. $L \in \mathbf{RS}$ and for every $Q \in \boldsymbol{S^n}$, if (X, U) is a KKT pair of the quadratic semidefinite programming problem (3.7), then both X and U commute with $L^T(X - U)$;
2. For each $Q \in \boldsymbol{S}^n$, if (X, U) is a KKT pair of the quadratic semidefinite programming problem (3.7), then X solves SDLCP(L, Q).

Proof
[$1 \Rightarrow 2$] First, we prove for a KKT pair of (3.7), $(X - U)L^T(X - U) \preceq 0$.
Since $X \succeq 0$ and $Q + L(X) + L^T(X - U) \succeq 0$,

$$\langle X, Q + L(X) + L^T(X - U)\rangle = 0 \Leftrightarrow X(Q + L(X) + L^T(X - U)) = 0.$$

Similarly, since $U \succeq 0$ and $L(X) + Q \succeq 0$,

$$\langle U, Q + L(X)\rangle = 0 \Leftrightarrow U(Q + L(X)) = 0.$$

Now since U and $L^T(X - U)$ commute, and from (3.8), (3.9) and (3.12), U and $Q + L(X)$ commute, thus, U and $Q + L(X) + L^T(X - U)$ commute. Since they are both positive semidefinite, we have

$$U(Q + L(X) + L^T(X - U)) \succeq 0.$$

Also since $U(Q + L(X)) = 0$ by (3.12), we have

$$U(L^T(X - U)) \succeq 0.$$

Finally since X and $L^T(X - U)$ commute, by (3.8), (3.10) and (3.11), X and $Q + L(X) + L^T(X - U)$ commute, we have X and $Q + L(X)$ commute. Since $X \succeq 0$, $Q + L(X) \succeq 0$, we have $X(Q + L(X)) \succeq 0$. Then by (3.11),

$$X(L^T(X - U)) \preceq 0.$$

Therefore, by the definition of the row sufficient property, we get

$$(X - U)L^T(X - U) \preceq 0 \Rightarrow (X - U)L^T(X - U) = 0.$$

Thus, $X(L^T(X - U)) = 0$, and by (3.11), we have

$$X(Q + L(X)) = 0.$$

This together with (3.8) and (3.9) shows that X solves SDLCP(L, Q).

[$2 \Rightarrow 1$] If statement 2 holds, we have X commutes both with $Q + L(X) + L^T(X - U)$ and $Q + L(X)$ commute, then we can decompose X and $Q + L(X)$ using the same decomposition matrix Λ:

$$X := \Lambda^T \tilde{X} \Lambda \text{ and } Q + L(X) := \Lambda^T \tilde{L} \Lambda,$$

where $\tilde{X}$, $\tilde{L}$ are diagonal matrices. Define

$$\tilde{U} := \Lambda U \Lambda^T \text{ and } \tilde{P} := \Lambda L^T(X-U)\Lambda^T,$$

Then, $U := \Lambda^T \tilde{U} \Lambda$ and $L^T(X-U) := \Lambda^T \tilde{P} \Lambda$, $\tilde{U}$ and $\tilde{P}$ are not necessary to be diagonal.

Now we consider $\tilde{X}$ in blocks, where $\tilde{X} = \begin{bmatrix} \tilde{X}_1 & \tilde{X}_2 \\ \tilde{X}_2^T & \tilde{X}_3 \end{bmatrix}$. Partition similarly for $\tilde{L}$, $\tilde{U}$, $\tilde{Q}$ in blocks as well.

Since $\tilde{X}$ is diagonal and positive semidefinite, write

$$\tilde{X} = \begin{bmatrix} 0 & 0 \\ 0 & \tilde{X}_3 \end{bmatrix},$$

where $\tilde{X}_3 \succ 0$ is a diagonal matrix. Then since $X(L(X)+Q) = 0$, we write

$$\tilde{L} = \begin{bmatrix} \tilde{L}_1 & 0 \\ 0 & 0 \end{bmatrix},$$

where $\tilde{L}_1 \succeq 0$ is a diagonal matrix. Again, since $U(L(X)+Q) = 0$, we write

$$\tilde{U} = \begin{bmatrix} 0 & 0 \\ 0 & \tilde{U}_3 \end{bmatrix},$$

where $\tilde{U}_3 \succeq 0$ is not necessary to be a diagonal matrix. And finally write

$$\tilde{P} = \begin{bmatrix} \tilde{P}_1 & \tilde{P}_2 \\ \tilde{P}_2^T & \tilde{P}_3 \end{bmatrix}.$$

Since $X(L^T(X-U)) = 0$, we have $\tilde{X}\tilde{P} = 0$. Also since $\tilde{X}_3 \succ 0$, $\tilde{P}_2 = 0$ and $\tilde{P}_3 = 0$, we have

$$\tilde{U}\tilde{P} = 0.$$

Therefore, $\tilde{U}$ and $\tilde{P}$ commute, thus, U and $L^T(X-U)$ commute.

Now Suppose that $L \notin \mathbf{RS}$. Then there exists a matrix $Y \in \boldsymbol{S^n}$ such that

$$Y \text{ and } L(Y) \text{ commute, } YL^T(Y) \preceq 0 \text{ and } YL^T(Y) \neq 0.$$

Since Y and $L^T(Y)$ commute, we can decompose Y and $L^T(Y)$ with the same decomposition matrix V:

$$Y = VDV^T \text{ and } L^T(Y) = VEV^T,$$

where D and E are diagonal matrices.

Let $X = VD^+V^T$ and $U = VD^-V^T$. Set

$$Q = -L(X) + VE^-V^T.$$

It is easy to verify that (X, U) satisfies the KKT conditions (3.8)-(3.12). Also since U and $L^T(Y) = L^T(X-U)$ commute, and U commutes with both $L^T(U)$ and $L^T(X)$. But

$$\langle X, L(X)+Q\rangle = \text{trace}(VD^+V^T)(VE^-V^T) > 0,$$

this contradicts the assumption that X solves SDLCP(L, Q). Therefore, $L \in \mathbf{RS}$, which completes the proof. $\square$

As in the previous case, we next justify the necessity of the additional conditions regarding the commutativity in Theorem 3.2.3. In particular, we show in the following example the existence of a linear transformation $L \in \mathbf{RS}$ and a KKT pair (X, U) for (3.7) for which X is not a solution to SDLCP(L, Q). This happens because neither X nor U commute with $L^T(X - U)$.

Example 3.2.2. Consider

$$L : \begin{pmatrix} x & y \\ y & z \end{pmatrix} \to \begin{pmatrix} 0 & \frac{x}{2} + y - z \\ \frac{x}{2} + y - z & 0 \end{pmatrix}.$$

Then

$$L^T : \begin{pmatrix} x & y \\ y & z \end{pmatrix} \to \begin{pmatrix} y & y \\ y & -2y \end{pmatrix}.$$

From the definition, it can be easily verified that $L \in \mathbf{RS}$.
Set

$$Q = \begin{pmatrix} 3 & -\frac{5}{2} \\ -\frac{5}{2} & 3 \end{pmatrix}.$$

Let

$$X = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$$

and

$$U = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Then

$$L(X) + Q = \begin{pmatrix} 3 & -3 \\ -3 & 3 \end{pmatrix},$$

$$L(X) + Q + L^T(X - U) = \begin{pmatrix} 4 & -2 \\ -2 & 1 \end{pmatrix},$$

and (X, U) satisfies the KKT conditions but $\langle X, L(X) + Q \rangle \neq 0$. Therefore, X is not a solution to SDLCP(L, Q). $\diamondsuit$

Since the linear transformation L in the preceding example belongs to $\mathbf{CS}$ as well, we see that even if a linear transformation L is sufficient (that is, $L \in \mathbf{CS} \cap \mathbf{RS}$), the KKT pair of (3.7) does not necessarily give a solution to SDLCP(L, Q).

On the other hand, the KKT pair (X, U) that leads to a solution to SDLCP does not always satisfies the commutativity condition $U(L^T(X-U)) = (L^T(X-U))U$. In the example below, $L \in \mathbf{RS}$ and (X, U) is a KKT pair of the quadratic semidefinite programming problem (3.7), yet U and $L^T(X - U)$ do not commute.

Example 3.2.3. Consider the linear transformation

$$L^T : \begin{pmatrix} x & y \\ y & z \end{pmatrix} \to \begin{pmatrix} -8z & y \\ y & x \end{pmatrix}.$$

By the definition, it can be easily verified that $L \in \mathbf{RS}$.

Let

$$Q = \begin{pmatrix} 8 & 4 \\ 4 & 2 \end{pmatrix},$$

$$X = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

and

$$U = \begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix}.$$

Then, (X, U) is a Karush-Kuhn-Tucker pair of the quadratic semidefinite programming problem (3.7). But

$$UL^T(X - U) = \begin{pmatrix} -28 & -4 \\ 56 & 8 \end{pmatrix},$$

which means that U and $L^T(X) - L^T(U)$ do not commute. ◊

Noncommutativity adds another layer of difficulties in dealing with the transpose of linear transformations. The problem is that commutativity of a linear transformation does not necessarily implies commutativity for its transpose. The example below shows that X and $L(X)$ commute, while X and $L^T(X)$ do not.

Example 3.2.4. Consider the linear transformation

$$L : \begin{pmatrix} x & y \\ y & z \end{pmatrix} \to \begin{pmatrix} y & 0 \\ 0 & z \end{pmatrix}.$$

Then

$$L^T : \begin{pmatrix} x & y \\ y & z \end{pmatrix} \to \begin{pmatrix} 0 & \frac{x}{2} \\ \frac{x}{2} & z \end{pmatrix}.$$

Let

$$X = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Then, X and $L(X)$ commute, yet X and $L^T(X)$ do not commute. ◊

Considering Proposition 3.1.3 and the Frank-Wolfe theorem (Frank and Wolfe, 1956a), which states that the quadratic programming problem

$$\begin{array}{ll} \min & z^T(q + Mz) \\ \text{subject to} & q + Mx \geq 0 \\ & x \geq 0 \end{array}$$

always has a solution, we have that the matrix class $\mathbf{RS}$ is in class $\mathbf{Q_0}$. Unfortunately, Frank-Wolfe theorem is not applicable to the quadratic semidefinite programming problem which was introduced in (3.7). This complicates the analysis of the relationships among the classes of linear transformations in the context of semidefinite linear complementarity problems. In the following example we introduce a quadratic semidefinite programming problem which is feasible and bounded but does not have a minimum point.

Example 3.2.5. Consider the linear transformation

$$L : \begin{pmatrix} x & y \\ y & z \end{pmatrix} \to \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}.$$

Set

$$Q = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then the quadratic semidefinite programming problem (3.7) can be written as

$$\begin{array}{ll} \min & x^2 + z(y-1) \\ \text{s.t.} & xz \geq y^2 \\ & x \geq 0 \\ & y \geq 1 \\ & z \geq 0. \end{array}$$

The problem is feasible and the objective value is bounded by zero, yet zero is not the minimum. Actually, if $x = \frac{1}{k}$, $y = 1$ and $z = k$, then $f(\frac{1}{k}, 1, k) = \frac{1}{k^2} \to 0$, as $k \to +\infty$. $\Diamond$

To analyze the row sufficient class of linear transformations and the $\mathbf{Q_0}$ class, we define a subclass of **RS** by adding the extra conditions in Theorem 3.2.3 and the requirement for the existence of a KKT pair to the quadratic semidefinite programming problem (3.7).

Definition 3.2.5. We say that a linear transformation $L : \boldsymbol{S^n} \to \boldsymbol{S^n}$ has the *row sufficient plus property*, if it has **RS**-property and for every $Q \in \boldsymbol{S^n}$, if the quadratic semidefinite programming problem (3.7) is feasible, then there exists a KKT pair for (3.7), with both X and U commute with $L^T(X - U)$. The class of linear transformations having such property is denoted by $\mathbf{RS^+}$.

The class $\mathbf{RS^+}$ is properly contained in **RS** as the following example shows.

Example 3.2.6. Consider the linear transformation

$$L : \begin{pmatrix} x & y \\ y & z \end{pmatrix} \to \begin{pmatrix} x - z & 0 \\ 0 & z - x \end{pmatrix}.$$

Then

$$L^T : \begin{pmatrix} x & y \\ y & z \end{pmatrix} \to \begin{pmatrix} x - z & 0 \\ 0 & z - x \end{pmatrix}.$$

$L \in \mathbf{RS}$. For any matrix $Q \in \boldsymbol{R^{n\times n}}$, if (X, U) is a KKT pair for (3.7), it can be shown that both X and U commute with $L^T(X - U)$.

However, if

$$Q = \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix},$$

SDLCP(L, Q) is feasible, but there exists no KKT pair for (3.7). Therefore, $L \notin \mathbf{RS^+}$. $\Diamond$

While unlike in LCP theory row sufficient linear transformations do not necessarily have the $\mathbf{Q_0}$ property, we show next that $\mathbf{RS^+}$ linear transformations do.

Lemma 3.2.1. $\mathbf{RS^+} \subseteq \mathbf{Q_0}$.

The proof of the lemma follows directly from the definition of $\mathbf{RS^+}$-property and Theorem 3.2.3.

We show in the following example, the restriction of the **RS** class in Lemma 3.2.1 is necessary as the **RS** class does not belong to the $\mathbf{Q_0}$ class.

Example 3.1. Consider the linear transformation

$$L : \begin{pmatrix} x & y \\ y & z \end{pmatrix} \rightarrow \begin{pmatrix} x-z & 0 \\ 0 & x-z \end{pmatrix}.$$

Then,

$$L^T : \begin{pmatrix} x & y \\ y & z \end{pmatrix} \rightarrow \begin{pmatrix} x-z & 0 \\ 0 & x-z \end{pmatrix}.$$

By the definitions, It can be easily verified that $L \in \mathbf{RS}$ and also $L \notin \mathbf{RS^+}$.

Set

$$Q = \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}.$$

Then, SDLCP(L, Q) is feasible, since

$$\bar{X} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$$

satisfies $\bar{X} \succeq 0$, $L(\bar{X})+Q \succeq 0$. Consider any $X = \begin{pmatrix} x & y \\ y & z \end{pmatrix}$, $L(X)+Q = \begin{pmatrix} x-z & 1 \\ 1 & x-z+2 \end{pmatrix}$. If $L(X)+Q \succeq 0$, then $L(X)+Q \succ 0$. Moreover if $X(L(X)+Q) = 0$, then $X = 0$, which leads to contradiction. Therefore, SDLCP(L, Q) has no solution. Thus, $L \notin \mathbf{Q_0}$. $\Diamond$

Chapter 4

Relationships between Classes

In LCP theory, the matrix classes relate to each other in a complex manner. Over the years, many authors have complied lists of matrix classes with special properties related to the solutions of the corresponding LCPs. Some of the authors have also presented graphs summarizing the inclusion relationships among matrix classes. A recent comprehensive paper (Cottle, 2009) presents a detailed graph consisting of over 50 classes of matrices related to the LCPs.

In this chapter, we present some known inclusion relationships among several classes of linear transformations. We then prove few more relationships usually motivated by their counterparts as presented in (Cottle et al., 1992). Finally, we summarize all of these relationships in a graph which is displayed at the end of the chapter.

In the LCPs, the $\mathbf{P}$ and $\mathbf{P_0}$ matrix classes are closely related, and this relationship is preserved when considering the classes $\mathbf{P}$ and $\mathbf{P_0}$ of linear transformations corresponding to SDLCPs. Following the definition of the class of $\mathbf{P_0}$-matrices in LCPs, we introduce the $\mathbf{P_0}$ class of linear transformations in SDLCPs, which contains the class $\mathbf{P}$ of linear transformations as a special case.

Definition 4.0.6. A linear transformation $L : \boldsymbol{S^n} \to \boldsymbol{S^n}$ has the $\mathbf{P_0}$-*property* if

$$[X \neq 0] \;\;\Rightarrow\;\; [\text{if } X \text{ and } L(X) \text{ commute}, \;\exists\, i \text{ such that } X(i) \neq 0,\;\; X(i) \times L(X)(i) \geq 0].$$

The class of linear transformations having such property is denoted by $\mathbf{P_0}$.

By their definitions, it is clear that the $\mathbf{P_0}$ and $\mathbf{P}$ classes are closely related. Motivated by a known results (see e.g. Cottle et al., 1992, Theorem 3.4.2) showing that $\mathbf{P_0}$-matrices arise as a perturbation of $\mathbf{P}$-matrices, we present in the next theorem a similar result for the analogous linear transformation classes. Here we let $I : \boldsymbol{S^n} \to \boldsymbol{S^n}$ denote the identity linear transformation.

Theorem 4.1. Given a linear transformation $L : \boldsymbol{S^n} \to \boldsymbol{S^n}$, the following statements are equivalent:

1. $L \in \mathbf{P_0}$;

2. $L + \epsilon I \in \mathbf{P}$, $\forall\, \epsilon > 0$.

Proof
[$\Rightarrow$] Assume that $L \in \mathbf{P_0}$, then by definition, we have,

$$\forall\, X \neq 0 \text{ such that } X \text{ and } L(X) \text{ commute},$$

and under the same decomposition matrix U,

$$\exists\ i \text{ such that } X(i) \neq 0 \text{ and } X(i) \times L(X)(i) \geq 0.$$

Since $X \neq 0$, X and $L(X)$ commute, we have for any $\epsilon > 0$, X and $(L + \epsilon I)(X) = L(X) + \epsilon X$ commute as well, since a matrix always commutes with itself. For the same i as above, we have

$$X(i) \times (L + \epsilon I)(X)(i) = X(i) \times L(X)(i) + X(i) \times \epsilon X(i) > 0.$$

This shows that $L + \epsilon I \in \mathbf{P}$, $\forall\ \epsilon > 0$.
[$\Leftarrow$] Assume $L + \epsilon I \in \mathbf{P}$, $\forall\ \epsilon > 0$. Then for any matrix $X \neq 0$ satisfying X and $(L + \epsilon I)(X)$ commute, under the same decomposition matrix U,

$$\exists\ i \text{ such that } X(i) \times (L + \epsilon I)(X)(i) > 0.$$

Suppose on the contrary, $L \notin \mathbf{P_0}$. This means that there exists an $X \neq 0$ such that X and $L(X)$ commute, yet under the same decomposition matrix U,

$$\forall\ i \text{ with } X(i) \neq 0,\ X(i) \times L(X)(i) < 0.$$

Take

$$0 < \epsilon_0 < \min_{i:X(i)\neq 0} \{-\frac{L(X)(i)}{X(i)}\},$$

then $\forall\ i$ with $X(i) \neq 0$,

$$X(i) \times (L + \epsilon_0 I)(X)(i) = X(i) \times L(X)(i) + \epsilon_0 X(i)^2 < 0.$$

This contradicts with the fact that $L + \epsilon I \in \mathbf{P}$, $\forall\ \epsilon > 0$. Therefore, $L \in \mathbf{P_0}$. □

Following similar argument as in Theorem 4.1, it is possible to extend it to the $\mathbf{E}$ and $\mathbf{E_0}$ classes of linear transformations in SDLCPs. (This extension is also valid with respect to the corresponding matrix classes $\mathbf{E}$ and $\mathbf{E_0}$ (Cottle and Dantzig, 1968b).)

Theorem 4.2. Given a linear transformation $L : \boldsymbol{S^n} \to \boldsymbol{S^n}$, the following statements are equivalent:

1. $L \in \mathbf{E_0}$;

2. $L + \epsilon I \in \mathbf{E}$, $\forall\ \epsilon > 0$.

We omit the proof of Thoerem 4.2 as it can be proved by using similar arguments as in Theorem 4.1.

Malik and Mohan (Malik and Mohan, 2006) introduced another two classes of linear transformations analogous to the copositive[1] and strictly copositive [2] classes in LCPs as follows.

[1]A matrix $M \in \boldsymbol{R^{n\times n}}$ is said to be *copositive* if $x^T Mx \geq 0$ for all nonnegative n-vector x.

[2]A matrix $M \in \boldsymbol{R^{n\times n}}$ is said to be *strictly copositive* if $x^T Mx > 0$ for all nonzero nonnegative n-vector x.

Definition 4.1. A linear transformation $L : \boldsymbol{S^n} \to \boldsymbol{S^n}$ is said to be *copositive* if

$$\langle X, L(X) \rangle \geq 0, \quad \forall\, X \succeq 0.$$

The class of linear transformations that have such property is denoted by $\mathbf{C_0}$.

Definition 4.2. A linear transformation $L : \boldsymbol{S^n} \to \boldsymbol{S^n}$ is said to be *strictly copositive* if

$$\langle X, L(X) \rangle > 0, \quad \forall\, 0 \neq X \succeq 0.$$

The class of linear transformations that have such property is denoted by $\mathbf{C}$.

Note that since the definitions of $\mathbf{C_0}$ and $\mathbf{C}$ above use only inner products, there is no need to add commutativity restriction to obtain results generalizing the LCP case to linear transformations.

In the same manner as is in Theorems 4.1 and 4.2 and analogous to the LCP results (see e.g. Cottle et al., 1992), we have

Theorem 4.3. Given a linear transformation $L : \boldsymbol{S^n} \to \boldsymbol{S^n}$, the following statements are equivalent:

- $L \in \mathbf{C_0}$;
- $L + \epsilon I \in \mathbf{C}$, $\forall\, \epsilon > 0$.

We omit the proof of Thoerem 4.3 as it can be proved by using similar arguments as in Theorem 4.1.

Gowda and Song (Gowda and Song, 2000) presented several additional relationships among some classes of linear transformations. In particular, they showed that $\mathbf{P} \subseteq \mathbf{E} \subseteq \mathbf{Q}$ (Gowda and Song, 2000, p.579), and that $\mathbf{E_0} \cap \mathbf{R_0} \subseteq \mathbf{Q}$ (Gowda and Song, 2000, Theorem 4).[3]

Also, Malik and Mohan (who introduced the linear transformations $\mathbf{C_0}$ and $\mathbf{C}$) showed that $\mathbf{C} \subseteq \mathbf{E}$, and $\mathbf{C_0} \subseteq \mathbf{E_0}$ (Malik and Mohan, 2006, Theorem 5).

In the following, we establish several more relationships among classes of linear transformations that we have introduced in earlier chapters. We also present and prove some additional relationships between these newly defined classes and the classes that have been defined prior to this dissertation.

Like the arguments in Chapter 2 and Chapter 3, the noncommutativity of linear transformations imposes some difficulties in trying to link different classes. Again, the cross commutative property has to be embraced at times in order to draw meaningful connections. Moreover, the noncommutativity makes the analysis of innnerconnection between a linear transformation L and its transpose L^T very hard, because first there is no known implicit expression for L^T, and second that for a given linear transformation L, the set $\{X \in \boldsymbol{S^n} \mid X \text{ and } L(X) \text{ commute}\}$ is not identical to $\{X \in \boldsymbol{S^n} \mid X \text{ and } L^T(X) \text{ commute}\}$. In general, classes involving transpose of linear transformations and also containing matrix products in their definitions (e.g., the $\mathbf{RS}$ class) are very hard to analyse. In fact, we provide some examples showing that for these classes, several existing connections among some matrix classes in LCPs are not valid among the corresponding classes of linear transformations in SDLCPs.

[3] $\mathbf{R_0}$ is the class of linear transformations $L : \boldsymbol{S^n} \to \boldsymbol{S^n}$ such that $\mathrm{SOL}(L, 0) = \{0\}$.

As was mentioned in Section 2.1.3, the matrix class $\boldsymbol{L} = \mathbf{E_0} \cap \mathbf{E_1}$ plays a key role in LCP theory. To initiate the investigation of its analogue for linear transformation classes, we begin by extending the definition of the matrix class $\mathbf{E_1}$ to linear transformations.

Definition 4.3. A linear transformation $L : \boldsymbol{S}^n \to \boldsymbol{S}^n$ is said to have the $\mathbf{E_1}$-property if for any $0 \neq Z \in \mathrm{SOL}(L, 0)$, there exist linear transformations $L_1 : \boldsymbol{S}^n \to \boldsymbol{S}^n$ and $L_2 : \boldsymbol{S^n} \to \boldsymbol{S}^n$, that satisfy

(a) $\forall\ X \in \boldsymbol{S}^n$, $\langle X, L_i(X)\rangle \geq 0$, and $\mathrm{rank}(L_i(X)) \leq \mathrm{rank}(X)$, for $i = 1, 2$.

(b) $L_2(Z) \neq 0$ and $L_1(L(Z)) + L^T(L_2(Z)) = 0$.

The class of linear transformations having such property is denoted by $\mathbf{E_1}$. As in LCP theorey, we now define $\boldsymbol{L}$ to be the intersection of $\mathbf{E_0}$ and $\mathbf{E_1}$.[4]

It is clear that, as is the case for the corresponding classes of matrices, $\mathbf{E_1} \subseteq \mathbf{E}$.

Before proceeding to the main results of this chapter, we present and prove an extension of the theorem of alternatives (see e.g. Cottle et al., 1992, Theorem 2.7.6) as applied to linear transformations. This lemma, as in LCP theory, plays an essential role in establishing relationships among classes of linear transformations.

Lemma 4.1. For $X, Y \in \boldsymbol{S^n}$ and a linear transformation $L : \boldsymbol{S^n} \to \boldsymbol{S^n}$.

The system $\left\{\begin{array}{l} 0 \neq X \succeq 0 \\ L(X) \succeq 0 \end{array}\right.$ has no solution. $\Leftrightarrow$ The system $\left\{\begin{array}{l} Y \succeq 0 \\ L^T(Y) \prec 0 \end{array}\right.$ has a solution.

Proof

[$\Leftarrow$] Suppose that there exists $\bar{X},\ \bar{Y} \in \boldsymbol{S^n}$ such that

$$\left\{\begin{array}{l} 0 \neq \bar{X} \succeq 0 \\ L(\bar{X}) \succeq 0 \end{array}\right.$$

and

$$\left\{\begin{array}{l} \bar{Y} \succeq 0 \\ L^T(\bar{Y}) \prec 0 \end{array}\right. .$$

Then,

$$0 > \langle \bar{X}, L^T(\bar{Y})\rangle = \langle L(\bar{X}), \bar{Y}\rangle <\geq 0,$$

a contradiction.

[$\Rightarrow$] Suppose that both the system

$$\left\{\begin{array}{l} 0 \neq X \succeq 0 \\ L(X) \succeq 0 \end{array}\right.$$

and the system

$$\left\{\begin{array}{l} Y \succeq 0 \\ L^T(Y) \prec 0 \end{array}\right.$$

has no solution. Consider the two disjoint convex sets:

$$C_1 = \{L^T(Y) \mid Y \succeq 0\},$$

[4] We use bold font to distinguish the class $\boldsymbol{L}$ from the notation for a generic linear transformation L.

$$C_2 = \{Y \mid Y \prec 0\}.$$

Then by the Hahn-Banach Separation Theorem (see e.g. Mangasarian, 1969), and also since C_2 is an open set, there is a hyperplane separate the set C_1 and C_2, *i.e.*, there exists an $0 \neq X \in \boldsymbol{S^n}$ such that

$$\begin{aligned}\langle X, Z\rangle &\geq 0, \quad \forall\, Z \in C_1\\ \langle X, Z\rangle &< 0, \quad \forall\, Z \in C_2,\end{aligned}$$

which can be written explicitly as

$$\langle X, L^T(Y)\rangle \geq 0, \quad \forall\, Y \succeq 0 \tag{4.1}$$

$$\langle X, Y\rangle < 0, \qquad \forall\, Y \prec 0. \tag{4.2}$$

(4.1) gives

$$\langle L(X), Y\rangle \geq 0,\ \forall\, Y \succeq 0,$$

thus, $L(X) \succeq 0$. (4.2) gives

$$0 \neq X \succeq 0.$$

Therefore, the system

$$\begin{cases} 0 \neq X \succeq 0 \\ L(X) \preceq 0 \end{cases}$$

has a solution, which leads to contradiction. □

Next, we summarize several relationships among some classes of linear transformations (several of which that had been introduced for the first time in this dissertation) in the following theorem.

Theorem 4.4.

1. $\mathbf{P_0} \subseteq \mathbf{E_0}$;
2. $\mathbf{P} \subseteq \mathbf{CS} \subseteq \mathbf{P_0}$;
3. $\hat{\mathbf{E}}^* \subseteq \mathbf{E_1}$;
4. $\hat{\mathbf{E}}^* \subseteq \boldsymbol{L}$.

Proof

1. Assume that $L \in \mathbf{P_0}$. Then by definition, for any matrix $0 \neq X \in \boldsymbol{S^n}$, with X and $L(X)$ commute, and under the same decomposition matrix U,

$$\exists\, i \text{ such that } X(i) \neq 0 \text{ and } X(i) \times L(X)(i) \geq 0.$$

Then for any matrix $0 \neq X \succeq 0$, with X and $L(X)$ commute, under the same decomposition matrix U,

$$\exists\, i \text{ such that } X(i) \neq 0 \text{ and } X(i) \times L(X)(i) \geq 0.$$

Since $X \succeq 0$, we have $X(i) > 0$ and thus $L(X)(i) \geq 0$. Thus, we can conclude that $L \in \mathbf{E_0}$.

2. The first inclusion can be derived from the definitions straightforwardly.

We prove the second inclusion relation by contradiction. Suppose on the contrary, that $L \in \mathbf{CS}$, but $L \notin \mathbf{P_0}$. Then, there exists an $X \neq 0$, with X and $L(X)$ commute, and under the same decomposition matrix U, for every the index i such that $X(i) \neq 0$,

$$X(i) \times L(X)(i) < 0.$$

This means that $X \neq 0$ and $XL(X) \preceq 0$, but $XL(X) \neq 0$. This contradicts the fact that $L \in \mathbf{CS}$. Therefore, $L \in \mathbf{P_0}$.

3. Consider a linear transformation $L \in \hat{\mathbf{E}}^{*}$. By Theorem 2.2.3, SDLCP(L, Q) has a unique solution, $\forall\, 0 \neq Q \succeq 0$. Assume that SDLCP$(L, 0)$ has multiple solutions, *i.e.*, there exists a matrix Z such that $0 \neq Z \in \mathrm{SOL}(L, 0)$. (If SDLCP$(L, 0)$ has a unique solution, then $L \in \mathbf{E_1}$ immediately satisfies.) Then

$$Z \succeq 0,\ \ L(Z) \succeq 0,$$
$$\langle Z, L(Z)\rangle = 0.$$

Thus, Z and $L(Z)$ commute, $ZL(Z) \preceq 0$, by the definition of the $\mathbf{E}^{*}$-property,

$$L(Z) = 0,\ \ Z \succ 0.$$

Since SDLCP(L, Q) has a unique solution for all $0 \neq Q \succeq 0$, The system

$$\begin{cases} Y \succeq 0 \\ L(Y) \prec 0 \end{cases}$$

has no solution. Then by Lemma 4.1, there exists $\bar{X}$ such that

$$\begin{cases} 0 \neq \bar{X} \succeq 0 \\ L^T(\bar{X}) \succeq 0 \end{cases}$$

Since $Z \succ 0$ and $\bar{X} \succeq 0$, there exist $L_2(\cdot)$ satisfies $L_2(Z) = \bar{X}$ and [a], then

$$L^T(L_2(Z)) = L^T(\bar{X}) \succeq 0.$$

Since $0 = \langle L(Z), L_2(Z)\rangle = \langle Z, L^T(L_2(Z))\rangle$ and $Z \succ 0$, we have

$$L^T(L_2(Z)) = 0.$$

Thus, $L \in \mathbf{E_1}$.
4. The comes directly from the definition of L class and 3. □

In LCP theory, the class of $\mathbf{P}$ is reflective, meaning that if a matrix is a $\mathbf{P}$-matrix, then its transpose is also a $\mathbf{P}$-matrix, which implies that a $\mathbf{P}$-matrix is also a $\mathbf{RS}$-matrix (since $\mathbf{P} \subseteq \mathbf{CS}$ by their definitions). This is in general not true for the classes of $\mathbf{P}$ and $\mathbf{RS}$ of linear transformations. The following is an example showing the existence of a linear transformation $L \in \mathbf{P}$ such that $L^T \notin \mathbf{P}$, moreover $L \notin \mathbf{RS}$.

Example 4.1. Consider the linear transformation

$$L : \begin{pmatrix} x & y \\ y & z \end{pmatrix} \to \begin{pmatrix} x + 4y & y \\ y & z + 6y \end{pmatrix}.$$

Then

$$XL(X) = \begin{pmatrix} x^2+4xy+y^2 & xy+yz+6y^2 \\ xy+yz+4y^2 & y^2+z^2+6yz \end{pmatrix} \preceq 0 \Rightarrow y=0,\ x=0,\ z=0.$$

Thus, $L \in \mathbf{P}$. Its transpose is

$$L^T : \begin{pmatrix} x & y \\ y & z \end{pmatrix} \to \begin{pmatrix} x & 2x+y+3z \\ 2x+y+3z & z \end{pmatrix}.$$

Then, $L^T \notin \mathbf{P}$ since

$$XL^T(X) = \begin{pmatrix} x^2+2xy+y^2+3yz & 2x^2+xy+3xz+yz \\ x^2+2xy+yz+3z^2 & 2xy+y^2+3yz+z^2 \end{pmatrix},$$

when $x=1,\ y=-1,\ z=1,$

$$XL^T(X) = \begin{pmatrix} -3 & 3 \\ 3 & -3 \end{pmatrix} \preceq 0.$$

Also, since $XL^T(X) \neq 0$, $L \notin \mathbf{RS}$ ♢

One of the main reason of the interest in the matrix class $\boldsymbol{L}$ in LCPs is the fact that $\boldsymbol{L} \subseteq \mathbf{Q_0}$. However, as the next example shows, this is not the case for the $\boldsymbol{L}$ class of linear transformations .

Example 4.2. Consider the linear transformation

$$L : \begin{pmatrix} x & y \\ y & z \end{pmatrix} \to \begin{pmatrix} x-z & 0 \\ 0 & z-x \end{pmatrix}.$$

Then,

$$L^T : \begin{pmatrix} x & y \\ y & z \end{pmatrix} \to \begin{pmatrix} x-z & 0 \\ 0 & z-x \end{pmatrix}.$$

By the definitions, It can be easily verified that $L \in \mathbf{E_0} \cap \mathbf{E_1} = \boldsymbol{L}$.

Set

$$Q = \begin{pmatrix} -1 & 1 \\ 1 & 3 \end{pmatrix}.$$

Then, SDLCP(L, Q) is feasible, since

$$X = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$$

is a feasible solution to SDLCP(L, Q). Now suppose $X = \begin{pmatrix} x & y \\ y & z \end{pmatrix} \in \text{SOL}(L, Q)$. Then, $L(X) + Q = \begin{pmatrix} x-z-1 & 1 \\ 1 & z-x+3 \end{pmatrix}$. Since $X(L(X)+Q) = 0$ and $L(X)+Q \neq 0$, we have

$$\det(X) = 0 \Rightarrow xz = y^2. \tag{4.3}$$

Similarly, since $X(L(X)+Q) = 0$ and $X \neq 0$, we have

$$\det(L(X)+Q) = 0 \Rightarrow (x-z-1)(z-x+3) = 1. \tag{4.4}$$

Also, since $\langle X, L(X) + Q \rangle = 0$, we have

$$x(x - z - 1) + z(z - x + 3) + 2y = 0. \tag{4.5}$$

Then by (4.3)-(4.5, we have

$$z(z + 2) = (z + 1)^2.$$

Since no z could satisfy the preceding equation, SDLCP(L, Q) has no solution, *i.e.*, $L \notin \mathbf{Q_0}$. $\Diamond$

The relationships among classes of linear transformations as established in this chapter are displayed. Because the commutativity does not follow when taking the transpose of linear transformations, the classes that involve the transpose of linear transformations can not be related to other classes as their corresponding classes of matrices in LCP theory.

In Figure 4, an arrow from class $\boldsymbol{X}$ to class $\boldsymbol{Y}$ means that $\boldsymbol{X} \subseteq \boldsymbol{Y}$.

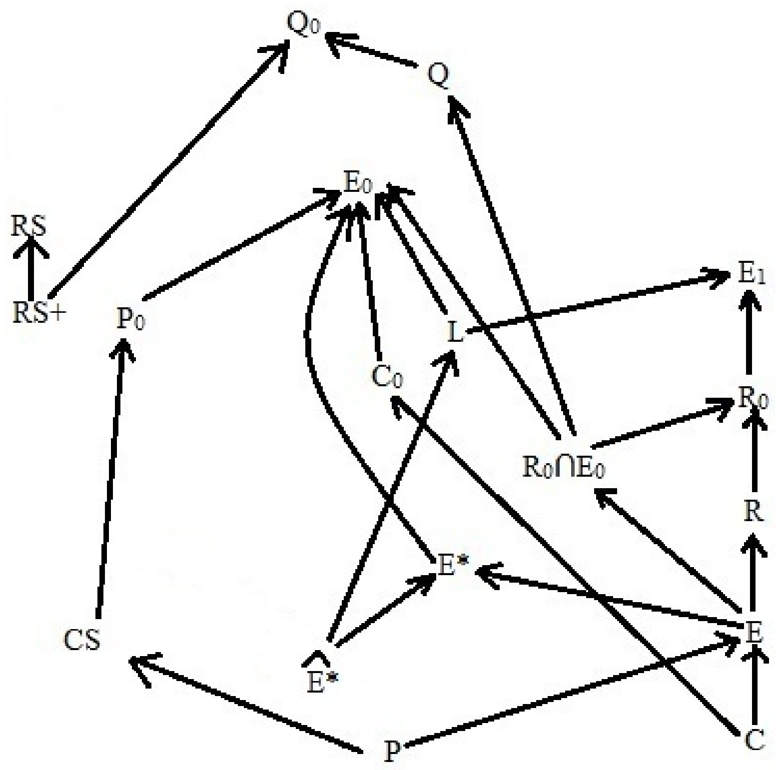

Figure 4.1: The property classes inclusion graph

Chapter 5

Conclusion

In this thesis, we extended some of the classes of matrices related to LCP to the classes of linear transformations related to SDLCP. Due to the noncommutativity of matrix products and the nonpolyhedrality of the positive semidefinite matrix cone, additional conditions have been introduced to extend some of the known result concerning classes of matrics and LCPs to classes of linear transformations and SDLCPs. With these additional restrictions (usually related to the commutativity of the product of certain matrices), we (and other authors before us) were able to extend some of the key (as well as other) LCP results to the SDLCPs.

Given the wealth of results for the LCPs that had been developed over the last 40 years and the importance of SDLCPs in optimization theory and practice, there are still many challenging open problems.

Bibliography

I Adler, R.W. Cottle, and S. Verma. New characterizations of row sufficient matrices. *Linear Algebra Applications*, 430, 2009.

R. Bellman and K. Fan. On systems of linear inequalities in hermitian matrix variables. *Proceedings of Symposia in Pure Mathematics*, 7, 1963.

A Borodin and I. Munro. *The Computational Complexity of Algebraic and Numeric Problems.* American Elsevier, New York, 1975.

S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan. *Linear Matrix Inequalities in System and Control Theory.* 1994.

S. Boyd and L. Vandenberghe. *Convex Optimization.* Cambridge University Press, Cambridge, 2004.

S.J. Chung. Np-completeness of the linear complementarity problem. *Journal of optimization theory and applications*, 60(3), 1989.

T.F. Conry and A. Seirge. A mathematical programming method for design for elastic bodies in contact. *Journal of Applied Mechanics*, 38(1293-1307), 1971.

R.W. Cottle. A field guide to matrix classes found in the linear complementarity problem. *Journal of Global Optimization*, 2009.

R.W Cottle and M.S. Coheen. *A special class of large quadratic programs*, volume 3. 1978.

R.W. Cottle and G.B. Dantzig. complementary pivot theory of mathematical programming. *Linear algebra and its applications*, 1:103–125, 1968a.

R.W. Cottle and G.D. Dantzig. Complementary pivot theory of mathematical programing. *Linear algebra applications*, 1:103–125, 1968b.

R.W. Cottle, J.S. Pang, and R.S. Stone. *The Linear Complementarity Problem.* IEEE Trans. Academic Press, 1992.

C.W. Cryer. A bibliography of free boundary problems. Technical summary report no.1973, Mathematics Reseach Center, University of Wisconsin, Madison, 1977.

R.A. Danao. On a class of semimonotone $\mathbf{Q_0}$-matrices in the linear complementarity problem. *Operations Research Letter*, 13:121–125, 1993.

R.A. Danao. A note on $\mathbf{E'}$-matrices. *Linear Algebra Applications*, 259:299–305, 1997.

O. Dedonato and G. Maier. Mathematical programming methods for the inelastic analysis of reinforced concrete frames allowing for limited rotation capacity. *International journal for numerical methods in engineering*, 4(307-329), 1972.

B.C. Eaves. The linear complementarity problem. *Management Science*, 17:612–634, 1971.

M.C. Ferris and J.S. Pang, editors. *Complementarity and Variational Problems (state of the art)*. Society for industrial and applied mathematics, Philadelpha, 1997.

M. Frank and P. Wolfe. An algorithm for quadratic programming. *Naval Research Logistics Quarterly*, 3:95–110, 1956a.

M. Frank and P. Wolfe. An algotithm for quadratic progrmming. *NAVAL RESEARCH LOGISTICS QUARTERLY*, 3:95–110, 1956b.

P.K. Goenka. Dynamically loaded journal bearings: finite element method analysis. *L. Lubr. Technol*, 106(429-439), 1984.

M.S. Gowda and Y. Song. On semidefinite linear complementarity problems. *Mathematical Programming*, 88(3):575–587, 2000.

T.H. Hansen and H. Scarf. On the approximation of nash equilibrium points in an n-person noncooperative game. *SIAM journal of applied mathematrics*, 26(622-637), 1974.

P.T. Harker and J.S. Pang. Finite-dimensional variational inequality and nonlinear complementarity problems: A survey of theory, algorithms and applications. *Mathematical Programming*, 48:161–220, 1990.

C. Hildreth. Point estimates of ordinate of concave functions. *Journal of American Statistian Association*, 49:598–619, 1954.

C. Hildreth. A quadratic programming procedure. *NAVAL RESEARCH LOGISTICS QUARTERLY*, 4:79–85, 1957.

R.A. Horn and C.R. Johnson. *Mstrix Analysis*. Cambridge Univeristy Press, Combridge, 1985.

G. Isac. *Complementarity Problems*. Lecture notes in Mathematics. Springer-Verlag, New York, 1992.

I. Kaneko. A mathematical programming method for the inelastic analysis of reinforced concrete frames. *International journal for numerical methods in engineering*, 11(1137-1154), 1977.

I. Kaneko. Piecewise linear elastic-plastic analysis. *International journal for numerical methods in engineering*, 14(757-767), 1979.

L.G. Khachian. A polynomial algorithm in linear programming. *Sociert. Math. Dokl.*, 20(1): 191–194, 1979.

M. Kojima, S. Shindoh, and S. Hara. Interior-point methods for the monotone semidefinite linear complementarity problems. *SIAM Journal of Optimization*, 7:86–125, 1997.

C.E. Lemke and J.T.Jr. Howson. Equlibrium porints of bimatrix games. *society of industrial and applied mathematics*, 12(2):413–423, 1964.

M Malik and S.R. Mohan. Some geometrical aspects of semidefinite linear complementarity problems. *Linear Multilinear Algebra*, 54(1):55–70, 2006.

O.L. Mangasarian. *Nonlinear Programming.* McGraw-Hill Book Company, USA, 1969.

A.S. Manne. Economic equilibrium: model formulation and solution. *Mathematical programming Society*, 23, 1985.

M. Mesbahi and G.P. Papavassilopoulos. A cone programming approach to the bilinear matrix inequality problem and its geometry. *Mathematical Programming*, 77:247–272, 1997.

J.J. Moré and G. Toraldo. On the solution of large quadratic programming problems with bound constraints. *SIAM journal of Oprimization*, 1(93-113), 1991.

K.G. Murty. *Linear Complementarity, Linear and Nonlinear Programming.* Helderman-Berlag, Berlin, 1988.

J. Oden and N. Kikuchi. Theory of variational inequalities with applications to problems of flow through porous media. *International jounral of engineering science*, 18(1173-1284), 1986.

L. Qin, L. Kong, and J. Han. Sufficiency of linear transformations on euclidean jordan algebras. *Optimization Letters*, 3(2), 2009.

M. V. Ramana. An exact duality theorem for semidefinite programming and its complexity implications. *Mathematical Programming*, 77:129–162, 1997.

L. Vandenberghe and S. Boyd. Semidefinite programming. *SIAM Review*, 38(1):49–95, 1996.

H. Wolkowicz, R. Saigal, and L. Vandenberghe, editors. *Handbook of semidefinite programming: theory, algorithms, and applications.* Kluwer Acedemic Publishers group, USA, 2000.