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ABSTRACT

Solutions to the Bethe-Salpeter bootstrap equations are found in the sense that the exchanged particle and the composite state have the same mass and coupling constant to the external (i.e. bound) particles. One solution predicts a (nonexistent) $\pi\pi$ bound state at 1.76 m_π. The method of calculation is numerical and utilizes the Schwinger variational principle.

I. MODEL AND RESULTS

The Bethe-Salpeter (BS) equation provides a possible dynamics for the binding of two elementary particles (m_1, m_2) through the exchange of a third particle (M). The general spirit of single channel bootstrap calculations in the context of the BS equation is to impose conditions which force the bound state (energy E) to be the "same" particle as the exchanged meson, M. A truly self-consistent bootstrap involves the solution of nonlinear integral equations such as are described in a recent article by Harte.[1] We have not attempted to solve these difficult equations, but instead restricted ourselves in this article to a much simpler model (also proposed by Harte[2]) which can be summarized in two steps.

1. Same Mass. Using the usual ϕ^3 BS equation, (Fig. 1) adjust the elementary coupling parameter $\lambda_e \left(= g_e{}^2/(4\pi)^2\right)$ to make the bound state and exchange masses equal:

$$E = M. \tag{1.1}$$

2. Same Coupling to m_1, m_2. Compute the effective coupling parameter

$$g_c = (2\pi)^2 G_0{}^{-1} \chi(P,p)\Big|_{\text{on shell}} . \tag{1.2}$$

G_0, the free two-body propagator, and $\chi(P,p)$, the BS wavefunction, are defined in Section II; and "on shell" means all three particles (m_1, m_2, and the composite) are on their respective mass shells. As the bound-state BS equation is homogeneous, the Cutkosky-Leon normalization condition[3] can be used to fix the absolute scale. The second bootstrap condition is

$$\lambda_c = \lambda_e \tag{1.3}$$

where $\lambda_c = g_c^2/(4\pi)^2$.

Using the Schwinger variational principle,[4] we have calculated λ_c and λ_e as a function of $E(=M)$ for three representative cases (all with $\ell = 0$):

(a) ground state with $m_1 = m_2 = 1$

(b) first excited state with $m_1 = m_2 = 1$

(c) ground state with $m_1 = 1.5$, $m_2 = 0.5$

Case (a) is shown in Fig. 2. A "bootstrap" solution (i.e. the intersection of λ_e and λ_c curves) is possible for all three cases. As case (a) corresponds to the $\pi\pi$ system interacting through an isosinglet scalar meson (σ), this model predicts the occurrence of a (nonexistent) scalar meson of mass $1.76\ m_\pi$ which must be stable under strong interactions.

We wish to emphasize that the conditions 1.1 and 1.3 are not sufficient to insure a complete bootstrap because the vertices involved in the BS equation may have all three legs off the mass shell; so we have implicitly assumed that λ_c is not strongly altered by off-mass-shell effects. We feel that this assumption is the weak link in this type of bootstrap procedure and look forward to a detailed study of the nonlinear equations of Ref. 1.

II. NUMERICAL DETAILS

In the following we use a Euclidean metric with $p^2 = \underset{\sim}{p}^2 + p_4^2$. The BS equation can be written in terms of the center-of-mass momentum (P) and the relative momentum (p) as

$$G_0^{-1}(P,p)\ \chi(P,p) = \lambda_e \int V(p-q)\ \chi(q,P)\ d^4q \equiv \lambda_e\ u(p,P), \tag{2.1}$$

where

$$V(p-q) = \left\{\pi^2\ [(p-q)^2 + M^2]\right\}^{-1} \tag{2.2}$$

and

$$G_0^{-1}(P,p) = [(\mu_1 P + p)^2 + m_1^2][(\mu_2 P - p)^2 + m_2^2]\ . \tag{2.3}$$

The parameters μ_i define the relative coordinate system and are taken to be

$$\mu_1 = \omega_1/(\omega_1 + \omega_2) \quad \text{and} \quad \mu_2 = \omega_2/(\omega_1 + \omega_2),$$

where $\omega_i = (\underset{\sim}{k}^2 + m_i^2)^{\frac{1}{2}}$ and $\omega_1 + \omega_2 = E$.

The bootstrap condition 1.3 can then be expressed as

$$\lambda_c = [\pi\ \lambda_e\ u(p,P)]^2 \quad |N|^2 \Big|_{\text{on shell}} = \lambda_e, \tag{2.4}$$

where the normalization coefficient N is fixed by[3]

$$|N|^2 \int \overline{\chi}(P,p) \left(\frac{\partial G_0^{-1}}{\partial E}\right) \chi(P,p)\ d^4p = 2E. \tag{2.5}$$

The adjoint wavefunction $\overline{\chi}(P,p)$ equals $\chi(P,p)$ for $m_1 = m_2$ and $\ell = 0$ (even time parity).

For equal mass kinematics "on shell" means

$$P = (0,\ iM) \quad \text{and} \quad p^2 = (E/2)^2 - m^2. \tag{2.6}$$

The Schwinger variational form of 2.1 is[4]

$$\frac{\lambda_e}{(2\pi)^4} = \frac{\int \chi(-x)\ V(x)\ \chi(x)\ d^4x}{\int u(p)\ G_0(p)\ u(p)\ d^4p} \equiv \frac{\langle V^{-1}\rangle}{\langle G_0\rangle}\ , \tag{2.7}$$

where we have suppressed P. The coordinate-space functions in the numerator are defined by

$$V(x) = \int \exp(-i\ q\cdot x)\ V(q)\ d^4q = \frac{4\ M\ K_1(MR)}{R} \tag{2.8}$$

$$R = (\underset{\sim}{x}^2 + x_4^{\ 2})^{\frac{1}{2}}$$

$$\chi(x) = \int \exp(-i\ q\cdot x)\ \chi(q)\ \frac{d^4q}{(2\pi)^4} \tag{2.9}$$

Combining 2.5 and 2.7 leads to

$$|N|^2 = \left[\frac{d\ \lambda_e}{d\ E^2}\ \langle V^{-1}\rangle\ (2\pi)^4\right]^{-1} \tag{2.10}$$

where $\langle V^{-1}\rangle$ is evaluated in terms of the unnormalized solution used to calculate u. Solution of 2.7 automatically gives u(p) and $\langle V^{-1}\rangle$, hence λ_c can be computed directly from 2.4. It should be pointed out that by evaluating u by use of the right hand side of

2.1, the continuation of p to the physical value is reliable, provided the state is not too deeply bound, because the approximate variational solution χ is used only in the Euclidean region, where it was originally calculated.

The factor $(d\,\lambda_e/dE^2)$ is computed numerically by fixing M and varying E^2 (see Fig. 3). The cusp at threshold, which is familiar from Schrodinger theory,[5] is in part responsible for the sharp drop of the λ_c curve of Fig. 3. This cusp feature also seems to occur for $\ell = 0$ excited states (except for the "abnormal" solutions) although we have not proven this. Choosing the coefficient of the leading trial function equal to unity, we find u and $\langle V^{-1} \rangle$ to be finite at threshold. The large slope at threshold then drives $|N|^2$ to zero and hence λ_c to zero. This leads us to conjecture that there exists a whole family of excited solutions to the bootstrap conditions.

III. FURTHER COMMENTS

A few more comments about the cusplike threshold behavior of λ_e as a function of E^2 may be in order. It is easily shown that the $\ell = 0$ Schrodinger solution with a single attractive Yukawa potential has this cusp. In fact, the result of a Schwinger variational calculation with a hydrogen atom trial function (a scale factor being the variational parameter) is approximately

$$\lambda_e - \lambda_0 = \frac{(2\, m_{red}\, E_B)^{\frac{1}{2}}}{M}\left(\frac{4\,\lambda_0}{3}\right), \tag{3.1}$$

where

$$\lambda_0 = (27\, M/32\, m_{red}),$$

and leads to $d\,\lambda_e/dE^2_{total} \sim E_B^{-\frac{1}{2}}$. In the course of the derivation of this formula it is found that the $(E_B)^{\frac{1}{2}}$ term comes from the matrix element of the Green's function (unitarity cut), and a similar factor arises in the BS calculation from the Schrodinger piece of the four-dimensional Green's function.[6] When M and E_B are small compared to the external masses, the BS equation reduces to the Schrodinger equation; but we find that the general form of 3.1 as a function of E_B persists even when M is no longer small (see Fig. 3).

As an application we briefly consider the calculation of the nucleon-deuteron coupling constant. With our conventions, the

Landau-Nauenberg N/D type calculation[7] of the coupling of a composite deuteron to its constituent nucleons leads to

$$\lambda_{N/D} = \left(\frac{8}{\pi}\right) m(m\, E_B)^{\frac{1}{2}} , \tag{3.2}$$

where E_B is approximately 2 MeV and m is the nucleon mass. We have taken spinless "nucleons" bound in a pure s-state. The derivation assumes there are no CDD poles (i.e. the deuteron is purely composite), the numerator function is not a strong function of energy near the deuteron pole, and that E_B is small. It is easy to see that 2.4 and 2.10 also imply this energy dependence. Because the dominant threshhold energy dependence of λ_c comes from $|N|^2$, the form 3.1 implies

$$\lambda_c \propto \left[\frac{d\,\lambda}{dE^2}\right]^{-1} \propto E_B^{\frac{1}{2}} . \tag{3.3}$$

We have done some numerical calculations of λ_c using the techniques of Section II. The results are summarized in Table I. Note the weak dependence of the coupling strength on the potential range (see also Fig. 4).

The σ meson has been of great utility as a phenomenological tool in the fitting of NN scattering data at low energies, where it is believed to summarize the contributions of multiple π exchange. Using the "Equivalent Potential Method",[8] Binstock[9] has shown that the potential constructed from the Feynman amplitude

$$\frac{\mathcal{M}}{16\pi} = \frac{\frac{1}{2}\left(g_{\sigma NN}\, M_N\right)^2}{m_\sigma^{\ 2} - t} + (\pi,\ \rho, \cdots,\ \text{exchange terms}), \qquad (3.4)$$

with $m_\sigma \simeq 2\, m_\pi$ and $g_{\sigma NN} \simeq 0.9$, provides a good fit to the low energy singlet data in the higher partial waves. In an attempt to get some correlation to our bootstrap result, we blindly put $\lambda_{NN\sigma} = \lambda_{\pi\pi\sigma}$, the latter being $3\, m_\pi^{\ 2}$ from our bootstrap result, to find

$$g_{\sigma NN} = \left[\frac{4\ \pi\ \lambda}{M_N^{\ 2}}\right]^{\frac{1}{2}} \simeq 0.8,$$

which is embarrassingly close to the phenomenological fit. We see no a priori justification for setting the $NN\sigma$ and π-π-σ couplings equal, however.

Finally, as the variational calculation gives us an approximate wavefunction, we could calculate the σ mean radius; but we have not at present undertaken this.

IV. CONCLUSION

If we assume that the general bootstrap approach is valid, the unphysical solutions which we have found can be due to several things.

(a) The external lines (π mesons) have been taken to be elementary with only the bound state (σ meson) as composite [Fig. 5(a)]. A true bootstrap would have to produce the π's as composite states as well. If π is mainly formed out of, say, bound $\bar{N}N$ systems, then our treatment of the π as elementary is not important in this context. If, however, the π is to a large extent a σ-π bound state, then for any reasonable self-consistent result we would have to include processes as shown in Fig. 5(b). We can easily see that the π exchange potential shown there is not sufficient to give a bootstrap solution because a bound state of σ-π with the mass of a π needs an (elementary) coupling of 7.15, while we have found a result of approximately 3. The next extension would be to include a σ-σ-σ coupling. This would require the solution of the system shown in Figs. 5(b), (c), (d). This problem can be treated by our methods, but we have not undertaken it.

(b) As we pointed out at the close of Section I, the assumption that λ_c is not strongly dependent on off-mass-shell effects is suspect. The coupling λ_c was evaluated for timelike values of the total momentum; however, the <u>exchanged</u> particle momentum is to a large extent spacelike. As the equations introduced in Ref. 1 do not suffer from this defect, we hope that further study of these equations (now in progress) will clarify this point.

V. ACKNOWLEDGMENTS

I would like to thank Dr. John Harte for suggesting this problem and for his help, Professor Charles Schwartz for suggesting the easy and interesting way of computing N, and Dr. Bert McInnis for his interest and comments.

FOOTNOTES AND REFERENCES

* This work was done under the auspices of the U.S. Atomic Energy Commission.

1. J. Harte, Crossing-Symmetric Bootstrap and Exponentially Falling Form Factors, Lawrence Radiation Laboratory Report UCRL-17735, August 1967. See also R.E. Cutkosky, Phys. Rev. 154, 1375 (1967).

2. J. Harte, Nuovo Cimento 45, 179 (1966).

3. R. Cutkosky and M. Leon, Phys. Rev. 138, B667 (1965). The potential V is not an explicit function of E. The condition $M = E$ merely serves to pick out a particular solution from an already normalized set of solutions.

4. C. Schwartz and C. Zemach, Phys. Rev. 141, 1454 (1966). The techniques developed here are easily generalized to include the case $m_1 \neq m_2$. The reader is referred to this article for a detailed discussion of the method of solution of the BS equation which we have used.

5. For example, look at the Bohr formula! For the BS equation we find empirically $(E_{thres}-E) \sim c(\lambda_e - \lambda_{thres})^2$ near threshold, where c is a constant. It follows that $d\lambda_e/dE^2 \sim [4\ c\ E(\lambda_e - \lambda_{thres})]^{-1}$.

6. See Ref. 4, Eq. 2.36 for the Green's function as the sum of a Schrodinger Green's function and a "correction" term. The correction term does not have a singularity at $E_B = 0$. Alternately we

can extract the $E_B^{\frac{1}{2}}$ behavior of G from the small p behavior of their momentum space expression B 15.

7. See, for example, G. Barton, Dispersion Techniques in Field Theory (W. A. Benjamin, Inc., New York, 1965), p. 196. The elementary coupling to bind a deuteron was also calculated by S. H. Vosko, J. Math. Phys. 1, 505 (1960).

8. See J. Finkelstein, (Ph.D. thesis), Lawrence Radiation Laboratory Report UCRL-17311, Jan. 1967, for the formulation used.

9. J. Binstock, private communication. I would like to thank Dr. Binstock for supplying these numbers. The conventions used are those of Ref. 7.

Table I. Results of the "deuteron" calculation. The parameters are $m = m_1 = m_2 = 0.938$, and $E_B = 0.002$. The exchange mass is M.

M	λ_e	λ_c	$\lambda_{N/D}$
0.14	0.112	0.19	0.125
0.25	0.183	0.16	0.125
0.50	0.364	0.13	0.125
0.76	0.578	0.12	0.125

FIGURE CAPTIONS

Fig. 1. The Bethe-Salpeter equation for a composite system of m_1 and m_2 bound by exchange of meson M. The total energy of the bound state is E.

Fig. 2. Coupling constant bootstrap condition. The BS equation is solved for various E subject to the condition $M = E$. The eigenvalue λ_e is seen to coincide with the composite coupling constant λ_c calculated from the eigenfunction and the normalization condition at $E \approx 1.8$.

Fig. 3. Elementary coupling constant as a function of E^2. Note that both the ground state and excited states have a cusp at threshold.

Fig. 4. Elementary and composite couplings for fixed exchange mass ($M = 1.6$). The intersection at $E = 1.8$ does not correspond to a bootstrap. The curve labeled $\lambda_{N/D}$ and λ_c are the same as threshold is approached. Due to the approximations made in the derivation of $\lambda_{N/D}$ the only reliable points are these threshold values; i.e. $E_B \ll m_{red}/2 = 0.25$.

Fig. 5. Some generalizations. We have found bootstrap solutions to (a). To include the π as composite, at least (b) and (a) would have to be solved simultaneously. This is incompatible with the bootstrap conditions. If a σ-σ-σ interaction is included, (b), (c), and (d) would have to solved simultaneously.

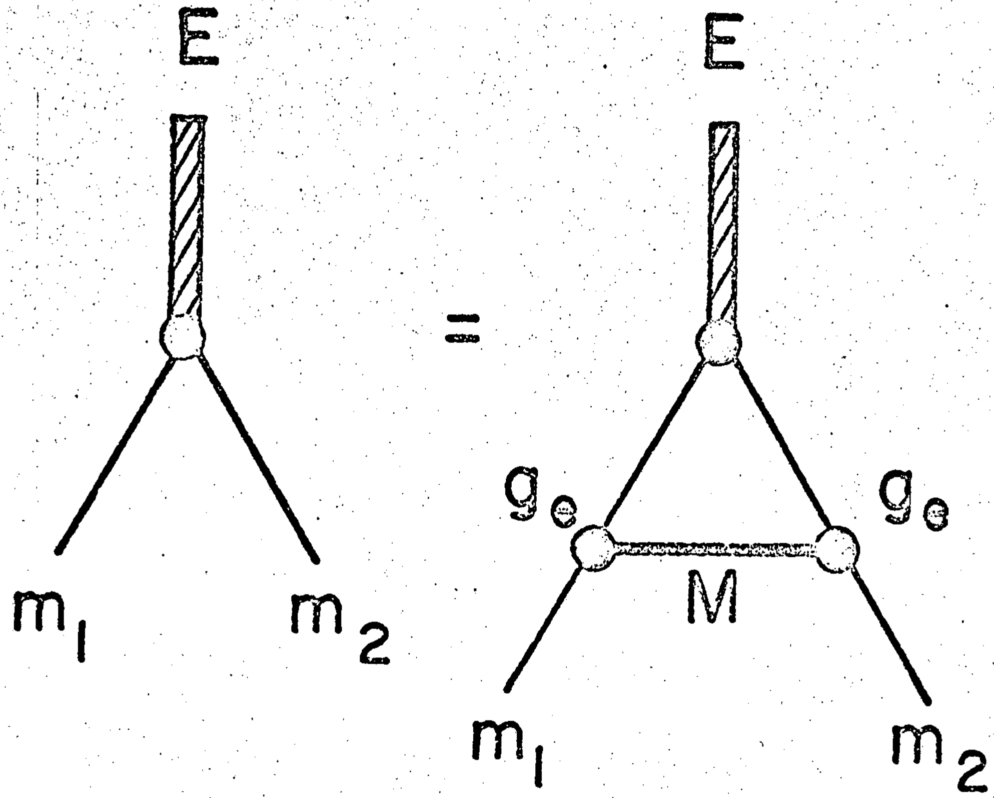

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Fig. 1

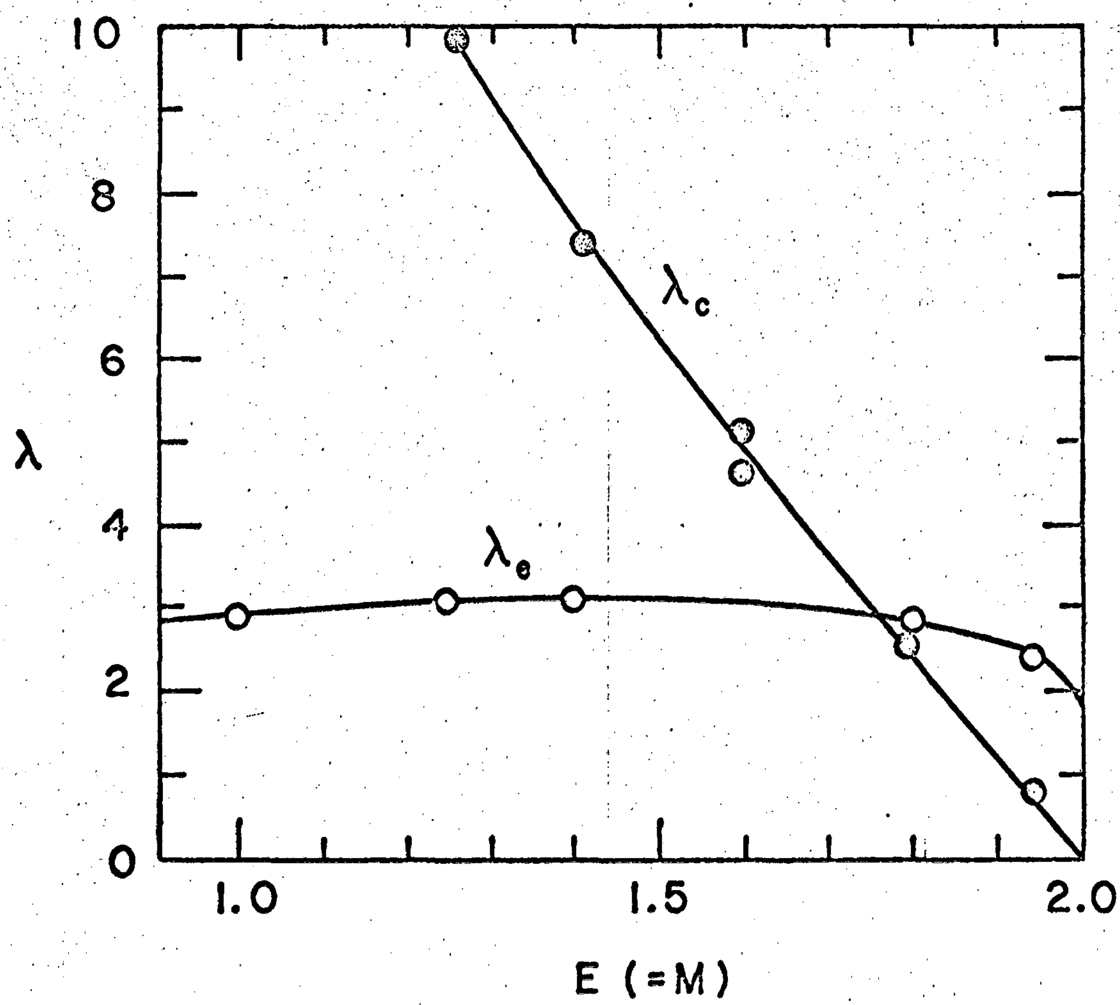

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Fig. 2

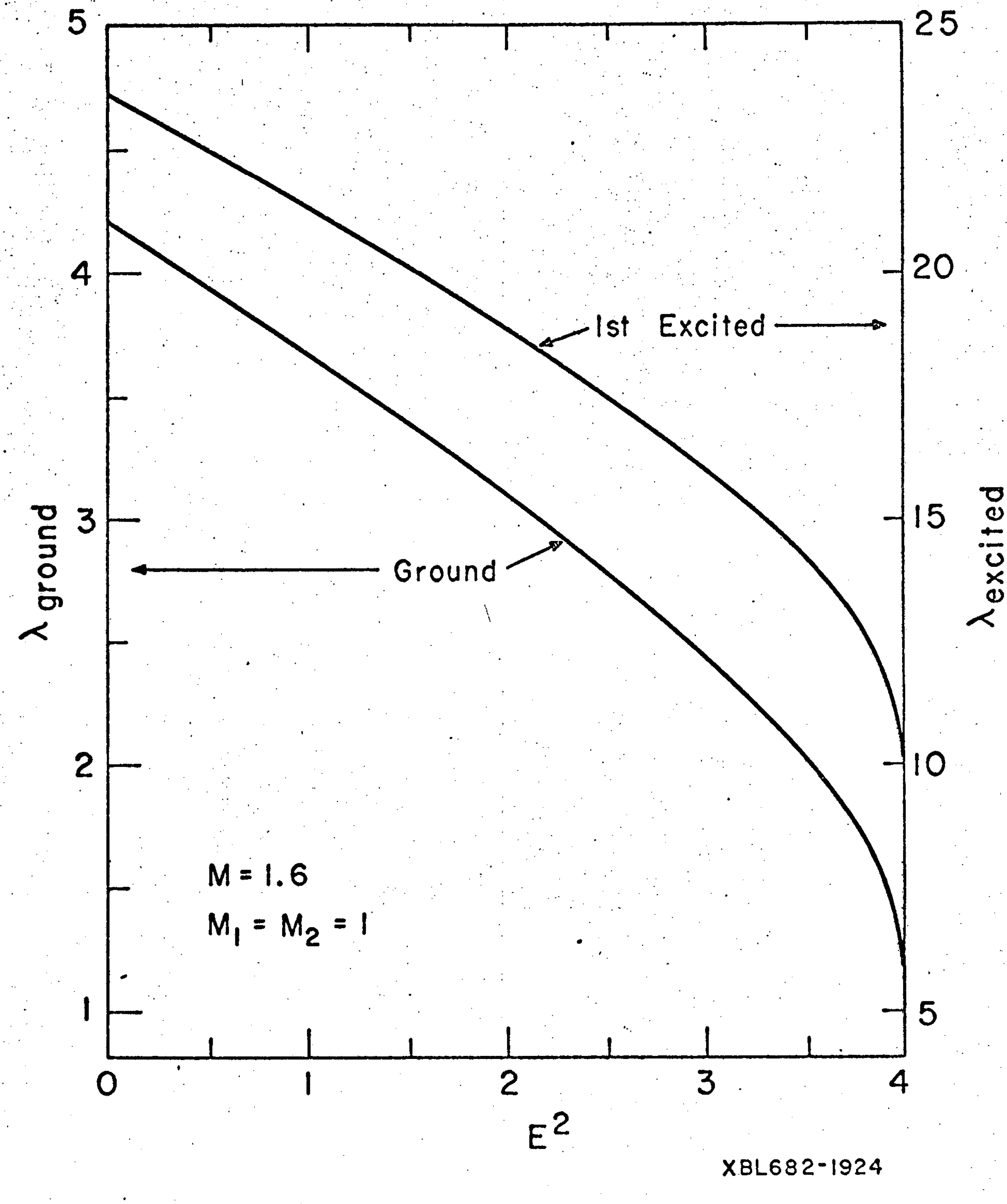
5
4
3
2
1
25
20
15
10
5
0
1
2
3
4
E^2
λ_{ground}
$\lambda_{excited}$
1st Excited
Ground
M = 1.6
$M_1 = M_2 = 1$
XBL682-1924

Fig. 3

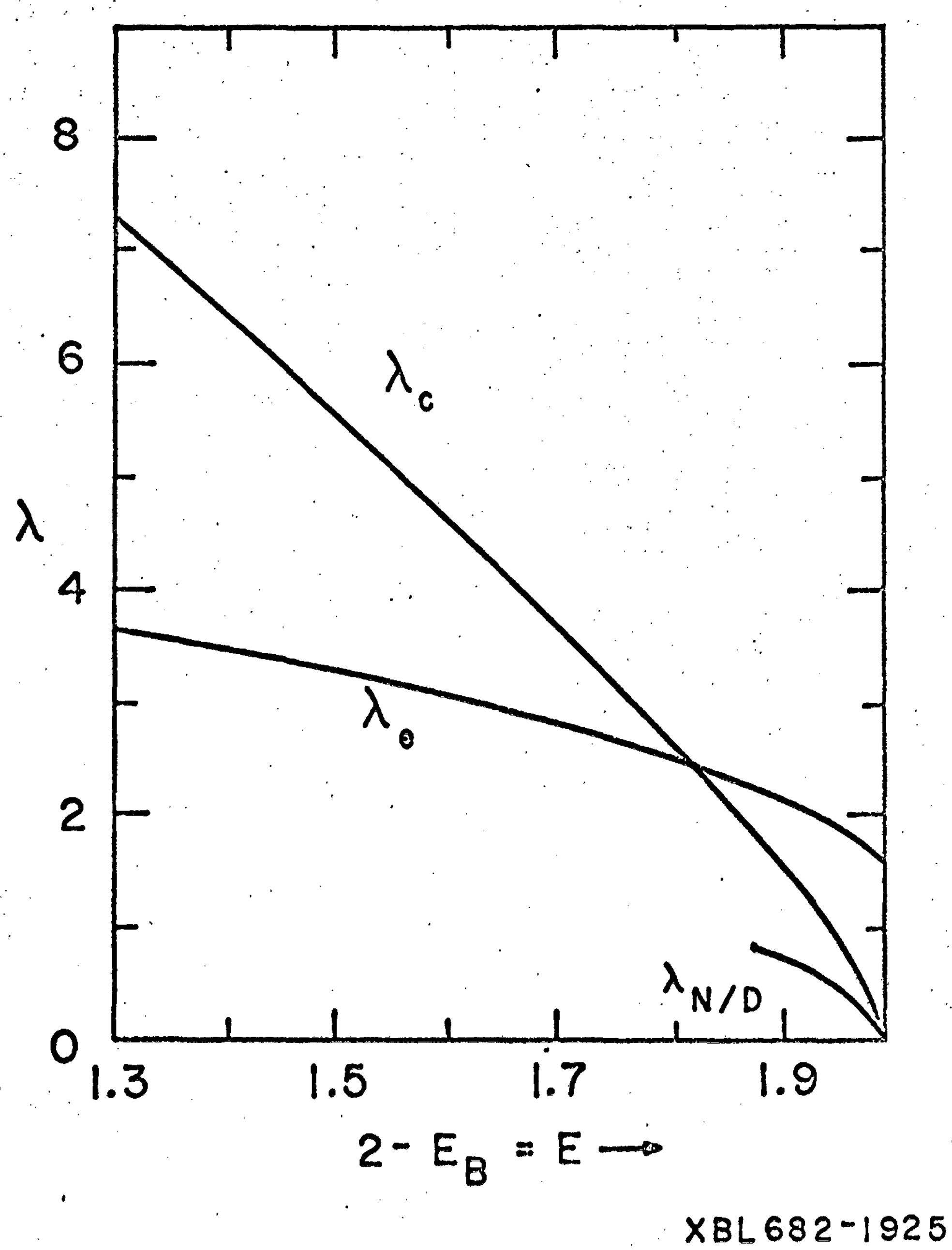

Fig. 4

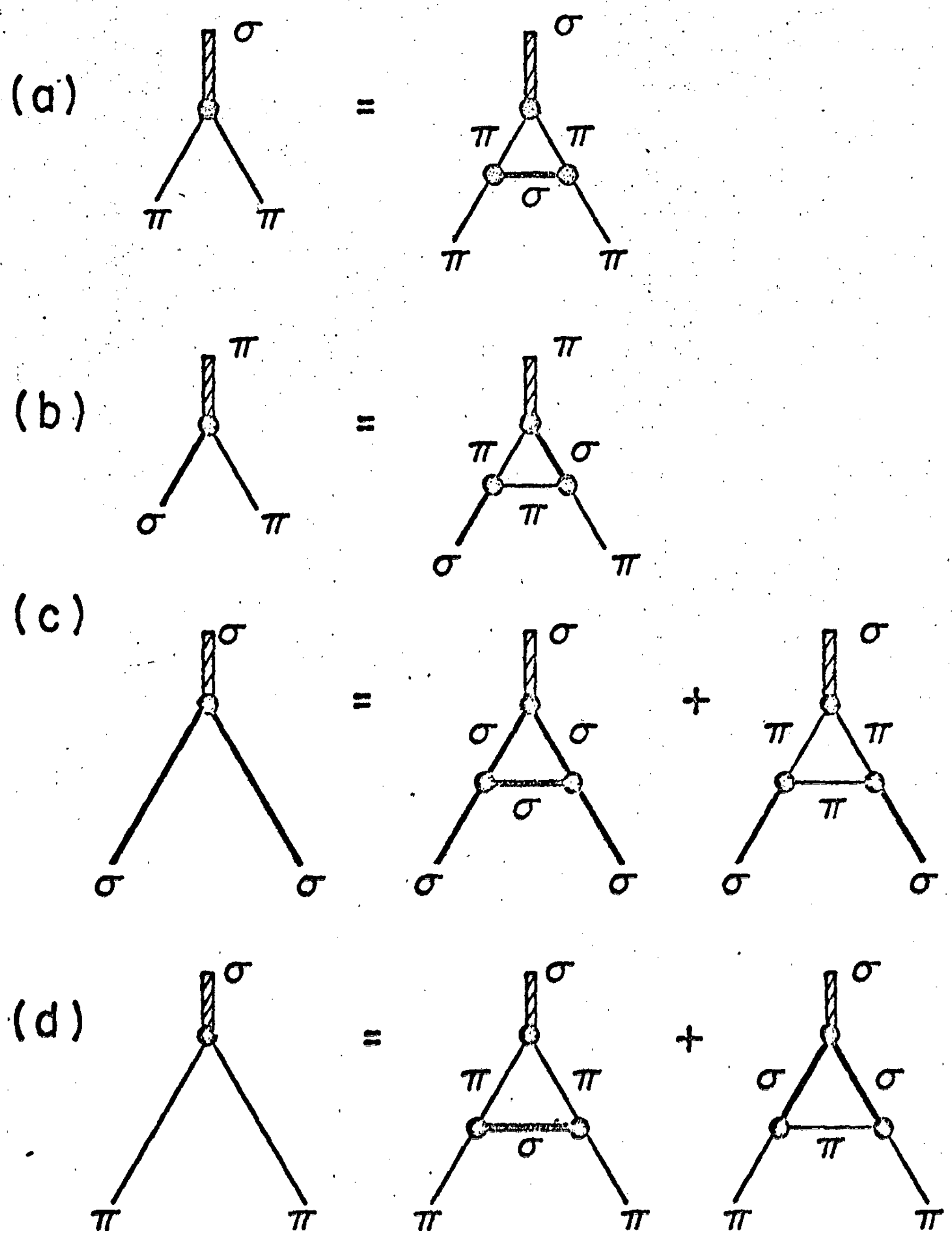

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Fig. 5