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E.M. Kwicklis

July 1989

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Absorption of Water into Porous Blocks of Various Shapes and Sizes

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ABSTRACT

Approximate solutions are presented for absorption of water into porous spherical, cylindrical, and slab-like blocks whose characteristic curves are of the van Genuchten type. The solutions are compared to numerical simulations of absorption into blocks of Topopah Spring tuff from the site of the proposed nuclear waste repository at Yucca Mountain, Nevada. Guided by these results, a scaling law, based on the ratio of surface area to volume, is then proposed for predicting the rate of absorption into irregularly-shaped blocks. This scaling law is tested against a numerical simulation of absorption into an irregularly-shaped two-dimensional polygonal block, and is shown to be a good approximation.

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Introduction

Yucca Mountain in Nevada is currently being considered for the location of an underground repository for the disposal of high-level radioactive waste [U.S. DOE, 1986]. The proposed repository would be constructed in the unsaturated zone above the water table, in a formation consisting of highly fractured volcanic tuff. As part of the process of characterizing the site for the purposes of determining its suitability for the repository, it is necessary to develop models for studying the flow of groundwater in a fractured rock mass with low (but nonzero) matrix permeability. The complex geology of the site, along with other factors, make a purely analytical treatment impractical, while numerical analysis is constrained by limitations of computational time. Hence it is desirable to combine both approaches in order to take advantage of the benefits of each. One of the processes which will be a factor in the overall evaluation of the site is that of absorption of water from the fracture network into a matrix block. In this paper, we study the effect of the shape and size of a matrix block on the rate at which it saturates.

Absorption of water into a porous block can be described by the following equation [Bear, 1988]:

$$\text{div}[\beta k_r(\psi) \text{grad } \psi] = \frac{\partial S}{\partial t}, \quad (1)$$

where ψ is the potential of the water in the porous medium, S is the fraction of the pore space that is filled with liquid, and k_r is a dimensionless relative permeability function. The parameter β is equal to $k/\mu\phi$, where k is the permeability under conditions of full saturation, μ is the viscosity of the water, and ϕ is the porosity of the medium. Equation (1) embodies the principle of conservation of mass for the liquid phase, along with a modified form of Darcy's law to relate the volumetric flux to the

potential gradient.

Each different porous medium has its own characteristic functions that describe k_r and S as functions of ψ . One commonly used form of the characteristic equations is that of *van Genuchten* [1980]:

$$k_r(\psi) = \frac{\{1 - (\alpha|\psi|)^n\}^{n-1} [1 + (\alpha|\psi|)^n]^{-m}}{[1 + (\alpha|\psi|)^n]^{m/2}}, \quad (2)$$

$$S(\psi) = S_r + (S_s - S_r)[1 + (\alpha|\psi|)^n]^{-m}, \quad (3)$$

where α is a parameter that has dimensions of 1/Pressure, and m and n are dimensionless parameters that are related by $m = 1 - 1/n$, $n > 1$. A more detailed discussion of these characteristic curves is given by *van Genuchten* [1980], and *Zimmerman and Bødvarsson* [1989a]. Equations (1-3) assume constant porosity, no hysteresis, and an infinitely mobile air phase which is always at atmospheric pressure. The implications and limitations of these standard assumptions are discussed by *Hillel* [1980] and *Bear* [1988]. An additional assumption implicit in Equation (1) is the neglect of gravity forces, which otherwise would contribute an additional term $\rho g z$ to the potential, where ρ is the density of water, g is the gravitational acceleration, and z is the vertical coordinate. Discussions of the relative magnitude of the effect of gravitational forces compared to that of capillary forces have been given by *Parlange* [1972], and *Lockington et al.* [1989]. Roughly speaking, gravity can be ignored if $\rho g L \ll \max |\psi|$, where L is a characteristic vertical dimension of the block, and $\max |\psi|$ is the characteristic pressure potential appearing in the problem. This point will be discussed further when specific cases are treated.

Approximate Solutions for Slabs, Cylinders and Spheres

Many approximate methods of analysis have been brought to bear on the problem of one-dimensional absorption of water into an unsaturated porous medium. A review and comparison of some of these approaches is given by *Brutsaert* [1976]. Most of these methods make use of the fact that the one-dimensional Richard's equation for the potential ψ admits a similarity solution of the form $\psi(x, t) = \psi(\eta)$, where $\eta = x/\sqrt{t}$. As pointed out by *Philip* [1955], transforming the equation into the new variable η will always reduce it from a partial to an ordinary differential equation, *regardless* of the specific form of the capillary pressure and relative permeability curves. Unfortunately, with the exception of certain problems such as line or point sources in an infinite media, the similarity approach does not work for most other geometries. This is because of the fact that for the problem of flow into an infinite medium, the "natural length scale" (*i.e.*, the penetration distance of the wetting front) grows with $\sqrt{t}$, and this growth is accounted for by the similarity variable. For flow into finite size blocks, however, there is always a natural length scale defined by the geometry of the problem.

As exact analytical solutions are extremely difficult to obtain for one-dimensional flow, one cannot expect to be able to derive them for more complicated geometries. However, other methods that do not rely upon the similarity transformation might be expected to provide approximate solutions to such problems. *Zimmerman and Bødvarsson* [1989a] used an integral method approach for one-dimensional absorption which does not assume a similarity solution at the outset, although it does lead to a solution in terms of $x/\sqrt{t}$ for this particular problem. For a semi-infinite medium, this method consists of using a trial saturation profile that depends on x as well as on a penetration parameter δ , where δ has an implicit dependence on t . Inserting the profile into the governing Richard's equation, and integrating the equation in the space variable from $x=0$ out to $x=\infty$, leads to a differential equation for δ as a function of t .

Integration of this ordinary differential equation provides the approximate solution of the problem. This method can be extended to geometries such as slabs, spheres, and cylinders with only slight modification of the methodology that was used for the semi-infinite case. The main idea behind this approach is to use a saturation profile similar to the 1-d solution for semi-infinite media at small times, until the wetting front reaches the center of the block. At this point the finite size of the block has made itself felt, and so for later times, the assumed saturation profiles must be chosen so as to reflect this fact. It will be seen that even if extremely simple trial profiles are chosen, the method leads to reasonably accurate predictions, particularly with respect to the cumulative flux, which is often the most important parameter.

Slabs

Consider the problem of absorption into a thin slab of thickness $2a$, with the x coordinate system defined so that $x=0$ and $x=2a$ at the outer boundaries of the slab. Both surfaces of the slab are assumed to be fully saturated and at zero potential, whereas the slab is initially partially saturated at some potential $\psi_i < 0$. By the symmetry of the problem, the midplane is a no-flow boundary. Neglecting gravity and end effects, the problem can be formulated as follows:

$$\frac{\partial}{\partial x} \left[\beta k_r(\psi) \frac{\partial \psi}{\partial x} \right] = \frac{\partial S}{\partial t}, \quad (4)$$

$$\psi(x=0, t) = \psi_w = 0, \quad (5)$$

$$\frac{\partial \psi}{\partial x}(x=a, t) = 0, \quad (6)$$

$$\psi(x, t=0) = \psi_i, \quad (7)$$

along with Equations (2) and (3) to describe the hydraulic properties of the medium. For sufficiently small periods of time, the wetting front will not have reached the mid-plane, and this problem is identical to that of 1-d flow into a semi-infinite medium. For this range of times, the following approximate solution developed by *Zimmerman and Bødvarsson* [1989a] can be used:

$$S = S_s - (S_s - S_i)(x/\delta)^n \quad \text{for } 0 < x < \delta, \quad (8a)$$

$$S = S_i \quad \text{for } x > \delta, \quad (8b)$$

$$\text{where } \delta = \left[\frac{2(n+1)\beta(S_s - S_i)^{-m} t}{\alpha n [m(S_s - S_r)]^{1/n}} \right]^{1/2}. \quad (9)$$

The time t^* at which the front reaches the midplane is found by setting $\delta = a$ and solving for t^* :

$$t^* = \frac{\alpha n a^2 [m(S_s - S_r)]^{1/n}}{2(n+1)\beta(S_s - S_i)^{-m}}. \quad (10)$$

For times greater than t^* , the simplest profile to use is

$$S = S_s - (S_s - S_c)(x/a)^n, \quad (11)$$

where S_c is the (unknown) saturation at the center of the slab. This particular trial profile has the x^n dependence near the saturated boundary that is required by the *van*

Genuchten equations [Zimmerman and Bødvarsson 1989a]. It does not satisfy the no-flow boundary condition (6) at the midplane; however, it was shown by Zimmerman and Bødvarsson [1989b] for the analogous saturated-flow problem that this discrepancy is of little importance. One can imagine a small tail on the saturation profile which adjusts the slope to zero at $x = a$, while being so localized that it contributes insignificantly to the overall mass balance.

The saturation S_c now plays a role analogous to that of δ , in that it is an open parameter whose evolution is found by integrating the governing equation over the spatial variable, using the assumed saturation profile (11). Integration of the left-hand side of Equation (4) yields

$$\begin{aligned} \int_0^a \frac{\partial}{\partial x} \left[\beta k_r(\psi) \frac{\partial \psi}{\partial x} \right] dx &= -\beta k_r(\psi_w) \frac{\partial \psi}{\partial x} \Big|_{x=0} \\ &= \beta k_r(\psi_w) \frac{\partial \psi}{\partial S} \Big|_{S=S_s} \frac{\partial S}{\partial x} \Big|_{x=0} = \frac{\beta k_r(\psi_w)}{\alpha a} \left[\frac{(S_s - S_c)}{m(S_s - S_r)} \right]^{1/n}. \end{aligned} \quad (12)$$

The right-hand side of Equation (4) integrates out to

$$\int_0^a \frac{\partial S}{\partial t} dx = \frac{d}{dt} \int_0^a [S_s - (S_s - S_c)(x/a)^n] dx = \frac{a}{n+1} \frac{dS_c}{dt}. \quad (13)$$

The evolution of S_c is therefore governed by the ordinary differential equation

$$\frac{(n+1)\beta k_r(\psi_w)}{\alpha a^2} \left[\frac{(S_s - S_c)}{m(S_s - S_r)} \right]^{1/n} = \frac{dS_c}{dt}. \quad (14)$$

The initial condition for this ODE is that S_c must equal S_i when $t=t^*$. The solution to Equation (14) that satisfies this condition is (recall that $m = 1 - 1/n$)

$$\frac{m(n+1)\beta k_r(\psi_w)[t-t^*]}{\alpha a^2[m(S_s - S_r)]^{1/n}} = [(S_s - S_i)^m - (S_s - S_c)^m]. \quad (15)$$

Note that in Equation (15), the relative permeability is evaluated at $\psi = \psi_w = 0$, which is the potential at the boundary of the slab. According to this solution, the decrease of k_r within the block due to the negative potential does not appreciably affect the rate of influx. This was found by *Zimmerman and Bödvarsson* [1989a] to be true for one-dimensional absorption into a semi-infinite region. The resistance to flow in that problem apparently occurs mainly near the imbibition surface, so to speak. When water is being absorbed into a finite size block such as the thin slab, the low relative permeability at the center seems to significantly retard the absorption process after the wetting front reaches the symmetry plane at the center of the slab. We have found, by comparison with numerical simulations (see below), that this effect can be partially accounted for by using a crude form of geometric averaging of k_r in Equation (15), in place of $k_r(\psi_w)$. If we first take a "geometric average" over space at $t=0$, from $x=0$ (where $\psi=0$) out to $x=a$ (where $\psi=\psi_i$), the effective relative permeability at $t=0$ is $[k_r(\psi_i)]^{1/2}$, since $k_r(0)=1$. At the end of the absorption process, $\psi=0$ throughout the block, and so the effective relative permeability is 1. Averaging again over time leads to an effective k_r value for the entire process of

$$\bar{k}_r = [k_r(\psi_i)]^{1/4}. \quad (16)$$

Although this averaging scheme is very simplistic, it is more accurate (see below) than

merely using $k_r(\psi_w)$ in Equation (15).

An interesting feature of this solution is that it predicts a finite time t_f at which the absorption process is complete. This is in contrast to heat or solute diffusion into a finite-size block, for which the flux tails off exponentially [Crank, 1975]. The time at which absorption is complete is found by setting $S_c = S_s$ in Equation (15), and recalling that $m = (n-1)/n$, which yields

$$t_f = t^* + \frac{\alpha a^2 [m(S_s - S_r)]^{1/n}}{m(n+1)\beta \bar{k}_r (S_s - S_i)^{-m}}$$

$$= \left[1 + \frac{2}{(n-1)\bar{k}_r} \right] \frac{\alpha n a^2 [m(S_s - S_r)]^{1/n}}{2(n+1)\beta (S_s - S_i)^{-m}}. \quad (17)$$

As would be predicted from dimensional analysis, the total time needed for complete absorption is proportional to a^2 . If the dimensionless terms in Equation (17) are neglected, we see that the time needed for full absorption is roughly on the order of $\alpha a^2/\beta = \alpha \mu \phi a^2/k$. Actually, it is difficult to determine from numerical simulations whether the absorption process ends abruptly, or asymptotically; nevertheless, the simulations discussed below show that expression (17) gives a reasonable estimation of the time at which the slab essentially reaches full saturation.

The total volume of water absorbed by the slab (per unit area) up to any time t is found by integrating that part of the saturation profile over and above the initial saturation S_i . For $t < t^*$ the saturation profile is given by Equation (8), and so

$$Q(t) = 2\phi \int_0^a [S(x,t) - S_i] dx = 2\phi(S_s - S_i) \int_0^\delta [1 - (x/\delta)^n] dx$$

$$= \frac{2n\phi(S_s - S_i)\delta(t)}{n+1} \quad \text{for } t < t^*, \quad (18)$$

where $\delta(t)$ is given by Equation (9). Note that since δ grows at a rate proportional to $\sqrt{t}$, Q also grows at this same rate. For $t > t^*$, the saturation profile is given by Equation (11), and so

$$\begin{aligned} Q(t) &= 2\phi \int_0^a [S(x,t) - S_i] dx = 2\phi \int_0^a \{[S_s - S_i] - [S_s - S_c(t)](x/a)^n\} dx \\ &= 2a\phi \left[(S_s - S_i) - \frac{S_s - S_c(t)}{n+1} \right] \quad \text{for } t > t^*, \end{aligned} \quad (19)$$

where $S_c(t)$ is given by Equation (15).

Cylinders and Spheres

Absorption into spheres or long cylinders (neglecting end effects) can be analyzed simultaneously, since the governing equations for both of these problems can be written in the following form:

$$\frac{1}{r^{d-1}} \frac{\partial}{\partial r} \left[\beta k_r(\psi) r^{d-1} \frac{\partial \psi}{\partial r} \right] = \frac{\partial S}{\partial t}, \quad (20)$$

$$\psi(r=a, t) = 0, \quad (21)$$

$$\frac{\partial \psi}{\partial r}(r=0, t) = 0, \quad (22)$$

$$\psi(r, t=0) = \psi_i, \quad (23)$$

where the dimension parameter d equals 2 for the cylinder and 3 for the sphere. Actually, the slab problem could also have been treated as a special case of Equation (20), by setting $d=1$ and re-orienting the x axis, but for clarity the slab solution was developed in its own right.

For times small enough so that the front has not reached the center of the block, the saturation profile (8) can again be used, modified so as to conform with the fact that r decreases away from the boundary:

$$S = S_s - (S_s - S_i) \left[\frac{a-r}{a-r_o} \right]^n \quad \text{for } r_o < r < a, \quad (24a)$$

$$S = S_i \quad \text{for } 0 < r < r_o, \quad (24b)$$

where $r=r_o$ is the location of the wetting front. If Equation (20) is multiplied through by r^{d-1} , and integrated from $r=0$ to $r=a$, the left-hand side becomes (ignoring the spurious flux emanating from the tail of the profile)

$$\int_0^a \frac{\partial}{\partial r} \left[r^{d-1} \beta k_r(\psi) \frac{\partial \psi}{\partial r} \right] dr = \frac{\beta a^{d-1}}{\alpha(a-r_o)} \left[\frac{(S_s - S_i)}{m(S_s - S_r)} \right]^{1/n}, \quad (25)$$

where we have used the fact that $k_r(\psi_w=0)=1$. The right-hand side integrates out to

$$\int_0^a \frac{\partial S}{\partial t} r^{d-1} dr = \int_0^{r_o} \frac{\partial S_i}{\partial t} r^{d-1} dr + \int_{r_o}^a \frac{\partial}{\partial t} \left\{ S_s - (S_s - S_i) \left[\frac{a-r}{a-r_o} \right]^n \right\} r^{d-1} dr$$

$$= n(S_s - S_i) \left[\frac{a^{d-1}}{n+1} - \frac{(d-1)(a-r_o)a^{d-2}}{n+2} + \frac{(d-2)(a-r_o)^2}{n+3} \right] \frac{dr_o}{dt}. \quad (26)$$

If we put $a - r_o = \delta$ and $\epsilon = \delta/a$ and equate (25) and (26), the result can be written as

$$\frac{\beta(S_s - S_i)^{-m}}{\alpha n a^2 [m(S_s - S_r)]^{1/n}} = \left[\frac{1}{n+1} \epsilon - \frac{(d-1)}{n+2} \epsilon^2 + \frac{(d-2)}{n+3} \epsilon^3 \right] \frac{d\epsilon}{dt}. \quad (27)$$

Now we separate the variables in Equation (27) and integrate, using the initial condition that $\epsilon = 0$ when $t = 0$:

$$\frac{2(n+1)\beta(S_s - S_i)^{-m} t}{\alpha n a^2 [m(S_s - S_r)]^{1/n}} = \epsilon^2 \left[1 - \frac{2(d-1)(n+1)}{3(n+2)} \epsilon + \frac{(d-2)(n+1)}{2(n+3)} \epsilon^2 \right]. \quad (28)$$

Equation (28) expresses the normalized penetration depth $\epsilon = \delta/a$ as an implicit function of time. Aside from the bracketed term on the right, Equation (28) is identical to Equation (9) for linear flow. For "small" times, when $\epsilon \ll 1$, the bracketed term is small, and so all three solutions (sphere, cylinder and slab) asymptotically coincide. The correction factor is always less than one, for all allowable values of n and d , which shows that a spherical or cylindrical block saturates faster than a slab of similar overall thickness. This reflects the fact that as the wetting front penetrates into the block, the cross-sectional area available to flow decreases as r^{d-1} , while the unsaturated volume remaining ahead of the front decreases at a faster rate, r^d . The time at which the front reaches the center of the block is found by setting $\epsilon = 1$, which yields

$$t^* [\text{cylinder, sphere}] = \left[1 - \frac{2(d-1)(n+1)}{3(n+2)} + \frac{(d-2)(n+1)}{2(n+3)} \right] t^* [\text{slab}]. \quad (29)$$

As n spans its allowable range from 1 to ∞ , the bracketed term in Equation (29) varies from 0.555 to 0.333 for a cylinder, and from 0.361 to 0.167 for a sphere.

After the front has reached the center of the block, a convenient form for the saturation profile can be found by analogy with Equations (8a), (11), and (24a):

$$S = S_s - (S_s - S_c) \left[\frac{a-r}{a} \right]^n \quad (30)$$

Following the previously described procedure, we multiply Equation (20) through by r^{d-1} and integrate from $r=0$ to $r=a$. The integral of the left-hand "flux" term follows immediately from Equation (25), with r_o replaced by 0 and S_i replaced by S_c :

$$\int_0^a \frac{\partial}{\partial r} \left[r^{d-1} \beta k_r(\psi) \frac{\partial \psi}{\partial r} \right] dr = \frac{\beta a^{d-2} k_r(\psi_w)}{\alpha} \left[\frac{(S_s - S_c)}{m(S_s - S_r)} \right]^{1/n} \quad (31)$$

The right-hand side of Equation (20) integrates out to

$$\begin{aligned} \int_0^a \frac{\partial S}{\partial t} r^{d-1} dr &= \int_0^a \frac{\partial}{\partial r} \left\{ S_s - (S_s - S_c) \left[\frac{a-r}{a} \right]^n \right\} r^{d-1} dr \\ &= \left[\frac{(d-1)! a^d}{(n+1)(n+2)\dots(n+d)} \right] \frac{dS_c}{dt}, \end{aligned} \quad (32)$$

where $(d-1)! = (d-1)(d-2)\dots(1)$ is the factorial function. Equating (31) and (32) leads to the following ODE for S_c as a function of time:

$$\frac{\beta k_r(\psi_w)(n+1)(n+2)\dots(n+d)}{\alpha a^2(d-1)!} \left[\frac{(S_s - S_c)}{m(S_s - S_r)} \right]^{1/n} = \frac{dS_c}{dt} \quad (33)$$

Following the results for absorption into a slab, at this point we replace $k_r(\psi_w)$ with $\bar{k}_r = [k_r(\psi_i)]^{1/4}$. The integral of Equation (33) that satisfies the initial condition $S_c(t^*) = S_i$ is

$$\frac{m \beta \bar{k}_r (n+1)(n+2)\dots(n+d)[t - t^*]}{\alpha a^2(d-1)! [m(S_s - S_r)]^{1/n}} = [(S_s - S_i)^m - (S_s - S_c)^m] \quad (34)$$

Equation (34) applies not only for cylinders and spheres, with $d=2$ or 3 , but also reproduces Equation (15) for the slab if d is set equal to 1 . (Note that most equations in this section are *not* written in a form that holds for $n=1$). The geometry-dependant term $(n+1)(n+2)\dots(n+d)/(d-1)!$ increases rapidly with d for the relevant values of $d=1-3$, so that a sphere reaches full saturation much faster than a cylinder of equal diameter, which in turn reaches full saturation faster than a slab of the same "diameter".

As was the case for the slab solution, the cumulative volumetric influx can be found by integrating the saturation profiles (24) and (30), in conjunction with Equations (28) and (34) which describe the evolution of the parameters δ and S_c with time. These calculations are straightforward, and similar to Equations (18) and (19) for the slab, so for considerations of space only the results will be given here:

$$\frac{Q(t)}{Q_\infty} = \frac{dn}{n+1} \epsilon - \frac{(2d-3)n}{n+2} \epsilon^2 + \frac{(d-2)n}{n+3} \epsilon^3 \quad \text{for } t < t^*, \quad (35)$$

$$\frac{Q(t)}{Q_\infty} = 1 - \left[\frac{S_s - S_c}{S_s - S_i} \right] \frac{d!}{(n+1)\dots(n+d)} \quad \text{for } t > t^*, \quad (36)$$

where $\epsilon(t)$ is given by Equation (28), $S_c(t)$ is given by Equation (34), and Q_∞ is the ultimate value of Q , i.e., $Q(t \rightarrow \infty)$.

Comparison with Numerical Simulations

In order to determine the accuracies of the three approximate solutions, we have compared their predictions to the results of numerical simulations. The simulations were carried out using TOUGH [Pruess, 1987], a program that uses integrated-finite-differences, and which is known to accurately model three-dimensional flow of water and air in porous media. The physical parameters used in these simulations are those that have been estimated for the Topopah Spring unit at Yucca Mountain, Nevada, the proposed site of a high-level radioactive waste repository [U.S. DOE, 1986; Peters and Klavetter, 1988]. The Topopah Spring unit is a welded volcanic tuff with an estimated matrix permeability of $3.9 \times 10^{-18} \text{ m}^2$, estimated matrix porosity of 14%, and estimated van Genuchten parameters of $n = 3.04$, $m = 0.671$, $S_s = 0.984$, $S_r = 0.318$, and $\alpha = 1.147 \times 10^{-5} \text{ Pa}^{-1}$ [Rulon et al., 1986].

The problems that were simulated correspond to Equations (4-7) for the slab, and Equations (20-23) for the sphere and cylinder. A value of $a = 0.20 \text{ m}$ was used for each geometry, which roughly corresponds to the fracture spacings of 0.22 - 0.48 m that were estimated by Wang and Narasimhan [1985]. The blocks are initially at a uniform saturation of 0.6765, after which the outer boundaries of the blocks are saturated with water at zero potential (i.e., zero excess pressure). This initial saturation seems to be in the range that has been estimated [Niemi and Bdvarsson, 1988] for the Topopah Spring unit. These simulations, however, are intended mainly as a test of the approximate solutions, and not as predictions of actual processes at Yucca Mountain, for which more accurate physical property data would be needed. The viscosity of water was taken to be 1 cp, or 0.001 kg/ms. In each case the block was divided into 20 grid blocks of equal volume. Sensitivity studies, as well as previous

experience with TOUGH, have shown that for problems such as these, further refinement of the grid leads to improvements in accuracy of only a few percent, which for the present purposes is not necessary. Note that when implementing the approximate solutions, $k_r(\psi_i)=0.061$ when $n=3.04$ and $S_i=0.6765$, and so $\bar{k}_r=[k_r(\psi_i)]^{1/4}=0.497$. In order to facilitate comparison with the approximate solutions, the gravitational acceleration was set to zero in the simulations.

The resulting saturation profiles from the approximate and numerical solutions for the slab, cylinder and sphere are shown in Figures 1-3. Note that the cylinder and sphere profiles are plotted against $(r/a)^2$ and $(r/a)^3$, respectively, so that equal distances along the abscissa represent equal pore volumes. In this way, the cumulative volumetric flux into the block is proportional to the area under the saturation curve. The cumulative volumetric liquid fluxes into the blocks are plotted in Figures 4-6 as functions of a dimensionless time $kt/\alpha\mu\phi a^2$ (a more general non-dimensionalization will be introduced below). The approximate solutions are seen to have fair accuracy in predicting the saturation profiles and the cumulative fluxes. The greatest discrepancy for the slab solution occurs near the time at which the presence of the "impermeable" boundary at the center of the block has begun to retard the absorption process. In the numerical solution this effect is seen to occur gradually, but in the approximate solution it occurs abruptly, and somewhat too late. Hence the instantaneous flux is discontinuous at $t = t^*$ in the approximate solutions, although it is smoothly varying in the numerical solutions. The approximate solutions do not properly account for the effect of the cross-sectional areas normal to the flow dropping off to zero as $r \rightarrow 0$ in the cylinder and sphere problems, and so the cumulative infiltration is increasingly overpredicted as time increases. Figures 4-6 show that the slab needs about twice as much time to reach full saturation as does the cylinder, and about three times as much time as does the sphere.

For matrix blocks of the size that are found at Yucca Mountain, it is easy to show that gravitational effects are small during the absorption process, and so their neglect was justified. Consider first the spherical block of radius a . An order-of-magnitude estimate of the gravitational potential would be ρga , while the order-of-magnitude of the pressure potential is $\max |\psi| = |\psi_i|$. Hence the ratio of gravitational forces to capillary forces can be estimated as $\rho ga / |\psi_i|$. In the problem simulated above, $a = 0.2\text{m}$ and $\psi_i = -1\text{ bar} = -1.0 \times 10^5\text{ kg/ms}^2$. Since $\rho \approx 1000\text{ kg/m}^3$, and $g = 9.8\text{ m/s}^2$, the ratio of the gravitational to capillary forces is on the order of 0.02. Blocks that are "infinitely" long in one direction, such as the one shown in Figure 8, would be used for modeling purposes only if the long axis was oriented horizontally, so that the characteristic vertical dimension would still be small enough so as not to violate the condition $\rho gL \ll |\psi_i|$. Since the magnitude of the initial capillary pressures at Yucca Mountain are thought to be at least as large as 1 bar [*Peters and Klavetter*, 1988], it is clear that gravity can safely be neglected in the analysis of absorption into matrix blocks. On a macroscopic scale, of course, gravity effects will be of major importance [*Rulon et al.*, 1986].

Scaling Law for Irregularly-Shaped Blocks

The approximate solutions that were derived for absorption into slabs, spheres, and cylinders were made possible by the fact that for these geometries, only one mathematical variable is effectively needed to describe the location of a point in the block. Even though flow in a cylinder or slab is physically three-dimensional, the symmetry of the problem reduces it to a mathematically one-dimensional problem. The problem was also simplified by the existence of appropriate coordinate systems (spherical and polar, cylindrical) that allowed the outer boundary of the block to be described by a single constant value of the mathematical variable, *i.e.*, $r = a$. For irregular blocks, such as the polygonal shapes that are formed by the intersection of

non-parallel fracture systems, the simplification allowed by these symmetries will not occur. Since approximate solutions analogous to those developed above are not obtainable for irregular blocks, it would be convenient if a rough rule-of-thumb could be developed to allow the rate of absorption into such blocks to be predicted without recourse to numerical simulation.

For diffusion problems with constant diffusivity, scaling laws for irregularly-shaped blocks have been proposed by *Barker* [1985] and *van Genuchten and Dalton* [1986]. These rules were suggested by manipulations in the Laplace transform domain that are not applicable to the present nonlinear absorption problems. A simple scaling rule relating the cumulative influx to the ratio of volume V to surface area A can be devised, however, by the following considerations. First, note that as $t \rightarrow \infty$, the amount of water absorbed by a block must be proportional to its volume. In particular, the final cumulative flux Q_∞ will equal $V\phi(S_s - S_i)$, where ϕ is the porosity and S_i is the initial liquid saturation. For very small times, on the other hand, the problem will be identical to linear one-dimensional absorption, since the penetration depth of the wetting front will be small enough that the curvature of the surface will not have made itself felt. This fact is demonstrated rigorously for absorption into spheres and cylinders with constant diffusivity by *Crank* [1975], and also pertains for the approximate solutions developed above. Hence for "small" times, $Q \approx A\sqrt{t}$, and $Q/Q_\infty \approx A\sqrt{t}/V = \sqrt{t}/(V/A)$. We now make the assumption that for larger times, Q/Q_∞ will continue to be a function of $\sqrt{t}/(V/A)$, and that this function is the *same* for all geometries. To see if this is true for the three regular shapes analyzed above, we can replot the *numerical* results for the spherical, cylindrical, and slab-like blocks of Topopah Spring welded tuff, as a function of the appropriately normalized time. A convenient form for the normalized time τ is suggested by Equation (10):

$$\tau = \frac{2(n+1)\beta(S_s - S_i)^{-m} t}{\alpha n [m(S_s - S_r)]^{1/n} (V/A)^2} = \frac{2(n+1)k(S_s - S_i)^{-m} t}{\alpha n \mu \phi [m(S_s - S_r)]^{1/n} (V/A)^2} \quad (37)$$

Note that $V/A = a$ for a slab of half-thickness a , $V/A = a/2$ for a cylinder of radius a , and $V/A = a/3$ for a sphere of radius a . Figure 7 shows that the normalized cumulative absorption curves nearly coalesce into a single curve when plotted against normalized time.

While the simulations shown in Figure 7 were all carried out under identical initial conditions, the scaling law is much more general, since the dimensionless time τ in Equation (37) incorporates all of the van Genuchten parameters, as well as the initial saturation. As an illustration of this fact, we have also run simulations for absorption into a 0.2 m thick slab of Topopah Spring welded tuff under different initial saturations. Figure 8 shows the fractional uptake curves for initial saturations of 0.50, 0.70, and 0.90, with equations (37) used to calculate the normalized time. Note that the three curves are almost exactly coincident, and so the scaling concept again shows itself to be very useful.

Since the scaling law works reasonably well for the three shapes treated above, it is worthwhile to test its validity for the type of irregular blocks which are formed by the intersection of non-parallel sets of fractures. Figure 9 shows a two-dimensional block that was chosen to have an "irregular" shape, in the sense that it does not appear slab-like or cylindrical, has both acute and obtuse angles, and has sides of various lengths. We have simulated absorption into this block with TOUGH [Pruess, 1987], using the same physical properties and boundary/initial conditions as in the simulations described above for the slab, cylinder and sphere. The dotted lines denote the boundaries of the 109 grid blocks that were used in the simulation. The volume of this block per width W in the third dimension is $1.00W \text{ m}^3$, its surface area is $4.41W \text{ m}^2$, and so its characteristic length V/A is 0.227 m. The normalized cumulative absorption rate for this block is plotted on Figure 7, where it is seen to lie extremely close to the curve for the cylinder. This suggests another rule-of-thumb, which would be to use the cylinder results for all two-dimensional prismatic blocks, and the sphere

results for all blocks that are roughly of equal dimensions in each direction. Obviously, whether or not any of the approximate solutions, or the scaling laws, would be used in a given problem depends on the accuracy required. For incorporation into double-porosity models, similar to that carried out by *Neretnieks and Rasmuson* [1984] for the related problem of radionuclide transport in fractured rock masses, it would mainly be necessary to distinguish between the different time scales at which various blocks become fully saturated. For these purposes it may well suffice to use one of the approximate solutions developed above, along with the scaling law given by Equation (37).

Conclusions

Approximate solutions have been derived for the absorption of water into porous blocks of spherical, cylindrical, and slab-like geometries. The relative permeability and capillary pressure curves were assumed to be of the van Genuchten type. The solution for the slab was found in closed-form, while the cylinder and sphere solutions require a very modest amount of iteration to find the location of the wetting fronts. The cumulative volumes of absorbed water that were predicted by these solutions compared reasonably well with numerical simulations using parameters believed to be appropriate for the Topopah Spring welded tuff from Yucca Mountain, Nevada. The predicted saturation profiles are in general fairly accurate for times up until $Q(t)/Q_{\infty} \approx 0.4$, but lose accuracy as absorption proceeds.

Based on the solutions for the three "regular" geometries, a scaling law was proposed that expresses the normalized cumulative influx $Q(t)/Q_{\infty}$ as a function of a dimensionless time parameter that is proportional to At/V , where A is the surface area of the block and V is its volume. This scaling law brings the absorption curves for the sphere, cylinder and slab into much closer agreement than when they are simply plotted as functions of time. The conjecture was made that absorption into any non-

pathological two (or three) dimensional block could be predicted by using the Q/Q_{∞} curve for a cylinder (or sphere), and then scaling the time according to the above-mentioned V/A rule. This law was tested against numerical results for absorption into a highly irregularly-shaped polygonal two-dimensional block, with extremely accurate results. These approximate solutions, extended to irregularly-shaped blocks via the scaling law, are intended to be used as "sink" terms for fracture-elements in numerical simulations of liquid flow through unsaturated fractured porous media.

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Notation

a	radius of cylinder or sphere, half-thickness of slab [m]
A	surface area of block [m ²]
d	dimension (1 for slab, 2 for cylinder, 3 for sphere)
g	gravitational acceleration [m/s ²]
k	absolute permeability [m ²]
k_r	relative permeability
$\bar{k}_r$	“average” relative permeability
L	characteristic vertical dimension of block [m]
m	van Genuchten parameter, $= 1 - 1/n$
n	van Genuchten parameter for characteristic curves
Q	cumulative liquid flux [m ³]
Q_∞	ultimate cumulative liquid flux [m ³]
r	radial coordinate for cylinder or sphere [m]
r_o	radial location of wetting front [m]
S	liquid saturation
S_c	saturation at center of block
S_i	initial saturation
S_r	residual liquid saturation
S_s	saturation at zero potential
t	time since start of absorption [s]
t^*	time when front reaches middle of block [s]
t_f	time when absorption is complete [s]
x	distance from outer face of slab [m]
V	volume of block [m ³]

Greek letters

α	van Genuchten parameter [ms^2/kg]
β	hydraulic conductivity parameter, $= k/\mu\phi$ [$\text{m}^3\text{s}/\text{kg}$]
δ	penetration depth of wetting front [m]
ε	normalized penetration depth, $= \delta/a$
η	similarity variable, $= x/\sqrt{t}$
μ	viscosity [kg/ms]
ρ	density of water [kg/m^3]
ϕ	porosity
ψ	potential [kg/ms^2]
ψ_i	initial potential [kg/ms^2]
ψ_w	potential at outer boundary [kg/ms^2]
τ	dimensionless time, defined by Eq. (37)

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Figure Captions

Fig. 1. Saturation profiles for absorption into a thin slab of Topopah Spring welded volcanic tuff. Slab thickness is 0.4 m, the initial saturation is 0.6765, and the potential at the outer boundary is zero. Physical properties of the tuff are listed in the text.

Fig. 2. Same as Figure 1, for a long cylinder of radius 0.2 m.

Fig. 3. Same as Figure 2, for a sphere of radius 0.2 m.

Fig. 4. Normalized cumulative liquid flux for slab absorption problem of Fig. 1.

Fig. 5. Normalized cumulative liquid flux for cylinder absorption problem of Fig. 2.

Fig. 6. Normalized cumulative liquid flux for sphere absorption problem of Fig. 3.

Fig. 7. Normalized cumulative liquid flux for variously-shaped blocks of Topopah Spring welded tuff, calculated using TOUGH. Time is normalized according to Equation (37). Irregular "block" is shown in Fig. 9.

Fig. 8. Normalized cumulative liquid flux into a thin slab of Topopah Spring welded tuff, for different initial saturations.

Fig. 9. Irregularly-shaped two-dimensional block used to test scaling law. Grid lines indicate the boundaries of the 109 blocks used in numerical simulation.

XBL 895-7595

Fig. 1. Saturation profiles for absorption into a thin slab of Topopah Spring welded volcanic tuff. Slab thickness is 0.4 m, the initial saturation is 0.6765, and the potential at the outer boundary is zero. Physical properties of the tuff are listed in the text.

XBL 895-7596

Fig. 2. Same as Figure 1, for a long cylinder of radius 0.2 m.

XBL 895-7597

Fig. 3. Same as Figure 2, for a sphere of radius 0.2 m.

XBL 897-7655

Fig. 4. Normalized cumulative liquid flux for slab absorption problem of Fig. 1.

XBL 897-7657

Fig. 5. Normalized cumulative liquid flux for cylinder absorption problem of Fig. 2.

XBL 897-7656

Fig. 6. Normalized cumulative liquid flux for sphere absorption problem of Fig. 3.

XBL 895-7601

Fig. 7. Normalized cumulative liquid flux for variously-shaped blocks of Topopah Spring welded tuff, calculated using TOUGH. Time is normalized according to Equation (37). Irregular "block" is shown in Fig. 9.

XBL 895-7602

Fig. 8. Normalized cumulative liquid flux into a thin slab of Topopah Spring welded tuff, for different initial saturations.

XBL 895-7603

Fig. 9. Irregularly-shaped two-dimensional block used to test scaling law. Grid lines indicate the boundaries of the 109 blocks used in numerical simulation.

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