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Polyfolds and Persistence

by

Benjamin Filippenko

A dissertation submitted in partial satisfaction of the

requirements for the degree of

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in

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of the

University of California, Berkeley

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Associate Professor Katrin Wehrheim, Chair

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Polyfolds and Persistence

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Abstract

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Doctor of Philosophy in Mathematics

University of California, Berkeley

Associate Professor Katrin Wehrheim, Chair

This thesis contains polyfold constructions with applications to symplectic topology, as well as Künneth formulas for persistent homology of various filtrations with applications to Topological Data Analysis.

In symplectic topology, we provide a polyfold version of the Piunikhin-Salamon-Schwarz proof of the Arnold conjecture – our proof holds for general closed symplectic manifolds. This proof relies on the polyfold regularization of moduli spaces with a finite dimensional constraint imposed on evaluation maps, for which we provide multiple polyfold constructions that can be applied as black boxes in general situations. One of these black boxes can be viewed as an implicit function theorem: we construct a polyfold structure on the subset of a polyfold cut out by a submersive finite dimensional constraint, and then we prove the sc-Fredholm property for the restriction of a sc-Fredholm section to this subset. We go on to further investigate when an implicit function theorem holds in polyfold theory: We give explicit counterexamples to a general implicit function theorem for sc-smooth maps, and we show how the novel notion of a sc-Fredholm map overcomes this difficulty, justifying the technical complexity of polyfold theory.

In Topological Data Analysis, we prove a Künneth formula in low homological dimensions for the persistent homology of a Cartesian product $X \times Y$ of finite metric spaces equipped with the sum metric $d_X + d_Y$. In all homological dimensions, we bound the interleaving distance between the prediction from the Künneth formula and the true persistent homology. As preliminary results of independent interest, we prove Künneth formulas in all homological dimensions for the persistent homology of the Cartesian product of $\mathbb{R}_+$ -filtered simplicial sets and also for the homology of the graded tensor product of simplicial $\kappa[\mathbb{R}_+]$ -modules.

To my parents, Reka and Ivan Filippenko.

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Chapter 1

Polyfold regularization of constrained moduli spaces

This chapter is a reproduction of the paper [15].

Abstract:

We introduce tame sc-Fredholm sections and slices of sc-Fredholm sections. A slice is a notion of subpolyfold that is compatible with the sc-Fredholm section and has finite locally constant codimension. We prove that a sc-Fredholm section restricted to a slice is a tame sc-Fredholm section with a drop in Fredholm index given by the codimension of the slice. Moreover, we prove that the subspace of a tame polyfold that satisfies a transverse sc-smooth constraint in a finite dimensional smooth manifold is a slice of any tame sc-Fredholm section that is compatible with the constraint. As a corollary, we obtain fiber products of tame sc-Fredholm sections. We describe applications to Gromov-Witten invariants, the construction [16] of the Piunikhin-Salamon-Schwarz maps for general closed symplectic manifolds, and avoiding sphere bubbles in moduli spaces of pseudoholomorphic curves of expected dimension 0 and 1.

1.1 Introduction

Polyfold theory, developed by Hofer-Wysocki-Zehnder [28] [29] [30] [31] [32] [33] [34], is an analog of classical nonlinear Fredholm theory designed to realize compact moduli spaces, e.g. Gromov-Witten moduli spaces and Floer trajectory spaces, as zero sets of sc-Fredholm sections of polyfold bundles. See the polyfold survey [14] for an overview and a discussion of applications. The abstract polyfold machinery provides perturbations such that the perturbed sc-Fredholm section is transverse to zero, and hence the perturbed solution space has smooth structure by a polyfold implicit function theorem. Crucially, the perturbations can be chosen so that the perturbed solution space remains compact. This process, beginning with the description of the compact moduli space as the zero set of a sc-Fredholm section and ending with the smooth compact perturbed solution space, is colloquially referred to as “polyfold regularization” of a moduli space.

Often in symplectic topology we wish to constrain moduli spaces of pseudoholomorphic curves to consist of those curves satisfying intersection conditions with submanifolds. For example, Gromov-Witten invariants can be defined as counts of curves whose marked points evaluate to submanifolds. Any fiber product of moduli spaces over evaluation maps is another example of such a constraint. The evaluation maps are usually not transverse on the moduli space, however they extend to the ambient polyfold and here they are submersive. Using this transversality, we construct in this paper the constrained polyfold and the constrained sc-Fredholm section. This provides an abstract tool to regularize constrained moduli spaces, whenever the original moduli space is given as the zero set of a sc-Fredholm section.

We state our theorems in Section 1.1.1. Then in Section 1.1.2 we explain applications to Gromov-Witten invariants (Section 1.1.2.1), constructing the Piunikhin-Salamon-Schwarz maps (Section 1.1.2.2) to prove the weak Arnold conjecture (see [16] for details), and avoiding sphere bubbles in perturbed moduli spaces of expected dimension 0 and 1 (Section 1.1.2.3).

1.1.1 Results

The main goal of this paper is to prove the M -polyfold and ep-groupoid (with boundary and corners) versions of the following classical Facts 1.1.1, 1.1.2 from non-linear Fredholm theory over Banach manifolds.

Fact 1.1.1. (Restrictions of Fredholm sections to sub-Banach manifolds) *Consider a Banach manifold B , a smooth Banach bundle $p : E \rightarrow B$, and a Fredholm section $s : B \rightarrow E$ with Fredholm index $\text{ind}_x(s)$ for $x \in B$. If $\tilde{B} \subset B$ is a codimension- n sub-Banach manifold, then the restriction $\tilde{p} : \tilde{E} := p^{-1}(\tilde{B}) \rightarrow \tilde{B}$ is a smooth Banach bundle and the restricted section $\tilde{s} := s|_{\tilde{B}} : \tilde{B} \rightarrow \tilde{E}$ is a Fredholm section with Fredholm index satisfying $\text{ind}_x(\tilde{s}) = \text{ind}_x(s) - n$ for $x \in \tilde{B}$. $\square$*

The proof of Fact 1.1.1 is an exercise in differential topology and functional analysis. Then Fact 1.1.2 follows from Fact 1.1.1 together with the codimension- n Banach manifold charts provided by the normal form of a C^1 local submersion to $\mathbb{R}^n$, for which we provide a proof (in the context of boundary and corners) in Lemma 1.2.1 for later use.

Fact 1.1.2. (Transverse preimages are sub-Banach manifolds) *Consider a Banach manifold B , a finite dimensional smooth manifold Y together with a codimension- n submanifold $N \subset Y$, and a smooth map $f : B \rightarrow Y$. Assume that f is transverse to N .*

Then, $\tilde{B} := f^{-1}(N)$ is a codimension- n sub-Banach manifold of B . In particular, if $s : B \rightarrow E$ is a Fredholm section of a smooth Banach bundle $p : E \rightarrow B$, then the restriction $\tilde{p} : \tilde{E} := p^{-1}(\tilde{B}) \rightarrow \tilde{B}$ is a smooth Banach bundle and the restricted section $\tilde{s} := s|_{\tilde{B}} : \tilde{B} \rightarrow \tilde{E}$ is a Fredholm section with Fredholm index satisfying $\text{ind}_x(\tilde{s}) = \text{ind}_x(s) - n$ for $x \in \tilde{B}$. $\square$

The M -polyfold versions (with boundary and corners) of the above facts are our main theorems, Theorem 1.1.3 and Theorem 1.1.5, which we state in this section and prove in Sections 1.5.2, 1.5.3. The generalizations of these theorems to the ep-groupoid case, which are required in applications to handle nontrivial isotropy groups, are Corollary 1.6.7 and Corollary 1.6.8. We obtain fiber products of tame sc-Fredholm sections as Corollary 1.7.3.

Throughout, we denote M -polyfolds $\mathcal{B}$, strong M -polyfold bundles $\rho : \mathcal{E} \rightarrow \mathcal{B}$, and sc-Fredholm sections $\sigma : \mathcal{B} \rightarrow \mathcal{E}$.

The central objects developed in this paper are tame sc-Fredholm sections $\sigma : \mathcal{B} \rightarrow \mathcal{E}$ (Definition 1.5.4) and slices $\tilde{\mathcal{B}} \subset \mathcal{B}$ (Definition 1.5.7) of sc-Fredholm sections. A tame sc-Fredholm section is a new notion that is stronger than a sc-Fredholm section (Definition 1.5.3). To be tame, we require the change of coordinates that brings the local sc-Fredholm fillers into basic germ form to be a linear sc-isomorphism on the base. A slice is a new notion of a finite codimension M -polyfold $\tilde{\mathcal{B}}$ embedded in $\mathcal{B}$. These notions are related by our main Theorems 1.1.3, 1.1.5. Roughly, given a tame sc-Fredholm section $\sigma : \mathcal{B} \rightarrow \mathcal{E}$ and a sc-smooth map $f : \mathcal{B} \rightarrow Y$ to a finite dimensional manifold Y that is σ -compatibly transverse (Definition 1.5.8) to a submanifold $N \subset Y$, then $f^{-1}(N)$ is a slice of σ . Moreover, given a slice $\tilde{\mathcal{B}} \subset \mathcal{B}$ of a sc-Fredholm section $\sigma : \mathcal{B} \rightarrow \mathcal{E}$, the restriction $\sigma|_{\tilde{\mathcal{B}}}$ is tame sc-Fredholm. In Section 1.5.1, we explain why the Cauchy-Riemann section $\bar{\partial}_J : \mathcal{B} \rightarrow \mathcal{E}$ is

tame sc-Fredholm and why evaluation maps $ev : \mathcal{B} \rightarrow Y$ at marked points are $\bar{\partial}_J$ -compatibly transverse to every submanifold $N \subset Y$.

Before we state the theorems, we briefly recall some polyfold notation. See Section 1.5 for more detail. Given a M -polyfold $\mathcal{B}$, there is a filtration $\mathcal{B} = \mathcal{B}_0 \supset \mathcal{B}_1 \supset \cdots$ induced by the sc-structures in local charts. Each filtration level $\mathcal{B}_m$ has its own topology which is not the subspace topology, but the inclusions are continuous and dense. The smooth points of $\mathcal{B}$ are the subset $\mathcal{B}_\infty := \cap_{m \geq 0} \mathcal{B}_m$, which is also dense in $\mathcal{B}$.

For $x \in \mathcal{B}$, the degeneracy index $d_{\mathcal{B}}(x)$ is the number of distinct local boundary faces of $\mathcal{B}$ intersecting at x .

The following result is the M -polyfold analog of Fact 1.1.1; see Section 1.5.2 for the proof. See Corollary 1.6.7 for the generalization to ep-groupoids.

Theorem 1.1.3. (Restrictions of sc-Fredholm sections to slices)

- (I) Consider a tame M -polyfold $\mathcal{B}$ and a slice $\tilde{\mathcal{B}} \subset \mathcal{B}$ (Definition 1.5.7). Then, $\tilde{\mathcal{B}}$ is a tame M -polyfold with atlas induced by the sliced charts with respect to $\tilde{\mathcal{B}} \subset \mathcal{B}$. For $x \in \tilde{\mathcal{B}}_1$, the codimension $\text{codim}_x(\tilde{\mathcal{B}} \subset \mathcal{B})$ is well-defined and locally constant in $\tilde{\mathcal{B}}$, i.e. it is equal to $\text{codim}_x(\tilde{\mathcal{B}} \subset \mathcal{B})$ in an open neighborhood of x in $\tilde{\mathcal{B}}$. For $x \in \tilde{\mathcal{B}}_\infty$, the degeneracy index satisfies $d_{\tilde{\mathcal{B}}}(x) = d_{\mathcal{B}}(x)$.
- (II) Consider, in addition, a tame strong bundle $\rho : \mathcal{E} \rightarrow \mathcal{B}$. If $\tilde{\mathcal{B}} \subset \mathcal{B}$ is a slice of ρ , then the restriction $\tilde{\rho} := \rho|_{\tilde{\mathcal{E}}} : \tilde{\mathcal{E}} := \rho^{-1}(\tilde{\mathcal{B}}) \rightarrow \tilde{\mathcal{B}}$ is a tame strong bundle with atlas induced by the sliced bundle charts for ρ with respect to $\tilde{\mathcal{B}} \subset \mathcal{B}$.
- (III) Consider, in addition, a sc-Fredholm section $\sigma : \mathcal{B} \rightarrow \mathcal{E}$. If $\tilde{\mathcal{B}} \subset \mathcal{B}$ is a slice of σ , then the restriction $\tilde{\sigma} = \sigma|_{\tilde{\mathcal{B}}} : \tilde{\mathcal{B}} \rightarrow \tilde{\mathcal{E}}$ is a tame sc-Fredholm section (Definition 1.5.4) of $\tilde{\rho}$ with tame sc-Fredholm charts induced by the sliced sc-Fredholm charts for σ with respect to $\tilde{\mathcal{B}} \subset \mathcal{B}$. For $x \in \tilde{\mathcal{B}}_\infty$, the index satisfies $\text{ind}_x(\tilde{\sigma}) = \text{ind}_x(\sigma) - \text{codim}_x(\tilde{\mathcal{B}} \subset \mathcal{B})$. If $\sigma^{-1}(0)$ is compact and $\tilde{\mathcal{B}}_\infty \subset \mathcal{B}_\infty$ is closed, then $\tilde{\sigma}^{-1}(0)$ is compact.

Remark 1.1.4. There are three notions of a slice $\tilde{\mathcal{B}} \subset \mathcal{B}$ (Definition 1.5.7) appearing in Theorem 1.1.3: (I) a slice $\tilde{\mathcal{B}} \subset \mathcal{B}$ of a tame M -polyfold, (II) a slice $\tilde{\mathcal{B}} \subset \mathcal{B}$ of a tame strong bundle $\rho : \mathcal{E} \rightarrow \mathcal{B}$, and (III) a slice $\tilde{\mathcal{B}} \subset \mathcal{B}$ of a sc-Fredholm section $\sigma : \mathcal{B} \rightarrow \mathcal{E}$. Each successive notion requires further compatibility of the subset $\tilde{\mathcal{B}} \subset \mathcal{B}$ with the additional structure. This is in contrast to the Banach manifold situation in Fact 1.1.1 where bundles and Fredholm sections automatically restrict to any finite codimension sub-Banach manifold $\tilde{B} \subset B$.

The required compatibilities are roughly as follows. (I) There are charts on $\mathcal{B}$ to $\mathbb{R}^n$ -sliced sc-retracts $\mathcal{O}$ (Definition 1.3.2) that locally identify $\tilde{\mathcal{B}} \subset \mathcal{B}$ with the induced tame sc-retract $\tilde{\mathcal{O}} \subset \mathcal{O}$ from Lemma 1.3.3. (II) There are bundle charts on ρ to $\mathbb{R}^n$ -sliced bundle retracts K (Definition 1.3.4) covering $\mathbb{R}^n$ -sliced sc-retracts $\mathcal{O}$. In this case, $\rho^{-1}(\tilde{\mathcal{B}})$ is locally identified with the induced tame bundle retract $\tilde{K} \subset K$ from Lemma 1.3.5, and the restriction

$\rho^{-1}(\tilde{\mathcal{B}}) \rightarrow \tilde{\mathcal{B}}$ is locally identified with the induced tame local bundle model $\tilde{K} \rightarrow \tilde{\mathcal{O}}$. (III) There are sc-Fredholm charts for σ at every $x \in \tilde{\mathcal{B}}_\infty$ to $\mathbb{R}^n$ -sliced sc-Fredholm germs $\mathcal{O} \rightarrow K$ (Definition 1.3.8). In this case, the restriction $\sigma|_{\tilde{\mathcal{B}}} : \tilde{\mathcal{B}} \rightarrow \tilde{\mathcal{E}}$ is locally identified with the induced tame sc-Fredholm germ $\tilde{\mathcal{O}} \rightarrow \tilde{K}$ from Lemma 1.3.10.

The reason for the further requirements in the M -polyfold setting is the non-trivial sc-retractions and sc-Fredholm fillings: compatibility of $\tilde{\mathcal{B}}$ with the sc-retractions on $\mathcal{B}$ does not imply compatibility of $\tilde{\mathcal{B}}$ with the bundle retractions on $\mathcal{E}$ or with the local fillings of σ .

We now state our main theorem, which is the M -polyfold analog of the classical Fact 1.1.2; see Section 1.5.3 for the proof, and Corollary 1.6.8 for the generalization to ep-groupoids. See the following Remark 1.1.6 for a discussion of the technicalities in the statement. Given a M -polyfold $\mathcal{B}$, there is a m -shifted M -polyfold $\mathcal{B}^m$ for each $m \geq 1$ which is obtained by forgetting about the filtration levels $\mathcal{B}_0, \dots, \mathcal{B}_{m-1}$ of $\mathcal{B}$ discussed above.

Theorem 1.1.5. (Transverse preimages are slices of sc-Fredholm sections)

(I) Consider a tame M -polyfold $\mathcal{B}$, a smooth manifold Y together with a codimension- n submanifold $N \subset Y$, and a sc-smooth map $f : \mathcal{B} \rightarrow Y$. Assume that f is transverse to N (Definition 1.5.8).

Then, there exists an open neighborhood

$$\tilde{\mathcal{B}} \subset f^{-1}(N) \cap \mathcal{B}_1$$

of $f^{-1}(N) \cap \mathcal{B}_\infty$ such that $\tilde{\mathcal{B}}$ is a slice of $\mathcal{B}^1$ with $\text{codim}_x(\tilde{\mathcal{B}} \subset \mathcal{B}^1) = n$ for every $x \in \tilde{\mathcal{B}}_1 = \tilde{\mathcal{B}} \cap \mathcal{B}_2$. In particular, $\tilde{\mathcal{B}}$ is a tame M -polyfold with degeneracy index satisfying $d_{\tilde{\mathcal{B}}}(x) = d_{\mathcal{B}}(x)$ for all $x \in \tilde{\mathcal{B}}_\infty$.

(II) Consider, in addition, a tame strong bundle $\rho : \mathcal{E} \rightarrow \mathcal{B}$. Then, there exists a possibly smaller neighborhood $\tilde{\mathcal{B}}$ in (I) that is a slice of the bundle $\rho|_{\mathcal{E}^1} : \mathcal{E}^1 \rightarrow \mathcal{B}^1$. In particular, the restriction

$$\tilde{\rho} := \rho|_{\tilde{\mathcal{E}}} : \tilde{\mathcal{E}} := (\rho|_{\mathcal{E}^1})^{-1}(\tilde{\mathcal{B}}) \rightarrow \tilde{\mathcal{B}}$$

is a tame strong bundle.

(III) Consider, in addition, a tame sc-Fredholm section $\sigma : \mathcal{B} \rightarrow \mathcal{E}$ (Definition 1.5.4) of ρ . Assume that f is σ -compatibly transverse to N (Definition 1.5.8). Then, there exists a possibly smaller neighborhood $\tilde{\mathcal{B}}$ in (II) that is a slice of the tame sc-Fredholm section $\sigma|_{\mathcal{B}^1} : \mathcal{B}^1 \rightarrow \mathcal{E}^1$. In particular, the restriction

$$\tilde{\sigma} := \sigma|_{\tilde{\mathcal{B}}} : \tilde{\mathcal{B}} \rightarrow \tilde{\mathcal{E}}$$

is a tame sc-Fredholm section of $\tilde{\rho}$ with index satisfying $\text{ind}_x(\tilde{\sigma}) = \text{ind}_x(\sigma) - n$ for all $x \in \tilde{\mathcal{B}}_\infty$. If N is closed as a subset of Y and $\sigma^{-1}(0)$ is compact, then $\tilde{\sigma}^{-1}(0)$ is compact.

Remark 1.1.6.

- (i) The notion of σ -compatibly transverse (Definition 1.5.8) requires compatibility between the tangent map $D_x f$ at $x \in f^{-1}(N) \cap \mathcal{B}_\infty$ with the change of coordinates on the base of the local sc-Fredholm filling of σ at x that brings the filling into basic germ form. See Section 1.5.1 for an explanation why evaluation maps $f = \text{ev}$ at marked points are compatible with the Cauchy-Riemann section $\sigma = \bar{\partial}_J$ in applications.
- (ii) The reason Theorem 1.1.5 holds only in some neighborhood $\tilde{\mathcal{B}}$ of the smooth points of the preimage $f^{-1}(N) \cap \mathcal{B}_\infty$ is as follows. The tame sc-retracts modeling $\tilde{\mathcal{B}}$ are built from the subspace of the tangent space $T_x \mathcal{B}$ at $x \in f^{-1}(N)$ that is mapped by the tangent map $D_x f : T_x \mathcal{B} \rightarrow T_{f(x)} Y$ onto $T_{f(x)} N$, and this tangent space $T_x \mathcal{B}$ has the structure of a sc-Banach space only at smooth points $x \in \mathcal{B}_\infty$. So we can only hope to construct a sc-retract modeling a neighborhood of x in $f^{-1}(N)$ around smooth points x .
- (iii) The neighborhood $\tilde{\mathcal{B}}$ is open only in the 1-level of the preimage $f^{-1}(N) \cap \mathcal{B}_1$ because, in the proof of the local submersion normal form in sc-calculus (Lemma 1.2.3), we must 1-shift the sc-Banach space to obtain C^1 regularity in order to use the classical C^1 local submersion normal form (Lemma 1.2.1).

Remark 1.1.7. All manifolds Y and submanifolds $N \subset Y$ in this paper are smooth, finite dimensional, and without boundary:

$$\partial N = \emptyset \quad \text{and} \quad \partial Y = \emptyset.$$

This suffices for our initial intended applications to evaluation maps with target a closed symplectic manifold Y .

It is possible to generalize our theorems to the case where both N and Y are smooth finite dimensional orbifolds with boundary and corners. This generalization will be useful in applications. For example, polyfolds $\mathcal{B}$ constructed for regularization of moduli spaces of pseudoholomorphic curves in symplectic topology come with an everywhere submersive sc-smooth forgetful map $\mathcal{B} \rightarrow Y$ to the Deligne-Mumford space Y consisting of all domains of curves in $\mathcal{B}$. The Deligne-Mumford space Y usually has an orbifold structure with non-trivial isotropy, and when the domains have boundary, Y has boundary and corners.

1.1.2 Applications

We discuss applications of our polyfold results to pseudoholomorphic curves in symplectic manifolds.

First, we provide an alternative construction of the Gromov-Witten invariants, defined using polyfold theory in [33] as integrals of differential forms over a perturbed moduli space, as counts of points in a 0-dimensional constrained moduli space, where the constraints are evaluation maps at marked points that are required to evaluate to submanifolds. Then we describe the construction of the Hamiltonian Piunikhin-Salamon-Schwarz maps for general

closed symplectic manifolds, which is carried out in detail in [16] using our theorems to construct the fiber product of Morse moduli spaces and Symplectic Field Theory polyfolds [18], providing a proof of the weak Arnold conjecture. Last, we describe a method for perturbing expected dimension 0 and 1 moduli spaces so that the perturbed moduli space does not contain any curves with a sphere bubble.

The applicable polyfold results in this paper are the ep-groupoid generalizations of the theorems presented above, because in applications there will be nontrivial isotropy groups. The ep-groupoid results are in Sections 1.6, 1.7.

Throughout, let (Y, ω) be a closed symplectic manifold of dimension $2n$.

1.1.2.1 Gromov-Witten invariants

The Gromov-Witten invariants are defined in [33] for general closed symplectic manifolds as integrals of differential forms over the solution set of a perturbed sc-Fredholm section. The results in this paper provide an alternative construction as counts of 0-dimensional perturbed moduli spaces of curves satisfying intersection conditions with submanifolds.

For a given homology class $A \in H_2(Y)$ and integers $g, m \geq 0$ satisfying $2g + m \geq 3$, the Gromov-Witten invariant with respect to the fundamental class $[\overline{\mathcal{M}}_{g,m}]$ of the Deligne-Mumford space $\overline{\mathcal{M}}_{g,m}$ of closed genus g curves with m marked points is a multilinear map

$$\Psi_{A,g,m} : H^*(Y; \mathbb{R})^{\otimes m} \rightarrow \mathbb{R}.$$

This map is defined for $\alpha_1, \dots, \alpha_m \in H^*(Y; \mathbb{R})$ with degrees satisfying $|\alpha_1| + \dots + |\alpha_m| = 2c_1(A) + (2n - 6)(1 - g) + 2m$ as follows, and for other choices of α_i it is defined to be 0. For an ω -compatible almost complex structure J on Y , a sc-Fredholm section

$$\overline{\partial}_J : X_{A,g,m} \rightarrow E$$

of a polyfold bundle $E \rightarrow X_{A,g,m}$ with Fredholm index $\text{ind}(\overline{\partial}_J) = 2c_1(A) + (2n - 6)(1 - g) + 2m$ is constructed in [33] such that the solution set

$$\overline{\partial}_J^{-1}(0) = \overline{\mathcal{M}}_{g,m}(Y, A, J)$$

is the Gromov compactified moduli space of J -holomorphic curves in Y of genus g with m marked points that represent the class A . Then the abstract perturbation theory in [34] provides a sc^+ -multisection Λ of the bundle so that the perturbed solution space $S(\overline{\partial}_J, \Lambda)$ is a smooth compact oriented weighted branched orbifold of dimension $\dim S(\overline{\partial}_J, \Lambda) = \text{ind}(\overline{\partial}_J)$ over which we can integrate differential forms using the integration theory from [31]. For $|\alpha_1| + \dots + |\alpha_m| = \text{ind}(\overline{\partial}_J)$, the Gromov-Witten invariant is defined by

$$\Psi_{A,g,m}(\alpha_1 \otimes \dots \otimes \alpha_m) := \int_{S(\overline{\partial}_J, \Lambda)} ev_1^*(\alpha_1) \wedge \dots \wedge ev_m^*(\alpha_m),$$

where

$$ev_k : X_{A,g,m} \rightarrow Y$$

is evaluation at the k -th marked point.

We now explain how to use the results in this paper to construct the Gromov-Witten invariant as a count of points in a 0-dimensional moduli space of curves evaluating to submanifolds of Y . For $k = 1, \dots, m$ let $L_k \subset Y$ be an oriented submanifold such that $[L_k] = PD(\alpha_k) \in H_*(Y; \mathbb{R})$, where PD denotes the Poincaré dual. Then the codimension of L_k in Y is equal to the degree $|\alpha_k|$, and so $L_1 \times \dots \times L_m$ is a codimension- $\text{ind}(\bar{\partial}_J)$ submanifold of Y^m . The total evaluation map

$$\begin{aligned} ev_1 \times \dots \times ev_m : X_{A,g,m} &\rightarrow Y^m \\ u &\mapsto (ev_1(u), \dots, ev_m(u)) \end{aligned}$$

records the positions of all the marked points. Consider the subspace

$$X_{A,g,m}^{L_1, \dots, L_m} := (ev_1 \times \dots \times ev_m)^{-1}(L_1 \times \dots \times L_m)$$

of $X_{A,g,m}$, which consists of those curves whose k -th marked point evaluates to L_k for every $k = 1, \dots, m$. Then

$$\overline{\mathcal{M}}_{g,m}^{L_1, \dots, L_m}(Y, A, J) = \bar{\partial}_J^{-1}(0) \cap X_{A,g,m}^{L_1, \dots, L_m}$$

is the compactified moduli space of J -holomorphic curves $u \in \overline{\mathcal{M}}_{g,m}(Y, A, J)$ satisfying the point constraints $ev_k(u) \in L_k$ for all $k = 1, \dots, m$.

To perturb this constrained moduli space $\overline{\mathcal{M}}_{g,m}^{L_1, \dots, L_m}(Y, A, J)$ so that it is cut out transversely, we first apply Corollary 1.6.8 to obtain a description of $\overline{\mathcal{M}}_{g,m}^{L_1, \dots, L_m}(Y, A, J)$ as the zero set of a sc-Fredholm section of a polyfold bundle, as follows. The hypotheses of Corollary 1.6.8 are satisfied because the total evaluation map $ev_1 \times \dots \times ev_m : X_{A,g,m} \rightarrow Y^m$ is submersive, and moreover it is $\bar{\partial}_J$ -compatibly transverse to $L_1 \times \dots \times L_m$ as explained in Section 1.5.1. Hence, applying the corollary, there exists an open neighborhood $\tilde{X} \subset X_{A,g,m}^{L_1, \dots, L_m}$ of the constrained moduli space $\overline{\mathcal{M}}_{g,m}^{L_1, \dots, L_m}(Y, A, J)$ such that the restricted section $\bar{\partial}_J|_{\tilde{X}} : \tilde{X} \rightarrow E|_{\tilde{X}}$ is sc-Fredholm and has Fredholm index 0. Moreover, the solution space

$$\bar{\partial}_J|_{\tilde{X}}^{-1}(0) = \overline{\mathcal{M}}_{g,m}^{L_1, \dots, L_m}(Y, A, J)$$

is the constrained moduli space. The perturbation theory in [34] then provides a sc^+ -multisection $\tilde{\Lambda}$ of the restricted bundle $E|_{\tilde{X}} \rightarrow \tilde{X}$ so that the perturbed solution space $S(\bar{\partial}_J|_{\tilde{X}}^{-1}(0), \tilde{\Lambda})$ is a smooth compact oriented weighted branched orbifold of dimension 0.

Notice that, even after perturbation by $\tilde{\Lambda}$, all curves $u \in S(\bar{\partial}_J|_{\tilde{X}}^{-1}(0), \tilde{\Lambda})$ are guaranteed to satisfy the constraints $ev_k(u) \in L_k$ for $k = 1, \dots, m$ since $S(\bar{\partial}_J|_{\tilde{X}}^{-1}(0), \tilde{\Lambda})$ is contained in $X_{A,g,m}^{L_1, \dots, L_m}$. Morally, the weighted count $\#S(\bar{\partial}_J|_{\tilde{X}}^{-1}(0), \tilde{\Lambda})$ is the Gromov-Witten invariant, however to prove the equality $\#S(\bar{\partial}_J|_{\tilde{X}}^{-1}(0), \tilde{\Lambda}) = \Psi_{A,g,m}(\alpha_1 \otimes \dots \otimes \alpha_m)$ one would need to construct the perturbations so that $\tilde{\Lambda} = \Lambda|_{\tilde{X}}$; see [46][47] for a precise treatment of this comparison.

1.1.2.2 The Piunikhin-Salamon-Schwarz morphism

Let $H : S^1 \times Y \rightarrow \mathbb{R}$ be a nondegenerate Hamiltonian and (f, g) a Morse-Smale pair of a Morse function $f : Y \rightarrow \mathbb{R}$ and a Riemannian metric g . The original application that motivates this project is the construction of the Piunikhin-Salamon-Schwarz (PSS) morphism

$$PSS : H_*^{Morse}(Y; f, g) \rightarrow H_*^{Floer}(Y; H),$$

where $H_*^{Morse}(Y; f, g)$ is the Morse homology of (f, g) and $H_*^{Floer}(Y; H)$ is the Floer homology of H . This was originally done under the assumption that (Y, ω) is semi-positive in [44], where it is proved that this map is an isomorphism, proving the weak Arnold conjecture. In [16], we carry out a version of this construction, joint with Katrin Wehrheim, for general closed symplectic manifolds, proving the Arnold conjecture in full generality.

The moduli spaces from which the PSS morphism is constructed are as follows. Consider a critical point p of f , a 1-periodic orbit γ of the Hamiltonian vector field associated to H , and a singular homology class $A \in H_2(Y)$. Let $\overline{\mathcal{M}}(p, Y)$ denote the compactified moduli space of half-infinite gradient flow lines $\tau : (-\infty, 0] \rightarrow Y$ that limit to p on their infinite end and evaluate to $ev(\tau) := \tau(0)$ in the unstable manifold of p . The compactification includes broken flow lines that start at p , break at finitely many other critical points, and end in a half-infinite flow line originating from the final critical point at which breaking occurs and evaluating to its unstable manifold. Then $\overline{\mathcal{M}}(p, Y)$ can be given the structure of a smooth compact manifold with boundary and corners (see for example [53]) equipped with a smooth evaluation map

$$ev_p : \overline{\mathcal{M}}(p, Y) \rightarrow Y.$$

Fix smooth capping discs on each periodic orbit of H and an ω -compatible almost complex structure J on Y . Then let $\mathcal{M}(\gamma, A)$ denote the moduli space appearing in Symplectic Field Theory [13] consisting of smooth maps $\mathbb{C} \rightarrow X$ satisfying the Cauchy-Riemann equation $\bar{\partial}_J$ near 0 and the Floer equation near ∞ (with a fixed interpolation in between given by a cutoff function that turns off the Hamiltonian term in Floer's equation near 0), and such that the map glued to the capping disc on γ represents the homology class A . The compactified moduli space $\overline{\mathcal{M}}(\gamma, A)$ also includes configurations with broken Floer trajectories and sphere bubble trees. There is an evaluation map

$$ev_\gamma : \overline{\mathcal{M}}(\gamma, A) \rightarrow Y$$

given by evaluating at $0 \in \mathbb{C}$. The PSS moduli spaces are then the fiber products

$$\overline{\mathcal{M}}(p, \gamma, A) := \overline{\mathcal{M}}(p, Y) \times_{ev_p \times ev_\gamma} \overline{\mathcal{M}}(\gamma, A).$$

See Figure 1.1 for a diagram of an element of $\overline{\mathcal{M}}(p, \gamma, A)$.

If all choices can be made so that $\overline{\mathcal{M}}(p, \gamma, A)$ is smooth (and of the expected dimension), then the coefficient of $PSS(p)$ on the generator (γ, A) is defined to be the count $\#\overline{\mathcal{M}}(p, \gamma, A)$ if the expected dimension is 0, and the coefficient is 0 otherwise. Now, for general Y , the

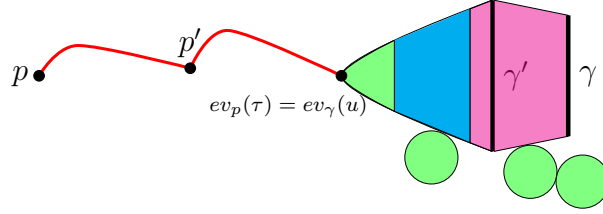


Figure 1.1: An element (τ, u) in the moduli space $\overline{\mathcal{M}}(p, \gamma, A)$. The red lines represent an element $\tau \in \overline{\mathcal{M}}(p, Y)$ consisting of a Morse trajectory from p to p' and a half-infinite Morse trajectory starting at p' and evaluating to $ev_p(\tau)$ in the unstable manifold of p' . The green near $ev_\gamma(u)$ represents the neighborhood of $0 \in \mathbb{C}$ on which the map $\mathbb{C} \rightarrow Y$ satisfies the J -holomorphic curve equation. As the map limits to the Hamiltonian orbit γ' , the J -curve equation interpolates in the blue region to Floer's equation represented in magenta. A Floer trajectory from γ' to γ has broken off. The green circles represent bubbled off J -holomorphic spheres. The evaluations $ev_p(\tau) = ev_\gamma(u)$ agree since $\overline{\mathcal{M}}(p, \gamma, A)$ is a fiber product.

compact moduli space $\overline{\mathcal{M}}(\gamma, A)$ will not be cut out transversely for any J , and hence has no reason to be smooth. Moreover, even if $\overline{\mathcal{M}}(\gamma, A)$ is cut out transversely, there is no reason to expect that the fiber product with $\overline{\mathcal{M}}(p, Y)$ is transverse.

In [16], these transversality issues are overcome using the results in this paper as follows. The Symplectic Field Theory polyfolds in [18] include polyfold bundles $E(\gamma, A) \rightarrow X(\gamma, A)$ and sc-Fredholm sections

$$\sigma(\gamma, A) : X(\gamma, A) \rightarrow E(\gamma, A)$$

with solution set the SFT moduli space

$$\sigma(\gamma, A)^{-1}(0) = \overline{\mathcal{M}}(\gamma, A).$$

Here $X(\gamma, A)$ is the polyfold of broken and nodal maps of the same form as those in $\overline{\mathcal{M}}(\gamma, A)$ but not necessarily satisfying any equation, and the section $\sigma(\gamma, A)$ is the equation that maps in $\overline{\mathcal{M}}(\gamma, A)$ are required to satisfy. Moreover there is a sc-smooth evaluation map

$$ev_\gamma : X(\gamma, A) \rightarrow Y$$

which evaluates at $0 \in \mathbb{C}$. This evaluation map is a submersion on the ambient space $X(\gamma, A)$, and it restricts to the evaluation map on the moduli space $ev_\gamma : \overline{\mathcal{M}}(\gamma, A) \rightarrow Y$.

Applying the fiber product result Corollary 1.7.3 to the zero section of the rank-0 bundle over the Morse moduli space $\overline{\mathcal{M}}(p, Y)$ and the sc-Fredholm section $\sigma(\gamma, A)$, we obtain an open neighborhood $X(p, \gamma, A)$ of the zero set of the fiber product section

$$\begin{aligned} \overline{\mathcal{M}}(p, Y) \times_{ev_p \times ev_\gamma} X(\gamma, A) &\rightarrow E(\gamma, A) \\ (\tau, u) &\mapsto \sigma(\gamma, A)(u) \end{aligned}$$

such that the restricted section

$$\sigma(p, \gamma, A) : X(p, \gamma, A) \rightarrow E(p, \gamma, A) := E(\gamma, A)|_{X(p, \gamma, A)}$$

is sc-Fredholm with index $\text{ind}(\sigma(p, \gamma, A)) = \dim \overline{\mathcal{M}}(p, Y) + \text{ind}(\sigma(\gamma, A)) - 2n$. Its zero set is compact and equal to the PSS moduli space

$$\sigma(p, \gamma, A)^{-1}(0) = \overline{\mathcal{M}}(p, \gamma, A).$$

The perturbation theory in [34] then provides a sc^+ -multisection Λ of the polyfold bundle $E(p, \gamma, A) \rightarrow X(p, \gamma, A)$ so that the perturbed solution space $S(\sigma(p, \gamma, A), \Lambda)$ is a smooth compact weighted branched orbifold; see Figure 1.2 for an element of $S(\sigma(p, \gamma, A), \Lambda)$. The weighted count of points in this perturbed moduli space provides the definition of the PSS map. That is, the coefficient of $PSS(p)$ on (γ, A) is given by $\langle PSS(p), (\gamma, A) \rangle := \#S(\sigma(p, \gamma, A), \Lambda)$.

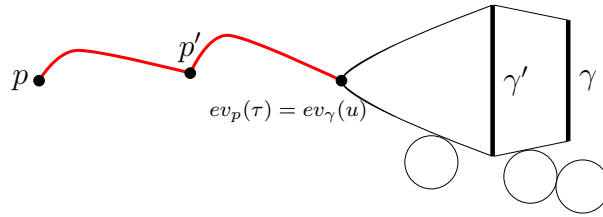


Figure 1.2: An element (τ, u) in the perturbed moduli space $S(\sigma(p, \gamma, A), \Lambda) \subset \overline{\mathcal{M}}(p, Y) \times_{\text{ev}_p \times \text{ev}_\gamma} X(\gamma, A)$. The red lines represent the broken Morse trajectory $\tau \in \overline{\mathcal{M}}(p, Y)$ with finite end evaluating to $\text{ev}_p(\tau)$. The region from $\text{ev}_\gamma(u)$ to γ' is a map $\mathbb{C} \rightarrow Y$ limiting to the Hamiltonian orbit γ' at ∞ . The region in between γ' and γ represents a cylinder limiting to these orbits on its two ends, and the circles represent attached sphere bubbles. Together the white regions represent the element $u \in X(\gamma, A)$. They are colored white, in contrast to Figure 1.1, to indicate that they do not necessarily satisfy any equation due to the perturbation Λ . The evaluations $\text{ev}_p(\tau) = \text{ev}_\gamma(u)$ still agree since $S(\sigma(p, \gamma, A), \Lambda)$ is contained in the fiber product.

1.1.2.3 Avoiding sphere bubbles in expected dimension 0 and 1

A common mantra in symplectic topology is that “sphere bubbling is a codimension-2 phenomenon” and hence sphere bubbles do not appear in regularized moduli spaces of dimension 0 and 1. The notions of a sliced sc-retract (Definition 1.3.2) and a sliced sc-Fredholm germ (Definition 1.3.8) introduced in this paper provide a method for making this precise in the context of polyfold theory.

For simplicity, we consider the case of a curve with 1 interior node, for example a curve with a single sphere bubble. This curve naturally sits inside a $\mathbb{R}^2$ -sliced sc-retract $(\mathcal{O}, \mathbb{R}^2 \times C, \mathbb{R}^2 \times \mathbb{E})$ which locally models a neighborhood of the curve in an ambient polyfold. The $\mathbb{R}^2$ -sliced sc-retraction $r : U \rightarrow U$ with image the sc-retract $r(U) = \mathcal{O}$ is (roughly) the splicing

[34, Def. 2.18] obtained from pregluing at the node. In particular, $\mathcal{O}$ is homeomorphic to the image of the pregluing map, so conceptually we identify them. Since pregluing with gluing parameter $0 \in \mathbb{R}^2$ preserves the node, the induced tame sc-retract $\tilde{\mathcal{O}} = \mathcal{O} \cap (\{0\} \times C)$ (Lemma 1.3.3) consists of the curves in $\mathcal{O}$ that have 1 interior node. This formalizes the notion that a curve with 1 interior node sits inside a codimension-2 stratum consisting of nearby nodal curves.

Moreover, the Cauchy-Riemann section $\bar{\partial}_J : \mathcal{O} \rightarrow K$ of the local bundle model $K \rightarrow \mathcal{O}$ is a $\mathbb{R}^2$ -sliced sc-Fredholm germ, and hence by Lemma 1.3.10 its restriction to $\tilde{\mathcal{O}}$ is sc-Fredholm with index satisfying $\text{ind}(\bar{\partial}_J|_{\tilde{\mathcal{O}}}) = \text{ind}(\bar{\partial}_J) - 2$. If the original section satisfies $\text{ind}(\bar{\partial}_J) \leq 1$, i.e. the expected dimension of the moduli space is ≤ 1 , then its restriction to the nodal curves in $\tilde{\mathcal{O}}$ satisfies $\text{ind}(\bar{\partial}_J|_{\tilde{\mathcal{O}}}) < 0$. So, after perturbing the restricted section, the transversely cut out zero set must be empty as a smooth object with negative dimension. Extending this perturbation over all of $\mathcal{O}$, the perturbed zero set will not intersect $\tilde{\mathcal{O}}$, meaning that the perturbed zero set will not contain any nodal curves.

This perturbation extension result will be part of a future work, which to be applied must include an inductive procedure that perturbs and extends starting with the highest codimension strata corresponding to solutions with the most nodes. Indeed, a curve with $k \geq 1$ nodes sits inside the intersection of k distinct codimension-2 strata, and this intersection is a codimension- $2k$ stratum. One must perform the local perturbations and extensions coherently with respect to the intersections of these nodal strata.

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1.2 Sc-calculus: the normal form of a local sc-smooth submersion to $\mathbb{R}^n$

The purpose of this section is to establish the normal form (Lemma 1.2.3) of a sc-smooth local submersion $f : [0, \infty)^s \times \mathbb{E} \rightarrow \mathbb{R}^n$, where $\mathbb{E}$ is a sc-Banach space. This is a sc-calculus analog of the classical local submersion normal form (Lemma 1.2.1) in the case where $\mathbb{E}$ is an ordinary Banach space E . In the Banach case, the normal form follows from the inverse function theorem for C^1 maps between open subsets of quadrants $[0, \infty)^s \times E$. The inverse function theorem does not hold in sc-calculus [17], however, using relationships between classical differentiability and sc-differentiability, we leverage the classical normal form to prove the normal form in sc-calculus. The key ingredient is that the target of these submersions is finite dimensional $\mathbb{R}^n$, on which all sc-structures are trivial (i.e. all levels are isomorphic to

the infinity level).

For completeness, we now provide a proof of the normal form in the classical Banach case. We view a quadrant $[0, \infty)^s \times E$ as a Banach space with boundary and corners. It suffices to consider neighborhoods in $[0, \infty)^s \times E$ of a point x that sits in the maximally degenerate corner $x \in \{0\} \times E$.

Lemma 1.2.1. (Normal form of a C^1 local submersion to $\mathbb{R}^n$) *Consider a Banach space E , an open subset $U \subset [0, \infty)^s \times E$ for some $s \geq 0$, and a C^1 map $f : U \rightarrow \mathbb{R}^n$. Suppose that, for some $x \in U \cap (\{0\}^s \times E)$, the tangent map $(d_x f)|_{\{0\}^s \times E} : E \rightarrow \mathbb{R}^n$ of the restriction to the corner $f|_{U \cap (\{0\}^s \times E)}$ is surjective.*

Then, for any complement L of $K := \ker(d_x f)|_{\{0\}^s \times E}$ in E , there exist open neighborhoods $x \in \hat{U} \subset U$ and $U' \subset \mathbb{R}^n \times [0, \infty)^s \times K$ such that, writing $v \in [0, \infty)^s$ and $e \in E$, the map

$$\begin{aligned} g : \hat{U} &\rightarrow U' \\ (v, e) &\mapsto (f(v, e), v, pr(e)) \end{aligned}$$

is a C^1 -diffeomorphism, where $pr : E = K \oplus L \rightarrow K$ is the projection along L .

Proof. Denote the restriction of f to the corner by

$$\underline{f} := f|_{U \cap (\{0\} \times E)} : U \cap (\{0\} \times E) \rightarrow \mathbb{R}^n.$$

Then $d_x \underline{f} : E \rightarrow \mathbb{R}^n$ is surjective by hypothesis. Let L be any complement of $K = \ker d_x \underline{f}$ in E . In particular, note that this means the restriction

$$d_x \underline{f}|_L : L \rightarrow \mathbb{R}^n \tag{1.1}$$

is an isomorphism. Writing $v \in [0, \infty)^s$ and $e \in E$, define the map

$$\begin{aligned} g : U &\rightarrow \mathbb{R}^n \times [0, \infty)^s \times K \\ (v, e) &\mapsto (f(v, e), v, pr(e)), \end{aligned}$$

where $pr : E = K \oplus L \rightarrow K$ is the projection along L . Note that g is C^1 since f is C^1 and pr is C^∞ .

We claim that the tangent map

$$\begin{aligned} d_x g : \mathbb{R}^s \times E &\rightarrow \mathbb{R}^n \times \mathbb{R}^s \times K \\ (v, e) &\mapsto (d_x f(v, e), v, pr(e)) \end{aligned}$$

is an isomorphism. To verify injectivity, suppose $(0, 0, 0) = (d_x f(v, e), v, pr(e))$. Then $e \in \ker(pr) = L$, and moreover $0 = d_x f(0, e) = d_x \underline{f}(e)$ means $e \in K$, hence $e = 0$. To verify surjectivity, let $(p, v, k) \in \mathbb{R}^n \times \mathbb{R}^s \times K$. Since the map (1.1) is an isomorphism, there exists $l \in L$ such that $d_x \underline{f}(l) = p - d_x f(v, k)$. It follows that $d_x f(v, k + l) = d_x f(v, k) + d_x \underline{f}(l) = p$

and hence $d_x g(v, k + l) = (d_x f(v, k + l), v, pr(k + l)) = (p, v, k)$. So $d_x g$ is an isomorphism, as claimed.

Since g is a C^1 map whose tangent map $d_x g$ at x is an isomorphism, the inverse function theorem¹ for C^1 maps between quadrants of Banach spaces applies: There exists an open neighborhood $\hat{U} \subset U$ of x and an open set $U' \subset \mathbb{R}^n \times [0, \infty)^s \times K$ such that the restriction $g|_{\hat{U}} : \hat{U} \rightarrow U'$ is a C^1 -diffeomorphism, as claimed. $\square$

We briefly review basics about sc-calculus on sc-Banach spaces from [34, Sec. 1.1] to prepare for Lemma 1.2.3. A *sc-Banach space* [34, Def. 1.1] is a sequence of Banach spaces and continuous linear injections

$$\mathbb{E} := (E_0 \hookleftarrow E_1 \hookleftarrow \cdots)$$

such that the map $E_{m+1} \hookrightarrow E_m$ is a compact operator for every $m \geq 0$ and the intersection

$$E_\infty := \bigcap_{m \geq 0} E_m$$

is dense in every E_m . We call E_∞ the *smooth points* of $\mathbb{E}$. By a *subset S of a sc-Banach space $\mathbb{E}$* we mean a subset S_0 of E_0 , which then induces subsets

$$\begin{aligned} S_m &:= S \cap E_m \subset E_m \text{ for all } m \geq 0, \\ S_\infty &:= S \cap E_\infty \subset E_\infty. \end{aligned}$$

In particular, if U is open in E_0 then $U_m \subset E_m$ is open for all $m \geq 0$, so we say that U is open in the sc-Banach space $\mathbb{E}$.

A *sc-subspace* $\mathbb{F} \subset \mathbb{E}$ [34, Def. 1.4] is a closed subspace $F \subset E_0$ such that the induced subsets $F_m := F \cap E_m$ define a sc-Banach space $\mathbb{F} = (F_0 \hookleftarrow F_1 \hookleftarrow \cdots)$. Given sc-subspaces $\mathbb{F}, \mathbb{F}' \subset \mathbb{E}$ such that, for every $m \geq 0$, the Banach space E_m splits as a direct sum $E_m = F_m \oplus F'_m$, we say that there is a *sc-splitting* $\mathbb{E} = \mathbb{F} \oplus \mathbb{F}'$ and that $\mathbb{F}, \mathbb{F}'$ are *sc-complements* in $\mathbb{E}$. Given a sc-subspace $\mathbb{F} \subset \mathbb{E}$, the quotient space $\mathbb{E}/\mathbb{F}$ has the structure of a sc-Banach space with m -level E_m/F_m ; see [34, Prop. 1.2]. The following fact is established in the proof of [34, Prop. 1.4]. We reproduce the proof here for completeness.

Lemma 1.2.2. *Consider a sc-Banach space $\mathbb{E}$ and a sc-subspace $\mathbb{F} \subset \mathbb{E}$ such that the quotient $\mathbb{E}/\mathbb{F}$ is finite dimensional. Then, there exists a sc-complement $\mathbb{F} \oplus L = \mathbb{E}$ and moreover $L \subset E_\infty$.*

Proof. Consider the sc-continuous quotient map $p : \mathbb{E} \rightarrow \mathbb{E}/\mathbb{F}$. Since $E_\infty \subset E_0$ is a dense linear subspace and $p : E_0 \rightarrow E_0/F_0$ is surjective, it follows that $p(E_\infty)$ is a dense linear subspace of the finite dimensional space E_0/F_0 , and so we have $p(E_\infty) = E_0/F_0$. Hence, choosing any basis of E_0/F_0 , there are preimages of the basis elements in E_∞ , and these preimages span a subspace $L \subset E_\infty$. We claim that L is sc-complementary to $\mathbb{F}$ in $\mathbb{E}$. Indeed, for every $m \geq 0$ the subspace $E_m \subset E_0$ is dense and so $p_m : E_m \rightarrow E_0/F_0$ is surjective. Moreover, we have $F_m = \ker(p_m)$ and the restriction $p_m : L \rightarrow E_0/F_0$ is an isomorphism, so $F_m \oplus L = E_m$ holds. $\square$

¹See, for example, [39, Thm. 2.2.4].

For every $l \geq 0$ there is a l -shifted sc-Banach space defined by

$$\mathbb{E}^l := (E_l \hookleftarrow E_{l+1} \hookleftarrow \cdots),$$

that is,

$$(\mathbb{E}^l)_m = E_{m+l} \text{ for all } m \geq 0.$$

Conceptually, we are forgetting about finitely many levels. Note that l -shifting does not change the ∞ -level. Analogously, for any subset $S \subset \mathbb{E}$, we define $S^l \subset \mathbb{E}^l$ by $(S^l)_m := S_{m+l}$. In particular, if $U \subset \mathbb{E}$ is open, then $U^l \subset \mathbb{E}^l$ is open for all $l \geq 0$.

A Cartesian product $\mathbb{E} \times \mathbb{F}$ of sc-Banach spaces has a natural sc-structure with m -level given by $(\mathbb{E} \times \mathbb{F})_m := E_m \times F_m$ equipped with any standard Banach norm on a Cartesian product. In this paper, we use the convention that all norms on Cartesian products are the sum norm $\|(\cdot, \cdot)\|_{E_m \times F_m} = \|\cdot\|_{E_m} + \|\cdot\|_{F_m}$, which is equivalent to all standard choices.

The finite dimensional space $\mathbb{E} = \mathbb{R}^n$ has a canonical sc-structure given by $E_m = \mathbb{R}^n$ equipped with the standard Euclidean norm for every $m \geq 0$, and where every inclusion $E_{m+1} \rightarrow E_m$ is the identity map.

The *tangent space* [34, Def. 1.8] of a sc-Banach space $\mathbb{E}$ is the sc-Banach space

$$T\mathbb{E} := \mathbb{E}^1 \times \mathbb{E},$$

with sc-structure given by $(T\mathbb{E})_m = E_{m+1} \times E_m$ for $m \geq 0$. Given an open subset $U \subset [0, \infty)^s \times \mathbb{E}$ for some $s \geq 0$, its tangent space is

$$TU := U^1 \times (\mathbb{R}^s \times \mathbb{E}).$$

Consider sc-Banach spaces $\mathbb{E}, \mathbb{F}$, and open subsets $U \subset [0, \infty)^s \times \mathbb{E}$ and $V \subset [0, \infty)^{s'} \times \mathbb{F}$. Then a map

$$f : U \rightarrow V$$

is called sc^0 or *sc-continuous* [34, Def. 1.7] if, for all $m \geq 0$, we have $f(U_m) \subset V_m$ and the map $f : U_m \rightarrow V_m$ is continuous. A sc^0 map $f : U \rightarrow V$ is called sc^1 with tangent map [34, Def. 1.9]

$$Tf : TU \rightarrow TV \tag{1.2}$$

defined by

$$\begin{aligned} Tf : U^1 \times (\mathbb{R}^s \times \mathbb{E}) &\rightarrow V^1 \times (\mathbb{R}^{s'} \times \mathbb{F}) \\ (x, \xi) &\mapsto (f(x), D_x f(\xi)) \end{aligned}$$

if, for every $x \in U^1$, there exists a bounded linear operator

$$D_x f : \mathbb{R}^s \times E_0 \rightarrow \mathbb{R}^{s'} \times F_0$$

such that, for $\xi \in E_1$ satisfying $x + \xi \in U_1$,

$$\lim_{|\xi|_1 \rightarrow 0} \frac{|f(x + \xi) - f(x) - D_x f(\xi)|_0}{|\xi|_1} = 0,$$

holds, and moreover such that Tf is sc^0 . Iterating the definition of sc^1 yields the notions of sc^k for $k \geq 0$ and sc -smooth (denoted sc^∞); see the discussion after [34, Def. 1.9].

An important note is that, for a sc^1 map $f : U \rightarrow V$ and $x \in U^1$, the bounded linear operator $D_x f : \mathbb{R}^s \times E_0 \rightarrow \mathbb{R}^{s'} \times F_0$ is not necessarily sc -continuous when considered as a map between sc -Banach spaces $D_x f : \mathbb{R}^s \times \mathbb{E} \rightarrow \mathbb{R}^{s'} \times \mathbb{F}$; that is, continuity on levels higher than 0 can fail. However, if $x \in U_\infty$ is a smooth point, then by [34, Prop. 1.5] the map $D_x f$ is indeed a *sc-operator* [34, Def. 1.2], i.e. a sc -continuous linear map. For this reason, we consider only smooth points x in the following lemma so that the kernel of $D_x f$ is a sc -Banach space as the kernel of a sc -operator.

Lemma 1.2.3. (Normal form of a sc -smooth local submersion to $\mathbb{R}^n$) *Consider a sc -Banach space $\mathbb{E}$, an open subset $U \subset [0, \infty)^s \times \mathbb{E}$ for some $s \geq 0$, and a sc -smooth map $f : U \rightarrow \mathbb{R}^n$. Suppose that, for some smooth point $x \in U_\infty \cap (\{0\} \times \mathbb{E})$, the tangent map $(D_x f)|_{\{0\} \times \mathbb{E}} : \mathbb{E} \rightarrow \mathbb{R}^n$ of the restriction to the corner $f|_{U \cap (\{0\} \times \mathbb{E})}$ is surjective.*

Then, for any sc -complement² L of $\mathbb{K} := \ker(D_x f)|_{\{0\} \times \mathbb{E}}$ in $\mathbb{E}$, there exist open neighborhoods $x \in \hat{U} \subset U^1$ and $U' \subset \mathbb{R}^n \times [0, \infty)^s \times \mathbb{K}^1$ such that, writing $v \in [0, \infty)^s$ and $e \in \mathbb{E}^1$, the map

$$\begin{aligned} g : \hat{U} &\rightarrow U' \\ (v, e) &\mapsto (f(v, e), v, pr(e)) \end{aligned}$$

is a sc -diffeomorphism, where $pr : \mathbb{E} = \mathbb{K} \oplus L \rightarrow \mathbb{K}$ is the projection along L . Moreover, for all $m \geq 0$, the map $g|_{\hat{U}_m} : \hat{U}_m \rightarrow U'_m$ is a C^{m+1} -diffeomorphism.

In particular, the following statements hold:

- *The composition $f \circ g^{-1} : U' \rightarrow \mathbb{R}^n$ is projection onto the $\mathbb{R}^n$ -coordinate.*
- *g preserves the $[0, \infty)^s$ -coordinate.*

Proof. We claim that the Banach space E_1 (the 1-level of the sc -Banach space $\mathbb{E}$), the open subset $U_1 \subset [0, \infty)^s \times E_1$, and the C^1 -map

$$f_1 := f|_{U_1} : U_1 \rightarrow \mathbb{R}^n$$

satisfy the hypotheses of Lemma 1.2.1 (the local submersion normal form in the classical Banach case) at the given point x . First of all, the map f_1 is indeed C^1 by [34, Prop. 1.7]. By [34, Prop. 1.5], we have $d_x f_1 = (D_x f)|_{\mathbb{R}^s \times E_1}$.

We now deduce the surjectivity of the tangent map

$$(d_x f_1)|_{\{0\} \times E_1} = (D_x f)|_{\{0\} \times E_1} : E_1 \rightarrow \mathbb{R}^n. \tag{1.3}$$

By hypothesis, the map $(D_x f)|_{\{0\} \times E_0}$ is surjective. Then since $E_1 \subset E_0$ is dense, it follows that $(D_x f)|_{\{0\} \times E_0}(E_1)$ is a dense linear subspace of $\mathbb{R}^n$ and hence is equal to $\mathbb{R}^n$, proving the claimed surjectivity of the map (1.3).

²A sc -complement L of $\mathbb{K}$ exists by Lemma 1.2.2, since the surjection $(D_x f)|_{\{0\} \times \mathbb{E}} : \mathbb{E} \rightarrow \mathbb{R}^n$ induces an isomorphism $\mathbb{E}/\mathbb{K} \cong \mathbb{R}^n$.

Let $L \subset E_\infty$ be any sc-complement of $\mathbb{K} = \ker(D_x f)|_{\{0\} \times \mathbb{E}}$ in $\mathbb{E}$, which exists by Lemma 1.2.2 since the surjection $(D_x f)|_{\{0\} \times \mathbb{E}} : \mathbb{E} \rightarrow \mathbb{R}^n$ induces an isomorphism $\mathbb{E}/\mathbb{K} \cong \mathbb{R}^n$. In particular, on the 1-level, we have $K_1 \oplus L = E_1$. Notice that K_1 is the kernel of the map (1.3).

We have shown that the map f_1 satisfies the hypotheses of the classical local submersion normal form (Lemma 1.2.1), yielding an open neighborhood $\hat{U} \subset U_1$ of x and an open subset $U' \subset \mathbb{R}^n \times [0, \infty)^s \times K_1$ such that, writing $v \in [0, \infty)^s$ and $e \in E_1$, the map

$$\begin{aligned} g : \hat{U} &\rightarrow U' \\ (v, e) &\mapsto (f_1(v, e), v, pr(e)) \end{aligned}$$

is a C^1 -diffeomorphism, where $pr : E_1 = K_1 \oplus L \rightarrow K_1$ is the projection along L .

We may view $\hat{U}$ and U' as open neighborhoods in the sc-calculus sense, i.e.

$$\hat{U} \subset [0, \infty)^s \times \mathbb{E}^1 \text{ and } U' \subset \mathbb{R}^n \times [0, \infty)^s \times \mathbb{K}^1.$$

We claim that g is a sc-diffeomorphism. First of all, it is sc-smooth since f_1 is sc-smooth by hypothesis and since the projection $pr : \mathbb{E}^1 = \mathbb{K}^1 \oplus L \rightarrow \mathbb{K}^1$ is sc-smooth.

To show that g^{-1} is sc-smooth, we show that it satisfies the conditions of [34, Prop. 1.8]. Let $m, l \geq 0$. We must show that g^{-1} induces a map $g^{-1}|_{U'_{m+l}} : U'_{m+l} \rightarrow \hat{U}_m$ that is C^{l+1} .

It suffices to show that, for all $m \geq 0$, g^{-1} restricts to a C^{m+1} -map $g^{-1}|_{U'_m} : U'_m \rightarrow \hat{U}_m$, because then given $m, l \geq 0$ the composition $U'_{m+l} \xrightarrow{g^{-1}} \hat{U}_{m+l} \hookrightarrow \hat{U}_m$ is C^{m+l+1} since the inclusion $\hat{U}_{m+l} \hookrightarrow \hat{U}_m$ is continuous and linear hence C^∞ . So, to complete the proof of the lemma, it suffices to show that

$$g : \hat{U}_m \rightarrow U'_m$$

is a C^{m+1} -diffeomorphism for all $m \geq 0$.

By [34, Prop. 1.7], the restriction

$$f|_{\hat{U}_m} : \hat{U}_m = \hat{U} \cap (\mathbb{R}^s \times E_{m+1}) \rightarrow \mathbb{R}^n$$

is C^{m+1} . It follows that the restriction

$$g|_{\hat{U}_m} : \hat{U}_m \rightarrow U'_m$$

is C^{m+1} .

To see that $g|_{\hat{U}_m} : \hat{U}_m \rightarrow U'_m$ is a bijection, note first that injectivity holds since it is a restriction of the bijection g . To see surjectivity, note first that, since g is surjective onto all of U' , it suffices to show that $g(v, e) \in U'_m \implies (v, e) \in \hat{U}_m$. Note that $U'_m \subset \mathbb{R}^n \times [0, \infty)^s \times K_{m+1}$. So, from the definition of g , we have $g(v, e) \in U'_m \implies pr(e) \in K_{m+1} \subset E_{m+1}$. Since $e - pr(e) \in L \subset E_\infty$, we conclude that $e \in E_{m+1}$. Hence indeed $(v, e) \in \hat{U}_m = \hat{U} \cap (\mathbb{R}^s \times E_{m+1})$ holds, proving surjectivity of $g|_{\hat{U}_m}$ onto U'_m . The same reasoning shows that the classical tangent map

$$d(g|_{\hat{U}_m}) : \hat{U}_m \times (\mathbb{R}^s \times E_{m+1}) \rightarrow U'_m \times (\mathbb{R}^n \times \mathbb{R}^s \times K_{m+1})$$

is bijective.

The inverse

$$(g^{-1})|_{U'_m} = (g|_{\hat{U}_m})^{-1} : U'_m \rightarrow \hat{U}_m$$

is C^{m+1} because it is the inverse of a C^{m+1} map with invertible derivative. We have shown that $g|_{\hat{U}_m} : \hat{U}_m \rightarrow U'_m$ is a C^{m+1} -diffeomorphism, completing the proof of the lemma. $\square$

We briefly review general partial quadrants, which up to a linear change of coordinates are the same as the standard quadrants (1.5), i.e. of the form $[0, \infty)^s \times \mathbb{E}$ for some sc-Banach space $\mathbb{E}$. This level of generality makes constructions more convenient and is equivalent to working with standard partial quadrants only.

A *partial quadrant* [34, Def. 1.6] in a sc-Banach space $\mathbb{E}$ is a closed convex subset $C \subset \mathbb{E}$ such that there exists another sc-Banach space $\mathbb{E}'$ and a linear sc-isomorphism

$$\Psi : \mathbb{E} \rightarrow \mathbb{R}^s \times \mathbb{E}' \text{ satisfying } \Psi(C) = [0, \infty)^s \times \mathbb{E}' \text{ for some } s \geq 0. \quad (1.4)$$

That is, all partial quadrants come from applying a linear change of coordinates to a partial quadrant in the *standard form*:

$$C = [0, \infty)^s \times \mathbb{E}' \subset \mathbb{R}^s \times \mathbb{E}' = \mathbb{E}. \quad (1.5)$$

The *degeneracy index* [34, Def. 1.10] $d_C : C \rightarrow \mathbb{N}_0$ is defined for $x \in C$ by

$$d_C(x) := \#\{i \in \{1, \dots, s\} \mid \text{the } i\text{-th coordinate of } \Psi(x) \text{ is } 0\},$$

which is independent of the choice of Ψ by [34, Lem. 1.1]. Conceptually, the degeneracy index of a point in a partial quadrant is the local notion of boundary and corner index in a M -polyfold.

Later, we need the following properties of the degeneracy index. Let C be a partial quadrant of a sc-Banach space $\mathbb{E}$ and $n \geq 0$. Then $\mathbb{R}^n \times C$ is a partial quadrant of $\mathbb{R}^n \times \mathbb{E}$ and

$$d_{\mathbb{R}^n \times C}(p, x) = d_C(x) \text{ for all } (p, x) \in \mathbb{R}^n \times C. \quad (1.6)$$

Let C_i be a partial quadrant of $\mathbb{E}_i$ for $i = 1, 2$. Then $C_1 \times C_2$ is a partial quadrant of $\mathbb{E}_1 \times \mathbb{E}_2$ and

$$d_{C_1 \times C_2}(x_1, x_2) = d_{C_1}(x_1) + d_{C_2}(x_2) \text{ for all } (x_1, x_2) \in C_1 \times C_2. \quad (1.7)$$

We recall from [34, Def. 2.16] the following linear sc-subspace $\mathbb{E}_x \subset \mathbb{E}$ associated to a point x in a partial quadrant $C \subset \mathbb{E}$. First assume that C is in the standard form (1.5) and write $x = (x_1, \dots, x_s, e_x) \in C$. Then, the sc-subspace

$$\mathbb{E}_x := \{(v_1, \dots, v_s, e) \in \mathbb{R}^s \times \mathbb{E}' \mid v_i = 0 \text{ if } x_i = 0\} \subset \mathbb{E} \quad (1.8)$$

conceptually is the tangent space of the intersection of all of the faces of C that contain x . For a general partial quadrant $C \subset \mathbb{E}$ and $x \in C$, the subspace $\mathbb{E}_x \subset \mathbb{E}$ is given by

$$\mathbb{E}_x := \Psi^{-1}((\mathbb{R}^s \times \mathbb{E}')_{\Psi(x)}), \quad (1.9)$$

where Ψ is any linear isomorphism of the form (1.4).

1.3 Slices: the local picture

1.3.1 Sliced sc-retracts

In this section, we introduce the new notion of $\mathbb{R}^n$ -sliced sc-retracts (Definition 1.3.2), which we use later as the local models in our definition of a slice $\tilde{\mathcal{B}} \subset \mathcal{B}$ (Definition 1.5.7) of a tame M -polyfold $\mathcal{B}$ (Definition 1.5.1). We prove in Lemma 1.3.3 that a $\mathbb{R}^n$ -sliced sc-retract $\mathcal{O}$ induces a tame sc-retract $\tilde{\mathcal{O}} \subset \mathcal{O}$ which has codimension- n tangent spaces $T_x \tilde{\mathcal{O}} \subset T_x \mathcal{O}$ at every $x \in \tilde{\mathcal{O}}_1$. The global definition of a slice $\tilde{\mathcal{B}} \subset \mathcal{B}$ is then a subspace such that around every point $x \in \tilde{\mathcal{B}}$ there is a M -polyfold chart to a $\mathbb{R}^{n_x}$ -sliced sc-retract $\mathcal{O}$ that locally identifies $\tilde{\mathcal{B}}$ with the induced tame sc-retract $\tilde{\mathcal{O}}$.

We first recall the local structure of tame M -polyfolds: tame sc-retracts (Definition 1.3.1). Consider a relatively open subset U of a partial quadrant C in a sc-Banach space $\mathbb{E}$. A sc-smooth map $r : U \rightarrow U$ satisfying $r \circ r = r$ is called a *sc-smooth retraction* (or *sc-retraction*) [34, Def. 2.1] on U , and the image $\mathcal{O} := r(U)$ of such a map is called a *sc-smooth retract* (or *sc-retract*). The triple $(\mathcal{O}, C, \mathbb{E})$ is also called a *sc-retract* [34, Def. 2.2].

We note that the notion of a smooth retract makes sense in the classical Banach space setting, i.e. given an ordinary Banach space E , we can define a smooth retract $\mathfrak{D}$ to be any image $\mathfrak{D} = r(U)$ of a smooth map $r : U \rightarrow U$ that satisfies $r \circ r = r$, where $U \subset [0, \infty)^s \times E$ is open. However, modeling spaces on these smooth retracts reproduces the definition of a Banach manifold because, by [34, Prop. 2.1], a smooth retract $\mathfrak{D}$ is a C^∞ -sub-Banach manifold of E . The sc-retracts can have much more complicated structure, including locally varying dimension. This is a key difference between classical differentiability and sc-differentiability which allows M -polyfolds to have local dimension jumps and other non-manifold-like structure. Polyfolds arising in applications have these local dimension jumps near broken and nodal curves.

A map $\varphi : \mathcal{O} \rightarrow \mathcal{O}'$ between sc-retracts $(\mathcal{O}, C, \mathbb{E})$ and $(\mathcal{O}', C', \mathbb{E}')$ is called *sc-smooth* [34, Def. 2.4] if the composition $\varphi \circ r : U \rightarrow \mathcal{O}' \subset \mathbb{E}'$ is sc-smooth as a map $U \rightarrow \mathbb{E}'$, where $U \subset C$ is open and $r : U \rightarrow U$ is any sc-retraction onto $r(U) = \mathcal{O}$. This definition is independent of the choice of open set U and sc-retraction r by [34, Prop. 2.3]. The chain rule holds for sc-smooth maps between sc-retracts; see [34, Thm. 2.1].

The *tangent space* [34, Def. 2.3] of a sc-retract $(\mathcal{O}, C, \mathbb{E})$ is the image

$$T\mathcal{O} := Tr(TU), \quad (1.10)$$

where $r : U \rightarrow U$ is any sc-retraction on some open subset $U \subset C$ with image $r(U) = \mathcal{O}$ and $Tr : TU \rightarrow TU$ is the tangent map (see (1.2)) of r . The tangent space $T\mathcal{O}$ is well-defined, i.e. independent of U and r , by [34, Prop. 2.2]. The tangent space at $x \in \mathcal{O}_1$ is

$$T_x \mathcal{O} = D_x r(T_x U). \quad (1.11)$$

For a smooth point $x \in \mathcal{O}_\infty$, the tangent space $T_x \mathcal{O}$ is a sc-Banach space since $D_x r$ is a sc-operator. The *reduced tangent space* [34, Def. 2.15] is the subspace of $T_x \mathcal{O}$ defined by

$$T_x^R \mathcal{O} := T_x \mathcal{O} \cap \mathbb{E}_x, \quad (1.12)$$

where $\mathbb{E}_x \subset \mathbb{E}$ is the subspace from (1.9). Conceptually, $T_x^R \mathcal{O}$ consists of those tangent vectors that point in directions that preserve the degeneracy index, i.e. they lie along the intersection of all of the local faces that contain x . Note that in [34, Def. 2.15] the reduced tangent space is only defined at smooth points $x \in \mathcal{O}_\infty$. This is because $T_x^R \mathcal{O}$ can be proven to be invariant under sc-diffeomorphisms $\varphi : \mathcal{O} \rightarrow \mathcal{O}'$, i.e. $D_x \varphi(T_x^R \mathcal{O}) = T_{\varphi(x)}^R \mathcal{O}'$ holds, only for smooth points x ; see [34, Prop. 2.8]. This invariance proves that the reduced tangent space at a smooth point in a M -polyfold is well-defined, i.e. independent of the chart. The invariance is proven using the characterization [34, Lem. 2.4] of the reduced tangent space $T_x^R \mathcal{O}$ at smooth points $x \in \mathcal{O}_\infty$ as the closure of the space of derivatives of sc-smooth paths $\gamma : (-\epsilon, \epsilon) \rightarrow \mathcal{O}$ satisfying $\gamma(0) = x$. This only works for smooth points $x \in \mathcal{O}_\infty$ since the image of any sc-continuous map $(-\epsilon, \epsilon) \rightarrow \mathcal{O}$ is contained in $\mathcal{O}_\infty$ because $(-\epsilon, \epsilon) \subset \mathbb{R}$ has the trivial sc-structure where all levels are equal.

As discussed in [34], we must require sc-retractions to be well-behaved with respect to the boundary faces of the partial quadrant C in the following way in order for the full polyfold machinery to work as required in applications.

Definition 1.3.1. [34, Def. 2.17] Consider an open subset U of a partial quadrant C of a sc-Banach space $\mathbb{E}$. A sc-retraction $r : U \rightarrow U$ is called a **tame sc-retraction** if it satisfies the following conditions:

- (1) $d_C(r(x)) = d_C(x)$ for all $x \in U$.
- (2) At every smooth point $x \in \mathcal{O}_\infty = \mathcal{O} \cap E_\infty$, there exists a sc-subspace $A \subset \mathbb{E}$ such that $\mathbb{E} = T_x \mathcal{O} \oplus A$ and $A \subset \mathbb{E}_x$ (see (1.9) for $\mathbb{E}_x$).

If so, then the sc-retract $\mathcal{O} = r(U)$ is called a **tame sc-retract** (and so is the triple $(\mathcal{O}, C, \mathbb{E})$).

We introduce the following new notions of $\mathbb{R}^n$ -sliced sc-retractions and $\mathbb{R}^n$ -sliced sc-retracts.

Definition 1.3.2. Consider a partial quadrant C of a sc-Banach space $\mathbb{E}$ and an open subset $U \subset \mathbb{R}^n \times C$ for some $n \geq 0$. A tame sc-retraction $r : U \rightarrow U$ is called a **$\mathbb{R}^n$ -sliced sc-retraction** if it satisfies

$$\pi_{\mathbb{R}^n} \circ r = \pi_{\mathbb{R}^n} \text{ on } U, \quad (1.13)$$

i.e. r preserves the $\mathbb{R}^n$ -coordinate.

If so, then the tame sc-retract $\mathcal{O} = r(U)$ (and the triple $(\mathcal{O}, \mathbb{R}^n \times C, \mathbb{R}^n \times \mathbb{E})$) is called a **$\mathbb{R}^n$ -sliced sc-retract**.

In the following lemma, we show that for any $\mathbb{R}^n$ -sliced sc-retract $\mathcal{O}$ in $\mathbb{R}^n \times C$, the set $\tilde{\mathcal{O}} := \mathcal{O} \cap (\{0\} \times C)$ is a tame sc-retract. Later, we use the inclusion $\tilde{\mathcal{O}} \subset \mathcal{O}$ to define the local models for a slice $\tilde{\mathcal{B}} \subset \mathcal{B}$ (Definition 1.5.7), which is our new notion of a M -polyfold $\tilde{\mathcal{B}}$ embedded with finite codimension in an ambient M -polyfold $\mathcal{B}$.

Lemma 1.3.3. Consider a partial quadrant C of a sc-Banach space $\mathbb{E}$ and a $\mathbb{R}^n$ -sliced sc-retract $(\mathcal{O}, \mathbb{R}^n \times C, \mathbb{R}^n \times \mathbb{E})$.

Then, for any open subset $U \subset \mathbb{R}^n \times C$ and $\mathbb{R}^n$ -sliced sc-retraction $r : U \rightarrow U$ such that $r(U) = \mathcal{O}$, the set $\tilde{U} := U \cap (\{0\} \times C)$ is open in C and the restriction

$$\tilde{r} := r|_{\tilde{U}} : \tilde{U} \rightarrow \tilde{U}$$

is a tame sc-retraction onto $\tilde{\mathcal{O}} := \tilde{r}(\tilde{U})$. We call $\tilde{r}$ the **tame sc-retraction induced by r** .

Moreover,

$$\tilde{\mathcal{O}} = \mathcal{O} \cap (\{0\} \times C) \quad (1.14)$$

holds, so in particular $\tilde{\mathcal{O}}$ does not depend on the choices of U and r . We may view $\tilde{\mathcal{O}}$ as a subset of C , and we call $(\tilde{\mathcal{O}}, C, \mathbb{E})$ the **tame sc-retract induced by the $\mathbb{R}^n$ -sliced sc-retract $(\mathcal{O}, \mathbb{R}^n \times C, \mathbb{R}^n \times \mathbb{E})$** .

At every $x \in \tilde{\mathcal{O}}_1$, the inclusion $\tilde{\mathcal{O}} \subset \mathcal{O}$ induces an inclusion of tangent spaces $T_x \tilde{\mathcal{O}} \subset T_x \mathcal{O}$ satisfying

$$T_x \tilde{\mathcal{O}} = T_x \mathcal{O} \cap (\{0\} \times \mathbb{E}), \quad (1.15)$$

$$T_x^R \tilde{\mathcal{O}} = T_x^R \mathcal{O} \cap (\{0\} \times \mathbb{E}), \quad (1.16)$$

and

$$T_x \mathcal{O} / T_x \tilde{\mathcal{O}} \cong \mathbb{R}^n. \quad (1.17)$$

We say that $\tilde{\mathcal{O}}$ is **codimension- n** in $\mathcal{O}$.

If $x \in \tilde{\mathcal{O}}_\infty$ is a smooth point, then the inclusion $T_x \tilde{\mathcal{O}} \hookrightarrow T_x \mathcal{O}$ induces a linear isomorphism

$$T_x \tilde{\mathcal{O}} / T_x^R \tilde{\mathcal{O}} \cong T_x \mathcal{O} / T_x^R \mathcal{O}. \quad (1.18)$$

Proof. The defining property (1.13) of the $\mathbb{R}^n$ -sliced sc-retraction r implies $r(\tilde{U}) \subset \tilde{U}$, so indeed the map $\tilde{r} := r|_{\tilde{U}} : \tilde{U} \rightarrow \tilde{U}$ takes values in $\tilde{U}$. Moreover, $\tilde{r}$ inherits sc-smoothness and the retraction property $\tilde{r} \circ \tilde{r} = \tilde{r}$ from the corresponding properties of r . So $\tilde{r}$ is a sc-retraction onto the sc-retract $\tilde{\mathcal{O}}$. We prove the other statements in the lemma before showing that $\tilde{r}$ is tame.

We now verify that (1.14) holds. The forwards inclusion is immediate from the definitions of the sets involved. To prove the reverse inclusion, let $x \in \mathcal{O} \cap (\{0\} \times C)$. Then since $\mathcal{O} \subset U$ we have $x \in \tilde{U}$ and so $\tilde{r}(x) \in \tilde{\mathcal{O}}$. We claim that $x = \tilde{r}(x)$, proving (1.14). Indeed, since $x \in \mathcal{O}$ and r is a retraction with image $\mathcal{O}$, it follows that $x = r(x) = \tilde{r}(x)$.

We now verify (1.15) and (1.17). Let $x \in \tilde{\mathcal{O}}_1$. By definition of tangent space (1.11) of a sc-retract, we have

$$T_x \mathcal{O} = D_x r(\mathbb{R}^n \times \mathbb{E}) \subset \mathbb{R}^n \times \mathbb{E}.$$

Since r preserves the $\mathbb{R}^n$ -coordinate by the sliced retraction property (1.13), the same is true for $D_x r$, from which it follows that $D_x r(\mathbb{R}^n \times \mathbb{E}) \cap (\{0\} \times \mathbb{E}) = D_x r(\{0\} \times \mathbb{E})$. Hence we have

$$T_x \mathcal{O} \cap (\{0\} \times \mathbb{E}) = D_x \tilde{r}(\{0\} \times \mathbb{E}) = T_x \tilde{\mathcal{O}},$$

proving (1.15). Moreover, the projection $\pi : T_x\mathcal{O} \rightarrow \mathbb{R}^n$ to the first factor of $\mathbb{R}^n \times \mathbb{E}$ is a surjection. Since $\ker(\pi) = T_x\mathcal{O} \cap (\{0\} \times \mathbb{E})$, we conclude that π induces an isomorphism $T_x\mathcal{O}/T_x\tilde{\mathcal{O}} \cong \mathbb{R}^n$, proving (1.17).

To verify (1.16), first note that by (1.8),(1.9), we have

$$(\mathbb{R}^n \times \mathbb{E})_x \cap (\{0\} \times \mathbb{E}) = (\{0\} \times \mathbb{E})_x, \quad (1.19)$$

and hence we have $T_x^R\tilde{\mathcal{O}} = T_x\tilde{\mathcal{O}} \cap (\{0\} \times \mathbb{E})_x = T_x\tilde{\mathcal{O}} \cap (\mathbb{R}^n \times \mathbb{E})_x \cap (\{0\} \times \mathbb{E}) = T_x\mathcal{O} \cap (\mathbb{R}^n \times \mathbb{E})_x \cap (\{0\} \times \mathbb{E}) = T_x^R\mathcal{O} \cap (\{0\} \times \mathbb{E})$, as required.

To verify (1.18), it suffices to consider the case $x = 0$ and $C = [0, \infty)^s \times \mathbb{E}' \subset \mathbb{R}^s \times \mathbb{E}' = \mathbb{E}$ is in standard form (1.5). By (1.8), we have $(\mathbb{R}^n \times \mathbb{R}^s \times \mathbb{E}')_x = \mathbb{R}^n \times \{0\}^s \times \mathbb{E}'$. Since r preserves the degeneracy index by Definition 1.3.1, we claim it follows that

$$D_x r(\mathbb{R}^n \times \{0\}^s \times \mathbb{E}') \subset \mathbb{R}^n \times \{0\}^s \times \mathbb{E}'.$$

Indeed, given a smooth point $\xi \in \mathbb{R}^n \times \{0\}^s \times \mathbb{E}'_\infty$, there exists a sc-smooth path $\alpha : (-\epsilon, \epsilon) \rightarrow U \cap (\mathbb{R}^n \times \{0\}^s \times \mathbb{E}')$ satisfying $\alpha(0) = x$ and $\alpha'(0) = \xi$. Since r preserves degeneracy index we have $r \circ \alpha((-\epsilon, \epsilon)) \subset \mathbb{R}^n \times \{0\}^s \times \mathbb{E}'$, and hence we have $D_x r(\xi) \in \mathbb{R}^n \times \{0\}^s \times \mathbb{E}'$. For an arbitrary point $\xi \in \mathbb{R}^n \times \{0\}^s \times \mathbb{E}'$ the result follows by considering a sequence $\{\xi_k\}_{k \geq 0}$ of smooth points that converges to ξ , which exists by density of the inclusion $\mathbb{E}'_\infty \subset \mathbb{E}'_0$. Note that this essentially uses the fact that x is a smooth point.

Since by definition (1.12) we have $T_x^R\mathcal{O} = D_x r(\mathbb{R}^n \times \mathbb{R}^s \times \mathbb{E}') \cap (\mathbb{R}^n \times \{0\}^s \times \mathbb{E}')$ and moreover the retraction property $D_x r \circ D_x r = D_x r$ holds, it follows that

$$T_x^R\mathcal{O} = D_x r(\mathbb{R}^n \times \{0\}^s \times \mathbb{E}').$$

Hence, since $D_x r$ preserves the $\mathbb{R}^n$ -coordinate, the projection $\pi : T_x\mathcal{O} \rightarrow \mathbb{R}^n$ restricts to a surjection $\pi : T_x^R\mathcal{O} \rightarrow \mathbb{R}^n$. The kernel of this surjection is $T_x^R\tilde{\mathcal{O}}$ by (1.16). So we have a short exact sequence of sc-Banach spaces $0 \rightarrow T_x^R\tilde{\mathcal{O}} \rightarrow T_x^R\mathcal{O} \xrightarrow{\pi} \mathbb{R}^n \rightarrow 0$ that includes into the short exact sequence $0 \rightarrow T_x\tilde{\mathcal{O}} \rightarrow T_x\mathcal{O} \xrightarrow{\pi} \mathbb{R}^n$. This implies (1.18).

To prove the lemma, it remains to show that $\tilde{r}$ is tame. The $\mathbb{R}^n$ -sliced sc-retraction r is tame by definition. Hence, for all $x \in \tilde{U}$, we compute, using (1.6) and property Definition 1.3.1(1) of r ,

$$d_{\{0\} \times C}(\tilde{r}(x)) = d_{\mathbb{R}^n \times C}(\tilde{r}(x)) = d_{\mathbb{R}^n \times C}(r(x)) = d_{\mathbb{R}^n \times C}(x) = d_{\{0\} \times C}(x),$$

verifying property Definition 1.3.1(1) for $\tilde{r}$.

To verify that $\tilde{r}$ satisfies property Definition 1.3.1(2), let $x \in \tilde{\mathcal{O}}_\infty$. Then $x \in \mathcal{O}_\infty$, and so by the corresponding property of r and by [34, Prop. 2.9], the sc-subspace $A := (id_{\mathbb{R}^n \times \mathbb{E}} - D_x r)(\mathbb{R}^n \times \mathbb{E})$ of $\mathbb{R}^n \times \mathbb{E}$ satisfies

$$\mathbb{R}^n \times \mathbb{E} = T_x\mathcal{O} \oplus A \quad (1.20)$$

and $A \subset (\mathbb{R}^n \times \mathbb{E})_x$. By the sliced retraction property (1.13) of r and the definition of A , we conclude that $A \subset \{0\} \times \mathbb{E}$ holds. Then we have $A \subset (\mathbb{R}^n \times \mathbb{E})_x \cap (\{0\} \times \mathbb{E}) = (\{0\} \times \mathbb{E})_x$

by (1.19). We claim that $\{0\} \times \mathbb{E} = T_x \tilde{\mathcal{O}} \oplus A$ holds, completing the proof that $\tilde{r}$ is tame. Indeed, it follows from (1.15), (1.20), and $A \subset \{0\} \times \mathbb{E}$ that we have

$$\{0\} \times \mathbb{E} = (T_x \mathcal{O} \cap (\{0\} \times \mathbb{E})) \oplus A = T_x \tilde{\mathcal{O}} \oplus A.$$

□

1.3.2 Sliced bundle retracts

In this section, we introduce the new notion of $\mathbb{R}^n$ -sliced bundle retracts (Definition 1.3.4). We prove in Lemma 1.3.5 that a $\mathbb{R}^n$ -sliced bundle retract K covering a $\mathbb{R}^n$ -sliced sc-retract $\mathcal{O}$ induces a tame bundle retract $\tilde{K} \subset K$ covering the induced tame sc-retract $\tilde{\mathcal{O}} \subset \mathcal{O}$ from Lemma 1.3.3. The global definition of a slice $\tilde{\mathcal{B}} \subset \mathcal{B}$ of a bundle $\rho : \mathcal{E} \rightarrow \mathcal{B}$ (Definition 1.5.7) is then a subspace such that around every point $x \in \tilde{\mathcal{B}}$ there is a bundle chart for ρ to a $\mathbb{R}^{n_x}$ -sliced bundle retract K that locally identifies $\rho^{-1}(\tilde{\mathcal{B}})$ with the induced tame bundle retract $\tilde{K}$.

We first recall the local structure of tame strong bundles: tame bundle retracts. Consider a relatively open subset U of a partial quadrant C of a sc-Banach space $\mathbb{E}$, and another sc-Banach space $\mathbb{F}$. Then the trivial bundle

$$U \triangleleft \mathbb{F} \rightarrow U \tag{1.21}$$

has total space $U \triangleleft \mathbb{F} = U \times \mathbb{F}$ as a set, and the map is projection onto U . The triangle $\triangleleft$ signifies the extra structure of a double filtration on the set $U \times \mathbb{F}$. That is, for $0 \leq k \leq m+1$, we have

$$(U \triangleleft \mathbb{F})_{m,k} := U_m \oplus F_k.$$

Then, for $i = 0, 1$, we define the sc-structure $(U \triangleleft \mathbb{F})[i]$ by

$$((U \triangleleft \mathbb{F})[i])_m := U_m \oplus F_{m+i}, \quad m \geq 0. \tag{1.22}$$

The purpose of defining these two filtrations is that they correspond to the two notions of smoothness of a section of a bundle that are important for polyfold theory. Precisely, a section $s : U \rightarrow U \triangleleft \mathbb{F}$ is called *sc-smooth* if it is sc-smooth as a map to $(U \triangleleft \mathbb{F})[0]$. If, moreover, we have $s(U) \subset (U \triangleleft \mathbb{F})[1]$ and the map $s : U \rightarrow (U \triangleleft \mathbb{F})[1]$ is sc-smooth, then s is called a *sc⁺-section*. See [34, Def. 2.24] for a detailed discussion.

A *strong bundle map* $\Phi : U \triangleleft \mathbb{F} \rightarrow U' \triangleleft \mathbb{F}'$ [34, Def. 2.22] is a map which preserves the double filtration and is of the form $\Phi(x, \xi) = (\varphi(x), \Gamma(x, \xi))$, where the map $\Gamma : U \triangleleft \mathbb{F} \rightarrow \mathbb{F}'$ is linear in ξ . In addition, for $i = 0, 1$, we require that the maps

$$\Phi : (U \triangleleft \mathbb{F})[i] \rightarrow (U' \triangleleft \mathbb{F}') [i]$$

are sc-smooth. A *strong bundle isomorphism* is an invertible strong bundle map whose inverse is also a strong bundle map.

To extend (1.21) to a notion of a trivial bundle over a sc-retract, we employ the following notion of a retraction in the fibers. A *strong bundle retraction* is a strong bundle map $R : U \triangleleft \mathbb{F} \rightarrow U \triangleleft \mathbb{F}$ satisfying $R \circ R = R$. As a consequence, the map R has the form

$$R(x, \xi) = (r(x), \Gamma(x, \xi)), \quad (1.23)$$

where $r : U \rightarrow U$ is a sc-smooth retraction and $\Gamma(x, \cdot) : \mathbb{F} \rightarrow \mathbb{F}$ is a linear projection for every $x \in U$. If r is tame, then R is called a *tame strong bundle retraction*. The image $K := R(U \triangleleft \mathbb{F})$ of R is called a *strong bundle retract* [34, Def. 2.23], as is the triple $(K, C \triangleleft \mathbb{F}, \mathbb{E} \triangleleft \mathbb{F})$. We say that K *covers the sc-retract* $\mathcal{O} = r(U)$. If R is tame, then K is called a *tame strong bundle retract*. The projection $U \triangleleft \mathbb{F} \rightarrow U$ induces a mapping $K \rightarrow \mathcal{O}$, which we call a *strong local bundle model*.

We now introduce the new notion of a $\mathbb{R}^n$ -sliced bundle retract.

Definition 1.3.4. Consider a partial quadrant C of a sc-Banach space $\mathbb{E}$, an open subset $U \subset \mathbb{R}^n \times C$ for some $n \geq 0$, and another sc-Banach space $\mathbb{F}$.

A tame bundle retraction $R : U \triangleleft \mathbb{F} \rightarrow U \triangleleft \mathbb{F}$ is called a **$\mathbb{R}^n$ -sliced bundle retraction** if the tame sc-retraction $r : U \rightarrow U$ covered by R (see (1.23)) is a $\mathbb{R}^n$ -sliced sc-retraction (Definition 1.3.2).

If so, then the tame bundle retract $K = R(U \triangleleft \mathbb{F})$ (and the triple $(K, \mathbb{R}^n \times C \triangleleft \mathbb{F}, \mathbb{R}^n \times \mathbb{E} \triangleleft \mathbb{F})$) is called a **$\mathbb{R}^n$ -sliced bundle retract** and the tame local bundle model $K \rightarrow \mathcal{O} := r(U)$ is called a **$\mathbb{R}^n$ -sliced local bundle model**.

In the following lemma, we show that for any $\mathbb{R}^n$ -sliced bundle retract K in $\mathbb{R}^n \times C \triangleleft \mathbb{F}$, the set $\tilde{K} := K \cap (\{0\} \times C \triangleleft \mathbb{F})$ is a tame bundle retract. Later, we use the inclusion $\tilde{K} \subset K$ to define the local models for the restriction of a bundle to a slice (Definition 1.5.7).

Lemma 1.3.5. Consider a partial quadrant C of a sc-Banach space $\mathbb{E}$, another sc-Banach space $\mathbb{F}$, and a $\mathbb{R}^n$ -sliced bundle retract $(K, \mathbb{R}^n \times C \triangleleft \mathbb{F}, \mathbb{R}^n \times \mathbb{E} \triangleleft \mathbb{F})$ covering a $\mathbb{R}^n$ -sliced sc-retract $(\mathcal{O}, \mathbb{R}^n \times C, \mathbb{R}^n \times \mathbb{E})$. Let $\pi : K \rightarrow \mathcal{O}$ denote the local bundle model given by restriction of the projection along the fiber $\mathbb{R}^n \times C \triangleleft \mathbb{F} \rightarrow \mathbb{R}^n \times C$.

Then, for any open subset $U \subset \mathbb{R}^n \times C$ and $\mathbb{R}^n$ -sliced bundle retraction $R : U \triangleleft \mathbb{F} \rightarrow U \triangleleft \mathbb{F}$ covering a $\mathbb{R}^n$ -sliced sc-retraction $r : U \rightarrow U$ such that $r(U) = \mathcal{O}$ and $R(U \triangleleft \mathbb{F}) = K$, the set $\tilde{U} := U \cap (\{0\} \times C)$ is open in C and the restriction

$$\tilde{R} := R|_{\tilde{U} \triangleleft \mathbb{F}} : \tilde{U} \triangleleft \mathbb{F} \rightarrow \tilde{U} \triangleleft \mathbb{F}$$

is a tame bundle retraction onto $\tilde{K} := \tilde{R}(\tilde{U} \triangleleft \mathbb{F})$ covering the induced tame sc-retraction $\tilde{r} : \tilde{U} \rightarrow \tilde{U}$ onto the induced tame sc-retract $\tilde{\mathcal{O}} = \mathcal{O} \cap (\{0\} \times C)$ (see Lemma 1.3.3). We call $\tilde{R}$ the **tame bundle retraction induced by R** .

Moreover,

$$\tilde{K} = K \cap (\{0\} \times C \triangleleft \mathbb{F}) = \pi^{-1}(\tilde{\mathcal{O}}) \quad (1.24)$$

holds, so in particular $\tilde{K}$ does not depend on the choices of U and R . We may view $\tilde{K}$ as a subset of $C \triangleleft \mathbb{F}$, and we call $(\tilde{K}, C \triangleleft \mathbb{F}, \mathbb{E} \triangleleft \mathbb{F})$ the **tame bundle retract induced by the $\mathbb{R}^n$ -sliced bundle retract** $(K, \mathbb{R}^n \times C \triangleleft \mathbb{F}, \mathbb{R}^n \times \mathbb{E} \triangleleft \mathbb{F})$.

In particular, the $\mathbb{R}^n$ -sliced local bundle model $\pi : K \rightarrow \mathcal{O}$ restricted to $\tilde{K}$ is a tame local bundle model

$$\tilde{\pi} := \pi|_{\tilde{K}} : \tilde{K} \rightarrow \tilde{\mathcal{O}}, \quad (1.25)$$

which we call the **induced tame local bundle model**.

Proof. The map $\tilde{R}$ is a strong bundle map and satisfies $\tilde{R} \circ \tilde{R} = \tilde{R}$, by the corresponding properties of R , so $\tilde{R}$ is a strong bundle retraction. Moreover, $\tilde{R}$ is tame because it covers the tame sc-retraction $\tilde{r}$.

We now verify that the first equality in (1.24) holds. The forwards inclusion is immediate from the definitions. To see the reverse inclusion, let $(x \triangleleft \xi) \in K \cap (\{0\} \times C \triangleleft \mathbb{F})$. Then $x = \pi(x \triangleleft \xi) \in \mathcal{O} \cap (\{0\} \times C) = \tilde{\mathcal{O}}$, where $\pi : K \rightarrow \mathcal{O}$ is the local bundle model given by restriction of the projection along the fiber $\pi : \mathbb{R}^n \times C \triangleleft \mathbb{F} \rightarrow \mathbb{R}^n \times C$. In particular, $x \triangleleft \xi$ is in the domain of $\tilde{R} = R|_{\tilde{U} \triangleleft \mathbb{F}}$. Since R is a retraction onto K , it follows that $(x \triangleleft \xi) = R(x \triangleleft \xi) = \tilde{R}(x \triangleleft \xi) \in \text{im}(\tilde{R}) = \tilde{K}$, as required. The second equality in (1.24) holds from the definitions and $\tilde{\mathcal{O}} = \mathcal{O} \cap (\{0\} \times C)$. $\square$

1.3.3 Sliced sc-Fredholm germs

In this section, we review the standard notion of a local sc-Fredholm germ (Definition 1.3.6) and we introduce the new notions of a tame sc-Fredholm germ (Definition 1.3.7) and a $\mathbb{R}^n$ -sliced sc-Fredholm germ (Definition 1.3.8). We later use tame sc-Fredholm germs as the local models for our new notion of a tame sc-Fredholm section $\sigma : \mathcal{B} \rightarrow \mathcal{E}$ (Definition 1.5.4) of a tame strong bundle $\rho : \mathcal{E} \rightarrow \mathcal{B}$, and we use $\mathbb{R}^n$ -sliced sc-Fredholm germs as the local models in our definition of a slice $\tilde{\mathcal{B}} \subset \mathcal{B}$ (Definition 1.5.7) of a sc-Fredholm section σ .

We prove in Lemma 1.3.10 that a $\mathbb{R}^n$ -sliced sc-Fredholm germ $\sigma : \mathcal{O} \rightarrow K$ of a tame local bundle model $K \rightarrow \mathcal{O}$ induces a tame sc-Fredholm germ $\tilde{\sigma} := \sigma|_{\tilde{\mathcal{O}}} : \tilde{\mathcal{O}} \rightarrow \tilde{K}$ of the induced tame local bundle model $\tilde{K} \rightarrow \tilde{\mathcal{O}}$ from (1.25). The sc-Fredholm index (1.27) satisfies $\text{ind}(\tilde{\sigma}) = \text{ind}(\sigma) - n$. The global definition of a slice $\tilde{\mathcal{B}} \subset \mathcal{B}$ of a sc-Fredholm section $\sigma : \mathcal{B} \rightarrow \mathcal{E}$ is then a slice of the bundle such that around every smooth point $x \in \tilde{\mathcal{B}}_\infty$ there exists a chart in which σ is a $\mathbb{R}^{n_x}$ -sliced sc-Fredholm germ and such that the restriction $\sigma|_{\tilde{\mathcal{B}}} : \tilde{\mathcal{B}} \rightarrow \rho^{-1}(\tilde{\mathcal{B}})$ is locally identified with the induced tame sc-Fredholm germ.

All currently known sc-Fredholm sections arising in applications are tame sc-Fredholm by construction; see Section 1.5.1 for a discussion of the tame sc-Fredholm property of the Cauchy-Riemann section $\bar{\partial}_J$.

We now briefly review sc-germ language. Let C be a partial quadrant of a sc-Banach space $\mathbb{E}$. Then a *sc-germ of neighborhoods around* $0 \in C$ [34, Def. 3.1], denoted

$$U(C, 0),$$

is a sequence

$$U = U(0) \supset U(1) \supset U(2) \supset \cdots$$

where $U(m)$ is a relatively open neighborhood of 0 in $C \cap E_m$. We often write $U = U(C, 0)$ for brevity. A sc^0 -germ $h : U(C, 0) \rightarrow \mathbb{F}$ [34, Def. 3.2] into the sc-Banach space $\mathbb{F}$ is a continuous map $h : U(0) \rightarrow \mathbb{F}$ such that $h(U(m)) \subset F_m$ and $h : U(m) \rightarrow F_m$ is continuous for all $m \geq 0$. A sc^1 -germ $h : U(C, 0) \rightarrow F$ is a sc^0 -germ which is of class sc^1 in the same sense as for a usual map except that the sc-differential $D_x h$ is required to exist only for $x \in U(1)$ (where $U(1)$ can be smaller than $U(0) \cap E_1$ in the germ case); see [34, Def. 3.2] for a precise definition. We can in turn define a tangent map on the tangent of a germ and then iterate the notion of sc^1 to define a *sc-smooth germ*.

It is convenient to denote any section of a trivial bundle by $h(x) = (x, \mathbf{h}(x)) : U \rightarrow U \triangleleft \mathbb{F}$. We refer to $\mathbf{h} : U \rightarrow \mathbb{F}$ as the *principal part* of h .

Recall the standard notion of a local sc-Fredholm germ.

Definition 1.3.6. [34, Defs. 3.5, 3.6, 3.7] Consider a tame strong bundle retract $(K, C \triangleleft \mathbb{F}, \mathbb{E} \triangleleft \mathbb{F})$ covering the tame sc-retract $(\mathcal{O}, C, \mathbb{E})$ and a sc-smooth section $\sigma : \mathcal{O} \rightarrow K$ of the local bundle model $K \rightarrow \mathcal{O}$. Assume $0 \in \mathcal{O}$.

Then σ is called a **local sc-Fredholm germ** if the following conditions hold:

- (a) There exists a sc-germ of neighborhoods $U(C, 0)$ around $0 \in C$ and a tame sc-retraction $r : U \rightarrow U$ onto $r(U) = \mathcal{O}$ covered by a tame bundle retraction $R : U \triangleleft \mathbb{F} \rightarrow U \triangleleft \mathbb{F}$ onto $R(U \triangleleft \mathbb{F}) = K$.
- (b) The principal part $\sigma : \mathcal{O} \rightarrow \mathbb{F}$ of σ has the property that the composition

$$\sigma \circ r : U(C, 0) \rightarrow \mathbb{F}$$

possesses a **filling**

$$h : U(C, 0) \rightarrow U(C, 0) \triangleleft \mathbb{F},$$

which is a section of the trivial bundle $U(C, 0) \triangleleft \mathbb{F} \rightarrow U(C, 0)$ whose principal part $\mathbf{h} : U(C, 0) \rightarrow \mathbb{F}$ is a sc-smooth germ and such that the following conditions (i) – (iii) hold. Recall that R is of the form $R(x, \xi) = (r(x), \Gamma(x, \xi))$ where $\Gamma(x, -) : \mathbb{F} \rightarrow \mathbb{F}$ is a linear projection.

- (i) $\sigma(x) = h(x)$ for $x \in \mathcal{O}$.
- (ii) If $x \in U$ and $\mathbf{h}(x) = \Gamma(r(x), \mathbf{h}(x))$, then $x \in \mathcal{O}$.
- (iii) The linearization of the map $x \mapsto (id_{\mathbb{F}} - \Gamma(r(x), \cdot))\mathbf{h}(x)$ at the point 0, restricted to $\ker D_0 r$, defines a linear sc-isomorphism $\ker D_0 r \rightarrow \ker \Gamma(0, \cdot)$.
- (c) There exists a sc^+ -section $\mathfrak{s} : U \rightarrow U \triangleleft \mathbb{F}$ satisfying $\mathfrak{s}(0) = h(0)$, a sc-Banach space $\mathbb{W}$, a sc-germ of neighborhoods

$$U' = U'(C', 0) \subset C' = [0, \infty)^s \times \mathbb{R}^{k-s} \times \mathbb{W}$$

centered around $0 \in C'$ for some $k \geq s \geq 0$, and a strong bundle isomorphism

$$\Psi : U \triangleleft \mathbb{F} \rightarrow U' \triangleleft \mathbb{R}^{k'} \times \mathbb{W}$$

(for some $k' \geq 0$) covering a sc-diffeomorphism

$$\psi : U \rightarrow U'$$

satisfying $\psi(0) = 0$ and such that the principal part of the section

$$b := \Psi \circ (h - \mathfrak{s}) \circ \psi^{-1} : U' \rightarrow U' \triangleleft \mathbb{R}^{k'} \times \mathbb{W}$$

is a **basic germ**. This means that the principal part

$$\mathbf{b} : U' \rightarrow \mathbb{R}^{k'} \times \mathbb{W}$$

of b is a sc-smooth germ satisfying $\mathbf{b}(0) = 0$ and having the following property: Letting $P : \mathbb{R}^{k'} \times \mathbb{W} \rightarrow \mathbb{W}$ denote projection onto $\mathbb{W}$, the germ $P \circ \mathbf{b} : U' \rightarrow \mathbb{W}$ is of the form

$$P \circ \mathbf{b}(a, w) = w - B(a, w),$$

for $(a, w) \in ([0, \infty)^s \times \mathbb{R}^{k-s}) \times \mathbb{W}$, where B is a sc-smooth germ and $B(0) = 0$. Moreover, B satisfies the contraction property: for every $\epsilon > 0$ and integer $m \geq 0$ there exists $\delta > 0$ such that

$$\|(a, w)\|_m, \|(a, w')\|_m < \delta \implies \|B(a, w) - B(a, w')\|_m \leq \epsilon \|w - w'\|_m. \quad (1.26)$$

The notation $\|\cdot\|_m$ means the Banach norm in the m -level of the sc-structure. We use the convention³ that the m -norm $\|\cdot\|_m$ on $(\mathbb{R}^k \times \mathbb{W})_m = \mathbb{R}^k \times W_m$ is the sum of the standard Euclidean norm on $\mathbb{R}^k$ plus the W_m -norm.

The **index of the local sc-Fredholm germ** σ is the integer

$$\text{ind}(\sigma) := k - k', \quad (1.27)$$

where k and k' are the dimensions of the finite dimensional spaces split off in the domain $U' \subset ([0, \infty)^s \times \mathbb{R}^{k-s}) \times \mathbb{W}$ and codomain $\mathbb{R}^{k'} \times \mathbb{W}$, respectively, of the basic germ $\mathbf{b}$.

We now introduce a new class of sc-Fredholm germs, called tame sc-Fredholm.

Definition 1.3.7. A local sc-Fredholm germ $\sigma : \mathcal{O} \rightarrow K$ is called a **tame sc-Fredholm germ** if the structures that exist by Definition 1.3.6 of local sc-Fredholm germ can be chosen such that the partial quadrant C is in the standard form (1.5), i.e. $C = [0, \infty)^s \times \mathbb{E}' \subset$

³This convention is equivalent to using any norm on $\mathbb{R}^k$ and any standard norm on a Cartesian product that is equivalent to the sum norm.

$\mathbb{R}^s \times \mathbb{E}' = \mathbb{E}$ for some *sc-Banach space* $\mathbb{E}'$ and integer $s \geq 0$, and such that the required *sc-diffeomorphism* $\psi : U \rightarrow U'$ in Definition 1.3.6(c) is the restriction of a linear *sc-isomorphism* of the form

$$\begin{aligned} \psi = id_{[0, \infty)^s} \times \bar{\psi} : [0, \infty)^s \times \mathbb{E}' &\rightarrow [0, \infty)^s \times \mathbb{R}^{k-s} \times \mathbb{W}, \\ (v, e) &\mapsto (v, \bar{\psi}(e)) \end{aligned}$$

for some linear *sc-isomorphism*

$$\bar{\psi} : \mathbb{E}' \rightarrow \mathbb{R}^{k-s} \times \mathbb{W}.$$

The following new class of *sc-Fredholm germs*, called $\mathbb{R}^n$ -sliced *sc-Fredholm*, is defined for sections of $\mathbb{R}^n$ -sliced local bundle models $K \rightarrow \mathcal{O}$ (Definition 1.3.4).

Definition 1.3.8. Consider a $\mathbb{R}^n$ -sliced bundle retract (Definition 1.3.4)

$$(K, \mathbb{R}^n \times C \triangleleft \mathbb{F}, \mathbb{R}^n \times \mathbb{E} \triangleleft \mathbb{F})$$

covering the $\mathbb{R}^n$ -sliced *sc-retract*

$$(\mathcal{O}, \mathbb{R}^n \times C, \mathbb{R}^n \times \mathbb{E}).$$

Assume $0 \in \mathcal{O}$ and that the partial quadrant is in the standard form (1.5), i.e. $C = [0, \infty)^s \times \mathbb{E}' \subset \mathbb{R}^s \times \mathbb{E}' = \mathbb{E}$ for some $s \geq 0$ and *sc-Banach space* $\mathbb{E}'$.

Then a local *sc-Fredholm germ*

$$\sigma : \mathcal{O} \rightarrow K$$

of the $\mathbb{R}^n$ -sliced local bundle model $K \rightarrow \mathcal{O}$ is called a **$\mathbb{R}^n$ -sliced *sc-Fredholm germ*** if the structures that exist by Definition 1.3.6 of local *sc-Fredholm germ* can be chosen such that the *sc-Banach space* $\mathbb{W}$ and *sc-diffeomorphism* $\psi : U \rightarrow U'$ from Definition 1.3.6(c) have the following form: First, we have

$$\mathbb{W} = \tilde{\mathbb{W}} \times \mathbb{R}^n$$

for some other *sc-Banach space* $\tilde{\mathbb{W}}$. Moreover, the *sc-diffeomorphism* $\psi : U \rightarrow U'$ is of the form

$$\begin{aligned} \psi : \mathbb{R}^n \times [0, \infty)^s \times \mathbb{E}' \supset U &\rightarrow U' \subset [0, \infty)^s \times \mathbb{R}^{k-s} \times \tilde{\mathbb{W}} \times \mathbb{R}^n \\ (p, v, e) &\mapsto (v, \bar{\psi}(e), \lambda(p, v, e)) \end{aligned} \tag{1.28}$$

for some linear *sc-isomorphism*

$$\bar{\psi} : \mathbb{E}' \rightarrow \mathbb{R}^{k-s} \times \tilde{\mathbb{W}}$$

and such that the map

$$(p, v, e) \mapsto \lambda(p, v, e) \in \mathbb{R}^n$$

is C^1 on all levels $\mathbb{R}^n \times [0, \infty)^s \times E'_m$ for $m \geq 0$.

Remark 1.3.9. *Given a local sc-Fredholm germ $\sigma : \mathcal{O} \rightarrow K$ of a local bundle model $K \rightarrow \mathcal{O}$, all essential properties of the setup are preserved after restricting to the m -level of the sc-structure for any $m \geq 0$. Precisely, the m -shifted map $K^m \rightarrow \mathcal{O}^m$ is a local bundle model and $\sigma|_{\mathcal{O}^m} : \mathcal{O}^m \rightarrow K^m$ is a local sc-Fredholm germ [34, Cor. 5.1] with the same index $\text{ind}(\sigma|_{\mathcal{O}^m}) = \text{ind}(\sigma)$.*

In particular, if a local sc-Fredholm germ $\sigma : \mathcal{O} \rightarrow K$ satisfies the properties required of a $\mathbb{R}^n$ -sliced sc-Fredholm germ except for the C^1 regularity of the map $(p, v, e) \mapsto \lambda(p, v, e)$, then the 1-shift $\sigma|_{\mathcal{O}^1} : \mathcal{O}^1 \rightarrow K^1$ is $\mathbb{R}^n$ -sliced sc-Fredholm since the map λ has the required C^1 regularity on all levels $m \geq 1$ by [34, Prop. 1.7].

In the following lemma, we prove that a $\mathbb{R}^n$ -sliced sc-Fredholm germ of a $\mathbb{R}^n$ -sliced local bundle model $K \rightarrow \mathcal{O}$ (Definition 1.3.4) restricts to a tame sc-Fredholm germ of the induced tame local bundle model $\tilde{K} \rightarrow \tilde{\mathcal{O}}$ from (1.25).

Lemma 1.3.10. *Consider a $\mathbb{R}^n$ -sliced bundle retract $(K, \mathbb{R}^n \times C \triangleleft \mathbb{F}, \mathbb{R}^n \times \mathbb{E} \triangleleft \mathbb{F})$ covering a $\mathbb{R}^n$ -sliced sc-retract $(\mathcal{O}, \mathbb{R}^n \times C, \mathbb{R}^n \times \mathbb{E})$. Recall the induced tame local bundle model $\tilde{K} \rightarrow \tilde{\mathcal{O}}$ from (1.25).*

Let $\sigma : \mathcal{O} \rightarrow K$ be a $\mathbb{R}^n$ -sliced sc-Fredholm germ. Then, the restriction $\tilde{\sigma} := \sigma|_{\tilde{\mathcal{O}}} : \tilde{\mathcal{O}} \rightarrow \tilde{K}$ is a tame sc-Fredholm germ with index satisfying $\text{ind}(\tilde{\sigma}) = \text{ind}(\sigma) - n$.

*We call $\tilde{\sigma}$ the **tame sc-Fredholm germ induced by the $\mathbb{R}^n$ -sliced sc-Fredholm germ σ** .*

Proof. Since σ is a $\mathbb{R}^n$ -sliced sc-Fredholm germ, the partial quadrant is in the standard form $C = [0, \infty)^s \times \mathbb{E}' \subset \mathbb{R}^s \times \mathbb{E}' = \mathbb{E}$ and there exists an open subset $U \subset \mathbb{R}^n \times C$ and a $\mathbb{R}^n$ -sliced sc-retraction $r : U \rightarrow U$ onto $r(U) = \mathcal{O}$ that satisfies the conditions of Definition 1.3.2 and Definition 1.3.8, which we recall as we use them.

Since r satisfies the properties in Definition 1.3.6 of local sc-Fredholm germ, we can assume that there exists a sc-germ of neighborhoods $U(\mathbb{R}^n \times C, 0)$ around 0 satisfying $U(0) = U$ (i.e. the 0-level open set in the germ is the open set U from above) such that r is covered by a tame bundle retraction $R : U \triangleleft \mathbb{F} \rightarrow U \triangleleft \mathbb{F}$ such that $\sigma \circ r : U(\mathbb{R}^n \times C, 0) \rightarrow \mathbb{F}$ possesses a filling

$$h : U(\mathbb{R}^n \times C, 0) \rightarrow U(\mathbb{R}^n \times C, 0) \triangleleft \mathbb{F}.$$

Since r is a $\mathbb{R}^n$ -sliced sc-retraction covered by R , it follows by Definition 1.3.4 that R is a $\mathbb{R}^n$ -sliced bundle retraction. As in Lemma 1.3.5, set

$$\tilde{U} := U \cap (\{0\}^n \times C)$$

and denote the induced tame sc-retraction and tame bundle retraction by

$$\begin{aligned} \tilde{r} : \tilde{U} &\rightarrow \tilde{U}, \\ \tilde{R} : \tilde{U} \triangleleft \mathbb{F} &\rightarrow \tilde{U} \triangleleft \mathbb{F}, \end{aligned}$$

and the induced tame sc-retract and tame bundle retract by

$$\begin{aligned}\tilde{\mathcal{O}} &= \tilde{r}(\tilde{U}) = \mathcal{O} \cap (\{0\}^n \times C), \\ \tilde{K} &= \tilde{R}(\tilde{U} \triangleleft \mathbb{F}) = K \cap (\{0\}^n \times C \triangleleft \mathbb{F}).\end{aligned}$$

Also, let $\tilde{U}(C, 0)$ be the sc-germ of neighborhoods given on level- m by $\tilde{U}(m) := U(m) \cap (\{0\}^n \times C)$, or more concisely

$$\tilde{U}(C, 0) := U(\mathbb{R}^n \times C, 0) \cap (\{0\}^n \times C).$$

We claim that the restriction

$$\tilde{h} := h|_{\tilde{U}(C, 0)} : \tilde{U}(C, 0) \rightarrow \tilde{U}(C, 0) \triangleleft \mathbb{F}$$

is a filling for the composition $\tilde{\sigma} \circ \tilde{r} : \tilde{U}(C, 0) \rightarrow \mathbb{F}$. It is a section of the trivial bundle $\tilde{U}(C, 0) \triangleleft \mathbb{F} \rightarrow \tilde{U}(C, 0)$ and its principal part $\tilde{\mathbf{h}}$ is a sc-smooth germ, because we have

$$\tilde{\mathbf{h}} = \mathbf{h}|_{\tilde{U}(C, 0)}$$

and the corresponding properties hold for $\tilde{\mathbf{h}}$. It remains to verify that the filler properties in Definition 1.3.6(b).(i)-(iii) hold for $\tilde{\sigma}$ and $\tilde{h}$. These follow from the corresponding properties of σ and h , as we now describe. Property (i) is immediate since $\tilde{\sigma}$ and $\tilde{h}$ are restrictions of σ and h , respectively. Write $R(x, \xi) = (r(x), \Gamma(x, \xi))$, as in (1.23). Consider the restriction $\tilde{I} := I|_{\tilde{U} \triangleleft \mathbb{F}}$. Then we have $\tilde{R}(x, \xi) = (\tilde{r}(x), \tilde{I}(x, \xi))$. To verify (ii), let $x \in \tilde{U}$ and assume $\tilde{\mathbf{h}}(x) = \tilde{I}(\tilde{r}(x), \tilde{\mathbf{h}}(x))$. It follows that $\mathbf{h}(x) = \Gamma(r(x), \mathbf{h}(x))$, which implies $x \in \mathcal{O}$ by property (ii) for $\mathbf{h}$. Hence $x \in \tilde{U} \cap \mathcal{O} = \tilde{\mathcal{O}}$, as required. It remains to verify (iii) for $\tilde{\mathbf{h}}$. We claim that

$$\ker D_0 \tilde{r} = \ker D_0 r. \tag{1.29}$$

The forwards inclusion follows from $\tilde{r}$ being the restriction of r . To see the reverse inclusion, let $\xi \in \ker D_0 r$. By the defining property (1.13) of a $\mathbb{R}^n$ -sliced sc-retraction, we see that $D_0 r$ preserves the $\mathbb{R}^n$ -coordinate of ξ and hence $\xi \in \{0\}^n \times \mathbb{R}^s \times \mathbb{E}'$. In particular, ξ is in the domain of $D_0 \tilde{r}$, and moreover $D_0 \tilde{r}(\xi) = D_0 r(\xi) = 0$. Hence (1.29) holds. We now verify that Definition 1.3.6(iii) holds for $\tilde{\mathbf{h}}$. From the corresponding property of $\mathbf{h}$, the linearization $D_0 L$ at 0 of the map

$$\begin{aligned}L : U &\rightarrow \mathbb{F} \\ x &\mapsto (id_{\mathbb{F}} - \Gamma(r(x), \cdot))\mathbf{h}(x)\end{aligned}$$

restricts to a linear sc-isomorphism $\ker D_0 r \rightarrow \ker \Gamma(0, \cdot)$. We must show that the linearization at 0 of the map

$$\begin{aligned}\tilde{L} : \tilde{U} &\rightarrow \mathbb{F} \\ x &\mapsto (id_{\mathbb{F}} - \tilde{I}(\tilde{r}(x), \cdot))\tilde{\mathbf{h}}(x)\end{aligned}$$

restricts to a linear sc-isomorphism $\ker D_0 \tilde{r} \rightarrow \ker \tilde{F}(0, \cdot) = \ker F(0, \cdot)$. This follows from (1.29) and since $\tilde{L}$ and $\tilde{r}$ are the restrictions of L and r , respectively, to $\tilde{U}$. This completes the proof that $\tilde{h}$ is a filling for $\tilde{\sigma} \circ \tilde{r}$, as claimed.

To show that $\tilde{\sigma}$ is a local sc-Fredholm germ, it remains to verify the properties in Definition 1.3.6(c). Since σ is a $\mathbb{R}^n$ -sliced sc-Fredholm germ, the corresponding properties in Definition 1.3.6(c) hold for σ , and in addition the stronger properties in the definition of $\mathbb{R}^n$ -sliced sc-Fredholm germ (Definition 1.3.8) hold: There exists a sc^+ -section $\mathfrak{s} : U \rightarrow U \triangleleft \mathbb{F}$ satisfying $\mathfrak{s}(0) = h(0)$, a sc-Banach space of the form $\mathbb{W} = \tilde{\mathbb{W}} \times \mathbb{R}^n$ for some other sc-Banach space $\tilde{\mathbb{W}}$, a sc-germ of neighborhoods

$$U' \text{ around } 0 \in ([0, \infty)^s \times \mathbb{R}^{k-s}) \times (\tilde{\mathbb{W}} \times \mathbb{R}^n),$$

and a strong bundle isomorphism

$$\Psi : U \triangleleft \mathbb{F} \rightarrow U' \triangleleft \mathbb{R}^{k'} \times (\tilde{\mathbb{W}} \times \mathbb{R}^n)$$

covering a sc-diffeomorphism $\psi : U \rightarrow U'$ satisfying $\psi(0) = 0$ and of the form

$$\begin{aligned} \psi : \mathbb{R}^n \times [0, \infty)^s \times \mathbb{E}' \supset U &\rightarrow U' \subset [0, \infty)^s \times \mathbb{R}^{k-s} \times \tilde{\mathbb{W}} \times \mathbb{R}^n \\ (p, v, e) &\mapsto (v, \bar{\psi}(e), \lambda(p, v, e)) \end{aligned}$$

for some linear sc-isomorphism

$$\bar{\psi} : \mathbb{E}' \rightarrow \mathbb{R}^{k-s} \times \tilde{\mathbb{W}}$$

such that, on all levels $\mathbb{R}^n \times [0, \infty)^s \times E'_m$ for $m \geq 0$, the map

$$(p, v, e) \mapsto \lambda(p, v, e) \in \mathbb{R}^n \tag{1.30}$$

is C^1 . Moreover, the principal part of the section

$$b := \Psi \circ (h - \mathfrak{s}) \circ \psi^{-1} : U' \rightarrow U' \triangleleft \mathbb{R}^{k'} \times (\tilde{\mathbb{W}} \times \mathbb{R}^n) \tag{1.31}$$

is a basic germ, which means that, for all $a \in [0, \infty)^s \times \mathbb{R}^{k-s}$ and $z \in \tilde{\mathbb{W}} \times \mathbb{R}^n$ such that $(a, z) \in U'$, we have

$$P \circ \mathbf{b}(a, z) = z - B(a, z), \tag{1.32}$$

where $P : \mathbb{R}^{k'} \times (\tilde{\mathbb{W}} \times \mathbb{R}^n) \rightarrow (\tilde{\mathbb{W}} \times \mathbb{R}^n)$ is projection onto $\tilde{\mathbb{W}} \times \mathbb{R}^n$ and $B : U' \rightarrow \tilde{\mathbb{W}} \times \mathbb{R}^n$ is sc-smooth, satisfies $B(0) = 0$, and satisfies the contraction property (1.26).

Now, to verify that $\tilde{\sigma} = \sigma|_{\tilde{\mathcal{O}}}$ inherits the local sc-Fredholm germ property, consider the restricted sc^+ section

$$\tilde{\mathfrak{s}} := \mathfrak{s}|_{\tilde{U}} : \tilde{U} \rightarrow \tilde{U} \triangleleft \mathbb{F},$$

the linear sc-isomorphism given by

$$\begin{aligned} \tilde{\psi} : [0, \infty)^s \times \mathbb{E}' &\rightarrow [0, \infty)^s \times \mathbb{R}^{k-s} \times \tilde{\mathbb{W}} \\ (v, e) &\mapsto (v, \bar{\psi}(e)), \end{aligned}$$

the open set

$$\tilde{U}' := \tilde{\psi}(\tilde{U}),$$

and the strong bundle isomorphism given by

$$\begin{aligned} \tilde{\Psi} : \tilde{U} \triangleleft \mathbb{F} &\rightarrow \tilde{U}' \triangleleft (\mathbb{R}^{k'} \times \mathbb{R}^n) \times \tilde{\mathbb{W}} \\ ((v, e) \triangleleft \xi) &\mapsto (\tilde{\psi}(v, e) \triangleleft \eta(\Psi((v, e) \triangleleft \xi))), \end{aligned}$$

where $\eta : \mathbb{R}^{k'} \times (\tilde{\mathbb{W}} \times \mathbb{R}^n) \rightarrow (\mathbb{R}^{k'} \times \mathbb{R}^n) \times \tilde{\mathbb{W}}$ is the reordering of factors map.

We claim that the principal part of the section

$$\tilde{b} := \tilde{\Psi} \circ (\tilde{h} - \tilde{\mathfrak{s}}) \circ \tilde{\psi}^{-1} : \tilde{U}' \rightarrow \tilde{U}' \triangleleft (\mathbb{R}^{k'} \times \mathbb{R}^n) \times \tilde{\mathbb{W}} \quad (1.33)$$

is a basic germ, where now $\tilde{\mathbb{W}}$ plays the role of the sc-Banach space $\mathbb{W}$ in Definition 1.3.6(c). Let the maps

$$\begin{aligned} \tilde{P} : (\mathbb{R}^{k'} \times \mathbb{R}^n) \times \tilde{\mathbb{W}} &\rightarrow \tilde{\mathbb{W}}, \\ \pi_{\tilde{\mathbb{W}}} : \mathbb{R}^n \times \tilde{\mathbb{W}} &\rightarrow \tilde{\mathbb{W}}, \end{aligned}$$

be the projections onto the $\tilde{\mathbb{W}}$ factor in their respective domains. Notice that we have $\tilde{P} = \pi_{\tilde{\mathbb{W}}} \circ P \circ \eta^{-1}$.

We write $a \in [0, \infty)^s \times \mathbb{R}^{k-s}$ and $w \in \tilde{\mathbb{W}}$. Define the map

$$\begin{aligned} \tau : \tilde{U}' &\rightarrow U' \\ (a, w) &\mapsto \psi(0, \tilde{\psi}^{-1}(a, w)), \end{aligned}$$

and from the definitions observe

$$\tau(a, w) = (a, w, \lambda(0, \tilde{\psi}^{-1}(a, w))). \quad (1.34)$$

From the definitions and (1.32),(1.34), we compute

$$\begin{aligned} \tilde{P} \circ \tilde{\mathbf{b}}(a, w) &= \pi_{\tilde{\mathbb{W}}} \circ \tilde{P} \circ \pi_{\mathbb{R}^{k'} \times \mathbb{R}^n \times \tilde{\mathbb{W}}} \circ \tilde{\Psi} \circ (\tilde{h} - \tilde{\mathfrak{s}}) \circ \tilde{\psi}^{-1}(a, w) \\ &= \pi_{\tilde{\mathbb{W}}} \circ P \circ \pi_{\mathbb{R}^{k'} \times \tilde{\mathbb{W}} \times \mathbb{R}^n} \circ \Psi \circ (h - \mathfrak{s})(0, \tilde{\psi}^{-1}(a, w)) \\ &= \pi_{\tilde{\mathbb{W}}} \circ P \circ \pi_{\mathbb{R}^{k'} \times \tilde{\mathbb{W}} \times \mathbb{R}^n} \circ \Psi \circ (h - \mathfrak{s}) \circ (\psi^{-1} \circ \psi)(0, \tilde{\psi}^{-1}(a, w)) \\ &= \pi_{\tilde{\mathbb{W}}} \circ P \circ \mathbf{b} \circ \psi(0, \tilde{\psi}^{-1}(a, w)) \\ &= \pi_{\tilde{\mathbb{W}}} \circ P \circ \mathbf{b} \circ \tau(a, w) \\ &= \pi_{\tilde{\mathbb{W}}}((w, \lambda(0, \tilde{\psi}^{-1}(a, w))) - B \circ \tau(a, w)) \\ &= w - \pi_{\tilde{\mathbb{W}}} \circ B \circ \tau(a, w). \end{aligned}$$

So, setting

$$\tilde{B} := \pi_{\tilde{\mathbb{W}}} \circ B \circ \tau : \tilde{U}' \rightarrow \tilde{\mathbb{W}},$$

we have $\tilde{P} \circ \tilde{\mathbf{b}}(a, w) = w - \tilde{B}(a, w)$. Hence to complete the proof that $\tilde{\mathbf{b}}$ is a basic germ it remains to show that $\tilde{B}$ satisfies the contraction property (1.26).

Recall from (1.34) the map

$$\begin{aligned} \tilde{\lambda} : \tilde{U}' &\rightarrow \mathbb{R}^n \\ (a, w) &\mapsto \lambda(0, \tilde{\psi}^{-1}(a, w)). \end{aligned}$$

For all $m \geq 0$, this map restricted to the m -level $\tilde{U}'_m \rightarrow \mathbb{R}^n$ is C^1 since the map λ from (1.30) is C^1 on every level by definition of $\mathbb{R}^n$ -sliced sc-Fredholm germ, and since $\tilde{\psi}$ is a linear sc-isomorphism and hence C^∞ on every level. So, there exist constants $\delta'_m > 0$ and $C_m > 0$ such that, for all $\|(a, w)\|_m, \|(a, w')\|_m < \delta'_m$, we have the C^1 -estimate

$$\|\tilde{\lambda}(a, w) - \tilde{\lambda}(a, w')\|_{\mathbb{R}^n} \leq C_m \cdot \|(a, w) - (a, w')\|_m = C_m \cdot \|w - w'\|_m. \quad (1.35)$$

Then by (1.34) and our convention that the norm on a Cartesian product is the sum norm (which is equivalent to any standard Banach norm on the product), we have the estimate

$$\begin{aligned} \|\tau(a, w) - \tau(a, w')\|_m &= \|(a, w, \tilde{\lambda}(a, w)) - (a, w', \tilde{\lambda}(a, w'))\|_m \\ &= \|\tilde{\lambda}(a, w) - \tilde{\lambda}(a, w')\|_{\mathbb{R}^n} + \|w - w'\|_m, \\ &\leq (C_m + 1) \cdot \|w - w'\|_m. \end{aligned} \quad (1.36)$$

We now verify the contraction property (1.26) for $\tilde{B}$. Let $\epsilon > 0$ and $m \geq 0$. By the contraction property which is satisfied for B , there exists $\delta'_m > 0$ such that (1.26) holds for B and

$$\epsilon'_m := \epsilon / (C_m + 1).$$

Shrink $\delta'_m > 0$ to be smaller than the δ'_m for which (1.36) holds. We claim that

$$\delta_m := \delta'_m / (C_m + 1)$$

satisfies (1.26) for $\tilde{B}$ and ϵ . Indeed, consider some $\|(a, w)\|_m, \|(a, w')\|_m < \delta_m$. Then by $\tau(0) = \psi(0, \tilde{\psi}^{-1}(0)) = 0$ and (1.36), we have

$$\|\tau(a, w)\|_m, \|\tau(a, w')\|_m < (C_m + 1) \cdot \delta_m = \delta'_m.$$

We compute, using the contraction property (1.26) for B and the estimate (1.36),

$$\begin{aligned}
 \|\tilde{B}(a, w) - \tilde{B}(a, w')\|_m &= \|\pi_{\tilde{\mathbb{W}}} \circ B \circ \tau(a, w) - \pi_{\tilde{\mathbb{W}}} \circ B \circ \tau(a, w')\|_m \\
 &\leq \|B \circ \tau(a, w) - B \circ \tau(a, w')\|_m \\
 &= \|B(a, w, \tilde{\lambda}(a, w)) - B(a, w', \tilde{\lambda}(a, w'))\|_m \\
 &\leq \epsilon'_m \cdot \|(w, \tilde{\lambda}(a, w)) - (w', \tilde{\lambda}(a, w'))\|_m \\
 &= \epsilon'_m \cdot \|(a, w, \tilde{\lambda}(a, w)) - (a, w', \tilde{\lambda}(a, w'))\|_m \\
 &= \epsilon'_m \cdot \|\tau(a, w) - \tau(a, w')\|_m \\
 &\leq (C_m + 1) \cdot \epsilon'_m \cdot \|w - w'\|_m \\
 &= \epsilon \cdot \|w - w'\|_m.
 \end{aligned}$$

This completes the proof that (1.26) holds for $\tilde{B}$, and hence that $\tilde{\mathbf{b}}$ is a basic germ.

We have shown that $\tilde{\sigma}$ is a local sc-Fredholm germ. Moreover, $\tilde{\sigma}$ is a tame sc-Fredholm germ because we have $\tilde{\psi} = id_{[0, \infty)^s} \times \bar{\psi}$ by definition of $\tilde{\psi}$, where $\bar{\psi} : \mathbb{E}' \rightarrow \mathbb{R}^{k-s} \times \tilde{\mathbb{W}}$ is the linear sc-isomorphism given by the $\mathbb{R}^n$ -sliced sc-Fredholm germ property of σ .

The claimed index formula holds because, by definition of the sc-Fredholm index (1.27) and the form of the basic germs $\mathbf{b}$ (1.31) and $\tilde{\mathbf{b}}$ (1.33), we have $ind(\sigma) = k - k'$ and $ind(\tilde{\sigma}) = k - (k' + n)$. $\square$

1.4 Slice coordinates for local submersions to $\mathbb{R}^n$

The purpose of this section is to prove Lemma 1.4.2, which generalizes the local submersion normal form (Lemma 1.2.3) for sc-smooth maps $f : U \rightarrow \mathbb{R}^n$ where the domain $U \subset [0, \infty)^s \times \mathbb{E}$ is open to maps $f : \mathcal{O} \rightarrow \mathbb{R}^n$ where the domain is a tame sc-retract $(\mathcal{O}, [0, \infty)^s \times \mathbb{E}, \mathbb{R}^s \times \mathbb{E})$. This means that the set $\mathcal{O} = r(U)$ is the image of a tame sc-retraction $r : U \rightarrow U$ (see Definition 1.3.1), which can have much more complicated local structure than the open set U , for example locally varying dimension. In this case, the local submersion normal form is obtained by a change of coordinates around any smooth point $x \in \mathcal{O}_\infty$ at which f is submersive and satisfies $f(x) = 0$ such that the sc-retract in the new coordinates is a $\mathbb{R}^n$ -sliced sc-retract (Definition 1.3.2) with induced tame sc-retract (see Lemma 1.3.3) identified with a neighborhood of x in $f^{-1}(0) \cap \mathcal{O}^1$. For this reason, we call the sc-diffeomorphism with this $\mathbb{R}^n$ -sliced sc-retract *slice coordinates around x* .

Moreover, given a tame local bundle model $K \rightarrow \mathcal{O}$ and a tame sc-Fredholm germ $\sigma : \mathcal{O} \rightarrow K$ (Definition 1.3.7), and assuming that f is compatible with σ as explained in Lemma 1.4.2(III), we prove that σ in the new slice coordinates around x is a $\mathbb{R}^n$ -sliced sc-Fredholm germ (Definition 1.3.8) and its induced tame sc-Fredholm germ is identified with the restriction of σ to $f^{-1}(0) \cap \mathcal{O}^1$. See Section 1.5.1 for a discussion of why evaluation maps $f = ev$ at marked points are compatible with the Cauchy-Riemann section $\sigma = \bar{\partial}_J$.

For simplicity, in this section we take all partial quadrants C to be in standard form $C = [0, \infty)^s \times \mathbb{E} \subset \mathbb{R}^s \times \mathbb{E}$. There is no loss of generality because all partial quadrants are

linearly sc-isomorphic to a standard partial quadrant by definition (see (1.4)) and hence all sc-retracts are sc-diffeomorphic to sc-retracts in standard partial quadrants.

The notion of submersion that we use in Lemma 1.4.2, as in the local submersion normal form in sc-calculus (Lemma 1.2.3), requires surjectivity of the tangent map relative to the boundary: For a point $x \in \mathcal{O}_\infty \cap (\{0\} \times \mathbb{E})$, we require that $D_x f(T_x \mathcal{O} \cap (\{0\} \times \mathbb{E})) = \mathbb{R}^n$. Recall that $T_x^R \mathcal{O} = T_x \mathcal{O} \cap (\{0\} \times \mathbb{E})$ is the reduced tangent space at x (see (1.12)). In the following lemma we interpret $T_x^R \mathcal{O}$ as the tangent space at x along the corner $\mathcal{O} \cap (\{0\} \times \mathbb{E})$.

Lemma 1.4.1. *Consider a standard partial quadrant $[0, \infty)^s \times \mathbb{E}$ and a tame sc-retract $(\mathcal{O}, [0, \infty)^s \times \mathbb{E}, \mathbb{R}^s \times \mathbb{E})$. Then, for any open subset $U \subset [0, \infty)^s \times \mathbb{E}$ and tame sc-retraction $r : U \rightarrow U$ onto $r(U) = \mathcal{O}$, the restriction of r to $U_\partial := U \cap (\{0\} \times \mathbb{E})$ is a tame sc-retraction*

$$r_\partial := r|_{U_\partial} : U_\partial \rightarrow U_\partial$$

onto

$$\mathcal{O}_\partial := \mathcal{O} \cap (\{0\} \times \mathbb{E}).$$

In particular, the tuple

$$(\mathcal{O}_\partial, \mathbb{E}, \mathbb{E})$$

is a tame sc-retract, which we call the **boundary sc-retract associated to the tame sc-retract $\mathcal{O}$** .

Moreover, for all $x \in \mathcal{O}_\partial^1$, we have

$$T_x \mathcal{O}_\partial = T_x \mathcal{O} \cap (\{0\} \times \mathbb{E}) = T_x^R \mathcal{O}. \quad (1.37)$$

Proof. Since r is tame, for $x \in U_\partial$ we have $d_{[0, \infty)^s \times \mathbb{E}}(r(x)) = d_{[0, \infty)^s \times \mathbb{E}}(x) = s$, and hence $r(x) \in U_\partial$. Thus $r_\partial(U_\partial) \subset U_\partial$ holds. It is then immediate that r_∂ is a sc-smooth retraction, as the restriction of the sc-smooth retraction r . Moreover, the domain U_∂ of r_∂ is an open subset of a cornerless partial quadrant, i.e. $U_\partial \subset \mathbb{E}$, and hence r trivially satisfies the tameness hypotheses.

We now prove that the image of r_∂ is $\mathcal{O}_\partial$. Indeed, if $x \in \text{im}(r_\partial)$ then $x \in \mathcal{O}$ and $x \in U_\partial \subset \{0\} \times \mathbb{E}$, hence $x \in \mathcal{O}_\partial$. For the reverse inclusion, if $x \in \mathcal{O} \cap (\{0\} \times \mathbb{E})$ then $x \in U_\partial$ and hence $x = r(x) = r_\partial(x)$.

It remains to prove that (1.37) holds. Let $\xi \in T_x \mathcal{O}_\partial$. Then $\xi \in \{0\} \times \mathbb{E}$ and $D_x r(\xi) = D_x r_\partial(\xi) = \xi$, so $\xi \in T_x \mathcal{O}$. To prove the reverse inclusion, let $\xi \in T_x \mathcal{O} \cap (\{0\} \times \mathbb{E})$. Then we have $D_x r_\partial(\xi) = D_x r(\xi) = \xi$, which implies $\xi \in T_x \mathcal{O}_\partial$, as required. $\square$

In the following lemma, we construct the slice coordinates around a smooth point x in a tame sc-retract $\mathcal{O}$ at which a sc-smooth map $f : \mathcal{O} \rightarrow \mathbb{R}^n$ is submersive on the tangent space $T_x \mathcal{O}_\partial$ at x to the associated boundary retract $\mathcal{O}_\partial$ from Lemma 1.4.1, i.e. $D_x f(T_x \mathcal{O}_\partial) = \mathbb{R}^n$.

The statement of the lemma is in three parts: (I) provides slice coordinates for $f^{-1}(0)$, (II) provides slice bundle coordinates for the restriction of a bundle retract to $f^{-1}(0)$, and (III) provides slice sc-Fredholm coordinates for the restriction of a sc-Fredholm section.

Lemma 1.4.2.

(I) Consider a tame sc-retract $(\mathcal{O}, [0, \infty)^s \times \mathbb{E}, \mathbb{R}^s \times \mathbb{E})$ and a sc-smooth map $f : \mathcal{O} \rightarrow \mathbb{R}^n$. Let $\mathcal{O}_\partial = \mathcal{O} \cap (\{0\}^s \times \mathbb{E})$ denote the boundary sc-retract associated to $\mathcal{O}$ (see Lemma 1.4.1), and $f_\partial := f|_{\mathcal{O}_\partial} : \mathcal{O}_\partial \rightarrow \mathbb{R}^n$ the restriction of f . Suppose that, at some $x \in (\mathcal{O}_\partial)_\infty$ satisfying $f(x) = 0$, the tangent map $D_x f_\partial : T_x \mathcal{O}_\partial \rightarrow \mathbb{R}^n$ is surjective.

Then, there exists an open subset

$$\hat{\mathcal{O}} \subset \mathcal{O}^1,$$

a sc-Banach space $\mathbb{K}$, a $\mathbb{R}^n$ -sliced sc-retract

$$(\mathcal{O}', \mathbb{R}^n \times [0, \infty)^s \times \mathbb{K}^1, \mathbb{R}^n \times \mathbb{R}^s \times \mathbb{K}^1), \quad (1.38)$$

and a sc-smooth diffeomorphism

$$g : \hat{\mathcal{O}} \rightarrow \mathcal{O}'$$

satisfying

$$g(f^{-1}(0) \cap \hat{\mathcal{O}}) = \mathcal{O}' \cap (\{0\}^n \times [0, \infty)^s \times \mathbb{K}^1) =: \tilde{\mathcal{O}}'. \quad (1.39)$$

Here, $\tilde{\mathcal{O}}'$ is the tame sc-retract induced by $\mathcal{O}'$ (see Lemma 1.3.3).

We view g as providing **slice coordinates** $\tilde{\mathcal{O}}' \subset \mathcal{O}'$ around x with respect to $(f^{-1}(0) \cap \mathcal{O}^1) \subset \mathcal{O}^1$.

(II) Consider, in addition, a tame strong bundle retract $(K, [0, \infty)^s \times \mathbb{E} \triangleleft \mathbb{F}, \mathbb{R}^s \times \mathbb{E} \triangleleft \mathbb{F})$ covering the tame sc-retract $\mathcal{O}$. Let $\hat{K} := \pi^{-1}(\hat{\mathcal{O}})$ where $\pi : K^1 \rightarrow \mathcal{O}^1$ is the 1-shifted local bundle model. Set

$$K' := (g \triangleleft id_{\mathbb{F}^1})(\hat{K}).$$

Then, the tuple

$$(K', \mathbb{R}^n \times [0, \infty)^s \times \mathbb{K}^1 \triangleleft \mathbb{F}^1, \mathbb{R}^n \times \mathbb{R}^s \times \mathbb{K}^1 \triangleleft \mathbb{F}^1) \quad (1.40)$$

is a $\mathbb{R}^n$ -sliced bundle retract covering the $\mathbb{R}^n$ -sliced sc-retract (1.38). In particular, the induced tame strong bundle retract $\tilde{K}'$ (see Lemma 1.3.5) covers the induced tame sc-retract $\tilde{\mathcal{O}}'$.

Moreover,

$$(g \triangleleft id_{\mathbb{F}^1})(\pi^{-1}(f^{-1}(0) \cap \hat{\mathcal{O}})) = \tilde{K}' \quad (1.41)$$

holds, so we view $(g \triangleleft id_{\mathbb{F}^1})$ as providing **slice bundle coordinates** $\tilde{K}' \subset K'$ for π around x with respect to $(f^{-1}(0) \cap \mathcal{O}^1) \subset \mathcal{O}^1$.

(III) Consider, in addition, a tame sc-Fredholm germ $\sigma : \mathcal{O} \rightarrow K$ (Definition 1.3.7). Assume that $x = 0$ and that σ has the following property: There exists a choice of sc-Banach

space $\mathbb{W}$ and linear sc-isomorphism $\bar{\psi} : \mathbb{E} \rightarrow \mathbb{R}^{k-s} \times \mathbb{W}$ satisfying the conditions in Definition 1.3.7 of tame sc-Fredholm germ for σ such that, in addition, there exists a sc-complement L of $\ker D_x f_{\partial}$ in $T_x \mathcal{O}_{\partial}$ satisfying

$$\bar{\psi}(L) \subset (\{0\}^{k-s} \times \mathbb{W}). \quad (1.42)$$

Then, the section

$$\sigma' := (g \triangleleft id_{\mathbb{R}^1}) \circ \sigma \circ g^{-1} : \mathcal{O}' \rightarrow K'$$

is a $\mathbb{R}^n$ -sliced sc-Fredholm germ. In particular, the restriction to the induced tame local bundle model

$$\tilde{\sigma}' := \sigma'|_{\tilde{\mathcal{O}}'} : \tilde{\mathcal{O}}' \rightarrow \tilde{K}'$$

is a tame sc-Fredholm germ with index satisfying $\text{ind}(\tilde{\sigma}') = \text{ind}(\sigma) - n$.

We view $\tilde{\sigma}'$ as being in **slice sc-Fredholm coordinates around x with respect to $(f^{-1}(0) \cap \mathcal{O}^1) \subset \mathcal{O}^1$** .

Proof. Proof of (I): Consider any open subset $U \subset [0, \infty)^s \times \mathbb{E}$, tame sc-retraction $r : U \rightarrow U$ onto $r(U) = \mathcal{O}$, and the associated retraction on the boundary (see Lemma 1.4.1) denoted by $r_{\partial} : U_{\partial} \rightarrow U_{\partial}$ where $U_{\partial} = U \cap (\{0\}^s \times \mathbb{E})$.

Set

$$\mathbb{K} := \ker D_x(f_{\partial} \circ r_{\partial}) \subset T_x U_{\partial} = \mathbb{E}.$$

Let $L \subset E_{\infty}$ be a sc-complement of $\ker D_x f_{\partial}$ in $T_x \mathcal{O}_{\partial}$, which exists by Lemma 1.2.2 since the surjection $D_x f_{\partial} : T_x \mathcal{O}_{\partial} \rightarrow \mathbb{R}^n$ induces an isomorphism $T_x \mathcal{O}_{\partial} / \ker D_x f_{\partial} \cong \mathbb{R}^n$. Note that the restriction

$$D_x f_{\partial}|_L : L \rightarrow \mathbb{R}^n \quad (1.43)$$

is an isomorphism.

We claim that L is a sc-complement of $\mathbb{K}$ in $\mathbb{E}$, i.e.

$$\mathbb{E} = \mathbb{K} \oplus L. \quad (1.44)$$

First, note that the sc-splitting $\mathbb{E} = T_x \mathcal{O}_{\partial} \oplus \ker D_x r_{\partial}$ holds since $D_x r_{\partial} : \mathbb{E} \rightarrow \mathbb{E}$ is a linear sc-retraction with image $T_x \mathcal{O}_{\partial}$. Hence we have

$$\mathbb{E} = (L \oplus \ker D_x f_{\partial}) \oplus \ker D_x r_{\partial}.$$

From this description of $\mathbb{E}$ together with the definition of $\mathbb{K}$, we claim that $\ker D_x f_{\partial} \oplus \ker D_x r_{\partial} = \mathbb{K}$ follows, which then implies (1.44). Indeed, write $\xi \in \mathbb{E}$ as $\xi = l + \eta + \nu$ for some $l \in L, \eta \in \ker D_x f_{\partial}$, and $\nu \in \ker D_x r_{\partial}$. Note that $D_x r_{\partial}(\eta) = \eta$ and $D_x r_{\partial}(l) = l$ since $\eta, l \in T_x \mathcal{O}_{\partial}$. Then if $\xi \in \mathbb{K}$ we have $0 = D_x(f_{\partial} \circ r_{\partial})(\xi) = D_x f_{\partial}(l) + D_x f_{\partial}(\eta) = D_x f_{\partial}(l)$, which implies $l = 0$ by (1.43), and hence $\xi = \eta + \nu \in \ker D_x f_{\partial} \oplus \ker D_x r_{\partial}$. To prove the reverse inclusion, observe that if $\xi = \eta + \nu$ then $D_x(f_{\partial} \circ r_{\partial})(\xi) = D_x f_{\partial}(\eta) = 0$ and so $\xi \in \mathbb{K}$. Hence (1.44) holds.

We now prepare to apply the normal form of a sc-smooth local submersion (Lemma 1.2.3) to the map

$$f \circ r : U \rightarrow \mathbb{R}^n.$$

Notice that the restriction of $f \circ r$ to $U_\partial = U \cap (\{0\}^s \times \mathbb{E})$ is equal to $f_\partial \circ r_\partial$, and hence by the hypothesized surjectivity of the map $D_x f_\partial : T_x \mathcal{O}_\partial \rightarrow \mathbb{R}^n$ and the definition of tangent space $T_x \mathcal{O}_\partial = D_x r_\partial(T_x U_\partial)$, it follows that the map

$$F := D_x(f \circ r|_{U \cap \{0\}^s \times \mathbb{E}}) : \mathbb{E} \rightarrow \mathbb{R}^n$$

is surjective with kernel $\mathbb{K} = \ker F$. Hence Lemma 1.2.3 applies to the map $f \circ r$ and the sc-complement L of $\mathbb{K}$ in $\mathbb{E}$. The lemma provides an open neighborhood

$$\hat{U} \subset U^1 \subset [0, \infty)^s \times \mathbb{E}^1$$

of x in U^1 , an open set

$$\underline{U}' \subset \mathbb{R}^n \times [0, \infty)^s \times \mathbb{K}^1,$$

and a sc-smooth diffeomorphism of the form

$$\begin{aligned} g : \hat{U} &\rightarrow \underline{U}' \\ (v, e) &\mapsto (f \circ r(v, e), v, pr(e)), \end{aligned}$$

where $pr : \mathbb{E} = L \oplus \mathbb{K} \rightarrow \mathbb{K}$ is the projection along L , such that on every level $m \geq 0$ the map $g : \hat{U}_m \rightarrow \underline{U}'_m$ is a C^{m+1} -diffeomorphism. In particular,

$$f \circ r \circ g^{-1} : \underline{U}' \rightarrow \mathbb{R}^n \text{ is projection onto the } \mathbb{R}^n\text{-coordinate} \quad (1.45)$$

and also g preserves the $[0, \infty)^s$ -coordinate.

We shrink the open sets $\hat{U}$ and $\underline{U}'$ in the following way, so that the smaller open set $\hat{U} \subset \hat{U}$ has the property $r(\hat{U}) \subset \hat{U}$, making $r|_{\hat{U}} : \hat{U} \rightarrow \hat{U}$ into a sc-retraction: Set

$$\begin{aligned} \hat{U} &:= r^{-1}(\hat{U} \cap \mathcal{O}) \cap \hat{U}, \\ U' &:= g(\hat{U}). \end{aligned}$$

The set $\hat{U}$ is open in U^1 because $\hat{U} \cap \mathcal{O}$ is open in $\mathcal{O}^1$ and $r : U \rightarrow \mathcal{O}$ is sc-continuous. We claim that $r(\hat{U}) \subset \hat{U}$, so r restricts to a tame sc-retraction $r : \hat{U} \rightarrow \hat{U}$ onto the tame sc-retract

$$\hat{\mathcal{O}} := r(\hat{U}).$$

Indeed, if $y \in \hat{U}$ then $r(r(y)) = r(y) \in \hat{U} \cap \mathcal{O}$, and so it follows that $r(y) \in \hat{U}$. Note that

$$U' \subset \mathbb{R}^n \times [0, \infty)^s \times \mathbb{K}^1$$

is open since g is a sc-diffeomorphism, and moreover the restricted map

$$g : \hat{U} \rightarrow U'$$

is a sc-diffeomorphism.

The map

$$r' := g \circ r \circ g^{-1} : U' \rightarrow U'$$

is a sc-smooth retraction since g is a sc-diffeomorphism and r is a sc-smooth retraction. Indeed, $r \circ r = r$ implies that $r' \circ r' = r'$. Set

$$\mathcal{O}' := r'(U').$$

Notice that

$$\hat{\mathcal{O}} = r(\hat{U}) = r \circ g^{-1}(U') = g^{-1} \circ r'(U') = g^{-1}(\mathcal{O}')$$

holds, so g restricts to a bijection

$$g : \hat{\mathcal{O}} \rightarrow \mathcal{O}'.$$

This restriction is a sc-smooth diffeomorphism, as required, since $g \circ r : \hat{U} \rightarrow U'$ is sc-smooth and $g^{-1} \circ r' : U' \rightarrow \hat{U}$ is sc-smooth. Indeed, the definition of a map between sc-retracts being sc-smooth is that these compositions with the sc-retractions are sc-smooth, where the domain and codomain are now open subsets of sc-Banach spaces and sc-smoothness is defined as usual.

We now prove that r' is a $\mathbb{R}^n$ -sliced sc-retraction, and so $\mathcal{O}'$ is the required $\mathbb{R}^n$ -sliced sc-retract (1.38). We must show that r is tame and satisfies the defining property (1.13) of a $\mathbb{R}^n$ -sliced retraction, i.e. that r preserves the $\mathbb{R}^n$ -coordinate. To prove (1.13), notice from the definitions and (1.45) that we have

$$(f \circ g^{-1}) \circ r' = f \circ r \circ g^{-1} = \pi_{\mathbb{R}^n}.$$

Since $r' \circ r' = r'$, we have

$$\pi_{\mathbb{R}^n} \circ r' = (f \circ g^{-1}) \circ r' \circ r' = (f \circ g^{-1}) \circ r' = \pi_{\mathbb{R}^n},$$

completing the proof of (1.13).

We now verify that r' is a tame sc-retraction. The property Definition 1.3.1(1) for $r' = g \circ r \circ g^{-1}$ holds because it holds for r and since g preserves the $[0, \infty)^s$ -coordinate. To verify property Definition 1.3.1(2) for r' , let $y' \in \mathcal{O}'_\infty$ and set $y := g^{-1}(y') \in \hat{\mathcal{O}}_\infty$. Since r is a tame sc-retraction, there exists a sc-subspace $A \subset \mathbb{R}^s \times \mathbb{E}^1$ such that $\mathbb{R}^s \times \mathbb{E}^1 = T_y \hat{\mathcal{O}} \oplus A$ and $A \subset (\mathbb{R}^s \times \mathbb{E}^1)_y$. We claim that $A' := D_y g(A)$ is the required sc-subspace of $\mathbb{R}^n \times \mathbb{R}^s \times \mathbb{K}^1$. First, since g is the identity on $[0, \infty)^s$, the tangent map $D_y g$ is the identity on $\mathbb{R}^s$, and thus $A' \subset (\mathbb{R}^n \times \mathbb{R}^s \times \mathbb{K}^1)_{y'}$ by (1.8). Moreover, by definition of r' and the tangent space of a retract we have $T_{y'} \mathcal{O}' = D_{y'} r'(T_{y'} U') = D_y g(D_y r(T_y \hat{U})) = D_y g(T_y \hat{\mathcal{O}})$. Hence, since $D_y g$ is a linear sc-isomorphism, we have $\mathbb{R}^n \times \mathbb{R}^s \times \mathbb{K}^1 = D_y g(\mathbb{R}^s \times \mathbb{E}^1) = D_y g(T_y \hat{\mathcal{O}} \oplus A) = T_{y'} \mathcal{O}' \oplus A'$, as required. This completes the verification that r' is a $\mathbb{R}^n$ -sliced sc-retraction and hence (1.38) is indeed a $\mathbb{R}^n$ -sliced sc-retract.

To complete the proof of (I), it remains to show that (1.39) holds. First, recall from above that $g(\hat{\mathcal{O}}) = \mathcal{O}'$. Let $y \in g(f^{-1}(0) \cap \hat{\mathcal{O}})$. Since $g^{-1}(y) \in \hat{\mathcal{O}}$ is in the image of the

retraction r , it follows that $0 = f \circ g^{-1}(y) = f \circ r \circ g^{-1}(y)$, and hence (1.45) implies $y \in \{0\}^n \times [0, \infty)^s \times \mathbb{K}^1$, proving the forwards inclusion in (1.39). For the reverse inclusion, let $y \in \mathcal{O}' \cap (\{0\}^n \times [0, \infty)^s \times \mathbb{K}^1)$. Then, again by (1.45), we have $f \circ g^{-1}(y) = f \circ r \circ g^{-1}(y) = \pi_{\mathbb{R}^n}(y) = 0$. This proves the reverse inclusion and hence the equality (1.39).

Proof of (II): After shrinking the original open set $U \subset [0, \infty)^s \times \mathbb{E}$, we can assume that there exists a tame bundle retraction

$$\begin{aligned} R : U \triangleleft \mathbb{F} &\rightarrow U \triangleleft \mathbb{F} \\ (y, \xi) &\mapsto (r(y), \Gamma(y, \xi)) \end{aligned}$$

onto $R(U \triangleleft \mathbb{F}) = K$ covering r . Since r restricts to a retraction $r : \hat{U} \rightarrow \hat{U}$ onto $\hat{\mathcal{O}}$ as discussed above, it follows that R restricts to a tame bundle retraction $R : \hat{U} \triangleleft \mathbb{F}^1 \rightarrow \hat{U} \triangleleft \mathbb{F}^1$ with image the tame bundle retract

$$\hat{K} = R(\hat{U} \triangleleft \mathbb{F}^1) = \pi^{-1}(\hat{\mathcal{O}}),$$

where $\pi : K^1 \rightarrow \mathcal{O}^1$ is the 1-shifted tame local bundle model.

Set

$$\begin{aligned} \Gamma' : U' \triangleleft \mathbb{F}^1 &\rightarrow \mathbb{F}^1 \\ (y', \xi) &\mapsto \Gamma(g^{-1}(y'), \xi) \end{aligned}$$

We claim that the map

$$\begin{aligned} R' := U' \triangleleft \mathbb{F}^1 &\rightarrow U' \triangleleft \mathbb{F}^1 \\ (y', \xi) &\mapsto (r'(y'), \Gamma'(y', \xi)) \end{aligned}$$

is a $\mathbb{R}^n$ -sliced bundle retraction onto

$$K' := (g \triangleleft id_{\mathbb{F}^1})(\hat{K}),$$

proving that (1.40) is a $\mathbb{R}^n$ -sliced bundle retract covering the $\mathbb{R}^n$ -sliced sc-retract (1.38). It is then immediate from Lemma 1.3.5 that the induced bundle retract $\tilde{K}'$ covers the induced sc-retract $\tilde{\mathcal{O}}'$, as claimed. Since $r' = g \circ r \circ g^{-1}$, we can write R' as a composition of strong bundle maps

$$R' = (g \triangleleft id_{\mathbb{F}^1}) \circ R \circ (g^{-1} \triangleleft id_{\mathbb{F}^1}),$$

and hence R' is a strong bundle map. In addition, $R' \circ R' = R'$ follows from $R \circ R = R$. Thus R' is a bundle retraction covering the $\mathbb{R}^n$ -sliced sc-retraction r' , which implies that R' is a $\mathbb{R}^n$ -sliced bundle retraction, as desired. Moreover, by definition of K' we have $R'(U' \triangleleft \mathbb{F}^1) = (g \triangleleft id_{\mathbb{F}^1}) \circ R(\hat{U} \triangleleft \mathbb{F}^1) = (g \triangleleft id_{\mathbb{F}^1})(\hat{K}) = K'$. Thus K' is indeed the desired $\mathbb{R}^n$ -sliced bundle retract (1.40).

We now prove (1.41). Let $\pi' : K' \rightarrow \mathcal{O}'$ denote the $\mathbb{R}^n$ -sliced local bundle model, which restricts to the induced tame local bundle model $\tilde{\pi} = \pi'|_{\tilde{K}'} : \tilde{K}' \rightarrow \tilde{\mathcal{O}}'$ (see (1.25)). Here $\tilde{K}' = \pi'^{-1}(\tilde{\mathcal{O}}')$ is the tame bundle retract induced by the $\mathbb{R}^n$ -sliced bundle retract K' (see (1.24)). To prove (1.41), let $y \in f^{-1}(0) \cap \hat{\mathcal{O}}$ and let $\xi \in \pi^{-1}(y)$. Then, by (1.39), we have $g(y) \in \tilde{\mathcal{O}}'$, and hence $(g \triangleleft id_{\mathbb{F}^1})(\xi) \in \pi'^{-1}(\tilde{\mathcal{O}}') = \tilde{K}'$, proving the forwards inclusion. To see the reverse inclusion, let $\xi \in \tilde{K}'$. Set $y = \pi'(\xi) \in \tilde{\mathcal{O}}'$. Then, again by (1.39), we have $g^{-1}(y) \in f^{-1}(0) \cap \hat{\mathcal{O}}$ and hence $(g^{-1} \triangleleft id_{\mathbb{F}^1})(\xi) \in \pi^{-1}(f^{-1}(0) \cap \hat{\mathcal{O}})$. This proves (1.41), completing the proof of the statements in (II).

Proof of (III): To prove that the section

$$\sigma' := (g \triangleleft id_{\mathbb{F}^1}) \circ \sigma \circ g^{-1} : \mathcal{O}' \rightarrow K'$$

is a $\mathbb{R}^n$ -sliced sc-Fredholm germ (Definition 1.3.8), we must first show that it is a local sc-Fredholm germ (Definition 1.3.6). Since the given section σ is a tame sc-Fredholm germ (Definition 1.3.7) by hypothesis, by shrinking the original open set $U \subset [0, \infty)^s \times \mathbb{E}$ we can assume that the tame sc-retraction $r : U \rightarrow U$ satisfies the properties guaranteed by the definition of a tame sc-Fredholm germ. Moreover, the 1-shifted section $\sigma|_{\mathcal{O}^1} : \mathcal{O}^1 \rightarrow K^1$ is also a tame sc-Fredholm germ (see Remark 1.3.9), and since $\hat{U} \subset U^1$ is an open subset of the 1-shift of U , the retraction $r : \hat{U} \rightarrow \hat{U}$ onto $r(\hat{U}) = \hat{\mathcal{O}}$ and the tame bundle retraction R covering r also satisfy the properties required in the definition of tame sc-Fredholm germ for $\sigma|_{\hat{\mathcal{O}}} : \hat{\mathcal{O}} \rightarrow \hat{K}$. We explain these properties as we use them.

There exists a sc-germ of neighborhoods $\hat{U}([0, \infty)^s \times \mathbb{E}^1, 0)$ around 0 such that the 0-level $\hat{U}(0)$ in the germ is equal to $\hat{U}$ and such that the composition $\sigma \circ r : \hat{U}([0, \infty)^s \times \mathbb{E}^1, 0) \rightarrow \mathbb{F}^1$ possesses a filling (see Definition 1.3.6(b))

$$h : \hat{U}([0, \infty)^s \times \mathbb{E}^1, 0) \rightarrow \hat{U}([0, \infty)^s \times \mathbb{E}^1, 0) \triangleleft \mathbb{F}^1,$$

where σ means the principal part in $\mathbb{F}^1$ of σ . By hypothesis and the construction of $\hat{\mathcal{O}}$, we have $x = 0 \in f^{-1}(0) \cap \hat{\mathcal{O}}$, and so it follows from the form of g above that

$$g(0) = 0 \in \mathcal{O}'. \quad (1.46)$$

The sc-germ of neighborhoods around 0 given by the image of $\hat{U}([0, \infty)^s \times \mathbb{E}^1, 0)$ under g , i.e.

$$U'(\mathbb{R}^n \times [0, \infty)^s \times \mathbb{K}^1, 0) := g(\hat{U}([0, \infty)^s \times \mathbb{E}^1, 0)),$$

has 0-level $U'(0)$ equal to the set open set $U' = g(\hat{U})$ above. From now on, we abbreviate the sc-germs of neighborhoods simply by their 0-level, i.e. by $\hat{U}$ and U' .

We claim that the map

$$h' := (g \triangleleft id_{\mathbb{F}^1}) \circ h \circ g^{-1} : U' \rightarrow U' \triangleleft \mathbb{F}^1.$$

is a filling of $\sigma' \circ r' : U' \rightarrow \mathbb{F}^1$. The required properties Definition 1.3.6(b).(i)-(iii) of a filling hold for h' because they hold for h , as we explain now. To verify (i) for h' , let $y' \in \mathcal{O}'$.

Then $y := g^{-1}(y') \in \hat{\mathcal{O}}$, so by (i) for h we have $\sigma(y) = h(y)$, which implies $\sigma'(y') = h'(y')$ by definition of σ' and h' , as required. To verify (ii), let $y' \in U'$ and assume $\mathbf{h}'(y') = \Gamma'(r'(y'), \mathbf{h}'(y'))$. Then, setting $y := g^{-1}(y') \in \hat{U}$, we have $\mathbf{h}(y) = \mathbf{h}'(y') = \Gamma(r(y), \mathbf{h}(y))$. Hence $y \in \hat{\mathcal{O}}$ by property (ii) for h , so $y' \in g(\hat{\mathcal{O}}) = \mathcal{O}'$, as required. We now verify property (iii) for h' . Property (iii) for h says that the linearization D_0A at 0 of the map

$$\begin{aligned} A : \hat{U} &\rightarrow \mathbb{F}^1 \\ y &\mapsto (id_{\mathbb{F}^1} - \Gamma(r(y), \cdot))\mathbf{h}(y) \end{aligned}$$

restricts to a linear sc-isomorphism $\ker D_0r \rightarrow \ker \Gamma(0, \cdot)$. We must show that the map

$$\begin{aligned} A' : U' &\rightarrow \mathbb{F}^1 \\ y' &\mapsto (id_{\mathbb{F}^1} - \Gamma'(r'(y'), \cdot))\mathbf{h}'(y') \end{aligned}$$

restricts to a linear sc-isomorphism $\ker D_0r' \rightarrow \ker \Gamma'(0, \cdot)$. Note that, by definition of Γ' and by (1.46), we have $\ker \Gamma'(0, \cdot) = \ker \Gamma(g^{-1}(0), \cdot) = \ker \Gamma(0, \cdot)$. Moreover, from the definition of h' we have

$$\mathbf{h}'(y') = \mathbf{h} \circ g^{-1}(y'),$$

and so by definition of r' we have

$$A' = A \circ g^{-1} : U' \rightarrow \mathbb{F}^1.$$

Hence D_0A' restricts to the desired isomorphism since D_0g^{-1} restricts to an isomorphism $\ker D_0r' \rightarrow \ker D_0r$ by definition of r' and (1.46). Hence h' is indeed a filling of $\sigma' \circ r'$, as claimed.

To verify that σ' is a local sc-Fredholm germ, it remains to check the properties in Definition 1.3.6(c). Since $\sigma : \hat{\mathcal{O}} \rightarrow \hat{K}$ is a tame sc-Fredholm germ, we have the following: There exists a sc^+ -section $\mathfrak{s} : \hat{U} \rightarrow \hat{U} \triangleleft \mathbb{F}^1$ satisfying $\mathfrak{s}(0) = h(0)$, a sc-Banach space $\mathbb{W}$, a sc-germ of neighborhoods

$$\hat{U}' \text{ around } 0 \text{ in } [0, \infty)^s \times \mathbb{R}^{k-s} \times \mathbb{W}$$

for some $k \geq s$, and a strong bundle isomorphism

$$\Psi : \hat{U} \triangleleft \mathbb{F}^1 \rightarrow \hat{U}' \triangleleft \mathbb{R}^{k'} \times \mathbb{W},$$

for some $k' \geq 0$, covering a linear sc-isomorphism

$$\psi : \hat{U} \rightarrow \hat{U}'$$

of the form

$$\psi = id_{[0, \infty)^s} \times \bar{\psi} : [0, \infty)^s \times \mathbb{E}^1 \rightarrow [0, \infty)^s \times \mathbb{R}^{k-s} \times \mathbb{W}$$

for some linear sc-isomorphism

$$\bar{\psi} : \mathbb{E}^1 \rightarrow \mathbb{R}^{k-s} \times \mathbb{W},$$

such that the principal part of the section

$$b := \Psi \circ (h - \mathfrak{s}) \circ \psi^{-1} : \hat{U}' \rightarrow \hat{U}' \triangleleft \mathbb{R}^{k'} \times \mathbb{W} \quad (1.47)$$

is a basic germ.

By the hypothesis (1.42), we can assume that the subspace $L \subset E_\infty$ satisfies $\bar{\psi}(L) \subset \{0\}^{k-s} \times \mathbb{W}$. We consider $\bar{\psi}(L)$ as a sc-subspace of $\mathbb{W}$ and let $\tilde{\mathbb{W}}$ be any sc-complement $\tilde{\mathbb{W}} \oplus \bar{\psi}(L) = \mathbb{W}$ which exists by [34, Prop. 1.1] since L is finite dimensional. After fixing any linear isomorphism $\bar{\psi}(L) \rightarrow \mathbb{R}^n$, we obtain a linear sc-isomorphism

$$\tau : \mathbb{W} = \tilde{\mathbb{W}} \oplus \bar{\psi}(L) \rightarrow \tilde{\mathbb{W}} \times \mathbb{R}^n.$$

Define

$$\begin{aligned} \psi' &:= (id_{[0,\infty)^s \times \mathbb{R}^{k-s}} \times \tau) \circ \psi \circ g^{-1} : U' \rightarrow [0,\infty)^s \times \mathbb{R}^{k-s} \times \tilde{\mathbb{W}} \times \mathbb{R}^n, \\ \mathfrak{s}' &:= (g \triangleleft id_{\mathbb{F}^1}) \circ \mathfrak{s} \circ g^{-1} : U' \rightarrow U' \triangleleft \mathbb{F}^1, \\ \hat{U}'' &:= \psi'(U') = (id_{[0,\infty)^s \times \mathbb{R}^{k-s}} \times \tau)(\hat{U}'), \\ \Psi' &:= ((id_{[0,\infty)^s \times \mathbb{R}^{k-s}} \times \tau) \triangleleft (id_{\mathbb{R}^{k'}} \times \tau)) \\ &\quad \circ \Psi \circ (g^{-1} \triangleleft id_{\mathbb{F}^1}) : U' \triangleleft \mathbb{F}^1 \rightarrow \hat{U}'' \triangleleft \mathbb{R}^{k'} \times \tilde{\mathbb{W}} \times \mathbb{R}^n. \end{aligned}$$

We claim that the principal part of the section

$$b' := \Psi' \circ (h' - \mathfrak{s}') \circ \psi'^{-1} : \hat{U}'' \rightarrow \hat{U}'' \triangleleft \mathbb{R}^{k'} \times (\tilde{\mathbb{W}} \times \mathbb{R}^n) \quad (1.48)$$

is a basic germ, where here $\tilde{\mathbb{W}} \times \mathbb{R}^n$ plays the role of the space $\mathbb{W}$ in Definition 1.3.6(c). By hypothesis, the principal part $\mathbf{b}$ of the map (1.47) is a basic germ, which means that, letting

$$P : \mathbb{R}^{k'} \times \mathbb{W} \rightarrow \mathbb{W}$$

denote projection onto $\mathbb{W}$, we have, for $a \in [0,\infty)^s \times \mathbb{R}^{k-s}$ and $w \in \mathbb{W}$,

$$P \circ \mathbf{b}(a, w) = w - B(a, w),$$

where $B : \hat{U}' \rightarrow \mathbb{W}$ is a sc-smooth germ satisfying $B(0) = 0$ and the contraction property (1.26). Let

$$P' : \mathbb{R}^{k'} \times (\tilde{\mathbb{W}} \times \mathbb{R}^n) \rightarrow \tilde{\mathbb{W}} \times \mathbb{R}^n$$

be projection onto $\tilde{\mathbb{W}} \times \mathbb{R}^n$. Define

$$B' := \tau \circ B \circ (id_{[0,\infty)^s \times \mathbb{R}^{k-s}} \times \tau)^{-1} : \hat{U}'' \rightarrow \tilde{\mathbb{W}} \times \mathbb{R}^n.$$

Notice from the definitions that we have

$$P' = \tau \circ P \times (id_{\mathbb{R}^{k'}} \times \tau)^{-1}.$$

Then, for $a \in [0, \infty)^s \times \mathbb{R}^{k-s}$ and $z \in \tilde{\mathbb{W}} \times \mathbb{R}^n$ such that $(a, z) \in \hat{U}''$, we compute

$$\begin{aligned} P' \circ \mathbf{b}'(a, z) &= \tau \circ P \circ (id_{\mathbb{R}^{k'}} \times \tau)^{-1} \circ \pi_{\mathbb{R}^{k'} \times \tilde{\mathbb{W}} \times \mathbb{R}^n} \circ \Psi' \circ (h' - \mathfrak{s}') \circ \psi'^{-1}(a, z) \\ &= \tau \circ P \circ \pi_{\mathbb{R}^{k'} \times \mathbb{W}} \circ \Psi \circ (h - \mathfrak{s}) \circ \psi^{-1}(a, \tau^{-1}(z)) \\ &= \tau \circ P \circ \mathbf{b}(a, \tau^{-1}(z)) \\ &= \tau(\tau^{-1}(z) - B(a, \tau^{-1}(z))) \\ &= z - B'(a, z), \end{aligned}$$

so to prove that $\mathbf{b}'$ is a basic germ, it remains to verify that B' satisfies the contraction property (1.26). Since τ is a linear sc-isomorphism, for every integer $m \geq 0$ there exist constants $c_m, C_m > 0$ such that, for all $w \in \mathbb{W}$, we have

$$\|\tau(w)\|_m \leq C_m \cdot \|w\|_m,$$

and for all $(a, z) \in ([0, \infty)^s \times \mathbb{R}^{k-s}) \times (\tilde{\mathbb{W}} \times \mathbb{R}^n)$ we have

$$\|(id_{\mathbb{R}^k} \times \tau)^{-1}(a, z)\|_m \leq c_m \cdot \|(a, z)\|_m.$$

Fix $\epsilon > 0$ and $m \geq 0$. Then since B satisfies the contraction property (1.26), there exists $\delta'_m > 0$ that suffices for B and $\epsilon'_m := \epsilon / (C_m \cdot c_m)$. We claim that $\delta_m := \delta'_m / c_m$ suffices for B' and ϵ . Indeed, given $a \in [0, \infty)^s \times \mathbb{R}^{k-s}$ and $z, z' \in \tilde{\mathbb{W}} \times \mathbb{R}^n$ satisfying $\|(a, z)\|_m, \|(a, z')\|_m < \delta_m$, we have $\|(a, \tau^{-1}(z))\|_m, \|(a, \tau^{-1}(z'))\|_m < c_m \cdot \delta_m = \delta'_m$. So, we compute

$$\begin{aligned} \|B'(a, z) - B'(a, z')\|_m &= \|\tau(B(a, \tau^{-1}(z)) - B(a, \tau^{-1}(z')))\|_m \\ &\leq C_m \cdot \|B(a, \tau^{-1}(z)) - B(a, \tau^{-1}(z'))\|_m \\ &\leq C_m \cdot \epsilon'_m \cdot \|\tau^{-1}(z) - \tau^{-1}(z')\|_m \\ &= \epsilon / c_m \cdot \|\tau^{-1}(z - z')\|_m \\ &\leq \epsilon \cdot \|z - z'\|_m, \end{aligned}$$

as required. This completes the proof that $\mathbf{b}'$ is a basic germ, and hence that σ' is a local sc-Fredholm germ.

We claim that the additional conditions required for σ' to be a $\mathbb{R}^n$ -sliced sc-Fredholm germ are satisfied by construction. We must check that $\psi' : U' \rightarrow \hat{U}''$ is of the required form (1.28). Consider the projection

$$T : \mathbb{W} = \tilde{\mathbb{W}} \oplus \overline{\psi}(L) \rightarrow \tilde{\mathbb{W}}$$

along $\overline{\psi}(L) \subset \{0\}^{k-s} \times \mathbb{W}$. We have $L \oplus \mathbb{K}^1 = \mathbb{E}^1$ by (1.44) and since $L \subset E_\infty$. Hence the map

$$\overline{\psi}' := (id_{\mathbb{R}^{k-s}} \times T) \circ \overline{\psi}|_{\{0\} \oplus \mathbb{K}^1} : \mathbb{K}^1 \rightarrow \mathbb{R}^{k-s} \times \tilde{\mathbb{W}}$$

is a linear sc-isomorphism. Let $(p, v, \kappa) \in U' \subset \mathbb{R}^n \times [0, \infty)^s \times \mathbb{K}^1$. By the form of the map g above, we can write

$$g^{-1}(p, v, \kappa) =: (v, e_{p,v,\kappa}) \in [0, \infty)^s \times \mathbb{E}^1$$

for some $e_{p,v,\kappa} \in \mathbb{E}^1$. Moreover, we have $pr(e_{p,v,\kappa}) = \kappa$, where pr is the projection $pr : \mathbb{E}^1 = L \oplus \mathbb{K}^1 \rightarrow \mathbb{K}^1$ along L . In particular, we have $e_{p,v,\kappa} - \kappa \in L$, from which we obtain a map into $\{0\} \times \mathbb{R}^n \subset \tilde{W} \times \mathbb{R}^n$ defined by

$$\begin{aligned} \lambda : U' &\rightarrow \mathbb{R}^n \\ (p, v, \kappa) &\mapsto \tau(\bar{\psi}(e_{p,v,\kappa} - \kappa)). \end{aligned}$$

This map is C^1 on every level $(U')_m \rightarrow \mathbb{R}^n$ for $m \geq 0$ because g is a C^1 -diffeomorphism on every level and the linear sc-isomorphisms τ and $\bar{\psi}$ are C^∞ on every level. Then we compute

$$\begin{aligned} \psi'(p, v, \kappa) &= (id_{[0,\infty)^s \times \mathbb{R}^{k-s}} \times \tau) \circ \psi \circ (v, e_{p,v,\kappa}) \\ &= (id_{[0,\infty)^s \times \mathbb{R}^{k-s}} \times \tau) \circ (id_{[0,\infty)^s} \times \bar{\psi}) \circ (v, e_{p,v,\kappa}) \\ &= (v, (id_{\mathbb{R}^{k-s}} \times \tau) \circ \bar{\psi}(e_{p,v,\kappa})) \\ &= (v, (id_{\mathbb{R}^{k-s}} \times \tau)(\bar{\psi}(\kappa) + \bar{\psi}(e_{p,v,\kappa} - \kappa))) \\ &= (v, (id_{\mathbb{R}^{k-s}} \times T)(\bar{\psi}(\kappa), \tau(\bar{\psi}(e_{p,v,\kappa} - \kappa)))) \\ &= (v, \bar{\psi}'(\kappa), \lambda(p, v, \kappa)). \end{aligned}$$

This proves that σ' is a $\mathbb{R}^n$ -sliced sc-Fredholm germ. Hence, by Lemma 1.3.10, the restriction $\tilde{\sigma}' = \sigma'|_{\tilde{\mathcal{O}}'} : \tilde{\mathcal{O}}' \rightarrow \tilde{K}'$ is a tame sc-Fredholm germ with sc-Fredholm index $ind(\tilde{\sigma}') = ind(\sigma') - n$. Moreover, by definition (1.27) of index and by the forms of the basic germs (1.47) and (1.48), we have $ind(\sigma') = ind(\sigma)$. Hence $ind(\tilde{\sigma}') = ind(\sigma) - n$. This completes the proof of the lemma. $\square$

Remark 1.4.3. We note some additional properties of the objects constructed in the proof of Lemma 1.4.2. These properties are required for the construction of quotients of polyfolds by group actions in [55].

The sc-Banach space $\mathbb{K}$ is $\ker D_x(f_\partial \circ r_\partial) \subset \mathbb{E}$, where $r_\partial : U \cap (\{0\} \times \mathbb{E}) \rightarrow U \cap (\{0\} \times \mathbb{E})$ is the boundary sc-retraction associated to a tame sc-retraction $r : U \rightarrow U$ onto $r(U) = \mathcal{O}$ from Lemma 1.4.1. There is a sc-complement L of $\ker D_x f_\partial$ in $T_x \mathcal{O}_\partial$, and moreover L is a sc-complement of $\mathbb{K}$ in $\mathbb{E}$. The change of coordinates $g : [0, \infty)^s \times \mathbb{E}^1 \rightarrow \mathbb{R}^n \times [0, \infty)^s \times \mathbb{K}^1$, which is defined in a neighborhood of x , is of the form $g(v, e) = (f \circ r(v, e), v, pr(e))$ where $pr : \mathbb{E} = L \oplus \mathbb{K} \rightarrow \mathbb{K}$ is projection along L . In particular, in the case $x = 0$, the sc-differential of g satisfies $D_0 g(\{0\} \times L) = \mathbb{R}^n \times \{0\} \times \{0\}$ and $D_0 g(\{0\} \times \mathbb{K}^1) = \{0\} \times \{0\} \times \mathbb{K}^1$.

1.5 Slicing tame sc-Fredholm sections with transverse constraints

The theory of sc-Fredholm sections $\sigma : \mathcal{B} \rightarrow \mathcal{E}$ (Definition 1.5.3) of tame strong bundles $\rho : \mathcal{E} \rightarrow \mathcal{B}$ (Definition 1.5.2) over tame M -polyfolds $\mathcal{B}$ (Definition 1.5.1) is developed in [34]. In this section, we introduce the new stronger notion of a tame sc-Fredholm section

(Definition 1.5.4), which is the same as a standard sc-Fredholm section with the stronger condition that the local sc-Fredholm germs (Definition 1.3.6) can be chosen to be the tame sc-Fredholm germs introduced in Definition 1.3.7: roughly, we require that the change of coordinates that brings the local fillers into basic germ form can be chosen to be a *linear sc-isomorphism* on the base. We also introduce the new notion of slices $\tilde{\mathcal{B}} \subset \mathcal{B}$ (Definition 1.5.7) of sc-Fredholm sections σ , which are subspaces of $\mathcal{B}$ in which the local sc-Fredholm germs are $\mathbb{R}^n$ -sliced (Definition 1.3.8): roughly, the dependence of the change of coordinates to basic germ form on the normal directions to $\tilde{\mathcal{B}}$ in $\mathcal{B}$ splits off into a $\mathbb{R}^n$ -factor in the codomain.

Tame sc-Fredholm sections and slices are the main objects of interest in this paper. In this section, we prove our main Theorems 1.1.3, 1.1.5. Roughly, given a tame sc-Fredholm section $\sigma : \mathcal{B} \rightarrow \mathcal{E}$ and a sc-smooth map $f : \mathcal{B} \rightarrow Y$ to a finite dimensional manifold Y that is σ -compatibly transverse (Definition 1.5.8) to a submanifold $N \subset Y$, then $f^{-1}(N)$ is a slice of σ . Moreover, given a slice $\tilde{\mathcal{B}} \subset \mathcal{B}$ of a sc-Fredholm section $\sigma : \mathcal{B} \rightarrow \mathcal{E}$, the restriction $\sigma|_{\tilde{\mathcal{B}}}$ is tame sc-Fredholm. The generalizations of the concepts and theorems in this section to the ep-groupoid case, which are required to handle isotropy in applications, are developed in Section 1.6.

In Section 1.5.1, we explain why the Cauchy-Riemann section $\bar{\partial}_J : \mathcal{B} \rightarrow \mathcal{E}$ is tame sc-Fredholm and why evaluation maps $ev : \mathcal{B} \rightarrow Y$ at marked points are $\bar{\partial}_J$ -compatibly transverse to every submanifold $N \subset Y$.

We begin with a review of standard M -polyfold notions.

Definition 1.5.1. [34, Defs. 2.6, 2.19]

- A **tame M -polyfold chart** on a topological space $\mathcal{B}$ is a tuple

$$(V, \varphi, (\mathcal{O}, C, \mathbb{E})),$$

where $V \subset \mathcal{B}$ is an open set, $\varphi : V \rightarrow \mathcal{O}$ is a homeomorphism, and $(\mathcal{O}, C, \mathbb{E})$ is a tame sc-retract (Definition 1.3.1).

- A **tame M -polyfold $\mathcal{B}$** is a Hausdorff paracompact topological space together with an equivalence class of tame sc-smooth atlases. A **tame sc-smooth atlas** consists of a cover of $\mathcal{B}$ by tame M -polyfold charts whose transition maps are sc-smooth. Atlases are equivalent if and only if their union is an atlas.

A tame M -polyfold $\mathcal{B}$ carries a filtration by a sequence of topological spaces

$$\mathcal{B} = (\mathcal{B}_0 \hookrightarrow \mathcal{B}_1 \hookrightarrow \dots)$$

where the maps $\mathcal{B}_m \rightarrow \mathcal{B}_{m+1}$ are continuous injections with dense image. This filtration is provided locally by the sc-structure on the sc-retracts in charts, and it is independent of the choice of charts due to sc-continuity of the transition maps. The smooth points $\mathcal{B}_\infty$ of $\mathcal{B}$ are the subset

$$\mathcal{B}_\infty := \bigcap_{m \geq 0} \mathcal{B}_m.$$

By forgetting about the first $m \geq 0$ levels of $\mathcal{B}$, we obtain the m -shift $\mathcal{B}^m$ of $\mathcal{B}$, which is the M -polyfold

$$\mathcal{B}^m := (\mathcal{B}_m \hookleftarrow \mathcal{B}_{m+1} \hookleftarrow \cdots)$$

with charts given by restricting charts on $\mathcal{B}$ to the m -shift $\varphi : V_m = V \cap \mathcal{B}_m \rightarrow \mathcal{O}_m = \mathcal{O} \cap E_m$. Notice that a m -shift does not affect the smooth points, i.e.

$$\mathcal{B}_\infty = \mathcal{B}_\infty^m.$$

The *tangent space* $T_x \mathcal{B}$ at a point $x \in \mathcal{B}_1$ is defined as the set of equivalence classes of tangent vectors in all charts around x , with the equivalence given by the tangent map of the transition map between charts, which is a linear isomorphism on every tangent space; see [34, Def. 2.11] for a precise treatment. The result is a Banach space structure on $T_x \mathcal{B}$ and, for any chart $(V, \varphi, (\mathcal{O}, C, \mathbb{E}))$ containing $x \in V$, a tangent map $D_x \varphi : T_x V = T_x \mathcal{B} \rightarrow T_{\varphi(x)} \mathcal{O}$ which is a bounded linear isomorphism. If $x \in \mathcal{B}_\infty$ is a smooth point then $\varphi(x) \in \mathcal{O}_\infty$ is a smooth point and $T_x \mathcal{B}$ inherits the structure of a sc-Banach space from the sc-Banach space $T_{\varphi(x)} \mathcal{O}$.

The *reduced tangent space* $T_x^R \mathcal{B}$ [34, Def. 2.20] is defined only for smooth points $x \in \mathcal{B}_\infty$ as the subspace of $T_x \mathcal{B}$ such that

$$D_x \varphi(T_x^R \mathcal{B}) = T_{\varphi(x)}^R \mathcal{O}, \quad (1.49)$$

where $T_{\varphi(x)}^R \mathcal{O}$ is the reduced tangent space to a sc-retract (1.12). Note that, since the reduced tangent space to a sc-retract is invariant under sc-diffeomorphisms of sc-retracts only at smooth points, this global notion of reduced tangent space $T_x^R \mathcal{B}$ is well-defined only for smooth points $x \in \mathcal{B}_\infty$.

The *degeneracy index* $d_{\mathcal{B}} : \mathcal{B} \rightarrow \mathbb{N}_0$ is defined for $x \in \mathcal{B}$ as the minimum over all charts $(V, \varphi, (\mathcal{O}, C, \mathbb{E}))$ containing $x \in V$ of the degeneracy index in the chart $d_C(\varphi(x))$; see [34, Def. 2.13]. For a smooth point $x \in \mathcal{B}_\infty$, it is explained in [34, Rmk. 2.4] that there is a global interpretation of this quantity:

$$d_{\mathcal{B}}(x) = \dim(T_x \mathcal{B} / T_x^R \mathcal{B}) \text{ for } x \in \mathcal{B}_\infty. \quad (1.50)$$

We now recall the definition of a tame strong bundle $\rho : \mathcal{E} \rightarrow \mathcal{B}$.

Definition 1.5.2. [34, Defs. 2.25, 2.26]

- Consider a continuous surjective map $\rho : \mathcal{E} \rightarrow \mathcal{B}$ from a topological space $\mathcal{E}$ onto a topological space $\mathcal{B}$ such that, for every $x \in \mathcal{B}$, the fiber $\rho^{-1}(x) = \mathcal{E}_x$ has the structure of a Banach space. A **tame strong bundle chart** for $\rho : \mathcal{E} \rightarrow \mathcal{B}$ is a tuple

$$(\rho^{-1}(V), \Phi, (K, C \triangleleft \mathbb{F}, \mathbb{E} \triangleleft \mathbb{F}))$$

in which $(K, C \triangleleft \mathbb{F}, \mathbb{E} \triangleleft \mathbb{F})$ is a tame strong bundle retract (Section 1.3.2) covering a tame sc-retract $(\mathcal{O}, C, \mathbb{E})$, i.e. $K \rightarrow \mathcal{O}$ is a tame strong local bundle model. In addition, $V \subset \mathcal{B}$ is the domain of a tame M -polyfold chart $(V, \varphi, (\mathcal{O}, C, \mathbb{E}))$ and

$$\Phi : \rho^{-1}(V) \rightarrow K$$

is a homeomorphism onto K that covers φ , i.e.

$$\pi_{\mathcal{O}} \circ \Phi = \varphi \circ \rho \quad (1.51)$$

holds on $\rho^{-1}(V)$. Moreover, Φ is linear on the fibers over V , i.e. for all $x \in V$ the map $\Phi : \rho^{-1}(x) \rightarrow \pi_{\mathcal{O}}^{-1}(\varphi(x))$ is a bounded linear isomorphism of Banach spaces.

- A **tame strong bundle** $\rho : \mathcal{E} \rightarrow \mathcal{B}$ is a continuous surjective map from a paracompact Hausdorff space $\mathcal{E}$ onto a tame M -polyfold $\mathcal{B}$ such that every fiber has the structure of a Banach space, together with an equivalence class of tame bundle atlases. A **tame bundle atlas** consists of a cover of $\mathcal{B}$ by tame strong bundle charts whose transition maps are sc-smooth. Atlases are equivalent if and only if their union is an atlas.

The tame strong bundle charts on a tame strong bundle $\rho : \mathcal{E} \rightarrow \mathcal{B}$ induce a double filtration $\mathcal{E}_{m,k}$ for $m \geq 0$ and $0 \leq k \leq m+1$ from the double filtration (1.22) on the bundle retracts in the charts. From this we distinguish the M -polyfolds $\mathcal{E}[i]$ for $i = 0, 1$, by the filtrations

$$\mathcal{E}[i]_m = \mathcal{E}_{m,m+i}, \quad m \geq 0. \quad (1.52)$$

Both projections

$$\rho[i] : \mathcal{E}[i] \rightarrow \mathcal{B}$$

are sc-smooth maps.

There are two notions of smoothness for a section of ρ (see [34, Def. 2.27]), which correspond to the two projections $\rho[i]$ for $i = 0, 1$. A section $s : \mathcal{B} \rightarrow \mathcal{E}$ is called a *sc-smooth section* of ρ if $s : \mathcal{B} \rightarrow \mathcal{E}[0]$ is a sc-smooth section of the bundle $\rho[0]$. The section s is called a *sc⁺-section* of ρ if $s(\mathcal{B}) \subset \mathcal{E}[1]$ and $s : \mathcal{B} \rightarrow \mathcal{E}[1]$ is a sc-smooth section of the bundle $\rho[1]$. The sc⁺-sections become important when perturbing sc-Fredholm sections. They are analogous to compact operators in classical Fredholm theory. In particular, adding sc⁺-sections to sc-Fredholm sections preserves the sc-Fredholm property [34, Thm. 3.2].

We now recall the definition of a sc-Fredholm section $\sigma : \mathcal{B} \rightarrow \mathcal{E}$.

Definition 1.5.3. [34, Def. 3.8]

- Consider a section $\sigma : \mathcal{B} \rightarrow \mathcal{E}$ of a tame strong bundle $\rho : \mathcal{E} \rightarrow \mathcal{B}$. Let $x \in \mathcal{B}_{\infty}$ be a smooth point. Then a tame strong bundle chart $(\rho^{-1}(V), \Phi, (K, C \triangleleft \mathbb{F}, \mathbb{E} \triangleleft \mathbb{F}))$ for ρ covering a tame M -polyfold chart $(V, \varphi, (\mathcal{O}, C, \mathbb{E}))$ on $\mathcal{B}$ is called a **sc-Fredholm chart for σ at x** if it satisfies $x \in V$ and $\varphi(x) = 0 \in \mathcal{O}$, and if the section

$$\Phi \circ \sigma \circ \varphi^{-1} : \mathcal{O} \rightarrow K$$

of the local bundle model $K \rightarrow \mathcal{O}$ is a local sc-Fredholm germ (Definition 1.3.6).

- A **sc-Fredholm section** $\sigma : \mathcal{B} \rightarrow \mathcal{E}$ of ρ is a section that has the following properties:
 - (1) σ is sc-smooth.
 - (2) σ is **regularizing**, which means that if $x \in \mathcal{B}_m$ and $\sigma(x) \in \mathcal{E}_{m,m+1}$ then $x \in \mathcal{B}_{m+1}$.

(3) For every smooth point $x \in \mathcal{B}_\infty$, there exists a sc-Fredholm chart for σ at x . The index $\text{ind}_x(\sigma)$ of σ at x is the index (1.27) of the local sc-Fredholm germ in any such chart.

We now introduce the new stronger notion of a tame sc-Fredholm section $\sigma : \mathcal{B} \rightarrow \mathcal{E}$ of a tame M -polyfold bundle $\rho : \mathcal{E} \rightarrow \mathcal{B}$.

Definition 1.5.4.

- Consider a section $\sigma : \mathcal{B} \rightarrow \mathcal{E}$ of a tame strong bundle $\rho : \mathcal{E} \rightarrow \mathcal{B}$. Let $x \in \mathcal{B}_\infty$ be a smooth point. Then a **tame sc-Fredholm chart for σ at x** is a sc-Fredholm chart (Definition 1.5.3) such that the local sc-Fredholm germ $\Phi \circ \sigma \circ \varphi^{-1}$ is in addition a tame sc-Fredholm germ (Definition 1.3.7).
- A **tame sc-Fredholm section** $\sigma : \mathcal{B} \rightarrow \mathcal{E}$ of ρ is a sc-Fredholm section (Definition 1.5.3) such that there exists a tame sc-Fredholm chart for σ at every $x \in \mathcal{B}_\infty$.

Remark 1.5.5. Given a strong bundle $\rho : \mathcal{E} \rightarrow \mathcal{B}$ and a sc-Fredholm section $\sigma : \mathcal{B} \rightarrow \mathcal{E}$, we obtain their m -shifts $\rho : \mathcal{E}^m \rightarrow \mathcal{B}^m$ and $\sigma : \mathcal{B}^m \rightarrow \mathcal{E}^m$ by restricting domains and codomains to the m -shifts $\mathcal{E}^m$ and $\mathcal{B}^m$. Performing this shift preserves all essential features of the polyfold setup. In particular, $\sigma : \mathcal{B}^m \rightarrow \mathcal{E}^m$ is a sc-Fredholm section of the strong bundle $\rho : \mathcal{E}^m \rightarrow \mathcal{B}^m$ and the sc-Fredholm index is preserved [34, Cor. 5.1].

Crucially, the zero set of a sc-Fredholm section is preserved under m -shifts. Indeed, since $\sigma^{-1}(0) \subset \mathcal{B}_\infty$ by the regularizing property Definition 1.5.3(2) of σ , it follows that $\sigma^{-1}(0) = \sigma^{-1}(0) \cap \mathcal{B}^m = \sigma|_{\mathcal{B}^m}^{-1}(0)$. Moreover, compactness of $\sigma^{-1}(0)$ in the topology of $\mathcal{B}_m$ for any fixed $m \geq 0$ is equivalent to compactness in the topology of $\mathcal{B}_\infty$ (see [34, Thm. 5.3, Cor. 5.1]). For this reason, we can unambiguously refer to compactness of $\sigma^{-1}(0)$ without reference to the levels, and m -shifting preserves this compactness.

We now introduce the new notions of a slice $\tilde{\mathcal{B}} \subset \mathcal{B}$ of a tame M -polyfold $\mathcal{B}$, a slice of a tame strong bundle $\rho : \mathcal{E} \rightarrow \mathcal{B}$, and a slice of a sc-Fredholm section $\sigma : \mathcal{B} \rightarrow \mathcal{E}$ (Definition 1.5.7). Roughly, a slice is a subspace $\tilde{\mathcal{B}}$ such that for every point $x \in \tilde{\mathcal{B}}$ there is a tame M -polyfold chart that identifies a neighborhood of x in $\mathcal{B}$ with a $\mathbb{R}^n$ -sliced sc-retract (Definition 1.3.2), and such that the induced tame sc-retract (1.14) is identified with a neighborhood of x in $\tilde{\mathcal{B}}$. To be a slice of the sc-Fredholm section σ , we additionally require that, at smooth points $x \in \tilde{\mathcal{B}}_\infty = \tilde{\mathcal{B}} \cap \mathcal{B}_\infty$, we can choose the chart on $\mathcal{B}$ such that it is covered by a bundle chart for ρ in which σ is a $\mathbb{R}^n$ -sliced sc-Fredholm germ (Definition 1.3.8). In such a chart, the induced tame sc-Fredholm germ (see Lemma 1.3.10) is identified with the restriction of σ to $\tilde{\mathcal{B}}$ near x . Globally, the result is that a slice of σ is a finite codimension M -polyfold $\tilde{\mathcal{B}}$ embedded in $\mathcal{B}$ that is compatible with the local fillers for σ in such a way that σ restricts to a tame sc-Fredholm section over $\tilde{\mathcal{B}}$ (Theorem 1.1.3(III)).

We first introduce the slice notions in charts.

Definition 1.5.6.

- Consider a subspace $\tilde{\mathcal{B}} \subset \mathcal{B}$ of a tame M -polyfold $\mathcal{B}$. A $\mathbb{R}^n$ -sliced chart with respect to $\tilde{\mathcal{B}} \subset \mathcal{B}$ is a tame M -polyfold chart on $\mathcal{B}$ of the form

$$(V, \varphi, (\mathcal{O}, \mathbb{R}^n \times C, \mathbb{R}^n \times \mathbb{E})) \quad (1.53)$$

where $(\mathcal{O}, \mathbb{R}^n \times C, \mathbb{R}^n \times \mathbb{E})$ is a $\mathbb{R}^n$ -sliced sc-retract (Definition 1.3.2) with induced tame sc-retract $\tilde{\mathcal{O}} = \mathcal{O} \cap (\{0\} \times C)$ (see (1.14)) satisfying

$$\tilde{\mathcal{O}} = \varphi(\tilde{\mathcal{B}} \cap V). \quad (1.54)$$

- Consider a tame strong bundle $\rho : \mathcal{E} \rightarrow \mathcal{B}$. A $\mathbb{R}^n$ -sliced bundle chart for ρ with respect to $\tilde{\mathcal{B}} \subset \mathcal{B}$ is a tame strong bundle chart for ρ of the form

$$(\rho^{-1}(V), \Phi, (K, \mathbb{R}^n \times C \triangleleft \mathbb{F}, \mathbb{R}^n \times \mathbb{E} \triangleleft \mathbb{F})) \quad (1.55)$$

covering a $\mathbb{R}^n$ -sliced chart with respect to $\tilde{\mathcal{B}} \subset \mathcal{B}$ as in (1.53). In particular, the tame bundle retract $(K, \mathbb{R}^n \times C \triangleleft \mathbb{F}, \mathbb{R}^n \times \mathbb{E} \triangleleft \mathbb{F})$ is a $\mathbb{R}^n$ -sliced bundle retract (Definition 1.3.4) covering the sc-retract $\mathcal{O}$ from (1.53).

- Consider a section $\sigma : \mathcal{B} \rightarrow \mathcal{E}$ of ρ and a smooth point $x \in \tilde{\mathcal{B}}_\infty = \tilde{\mathcal{B}} \cap \mathcal{B}_\infty$. Then a $\mathbb{R}^n$ -sliced sc-Fredholm chart for σ at x with respect to $\tilde{\mathcal{B}} \subset \mathcal{B}$ is a $\mathbb{R}^n$ -sliced bundle chart with respect to $\tilde{\mathcal{B}} \subset \mathcal{B}$ as in (1.55) that satisfies $x \in V$ and $\varphi(x) = 0$, and such that the section

$$\Phi \circ \sigma \circ \varphi^{-1} : \mathcal{O} \rightarrow K$$

is a $\mathbb{R}^n$ -sliced sc-Fredholm germ (Definition 1.3.8).

The following global notions of slices of tame M -polyfolds, tame strong bundles, and sc-Fredholm sections require covers of $\tilde{\mathcal{B}}$ by $\mathbb{R}^n$ -sliced charts.

Definition 1.5.7.

- Consider a tame M -polyfold $\mathcal{B}$. A subspace $\tilde{\mathcal{B}} \subset \mathcal{B}$ is called a **slice of $\mathcal{B}$** if for every $x \in \tilde{\mathcal{B}}$ there exists an integer $n_x = \text{codim}_x(\tilde{\mathcal{B}} \subset \mathcal{B}) \geq 0$ and a $\mathbb{R}^{n_x}$ -sliced chart with respect to $\tilde{\mathcal{B}} \subset \mathcal{B}$ that contains x . For $x \in \tilde{\mathcal{B}}_1 = \tilde{\mathcal{B}} \cap \mathcal{B}_1$, the integer $\text{codim}_x(\tilde{\mathcal{B}} \subset \mathcal{B})$ is called the **codimension of the slice at x** (and it is well-defined at x and locally constant in $\tilde{\mathcal{B}}$, by Theorem 1.1.3).
- Consider a tame strong bundle $\rho : \mathcal{E} \rightarrow \mathcal{B}$. A slice $\tilde{\mathcal{B}} \subset \mathcal{B}$ of $\mathcal{B}$ is called a **slice of the bundle ρ** if for every $x \in \tilde{\mathcal{B}}$ there exists a $\mathbb{R}^{n_x}$ -sliced bundle chart for ρ with respect to $\tilde{\mathcal{B}} \subset \mathcal{B}$ that contains x .
- Consider a sc-Fredholm section $\sigma : \mathcal{B} \rightarrow \mathcal{E}$ of ρ . A slice $\tilde{\mathcal{B}} \subset \mathcal{B}$ of the bundle ρ is called a **slice of the sc-Fredholm section σ** if for every $x \in \tilde{\mathcal{B}}_\infty = \tilde{\mathcal{B}} \cap \mathcal{B}_\infty$ there exists a $\mathbb{R}^{n_x}$ -sliced sc-Fredholm chart for σ at x with respect to $\tilde{\mathcal{B}} \subset \mathcal{B}$.

We are now equipped to prove one of the main theorems (Theorem 1.1.3), which says that slices of tame M -polyfolds are tame M -polyfolds with locally constant codimension, bundles restrict to slices, and sc-Fredholm sections restrict to slices with a drop in Fredholm index given by the codimension. See Section 1.5.2 for the proof. The essential point in the proof is that sliced charts with respect to $\tilde{\mathcal{B}} \subset \mathcal{B}$ induce tame M -polyfold charts on $\tilde{\mathcal{B}}$, sliced bundle charts induce tame bundle charts on the restricted bundle map, and sliced sc-Fredholm charts induce tame sc-Fredholm charts on the restricted section. This is the M -polyfold analog of the classical Fact 1.1.1 (restrictions of Fredholm sections to sub-Banach manifolds).

We now prepare for the proof of our main theorem (Theorem 1.1.5), which is the M -polyfold analog of the classical Fact 1.1.2 (transverse preimages are sub-Banach manifolds). The result is roughly as follows. Consider a tame sc-Fredholm section $\sigma : \mathcal{B} \rightarrow \mathcal{E}$ of a tame M -polyfold bundle $\rho : \mathcal{E} \rightarrow \mathcal{B}$. Given a sc-smooth map $f : \mathcal{B} \rightarrow Y$ to a finite dimensional smooth manifold Y that is σ -compatibly transverse (Definition 1.5.8) to a submanifold $N \subset Y$, we prove that the restriction of σ to the preimage $f^{-1}(N)$ is a tame sc-Fredholm section. The notion of σ -compatible transversality requires compatibility between the map f , the submanifold N , and the coordinate changes that bring the local fillers of σ into basic germ form. All necessary compatibility is satisfied in applications. In particular, evaluation maps $f = ev$ at marked points are $\bar{\partial}_J$ -compatibly transverse to every submanifold, where $\bar{\partial}_J$ is the Cauchy-Riemann section; see Section 1.5.1. We now introduce the precise definition of σ -compatibly transverse.

Definition 1.5.8.

- Consider a tame M -polyfold $\mathcal{B}$, a smooth manifold Y together with a codimension- n submanifold $N \subset Y$, and a sc-smooth map $f : \mathcal{B} \rightarrow Y$. Then f is called **transverse to N** if, for all $x \in f^{-1}(N) \cap \mathcal{B}_\infty$,

$$D_x f(T_x^R \mathcal{B}) + T_{f(x)} N = T_{f(x)} Y \quad (1.56)$$

holds.

- Consider a tame sc-Fredholm section $\sigma : \mathcal{B} \rightarrow \mathcal{E}$ (Definition 1.5.4) of a tame strong bundle $\rho : \mathcal{E} \rightarrow \mathcal{B}$. Then f is called **σ -compatibly transverse to N** if, for all $x \in f^{-1}(N) \cap \mathcal{B}_\infty$, condition (1.56) holds and moreover there exists a tame sc-Fredholm chart (Definition 1.5.4) for σ at x , denoted $(\rho^{-1}(V), \Phi, (K, [0, \infty)^s \times \mathbb{E} \triangleleft \mathbb{F}, \mathbb{R}^s \times \mathbb{E} \triangleleft \mathbb{F}))$, covering a tame M -polyfold chart $(V, \varphi, (\mathcal{O}, [0, \infty)^s \times \mathbb{E}, \mathbb{R}^s \times \mathbb{E}))$ on $\mathcal{B}$ such that the tame sc-Fredholm germ $\Phi \circ \sigma \circ \varphi^{-1} : \mathcal{O} \rightarrow K$ (Definition 1.3.7) satisfies the following property: There exists a choice of sc-Banach space $\mathbb{W}$ and linear sc-isomorphism $\bar{\psi} : \mathbb{E} \rightarrow \mathbb{R}^{k-s} \times \mathbb{W}$ satisfying the conditions in Definition 1.3.7 of tame sc-Fredholm germ such that, in addition, there exists a sc-complement L of $(D_x f)^{-1}(T_{f(x)} N) \cap T_x^R \mathcal{B}$ in $T_x^R \mathcal{B}$ satisfying

$$\bar{\psi} \circ D_x \varphi(L) \subset \{0\}^{k-s} \times \mathbb{W}. \quad (1.57)$$

We are now equipped to prove the main theorem (Theorem 1.1.5). See Section 1.5.3 for the proof.

1.5.1 Example: The Cauchy-Riemann section and evaluation maps at marked points

Consider a symplectic manifold (Y, ω) , a codimension- n submanifold $N \subset Y$, and an ω -compatible almost complex structure J on Y . In applications, the M -polyfold $\mathcal{B}$ in Theorem 1.1.5 (or, in the presence of isotropy, the ep-groupoid X in Corollary 1.6.8) is a space of maps $\Sigma \rightarrow Y$ modulo reparameterization of the domain Σ , where Σ varies in some Deligne-Mumford space of Riemann surfaces, and $\sigma = \bar{\partial}_J : \mathcal{B} \rightarrow \mathcal{E}$ is the Cauchy-Riemann section associated to J ; see for example the Gromov-Witten polyfolds constructed in [33]. We consider the evaluation map $f = ev : \mathcal{B} \rightarrow Y$ at a marked point⁴ that varies in the domains Σ . The purpose of this section is to explain the following properties of this setup:

- σ is a tame sc-Fredholm section (Definition 1.5.4),
- f is σ -compatibly transverse to N (Definition 1.5.8).

To see that σ is a tame sc-Fredholm section, we consider a smooth point $x \in \mathcal{B}_\infty$ and explain why σ is a tame sc-Fredholm germ (Definition 1.3.7) in the chart constructed around x when building the polyfold $\mathcal{B}$. The tame sc-retract

$$\mathcal{O} \subset [0, \infty)^s \times \mathbb{R}^t \times D \times \mathbb{E}$$

is homeomorphic to the image of the pregluing map near x . Here the $[0, \infty)^s$ -factor is gluing parameters near the broken points of x , the $\mathbb{R}^t$ -factor is gluing parameters near the nodal points of x , the space D is variations of the complex structure on the domain Σ of x (i.e. tangent directions to the Deligne-Mumford space), and the sc-Banach space $\mathbb{E}$ corresponds to varying the map x while preserving the matching conditions at nodes and breaking points. The tame bundle retract

$$K \subset [0, \infty)^s \times \mathbb{R}^t \times D \times \mathbb{E} \triangleleft \mathbb{F}$$

is homeomorphic to the image of the pregluing map in the fibers near x . From this local bundle model $K \rightarrow \mathcal{O}$ we obtain a tame M -polyfold chart on $\mathcal{B}$ with domain $V \subset \mathcal{B}$ and chart map $\varphi : V \rightarrow \mathcal{O}$ where $x \in V$ is identified with $\varphi(x) = 0 \in \mathcal{O}$. It is covered by a tame strong bundle chart $\Phi : \mathcal{E}|_V \rightarrow K$.

The section $\Phi \circ \sigma \circ \varphi^{-1} : \mathcal{O} \rightarrow K$ is a local sc-Fredholm germ (Definition 1.3.6), and moreover the natural change of coordinates of the filling to basic germ form satisfies the required conditions of a tame sc-Fredholm germ (Definition 1.3.7). Indeed, a filling

$$h : [0, \infty)^s \times \mathbb{R}^t \times D \times \mathbb{E} \rightarrow [0, \infty)^s \times \mathbb{R}^t \times D \times \mathbb{E} \triangleleft \mathbb{F}$$

of $\Phi \circ \sigma \circ \varphi^{-1}$ is constructed as in [33, Sec. 4.5]. The change of coordinates ψ on the domain of the filling h that brings h into basic germ form (Definition 1.3.6(c)) is obtained as follows;

⁴The evaluation map at k marked points $\mathcal{B} \rightarrow Y^k$ can be treated similarly; we consider a single marked point here for simplicity.

see [33, Prop. 4.8] for details. Let

$$P := \ker(D_0 h) \cap (\{0\}^{s+t} \times \{0\} \times \mathbb{E})$$

denote the kernel of the linearization of h at 0 in the directions $\mathbb{E}$. Note that P is finite dimensional since $D_0 h$ is a sc-Fredholm operator. Then, the sc-Banach space $\mathbb{W}$ and linear sc-isomorphism

$$\bar{\psi} : \mathbb{E}' := \mathbb{R}^t \times D \times \mathbb{E} \rightarrow \mathbb{R}^{k-s} \times \mathbb{W}$$

required in Definition 1.3.7 are obtained by choosing any sc-splitting $\mathbb{E} = P \oplus \mathbb{W}$ and linear isomorphism $D \times P \cong \mathbb{R}^{k-t-s}$ to obtain the linear sc-isomorphism $\bar{\psi} : \mathbb{E}' = \mathbb{R}^t \times D \times P \oplus \mathbb{W} \rightarrow \mathbb{R}^t \times \mathbb{R}^{k-t-s} \times \mathbb{W}$. That is, the linear sc-isomorphism $\psi = id_{[0,\infty)^s} \times \bar{\psi}$ is a suitable choice of the sc-diffeomorphism in Definition 1.3.6(c) of local sc-Fredholm germ, so in particular $\Phi \circ \sigma \circ \varphi^{-1}$ is a tame sc-Fredholm germ. It follows that $\sigma = \bar{\partial}_J$ is a tame sc-Fredholm section.

Suppose $x \in f^{-1}(N) \cap \mathcal{B}_\infty$. To show that f is σ -compatibly transverse to N at x , we now construct a sc-subspace $L \subset T_x^R \mathcal{B}$ which satisfies the required conditions, in particular condition (1.57).

Let Z be any complement of $T_{f(x)} N$ in $T_{f(x)} Y$. First we claim that it suffices to construct $L \subset T_x^R \mathcal{B}$ satisfying

- (i) $D_x f(L) = Z$ and $D_x f : L \rightarrow Z$ is an isomorphism,
- (ii) $D_x \varphi(L) \subset \{0\}^{s+t} \times \{0\} \times \mathbb{E}$,
- (iii) $D_x \varphi(L) \cap P = \{0\}$.

Observe that f is transverse to N at x by (i); indeed, the required spanning property (1.56) holds since $D_x f(L) + T_{f(x)} N = T_{f(x)} Y$. Moreover, we claim that L is a sc-complement of $\mathbb{A} := (D_x f)^{-1}(T_{f(x)} N) \cap T_x^R \mathcal{B}$ in $T_x^R \mathcal{B}$. Note that the linear isomorphism $D_x f|_L : L \rightarrow Z$ is automatically sc-continuous because all norms on finite dimensional spaces are equivalent. Consider the projection $\pi : T_{f(x)} Y = Z \oplus T_{f(x)} N \rightarrow Z$ and the composition $\pi \circ D_x f : T_x^R \mathcal{B} \rightarrow Z$. Then we have $\mathbb{A} = \ker(\pi \circ D_x f)$, and moreover $\pi \circ D_x f$ maps L isomorphically onto Z . Hence the sc-splitting $L \oplus \mathbb{A} = T_x^R \mathcal{B}$ holds because the coordinate projections are given by the sc-operators $\Pi_L := (D_x f)|_L^{-1} \circ (\pi \circ D_x f) : T_x^R \mathcal{B} \rightarrow L$ and $(id_{T_x^R \mathcal{B}} - \Pi_L) : T_x^R \mathcal{B} \rightarrow \mathbb{A}$.

Assuming properties (ii) and (iii) of L , we now explain how to choose $\mathbb{W}$ and $\bar{\psi}$ so that the required property (1.57) for σ -compatible transversality holds. By (ii) and the definition of P , we view both $D_x \varphi(L)$ and P as subspaces of $\mathbb{E}$. By (iii) and since both $D_x \varphi(L)$ and P are finite dimensional sc-subspaces of $\mathbb{E}$, their span has a sc-splitting $D_x \varphi(L) \oplus P$. Let $\mathbb{W}'$ be any sc-complement of $D_x \varphi(L) \oplus P$ in $\mathbb{E}$. Then choose any linear isomorphism $D \times P \cong \mathbb{R}^{k-t-s}$ and set $\mathbb{W} = D_x \varphi(L) \oplus \mathbb{W}'$. Then, the linear sc-isomorphism $\bar{\psi} : \mathbb{E}' = \mathbb{R}^t \times D \times P \oplus \mathbb{W} \rightarrow \mathbb{R}^t \times \mathbb{R}^{k-t-s} \times \mathbb{W}$ indeed satisfies $\bar{\psi} \circ D_x \varphi(L) \subset \{0\}^{k-s} \times \mathbb{W}$, as required.

It remains to construct the sc-subspace $L \subset T_x^R \mathcal{B}$ that satisfies the conditions (i)-(iii). The space $\mathbb{E}$ consists of sections of the pullback tangent bundle of Y along x that have matching asymptotic conditions at nodes and breaking points. Since $P \subset \mathbb{E}$ is a finite

dimensional subspace, by Lemma 1.5.9 (applied to the smooth component of Σ on which the marked point lies) there exists a neighborhood U of the marked point in Σ small enough such that the following holds: if $\xi \in P$ is a section supported in U , then $\xi = 0$. Moreover, the marked point is always in the complement of the nodes, breaking, or any other “special points,” and so we can choose U disjoint from all special points.

Choose any basis $\{z_1, \dots, z_n\}$ of the complement Z of $T_{f(x)}N$ in $T_{f(x)}Y$. For each $i = 1, \dots, n$, consider a sc-smooth path $\gamma_i : (-\epsilon, \epsilon) \rightarrow \mathcal{B}$ through $x = \gamma_i(0)$ obtained by deforming x to move the image $f(x)$ of the marked point in the direction z_i , i.e. $D_x f(\gamma'_i(0)) = z_i$, while only changing x in the neighborhood U . The result is that all special points are preserved along the path, i.e. nodes and breakings do not get glued, and moreover the complex structure on the domain of x is not varying along the path. Hence we have $D_x \varphi(\gamma'_i(0)) \in \{0\}^{s+t} \times \{0\} \times \mathbb{E}$. Define

$$L := \text{Span}(\{\gamma'_1(0), \dots, \gamma'_n(0)\}).$$

Then it is clear from the construction that (i) and (ii) hold. Moreover (iii) holds because every $\xi \in D_x \varphi(L)$ is supported in U by construction of the γ_i and hence if $\xi \in D_x \varphi(L) \cap P$ then $\xi = 0$ by our choice of U . Note that $L \subset T_x^R \mathcal{B}$ holds by (ii), since the reduced tangent space $T_x^R \mathcal{B}$ is defined by $D_x \varphi(T_x^R \mathcal{B}) = T_0^R \mathcal{O} \cap (\{0\}^s \times \mathbb{R}^t \times D \times \mathbb{E})$.

The following lemma was used in the preceding arguments.

Lemma 1.5.9. *Consider a smooth manifold Σ and a finite rank vector bundle $V \rightarrow \Sigma$. Let $\Gamma(V)$ denote smooth sections and let $q \in \Sigma$. Then, if $P \subset \Gamma(V)$ is a finite dimensional subspace, there exists an open neighborhood U of q in Σ such that, if $\xi \in P$ is supported in U , then $\xi = 0$.*

Proof. We assume that U does not exist and prove that P must then be infinite dimensional. We will construct a countable set of linearly independent elements of P .

Consider any open neighborhood U_0 of q . Then there exists some $\xi_0 \in P$ supported in U_0 such that $\xi_0 \neq 0$. Since $\xi_0 \neq 0$, there exists an open neighborhood U_1 of q such that ξ_0 is not supported in U_1 (because if a smooth section is supported in every neighborhood of a point then it is identically 0). Note that $U_0 \supset U_1$ necessarily holds.

Inductively, assume that for some $n \geq 0$ we have constructed a nested sequence of open neighborhoods $U_0 \supset \dots \supset U_{n+1}$ and sections $\xi_0, \dots, \xi_n$ such that, for all $0 \leq i \leq n$, the section ξ_i is supported in U_i but is not supported in U_{i+1} . Then by our assumption that the claimed open set U does not exist, there must exist some $\xi_{n+1} \in P$ supported in U_{n+1} such that $\xi_{n+1} \neq 0$. Then let U_{n+2} be a neighborhood of q on which ξ_{n+1} is not supported. Hence the inductive hypothesis holds for $n + 1$.

This inductive process constructs a section $\xi_n \in P$ for all $n \geq 0$ with support lying in U_n but not in U_{n+1} . We claim that the collection $\{\xi_n \mid n \geq 0\}$ is linearly independent, proving the lemma. Indeed, suppose $\sum_{n=0}^{\infty} c_n \cdot \xi_n = 0$ for some $c_n \in \mathbb{R}$. Then for every $n \geq 1$ the section ξ_n is supported in U_1 , but ξ_0 is not supported in U_1 , hence $c_0 = 0$. Hence $\sum_{n=1}^{\infty} c_n \cdot \xi_n = 0$. Inductively, we conclude that $c_n = 0$ for all $n \geq 0$. $\square$

1.5.2 Proof of Theorem 1.1.3

Proof of Theorem 1.1.3. Proof of (I): By Definition 1.5.7 of a slice, for every $x \in \tilde{\mathcal{B}}$ there is an integer $n_x \geq 0$ and a $\mathbb{R}^{n_x}$ -sliced chart with respect to $\tilde{\mathcal{B}} \subset \mathcal{B}$ that contains x . We will show that each of these sliced charts induces a tame M -polyfold chart on $\tilde{\mathcal{B}}$. The transition maps between the tame charts on $\tilde{\mathcal{B}}$ constructed in this way are sc-smooth because they are restrictions of the sc-smooth transition maps between the sliced charts on $\mathcal{B}$. Hence we have a tame sc-smooth atlas on $\tilde{\mathcal{B}}$, providing the claimed tame M -polyfold structure.

Given a $\mathbb{R}^n$ -sliced chart

$$(V, \varphi, (\mathcal{O}, \mathbb{R}^n \times C, \mathbb{R}^n \times \mathbb{E})) \quad (1.58)$$

with respect to $\tilde{\mathcal{B}} \subset \mathcal{B}$, we must construct the claimed induced tame M -polyfold chart on $\tilde{\mathcal{B}}$. Use the notation

$$\tilde{V} := \tilde{\mathcal{B}} \cap V \text{ and } \tilde{\varphi} := \varphi|_{\tilde{V}} : \tilde{V} \rightarrow \varphi(\tilde{V}),$$

and recall the induced tame sc-retract $(\tilde{\mathcal{O}}, C, \mathbb{E})$ from Lemma 1.3.3. We claim that the tuple

$$(\tilde{V}, \tilde{\varphi}, (\tilde{\mathcal{O}}, C, \mathbb{E})) \quad (1.59)$$

is a tame M -polyfold chart on $\tilde{\mathcal{B}}$. Indeed, the set $\tilde{V}$ is open in $\tilde{\mathcal{B}}$ since $V \subset \mathcal{B}$ is open and $\tilde{\mathcal{B}} \subset \mathcal{B}$ has the subspace topology. Since φ is a homeomorphism, it follows that its restriction $\tilde{\varphi}$ is a homeomorphism onto its image, which is $\tilde{\mathcal{O}}$ by (1.54) and definition of $\tilde{V}$. This completes the construction of the tame M -polyfold structure on $\tilde{\mathcal{B}}$.

Let $x \in \tilde{\mathcal{B}}_1$. We now verify that the codimension $n_x = \text{codim}_x(\tilde{\mathcal{B}} \subset \mathcal{B})$ of the slice $\tilde{\mathcal{B}} \subset \mathcal{B}$ at x is well-defined and is locally constant in $\tilde{\mathcal{B}}$. There exists a $\mathbb{R}^{n_x}$ -sliced chart as in (1.58) with $x \in V$. It suffices to show that, for all $y \in \tilde{V}_1$, we have $T_y \mathcal{B}/T_y \tilde{\mathcal{B}} \cong \mathbb{R}^{n_x}$. Indeed, this proves that the codimension n_x is independent of the sliced chart around x , hence is well-defined, and moreover that it is constant in the open neighborhood $\tilde{V}$ of x in $\tilde{\mathcal{B}}$. For every $y \in \tilde{V}_1$, the tangent map $D_y \varphi : T_y V \rightarrow T_{\varphi(y)} \mathcal{O}$ of the chart map φ at y is a linear isomorphism. Moreover, $D_y \varphi$ restricts to a linear isomorphism $D_y \tilde{\varphi} : T_x \tilde{V} \rightarrow T_{\varphi(y)} \tilde{\mathcal{O}}$. Hence $D_y \varphi$ induces an isomorphism $T_y V/T_y \tilde{V} \cong T_{\varphi(y)} \mathcal{O}/T_{\varphi(y)} \tilde{\mathcal{O}}$. Thus $T_y \mathcal{B}/T_y \tilde{\mathcal{B}} = T_y V/T_y \tilde{V} \cong \mathbb{R}^{n_x}$ holds by (1.17). Hence $\text{codim}_y(\tilde{\mathcal{B}} \subset \mathcal{B}) = n_x$ holds for all $y \in \tilde{V}_1$.

We now prove the claimed degeneracy index formula. Let $x \in \tilde{\mathcal{B}}_\infty$. Since x is a smooth point, the reduced tangent space $T_x^R \mathcal{B}$ is well-defined and $D_x \varphi$ restricts to a sc-isomorphism $T_x^R \mathcal{B} \rightarrow T_{\varphi(x)}^R \mathcal{O}$. Hence $D_x \varphi$ induces a sc-isomorphism $T_x \mathcal{B}/T_x^R \mathcal{B} \cong T_{\varphi(x)} \mathcal{O}/T_{\varphi(x)}^R \mathcal{O}$. Similarly, $D_x \tilde{\varphi}$ induces a sc-isomorphism $T_x \tilde{\mathcal{B}}/T_x^R \tilde{\mathcal{B}} \cong T_{\varphi(x)} \tilde{\mathcal{O}}/T_{\varphi(x)}^R \tilde{\mathcal{O}}$. Then by (1.18), the inclusion $\tilde{\mathcal{O}} \subset \mathcal{O}$ induces an isomorphism $T_{\varphi(x)} \tilde{\mathcal{O}}/T_{\varphi(x)}^R \tilde{\mathcal{O}} \cong T_{\varphi(x)} \mathcal{O}/T_{\varphi(x)}^R \mathcal{O}$. It follows that the inclusion $\tilde{\mathcal{B}} \subset \mathcal{B}$ induces an isomorphism $T_x \tilde{\mathcal{B}}/T_x^R \tilde{\mathcal{B}} \cong T_x \mathcal{B}/T_x^R \mathcal{B}$. So by the global description (1.50) of degeneracy index at smooth points, we conclude $d_{\tilde{\mathcal{B}}}(x) = \dim(T_x \tilde{\mathcal{B}}/T_x^R \tilde{\mathcal{B}}) = \dim(T_x \mathcal{B}/T_x^R \mathcal{B}) = d_{\mathcal{B}}(x)$. This completes the proof of the statements in (I).

Proof of (II): By definition of a slice of a bundle, there is a cover of $\tilde{\mathcal{B}}$ by sliced bundle charts with respect to $\tilde{\mathcal{B}} \subset \mathcal{B}$. We will show that each of these sliced bundle charts induces a tame strong bundle chart for $\tilde{\rho} = \rho|_{\tilde{\mathcal{E}}} : \tilde{\mathcal{E}} := \rho^{-1}(\tilde{\mathcal{B}}) \rightarrow \tilde{\mathcal{B}}$. The transition maps between the tame charts for $\tilde{\rho}$ constructed in this way are sc-smooth because they are restrictions of the transition maps between the sliced bundle charts for ρ . Hence we have a bundle atlas for $\tilde{\rho}$, providing the claimed tame strong bundle structure.

Given a $\mathbb{R}^n$ -sliced bundle chart

$$(\rho^{-1}(V), \Phi, (K, \mathbb{R}^n \times C \triangleleft \mathbb{F}, \mathbb{R}^n \times \mathbb{E} \triangleleft \mathbb{F})) \quad (1.60)$$

for ρ with respect to $\tilde{\mathcal{B}} \subset \mathcal{B}$ that covers a $\mathbb{R}^n$ -sliced chart with respect to $\tilde{\mathcal{B}} \subset \mathcal{B}$ as in (1.58), we must construct the claimed induced tame strong bundle chart on $\tilde{\rho}$. Let $(\tilde{K}, C \triangleleft \mathbb{F}, \mathbb{E} \triangleleft \mathbb{F})$ denote the induced tame bundle retract from Lemma 1.3.5 and use the notation

$$\tilde{\Phi} := \Phi|_{\tilde{\rho}^{-1}(\tilde{V})} : \tilde{\rho}^{-1}(\tilde{V}) \rightarrow \Phi(\tilde{\rho}^{-1}(\tilde{V})).$$

We claim that the tuple

$$(\tilde{\rho}^{-1}(\tilde{V}), \tilde{\Phi}, (\tilde{K}, C \triangleleft \mathbb{F}, \mathbb{E} \triangleleft \mathbb{F})) \quad (1.61)$$

is a tame strong bundle chart for $\tilde{\rho}$ that covers the induced M -polyfold chart (1.59). Since $\pi_{\mathcal{O}} \circ \Phi = \varphi \circ \rho$ by (1.51) it follows by restriction that $\pi_{\tilde{\mathcal{O}}} \circ \tilde{\Phi} = \tilde{\varphi} \circ \tilde{\rho}$ on $\tilde{\rho}^{-1}(\tilde{V})$. It follows that $\text{im}(\tilde{\Phi}) = \pi_{\tilde{\mathcal{O}}}^{-1}(\tilde{\mathcal{O}}) = \tilde{K}$, where the second equality holds by (1.24). Hence $\tilde{\Phi}$ is a homeomorphism onto its image $\tilde{K}$ because it is the restriction of the homeomorphism Φ , and similarly $\tilde{\Phi}$ is linear on fibers as the restriction of Φ . This completes the proof of (II).

Proof of (III): The final statement about compactness of the zero sets holds because $\tilde{\sigma}^{-1}(0) = \sigma^{-1}(0) \cap \tilde{\mathcal{B}}_{\infty}$ is the intersection of a compact set and a closed subset of the Hausdorff space $\mathcal{B}_{\infty}$.

We proceed to verify that $\tilde{\sigma} = \sigma|_{\tilde{\mathcal{B}}} : \tilde{\mathcal{B}} \rightarrow \tilde{\mathcal{E}}$ is a tame sc-Fredholm section of $\tilde{\rho}$. The section $\tilde{\sigma}$ is sc-smooth and regularizing because it is the restriction of the section σ . So to prove that $\tilde{\sigma}$ is a tame sc-Fredholm section it remains to show that there exists a tame sc-Fredholm chart (Definition 1.5.4) for $\tilde{\sigma}$ at every smooth point $x \in \tilde{\mathcal{B}}_{\infty}$. Since $\tilde{\mathcal{B}}$ is a slice of the sc-Fredholm section σ , there exists a $\mathbb{R}^{n_x}$ -sliced sc-Fredholm chart for σ at x with respect to $\tilde{\mathcal{B}} \subset \mathcal{B}$. This is a $\mathbb{R}^{n_x}$ -sliced bundle chart as in (1.60) such that the local sc-Fredholm germ $\Phi \circ \sigma \circ \varphi^{-1}$ is $\mathbb{R}^{n_x}$ -sliced and such that $\varphi(x) = 0$.

We claim that the tame bundle chart (1.61) is a tame sc-Fredholm chart for $\tilde{\sigma}$ at x . This requires that the local section $\tilde{\Phi} \circ \tilde{\sigma} \circ \tilde{\varphi}^{-1} : \tilde{\mathcal{O}} \rightarrow \tilde{K}$ is a tame sc-Fredholm germ, which follows from Lemma 1.3.10 since $\Phi \circ \sigma \circ \varphi^{-1} : \mathcal{O} \rightarrow K$ is a $\mathbb{R}^{n_x}$ -sliced sc-Fredholm germ. Moreover, Lemma 1.3.10 and Definition 1.5.3(3) provide the sc-Fredholm index formula $\text{ind}_x(\tilde{\sigma}) = \text{ind}(\tilde{\Phi} \circ \tilde{\sigma} \circ \tilde{\varphi}^{-1}) = \text{ind}(\Phi \circ \sigma \circ \varphi^{-1}) - n_x = \text{ind}_x(\sigma) - n_x$. This completes the proof that $\tilde{\sigma}$ is a tame sc-Fredholm section satisfying the claimed index formula. $\square$

1.5.3 Proof of Theorem 1.1.5

Proof of Theorem 1.1.5. Proof of (I): For every $x \in f^{-1}(N) \cap \mathcal{B}_\infty$ we will construct a $\mathbb{R}^n$ -sliced chart (Definition 1.5.6) with respect to $f^{-1}(N) \cap \mathcal{B}^1 \subset \mathcal{B}^1$ that contains x . Then we define $\tilde{\mathcal{B}}$ to be the union of the domains of these charts intersected with $f^{-1}(N) \cap \mathcal{B}^1$.

Let $x \in f^{-1}(N) \cap \mathcal{B}_\infty$. Let $Z \subset Y$ be the domain of a manifold chart $Z \xrightarrow{\sim} \mathbb{R}^{n+\dim N}$ on Y containing $f(x) \in N$ that identifies $N \cap Z$ with $\{0\}^n \times \mathbb{R}^{\dim N}$. Let $\gamma : Z \rightarrow \mathbb{R}^n$ be the smooth map given in the chart by the projection $\mathbb{R}^n \times \mathbb{R}^{\dim N} \rightarrow \mathbb{R}^n$. Then $0 = \gamma(f(x))$ is a regular value of γ and

$$\gamma^{-1}(0) = N \cap Z. \quad (1.62)$$

In particular, we have

$$T_{f(x)}N = \ker D_{f(x)}\gamma. \quad (1.63)$$

Then $f^{-1}(Z)$ is an open neighborhood of x in $\mathcal{B}$. Choose $V \subset f^{-1}(Z)$ so that, in addition, there is a tame M -polyfold chart

$$(V, \varphi, (\mathcal{O}, [0, \infty)^s \times \mathbb{E}, \mathbb{R}^s \times \mathbb{E})) \quad (1.64)$$

on $\mathcal{B}$ satisfying $\varphi(x) \in \{0\}^s \times \mathbb{E}$. Set

$$\underline{x} := \varphi(x).$$

By definition (1.49) of the reduced tangent space $T_x^R \mathcal{B}$, we have $D_x \varphi(T_x^R \mathcal{B}) = T_{\underline{x}}^R \mathcal{O}$. So since f is transverse to N , we conclude from (1.56) that we have

$$D_x f \circ D_{\underline{x}} \varphi^{-1}(T_{\underline{x}}^R \mathcal{O}) + T_{f(x)}N = T_{f(x)}Y. \quad (1.65)$$

We claim that the map

$$\underline{f} := \gamma \circ f \circ \varphi^{-1} : \mathcal{O} \rightarrow \mathbb{R}^n$$

satisfies the hypotheses of Lemma 1.4.2(I) at $\underline{x}$. Let $\mathcal{O}_\partial := \mathcal{O} \cap (\{0\}^s \times \mathbb{E})$ denote the boundary sc-retract associated to $\mathcal{O}$ (see Lemma 1.4.1) and consider the restriction $\underline{f}_\partial := \underline{f}|_{\mathcal{O}_\partial}$. Recall from (1.37) that we have $T_{\underline{x}}^R \mathcal{O} = T_{\underline{x}} \mathcal{O}_\partial$. Then the tangent map $D_{\underline{x}} \underline{f}_\partial : T_{\underline{x}} \mathcal{O}_\partial \rightarrow T_0 \mathbb{R}^n = \mathbb{R}^n$ is surjective by (1.65), (1.63), and since 0 is a regular value of γ . Hence, the hypotheses of Lemma 1.4.2(I) are indeed satisfied, yielding slice coordinates around $\underline{x}$ with respect to $(\underline{f}^{-1}(0) \cap \mathcal{O}^1) \subset \mathcal{O}^1$. Precisely, this means that we obtain a sc-Banach space $\mathbb{K}$, a $\mathbb{R}^n$ -sliced sc-retract (Definition 1.3.2)

$$(\mathcal{O}', \mathbb{R}^n \times [0, \infty)^s \times \mathbb{K}^1, \mathbb{R}^n \times \mathbb{R}^s \times \mathbb{K}^1), \quad (1.66)$$

an open set $\hat{\mathcal{O}} \subset \mathcal{O}^1$, and a sc-smooth diffeomorphism

$$g : \hat{\mathcal{O}} \rightarrow \mathcal{O}'$$

such that the induced tame sc-retract

$$\tilde{\mathcal{O}}' = \mathcal{O}' \cap (\{0\}^n \times [0, \infty)^s \times \mathbb{K}^1)$$

satisfies

$$g(\underline{f}^{-1}(0) \cap \hat{\mathcal{O}}) = \tilde{\mathcal{O}}'. \quad (1.67)$$

Set

$$\hat{V} := \varphi^{-1}(\hat{\mathcal{O}}) \text{ and } \hat{\varphi} := g \circ \varphi|_{\hat{V}} : \hat{V} \rightarrow \mathcal{O}'.$$

Then

$$(\hat{V}, \hat{\varphi}, (\mathcal{O}', \mathbb{R}^n \times [0, \infty)^s \times \mathbb{K}^1, \mathbb{R}^n \times \mathbb{R}^s \times \mathbb{K}^1)) \quad (1.68)$$

is a M -polyfold chart on $\mathcal{B}^1$.

At this point, for every $x \in f^{-1}(N) \cap \mathcal{B}_\infty$, we have constructed a M -polyfold chart on $\mathcal{B}^1$ with domain $\hat{V}_x \subset \mathcal{B}^1$ containing $x \in \hat{V}_x$. Define

$$\tilde{\mathcal{B}} := \bigcup_{x \in f^{-1}(N) \cap \mathcal{B}_\infty} \hat{V}_x \cap f^{-1}(N). \quad (1.69)$$

We claim that $\tilde{\mathcal{B}} \subset \mathcal{B}^1$ is a slice. We must show that the chart (1.68) is a $\mathbb{R}^n$ -sliced chart (Definition 1.5.6) with respect to $\tilde{\mathcal{B}} \subset \mathcal{B}^1$. The sc-retract (1.66) is given as $\mathbb{R}^n$ -sliced by Lemma 1.4.2(I), so it remains to show that (1.54) holds, which in the notation of this proof is the statement $\tilde{\mathcal{O}}' = \hat{\varphi}(\tilde{\mathcal{B}} \cap \hat{V})$. Let $p \in \tilde{\mathcal{O}}'$. By (1.67) and the definition of $\underline{f}$, we have $\hat{\varphi}^{-1}(p) = \varphi^{-1} \circ g^{-1}(p) \in (\gamma \circ f)^{-1}(0) \subset f^{-1}(N)$. Hence $\hat{\varphi}^{-1}(p) \in \hat{V} \cap f^{-1}(N) \subset \tilde{\mathcal{B}} \cap \hat{V}$. To see the reverse inclusion, let $p \in \hat{\varphi}(\tilde{\mathcal{B}} \cap \hat{V})$. By (1.67), we must show that $\underline{f}(g^{-1}(p)) = 0$. From the definitions we compute $\underline{f} \circ g^{-1}(p) = \gamma \circ f \circ \hat{\varphi}^{-1}(p) \in \gamma(f(\tilde{\mathcal{B}} \cap \hat{V})) \subset \gamma(N \cap Z) = \{0\}$, as required. Hence (1.68) is indeed a $\mathbb{R}^n$ -sliced chart with respect to $\tilde{\mathcal{B}} \subset \mathcal{B}^1$. This proves that $\tilde{\mathcal{B}} \subset \mathcal{B}^1$ is a slice (Definition 1.5.7). Hence $\tilde{\mathcal{B}}$ is a tame M -polyfold with the claimed degeneracy index by Theorem 1.1.3. Moreover, the existence of a $\mathbb{R}^n$ -sliced chart around every $y \in \tilde{\mathcal{B}}$ proves the claim that $\text{codim}_y(\tilde{\mathcal{B}} \subset \mathcal{B}^1) = n$ for every $y \in \tilde{\mathcal{B}}_1$. This completes the proof of the statements (I).

Proof of (II): To prove that $\tilde{\mathcal{B}}$ is a slice of the bundle $\rho|_{\mathcal{E}^1} : \mathcal{E}^1 \rightarrow \mathcal{B}^1$, for every $x \in \tilde{\mathcal{B}}$ we must construct a sliced bundle chart for $\rho|_{\mathcal{E}^1}$ with respect to $\tilde{\mathcal{B}} \subset \mathcal{B}^1$. We can choose the tame M -polyfold chart (1.64) so that it is covered by a tame strong bundle chart

$$(\rho^{-1}(V), \Phi, (K, C \triangleleft \mathbb{F}, \mathbb{E} \triangleleft \mathbb{F})) \quad (1.70)$$

for ρ . Note that this may require shrinking the open neighborhood $V \subset \mathcal{B}$ of x , and so the resulting slice $\tilde{\mathcal{B}}$ of $\mathcal{B}^1$ defined by the formula (1.69) may be smaller than in part (I).

The tame bundle retract $(K, C \triangleleft \mathbb{F}, \mathbb{E} \triangleleft \mathbb{F})$ covers the sc-retract $\mathcal{O}$. Hence Lemma 1.4.2(II) yields a $\mathbb{R}^n$ -sliced bundle retract

$$(K', \mathbb{R}^n \times [0, \infty)^s \times \mathbb{K}^1 \triangleleft \mathbb{F}^1, \mathbb{R}^n \times \mathbb{R}^s \times \mathbb{K}^1 \triangleleft \mathbb{F}^1)$$

that covers the sliced sc-retract (1.66) and satisfies

$$(g \triangleleft \text{id}_{\mathbb{F}^1})(\hat{K}) = K',$$

where $\hat{K} \subset K$ is the preimage of $\hat{\mathcal{O}}$ in the local bundle model $K^1 \rightarrow \mathcal{O}^1$. Set

$$\hat{\Phi} := (g \triangleleft id_{\mathbb{F}^1}) \circ \Phi|_{\rho|_{\mathcal{E}^1}^{-1}(\hat{V})} : \rho|_{\mathcal{E}^1}^{-1}(\hat{V}) \rightarrow K'.$$

Then the desired sliced bundle chart for $\rho|_{\mathcal{E}^1}$ with respect to $\tilde{\mathcal{B}} \subset \mathcal{B}^1$ is the tuple

$$(\rho|_{\mathcal{E}^1}^{-1}(\hat{V}), \hat{\Phi}, (K', \mathbb{R}^n \times [0, \infty)^s \times \mathbb{K}^1 \triangleleft \mathbb{F}^1, \mathbb{R}^n \times \mathbb{R}^s \times \mathbb{K}^1 \triangleleft \mathbb{F}^1)). \quad (1.71)$$

This proves that $\tilde{\mathcal{B}} \subset \mathcal{B}^1$ is a slice of the bundle $\rho|_{\mathcal{E}^1}$. The restriction $\tilde{\rho} : \rho|_{\mathcal{E}^1}^{-1}(\tilde{\mathcal{B}}) \rightarrow \tilde{\mathcal{B}}$ is then a tame strong bundle by Theorem 1.1.3(II).

Proof of (III): To prove that $\tilde{\mathcal{B}}$ is a slice of the tame sc-Fredholm section $\sigma|_{\mathcal{B}^1} : \mathcal{B}^1 \rightarrow \mathcal{E}^1$, for every $x \in \tilde{\mathcal{B}}_\infty$ we must construct a sliced sc-Fredholm chart for $\sigma|_{\mathcal{B}^1}$ at x with respect to $\tilde{\mathcal{B}} \subset \mathcal{B}^1$. Since f is σ -compatibly transverse to N , we can assume that the tame strong bundle chart (1.70) is also a tame sc-Fredholm chart for σ at x such that we can choose $\bar{\psi}$, $\mathbb{W}$, and $L \subset T_x^R \mathcal{B}$ satisfying (1.57). Note that this may require shrinking the open neighborhood V of x , and so the resulting slice $\tilde{\mathcal{B}}$ of $\rho|_{\mathcal{E}^1}$ defined by the formula (1.69) may be smaller than in parts (I) and (II).

We now show that $\mathbb{W}$, $\bar{\psi}$, and $L' := D_x \varphi(L)$ satisfy the hypotheses of Lemma 1.4.2(III). Indeed, by (1.57), L is a sc-complement of the sc-subspace

$$\mathbb{A} := (D_x f)^{-1}(T_{f(x)} N) \cap T_x^R \mathcal{B}$$

in $T_x^R \mathcal{B}$, and so we have the sc-splitting

$$D_x \varphi(L) \oplus D_x \varphi(\mathbb{A}) = T_{\underline{x}}^R \mathcal{O}. \quad (1.72)$$

Recall from (1.37) that we have $T_{\underline{x}}^R \mathcal{O} = T_{\underline{x}} \mathcal{O}_\partial$. Then from the definitions and (1.63), we compute

$$\begin{aligned} D_x \varphi(\mathbb{A}) &= D_x \varphi((D_x f)^{-1}(T_{f(x)} N) \cap T_x^R \mathcal{B}) \\ &= D_x \varphi((D_x f)^{-1}(\ker D_{f(x)} \gamma)) \cap T_{\underline{x}}^R \mathcal{O} \\ &= \ker D_{\underline{x}} f \cap T_{\underline{x}}^R \mathcal{O} \\ &= \ker D_{\underline{x}} f_\partial, \end{aligned}$$

and so by (1.72) the space $L' = D_x \varphi(L)$ is indeed a sc-complement of $\ker D_{\underline{x}} f_\partial$ in $T_{\underline{x}} \mathcal{O}_\partial$, as required. Moreover, the required condition (1.42) holds for L' by (1.57).

Hence, Lemma 1.4.2(III) asserts that the section $\sigma' := (g \triangleleft id_{\mathbb{F}^1}) \circ \sigma \circ g^{-1} : \mathcal{O}' \rightarrow K'$ is a $\mathbb{R}^n$ -sliced sc-Fredholm germ, which means that (1.71) is a $\mathbb{R}^n$ -sliced sc-Fredholm chart. This proves that $\tilde{\mathcal{B}}$ is a slice of $\sigma|_{\mathcal{B}^1}$. The restriction $\tilde{\sigma} : \tilde{\mathcal{B}} \rightarrow \tilde{\mathcal{E}}$ is then a tame sc-Fredholm section satisfying the claimed index formula by Theorem 1.1.3(III).

Since $N \subset Y$ is closed it follows that $f^{-1}(N) \subset \mathcal{B}$ is closed, so $\tilde{\mathcal{B}}_\infty = f^{-1}(N) \cap \mathcal{B}_\infty$ is closed in $\mathcal{B}_\infty$. Hence if $\sigma^{-1}(0)$ is compact then it follows from the final statement of Theorem 1.1.3(III) that $\tilde{\sigma}^{-1}(0)$ is compact. $\square$

1.6 Handling isotropy: the ep-groupoid case

In this section, we generalize the main theorems (Theorem 1.1.3 and Theorem 1.1.5) to the case of ep-groupoids in Corollary 1.6.7 and Corollary 1.6.8. An ep-groupoid is the “orbifold version” of a M -polyfold. In applications, we must keep track of finite isotropy groups of points in the polyfold. The theory of sc-Fredholm sections over ep-groupoids is the generalization of sc-Fredholm sections over M -polyfolds that incorporates this isotropy; see [34] for a detailed treatment.

To perform these constructions on polyfolds, it suffices to take representative ep-groupoids and then perform the constructions to obtain a representative ep-groupoid of a new polyfold. A polyfold [34, Def. 16.3] is a topological space Z together with an equivalence class of polyfold structures [34, Def. 16.1]. A polyfold structure for Z is an ep-groupoid together with a homeomorphism of its orbit space with Z . Polyfold structures are equivalent [34, Def. 16.2] if they are related by a generalized isomorphism [34, Def. 10.8] compatible with the homeomorphisms with Z . In particular, every ep-groupoid defines a canonical polyfold structure on its orbit space.

All of the results in this section follow from the results in the M -polyfold case in the previous sections. In fact, this is a general polyfold philosophy: the M -polyfold situation is where all of the analytic data (e.g. the sc-structures on base and bundle and the sc-Fredholm properties) is stored. So, if an application of polyfold theory works assuming that everything in sight is a M -polyfold, we expect that there is a suitable upgrade to ep-groupoids by keeping track of the isotropy using the ep-groupoid machinery.

This philosophy goes deeper: a polyfold theorist thinks through any construction first assuming that everything is a finite dimensional smooth manifold with boundary and corners. In particular, sc-Fredholm sections in the finite dimensional setting are the same as ordinary smooth sections. We expect that any construction *that is motivated by regularization of some moduli space of pseudoholomorphic curves*, and that works assuming everything is finite dimensional, will go through in the M -polyfold and ep-groupoid case.

We now review tame ep-groupoids (Definition 1.6.1), bundles over them (Definition 1.6.3), and their sc-Fredholm section functors (Definition 1.6.5) before we introduce the new notion of tame sc-Fredholm section functors (Definition 1.6.6) and prove Corollary 1.6.7 and Corollary 1.6.8.

A *groupoid* $\mathcal{X} = (X, \mathbf{X})$ is a small category with object set X and morphism set $\mathbf{X}$ such that all morphisms are invertible. Associated to any groupoid are the following structure maps. For a detailed description, see for example [34, Def. 7.1]. The *source map*

$$s : \mathbf{X} \rightarrow X$$

and the *target map*

$$t : \mathbf{X} \rightarrow X$$

send a morphism to its source and target, respectively. The *multiplication map*

$$m : \mathbf{X} \times_{s \times t} \mathbf{X} \rightarrow \mathbf{X}$$

composes any pair of morphisms such that the source of the first is the target of the second, and hence m is defined on the fiber product $\mathbf{X}_s \times_t \mathbf{X}$. The *unit map*

$$u : X \rightarrow \mathbf{X}$$

sends an object to the identity morphism from that object to itself, which exists and is unique since each self-morphism set is a group. The *inverse map*

$$\iota : \mathbf{X} \rightarrow \mathbf{X}$$

inverts morphisms.

Definition 1.6.1. [34, Defs. 7.3, 7.6] A **tame ep-groupoid** $\mathcal{X} = (X, \mathbf{X})$ is a groupoid equipped with tame M -polyfold structures on the object space X and on the morphism space $\mathbf{X}$ satisfying the following properties:

- (i) **(étale)** The source s and target t maps are surjective local sc-diffeomorphisms,
- (ii) The unit map u and the inverse map ι are sc-smooth.
- (iii) **(proper)** Every $x \in X$ possesses an open neighborhood $V(x) \subset X$ of x such that the map

$$t : s^{-1}(\overline{V(x)}) \rightarrow X$$
 is proper.
- (iv) The multiplication map m is sc-smooth, where the fiber product $\mathbf{X}_s \times_t \mathbf{X}$ is equipped with the M -polyfold structure given by [34, Prop. 2.15]. (See Remark 1.6.2 to compare with the fiber product results in this paper.)

Remark 1.6.2. There is an essential difference between the fiber product results in this paper and the fiber product in [34, Prop. 2.15] that is used to give $\mathbf{X}_s \times_t \mathbf{X}$ a M -polyfold structure in Definition 1.6.1(iv).

The result in [34, Prop. 2.15] requires one of the maps in the fiber product to be a local sc-diffeomorphism. In the case of the fiber product $\mathbf{X}_s \times_t \mathbf{X}$ from Definition 1.6.1, both the source map s and the target map t are local sc-diffeomorphisms.

In this paper, we construct fiber products (Corollary 1.7.3) over maps to a finite dimensional smooth manifold. In applications, these maps will never be local sc-diffeomorphisms because the M -polyfolds have infinite dimensional tangent spaces everywhere.

For an ep-groupoid $\mathcal{X} = (X, \mathbf{X})$, the *orbit space*

$$|X| = X / \sim \quad (1.73)$$

is the quotient of the object space X by the equivalence relation defined by $x \sim y$ if and only if there exists $\phi \in \mathbf{X}$ such that $s(\phi) = x$ and $t(\phi) = y$. That is, to obtain the orbit space, we identify any two objects that have a morphism between them.

The degeneracy indices $d_X : X \rightarrow \mathbb{N}_0$ and $d_{\mathbf{X}} : \mathbf{X} \rightarrow \mathbb{N}_0$ are defined as usual on the M -polyfolds X and $\mathbf{X}$. The *induced degeneracy index* [34, Def 7.5]

$$d_{|X|} : |X| \rightarrow \mathbb{N}_0$$

on the orbit space $|X|$ is defined by $d_{|X|}(|x|) = d_X(x)$, and is well-defined by [34, Prop. 2.7] as discussed above the definition [34, Def 7.5].

Now we review the notion of a strong bundle over an ep-groupoid $\mathcal{X} = (X, \mathbf{X})$; see [34, Sec. 8.3] for more detail. Consider a strong M -polyfold bundle

$$P : E \rightarrow X$$

over the object space X of $\mathcal{X}$. Since the source map s is a local sc-diffeomorphism (by definition), a M -polyfold structure on the fiber product $\mathbf{X}_{s \times_P} E$ is provided by [34, Prop. 2.15]. Moreover, the projection onto the first factor

$$\pi_1 : \mathbf{E} := \mathbf{X}_{s \times_P} E \rightarrow \mathbf{X}$$

is a strong M -polyfold bundle.

Definition 1.6.3. [34, Def. 8.4] A **tame strong bundle** over an ep-groupoid $\mathcal{X} = (X, \mathbf{X})$ is a pair (P, μ) of a tame strong bundle

$$P : E \rightarrow X$$

over the object M -polyfold X and a strong bundle map

$$\mu : \mathbf{X}_{s \times_P} E \rightarrow E$$

covering the target map $t : \mathbf{X} \rightarrow X$, i.e.

$$P \circ \mu = t \circ \pi_1,$$

and which satisfies

$$(i) \quad \mu(1_x, e) = e \text{ for all } x \in X \text{ and } e \in E_x,$$

$$(ii) \quad \mu(g \circ h, e) = \mu(g, \mu(h, e)) \text{ for all } g, h \in \mathbf{X} \text{ and } e \in E \text{ satisfying } s(h) = P(e) \text{ and } t(h) = s(g) = P(\mu(h, e)).$$

We call μ the **strong bundle structure map**.

Remark 1.6.4. The standard definition [34, Def. 8.4] of a tame strong bundle requires, in addition to the conditions in Definition 1.6.3, that the structure map μ is a surjective local sc-diffeomorphism. However, this condition is automatically satisfied, as noted in [33] on page 37.

Let $(P : E \rightarrow X, \mu)$ be a strong bundle over the ep-groupoid $\mathcal{X} = (X, \mathbf{X})$. A sc-smooth section functor σ [34, Def. 8.7] of (P, μ) is a sc-smooth section $\sigma : X \rightarrow E$ of the strong bundle $P : E \rightarrow X$ over the object M -polyfold X satisfying the following compatibility with μ : For all morphisms $\phi \in \mathbf{X}$,

$$\sigma(t(\phi)) = \mu(\phi, \sigma(s(\phi))) \quad (1.74)$$

holds.

Definition 1.6.5. [34, Def. 8.7] A **sc-Fredholm section functor** of a strong bundle $(P : E \rightarrow X, \mu)$ over an ep-groupoid $\mathcal{X} = (X, \mathbf{X})$ is a sc-smooth section functor $\sigma : X \rightarrow E$ such that, when viewed as a section of the strong bundle M -polyfold bundle P , it is sc-Fredholm in the M -polyfold sense (Definition 1.5.3).

We introduce the following new class of sc-Fredholm section functors.

Definition 1.6.6. A **tame sc-Fredholm section functor** of a strong bundle $(P : E \rightarrow X, \mu)$ over an ep-groupoid $\mathcal{X} = (X, \mathbf{X})$ is a sc-Fredholm section functor such that, when viewed as a section of the strong bundle M -polyfold bundle P , it is tame sc-Fredholm in the M -polyfold sense (Definition 1.5.4).

We now generalize Theorem 1.1.3 to the ep-groupoid case.

Corollary 1.6.7.

(I) Consider a tame ep-groupoid $\mathcal{X} = (X, \mathbf{X})$ and a slice $\tilde{X} \subset X$ of the object M -polyfold X , in the sense of Definition 1.5.7. Assume that $\tilde{X}$ is closed under morphisms, i.e. for all $\phi \in \mathbf{X}$ we have

$$s(\phi) \in \tilde{X} \iff t(\phi) \in \tilde{X}, \quad (1.75)$$

or equivalently,

$$s^{-1}(\tilde{X}) = t^{-1}(\tilde{X}).$$

Then, denoting the subset of morphisms $\mathbf{X}$ that have source and target in $\tilde{X}$ by

$$\tilde{\mathbf{X}} := s^{-1}(\tilde{X}) = t^{-1}(\tilde{X}), \quad (1.76)$$

the tuple $\tilde{\mathcal{X}} = (\tilde{X}, \tilde{\mathbf{X}})$ is a tame ep-groupoid with the tame M -polyfold atlas on $\tilde{X}$ induced by the sliced charts (Definition 1.5.6) with respect to $\tilde{X} \subset X$. Note that $\tilde{\mathcal{X}}$ is the full subcategory of $\mathcal{X}$ with object space $\tilde{X}$.

For $x \in \tilde{X}_1$, the codimension $\text{codim}_x(\tilde{X} \subset X)$ (Definition 1.5.7) is well-defined and locally constant in $\tilde{X}$, i.e. it is equal to $\text{codim}_x(\tilde{X} \subset X)$ in an open neighborhood of x in $\tilde{X}$. For $x \in \tilde{X}_\infty$, the degeneracy index satisfies $d_{\tilde{X}}(x) = d_X(x)$.

(II) Consider, in addition, a tame strong bundle $(P : E \rightarrow X, \mu)$ over $\mathcal{X}$ and suppose that $\tilde{X}$ is a slice of the bundle P in the sense of Definition 1.5.7.

Then, the tuple $(\tilde{P}, \tilde{\mu})$ consisting of the restrictions

$$\begin{aligned}\tilde{E} &:= P^{-1}(\tilde{X}) \\ \tilde{P} &:= P|_{\tilde{E}} : \tilde{E} \rightarrow \tilde{X}, \\ \tilde{\mu} &:= \mu|_{\tilde{\mathbf{X}}_s \times_P \tilde{E}} : \tilde{\mathbf{X}}_s \times_P \tilde{E} \rightarrow \tilde{E},\end{aligned}$$

is a tame strong bundle over $\tilde{\mathcal{X}}$, where the bundle atlas for $\tilde{P}$ is induced by the sliced bundle charts for P with respect to $\tilde{X} \subset X$.

(III) Consider, in addition, a sc-Fredholm section functor $\sigma : X \rightarrow E$ of (P, μ) and suppose that $\tilde{X}$ is a slice of σ in the sense of Definition 1.5.7.

Then, the restriction $\tilde{\sigma} := \sigma|_{\tilde{X}} : \tilde{X} \rightarrow \tilde{E}$ is a tame sc-Fredholm section functor of the bundle $(\tilde{P}, \tilde{\mu})$ with tame sc-Fredholm charts induced by the sliced sc-Fredholm charts for σ with respect to $\tilde{X} \subset X$. For $x \in \tilde{X}_\infty$, the index satisfies $\text{ind}_x(\tilde{\sigma}) = \text{ind}_x(\sigma) - \text{codim}_x(\tilde{X} \subset X)$. If $|\sigma^{-1}(0)|$ is compact and $|\tilde{X}_\infty| \subset |X_\infty|$ is closed, then $|\tilde{\sigma}^{-1}(0)|$ is compact.

Proof. Proof of (I): First we claim that $\tilde{X} = (\tilde{X}, \tilde{\mathbf{X}})$ is a groupoid. Given any subset A of the object set of a groupoid, say $A \subset X$, we obtain a sub-groupoid $(A, \mathbf{A})$ of $(X, \mathbf{X})$ by defining the morphism set to be $\mathbf{A} := s^{-1}(A) \cap t^{-1}(A)$. Hence, with the set of morphisms $\tilde{\mathbf{X}}$ as defined in (1.76), the tuple $(\tilde{X}, \tilde{\mathbf{X}})$ is a groupoid.

Since $\tilde{X} \subset X$ is a slice of the tame M -polyfold X , Theorem 1.1.3(I) provides a tame M -polyfold structure on $\tilde{X}$ with the claimed degeneracy index for $x \in \tilde{X}_\infty$ and the claimed locally constant codimension $\text{codim}_x(\tilde{X} \subset X)$ for $x \in \tilde{X}_1$.

We now equip $\tilde{\mathbf{X}}$ with a tame M -polyfold structure by pulling back the tame M -polyfold charts on $\tilde{X}$ through the source and target maps $s, t : \mathbf{X} \rightarrow X$ on $\mathcal{X}$. That is, since s and t are local sc-diffeomorphisms by the étale property (Definition 1.6.1(i)) of $\mathcal{X}$, for every $z \in \tilde{\mathbf{X}}$ there exists a neighborhood $\mathbf{V} \subset \mathbf{X}$ of z such that $s|_{\mathbf{V}} : \mathbf{V} \rightarrow V$ is a sc-diffeomorphism, where $V := s(\mathbf{V})$. Then V is a neighborhood of $s(z)$ in X . Since $\tilde{X} \subset X$ is a slice and $s(z) \in \tilde{X}$, after shrinking V we can assume that it is the domain of a sliced chart with respect to $\tilde{X} \subset X$ (Definition 1.5.6). This induces a tame M -polyfold chart on $\tilde{X}$ with domain $V \cap \tilde{X}$, as in the proof of Theorem 1.1.3. Pulling this back through s , we obtain a sliced chart with respect to $\tilde{\mathbf{X}} \subset \mathbf{X}$ with domain $\mathbf{V}$, and since $s(\mathbf{V} \cap \tilde{\mathbf{X}}) = V \cap \tilde{X}$, its induced tame M -polyfold chart has domain $\mathbf{V} \cap \tilde{\mathbf{X}}$. So, we have constructed a tame M -polyfold chart with domain an open neighborhood of z in $\tilde{\mathbf{X}}$. The analogous construction using the target map t instead of the source map s produces another such chart. On overlaps between any sliced charts with

respect to $\tilde{\mathbf{X}} \subset \mathbf{X}$ produced in this way, the transition maps are sc-smooth because they are compositions of the sc-smooth chart maps on X with the sc-smooth source and target maps (and their local inverses). Hence the transitions between the induced tame M -polyfold charts on $\tilde{\mathbf{X}}$ are also sc-smooth, as restrictions of the transitions on $\mathbf{X}$. Hence covering $\tilde{\mathbf{X}}$ with these M -polyfold charts provides the claimed M -polyfold structure. Moreover, observe from the construction of the charts that the source $\tilde{s} = s|_{\tilde{\mathbf{X}}}$ and target $\tilde{t} = t|_{\tilde{\mathbf{X}}}$ maps on $\tilde{\mathcal{X}} = (\tilde{X}, \tilde{\mathbf{X}})$ are local sc-diffeomorphisms, i.e. $\tilde{\mathcal{X}}$ satisfies the étale property (Definition 1.6.1(i)). Similarly, the unit map, the inverse map, and the multiplication map on $\tilde{\mathcal{X}}$ are sc-smooth since they are restrictions of the corresponding maps on $\mathcal{X}$. This verifies properties Definition 1.6.1(ii) and Definition 1.6.1(iv) for $\tilde{\mathcal{X}}$.

To verify that $\tilde{\mathcal{X}}$ is a tame ep-groupoid, it remains to check properness Definition 1.6.1(iii). Let $x \in \tilde{X}$. By properness of $\mathcal{X}$, there exists an open neighborhood $V(x)$ of x in X such that the map $t : s^{-1}(cl_X(V(x))) \rightarrow X$ is a proper map. Shrink $V(x)$ so that it is the domain of a sliced chart (Definition 1.5.6) with respect to $\tilde{X} \subset X$. Then $V(x) \cap \tilde{X}$ is closed in $V(x)$ because the chart map that homeomorphically identifies $V(x)$ with the sliced sc-retract $(\mathcal{O}, \mathbb{R}^n \times C, \mathbb{R}^n \times \mathbb{E})$ sends $V(x) \cap \tilde{X}$ to the induced tame sc-retract $\mathcal{O} \cap (\{0\} \times C)$, which is closed in $\mathcal{O}$. Let $V'(x) \subset V(x)$ be a smaller open neighborhood of x such that $cl_X(V'(x)) \subset V(x)$. Then we have $cl_X(V'(x) \cap \tilde{X}) = cl_{V(x)}(V'(x) \cap \tilde{X}) \subset cl_{V(x)}(V(x) \cap \tilde{X}) = V(x) \cap \tilde{X}$. In particular, the closure of $V'(x) \cap \tilde{X}$ in $\tilde{X}$ and in X agree, i.e.

$$cl_X(V'(x) \cap \tilde{X}) = cl_{\tilde{X}}(V'(x) \cap \tilde{X}). \quad (1.77)$$

We claim that the open neighborhood $V'(x) \cap \tilde{X}$ of x in $\tilde{X}$ satisfies the required condition, i.e. the map $t|_{\tilde{\mathbf{X}}} : s|_{\tilde{\mathbf{X}}}^{-1}(cl_{\tilde{X}}(V'(x) \cap \tilde{X})) \rightarrow \tilde{X}$ is proper. Let $K \subset \tilde{X}$ be compact. We must show that the set $A := t|_{\tilde{\mathbf{X}}}^{-1}(K) \cap s|_{\tilde{\mathbf{X}}}^{-1}(cl_{\tilde{X}}(V'(x) \cap \tilde{X}))$ is compact. We have

$$A = t^{-1}(K) \cap s^{-1}(cl_X(V'(x) \cap \tilde{X}))$$

by (1.77) and since by definition $\tilde{\mathbf{X}} = t^{-1}(\tilde{X}) = s^{-1}(\tilde{X})$. Observe that A is closed in $\mathbf{X}$ since $K \subset X$ is closed as a compact subset of the Hausdorff space X . By our choice of $V(x)$, the set $t^{-1}(K) \cap s^{-1}(cl_X(V(x)))$ is compact. Hence A is compact as a closed subset of the compact set $t^{-1}(K) \cap s^{-1}(cl_X(V(x)))$. This completes the proof that $\tilde{\mathcal{X}}$ is a tame ep-groupoid.

Proof of (II): Since $\tilde{X}$ is a slice of the tame strong bundle $P : E \rightarrow X$ in the M -polyfold sense, Theorem 1.1.3(II) provides a tame strong bundle structure on the restriction $\tilde{P}$. To prove that the tuple $(\tilde{P}, \tilde{\mu})$ is a tame strong bundle over the ep-groupoid $\tilde{\mathcal{X}}$, we must show that the map $\tilde{\mu}$ has the required properties. Recall from the lemma statement that $\tilde{\mu}$ is the restriction of μ , i.e.

$$\tilde{\mu} := \mu|_{\tilde{\mathbf{X}}_s \times_P \tilde{E}} : \tilde{\mathbf{X}}_s \times_P \tilde{E} \rightarrow \tilde{E}.$$

First of all, $\tilde{\mu}$ indeed takes values in $\tilde{E}$ because, since μ covers t , for any $(\phi, e) \in \tilde{\mathbf{X}}_s \times_P \tilde{E}$ we have $P \circ \mu(\phi, e) = t \circ \pi_1(\phi, e) = t(\phi) \in \tilde{X}$ and hence $\mu(\phi, e) \in P^{-1}(\tilde{X}) = \tilde{E}$. Moreover, $\tilde{\mu}$

covers the target map on $\tilde{\mathcal{X}}$ since $\tilde{\mu}$ is the restriction of μ which covers the target map t on $\mathcal{X}$. Finally, the required properties Definition 1.6.3(i)(ii) of $\tilde{\mu}$ follow immediately from the corresponding properties of μ , since $\tilde{\mu}$ is the restriction of μ . This completes the proof that $(\tilde{P}, \tilde{\mu})$ is a tame strong bundle over $\tilde{\mathcal{X}}$.

Proof of (III): Since $\tilde{X}$ is a slice of the sc-Fredholm section $\sigma : X \rightarrow E$ in the M -polyfold sense, Theorem 1.1.3(III) provides the structure of a tame sc-Fredholm section on the restricted section $\tilde{\sigma} := \sigma|_{\tilde{X}} : \tilde{X} \rightarrow \tilde{E}$ with the claimed sc-Fredholm index. Moreover, the section $\tilde{\sigma}$ is a section functor because the required compatibility (1.74) with $\tilde{\mu}$ is immediate from the compatibility of σ with μ . Hence $\tilde{\sigma}$ is a tame sc-Fredholm section functor.

The final statement about compactness of the zero sets holds because $|\tilde{\sigma}^{-1}(0)| = |\sigma^{-1}(0)| \cap |\tilde{X}_\infty|$ is the intersection of a compact set and a closed subset of the Hausdorff space $|X_\infty|$ (recall that the orbit space of an ep-groupoid is Hausdorff by [34, Thm. 7.2]). $\square$

The generalization of Theorem 1.1.5 to the ep-groupoid case now easily follows by combining Theorem 1.1.5 with Corollary 1.6.7.

Corollary 1.6.8.

(I) Consider a tame ep-groupoid $\mathcal{X} = (X, \mathbf{X})$, a smooth manifold Y with a codimension- n submanifold $N \subset Y$, and a sc-smooth map $f : X \rightarrow Y$ that satisfies the compatibility with morphisms

$$f(s(\phi)) = f(t(\phi)) \text{ for all } \phi \in \mathbf{X}. \quad (1.78)$$

Assume that f is transverse to N (Definition 1.5.8).

Then, there exists an open neighborhood

$$\tilde{X} \subset f^{-1}(N) \cap X_1$$

of $f^{-1}(N) \cap X_\infty$ such that $\tilde{X}$ is a slice of X^1 in the M -polyfold sense (Definition 1.5.7) satisfying $\text{codim}_x(\tilde{X} \subset X^1) = n$ for every $x \in \tilde{X}_1 = \tilde{X} \cap X_2$. In particular, the full subcategory $\tilde{\mathcal{X}} = (\tilde{X}, \tilde{\mathbf{X}})$ of $\mathcal{X}^1$ with object space $\tilde{X}$ is a tame ep-groupoid with degeneracy index satisfying $d_{\tilde{\mathcal{X}}}(x) = d_X(x)$ for all $x \in \tilde{X}_\infty$.

(II) Consider, in addition, a tame strong bundle $(P : E \rightarrow X, \mu)$ over $\mathcal{X}$. Then, there exists a possibly smaller neighborhood $\tilde{X}$ in (I) that is a slice of the bundle $P|_{E^1} : E^1 \rightarrow X^1$ in the M -polyfold sense (Definition 1.5.7). In particular, the tuple $(\tilde{P}, \tilde{\mu})$ consisting of the restrictions

$$\begin{aligned} \tilde{E} &:= P|_{E^1}^{-1}(\tilde{X}) \\ \tilde{P} &:= P|_{\tilde{E}} : \tilde{E} \rightarrow \tilde{X}, \\ \tilde{\mu} &:= \mu|_{\tilde{\mathbf{X}}_s \times_P \tilde{E}} : \tilde{\mathbf{X}}_s \times_P \tilde{E} \rightarrow \tilde{E}, \end{aligned}$$

is a tame strong bundle over $\tilde{\mathcal{X}}$.

(III) Consider, in addition, a tame sc-Fredholm section functor $\sigma : X \rightarrow E$ of (P, μ) . Assume that f is σ -compatibly transverse to N (Definition 1.5.8).

Then, there exists a possibly smaller neighborhood $\tilde{X}$ in (II) that is a slice of the tame sc-Fredholm section $\sigma|_{X^1} : X^1 \rightarrow E^1$ in the M -polyfold sense (Definition 1.5.7). In particular, the restriction

$$\tilde{\sigma} := \sigma|_{\tilde{X}} : \tilde{X} \rightarrow \tilde{E}$$

is a tame sc-Fredholm section functor of the bundle $(\tilde{P}, \tilde{\mu})$ with index satisfying $\text{ind}_x(\tilde{\sigma}) = \text{ind}_x(\sigma) - n$ for all $x \in \tilde{X}_\infty$. If N is closed as a subset of Y and $|\sigma^{-1}(0)|$ is compact, then $|\tilde{\sigma}^{-1}(0)|$ is compact.

Proof. We first prove the statements in (I). Since f is transverse to N , Theorem 1.1.5(I) provides an open neighborhood $\tilde{X} \subset f^{-1}(N) \cap X_1$ of $f^{-1}(N) \cap X_\infty$ that is a slice of X^1 with the claimed codimension at every point in $\tilde{X}_1$. The compatibility (1.78) of f with the morphisms $\mathbf{X}$ implies that $s^{-1}(\tilde{X}) = t^{-1}(\tilde{X})$ holds. Hence applying Corollary 1.6.7(I) to the tame ep-groupoid $\mathcal{X}^1 = (X^1, \mathbf{X}^1)$ and the slice $\tilde{X} \subset X^1$, we conclude that the full subcategory $\tilde{\mathcal{X}} = (\tilde{X}, \tilde{\mathbf{X}})$ of $\mathcal{X}^1$ with object space $\tilde{X}$ is a tame ep-groupoid with the claimed degeneracy index.

We now prove the statements in (II). Theorem 1.1.5(II) provides a choice of $\tilde{X}$ in (I) that is in addition a slice of the bundle $P|_{E^1} : E^1 \rightarrow X^1$. Hence applying Corollary 1.6.7(II), we conclude that the tuple $(\tilde{P}, \tilde{\mu})$ is indeed a tame strong bundle over $\tilde{\mathcal{X}}$, as claimed.

We now prove the statements in (III). Since f is σ -compatibly transverse to N , Theorem 1.1.5(III) provides a choice of $\tilde{X}$ in (II) that is in addition a slice of the tame sc-Fredholm section $\sigma|_{X^1} : X^1 \rightarrow E^1$. Hence applying Corollary 1.6.7(III), we conclude that the restriction $\tilde{\sigma} = \sigma|_{\tilde{X}}$ is indeed a tame sc-Fredholm section functor of the bundle $(\tilde{P}, \tilde{\mu})$ with the claimed index.

It remains to prove the final statement about compactness of the orbit spaces of the zero sets. Assume that $|\sigma^{-1}(0)|$ is compact. It suffices to show that the inclusion of orbit spaces $|\tilde{X}_\infty| \subset |(X^1)_\infty| = |X_\infty|$ is closed, because then Corollary 1.6.7(III) implies compactness of $|\tilde{\sigma}^{-1}(0)|$, as required. Notice that, by (1.78), the map f descends to a map on the orbit space $|f| : |X_\infty| \rightarrow Y$. Then $|\tilde{X}_\infty| = |f|^{-1}(N)$ is closed in $|X_\infty|$ by continuity of $|f|$, as required. This completes the proof. $\square$

1.7 Fiber products of tame sc-Fredholm sections

The main result of this section is the construction of fiber products of tame sc-Fredholm section functors (Corollary 1.7.3). This is a corollary of the construction of restrictions of tame sc-Fredholm section functors to transverse preimages of sc-smooth maps, which is the

main result of this paper; see Theorem 1.1.5 for the M -polyfold case and Corollary 1.6.8 for the ep-groupoid generalization.

In this section, we index the M -polyfolds and ep-groupoids with parenthesis around the subscript, i.e. $\mathcal{B}_{(i)}$ for $i = 1, 2$, to avoid confusion with the standard notation $\mathcal{B}_m$ for the m -level of a M -polyfold $\mathcal{B}$.

We first describe the Cartesian product of tame sc-Fredholm sections over M -polyfolds.

Lemma 1.7.1.

- (I) *Consider tame M -polyfolds $\mathcal{B}_{(i)}$ for $i = 1, 2$. Then, the Cartesian product $\mathcal{B}_{(1)} \times \mathcal{B}_{(2)}$ is a tame M -polyfold with charts given by products of charts on the factors, and with degeneracy index satisfying $d_{\mathcal{B}_{(1)} \times \mathcal{B}_{(2)}}(x_1, x_2) = d_{\mathcal{B}_{(1)}}(x_1) + d_{\mathcal{B}_{(2)}}(x_2)$ for all $(x_1, x_2) \in (\mathcal{B}_{(1)})_\infty \times (\mathcal{B}_{(2)})_\infty$.*
- (II) *Consider tame strong bundles $\rho_i : \mathcal{E}_{(i)} \rightarrow \mathcal{B}_{(i)}$ over $\mathcal{B}_{(i)}$ for $i = 1, 2$. Then, the product map $\rho_1 \times \rho_2 : \mathcal{E}_{(1)} \times \mathcal{E}_{(2)} \rightarrow \mathcal{B}_{(1)} \times \mathcal{B}_{(2)}$ is a tame strong bundle over $\mathcal{B}_{(1)} \times \mathcal{B}_{(2)}$ with bundle charts given by products of bundle charts on the factors.*
- (III) *Consider tame sc-Fredholm sections $\sigma_i : \mathcal{B}_{(i)} \rightarrow \mathcal{E}_{(i)}$ (Definition 1.5.4) of ρ_i for $i = 1, 2$. Then, the product map $\sigma_1 \times \sigma_2 : \mathcal{B}_{(1)} \times \mathcal{B}_{(2)} \rightarrow \mathcal{E}_{(1)} \times \mathcal{E}_{(2)}$ is a tame sc-Fredholm section of $\rho_1 \times \rho_2$ with tame sc-Fredholm charts given by products of tame sc-Fredholm charts on the factors (after reordering factors in the charts), and with index satisfying $\text{ind}_{(x_1, x_2)}(\sigma_1 \times \sigma_2) = \text{ind}_{x_1}(\sigma_1) + \text{ind}_{x_2}(\sigma_2)$ for all $(x_1, x_2) \in (\mathcal{B}_{(1)})_\infty \times (\mathcal{B}_{(2)})_\infty$. If $\sigma_i^{-1}(0)$ is compact for $i = 1, 2$, then $(\sigma_1 \times \sigma_2)^{-1}(0)$ is compact.*

Proof. Proof of (I): We first show that products of tame M -polyfold charts (Definition 1.5.1) on the factors $\mathcal{B}_{(i)}$, $i = 1, 2$, are indeed tame M -polyfold charts on the product $\mathcal{B}_{(1)} \times \mathcal{B}_{(2)}$, as claimed. For $i = 1, 2$, consider a tame M -polyfold chart

$$(V_i, \varphi_i, (\mathcal{O}_i, C_i, \mathbb{E}_i)) \tag{1.79}$$

on $\mathcal{B}_{(i)}$, an open subset $U_i \subset C_i$, and a tame sc-retraction $r_i : U_i \rightarrow U_i$ with image $\mathcal{O}_i = r_i(U_i)$. We claim that the tuple

$$(V_1 \times V_2, \varphi_1 \times \varphi_2, (\mathcal{O}_1 \times \mathcal{O}_2, C_1 \times C_2, \mathbb{E}_1 \times \mathbb{E}_2)) \tag{1.80}$$

is a tame M -polyfold chart on $\mathcal{B}_{(1)} \times \mathcal{B}_{(2)}$. We equip $\mathcal{B}_{(1)} \times \mathcal{B}_{(2)}$ with the product topology on every level. Recall that the product sc-structure is given by $(\mathbb{E}_1 \times \mathbb{E}_2)_m = (\mathbb{E}_1)_m \times (\mathbb{E}_2)_m$ for all $m \geq 0$. Then the product set $V_1 \times V_2$ is open in $\mathcal{B}_{(1)} \times \mathcal{B}_{(2)}$ and the product map $\varphi_1 \times \varphi_2 : V_1 \times V_2 \rightarrow \mathcal{O}_1 \times \mathcal{O}_2$ is a homeomorphism. It remains to show that $\mathcal{O}_1 \times \mathcal{O}_2$ is a tame sc-retract. The product map $r_1 \times r_2 : U_1 \times U_2 \rightarrow U_1 \times U_2$ is a sc-smooth retraction with image $\mathcal{O}_1 \times \mathcal{O}_2$. We now show that $r_1 \times r_2$ is tame. To check Definition 1.3.1(1) for $r_1 \times r_2$, let $(x_1 \times x_2) \in U_1 \times U_2$ and then compute using the corresponding property of each r_i together with (1.7) that the degeneracy index satisfies $d_{C_1 \times C_2}((r_1 \times r_2)(x_1, x_2)) =$

$d_{C_1 \times C_2}(r_1(x_1), r_2(x_2)) = d_{C_1}(r_1(x_1)) + d_{C_2}(r_2(x_2)) = d_{C_1}(x_1) + d_{C_2}(x_2) = d_{C_1 \times C_2}(x_1, x_2)$, as required. To check Definition 1.3.1(2) for $r_1 \times r_2$, let $(x_1, x_2) \in (\mathcal{O}_1 \times \mathcal{O}_2)_\infty = (\mathcal{O}_1)_\infty \times (\mathcal{O}_2)_\infty$ be a smooth point. Then by the corresponding property of each r_i , there exist sc-subspaces $A_i \subset \mathbb{E}_i$ such that $\mathbb{E}_i = T_{x_i} \mathcal{O}_i \oplus A_i$ and $A_i \subset (\mathbb{E}_i)_{x_i}$, where $(\mathbb{E}_i)_{x_i} \subset \mathbb{E}_i$ is the sc-subspace (1.9). From the definition (1.9) we have

$$(\mathbb{E}_1)_{x_1} \times (\mathbb{E}_2)_{x_2} = (\mathbb{E}_1 \times \mathbb{E}_2)_{(x_1, x_2)}. \quad (1.81)$$

So we have $A_1 \times A_2 \subset (\mathbb{E}_1 \times \mathbb{E}_2)_{(x_1, x_2)}$. Moreover, we have the sc-splitting

$$\begin{aligned} \mathbb{E}_1 \times \mathbb{E}_2 &= (T_{x_1} \mathcal{O}_1 \oplus A_1) \times (T_{x_2} \mathcal{O}_2 \oplus A_2) \\ &= T_{(x_1, x_2)}(\mathcal{O}_1 \times \mathcal{O}_2) \oplus (A_1 \times A_2) \end{aligned}$$

because, by definition (1.10) of tangent space to a sc-retract, the tangent space to a product satisfies

$$\begin{aligned} T_{(x_1, x_2)}(\mathcal{O}_1 \times \mathcal{O}_2) &= D_{(x_1, x_2)}(r_1 \times r_2)(T_{(x_1, x_2)}(U_1 \times U_2)) \\ &= D_{x_1} r_1(T_{x_1} U_1) \times D_{x_2} r_2(T_{x_2} U_2) \\ &= T_{x_1} \mathcal{O}_1 \times T_{x_2} \mathcal{O}_2. \end{aligned} \quad (1.82)$$

This completes the proof that $r_1 \times r_2$ is a tame sc-retraction and hence $\mathcal{O}_1 \times \mathcal{O}_2$ is a tame sc-retract.

We have shown that the product chart (1.80) is indeed a tame M -polyfold chart on $\mathcal{B}_{(1)} \times \mathcal{B}_{(2)}$. Since product charts of this form cover $\mathcal{B}_{(1)} \times \mathcal{B}_{(2)}$ and have sc-smooth transition maps due to the transitions on each factor being sc-smooth, the collection of these product charts forms a tame atlas on $\mathcal{B}_{(1)} \times \mathcal{B}_{(2)}$. Then $\mathcal{B}_{(1)} \times \mathcal{B}_{(2)}$ equipped with this atlas is a tame M -polyfold.

We now prove the claimed degeneracy index formula. Let $(x_1, x_2) \in (\mathcal{B}_{(1)})_\infty \times (\mathcal{B}_{(2)})_\infty$. From the splitting (1.82) of the tangent spaces in the product retract, we deduce the splitting $T_{(x_1, x_2)}(\mathcal{B}_{(1)} \times \mathcal{B}_{(2)}) \cong T_{x_1} \mathcal{B}_{(1)} \times T_{x_2} \mathcal{B}_{(2)}$. Furthermore, by definition of reduced tangent space in a retract (1.12) together with (1.81) and (1.82), we conclude $T_{(\varphi_1(x_1), \varphi_2(x_2))}^R(\mathcal{O}_1 \times \mathcal{O}_2) = T_{\varphi_1(x_1)}^R \mathcal{O}_1 \times T_{\varphi_2(x_2)}^R \mathcal{O}_2$, which implies by the global definition of reduced tangent space (1.49) that $T_{(x_1, x_2)}^R(\mathcal{B}_{(1)} \times \mathcal{B}_{(2)}) \cong T_{x_1}^R \mathcal{B}_{(1)} \times T_{x_2}^R \mathcal{B}_{(2)}$ by examining any product chart. Hence, by the global description (1.50) of degeneracy index at smooth points, we have $d_{\mathcal{B}_{(1)} \times \mathcal{B}_{(2)}}(x_1, x_2) = \dim(T_{(x_1, x_2)}(\mathcal{B}_{(1)} \times \mathcal{B}_{(2)}) / T_{(x_1, x_2)}^R(\mathcal{B}_{(1)} \times \mathcal{B}_{(2)})) = \dim(T_{x_1} \mathcal{B}_{(1)} / T_{x_1}^R \mathcal{B}_{(1)}) + \dim(T_{x_2} \mathcal{B}_{(2)} / T_{x_2}^R \mathcal{B}_{(2)}) = d_{\mathcal{B}_{(1)}}(x_1) + d_{\mathcal{B}_{(2)}}(x_2)$. This completes the proof of (I).

Proof of (II): For $i = 1, 2$, consider a tame M -polyfold chart (1.79) on $\mathcal{B}_{(i)}$ covered by a tame strong bundle chart (Definition 1.5.2)

$$(\rho_i^{-1}(V_i), \Phi_i, (K_i, C_i \triangleleft \mathbb{F}_i, \mathbb{E}_i \triangleleft \mathbb{F}_i)) \quad (1.83)$$

for ρ_i . We now construct a tame strong bundle chart for $\rho_1 \times \rho_2$ over the open set $V_1 \times V_2$. For $i = 1, 2$, consider a tame strong bundle retraction (1.23)

$$\begin{aligned} R_i : U_i \triangleleft \mathbb{F}_i &\rightarrow U_i \triangleleft \mathbb{F}_i \\ (x, \xi) &\mapsto (r_i(x), \Gamma_i(x, \xi)) \end{aligned}$$

with image $K_i = R_i(U_i \triangleleft \mathbb{F}_i)$, where

$$\Gamma_i : U_i \triangleleft \mathbb{F} \rightarrow \mathbb{F}$$

is a linear projection $\Gamma_i(x, \cdot)$ for every $x \in U_i$. Denote the reordering of factors map by

$$l : (\mathbb{E}_1 \times \mathbb{F}_1) \times (\mathbb{E}_2 \times \mathbb{F}_2) \rightarrow (\mathbb{E}_1 \times \mathbb{E}_2) \times (\mathbb{F}_1 \times \mathbb{F}_2),$$

and note that l is a linear sc-isomorphism. We claim that the tuple

$$((\rho_1 \times \rho_2)^{-1}(V_1 \times V_2), l \circ (\Phi_1 \times \Phi_2), (l(K_1 \times K_2), C_1 \times C_2 \triangleleft \mathbb{F}_1 \times \mathbb{F}_2, \mathbb{E}_1 \times \mathbb{E}_2 \triangleleft \mathbb{F}_1 \times \mathbb{F}_2)) \quad (1.84)$$

is a tame strong bundle chart for $\rho_1 \times \rho_2$ covering the product M -polyfold chart (1.80). Since for $i = 1, 2$, the map $\Phi_i : \rho_i^{-1}(V_i) \rightarrow K_i$ is a homeomorphism covering φ_i that is linear on fibers, it follows that the map $l \circ (\Phi_1 \times \Phi_2)$ is a homeomorphism from $(\rho_1 \times \rho_2)^{-1}(V_1 \times V_2)$ to $l(K_1 \times K_2)$ covering $\varphi_1 \times \varphi_2$ and is linear on fibers. Moreover, the set $l(K_1 \times K_2)$ is a tame strong bundle retract because it is the image of the tame strong bundle retraction

$$\begin{aligned} l \circ (R_1 \times R_2) \circ l^{-1} : (U_1 \times U_2) \triangleleft (\mathbb{F}_1 \times \mathbb{F}_2) &\rightarrow (U_1 \times U_2) \triangleleft (\mathbb{F}_1 \times \mathbb{F}_2) \\ (x_1, x_2, \xi_1, \xi_2) &\mapsto (r_1(x_1), r_2(x_2), \Gamma_1(x_1, \xi_1), \Gamma_2(x_2, \xi_2)). \end{aligned}$$

We have shown that the chart (1.84) is indeed a tame strong bundle chart for $\rho_1 \times \rho_2$. Sc-smoothness of bundle transitions between charts constructed in this way follows from sc-smoothness in each factor. Hence the collection of these charts forms a tame bundle atlas for $\rho_1 \times \rho_2$. This completes the proof of (II).

Proof of (III): To prove that $\sigma_1 \times \sigma_2$ is a tame sc-Fredholm section (Definition 1.5.4) of $\rho_1 \times \rho_2$, first observe that it is sc-smooth and regularizing because σ_1 and σ_2 are sc-smooth and regularizing, and the double filtration satisfies $(\mathcal{B}_{(1)} \times \mathcal{B}_{(2)})_m = (\mathcal{B}_{(1)})_m \times (\mathcal{B}_{(2)})_m$ and $(\mathcal{E}_{(1)} \times \mathcal{E}_{(2)})_{m,m+1} = (\mathcal{E}_{(1)})_{m,m+1} \times (\mathcal{E}_{(2)})_{m,m+1}$ for all $m \geq 0$ (see (1.52)).

Consider a smooth point $(x_1, x_2) \in (\mathcal{B}_{(1)})_\infty \times (\mathcal{B}_{(2)})_\infty$. Then since each σ_i is tame sc-Fredholm there exist tame bundle charts as in (1.83) that are in addition tame sc-Fredholm charts for σ_i at x_i . First of all, this means that the partial quadrants $C_i \subset \mathbb{E}_i$ are in the standard form (1.5), i.e. $C_i = [0, \infty)^{s_i} \times \mathbb{E}'_i \subset \mathbb{R}^{s_i} \times \mathbb{E}'_i = \mathbb{E}_i$. Now, we will prove that a chart similar to the tame bundle chart (1.84) is a tame sc-Fredholm chart for $\sigma_1 \times \sigma_2$ at (x_1, x_2) . The only issue with (1.84) is that the partial quadrant $C_1 \times C_2 \subset \mathbb{E}_1 \times \mathbb{E}_2$ is not in the standard form. This is easily remedied as follows. Consider the partial quadrant

$$C := [0, \infty)^{s_1 \times s_2} \times \mathbb{E}'_1 \times \mathbb{E}'_2$$

of the sc-Banach space

$$\mathbb{E} := \mathbb{R}^{s_1 \times s_2} \times \mathbb{E}'_1 \times \mathbb{E}'_2$$

Then the reordering of factors map

$$f : \mathbb{R}^{s_1} \times \mathbb{E}'_1 \times \mathbb{R}^{s_2} \times \mathbb{E}'_2 \rightarrow \mathbb{R}^{s_1+s_2} \times \mathbb{E}'_1 \times \mathbb{E}'_2$$

is a linear sc-isomorphism $\mathbb{E}_1 \times \mathbb{E}_2 \rightarrow \mathbb{E}$ that restricts to an isomorphism $C_1 \times C_2 \rightarrow C$. Applying f to the sc-retract $\mathcal{O}_1 \times \mathcal{O}_2$ produces a tame sc-retract

$$\mathcal{O} := f(\mathcal{O}_1 \times \mathcal{O}_2) \subset C$$

with tame sc-retraction

$$\begin{aligned} r &:= f \circ (r_1 \times r_2) \circ f^{-1} : U \rightarrow U, \\ U &:= f(U_1 \times U_2) \subset C, \end{aligned}$$

onto $r(U) = \mathcal{O}$. Moreover, the map

$$\varphi := f \circ (\varphi_1 \times \varphi_2) : V_1 \times V_2 \rightarrow \mathcal{O}$$

is a homeomorphism, and so from the tame M -polyfold chart (1.80) we obtain another tame M -polyfold chart

$$(V_1 \times V_2, \varphi, (\mathcal{O}, C, \mathbb{E})).$$

Similarly, applying $f \times id_{\mathbb{F}_1 \times \mathbb{F}_2}$ to the tame bundle retract $l(K_1 \times K_2)$ produces another tame bundle retract

$$K := (f \times id_{\mathbb{F}_1 \times \mathbb{F}_2}) \circ l(K_1 \times K_2)$$

with tame bundle retraction

$$\begin{aligned} R &:= (f \times id_{\mathbb{F}_1 \times \mathbb{F}_2}) \circ l \circ (R_1 \times R_2) \circ l^{-1} \circ (f \times id_{\mathbb{F}_1 \times \mathbb{F}_2})^{-1} \\ &: U \triangleleft \mathbb{F}_1 \times \mathbb{F}_2 \rightarrow U \triangleleft \mathbb{F}_1 \times \mathbb{F}_2 \end{aligned}$$

onto $R(U \triangleleft \mathbb{F}_1 \times \mathbb{F}_2) = K$. Consider the map

$$\Gamma := (\Gamma_1 \times \Gamma_2) \circ l^{-1} \circ (f \times id_{\mathbb{F}_1 \times \mathbb{F}_2})^{-1},$$

and observe that, for $(y, \xi_1, \xi_2) \in U \triangleleft \mathbb{F}_1 \times \mathbb{F}_2$, we have

$$R(y, \xi_1, \xi_2) = (r(y), \Gamma(y, \xi_1, \xi_2)).$$

The map

$$\Phi := (f \times id_{\mathbb{F}_1 \times \mathbb{F}_2}) \circ l \circ (\Phi_1 \times \Phi_2) : (\rho_1 \times \rho_2)^{-1}(V_1 \times V_2) \rightarrow K$$

is a homeomorphism that is linear on the fibers and covers φ . Hence from the tame bundle chart (1.84) we obtain another tame bundle chart

$$((\rho_1 \times \rho_2)^{-1}(V_1 \times V_2), \Phi, (K, C \triangleleft \mathbb{F}_1 \times \mathbb{F}_2, \mathbb{E} \triangleleft \mathbb{F}_1 \times \mathbb{F}_2)). \quad (1.85)$$

We claim that the bundle chart (1.85) is a tame sc-Fredholm chart for $\sigma_1 \times \sigma_2$ at (x_1, x_2) . We must show that the section

$$\tau := \Phi \circ (\sigma_1 \times \sigma_2) \circ \varphi^{-1} : \mathcal{O} \rightarrow K \quad (1.86)$$

is a tame sc-Fredholm germ. Since for each $i = 1, 2$, the section

$$\tau_i := \Phi_i \circ \sigma_i \circ \varphi_i^{-1} : \mathcal{O}_i \rightarrow K_i$$

is a tame sc-Fredholm germ, we can assume that there is a filling

$$h_i : U_i \rightarrow U_i \triangleleft \mathbb{F}_i$$

of $\tau_i \circ r_i : U_i \rightarrow \mathbb{F}_i$, where $\tau_i : \mathcal{O}_i \rightarrow \mathbb{F}_i$ denotes the principal part of τ_i in the fiber $\mathbb{F}_i$. We claim that

$$h := (f \times id_{\mathbb{F}_1 \times \mathbb{F}_2}) \circ l \circ (h_1 \times h_2) \circ f^{-1} : U \rightarrow U \triangleleft (\mathbb{F}_1 \times \mathbb{F}_2)$$

is a filling of $\tau \circ r : U \rightarrow \mathbb{F}_1 \times \mathbb{F}_2$. The required properties in Definition 1.3.6(b).(i)-(iii) of a filling follow from the corresponding properties for each h_i , as we now verify. Given $y \in \mathcal{O}$, we compute $\tau(y) = (f \times id_{\mathbb{F}_1 \times \mathbb{F}_2}) \circ l \circ (\Phi_1 \times \Phi_2) \circ (\sigma_1 \times \sigma_2) \circ (\varphi_1 \circ \varphi_2)^{-1} \circ f^{-1}(y) = (f \times id_{\mathbb{F}_1 \times \mathbb{F}_2}) \circ l \circ (\tau_1 \times \tau_2) \circ f^{-1}(y) = (f \times id_{\mathbb{F}_1 \times \mathbb{F}_2}) \circ l \circ (h_1 \times h_2) \circ f^{-1}(y) = h(y)$, which verifies property (i) for τ . To see property (ii), let $y \in U$ and assume $\mathbf{h}(y) = \Gamma(r(y), \mathbf{h}(y))$. Then since the principal parts satisfy $\mathbf{h} = (\mathbf{h}_1 \times \mathbf{h}_2) \circ f^{-1}$, we compute that $(\mathbf{h}_1 \times \mathbf{h}_2) \circ f^{-1}(y) = \Gamma(r(y), \mathbf{h}(y)) = (\Gamma_1 \times \Gamma_2) \circ l^{-1} \circ (f \times id_{\mathbb{F}_1 \times \mathbb{F}_2})^{-1}(f \circ (r_1 \times r_2) \circ f^{-1}(y), (\mathbf{h}_1 \times \mathbf{h}_2) \circ f^{-1}(y)) = (\Gamma_1 \times \Gamma_2) \circ l^{-1}((r_1 \times r_2) \circ f^{-1}(y), (\mathbf{h}_1 \times \mathbf{h}_2) \circ f^{-1}(y))$, which implies $f^{-1}(y) \in \mathcal{O}_1 \times \mathcal{O}_2$ by the corresponding property of τ_1 and τ_2 . Hence $y \in \mathcal{O}$, proving property (ii) for τ . We now verify (iii) for τ . By the corresponding property for each τ_i , the linearization $D_0 L_i$ at 0 of

$$\begin{aligned} L_i : U_i &\rightarrow \mathbb{F}_i \\ y_i &\mapsto (id_{\mathbb{F}_i} - \Gamma_i(r_i(y_i), \cdot))\mathbf{h}_i(y_i) \end{aligned}$$

restricts to a linear sc-isomorphism from $\ker D_0 r_i$ to $\ker \Gamma_i(0, \cdot)$. We must show that the linearization at 0 of the map

$$\begin{aligned} L : U &\rightarrow \mathbb{F}_1 \times \mathbb{F}_2 \\ y &\mapsto (id_{\mathbb{F}_1 \times \mathbb{F}_2} - \Gamma(r(y), \cdot))\mathbf{h}(y) \end{aligned}$$

restricts to a linear sc-isomorphism from $\ker D_0 r$ to $\ker \Gamma(0, \cdot)$. This follows from the observations that $L = (L_1 \times L_2) \circ f^{-1}$ holds, the map f^{-1} is a linear sc-isomorphism that restricts to an isomorphism $\ker D_0 r \rightarrow \ker D_0(f \circ (r_1 \times r_2)) = \ker D_0(r_1 \times r_2) = \ker(D_0 r_1) \times \ker(D_0 r_2)$, and $\ker \Gamma(0, \cdot) = \ker \Gamma_1(0, \cdot) \times \ker \Gamma_2(0, \cdot)$. Thus h is a filling of $\tau \circ r$, as claimed.

We now verify the properties of τ required in Definition 1.3.6(c) of a local sc-Fredholm germ. We in addition show that τ satisfies the stronger properties of a tame sc-Fredholm germ (Definition 1.3.7).

Since each τ_i is a tame sc-Fredholm germ, the corresponding properties hold: There exists a sc^+ -section

$$\mathfrak{s}_i : U_i \rightarrow U_i \triangleleft \mathbb{F}_i$$

satisfying $\mathfrak{s}_i(0) = h_i(0)$, a sc-Banach space $\mathbb{W}_i$, a sc-germ of neighborhoods

$$U'_i \text{ around } 0 \text{ in } [0, \infty)^{s_i} \times \mathbb{R}^{k_i - s_i} \times \mathbb{W}_i$$

for some $k_i \geq s_i \geq 0$, and a strong bundle isomorphism

$$\Psi_i : U_i \triangleleft \mathbb{F}_i \rightarrow U'_i \triangleleft \mathbb{R}^{k'_i} \times \mathbb{W}_i$$

covering a linear sc-isomorphism

$$\psi_i = id_{[0, \infty)^{s_i}} \times \bar{\psi}_i : U_i \rightarrow U'_i$$

satisfying $\psi_i(0) = 0$, where

$$\bar{\psi}_i : \mathbb{E}'_i \rightarrow \mathbb{R}^{k_i - s_i} \times \mathbb{W}_i$$

is a linear sc-isomorphism, and such that the principal part of the section

$$b_i := \Psi_i \circ (h_i - \mathfrak{s}_i) \circ \psi_i^{-1} : U'_i \rightarrow U'_i \triangleleft \mathbb{R}^{k'_i} \times \mathbb{W}_i \quad (1.87)$$

is a basic germ. This basic germ property means that the principal part

$$\mathbf{b}_i : U'_i \rightarrow \mathbb{R}^{k'_i} \times \mathbb{W}_i$$

is a sc-smooth germ satisfying $\mathbf{b}_i(0) = 0$ and such that, for $a_i \in [0, \infty)^{s_i}$, $d_i \in \mathbb{R}^{k_i - s_i}$, $w_i \in \mathbb{W}_i$, we have

$$P_i \circ \mathbf{b}_i(a_i, d_i, w_i) = w_i - B_i(a_i, d_i, w_i)$$

where $P_i : \mathbb{R}^{k'_i} \times \mathbb{W}_i \rightarrow \mathbb{W}_i$ is projection onto $\mathbb{W}_i$ and B_i is a sc-smooth germ satisfying $B_i(0) = 0$ and the contraction property (1.26).

The section

$$\mathfrak{s} := (f \times id_{\mathbb{F}_1 \times \mathbb{F}_2}) \circ l \circ (\mathfrak{s}_1 \times \mathfrak{s}_2) \circ f^{-1} : U \rightarrow U \triangleleft (\mathbb{F}_1 \times \mathbb{F}_2)$$

is sc^+ because each $\mathfrak{s}_i$ is sc^+ and the other maps in the composition are linear sc-isomorphisms. Denote the linear sc-isomorphisms given by reordering the factors by

$$\begin{aligned} q &: ([0, \infty)^{s_1} \times \mathbb{R}^{k_1 - s_1} \times \mathbb{W}_1) \times ([0, \infty)^{s_2} \times \mathbb{R}^{k_2 - s_2} \times \mathbb{W}_2) \\ &\rightarrow [0, \infty)^{s_1 + s_2} \times \mathbb{R}^{k_1 - s_1 + k_2 - s_2} \times \mathbb{W}_1 \times \mathbb{W}_2, \\ Q &: (\mathbb{R}^{k'_1} \times \mathbb{W}_1) \times (\mathbb{R}^{k'_2} \times \mathbb{W}_2) \rightarrow \mathbb{R}^{k'_1 + k'_2} \times \mathbb{W}_1 \times \mathbb{W}_2, \\ l' &: (U'_1 \triangleleft \mathbb{R}^{k'_1} \times \mathbb{W}_1) \times (U'_2 \triangleleft \mathbb{R}^{k'_2} \times \mathbb{W}_2) \\ &\rightarrow (U'_1 \times U'_2) \triangleleft (\mathbb{R}^{k'_1} \times \mathbb{W}_1 \times \mathbb{R}^{k'_2} \times \mathbb{W}_2). \end{aligned}$$

Set

$$\begin{aligned} U' &:= q(U'_1 \times U'_2), \\ \psi &:= q \circ (\psi_1 \times \psi_2) \circ f^{-1} : U \rightarrow U', \\ \Psi &:= (q \triangleleft Q) \circ l' \circ (\Psi_1 \times \Psi_2) \circ l^{-1} \circ (f \times id_{\mathbb{F}_1 \times \mathbb{F}_2})^{-1} \\ &: U \triangleleft (\mathbb{F}_1 \times \mathbb{F}_2) \rightarrow U' \triangleleft (\mathbb{R}^{k'_1+k'_2} \times \mathbb{W}_1 \times \mathbb{W}_2). \end{aligned}$$

We claim that the principal part $\mathbf{b}$ of the section

$$b := \Psi \circ (h - \mathfrak{s}) \circ \psi^{-1} : U' \rightarrow U' \triangleleft (\mathbb{R}^{k'_1+k'_2} \times \mathbb{W}_1 \times \mathbb{W}_2) \quad (1.88)$$

is a basic germ. Observe that the principal parts satisfy

$$\begin{aligned} \mathbf{b} &= Q \circ \pi_{\mathbb{R}^{k'_1} \times \mathbb{W}_1 \times \mathbb{R}^{k'_2} \times \mathbb{W}_2} \circ (\Psi_1 \times \Psi_2) \circ ((h_1 \times h_2) - (\mathfrak{s}_1 \times \mathfrak{s}_2)) \circ (\psi_1 \times \psi_2)^{-1} \circ q^{-1} \\ &= Q \circ (\mathbf{b}_1 \times \mathbf{b}_2) \circ q^{-1}. \end{aligned}$$

Let $P : \mathbb{R}^{k'_1+k'_2} \times \mathbb{W}_1 \times \mathbb{W}_2 \rightarrow \mathbb{W}_1 \times \mathbb{W}_2$ denote the projection onto $\mathbb{W}_1 \times \mathbb{W}_2$. Notice that $P = (P_1 \times P_2) \circ Q^{-1}$. Then, for $(a_1, a_2) \in [0, \infty)^{s_1} \times [0, \infty)^{s_2}$, $(d_1, d_2) \in \mathbb{R}^{k_1-s_1} \times \mathbb{R}^{k_2-s_2}$, and $(w_1, w_2) \in \mathbb{W}_1 \times \mathbb{W}_2$, we compute

$$\begin{aligned} P \circ \mathbf{b}(a_1, a_2, d_1, d_2, w_1, w_2) &= P \circ Q \circ (\mathbf{b}_1 \times \mathbf{b}_2) \circ q^{-1}(a_1, a_2, d_1, d_2, w_1, w_2) \\ &= (P_1 \times P_2) \circ (\mathbf{b}_1 \times \mathbf{b}_2)(a_1, d_1, w_1, a_2, d_2, w_2) \\ &= (w_1 - B_1(a_1, d_1, w_1), w_2 - B_2(a_2, d_2, w_2)) \\ &= (w_1, w_2) - (B_1(a_1, d_1, w_1), B_2(a_2, d_2, w_2)) \\ &= (w_1, w_2) - (B_1 \times B_2) \circ q^{-1}(a_1, a_2, d_1, d_2, w_1, w_2), \end{aligned}$$

so to prove that $\mathbf{b}$ is a basic germ, it remains to verify that the map

$$B := (B_1 \times B_2) \circ q^{-1} : [0, \infty)^{s_1+s_2} \times \mathbb{R}^{k_1-s_1+k_2-s_2} \times \mathbb{W}_1 \times \mathbb{W}_2 \rightarrow \mathbb{W}_1 \times \mathbb{W}_2$$

satisfies the contraction property (1.26). Recall our convention that Banach norms on Cartesian products are chosen to be the sum norm (which is equivalent to choosing any standard choice of norm on a product). The contraction property (1.26) for B then follows from the contraction property for each B_i , as we now verify. Let $\epsilon > 0$ and $m \geq 0$. Then, for $i = 1, 2$, there exists $\delta_i > 0$ such that (1.26) holds for B_i with the same choice of ϵ . Set $\delta := \min(\delta_1, \delta_2)$. Then, given $\|(a_1, a_2, d_1, d_2, w_1, w_2)\|_m, \|(a_1, a_2, d_1, d_2, w'_1, w'_2)\|_m < \delta$ we have

$$\|(a_i, d_i, w_i)\|_m, \|(a_i, d_i, w'_i)\|_m < \delta_i$$

for $i = 1, 2$, from which we compute, using property (1.26) for the B_i ,

$$\begin{aligned}
 & \|B(a_1, a_2, d_1, d_2, w_1, w_2) - B(a_1, a_2, d_1, d_2, w'_1, w'_2)\|_m \\
 &= \|(B_1(a_1, d_1, w_1), B_2(a_2, d_2, w_2)) - (B_1(a_1, d_1, w'_1), B_2(a_2, d_2, w'_2))\|_m \\
 &= \|(B_1(a_1, d_1, w_1) - B_1(a_1, d_1, w'_1), B_2(a_2, d_2, w_2) - B_2(a_2, d_2, w'_2))\|_m \\
 &= \|B_1(a_1, d_1, w_1) - B_1(a_1, d_1, w'_1)\|_m + \|B_2(a_2, d_2, w_2) - B_2(a_2, d_2, w'_2)\|_m \\
 &\leq \epsilon(\|w_1 - w'_1\|_m + \|w_2 - w'_2\|_m) \\
 &= \epsilon \cdot \|(w_1 - w'_1, w_2 - w'_2)\|_m \\
 &= \epsilon \cdot \|(w_1, w_2) - (w'_1, w'_2)\|_m,
 \end{aligned}$$

as required. This completes the proof that τ is a local sc-Fredholm germ.

We claim that, in addition, τ is a tame sc-Fredholm germ. We must show that ψ is in the form required by Definition 1.3.7. Given $(a_i, e_i) \in [0, \infty)^{s_i} \times \mathbb{E}'_i$ for $i = 1, 2$, write $\bar{\psi}_i(e_i) = (d_i, w_i) \in \mathbb{R}^{k_i - s_i} \times \mathbb{W}_i$ and compute

$$\begin{aligned}
 \psi(a_1, a_2, e_1, e_2) &= q \circ (\psi_1 \times \psi_2)(a_1, e_1, a_2, e_2) \\
 &= q \circ (id_{[0, \infty)^{s_1}} \times \bar{\psi}_1 \times id_{[0, \infty)^{s_2}} \times \bar{\psi}_2)(a_1, e_1, a_2, e_2) \\
 &= (a_1, a_2, d_1, d_2, w_1, w_2).
 \end{aligned}$$

So indeed $\psi = id_{[0, \infty)^{s_1 + s_2}} \times \bar{\psi}$ is of the required form, where $\bar{\psi} : \mathbb{E}_1 \times \mathbb{E}_2 \rightarrow \mathbb{R}^{k_1 - s_1 + k_2 - s_2} \times \mathbb{W}_1 \times \mathbb{W}_2$ is the linear sc-isomorphism given by $\bar{\psi}(e_1, e_2) = (d_1, d_2, w_1, w_2)$.

We have verified that $\sigma_1 \times \sigma_2$ is a tame sc-Fredholm section of $\rho_1 \times \rho_2$. To verify the claimed index formula, note first that by definition of sc-Fredholm index (1.27) of a local sc-Fredholm germ and the forms of the basic germs (1.87) and (1.88) we have $ind(\tau_i) = k_i - k'_i$ and $ind(\tau) = (k_1 + k_2) - (k'_1 + k'_2) = ind(\tau_1) + ind(\tau_2)$. Then by definition Definition 1.5.3(3) of index, for $(x_1, x_2) \in (\mathcal{B}_{(1)})_\infty \times (\mathcal{B}_{(2)})_\infty$ we have $ind_{(x_1, x_2)}(\sigma_1 \times \sigma_2) = ind(\tau) = ind(\tau_1) + ind(\tau_2) = ind_{x_1}(\sigma_1) + ind_{x_2}(\sigma_2)$.

The final statement about compactness holds because the zero set $(\sigma_1 \times \sigma_2)^{-1}(0) = \sigma_1^{-1}(0) \times \sigma_2^{-1}(0) \subset (\mathcal{B}_{(1)})_\infty \times (\mathcal{B}_{(2)})_\infty$ is equipped with the product topology in every level $(\mathcal{B}_{(1)} \times \mathcal{B}_{(2)})_m = (\mathcal{B}_{(1)})_m \times (\mathcal{B}_{(2)})_m$, and for $i = 1, 2$, the subspace $\sigma_i^{-1}(0) \subset (\mathcal{B}_{(i)})_m$ is compact for all $m \geq 0$ (see Remark 1.5.5 for a further discussion about compactness in the different levels). $\square$

We now generalize the above Cartesian product construction to the ep-groupoid setting.

Lemma 1.7.2.

(I) Consider tame ep-groupoids $\mathcal{X}_{(i)} = (X_{(i)}, \mathbf{X}_{(i)})$ for $i = 1, 2$. Then, the Cartesian product $\mathcal{X}_{(1)} \times \mathcal{X}_{(2)} = (X_{(1)} \times X_{(2)}, \mathbf{X}_{(1)} \times \mathbf{X}_{(2)})$ is a tame ep-groupoid with degeneracy index satisfying $d_{X_{(1)} \times X_{(2)}}(x_1, x_2) = d_{X_{(1)}}(x_1) + d_{X_{(2)}}(x_2)$ for all $(x_1, x_2) \in (X_{(1)})_\infty \times (X_{(2)})_\infty$.

(II) Consider tame strong bundles $(P_i : E_{(i)} \rightarrow X_{(i)}, \mu_i)$ over $\mathcal{X}_{(i)}$ for $i = 1, 2$. Denote the reordering of factors map by

$$l : (\mathbf{X}_{(1)} \times \mathbf{X}_{(2)})_{s_1 \times s_2 \times P_1 \times P_2} (E_{(1)} \times E_{(2)}) \rightarrow (\mathbf{X}_{(1)}_{s_1 \times P_1} E_{(1)}) \times (\mathbf{X}_{(2)}_{s_2 \times P_2} E_{(2)})$$

and set

$$\mu := (\mu_1 \times \mu_2) \circ l : (\mathbf{X}_{(1)} \times \mathbf{X}_{(2)})_{s_1 \times s_2 \times P_1 \times P_2} (E_{(1)} \times E_{(2)}) \rightarrow E_{(1)} \times E_{(2)}.$$

Then, the tuple $(P_1 \times P_2 : E_{(1)} \times E_{(2)} \rightarrow X_{(1)} \times X_{(2)}, \mu)$ is a tame strong bundle over $\mathcal{X}_{(1)} \times \mathcal{X}_{(2)}$.

(III) Consider tame sc-Fredholm section functors $\sigma_i : X_{(i)} \rightarrow E_{(i)}$ (Definition 1.6.6) of (P_i, μ_i) for $i = 1, 2$. Then, the product map $\sigma_1 \times \sigma_2 : X_{(1)} \times X_{(2)} \rightarrow E_{(1)} \times E_{(2)}$ is a tame sc-Fredholm section functor of $(P_1 \times P_2, \mu)$ with index satisfying $\text{ind}_{(x_1, x_2)}(\sigma_1 \times \sigma_2) = \text{ind}_{x_1}(\sigma_1) + \text{ind}_{x_2}(\sigma_2)$ for all $(x_1, x_2) \in (X_{(1)})_\infty \times (X_{(2)})_\infty$. If $|\sigma_i^{-1}(0)|$ is compact for $i = 1, 2$, then $|(\sigma_1 \times \sigma_2)^{-1}(0)| = |\sigma_1^{-1}(0)| \times |\sigma_2^{-1}(0)|$ is compact.

Proof. We prove the statements in (I). The product $\mathcal{X}_{(1)} \times \mathcal{X}_{(2)}$ is a groupoid with structure maps (source, target, multiplication, unit, and inverse) as we describe below. First note that Lemma 1.7.1(I) provides tame M -polyfold structures on the object space $X_{(1)} \times X_{(2)}$ and on the morphism space $\mathbf{X}_{(1)} \times \mathbf{X}_{(2)}$ with the claimed degeneracy index. So, to prove (I), it remains to describe the structure maps on $\mathcal{X}_{(1)} \times \mathcal{X}_{(2)}$, verify that they are sc-smooth, and verify the étale property Definition 1.6.1(i) and properness Definition 1.6.1(iii).

For $i = 1, 2$, let $(s_i, t_i, m_i, u_i, \iota_i)$ denote the structure maps on $\mathcal{X}_{(i)}$. The source $s_1 \times s_2 : \mathbf{X}_{(1)} \times \mathbf{X}_{(2)} \rightarrow X_{(1)} \times X_{(2)}$, target $t_1 \times t_2 : \mathbf{X}_{(1)} \times \mathbf{X}_{(2)} \rightarrow X_{(1)} \times X_{(2)}$, unit $u_1 \times u_2 : X_{(1)} \times X_{(2)} \rightarrow \mathbf{X}_{(1)} \times \mathbf{X}_{(2)}$, and inversion $\iota_1 \times \iota_2 : \mathbf{X}_{(1)} \times \mathbf{X}_{(2)} \rightarrow \mathbf{X}_{(1)} \times \mathbf{X}_{(2)}$ maps on the product are products of those in each factor, so sc-smoothness follows from sc-smoothness in each factor. Moreover, the étale property Definition 1.6.1(i) holds because products of surjective local sc-diffeomorphisms are surjective local sc-diffeomorphisms.

To see that the multiplication map on the product is sc-smooth, first note that the reordering of factors map $(\mathbf{X}_{(1)} \times \mathbf{X}_{(2)}) \times (\mathbf{X}_{(1)} \times \mathbf{X}_{(2)}) \rightarrow (\mathbf{X}_{(1)} \times \mathbf{X}_{(1)}) \times (\mathbf{X}_{(2)} \times \mathbf{X}_{(2)})$ is sc-smooth. It restricts to a bijection

$$q : (\mathbf{X}_{(1)} \times \mathbf{X}_{(2)})_{s_1 \times s_2 \times t_1 \times t_2} (\mathbf{X}_{(1)} \times \mathbf{X}_{(2)}) \rightarrow (\mathbf{X}_{(1)}_{s_1 \times t_1} \mathbf{X}_{(1)}) \times (\mathbf{X}_{(2)}_{s_2 \times t_2} \mathbf{X}_{(2)}),$$

which is sc-smooth by [34, Prop. 2.15] and [34, Prop. 2.6(1)]. The multiplication map on $\mathcal{X}_{(1)} \times \mathcal{X}_{(2)}$ is the composition

$$(m_1 \times m_2) \circ q : (\mathbf{X}_{(1)} \times \mathbf{X}_{(2)})_{s_1 \times s_2 \times t_1 \times t_2} (\mathbf{X}_{(1)} \times \mathbf{X}_{(2)}) \rightarrow \mathbf{X}_{(1)} \times \mathbf{X}_{(2)},$$

hence is sc-smooth.

We now check properness Definition 1.6.1(iii). Let $(x_1, x_2) \in X_{(1)} \times X_{(2)}$ and let $V(x_i) \subset X_{(i)}$ be open neighborhoods of x_i such that $t_i : s_i^{-1}(\overline{V(x_i)}) \rightarrow X_{(i)}$ are proper maps. Then $V(x_1) \times V(x_2)$ is an open neighborhood of (x_1, x_2) in $X_{(1)} \times X_{(2)}$ and we have

$$(s_1 \times s_2)^{-1}(\overline{V(x_1) \times V(x_2)}) = s_1^{-1}(\overline{V(x_1)}) \times s_2^{-1}(\overline{V(x_2)}).$$

Hence $t_1 \times t_2 : (s_1 \times s_2)^{-1}(\overline{V(x_1) \times V(x_2)}) \rightarrow X_{(1)} \times X_{(2)}$ is proper, as the product of proper maps. This completes the proof that $\mathcal{X}_{(1)} \times \mathcal{X}_{(2)}$ is a tame ep-groupoid, and so the statements in (I) are proved.

We now prove the statements in (II). For $i = 1, 2$, the map $P_i : E_{(i)} \rightarrow X_{(i)}$ is a tame strong bundle over the M -polyfold $X_{(i)}$, so Lemma 1.7.1(II) provides a tame strong bundle structure on the product map $P_1 \times P_2 : E_{(1)} \times E_{(2)} \rightarrow X_{(1)} \times X_{(2)}$. The map μ is a strong bundle map as the composition of the strong bundle maps $\mu_1 \times \mu_2$ and l , and the required properties Definition 1.6.3(i)(ii) of μ follow immediately from those of μ_i for $i = 1, 2$. Hence $(P_1 \times P_2, \mu)$ is a tame strong bundle over $\mathcal{X}_{(1)} \times \mathcal{X}_{(2)}$, as claimed.

We now prove the statements in (III). For $i = 1, 2$, the tame sc-Fredholm section functor $\sigma_i : X_{(i)} \rightarrow E_{(i)}$ is in particular a tame sc-Fredholm section of the bundle P_i , in the M -polyfold sense. So Lemma 1.7.1(III) provides the product map $\sigma_1 \times \sigma_2$ with the structure of a tame sc-Fredholm section of the bundle $P_1 \times P_2$ with the claimed sc-Fredholm index. Moreover, $\sigma_1 \times \sigma_2$ satisfies the required property (1.74) of a section functor of $(P_1 \times P_2, \mu)$ by the corresponding property of the sections functors σ_i of (P_i, μ_i) . Indeed, for all morphisms $(\phi_1 \times \phi_2) \in \mathbf{X}_{(1)} \times \mathbf{X}_{(2)}$, we compute

$$\begin{aligned} (\sigma_1 \times \sigma_2) \circ (t_1 \times t_2)(\phi_1, \phi_2) &= (\sigma_1(t_1(\phi_1)), \sigma_2(t_2(\phi_2))) \\ &= (\mu_1(\phi_1, \sigma_1(s_1(\phi_1))), \mu_2(\phi_2, \sigma_2(s_2(\phi_2)))) \\ &= (\mu_1 \times \mu_2) \circ l(\phi_1, \phi_2, \sigma_1(s_1(\phi_1)), \sigma_2(s_2(\phi_2))) \\ &= \mu(\phi_1, \phi_2, (\sigma_1 \times \sigma_2) \circ (s_1 \times s_2)(\phi_1, \phi_2)). \end{aligned}$$

This completes the proof that $\sigma_1 \times \sigma_2$ is a tame sc-Fredholm section functor of $(P_1 \times P_2, \mu)$.

The final statement about compactness holds because the orbit space of $\mathcal{X}_{(1)} \times \mathcal{X}_{(2)}$ is equal to the Cartesian product of the orbit spaces of the $\mathcal{X}_{(i)}$ equipped with the product topology. $\square$

We proceed to construct fiber products of tame sc-Fredholm section functors over ep-groupoids. As usual, the result specializes to the case of M -polyfolds by considering a M -polyfold $\mathcal{B}$ as an ep-groupoid with the trivial groupoid structure: the object space is $\mathcal{B}$ and the morphism space consists of the identity morphisms $\{1_x \mid x \in \mathcal{B}\}$ which is a M -polyfold by declaring the bijection $x \mapsto 1_x$ with $\mathcal{B}$ to be a sc-diffeomorphism.

Corollary 1.7.3.

(I) Consider tame ep-groupoids $\mathcal{X}_{(i)} = (X_{(i)}, \mathbf{X}_{(i)})$ for $i = 1, 2$, a smooth manifold Y , and sc-smooth maps $f_i : X_{(i)} \rightarrow Y$ for $i = 1, 2$, that satisfy the compatibility with morphisms

$$f_i(s_i(\phi)) = f_i(t_i(\phi)) \text{ for all } \phi \in \mathbf{X}_{(i)}.$$

Assume that the product map $f_1 \times f_2 : X_{(1)} \times X_{(2)} \rightarrow Y \times Y$ is transverse (Definition 1.5.8) to the diagonal $\Delta = \{(y, y) \mid y \in Y\} \subset Y \times Y$, and denote the fiber product of object spaces by

$$X_{(1)} \times_{f_1 \times f_2} X_{(2)} := \{(x_1, x_2) \in X_{(1)} \times X_{(2)} \mid f_1(x_1) = f_2(x_2)\} \subset X_{(1)} \times X_{(2)}.$$

Then, there exists an open neighborhood

$$\tilde{X} \subset (X_{(1)} \times_{f_1} X_{(2)}) \cap ((X_{(1)})_1 \times (X_{(2)})_1)$$

of $(X_{(1)} \times_{f_1} X_{(2)}) \cap ((X_{(1)})_\infty \times (X_{(2)})_\infty)$ such that $\tilde{X}$ is a slice of $X_{(1)}^1 \times X_{(2)}^1$ in the M -polyfold sense (Definition 1.5.7) satisfying $\text{codim}_x(\tilde{X} \subset X_{(1)}^1 \times X_{(2)}^1) = \dim Y$ for every $x \in \tilde{X}_1 = \tilde{X} \cap ((X_{(1)})_2 \times (X_{(2)})_2)$. In particular, the full subcategory $\tilde{\mathcal{X}} := (\tilde{X}, \tilde{\mathbf{X}})$ of $\mathcal{X}_{(1)}^1 \times \mathcal{X}_{(2)}^1$ with object space $\tilde{X}$ is a tame ep-groupoid with degeneracy index satisfying $d_{\tilde{X}}(x_1, x_2) = d_{X_{(1)}}(x_1) + d_{X_{(2)}}(x_2)$ for all $(x_1, x_2) \in \tilde{X}_\infty$.

(II) Consider, in addition, tame strong bundles $(P_i : E_{(i)} \rightarrow X_{(i)}, \mu_i)$ over $\mathcal{X}_{(i)}$ for $i = 1, 2$. Then, there exists a possibly smaller neighborhood $\tilde{X}$ in (I) that is a slice of the bundle $P_1|_{E_{(1)}^1} \times P_2|_{E_{(2)}^1} : E_{(1)}^1 \times E_{(2)}^1 \rightarrow X_{(1)}^1 \times X_{(2)}^1$ in the M -polyfold sense (Definition 1.5.7). In particular, the tuple $(\tilde{P}, \tilde{\mu})$ consisting of the restrictions

$$\begin{aligned} \tilde{E} &:= (P_1|_{E_{(1)}^1} \times P_2|_{E_{(2)}^1})^{-1}(\tilde{X}) \\ \tilde{P} &:= (P_1 \times P_2)|_{\tilde{E}} : \tilde{E} \rightarrow \tilde{X}, \\ \tilde{\mu} &:= (\mu_1 \times \mu_2) \circ l|_{\tilde{\mathbf{X}}_{s_1 \times s_2} \times_{\tilde{P}} \tilde{E}} : \tilde{\mathbf{X}}_{s_1 \times s_2} \times_{\tilde{P}} \tilde{E} \rightarrow \tilde{E}, \end{aligned}$$

is a tame strong bundle over $\tilde{\mathcal{X}}$, where $l : (\mathbf{X}_{(1)} \times \mathbf{X}_{(2)}) \times (E_{(1)} \times E_{(2)}) \rightarrow (\mathbf{X}_{(1)} \times E_{(1)}) \times (\mathbf{X}_{(2)} \times E_{(2)})$ is the reordering of factors map.

(III) Consider, in addition, tame sc-Fredholm section functors (Definition 1.6.6) $\sigma_i : X_{(i)} \rightarrow E_{(i)}$ of (P_i, μ_i) for $i = 1, 2$. Assume that $f_1 \times f_2$ is $(\sigma_1 \times \sigma_2)$ -compatibly transverse to Δ (Definition 1.5.8).

Then, there exists a possibly smaller neighborhood $\tilde{X}$ in (II) that is a slice of the tame sc-Fredholm section $(\sigma_1 \times \sigma_2)|_{X_{(1)}^1 \times X_{(2)}^1} : X_{(1)}^1 \times X_{(2)}^1 \rightarrow E_{(1)}^1 \times E_{(2)}^1$ in the M -polyfold sense (Definition 1.5.7). In particular, the restriction

$$\tilde{\sigma} := (\sigma_1 \times \sigma_2)|_{\tilde{X}} : \tilde{X} \rightarrow \tilde{E}$$

is a tame sc-Fredholm section functor of the bundle $(\tilde{P}, \tilde{\mu})$ with index satisfying

$$\text{ind}_{(x_1, x_2)}(\tilde{\sigma}) = \text{ind}_{x_1}(\sigma_1) + \text{ind}_{x_2}(\sigma_2) - \dim Y$$

for all $(x_1, x_2) \in \tilde{X}_\infty$. If $|\sigma_i^{-1}(0)|$ is compact for $i = 1, 2$, then $|\tilde{\sigma}^{-1}(0)|$ is compact.

Proof. We prove the statements in (I). Lemma 1.7.2(I) provides a tame ep-groupoid structure on $\mathcal{X}_{(1)} \times \mathcal{X}_{(2)}$ with degeneracy index satisfying $d_{X_{(1)} \times X_{(2)}}(x_1, x_2) = d_{X_{(1)}}(x_1) + d_{X_{(2)}}(x_2)$ for $(x_1, x_2) \in (X_{(1)})_\infty \times (X_{(2)})_\infty$. We claim that Corollary 1.6.8(I) applies to the product map

$f_1 \times f_2 : X_{(1)} \times X_{(2)} \rightarrow Y \times Y$ and the codimension $\dim Y$ submanifold $\Delta \subset Y \times Y$. Indeed, $f_1 \times f_2$ is transverse to Δ by hypothesis and the required morphism compatibility $(f_1 \times f_2) \circ (s_1 \times s_2)(\phi_1, \phi_2) = (f_1 \times f_2) \circ (t_1 \times t_2)(\phi_1, \phi_2)$ holds by the hypothesis $f_i(s_i(\phi_i)) = f_i(t_i(\phi_i))$. Since the fiber product is the preimage of the diagonal

$$X_{(1)} \times_{f_1 \times f_2} X_{(2)} = (f_1 \times f_2)^{-1}(\Delta),$$

the result of Corollary 1.6.8(I) is exactly the assertions in (I).

Similarly, to prove (II), we note that Lemma 1.7.2(II) provides a tame strong bundle structure on $(P_1 \times P_2, (\mu_1 \times \mu_2) \circ l)$, and then Corollary 1.6.8(II) provides the desired result.

We now prove (III). Lemma 1.7.2(III) shows that $\sigma_1 \times \sigma_2 : X_{(1)} \times X_{(2)} \rightarrow E_{(1)} \times E_{(2)}$ is a tame sc-Fredholm section functor of $(P_1 \times P_2, (\mu_1 \times \mu_2) \circ l)$ with index satisfying $\text{ind}_{(x_1, x_2)}(\sigma_1 \times \sigma_2) = \text{ind}_{x_1}(\sigma_1) + \text{ind}_{x_2}(\sigma_2)$ and with $|\sigma_1^{-1}(0)| \times |\sigma_2^{-1}(0)|$ compact. Then by Corollary 1.6.8(III), we conclude that $\tilde{\sigma}$ is a tame sc-Fredholm section functor with index satisfying $\text{ind}_{(x_1, x_2)}(\tilde{\sigma}) = \text{ind}_{(x_1, x_2)}(\sigma_1 \times \sigma_2) - \dim Y = \text{ind}_{x_1}(\sigma_1) + \text{ind}_{x_2}(\sigma_2) - \dim Y$ and such that $|\tilde{\sigma}^{-1}(0)|$ is compact. $\square$

Chapter 2

A polyfold proof of the Arnold conjecture

This chapter is a reproduction of the paper [16], co-authored with Katrin Wehrheim.

Abstract:

We give a detailed proof of the homological Arnold conjecture for nondegenerate periodic Hamiltonians on general closed symplectic manifolds M via a direct Piunikhin-Salamon-Schwarz morphism. Our constructions are based on a coherent polyfold description for moduli spaces of pseudoholomorphic curves in a family of symplectic manifolds degenerating from $\mathbb{CP}^1 \times M$ to $\mathbb{C}^+ \times M$ and $\mathbb{C}^- \times M$, as developed by Fish-Hofer-Wysocki-Zehnder as part of the Symplectic Field Theory package. To make the paper self-contained we include all polyfold assumptions, describe the coherent perturbation iteration in detail, and prove an abstract regularization theorem for moduli spaces with evaluation maps relative to a countable collection of submanifolds.

The 2011 sketch of this proof was joint work with Peter Albers, Joel Fish.

2.1 Introduction

Let (M, ω) be a closed symplectic manifold and $H : S^1 \times M \rightarrow \mathbb{R}$ a periodic Hamiltonian function. It induces a time-dependent Hamiltonian vector field $X_H : S^1 \times M \rightarrow TM$ given by $\omega(X_H(t, x), \cdot) = dH(t, \cdot)$. We denote the set of contractible periodic orbits by

$$\mathcal{P}(H) := \{ \gamma : S^1 \rightarrow M \mid \dot{\gamma}(t) = X_H(t, \gamma(t)) \text{ and } \gamma \text{ is contractible} \} \quad (2.1)$$

and note that periodic orbits can be identified with the fixed points of the time 1 flow $\phi_H^1 : M \rightarrow M$ of X_H . We call this Hamiltonian system nondegenerate if $\phi_H^1 \times \text{id}_M$ is transverse to the diagonal and hence cuts out the fixed points transversely. In particular, this guarantees a finite set of periodic orbits. Arnold [1] conjectured in the 1960s that the minimal number of critical points of a Morse function on M is also a lower bound for the number of periodic orbits of a nondegenerate Hamiltonian system as above. In this strict form, the Arnold conjecture has been confirmed for Riemann surfaces [12] and tori [10]. A weaker form is accessible by Floer theory, introduced by Floer [21, 22] in the 1980s. It constructs a chain complex generated by $\mathcal{P}(H)$ that can be compared with the Morse complex generated by the critical points of a Morse function. When Floer homology is well-defined, it is usually independent of the Hamiltonian, and on a compact symplectic manifold can in fact be identified with Morse homology, which is also independent of the Morse function and computes the singular homology. Using this approach, the following nondegenerate homological form of the Arnold conjecture was first proven by Floer [20, 23] in the absence of pseudoholomorphic spheres.

Theorem 2.1.1. *Let (M, ω) be a closed symplectic manifold and $H : S^1 \times M \rightarrow \mathbb{R}$ a nondegenerate periodic Hamiltonian function. Then*

$$\#\mathcal{P}(H) \geq \sum_{i=0}^{\dim M} \dim H_i(M; \mathbb{Q}).$$

Floer's proof was later extended to general closed symplectic manifolds [27, 43, 24, 38], and in the presence of pseudoholomorphic spheres of negative Chern number requires abstract regularizations of the moduli spaces of Floer trajectories since perturbations of the geometric structures may not yield regular moduli spaces; see e.g. [41]. Further generalizations and alternative proofs have been published in the meantime, using a variety of regularization methods. The purpose of this note is to provide a general and maximally accessible proof of Theorem 2.1.1 – using an abstract perturbation scheme provided by the polyfold theory of Hofer-Wysocki-Zehnder [34], following an approach by Piunikhin-Salamon-Schwarz [44] based on [49], and building on polyfold descriptions of Gromov-Witten moduli spaces [33] as well as their degenerations in Symplectic Field Theory [13, 19].

To be more precise, let $CF = \bigoplus_{\gamma \in \mathcal{P}(H)} \Lambda \langle \gamma \rangle$ be the Floer chain group of the Hamiltonian H with coefficients in the Novikov field Λ (see §2.2). Let (CM, d) be the Morse complex with coefficients in Λ associated to a Morse function $f : M \rightarrow \mathbb{R}$ and a suitable metric on M (see §2.3). We obtain a proof of Theorem 2.1.1 as consequence of Lemma 2.4.9, Definition 2.5.7, Lemma 2.6.4, 2.6.5, 2.6.6, summarized here.

Theorem 2.1.2. *There exist Λ -linear maps $PSS : CM \rightarrow CF$, $SSP : CF \rightarrow CM$, $\iota : CM \rightarrow CM$, and $h : CM \rightarrow CM$ such that the following holds.*

- (i) ι is a chain map, that is $\iota \circ d = d \circ \iota$.
- (ii) ι is a Λ -module isomorphism.
- (iii) h is a chain homotopy between $SSP \circ PSS$ and ι , that is $\iota - SSP \circ PSS = d \circ h + h \circ d$.

Here we view the Floer chain group CF as a vector space over Λ – not as a chain complex, and in particular do not consider a Floer differential. Thus we are neither constructing a Floer homology for H , nor identifying it with the Morse homology of f . However, the algebraic structures in Theorem 2.1.2 suffice to deduce the homological Arnold conjecture for the Hamiltonian H as follows.

Proof of Theorem 2.1.1. Denote the sum of the Betti numbers $k := \sum_{i=0}^{\dim M} \dim H_i(M; \mathbb{Q})$. Let $(CM_{\mathbb{Q}}, d_{\mathbb{Q}})$ be the Morse complex over $\mathbb{Q}$ as defined in §2.3. Then by the isomorphism of singular and Morse homology there exist $c_1, \dots, c_k \in CM_{\mathbb{Q}}$ that are cycles, $d_{\mathbb{Q}} c_i = 0$, and linearly independent in the Morse homology over $\mathbb{Q}$. Since the Morse differential $d : CM \rightarrow CM$ is given by Λ -linear extension of $d_{\mathbb{Q}}$ from $CM_{\mathbb{Q}} \subset CM$ the chains $c_1, \dots, c_k \in CM$ are also cycles $dc_i = d_{\mathbb{Q}} c_i = 0$ and linearly independent in the Morse homology over Λ . By Theorem 2.1.2 (i), (ii), ι induces an isomorphism $H\iota : HM \rightarrow HM$ on homology. This in particular implies that $[\iota(c_1)], \dots, [\iota(c_k)] \in HM$ are also linearly independent in homology, that is for any $\lambda_1, \dots, \lambda_k \in \Lambda$ we have

$$\sum_{i=1}^k \lambda_i \cdot \iota(c_i) \in \text{im } d \implies \lambda_1 = \dots = \lambda_k = 0. \quad (2.2)$$

We now show that $PSS(c_1), \dots, PSS(c_k) \in CF$ are Λ -linearly independent, proving $\#\mathcal{P}(H) \geq k$ since the elements of $\mathcal{P}(H)$ generate CF by definition. This proves the theorem.

Let $\lambda_1, \dots, \lambda_k \in \Lambda$ be a tuple such that

$$\sum_{i=0}^k \lambda_i \cdot PSS(c_i) = 0.$$

Then we deduce from Theorem 2.1.2 (iii) that

$$\begin{aligned} \sum_{i=0}^k \lambda_i \cdot \iota(c_i) &= \sum_{i=0}^k \lambda_i \cdot (SSP(PSS(c_i)) + dh(c_i) + h(dc_i)) \\ &= SSP\left(\sum_{i=0}^k \lambda_i \cdot PSS(c_i)\right) + \sum_{i=0}^k \lambda_i \cdot dh(c_i) = d\left(\sum_{i=0}^k \lambda_i \cdot h(c_i)\right), \end{aligned}$$

which implies $\lambda_1 = \dots = \lambda_k = 0$ by (2.2). $\square$

This algebraically minimalistic approach of deducing the homological Arnold conjecture from the existence of maps PSS and SSP whose composition is chain homotopic to an isomorphism on the Morse complex was developed in 2011 discussions of the second author, Peter Albers, and Joel Fish with Mohammed Abouzaid and Thomas Kragh. These were prompted by our observation that any rigorous proof of “Floer homology equals Morse homology” seemed to require equivariant transversality; see Remark 2.1.3. Since equivariant transversality is generally obstructed – even for simple equivariant sections of finite rank bundles – we were looking for a proof that requires the least amount of specific geometric insights or new abstract tools. We ultimately expect the [44]-approach to yield an isomorphism between Floer and Morse homology, as well as generalizations of spectral invariants [50] to all closed symplectic manifolds, but the purpose of the present manuscript is to exemplify the use of existing polyfold results to obtain detailed, rigorous, and transparent proofs.

For that purpose we include Appendix 2.7, which summarizes all necessary polyfold notions and abstract tools. Here we moreover establish in Theorem 2.7.9 a relative perturbation result that should be of independent interest: It allows to bring moduli spaces with an evaluation map into general position to a countable collection of submanifolds. Besides these 10 pages of background, there are a few more technical complications due to the current lack of polyfold publications: Since the polyfold description of Floer-theoretic moduli spaces – while evident to experts – is not published apart from an outline in [52], we reformulate all moduli spaces in our application into SFT moduli spaces, using the fact that graphs of perturbed pseudoholomorphic maps are J -curves for an appropriate J . Since the polyfold description of SFT moduli spaces [19, 18] is also not completely published yet, we summarize the anticipated results in Assumptions 2.4.3, 2.5.5, 2.6.3. Finally, we give a detailed account of the iterative construction of coherent perturbations in the proofs of Lemma 2.6.4 and 2.6.6. To strike a balance between supplying technical details that are not easily available in the literature and maximal accessibility, we have clearly labeled all such technical work. Readers willing to view polyfold theory as a black box can save 20 pages by skipping these parts.

For accessibility we begin with reviews of the pertinent facts on the Novikov field, §2.2, and Morse trajectories, §2.3. The proof of Theorem 2.1.2 then proceeds by constructing the PSS and SSP maps in §2.4 from curves in $\mathbb{C}^\pm \times M$, constructing the isomorphism ι and

chain homotopy h in §2.5 from curves in $\mathbb{CP}^1 \times M$ and its degeneration into $\mathbb{C}^- \times M$ and $\mathbb{C}^+ \times M$, and proving their algebraic relations in §2.6 by constructing coherent perturbations.

Remark 2.1.3. (i) There are essentially two approaches to the general Arnold conjecture as stated in Theorem 2.1.1. The first – developed by [23] and used verbatim in [27, 43, 24, 38] – is to establish the independence of Floer homology from the Hamiltonian function, and to identify the Floer complex for a $\mathcal{C}^2$ -small S^1 -invariant Hamiltonian $H : M \rightarrow \mathbb{R}$ with the Morse complex for H . This requires S^1 -equivariant transversality to argue that isolated Floer trajectories must be S^1 -invariant, hence Morse trajectories. A conceptually transparent construction of equivariant and transverse perturbations – under transversality assumptions at the fixed point set which are met in this setting – can be found in [55], assuming a polyfold description of Floer trajectories.

(ii) The second approach to Theorem 2.1.1 by [44] is to construct a direct isomorphism between the Floer homology of the given Hamiltonian and the Morse homology for some unrelated Morse function. Two chain maps $PSS : CM \rightarrow CF$, $SSP : CF \rightarrow CM$ between the Morse and Floer complexes are constructed from moduli spaces of once punctured perturbed holomorphic spheres with one marking evaluating to the unstable resp. stable manifold of a Morse critical point, and with the given Hamiltonian perturbation of the Cauchy-Riemann operator on a cylindrical neighborhood of the puncture. Then gluing and degeneration arguments are used to argue that both $PSS \circ SSP$ and $SSP \circ PSS$ are chain homotopic to the identity, and hence SSP is the inverse of PSS on homology. However, sphere bubbling can obstruct these arguments: In the first chain homotopy it creates an ambiguity in the choice of nodal gluing when the intermediate Morse trajectory shrinks to zero length. (We expect to be able to avoid this by arguing that “index 1 solutions generically avoid codimension 2 strata” – another classical fact in differential geometry that should generalize to polyfold theory.) The second chain homotopy is as claimed in Theorem 2.1.2 (iii) but with $\iota = \text{id}$, which requires arguing that the only isolated holomorphic spheres with two marked points evaluating to an unstable and stable manifold are constant. This again requires S^1 -equivariant transversality (which we expect to be able to achieve with the techniques in [55]).

(iii) Theorem 2.1.2 is proven by following the [44]-approach as above but avoiding the use of new polyfold technology such as equivariant or strata-avoiding perturbations. In particular, ι is the map that results from counting holomorphic spheres that intersect an unstable and stable manifold; its invertibility is deduced from an “upper triangular” argument.

(iv) The techniques in this paper – combining existing perturbation technology with the polyfold descriptions of SFT moduli spaces – would also allow to define the Floer differential, prove $d^2 = 0$, establish independence of Floer homology from the Hamiltonian (and other geometric data), and prove that PSS and SSP are chain maps. Then the chain homotopy between $SSP \circ PSS$ and the isomorphism ι implies that PSS is injective and SSP surjective on homology. However, proving that PSS and SSP are isomorphisms on homology, or directly identifying the Floer complex of a small S^1 -invariant Hamiltonian with its Morse complex, requires the techniques discussed in (ii).

Moreover, a proof of independence of Floer homology from the choice of abstract perturbation would require a study of the algebraic consequences of self-gluing Floer trajectories in expected dimension -1 during a homotopy of perturbations, as developed in the A_∞ -setting in [37].

We thank Peter Albers and Joel Fish for helping develop the outline of this project – and Edi Zehnder for asking the initial question. The project was further supported by various discussions with Mohammed Abouzaid, Helmut Hofer, Thomas Kragh, Kris Wysocki, and Zhengyi Zhou.

2.2 The Novikov field

We use the following Novikov field Λ associated to the symplectic manifold (M, ω) . Let $H_2(M)$ denote integral homology and consider the map $\omega : H_2(M) \rightarrow \mathbb{R}$ given by the pairing $\omega(A) := \langle \omega, A \rangle$ for $A \in H_2(M)$. The image of this pairing is a finitely generated additive subgroup of the real numbers denoted

$$\Gamma := \text{im } \omega = \omega(H_2(M)) \subset \mathbb{R}.$$

The *Novikov field* Λ is the set of formal sums

$$\lambda = \sum_{r \in \Gamma} \lambda_r T^r,$$

where T is a formal variable, with rational coefficients $\lambda_r \in \mathbb{Q}$ which satisfy the finiteness condition

$$\forall c \in \mathbb{R} \quad \#\{r \in \Gamma \mid \lambda_r \neq 0, r \leq c\} < \infty.$$

The multiplication is given by

$$\lambda \cdot \mu = \left(\sum_{r \in \Gamma} \lambda_r T^r \right) \cdot \left(\sum_{s \in \Gamma} \mu_s T^s \right) := \sum_{t \in \Gamma} \left(\sum_{r+s=t} \lambda_r \mu_s \right) T^t.$$

This defines a field Λ by [27, Thm.4.1] and the discussion preceding the theorem in [27, §4], the key being that Γ is a finitely generated subgroup of $\mathbb{R}$.

We will moreover make use of the following generalization of the invertibility of triangular matrices with nonzero diagonal entries.

Lemma 2.2.1. *Let $M = (\lambda^{ij})_{1 \leq i, j \leq \ell} \in \Lambda^{\ell \times \ell}$ be a square matrix with entries $\lambda^{ij} \in \Lambda$ in the Novikov field. Suppose that $\lambda^{ij} = \sum_{r \in \Gamma, r \geq 0} \lambda_r^{ij} T^r$ with $\lambda_0^{ij} = 0$ for $i \neq j$ and $\lambda_0^{ii} \neq 0$. Then M is invertible.*

Proof. Since Λ is a field, invertibility of M is equivalent to $\det(M) \neq 0$. Write $\det(M) = \sum_{r \in \Gamma} \mu_r T^r \in \Lambda$ for some $\mu_r \in \mathbb{Q}$. It suffices to show that $\mu_0 \neq 0$.

We proceed by induction on the size of the matrix M . In the $\ell = 1$ base case, when M is a 1×1 matrix $M = [\lambda^{11}]$, we have $\det(M) = \lambda^{11} = \sum_{r \in \Gamma} \mu_r T^r$ with $\mu_r = \lambda_r^{11}$ so $\mu_0 = \lambda_0^{11} \neq 0$.

Now suppose that M is size $\ell \times \ell$ for some $\ell > 1$ and inductively assume that, for any size $(\ell - 1) \times (\ell - 1)$ matrix N satisfying the hypotheses of the lemma, we have $\det(N) = \sum_{r \in \Gamma} \mu_r^N T^r$ with $\mu_0^N \neq 0$. For $1 \leq j \leq \ell$, let C_{1j} denote the matrix obtained by deleting the first row and j -th column of M . Then $N := C_{11}$ is an $(\ell - 1) \times (\ell - 1)$ matrix that satisfies the hypotheses of the lemma, and the cofactor expansion of the determinant yields

$$\det(M) = \lambda^{11} \det(N) + \sum_{j=2}^{\ell} (-1)^{1+j} \lambda^{1j} \det(C_{1j}).$$

By hypothesis, all entries of M are of the form $\lambda^{ij} = \sum_{r \geq 0} \lambda_r^{ij} T^r$. Since the determinants $\det(N)$ and $\det(C_{1j})$ are polynomials of those entries, they are of the same form – with zero coefficients for T^r with $r < 0$. Since we moreover have $\lambda_0^{1j} = 0$ for $j \geq 2$ by hypothesis, it follows that the constant term (i.e. the coefficient on T^0) of $\lambda^{1j} \det(C_{1j})$ is 0. Hence the constant term of $\det(M) = \sum \mu_r T^r$ is $\mu_0 = \lambda_0^{11} \cdot \mu_0^N$, where $\mu_0^N \neq 0$ by induction and $\lambda_0^{11} \neq 0$ by hypothesis. This implies $\det(M) = \mu_0 + \dots \neq 0$ and thus finishes the proof. $\square$

2.3 The Morse complex and half-infinite Morse trajectories

This section reviews the construction of the Morse complex as well as the compactified spaces of half-infinite Morse trajectories which will appear in all our moduli spaces.

2.3.1 Euclidean Morse-Smale pairs

The Morse complex can be constructed for any Morse-Smale pair of function and metric on a closed smooth manifold M (and more general spaces). However, we will also work with half-infinite Morse trajectories, and to obtain natural manifold with boundary and corner structures on these, we will restrict ourselves to the following special setting.

Definition 2.3.1. *A Euclidean Morse-Smale pair on a closed manifold M is a pair (f, g) consisting of a smooth function $f \in C^\infty(M, \mathbb{R})$ and a Riemannian metric g on M satisfying a normal form and transversality condition as follows.*

- (i) *For every critical point $p \in \text{Crit}(f)$ of index $|p| \in \mathbb{N}_0$ there exists a local chart ϕ to a neighborhood of $0 \in \mathbb{R}^n$ such that*

$$\begin{aligned} \phi^* f(x_1, \dots, x_n) &= f(p) - \frac{1}{2}(x_1^2 + \dots + x_{|p|}^2) + \frac{1}{2}(x_{|p|+1}^2 + \dots + x_n^2), \\ \phi^* g &= dx_1 \otimes dx_1 + \dots + dx_n \otimes dx_n. \end{aligned}$$

- (ii) *For every pair of critical points $p, q \in \text{Crit}(f)$ the intersection of unstable and stable manifolds is transverse, $W_p^- \pitchfork W_q^+$.*

Remark 2.3.2. One can check that given any Morse function f and metric g (e.g. one that is Euclidean in Morse normal coordinates around the critical points), any generic perturbation of g on annuli around the critical points yields a Euclidean Morse-Smale pair, see e.g. [4, Prp.2] or [48, Prp.2.24]. Hence Euclidean Morse-Smale pairs exist on every closed manifold, and for any given Morse function.

2.3.2 The Morse complex

For distinct critical points $p_- \neq p_+ \in \text{Crit}(f)$ the space of unbroken Morse trajectories (which are necessarily nonconstant) is

$$\begin{aligned} \mathcal{M}(p_-, p_+) &:= \left\{ \tau : \mathbb{R} \rightarrow M \mid \dot{\tau} = -\nabla f(\tau), \lim_{s \rightarrow \pm\infty} \tau(s) = p_{\pm} \right\} / \mathbb{R} \\ &\cong (W_{p_-}^- \cap W_{p_+}^+) / \mathbb{R} \cong W_{p_-}^- \cap W_{p_+}^+ \cap f^{-1}(c). \end{aligned} \quad (2.3)$$

It is canonically identified with the intersection of unstable and stable manifold modulo the $\mathbb{R}$ -action given by the flow of $-\nabla f$, or their intersection with a level set for any regular value $c \in (f(p_+), f(p_-))$. Both formulations equip it with a canonical smooth structure of dimension $|p_-| - |p_+| - 1$, see e.g. [48, §2.4.1]. Moreover, any choice of orientation of the unstable manifolds W_p^- for all $p \in \text{Crit}(f)$ induces orientations on the trajectory spaces $\mathcal{M}(p_-, p_+)$ by e.g. [51, §3.4]. Then the Morse chain complex of (f, g) is obtained by counting (with signs induced by the orientations) the zero dimensional spaces of unbroken trajectories,

$$CM_{\mathbb{Q}} := \bigoplus_{p \in \text{Crit}(f)} \mathbb{Q}\langle p \rangle, \quad d_{\mathbb{Q}} \langle p_- \rangle := \sum_{|p_+| = |p_-| - 1} \# \mathcal{M}(p_-, p_+) \langle p_+ \rangle. \quad (2.4)$$

It computes the singular homology of M ; see e.g. [48, §4.3]. More precisely, the Morse complex is graded $CM_{\mathbb{Q}} = \bigoplus_{i=0, \dots, \dim M} C_i M$ by Morse indices $C_i M = \bigoplus_{|p|=i} \mathbb{Q}\langle p \rangle$, and with $d_i := d_{\mathbb{Q}}|_{C_i M}$ we have $H_i(M; \mathbb{Q}) \cong \ker d_i / \text{im } d_{i+1}$.

The PSS and SSP morphisms will be constructed on the Morse complex with coefficients in the Novikov field Λ from Section 2.2,

$$CM = CM_{\Lambda} := CM_{\mathbb{Q}} \otimes \Lambda = \bigoplus_{p \in \text{Crit}(f)} \Lambda \langle p \rangle, \quad (2.5)$$

with differential $d = d_{\Lambda}$ the Λ -linear extension of $d_{\mathbb{Q}}$ (defined as above on generators). This complex is naturally graded with differential of degree 1,

$$C_* M = \bigoplus_{i=0}^{\dim M} C_i M, \quad C_i M = \bigoplus_{|p|=i} \Lambda \langle p \rangle, \quad d : C_i M \rightarrow C_{i-1} M. \quad (2.6)$$

2.3.3 Compactified spaces of Morse trajectories

Our construction of moduli spaces will also make use of the following spaces of half-infinite unbroken Morse trajectories for $p_{\pm} \in \text{Crit}(f)$

$$\begin{aligned} \mathcal{M}(M, p_+) &:= \left\{ \tau : [0, \infty) \rightarrow M \mid \dot{\tau} = -\nabla f(\tau), \lim_{s \rightarrow \infty} \tau(s) = p_+ \right\}, \\ \mathcal{M}(p_-, M) &:= \left\{ \tau : (-\infty, 0] \rightarrow M \mid \dot{\tau} = -\nabla f(\tau), \lim_{s \rightarrow -\infty} \tau(s) = p_- \right\}. \end{aligned}$$

These will be equipped with smooth structures of dimension $\dim \mathcal{M}(M, p_+) = \dim M - |p_+|$ resp. $\dim \mathcal{M}(p_-, M) = |p_-|$ by the evaluation maps

$$\text{ev} : \mathcal{M}(M, p_+) \rightarrow M, \quad \tau \mapsto \tau(0), \quad \text{ev} : \mathcal{M}(p_-, M) \rightarrow M, \quad \tau \mapsto \tau(0),$$

which identify the trajectory spaces with the unstable and stable manifolds $\mathcal{M}(M, p_+) \cong W_{p_+}^+$ resp. $\mathcal{M}(p_-, M) \cong W_{p_-}^-$. Note that these spaces contain constant trajectories at a critical point, $\{\tau \equiv p_+\} \in \mathcal{M}(M, p_+)$ and $\{\tau \equiv p_-\} \in \mathcal{M}(p_-, M)$. To compactify these trajectory spaces in a manner compatible with Morse theory, we cannot simply take the closure of the unstable or stable manifold $W_{p_\pm}^\pm \subset M$, but must add broken trajectories involving the bi-infinite Morse trajectories. The bi-infinite trajectories from (2.3) which appear in such a compactification are always nonconstant, i.e. between distinct critical points $p_- \neq p_+$. So, unlike constant half-infinite length trajectories, our constructions will not involve constant bi-infinite trajectories, and we simplify subsequent notation by setting $\mathcal{M}(p, p) := \emptyset$ for all $p \in \text{Crit}(f)$. With that we first introduce spaces of k -fold broken half- or bi-infinite Morse trajectories for $k \in \mathbb{N}_0$ and $p_\pm \in \text{Crit}(f)$,

$$\begin{aligned} \overline{\mathcal{M}}(M, p_+)_k &:= \bigcup_{p_1, \dots, p_k \in \text{Crit}(f)} \mathcal{M}(M, p_1) \times \mathcal{M}(p_1, p_2) \dots \times \mathcal{M}(p_k, p_+), \\ \overline{\mathcal{M}}(p_-, M)_k &:= \bigcup_{p_1, \dots, p_k \in \text{Crit}(f)} \mathcal{M}(p_-, p_1) \times \mathcal{M}(p_1, p_2) \dots \times \mathcal{M}(p_k, M), \\ \overline{\mathcal{M}}(p_-, p_+)_k &:= \bigcup_{p_1, \dots, p_k \in \text{Crit}(f)} \mathcal{M}(p_-, p_1) \times \mathcal{M}(p_1, p_2) \dots \times \mathcal{M}(p_k, p_+). \end{aligned} \quad (2.7)$$

Now the compactifications of the spaces of half- or bi-infinite Morse trajectories are given by

$$\begin{aligned} \overline{\mathcal{M}}(M, p_+) &:= \bigcup_{k \in \mathbb{N}_0} \overline{\mathcal{M}}(M, p_+)_k, \quad \overline{\mathcal{M}}(p_-, M) := \bigcup_{k \in \mathbb{N}_0} \overline{\mathcal{M}}(p_-, M)_k, \\ \overline{\mathcal{M}}(p_-, p_+) &:= \bigcup_{k \in \mathbb{N}_0} \overline{\mathcal{M}}(p_-, p_+)_k, \end{aligned}$$

with topology given by the Hausdorff distance between the images of the broken or unbroken trajectories. Compactness of these spaces is proven analogously to the bi-infinite Morse trajectory spaces in e.g. [4, Prp.3], using [53, Lemma 3.5]. Moreover, [53, Lemma 3.3] shows that the evaluation maps extend continuously to

$$\begin{aligned} \text{ev} : \overline{\mathcal{M}}(M, p_+) &\rightarrow M, \quad (\tau_0, [\tau_1], \dots, [\tau_k]) \mapsto \tau_0(0), \\ \text{ev} : \overline{\mathcal{M}}(p_-, M) &\rightarrow M, \quad ([\tau_0], \dots, [\tau_{k-1}], \tau_k) \mapsto \tau_k(0). \end{aligned} \quad (2.8)$$

Smooth structures on these spaces are obtained by the following variation of a folk theorem, which is proven in [53], using techniques similar to those of [4] for the bi-infinite trajectory spaces.

Theorem 2.3.3. *Let (f, g) be a Euclidean Morse-Smale pair and $p_\pm \in \text{Crit}(f)$. Then $\overline{\mathcal{M}}(M, p_+)$, $\overline{\mathcal{M}}(p_-, M)$, and $\overline{\mathcal{M}}(p_-, p_+)$ are compact, separable metric spaces and carry the structure of a smooth manifold with corners of dimension $\dim \overline{\mathcal{M}}(M, p_+) = \dim M - |p_+|$, $\dim \overline{\mathcal{M}}(p_-, M) = |p_-|$, and $\dim \overline{\mathcal{M}}(p_-, p_+) = |p_-| - |p_+| - 1$. Their k -th boundary stratum is $\partial_k \overline{\mathcal{M}}(\dots) = \overline{\mathcal{M}}(\dots)_k$. Moreover, the evaluation maps (2.8) are smooth.*

For reference, we recall the definition of a manifold with (boundary and) corners and its strata.

Definition 2.3.4. *A smooth manifold with corners of dimension $n \in \mathbb{N}_0$ is a second countable Hausdorff space M together with a maximal atlas of charts $\phi_i : M \supset U_i \rightarrow V_i \subset [0, \infty)^n$ (i.e. homeomorphisms between open sets such that $\cup_i U_i = M$) whose transition maps are smooth.*

For $k = 0, \dots, n$ the k -th boundary stratum $\partial_k M$ is the set of all $x \in M$ such that for some (and hence every) chart the point $\phi_i(x) \in [0, \infty)^n$ has k components equal to 0.

Remark 2.3.5. (i) To orient the Morse trajectory spaces in Theorem 2.3.3 we fix a choice of orientation on each unstable manifold $W_p^- \cong \mathcal{M}(p, M)$ for $p \in \text{Crit}(f)$, and orient $W_p^+ \cong \mathcal{M}(M, p)$ such that $T_p M = T_p W^- \oplus T_p W^+$ induces the orientation on M given by the symplectic form. This also induces orientations on $\mathcal{M}(p_-, p_+) = W_{p_-}^- \cap W_{p_+}^+ / \mathbb{R}$ that are coherent (by e.g. [51, §3.4]) in the sense that the top strata of the oriented boundaries of the compactified Morse trajectory spaces are products $\partial_1 \overline{\mathcal{M}}(\cdot, \cdot) = \bigcup_{q \in \text{Crit}(f)} o(\cdot, q, \cdot) \mathcal{M}(\cdot, q) \times \mathcal{M}(q, \cdot)$ with universal signs $o(\cdot, q, \cdot) = \pm 1$. We compute the relevant cases: For $\mathcal{M}(M, q) \times \mathcal{M}(q, p_+) \hookrightarrow \partial_1 \overline{\mathcal{M}}(M, p_+)$ with $\dim \mathcal{M}(q, p_+) = 0$ the sign is $o(M, q, p_+) = (-1)^{|p_+|+1}$. Indeed, a point in $\mathcal{M}(q, p_+)$ is positively oriented if $TW_q^- \cong \langle -\nabla f \rangle \times NW_{p_+}^+$. Here we identify $N_{p_+} W_{p_+}^+ \cong T_{p_+} W_{p_+}^-$, and the outer normal direction is represented by ∇f , so that the sign arises from

$$\begin{aligned} TW_{p_+}^- \times TW_{p_+}^+ &\cong TW_q^- \times TW_q^+ \cong \langle -\nabla f \rangle \times TW_{p_+}^- \times TW_q^+ \\ &\cong TW_{p_+}^- \times \langle (-1)^{1+|p_+|} \nabla f \rangle \times TW_q^+ \times T\mathcal{M}(q, p_+). \end{aligned}$$

Similarly, for $\mathcal{M}(p_-, q) \times \mathcal{M}(q, M) \hookrightarrow \partial_1 \overline{\mathcal{M}}(p_-, M)$ with $\dim \mathcal{M}(p_-, q) = 0$ the sign is $o(p_-, q, M) = +1$ since $-\nabla f$ is an outer normal and $TW_{p_-}^- \cong \langle -\nabla f \rangle \times TW_q^-$ when $T\mathcal{M}(p_-, q) = +\{0\}$.

(ii) For computational purposes in §2.6.3 we moreover determine the fiber products of the compactified Morse trajectory spaces of critical points $p_-, p_+ \in \text{Crit}(f)$ with the same Morse index $|p_-| = |p_+|$,

$$\begin{aligned} \overline{\mathcal{M}}(p_-, M)_{\text{ev} \times \text{ev}} \overline{\mathcal{M}}(M, p_+) &= \{(\tau^-, \tau^+) \in \overline{\mathcal{M}}(p_-, M) \times \overline{\mathcal{M}}(M, p_+) \mid \text{ev}(\tau^-) = \text{ev}(\tau^+)\} \\ &= \begin{cases} \emptyset & ; p_- \neq p_+, \\ (\tau^- \equiv p_-, \tau^+ \equiv p_+) & ; p_- = p_+. \end{cases} \end{aligned}$$

To verify this recall that the compactifications $\overline{\mathcal{M}}(p_-, M)$ and $\overline{\mathcal{M}}(M, p_+)$ are constructed in (2.7) via broken flow lines involving bi-infinite Morse trajectories in $\mathcal{M}(p_i, p_{i+1})$, which are (defined to be) nonempty only for $|p_i| > |p_{i+1}|$. So we have $\mathcal{M}(p_-, p_1) \times \dots \times \mathcal{M}(p_k, M) \subset \overline{\mathcal{M}}(p_-, M)$ only for $|p_k| < |p_-|$ and $\mathcal{M}(M, p_1) \times \dots \times \mathcal{M}(p_k, p_+) \subset \overline{\mathcal{M}}(M, p_+)$ only for $|p_1| >$

$|p_+|$, and thus the image of the evaluation maps are contained in unions of unstable/stable manifolds

$$\text{ev}(\overline{\mathcal{M}}(p_-, M)) \subset W_{p_-}^- \cup \bigcup_{|q_-| < |p_-|} W_{q_-}^-, \quad \text{ev}(\overline{\mathcal{M}}(M, p_+)) \subset W_{p_+}^+ \cup \bigcup_{|q_+| > |p_+|} W_{q_+}^+.$$

Since the intersections $W_{q_-}^- \cap W_{q_+}^+$ are transverse by the Morse-Smale condition, they can be nonempty only for $|q_-| + \dim M - |q_+| \geq \dim M$. So this intersection is empty whenever $|q_+| > |q_-|$. Thus for $|q_-| < |p_-| = |p_+| < |q_+|$ in the above images we have empty intersections $W_{q_-}^- \cap W_{q_+}^+ = \emptyset$ as well as $W_{q_-}^- \cap W_{p_+}^+ = \emptyset$ and $W_{p_-}^- \cap W_{q_+}^+ = \emptyset$. This proves $\text{ev}(\overline{\mathcal{M}}(p_-, M)) \cap \text{ev}(\overline{\mathcal{M}}(M, p_+)) = W_{p_-}^- \cap W_{p_+}^+$, and for $p_- \neq p_+$ this intersection is empty by transversality in (2.3). Lastly, for $p_{\pm} = p$ we have $W_p^- \cap W_p^+ = \{p\}$ since gradient flows do not allow for nontrivial self-connecting trajectories. This proves $\overline{\mathcal{M}}(p, M)_{\text{ev}} \times_{\text{ev}} \overline{\mathcal{M}}(M, p) = \{(p, p)\}$.

2.4 The PSS and SSP maps

In this section we construct the PSS and SSP morphisms in Theorem 2.1.2 between Morse and Floer complexes. As in the introduction, we fix a closed symplectic manifold (M, ω) and a smooth function $H : S^1 \times M \rightarrow \mathbb{R}$. This induces a time-dependent Hamiltonian vector field $X_H : S^1 \rightarrow \Gamma(TM)$, which we assume to be nondegenerate. Thus it has a finite set of contractible periodic orbits, denoted by $\mathcal{P}(H)$ as in (2.1). We moreover pick a Morse function $f : M \rightarrow \mathbb{R}$ and denote its – again finite – set of critical points by $\text{Crit}(f)$. Then we will work with the Floer and Morse complexes over the Novikov field from Section 2.2,

$$CF = \bigoplus_{\gamma \in \mathcal{P}(H)} \Lambda \langle \gamma \rangle, \quad CM = \bigoplus_{p \in \text{Crit}(f)} \Lambda \langle p \rangle,$$

and construct the Λ -linear maps $PSS : CM \rightarrow CF$, $SSP : CF \rightarrow CM$ from moduli spaces which we introduce in §2.4.1. We provide these moduli spaces with a compactification and polyfold description in §2.4.2, and in §2.4.3 rigorously construct the PSS/SSP map by using polyfold perturbations to obtain well defined (but still choice dependent) counts of compactified-and-perturbed moduli spaces.

2.4.1 The Piunikhin-Salamon-Schwarz moduli spaces

To construct the moduli spaces, we need to make further choices as follows.

- Let J be an ω -compatible almost complex structure on M .

Then the Cauchy-Riemann operator on maps $u : \Sigma \rightarrow M$ parametrized by a Riemann surface Σ with complex structure j is $\bar{\partial}_J u := \frac{1}{2}(du + J(u) \circ du \circ j) \in \Omega^{0,1}(\Sigma, u^*TM)$.

- Let g be a metric on M such that (f, g) is a Euclidean Morse-Smale pair as in Definition 2.3.1. It exists by Remark 2.3.2.

- Let $\beta : [0, \infty) \rightarrow [0, 1]$ be a smooth cutoff function with $\beta|_{[0,1]} \equiv 0$, $\beta' \geq 0$, and $\beta|_{[e,\infty)} \equiv 1$. Then we define the anti-holomorphic vector-field-valued 1-form $Y_H \in \Omega^{0,1}(\mathbb{C}, \Gamma(TM))$ in polar coordinates

$$Y_H(re^{i\theta}, x) := \frac{1}{2}\beta(r)(JX_H(\theta, x)r^{-1}dr + X_H(\theta, x)d\theta).$$

In the notation of [40, §8.1], we have $Y_H = -(X_{H_\beta})^{0,1}$ given by the anti-holomorphic part of the 1-form with values in Hamiltonian vector fields X_{H_β} which arises from the 1-form with values in smooth functions $H_\beta \in \Omega^1(\mathbb{C}, \mathcal{C}^\infty(M))$ given by $H_\beta(re^{i\theta}) = \beta(r)H(\theta, \cdot)d\theta$.

The vector-field-valued 1-form Y_H encodes the Floer equation on both the positive cylindrical end $\{z \in \mathbb{C} \mid |z| \geq e\} \cong [1, \infty) \times S^1$ and the negative end $\{|z| \geq e\} \cong (-\infty, -1] \times S^1$ (where $\beta \equiv 1$) as follows: The reparametrization $v(s, t) := u(e^{\pm(s+it)})$ of a map $u : \mathbb{C} \rightarrow M$ satisfies the Floer equation $(\partial_s + J\partial_t)v(s, t) = JX_H(t, v(s, t))$ iff $\bar{\partial}_J u(z) = Y_H(z, u(z))$.

- For each $\gamma \in \mathcal{P}(H)$, fix a smooth disk $u_\gamma : D^2 \rightarrow M$ with $u_\gamma|_{\partial D^2}(e^{it}) = \gamma(t)$.

Then for $u : \mathbb{C}^\pm \rightarrow M$ with $\lim_{R \rightarrow \infty} u(Re^{\pm it}) = \gamma(t)$, denote by $u \# u_\gamma : \mathbb{CP}^1 \rightarrow M$ the continuous map given by gluing u to $u_\gamma^\pm$ (where the $\pm$ denotes the orientation of D^2). By abuse of language, we will call $A := [u \# u_\gamma] = (u \# u_\gamma)_*[\mathbb{CP}^1] \in H_2(M)$ the homology class represented by u . Moreover, we denote by $\tilde{u}_\gamma : D^2 \rightarrow D^2 \times M$ the graph of u_γ . Then the graph $\tilde{u} : \mathbb{C} \rightarrow \mathbb{C} \times M, z \mapsto (z, u(z))$ glues with $\tilde{u}_\gamma^\pm$ to a continuous map representing $[\tilde{u} \# \tilde{u}_\gamma] = \tilde{A} := [\mathbb{CP}^1] + A \in H_2(\mathbb{CP}^1 \times M)$, or more precisely $\tilde{A} = [\mathbb{CP}^1] \times [\text{pt}] + [\text{pt}] \times A$. Now the condition $[v \# \tilde{u}_\gamma] = \tilde{A}$ makes sense for other maps $v : \mathbb{C} \rightarrow \mathbb{C} \times M$ with the same asymptotic behaviour, and we say v represents $\tilde{A}$. In fact, we will suppress the notation $\tilde{A}$ and label spaces with A – as this specifies the topological type of v .

Given such choices, the (choice-dependent) morphisms $PSS : CM \rightarrow CF$ and $SSP : CF \rightarrow CM$ will be constructed from the following moduli spaces for critical points $p \in \text{Crit}(f)$, periodic orbits $\gamma \in \mathcal{P}(H)$, and $A \in H_2(M)$,

$$\begin{aligned} \mathcal{M}(p, \gamma; A) &= \{u : \mathbb{C}^+ \rightarrow M \mid u(0) \in W_p^-, \bar{\partial}_J u = Y_H(u), \lim_{R \rightarrow \infty} u(Re^{it}) = \gamma(t), [u \# u_\gamma] = A\}, \\ \mathcal{M}(\gamma, p; A) &= \{u : \mathbb{C}^- \rightarrow M \mid u(0) \in W_p^+, \bar{\partial}_J u = Y_H(u), \lim_{R \rightarrow \infty} u(Re^{-it}) = \gamma(t), [u \# u_\gamma] = A\}. \end{aligned}$$

Each of these moduli spaces can be described as the zero set of a Fredholm section $\bar{\partial}_J - Y_H : \mathcal{B}_\pm \rightarrow \mathcal{E}_\pm$. Here the Banach manifolds $\mathcal{B}_\pm$ are given by a weighted Sobolev closure of the set of smooth maps $u : \mathbb{C}^\pm \rightarrow M$ representing the homology class A with point constraint $u(0) \in W_p^\mp$ and satisfying a decay condition $\lim_{R \rightarrow \infty} u(Re^{\pm it}) = \gamma(t)$, but not necessarily satisfying the perturbed Cauchy-Riemann equation $\bar{\partial}_J u = Y_H(u)$. Then $\bar{\partial}_J - Y_H$ is a Fredholm section of index

$$\begin{aligned} I(p, \gamma; A) &= CZ(\gamma) + 2c_1(A) - \frac{\dim M}{2} + |p|, \\ I(\gamma, p; A) &= -CZ(\gamma) + 2c_1(A) + \frac{\dim M}{2} - |p|, \end{aligned} \tag{2.9}$$

where $CZ(\gamma)$ is the Conley-Zehnder index with respect to a trivialization of u_γ^*TM as in e.g. [49], $c_1(A)$ is the first Chern class of (TM, J) paired with A , and $|p|$ is the Morse index of $p \in \text{Crit}(f)$.

If the moduli spaces were compact oriented manifolds, then we could define PSS (and analogously SSP) by a signed count of the index 0 solutions,

$$PSS\langle p \rangle := \#\mathcal{M}(p, \gamma; A) \cdot T^{\omega(A)}\langle \gamma \rangle,$$

where the sum is over $\gamma \in \mathcal{P}(H)$ and $A \in H_2(M)$ with $I(p, \gamma; A) = 0$. In many cases – if sphere bubbles of negative Chern number can be excluded – this compactness and regularity can be achieved by a geometric perturbation of the equation, e.g. in the choice of almost complex structure. In general, obtaining well defined “counts” of the moduli spaces requires an abstract regularization scheme. We will use polyfold theory to replace “ $\#\mathcal{M}(p, \gamma; A)$ ” by a count of 0-dimensional perturbed moduli spaces. In the presence of sphere bubbles with nontrivial isotropy, the perturbations will be multi-valued, yielding rational counts.

Remark 2.4.1. Compactness, or rather Gromov-compactifications, of the moduli spaces $\mathcal{M}(p, \gamma; A)$ and $\mathcal{M}(\gamma, p; A)$ will result from energy estimates [40, Remark 8.1.7] for solutions of $\bar{\partial}_J u = Y_H(u)$,

$$E(u) := \frac{1}{2} \int_{\mathbb{C}} |du + X_{H_\beta}(u)| \leq \int_{\mathbb{C}} u^* \omega + \|R_{H_\beta}\| \leq \omega([u \# u_\gamma]) + K. \quad (2.10)$$

Here the curvature $R_{H_\beta} \text{dvol}_{\mathbb{C}} = dH_\beta + \frac{1}{2} H_\beta \wedge H_\beta = \beta' H \, dr \wedge d\theta$ has finite Hofer norm

$$\|R_{H_\beta}\| = \int_{\mathbb{C}} (\max R_{H_\beta} - \min R_{H_\beta}) = \int_0^\infty \int_{S^1} |\beta'(r)| (\max_{x \in M} H(\theta, x) - \min_{x \in M} H(\theta, x)) \, d\theta \, dr$$

since β' has compact support in $[1, e]$. Since moreover $\mathcal{P}(H)$ is a finite set, we obtain the above estimate with a finite constant $K := \|R_{H_\beta}\| + \max_{\gamma \in \mathcal{P}(H)} \int_{D^2} u_\gamma^* \omega$. Thus the energy of the perturbed pseudoholomorphic maps in each of our moduli spaces will be bounded since we fix $[u \# u_\gamma] = A$.

Now SFT-compactness [2] asserts that for any $C > 0$ the set of solutions of bounded energy $\{u : \mathbb{C} \rightarrow M \mid \bar{\partial}_J u = Y_H(u), \lim_{R \rightarrow \infty} u(Re^{\pm it}) = \gamma(t), E(u) \leq C\}$ is compact up to breaking and bubbling. This compactness will be stated rigorously in polyfold terms in Assumption 2.5.5 (ii).

2.4.2 Polyfold description of moduli spaces

We will obtain a polyfold description for the moduli spaces in §2.4.1 by a fiber product construction motivated by the natural identifications

$$\mathcal{M}(p, \gamma; A) \cong \mathcal{M}(p, M)_{\text{ev} \times \text{ev}} \mathcal{M}^-(\gamma; A), \quad \mathcal{M}(\gamma, p; A) \cong \mathcal{M}^+(\gamma; A)_{\text{ev} \times \text{ev}} \mathcal{M}(M, p). \quad (2.11)$$

This couples the half-infinite Morse trajectory spaces from §2.3.3 with a space of perturbed pseudoholomorphic maps

$$\mathcal{M}^\pm(\gamma; A) := \{u : \mathbb{C}^\pm \rightarrow M \mid \bar{\partial}_J u = Y_H(u), \lim_{R \rightarrow \infty} u(Re^{\pm it}) = \gamma(t), [u \# u_\gamma] = A\}, \quad (2.12)$$

via the evaluation maps (2.8) and

$$\text{ev} : \mathcal{M}^\pm(\gamma; A) \rightarrow M, \quad u \mapsto u(0). \quad (2.13)$$

More precisely, the general approach to obtaining counts or more general invariants from moduli spaces such as (2.11) is to replace them by compact manifolds – or more general ‘regularizations’ which still carry ‘virtual fundamental classes’. Polyfold theory offers a universal regularization approach after requiring a compactification $\overline{\mathcal{M}}(\dots) \supset \mathcal{M}(\dots)$ of the moduli space and a description of the compact moduli space $\overline{\mathcal{M}}(\dots) = \sigma^{-1}(0)$ as zero set of a sc-Fredholm section $\sigma : \mathcal{B}(\dots) \rightarrow \mathcal{E}(\dots)$ of a strong polyfold bundle. For an introduction to the language [34] used here see Appendix 2.7.

The Morse trajectory spaces are compactified and given a smooth structure in Theorem 2.3.3. The Gromov compactification and perturbation theory for (2.12) will be achieved by identifying these spaces with moduli spaces that appear in Symplectic Field Theory (SFT) as introduced in [13], compactified in [2, 8], and given a polyfold description in [18]. Here we identify $u : \mathbb{C} \rightarrow M$ with the map to its graph $\tilde{u} : \mathbb{C} \rightarrow \mathbb{C} \times M, z \mapsto (z, u(z))$ as in [40, §8.1] to obtain a homeomorphism (in appropriate topologies) $\mathcal{M}^\pm(\gamma; A) \cong \widetilde{\mathcal{M}}_{\text{SFT}}^\pm(\tilde{\gamma}; A)/\text{Aut}(\mathbb{C}^\pm)$ to an SFT moduli space for the symplectic cobordism¹ $\mathbb{C}^\pm \times M$ between $\emptyset$ and $S^1 \times M$. Here $S^1 \times M$ is equipped with the stable Hamiltonian structure $(\pm dt, \omega + dH_t \wedge dt)$ whose Reeb field $\pm \partial_t + X_{H_t}$ has simply covered Reeb orbits² given by the graphs $\tilde{\gamma} : t \mapsto (\pm t, \gamma(t))$ of the 1-periodic orbits $\gamma \in \mathcal{P}(H)$. Moreover, $\text{Aut}(\mathbb{C}^\pm)$ is the action of biholomorphisms $\phi : \mathbb{C} \rightarrow \mathbb{C}$ by reparametrization $v \mapsto v \circ \phi$ on the SFT space for an almost complex structure $\tilde{J}_H^\pm$ on $\mathbb{C}^\pm \times M$ induced by J, X_H , and $j = \pm i$ on $\mathbb{C}^\pm$,

$$\widetilde{\mathcal{M}}_{\text{SFT}}^\pm(\tilde{\gamma}; A) := \{v : \mathbb{C}^\pm \rightarrow \mathbb{C}^\pm \times M \mid \bar{\partial}_{\tilde{J}_H^\pm} v = 0, v(Re^{\pm it}) \sim \tilde{\gamma}_R(t), [v \# \tilde{u}_\gamma] = [\mathbb{CP}^1] + A\}.$$

More precisely, the asymptotic requirement is $d_{\mathbb{C} \times M}(v(Re^{\pm i(t+t_0)}), \tilde{\gamma}_R(t)) \rightarrow 0$ for some $t_0 \in S^1$ as $R \rightarrow \infty$ for the graphs $\tilde{\gamma}_R(t) = (Re^{\pm it}, \gamma(t))$ of the orbit γ parametrized by $S^1 \cong \{|z| = R\} \subset \mathbb{C}^\pm$.

To express the evaluation (2.13) in SFT terms note that a holomorphic map in the given homology class intersects the holomorphic submanifold $\{0\} \times M$ in a unique point³, so we can fix the point $0 \in \mathbb{C}^\pm$ in the domain where this intersection occurs and rewrite the moduli space $\mathcal{M}^\pm(\gamma; A) \cong \{v \in \widetilde{\mathcal{M}}_{\text{SFT}}^\pm(\tilde{\gamma}; A) \mid v(0) \in \{0\} \times M\}/\text{Aut}(\mathbb{C}^\pm, 0)$ with a slicing condition and quotient by the biholomorphisms which fix $0 \in \mathbb{C}^\pm$. Thus we rewrite (2.11) into the

¹For definitions of these notions see [8, §2]. For $\mathbb{C} \times M$ the positive symplectization end is $\mathbb{R}^+ \times S^1 \times M \rightarrow \mathbb{C} \times M, (r, \theta, x) \mapsto (e^{r+i\theta}, x)$. After reversing orientation on $\mathbb{C}$ there is an analogous negative end $\mathbb{R}^- \times S^1 \times M \hookrightarrow \mathbb{C}^- \times M$.

²Here we have implicitly chosen asymptotic markers that fix a parametrization of each Reeb orbit.

³For solutions in $\widetilde{\mathcal{M}}_{\text{SFT}}^\pm(\tilde{\gamma}; A)$ this follows from $\text{pr}_{\mathbb{C}^\pm} \circ v : \mathbb{C}^\pm \rightarrow \mathbb{C}^\pm$ being an entire function with a pole of order 1 at infinity (prescribed by the asymptotics). For $\tilde{J}_H^\pm$ -holomorphic curves in the compactification, it follows from positivity of intersections, see e.g. [9, Prop.7.1].

fiber products over $\mathbb{C}^\pm \times M$

$$\begin{aligned}\mathcal{M}(p, \gamma; A) &\cong \mathcal{M}(p, M)_{\{0\} \times \text{ev} \times \text{ev}^+} \mathcal{M}_{\text{SFT}}^+(\gamma; A), \\ \mathcal{M}(\gamma, p; A) &\cong \mathcal{M}_{\text{SFT}}^-(\gamma; A)_{\text{ev}^- \times \{0\} \times \text{ev}} \mathcal{M}(M, p)\end{aligned}\quad (2.14)$$

using evaluation maps on the SFT moduli space with one marked point

$$\text{ev}^\pm : \mathcal{M}_{\text{SFT}}^\pm(\gamma; A) := \widetilde{\mathcal{M}}_{\text{SFT}}^\pm(\tilde{\gamma}; A) / \text{Aut}(\mathbb{C}^\pm, 0) \rightarrow \mathbb{C}^\pm \times M, \quad [v] \mapsto v(0). \quad (2.15)$$

Now we will obtain a polyfold description of the PSS/SSP moduli spaces (2.14) by the slicing construction of [15] applied to polyfold descriptions of the SFT-moduli spaces $\mathcal{M}_{\text{SFT}}^\pm(\tilde{\gamma}; A)$ (compactified as space of pseudoholomorphic buildings with one marked point). This result is outlined in [19], but to enable a self-contained proof of our results, we formulate it as assumption, where we use

$$\overline{\mathbb{C}}^\pm := \mathbb{C}^\pm \cup S_1 \cong \{z \in \mathbb{C}^\pm \mid |z| \leq 1\}$$

as target factor for a simplified evaluation map, as explained in the following remark.

Remark 2.4.2. Note that the compactified moduli space $\overline{\mathcal{M}}_{\text{SFT}}^\pm(\gamma; A)$ – in view of the noncompact target $\mathbb{C}^\pm \times M$ – contains broken curves $\underline{v} : \Sigma = \mathbb{C}^\pm \sqcup \mathbb{R} \times S^1 \sqcup \dots \sqcup \mathbb{R} \times S^1 \rightarrow \Sigma \times M$. We do not need a precise description of this compactification (beyond the fact that it exists and is cut out by a sc-Fredholm section), but it affects the formulation of the evaluation maps $[\underline{v}, z_0] \mapsto \underline{v}(z_0)$ for a marked point $z_0 \in \Sigma$ that $\underline{v}$ might map to a cylinder factor $\mathbb{R} \times S^1 \times M \subset \Sigma \times M$. We will simplify the resulting sc^∞ evaluation with varying target – being developed in [18] – to a continuous evaluation map $\overline{\text{ev}}^\pm : \overline{\mathcal{M}}_{\text{SFT}}^\pm(\gamma; A) \rightarrow \overline{\mathbb{C}}^\pm$ into the compactified target $\overline{\mathbb{C}}^\pm$.

For that purpose we topologize $\overline{\mathbb{C}}^\pm \cong \{|z| \leq 1\}$ as a disk via a diffeomorphism $\mathbb{C}^\pm \rightarrow \{|z| < 1\}$, $re^{i\theta} \mapsto f(r)e^{i\theta}$ induced by a diffeomorphism $f : [0, \infty) \rightarrow [0, 1)$ that is the identity near 0, and its extension to a homeomorphism $\overline{\mathbb{C}}^\pm \rightarrow \{|z| \leq 1\}$ via $S^1 = \mathbb{R}/2\pi\mathbb{Z} \rightarrow \{|z| = 1\}$, $\theta \mapsto e^{\pm i\theta}$. Then for any marked point $z_0 \in \mathbb{R} \times S^1$ on a cylinder we project the evaluation $\underline{v}(z_0) \in \mathbb{R} \times S^1 \times M$ to $S^1 \times M = \partial \overline{\mathbb{C}}^\pm \times M$ by forgetting the $\mathbb{R}$ -factor. The resulting simplified evaluation map will be unchanged and thus still sc^∞ when restricted to the open subset $(\overline{\text{ev}}^\pm)^{-1}(\mathbb{C}^\pm \times M)$ of the ambient polyfold – as stated in (v) below. This open subset inherits a scale-smooth structure, and still contains some broken curves – just not those on which the marked point leaves the main component. This suffices for our purposes since the fiber product construction uses the evaluation map only in an open set of curves $[\underline{v}, z_0]$ with $\underline{v}(z_0) \approx 0 \in \mathbb{C}^\pm$.

Assumption 2.4.3. *There is a collection of oriented sc-Fredholm sections of strong polyfold bundles $\sigma_{\text{SFT}}^\pm : \mathcal{B}_{\text{SFT}}^\pm(\gamma; A) \rightarrow \mathcal{E}_{\text{SFT}}^\pm(\gamma; A)$ and continuous maps $\overline{\text{ev}}^\pm : \mathcal{B}_{\text{SFT}}^\pm(\gamma; A) \rightarrow \overline{\mathbb{C}}^\pm \times M$, indexed by $\gamma \in \mathcal{P}(H)$ and $A \in H_2(M)$, with the following properties.*

- (i) The sections have Fredholm index $\text{ind}(\sigma_{\text{SFT}}) = CZ(\gamma) + 2c_1(A) + \frac{\dim M}{2} + 2$ on $\mathcal{B}_{\text{SFT}}^+(\gamma; A)$, resp. $\text{ind}(\sigma_{\text{SFT}}) = -CZ(\gamma) + 2c_1(A) + \frac{\dim M}{2} + 2$ on $\mathcal{B}_{\text{SFT}}^-(\gamma; A)$.
- (ii) Each zero set $\overline{\mathcal{M}}_{\text{SFT}}^\pm(\tilde{\gamma}; A) := \sigma_{\text{SFT}}^{-1}(0)$ is compact, and given any $C \in \mathbb{R}$ there are only finitely many $A \in H_2(M)$ with $\omega(A) \leq C$ and nonempty zero set $\sigma_{\text{SFT}}^{-1}(0) \cap \mathcal{B}_{\text{SFT}}^\pm(\gamma; A) \neq \emptyset$.
- (iii) Equivalence classes under reparametrization of $\text{Aut}(\mathbb{C}^\pm, 0)$ of smooth maps $v : \mathbb{C}^\pm \rightarrow \mathbb{C}^\pm \times M$ that satisfy $v(Re^{\pm it}) = (Re^{\pm it}, \gamma(t))$ for sufficiently large $R > 1$ and represent the class $[v \# \tilde{u}_\gamma] = [\mathbb{CP}^1] + A$ form a dense subset $\mathcal{B}_{\text{dense}}^\pm(\gamma; A) \subset \mathcal{B}_{\text{SFT}}^\pm(\gamma; A)$ contained in the interior. On this subset, the section is $\sigma_{\text{SFT}}([v]) = [(v, \bar{\partial}_{\tilde{J}_H^\pm} v)]$ and $\bar{\text{ev}}^\pm([v])$ is evaluation as in (2.15).
- (iv) The intersection of the zero set of σ_{SFT} with the dense subset $\mathcal{B}_{\text{dense}}^\pm(\gamma; A)$ is naturally identified with the SFT moduli space in (2.15), i.e., $\sigma_{\text{SFT}}^{-1}(0) \cap \mathcal{B}_{\text{dense}}^\pm(\gamma; A) \cong \mathcal{M}_{\text{SFT}}^\pm(\tilde{\gamma}; A)$.
- (v) The sections σ_{SFT} have tame sc-Fredholm representatives in the sense of [15, Def.5.4], and the evaluation maps $\bar{\text{ev}}^\pm$ restrict on the open subsets $\mathcal{B}_{\text{SFT}}^{\pm, \mathbb{C}}(\gamma; A) := (\bar{\text{ev}}^\pm)^{-1}(\mathbb{C}^\pm \times M) \subset \mathcal{B}_{\text{SFT}}^\pm(\gamma; A)$ to sc^∞ maps $\text{ev}^\pm : \mathcal{B}_{\text{SFT}}^{\pm, \mathbb{C}}(\gamma; A) \rightarrow \mathbb{C}^\pm \times M$, which are σ_{SFT} -compatibly submersive in the sense of Definition 2.7.4. Finally, this open subset contains the interior, $\partial_0 \mathcal{B}_{\text{SFT}}^\pm(\gamma; A) \subset \mathcal{B}_{\text{SFT}}^{\pm, \mathbb{C}}(\gamma; A)$.

Remark 2.4.4. The properties (iii),(iv) in this assumption are stated only to give readers an intuitive sense of what spaces we are working with. For the specific application in this paper it would be sufficient to assume the existence of sc-Fredholm sections and submersive maps, along with the further sections, maps, and boundary stratifications stated in Assumptions 2.5.5 and 2.6.3. Property (v), which is used to construct fiber products in Lemma 2.4.5, should follow similarly to the explanation given in [15, Ex.5.1] for the Gromov-Witten polyfolds [33].

We also expect the existence of a direct polyfold description of the moduli space (2.12) in terms of a collection of sc-Fredholm sections $\sigma : \mathcal{B}^\pm(\gamma; A) \rightarrow \mathcal{E}^\pm(\gamma; A)$ with the same indices, and submersive sc^∞ maps $\text{ev}^\pm : \mathcal{B}^\pm(\gamma; A) \rightarrow M$ with the following simplified properties.

- (iii') The smooth maps $u : \mathbb{C} \rightarrow M$ which equal $u(Re^{\pm it}) = \gamma(t)$ for sufficiently large $R > 1$ and represent the class A form a dense subset of $\mathcal{B}^\pm(\gamma; A)$ that is contained in the interior. On this subset, the section is $\sigma(u) = \bar{\partial}_J u - Y_H(u)$, and the evaluation is $\text{ev}^\pm(u) = u(0)$.
- (iv') The intersection of $\overline{\mathcal{M}}^\pm(\gamma; A) := \sigma^{-1}(0)$ with the dense subset from (iii') is naturally identified with the moduli space $\mathcal{M}^\pm(\gamma; A)$ in (2.12).

While such a construction should follow from the same construction principles as in [19], there is presently no writeup beyond [52], which proves the Fredholm property in a model case. Alternatively, one could abstractly obtain this construction from restricting the setup in Assumption 2.4.3 to subsets consisting of maps of the form $v(z) = (z, u(z))$. Thus there is no harm in using (iii') and (iv') as intuitive guide for our work with the abstract setup.

Given one or another polyfold description of the naturally identified moduli spaces (2.12) or (2.15) and corresponding evaluation maps, we will now extend the identifications (2.11) or (2.14) to a fiber product construction of polyfolds which will contain these PSS/SSP moduli spaces. For $p \in \text{Crit}(f)$, $\gamma \in \mathcal{P}(H)$, and $A \in H_2(M)$ we define the topological spaces

$$\begin{aligned}\tilde{\mathcal{B}}^+(p, \gamma; A) &:= \{(\underline{\tau}, \underline{v}) \in \overline{\mathcal{M}}(p, M) \times \mathcal{B}_{\text{SFT}}^+(\gamma; A) \mid (0, \text{ev}(\underline{\tau})) = \overline{\text{ev}}^+(\underline{v})\} \\ &= \{(\underline{\tau}, \underline{v}) \in \overline{\mathcal{M}}(p, M) \times \mathcal{B}_{\text{SFT}}^{+, \mathbb{C}}(\gamma; A) \mid (0, \text{ev}(\underline{\tau})) = \text{ev}^+(\underline{v})\}, \\ \tilde{\mathcal{B}}^-(\gamma, p; A) &:= \{(\underline{v}, \underline{\tau}) \in \mathcal{B}_{\text{SFT}}^-(\gamma; A) \times \overline{\mathcal{M}}(M, p) \mid (0, \text{ev}(\underline{\tau})) = \overline{\text{ev}}^-(\underline{v})\} \\ &= \{(\underline{v}, \underline{\tau}) \in \mathcal{B}_{\text{SFT}}^{-, \mathbb{C}}(\gamma; A) \times \overline{\mathcal{M}}(M, p) \mid (0, \text{ev}(\underline{\tau})) = \text{ev}^-(\underline{v})\}.\end{aligned}\tag{2.16}$$

We will use [15] to equip these spaces with natural polyfold structures and show that the pullbacks of the sections σ_{SFT} by the projections to $\mathcal{B}_{\text{SFT}}^\pm(\gamma; A)$ yield sc-Fredholm sections whose zero sets are compactifications of the PSS/SSP moduli spaces. This will require a shift in levels which is of technical nature as each m -level $\mathcal{B}_m \subset \mathcal{B}$ contains the dense “smooth level” $\mathcal{B}_\infty \subset \mathcal{B}_m$, which itself contains the moduli space $\overline{\mathcal{M}} = \sigma^{-1}(0) \subset \mathcal{B}_\infty$; see Remark 2.7.3.

Lemma 2.4.5. *For any $p \in \text{Crit}(f)$, $\gamma \in \mathcal{P}(H)$, and $A \in H_2(M)$ there exist open subsets $\mathcal{B}^+(p, \gamma; A) \subset \tilde{\mathcal{B}}^+(p, \gamma; A)_1$ and $\mathcal{B}^-(\gamma, p; A) \subset \tilde{\mathcal{B}}^-(\gamma, p; A)_1$ which contain the smooth levels $\tilde{\mathcal{B}}^\pm(\dots; A)_\infty$ of the fiber products (2.16) and inherit natural polyfold structures. The smooth level of their interior is⁴*

$$\begin{aligned}\partial_0 \mathcal{B}^+(p, \gamma; A)_\infty &= \mathcal{M}(p, M)_{\{0\} \times \text{ev} \times \text{ev}^+} \partial_0 \mathcal{B}_{\text{SFT}}^{+, \mathbb{C}}(\gamma; A)_\infty, \\ \partial_0 \mathcal{B}^-(\gamma, p; A)_\infty &= \partial_0 \mathcal{B}_{\text{SFT}}^{-, \mathbb{C}}(\gamma; A)_\infty_{\text{ev}^- \times \{0\} \times \text{ev}} \mathcal{M}(M, p).\end{aligned}$$

Moreover, pullback of the sc-Fredholm sections of strong polyfold bundles $\sigma_{\text{SFT}}^\pm : \mathcal{B}_{\text{SFT}}^\pm(\gamma; A) \rightarrow \mathcal{E}_{\text{SFT}}^\pm(\gamma; A)$ under the projection $\mathcal{B}^\pm(\dots; A) \rightarrow \mathcal{B}_{\text{SFT}}^\pm(\dots; A)$ induces sc-Fredholm sections of strong polyfold bundles $\sigma_{(\gamma, p; A)}^+ : \mathcal{B}^+(\gamma, p; A) \rightarrow \mathcal{E}^+(\gamma, p; A)$ resp. $\sigma_{(p, \gamma; A)}^- : \mathcal{B}^-(p, \gamma; A) \rightarrow \mathcal{E}^-(p, \gamma; A)$ of index $I(p, \gamma; A)$ resp. $I(\gamma, p; A)$ given in (2.9). Their zero sets contain⁵ the moduli spaces from §2.4.1,

$$\begin{aligned}\sigma_{(p, \gamma; A)}^{+,-1}(0) &= \overline{\mathcal{M}}(p, M)_{\{0\} \times \text{ev} \times \overline{\text{ev}}^+} \sigma_{\text{SFT}}^{+,-1}(0) \supset \mathcal{M}(p, \gamma; A), \\ \sigma_{(\gamma, p; A)}^{-,-1}(0) &= \sigma_{\text{SFT}}^{-,-1}(0)_{\overline{\text{ev}}^- \times \{0\} \times \text{ev}} \overline{\mathcal{M}}(M, p) \supset \mathcal{M}(\gamma, p; A).\end{aligned}$$

Finally, each zero set $\sigma_{(\dots; A)}^{\pm,-1}(0)$ is compact, and given any $p \in \text{Crit}(f)$, $\gamma \in \mathcal{P}(H)$, and $C \in \mathbb{R}$, there are only finitely many $A \in H_2(M)$ with $\omega(A) \leq C$ and nonempty zero set $\sigma_{(\dots; A)}^{\pm,-1}(0) \neq \emptyset$.

⁴ Here we can only make statements about the smooth level because we do not know what points of other levels are included in the fiber products. This is sufficient for applications as the zero set of any sc-Fredholm section (and its admissible perturbations) is contained in the smooth level.

⁵ As in Remark 2.4.4, this identification is stated for intuition and will ultimately not be used in our proofs.

Proof. We will follow [15, Cor.7.3] to construct the PSS polyfold, bundle, and sc-Fredholm section $\sigma_{p,\gamma;A}^+$ in detail, and note that the construction of the SSP section $\sigma_{\gamma,p;A}^-$ is analogous.

Consider an ep-groupoid representative $\mathcal{X} = (X, \mathbf{X})$ of the polyfold $\mathcal{B}_{\text{SFT}}^+(\gamma; A)$ with source and target maps denoted $s, t : \mathbf{X} \rightarrow X$ together with a strong bundle $P : W \rightarrow X$ over the M -polyfold X and a structure map $\mu : \mathbf{X}_s \times_P W \rightarrow X$ such that the pair (P, μ) is a strong bundle over $\mathcal{X}$ representing the polyfold bundle $\mathcal{E}_{\text{SFT}}^+(\gamma; A) \rightarrow \mathcal{B}_{\text{SFT}}^+(\gamma; A)$. In addition, consider a sc-Fredholm section functor $S_{\text{SFT}} : X \rightarrow W$ of (P, μ) that represents σ_{SFT}^+ . The ep-groupoid $\mathcal{X}$ and the bundle (P, μ) are tame, since they represent a tame polyfold and a tame bundle, respectively. Moreover, S_{SFT} is a tame sc-Fredholm section in the sense of [15, Def.5.4] by Assumption 2.4.3(v).

We view the Morse moduli space $\overline{\mathcal{M}}(p, M)$ as the object space of an ep-groupoid with morphism space another copy of $\overline{\mathcal{M}}(p, M)$ and with unit map a diffeomorphism; that is, the only morphisms are the identity morphisms. The unique rank-0 bundle over $\overline{\mathcal{M}}(p, M)$ is a strong bundle in the ep-groupoid sense, and the zero section of this bundle is a tame sc-Fredholm section functor. Next, note that $\tilde{\mathcal{B}}^+(p, \gamma; A) \subset \{(\underline{\tau}, \underline{v}) \in \overline{\mathcal{M}}(p, M) \times |X| \mid \overline{\text{ev}}^+(\underline{v}) \in \{0\} \times M\} \subset \overline{\mathcal{M}}(p, M) \times |X^{\text{ev}}|$ is represented within the open subset $X^{\text{ev}} := (\overline{\text{ev}}^+)^{-1}(\mathbb{C} \times M) \subset X$ and the corresponding full ep-subgroupoid $\mathcal{X}^{\text{ev}}$ of $\mathcal{X}$, which represent the open subset $\mathcal{B}_{\text{SFT}}^{+, \mathbb{C}}(\gamma, A) \subset |X|$, and by Assumption 2.4.3(v) the restricted evaluation $\text{ev}^+ : X^{\text{ev}} \rightarrow \mathbb{C} \times M$ is sc^∞ and S_{SFT} -compatibly submersive (see Definition 2.7.4). Denote by $\text{ev}_0 : \overline{\mathcal{M}}(p, M) \rightarrow \mathbb{C} \times M, \underline{\tau} \mapsto (0, \text{ev}(\underline{\tau}))$ the product of the trivial map to $0 \in \mathbb{C}$ and the Morse evaluation map. We claim that the product map $\text{ev}_0 \times \text{ev}^+ : \overline{\mathcal{M}}(p, M) \times X^{\text{ev}} \rightarrow (\mathbb{C} \times M) \times (\mathbb{C} \times M)$ is S_{SFT} -compatibly transverse to the diagonal $\Delta \subset (\mathbb{C} \times M) \times (\mathbb{C} \times M)$. Indeed, given $(\underline{\tau}, \underline{v}) \in (\text{ev}_0 \times \text{ev}^+)^{-1}(\Delta)$ let $L \subset T_{\underline{v}}^R X^{\text{ev}}$ be a sc-complement of the kernel of the linearization of ev^+ at some $\underline{v} \in X_\infty^{\text{ev}}$ that satisfies the conditions for S_{SFT} -compatible submersivity in Definition 2.7.4 w.r.t. a coordinate change ψ^{ev} on a chart of X^{ev} . Then the subspace $\{0\} \times L \subset T_{\underline{\tau}}^R \overline{\mathcal{M}}(p, M) \times T_{\underline{v}}^R X^{\text{ev}}$ satisfies the conditions for S_{SFT} -compatible transversality of $\text{ev}_0 \times \text{ev}^+$ with Δ at $(\underline{\tau}, \underline{v})$ w.r.t. the product change of coordinates $\text{id} \times \psi^{\text{ev}}$ in a product chart on the Cartesian product $\overline{\mathcal{M}}(p, M) \times X^{\text{ev}}$. (See [15, Lem.7.1] for a discussion of the sc-Fredholm property on Cartesian products.)

Next, note that $\overline{\mathcal{M}}(p, x)_{\text{ev}_0 \times \text{ev}^+} X_\infty^{\text{ev}}$ represents the smooth level of the fiber product topological space $\tilde{\mathcal{B}}^+(p, \gamma; A)$. So [15, Cor.7.3] yields an open subset $X' \subset \overline{\mathcal{M}}(p, M)_{\text{ev}_0 \times \text{ev}^+} X_1^{\text{ev}}$ containing the smooth level $\overline{\mathcal{M}}(p, x)_{\text{ev}_0 \times \text{ev}^+} X_\infty^{\text{ev}}$ such that the full subcategory $\mathcal{X}' := (X', \mathbf{X}')$ of $\overline{\mathcal{M}}(p, M) \times \mathcal{X}_1^{\text{ev}}$ is a tame ep-groupoid and the pullbacks of (P, μ) and S_{SFT} to $\mathcal{X}'$ are a tame bundle and tame sc-Fredholm section. Here we used the fact that the smooth level $\overline{\mathcal{M}}(p, x)_\infty = \overline{\mathcal{M}}(p, x)$ of any finite dimensional manifold is the manifold itself; see Remark 2.7.3.

The tame ep-groupoid $\mathcal{X}'$ yields the claimed polyfold $\mathcal{B}^+(p, \gamma; A) := |\mathcal{X}'|$, and similarly the pullbacks of (P, μ) and S_{SFT} through the projection $X' \rightarrow X_1$ define the claimed bundle and sc-Fredholm section $\sigma_{(p,\gamma;A)}^+ : \mathcal{B}^+(p, \gamma; A) \rightarrow \mathcal{E}^+(p, \gamma; A)$. The identification of the interior $\partial_0 \mathcal{B}^+(p, \gamma; A)_\infty$ follows from the degeneracy index formula $d_{\mathcal{X}'}(x_1, x_2) = d_{\overline{\mathcal{M}}(p,M)}(x_1) + d_{\mathcal{X}}(x_2)$ in [15, Cor.7.3] and the interior of the Morse trajectory spaces $\partial_0 \overline{\mathcal{M}}(p, M) = \mathcal{M}(p, M)$ from

Theorem 2.3.3.

The index formula in [15, Cor.7.3] yields $\text{ind}(\sigma_{(p,\gamma;A)}^+) = \text{ind}(\sigma_{\text{SFT}}) + |p| - \dim(\mathbb{C} \times M) = I(p, \gamma; A)$ since $\dim \overline{\mathcal{M}}(p, M) = |p|$ and $\text{ind}(\sigma_{\text{SFT}}) = CZ(\gamma) + 2c_1(A) + \frac{1}{2} \dim M + 2$.

Finally, the zero set $\sigma_{(p,\gamma;A)}^{+,-1}(0)$ is the fiber product of the zero sets as claimed, as these are contained in the smooth level, and the restriction to $\overline{\text{ev}}^{-1}(\{0\} \times M)$ already restricts considerations to the domain X^{ev} from which the fiber product polyfold is constructed. Moreover, $\sigma_{(p,\gamma;A)}^{+,-1}(0)$ is compact as in [15, Cor.7.3], since both $\overline{\mathcal{M}}(p, M)$ and $\sigma_{\text{SFT}}^{+,-1}(0)$ are compact and both ev_0 and $\overline{\text{ev}}^+$ are continuous. The final statement then follows from Assumption 2.4.3(ii). $\square$

2.4.3 Construction of the morphisms

To construct the Λ -linear maps PSS and SSP in Theorem 2.1.2 with relatively compact notation we index all moduli spaces from §2.4.1 by the two sets

$$\begin{aligned} \mathcal{I}^+ &:= \{ \alpha = (p, \gamma; A) \mid p \in \text{Crit}(f), \gamma \in \mathcal{P}(H), A \in H_2(M) \}, \\ \mathcal{I}^- &:= \{ \alpha = (\gamma, p; A) \mid p \in \text{Crit}(f), \gamma \in \mathcal{P}(H), A \in H_2(M) \}. \end{aligned}$$

To simplify notation we then denote $\mathcal{I} := \mathcal{I}^- \cup \mathcal{I}^+$ and drop the superscripts from the polyfolds $\mathcal{B}(\alpha) = \mathcal{B}^\pm(\alpha)$. Since Lemma 2.4.5 provides each moduli space $\mathcal{M}(\alpha)$ for $\alpha \in \mathcal{I}$ with a compactification and polyfold description $\mathcal{M}(\alpha) \subset \sigma_\alpha^{-1}(0)$, we can apply [34, Theorems 18.2, 18.3, 18.8] to obtain admissible regularizations of the moduli spaces, and counts of the 0-dimensional perturbed solution spaces [34, §15.4], in the following sense. Here we denote by $\mathbb{Q}^+ := \mathbb{Q} \cap [0, \infty)$ the groupoid with only identity morphisms.

Corollary 2.4.6. *(i) For every $\alpha \in \mathcal{I}$, choice of neighborhood of the zero sets $\sigma_\alpha^{-1}(0) \subset \mathcal{V}_\alpha \subset \mathcal{B}(\alpha)$, and choice of sc-Fredholm section functor $S_\alpha : \mathcal{X}_\alpha \rightarrow \mathcal{W}_\alpha$ representing $\sigma_\alpha|_{\mathcal{V}_\alpha}$, there exists a pair $(N_\alpha, \mathcal{U}_\alpha)$ controlling compactness in the sense of Definition 2.7.5 with $|S_\alpha^{-1}(0)| \subset |\mathcal{U}_\alpha| \subset \mathcal{V}_\alpha$.*

For $\alpha \in \mathcal{I}$ with $\sigma_\alpha^{-1}(0) = \emptyset$ we can choose $\mathcal{U}_\alpha = \emptyset$.

(ii) For every collection $(N_\alpha, \mathcal{U}_\alpha)_{\alpha \in \mathcal{I}}$ of pairs controlling compactness, there exists a collection $\underline{\kappa} = (\kappa_\alpha : \mathcal{W}_\alpha \rightarrow \mathbb{Q}^+)_{\alpha \in \mathcal{I}}$ of $(N_\alpha, \mathcal{U}_\alpha)$ -admissible sc^+ -multisections in the sense of [34, Definitions 13.4, 15.5] that are in general position relative to $(S_\alpha)_{\alpha \in \mathcal{I}}$ in the sense that each pair $(S_\alpha, \kappa_\alpha)$ is in general position as per [34, Def.15.6].

Here admissibility in particular implies $\kappa_\alpha \circ S_\alpha|_{\mathcal{X}_\alpha \setminus \mathcal{U}_\alpha} \equiv 0$ and thus $\kappa_\alpha \circ S_\alpha \equiv 0$ when $\sigma_\alpha^{-1}(0) = \emptyset$.

(iii) Every collection $\underline{\kappa}$ of admissible sc^+ -multisections in general position from (ii) induces a collection of compact, tame, branched ep^+ -groupoids $(\kappa_\alpha \circ S_\alpha : \mathcal{X}_\alpha \rightarrow \mathbb{Q}^+)_{\alpha \in \mathcal{I}}$. In particular, each perturbed zero set

$$Z^\kappa(\alpha) := |\{x \in X_\alpha \mid \kappa_\alpha(S_\alpha(x)) > 0\}| \subset |\mathcal{U}_\alpha|_\infty \subset |\mathcal{X}_\alpha|_\infty \cong \mathcal{B}(\alpha)_\infty$$

is compact, contained in the smooth level, and carries the structure of a weighted branched orbifold of dimension $I(\alpha)$ as in (2.9). Moreover, the inclusion in $|\mathcal{U}_\alpha|$ and general position of $\underline{\kappa}$ implies that for $I(\alpha) < 0$ or $\sigma_\alpha^{-1}(0) = \emptyset$ the perturbed zero set $Z^\kappa(\alpha) = \emptyset$ is empty.

(iv) For $\alpha \in \mathcal{I}$ with Fredholm index $I(\alpha) = 0$ and $\kappa_\alpha : \mathcal{W}_\alpha \rightarrow \mathbb{Q}^+$ as in (ii) the perturbed zero set is contained in the interior $Z^\kappa(\alpha) \subset \partial_0 \mathcal{B}(\alpha)_\infty$ and yields a well defined count

$$\#Z^\kappa(\alpha) := \sum_{|x| \in Z^\kappa(\alpha)} o_{\sigma_\alpha}(x) \kappa_\alpha(S_\alpha(x)) \in \mathbb{Q}.$$

Here $o_{\sigma_\alpha}(x) \in \{\pm 1\}$ is determined by the orientation of σ_α as in [34, Thm.6.3]. If $|\mathcal{U}_\alpha| \cap \partial \mathcal{B}(\alpha) = \emptyset$ then this count is independent of the choice of admissible sc^+ -multisection κ_α .

(v) For every $\alpha \in \mathcal{I}$ with Fredholm index $I(\alpha) = 1$ and $\kappa_\alpha : \mathcal{W}_\alpha \rightarrow \mathbb{Q}^+$ as in (ii) the boundary of the perturbed zero set is given by its intersection with the first boundary stratum of the polyfold,

$$\partial Z^\kappa(\alpha) = Z^\kappa(\alpha) \cap \partial_1 \mathcal{B}(\alpha)_\infty.$$

With orientations $o_{\sigma_\alpha|_{\partial \mathcal{B}(\alpha)}}(x) \in \{\pm 1\}$ induced by the boundary restriction $\sigma_\alpha|_{\mathcal{B}(\alpha)}$ this implies

$$\#\partial Z^\kappa(\alpha) = \sum_{|x| \in \partial Z^\kappa(\alpha)} o_{\sigma_\alpha|_{\partial \mathcal{B}(\alpha)}}(x) \kappa_\alpha(S_\alpha(x)) = 0.$$

Remark 2.4.7. (i) The statements in (iv) and (v) of Corollary 2.4.6 require orientations of the sections σ_α for $\alpha \in \mathcal{I}$. By the fiber product construction in Lemma 2.4.5 they do indeed inherit orientations from the orientations of the Morse trajectory spaces in Remark 2.3.5, the orientations of $\sigma_{\text{SFT}}^\pm$ given in Assumption 2.4.3, and an orientation convention for fiber products.

In practice, we will construct the perturbations $\underline{\kappa}$ in Corollary 2.4.6 by pullback of perturbations $\underline{\lambda} = (\lambda_{\gamma,A}^\pm)_{\gamma \in \mathcal{P}, A \in H_2(M)}$ of the oriented SFT-sections $\sigma_{\text{SFT}}^\pm$. Thus it suffices to specify the orientations of the regularized zero sets, which is implicit in their identification with transverse fiber products of oriented spaces over the oriented manifold M ,

$$Z^\kappa(p, \gamma; A) = \overline{\mathcal{M}}(p, M)_{\text{ev}_0 \times \text{ev}^+} Z^\lambda(\gamma; A), \quad Z^\kappa(\gamma, p; A) = Z^\lambda(\gamma; A)_{\text{ev}^- \times \text{ev}_0} \overline{\mathcal{M}}(M, p).$$

Orientations of the boundary restrictions in (v) are then induced by the orientations of $Z^\kappa(\alpha)$, via oriented isomorphisms of the tangent spaces $\mathbb{R}\nu(z) \times T_z \partial Z^\kappa(\alpha) \cong T_z Z^\kappa(\alpha)$, where $\nu(z) \in T_z Z^\kappa(\alpha)$ is an exterior normal vector at $z \in \partial Z^\kappa(\alpha)$.

(ii) Note that the counts in part (iv) of this Corollary may well depend on the choice of the multi-valued perturbations κ_α – unless the ambient polyfold has no boundary, $\partial \mathcal{B}(\alpha) = \emptyset$. Indeed, although the moduli space $\mathcal{M}(\alpha)$ is expected to have dimension 0, it may not be cut out transversely from the ambient polyfold $\mathcal{B}(\alpha)$, and moreover it may not be compact. Assumption 2.4.3 provides an inclusion in a compact set $\mathcal{M}(\alpha) \subset \sigma_\alpha^{-1}(0)$, and the perturbation theory for sc -Fredholm sections of strong bundles then associates to $\sigma_\alpha^{-1}(0)$ a perturbed zero set $Z^\kappa(\alpha) \subset \mathcal{B}(\alpha)$ with weight function $\kappa_\alpha \circ S_\alpha : Z^\kappa(\alpha) \rightarrow \mathbb{Q} \cap (0, \infty)$. This process generally adds points on the boundary $\sigma_\alpha^{-1}(0) \setminus \mathcal{M}(\alpha) \subset \mathcal{B}(\alpha) \setminus \partial_0 \mathcal{B}(\alpha)$, which may or may not persist under variations of the perturbation κ_α .

The following construction of morphisms will depend on the choices of perturbations and orientation convention (see the previous remark) as well as geometric data fixed in §2.4.1, and possibly the choice of polyfold construction in Assumption 2.4.3 and ep-groupoid representation in Remark 2.7.2. The algebraic properties in Theorem 2.1.2 will be achieved in §2.6 – for any given choice of geometric data – by particular choices of ep-groupoids and perturbations $\underline{\kappa}^\pm$, and an overall sign adjustment.

Definition 2.4.8. *Given collections $\underline{\kappa}^\pm = (\kappa_\alpha^\pm)_{\alpha \in \mathcal{I}^\pm}$ of admissible sc^+ -multisections in general position as in Corollary 2.4.6, we define the maps $PSS_{\underline{\kappa}^+} : CM \rightarrow CF$ and $SSP_{\underline{\kappa}^-} : CF \rightarrow CM$ to be the Λ -linear extension of*

$$PSS_{\underline{\kappa}^+} \langle p \rangle := \sum_{\substack{\gamma, A \\ I(p, \gamma; A) = 0}} \#Z^{\underline{\kappa}^+}(p, \gamma; A) \cdot T^{\omega(A)} \langle \gamma \rangle, \quad SSP_{\underline{\kappa}^-} \langle \gamma \rangle := \sum_{\substack{p, A \\ I(\gamma, p; A) = 0}} \#Z^{\underline{\kappa}^-}(\gamma, p; A) \cdot T^{\omega(A)} \langle p \rangle.$$

Lemma 2.4.9. *The maps $PSS_{\underline{\kappa}^+} : CM \rightarrow CF$ and $SSP_{\underline{\kappa}^-} : CF \rightarrow CM$ in Definition 2.4.8 are well defined, i.e. the coefficients take values in the Novikov field Λ defined in §2.2.*

Proof. To prove that $PSS_{\underline{\kappa}^+}$ is well defined we need to check finiteness of the following set for any $p \in \text{Crit}(f)$, $\gamma \in \mathcal{P}(H)$, and $c \in \mathbb{R}$,

$$\left\{ r \in \omega(H_2(M)) \cap (-\infty, c] \mid \sum_{\substack{A \in H_2(M) \\ \omega(A) = r}} \#Z^{\underline{\kappa}^+}(p, \gamma; A) \neq 0 \right\}.$$

Here $\omega : H_2(M) \rightarrow \mathbb{R}$ is given by pairing with the symplectic form on M , and recall from Lemma 2.4.5 that there are only finitely many homology classes $A \in H_2(M)$ with $\omega(A) \leq c$ and $\sigma_\alpha^{-1}(0) \neq \emptyset$. On the other hand, the perturbations $\underline{\kappa}^+$ were chosen in Corollary 2.4.6 (iii), (iv) so that $\#Z^{\underline{\kappa}^+}(\dots; A) = 0$ whenever $\sigma_\alpha^{-1}(0) = \emptyset$. Thus there are in fact only finitely many $A \in H_2(M)$ with $\omega(A) \leq c$ and $\#Z^{\underline{\kappa}^+}(\dots; A) \neq 0$, which proves the required finiteness. The proof for $SSP_{\underline{\kappa}^-}$ is analogous. $\square$

2.5 The chain homotopy maps

In this section we construct Λ -linear maps $\iota : CM \rightarrow CM$ and $h : CM \rightarrow CM$ on the Morse complex over the Novikov field Λ given in (2.5), which appear in Theorem 2.1.2. For that purpose we again fix a choice of geometric data as in §2.4.1 to construct moduli spaces in §2.5.1 and §2.5.2. We equip these with polyfold descriptions in §2.5.3, and define the maps ι, h for admissible regular choices of perturbations in Definitions 2.5.7. To obtain the algebraic properties claimed in Theorem 2.1.2 (i)–(iii) we will then construct particular “coherent” choices of perturbations in §2.6.

2.5.1 Moduli spaces for the isomorphism ι

We will construct $\iota : CM \rightarrow CM$ from the following moduli spaces for critical points $p_-, p_+ \in \text{Crit}(f)$, $A \in H_2(M)$, using the almost complex structure J and the unstable/stable

manifolds (see §2.3.3) of the Morse-Smale pair (f, g) chosen in §2.4.1,

$$\mathcal{M}^{\iota}(p_-, p_+; A) := \{u : \mathbb{CP}^1 \rightarrow M \mid u([1 : 0]) \in W_{p_-}^-, u([0 : 1]) \in W_{p_+}^+, \bar{\partial}_J u = 0, [u] = A\}. \quad (2.17)$$

Note that a cylinder acts on this moduli space by reparametrization with biholomorphisms of $\mathbb{CP}^1$ that fix the two points $[1 : 0], [0 : 1]$. However, we do not quotient out this symmetry so describe these moduli spaces as the zero set of a Fredholm section over a Sobolev closure of the set of smooth maps $u : \mathbb{CP}^1 \rightarrow M$ in the homology class $[u] = A$ satisfying the point constraints $u([1 : 0]) \in W_{p_-}^-$ and $u([0 : 1]) \in W_{p_+}^+$. This determines the Fredholm index as

$$I^{\iota}(p_-, p_+; A) = 2c_1(A) + |p_-| - |p_+|. \quad (2.18)$$

As in §2.4.2 we will obtain a compactification and polyfold description of this moduli space by identifying it with a fiber product of Morse trajectory spaces and a space of pseudoholomorphic curves, in this case the space of parametrized J -holomorphic spheres with evaluation maps for $z_0 \in \mathbb{CP}^1$,

$$\text{ev}_{z_0} : \mathcal{M}(A) := \{u : \mathbb{CP}^1 \rightarrow M \mid \bar{\partial}_J u = 0, [u] = A\} \rightarrow M, \quad u \mapsto u(z_0).$$

With this we can describe the moduli space (2.17) as fiber product with the half-infinite Morse trajectory spaces from §2.3.3, using $z_0^+ := [1 : 0]$ and $z_0^- := [0 : 1]$

$$\mathcal{M}^{\iota}(p_-, p_+; A) \cong \mathcal{M}(p_-, M)_{\text{ev} \times \text{ev}_{z_0^+}} \mathcal{M}(A)_{\text{ev}_{z_0^-} \times \text{ev}} \mathcal{M}(M, p_+). \quad (2.19)$$

Note here that we are not working with a Gromov-Witten moduli space, as we do not quotient by $\text{Aut}(\mathbb{CP}^1)$. This is due to the chain homotopy in Theorem 2.1.2 (iii), which will result from identifying a compactification of $\mathcal{M}(A)$ with a boundary of the neck-stretching moduli space $\mathcal{M}_{\text{SFT}}(A)$ in (2.26) that appears in Symplectic Field Theory [13]. For that purpose we identify a solution $u : \mathbb{CP}^1 \rightarrow M$ with the map to its graph $\tilde{u} : \mathbb{CP}^1 \rightarrow \mathbb{CP}^1 \times M, z \mapsto (z, u(z))$ as in [40, §8.1]. This yields a bijection (and homeomorphism in appropriate topologies)

$$\mathcal{M}(A) \cong \frac{\widetilde{\mathcal{M}}_{\text{GW}}([\mathbb{CP}^1] + A) := \{v : \mathbb{CP}^1 \rightarrow \mathbb{CP}^1 \times M \mid \bar{\partial}_{\tilde{J}} v = 0, [v] = [\mathbb{CP}^1] + A\}}{\text{Aut}(\mathbb{CP}^1)}$$

between the Cauchy-Riemann solution space for M and the Gromov-Witten moduli space for $\mathbb{CP}^1 \times M$ in class $[\mathbb{CP}^1] + A$ for the split almost complex structure $\tilde{J} := i \times J$ on $\mathbb{CP}^1 \times M$. To transfer the evaluation maps at $z_0^+ = [1 : 0]$ and $z_0^- = [0 : 1]$ we keep track of these as (unique) marked points mapping to $\{z_0^{\pm}\} \times M$ and thus replace (2.19) by a fiber product over $\mathbb{CP}^1 \times M$,

$$\mathcal{M}^{\iota}(p_-, p_+; A) \cong \mathcal{M}(p_-, M)_{\{z_0^+\} \times \text{ev} \times \text{ev}^+} \mathcal{M}_{\text{GW}}(A)_{\text{ev}^- \times \{z_0^-\} \times \text{ev}} \mathcal{M}(M, p_+). \quad (2.20)$$

This uses the evaluation maps from a Gromov-Witten moduli space with two marked points,

$$\text{ev}^{\pm} : \mathcal{M}_{\text{GW}}(A) := \widetilde{\mathcal{M}}_{\text{GW}}([\mathbb{CP}^1] + A) / \text{Aut}(\mathbb{CP}^1, z_0^-, z_0^+) \rightarrow \mathbb{CP}^1 \times M, \quad [v] \mapsto v(z_0^{\pm}), \quad (2.21)$$

where $\text{Aut}(\mathbb{CP}^1, z_0^-, z_0^+)$ denotes the set of biholomorphisms $\phi : \mathbb{CP}^1 \rightarrow \mathbb{CP}^1$ which fix $\phi(z_0^\pm) = z_0^\pm$. The polyfold setup in [33, Theorems 1.7, 1.10, 1.11] for Gromov-Witten moduli spaces now provides a strong polyfold bundle $\mathcal{E}_{\text{GW}}(A) \rightarrow \mathcal{B}_{\text{GW}}(A)$, and oriented sc-Fredholm section $\sigma_{\text{GW}} : \mathcal{B}_{\text{GW}}(A) \rightarrow \mathcal{E}_{\text{GW}}(A)$ that cuts out a compactification $\overline{\mathcal{M}}_{\text{GW}}(A) = \sigma_{\text{GW}}^{-1}(0)$ of $\mathcal{M}_{\text{GW}}(A)$. Here a dense subset of the base polyfold $\mathcal{B}_{\text{GW}}(A)$ consists of $\text{Aut}(\mathbb{CP}^1, z_0^-, z_0^+)$ -orbits of smooth maps $v : \mathbb{CP}^1 \rightarrow \mathbb{CP}^1 \times M$ in the homology class $[v] = [\mathbb{CP}^1] + A$, which implicitly carries the two marked points $z_0^\pm \in \mathbb{CP}^1$. Nodal curves in $\overline{\mathcal{M}}_{\text{GW}}(A)$ then explicitly come with the data of two marked points on their domain. On the dense subset the section is given by $\sigma_{\text{GW}}([v]) = [(v, \bar{\partial}_J v)]$. The setup in [33, Theorem 1.8] moreover provides sc^∞ evaluation maps $\text{ev}^\pm : \mathcal{B}_{\text{GW}}(A) \rightarrow \mathbb{CP}^1 \times M$ at the marked points, which on the dense subset are given by $\text{ev}^\pm([v]) = v(z_0^\pm)$.

Thus we have given each factor in the fiber product (2.20) a compactification⁶ that is either a manifold with corners given by the compactified Morse trajectory spaces in Theorem 2.3.3, or the compact zero set $\overline{\mathcal{M}}_{\text{GW}}(A) = \sigma_{\text{GW}}^{-1}(0)$ of a sc-Fredholm section. In §2.5.3 we will combine the polyfold description of the Gromov-compactification of (2.21) with an abstract construction of fiber products in polyfold theory [15] to obtain compactifications and polyfold descriptions of the moduli spaces. Then the construction of $\iota : CM \rightarrow CM$ proceeds as in §2.4.3. The algebraic properties of ι in Theorem 2.1.2 (i) and (ii) will follow from the boundary stratifications of the Morse trajectory spaces $\overline{\mathcal{M}}(p_-, M)$ and $\overline{\mathcal{M}}(M, p_+)$ since the ambient polyfold $\mathcal{B}_{\text{GW}}(A)$ has no boundary. However, this requires specific “coherent” choices of perturbations in §2.6.

Remark 2.5.1. Gromov-compactifications of the moduli spaces $\mathcal{M}^\iota(p_-, p_+; A)$ will result from the energy identity [40, Lemma 2.2.1] for solutions of $\bar{\partial}_J u = 0$,

$$E(u) := \frac{1}{2} \int_{\mathbb{C}} |du|^2 = \int_{\mathbb{CP}^1} u^* \omega = \omega([u]). \quad (2.22)$$

This fixes the energy of solutions on each solution space $\mathcal{M}(A)$, and Gromov compactness asserts that $\{u : \mathbb{CP}^1 \rightarrow M \mid \bar{\partial}_J u = 0, E(u) \leq C\}$ is compact up to bubbling for any $C > 0$.

Another consequence of (2.22) is that for $\omega(A) \leq 0$ we have no solutions $\mathcal{M}(A) = \emptyset$ except for $A = 0 \in H_2(M)$ when the solution space is the space of constant maps

$$\mathcal{M}(0) = \{u \equiv x \mid x \in M\} \simeq M,$$

which is compact and cut out transversely.

Translated to graphs in $\mathbb{CP}^1 \times M$ with two marked points, this means $\overline{\mathcal{M}}_{\text{GW}}(0) \simeq \mathbb{CP}^1 \times \mathbb{CP}^1 \times M$ by adding two marked points in the domain. That is, $(z^-, z^+, x) \in \mathbb{CP}^1 \times \mathbb{CP}^1 \times M$ corresponds to the (equivalence class of) graphs $\tilde{u}_x : z \mapsto (z, x)$ with two marked points $z^-, z^+ \in \mathbb{CP}^1$. For $z^- \neq z^+$ this tuple can be reparametrized to the fixed marked points $z_0^-, z_0^+ \in \mathbb{CP}^1$ and then represents an $\text{Aut}(\mathbb{CP}^1, z_0^-, z_0^+)$ -orbit. For $z^- = z^+$ the tuple (z^-, z^+, x) corresponds to a stable map in $\overline{\mathcal{M}}_{\text{GW}}(0)$, given by the graph $\tilde{u}_x$ with a node

⁶The term ‘compactification’ applied to spaces of pseudoholomorphic curves is always to be understood as Gromov-compactification, as $\mathcal{M}_{\text{GW}}(A) \subset \overline{\mathcal{M}}_{\text{GW}}(A)$ may not be dense.

at $z^- = z^+$ attached to a constant sphere with two distinct marked points. This will be stated in polyfold terms in Assumption 2.5.5 (ii).

2.5.2 Moduli spaces for the chain homotopy h

To construct the moduli spaces from which we will obtain $h : CM \rightarrow CM$, we again use the almost complex structure J and Morse-Smale pair (f, g) chosen in §2.4.1. In addition, we fixed an anti-holomorphic vector-field-valued 1-form $Y_H \in \Omega^{0,1}(\mathbb{C}, \Gamma(TM))$ that arises from the fixed Hamiltonian function $H : S^1 \times M \rightarrow \mathbb{R}$ and a choice of smooth cutoff function $\beta : [0, \infty) \rightarrow [0, 1]$ with $\beta|_{[0,1]} \equiv 0$, $\beta' \geq 0$, and $\beta|_{[e, \infty)} \equiv 1$. Gluing this 1-form to another copy of Y_H over $\mathbb{C}^-$ with neck length $R > 0$ in exponential coordinates yields the anti-holomorphic vector-field-valued 1-form $Y_H^R \in \Omega^{0,1}(\mathbb{CP}^1, \Gamma(TM))$ that vanishes near $[1 : 0], [0 : 1]$ and on $\mathbb{CP}^1 \setminus \{[1 : 0], [0 : 1]\} = \{[1 : re^{i\theta}] \mid (r, \theta) \in (0, \infty) \times S^1\}$ is given by

$$Y_H^R([1 : re^{i\theta}], x) := \frac{1}{2}\beta_R(r)(JX_H(\theta, x)r^{-1}dr + X_H(\theta, x)d\theta).$$

Here $\beta_R(r) := \beta(re^{\frac{R}{2}})\beta(r^{-1}e^{\frac{R}{2}})$ is a smooth cutoff function $\beta_R : (0, \infty) \rightarrow [0, 1]$ that is identical to 1 on $[e^{1-\frac{R}{2}}, e^{\frac{R}{2}-1}]$ and identical to 0 on $(0, e^{-\frac{R}{2}}) \cup (e^{\frac{R}{2}}, \infty)$. Now perturbing the Cauchy-Riemann operator on $\mathbb{CP}^1$ by Y_H^R yields the following moduli spaces for critical points $p_-, p_+ \in \text{Crit}(f)$, $A \in H_2(M)$, and $R \in [0, \infty)$,

$$\begin{aligned} \mathcal{M}_R(p_-, p_+; A) := \{u : \mathbb{CP}^1 \rightarrow M \mid u([1 : 0]) \in W_{p_-}^-, u([0 : 1]) \in W_{p_+}^+, \\ \bar{\partial}_J u = Y_H^R(u), [u] = A\}, \end{aligned}$$

and we will construct h from their union

$$\mathcal{M}(p_-, p_+; A) := \bigsqcup_{R \in [0, \infty)} \mathcal{M}_R(p_-, p_+; A). \quad (2.23)$$

Remark 2.5.2. Each vector-field-valued 1-form $Y_H^R = -(X_{H_\beta^R})^{0,1}$ is in the notation of [40, §8.1] induced from the 1-form with values in smooth functions $H_\beta^R \in \Omega^1(\mathbb{CP}^1, \mathcal{C}^\infty(M))$ given by $H_\beta^R(re^{i\theta}) = \beta_R(r)H(\theta, \cdot)d\theta$. It is constructed so that it has the following properties:

- (i) For $R = 0$ we have $Y_H^0 \equiv 0$ so that the moduli space $\mathcal{M}_0(p_-, p_+; A) = \mathcal{M}^\iota(p_-, p_+; A)$ is the same moduli space (2.17) from which ι will be constructed.
- (ii) The restriction of any solution $u \in \mathcal{M}_R(p_-, p_+; A)$ to the middle portion $\{[1 : z] \in \mathbb{CP}^1 \mid e^{1-\frac{R}{2}} < |z| < e^{\frac{R}{2}-1}\} \cong (1 - \frac{R}{2}, \frac{R}{2} - 1) \times S^1$ satisfies the Floer equation $\partial_s v + J\partial_t v = JX_H(t, v)$ after reparametrization $v(s, t) := u([1 : e^{s+it}])$.
- (iii) The shifts $u_-(z) := u([1 : e^{-\frac{R}{2}}z])$ and $u_+(z) := u([e^{\frac{R}{2}}z : 1]) = u([1 : e^{\frac{R}{2}}z^{-1}])$ of any solution $u \in \mathcal{M}_R(p_-, p_+; A)$, restricted to $\{z \in \mathbb{C} \mid |z| < e^{R-1}\}$, satisfy $\bar{\partial}_J u_\pm = Y_H(u_\pm)$ as in the PSS/SSP moduli spaces in §2.4.1.

The moduli space $\mathcal{M}(p_-, p_+; A)$ is the zero set of a Fredholm section over a Banach manifold $[0, \infty) \times \mathcal{B}$, where $\mathcal{B}$ is the same Sobolev closure as in §2.5.1 of the set of smooth maps $u : \mathbb{CP}^1 \rightarrow M$ in the homology class $[u] = A$ satisfying the point constraints $u([1 : 0]) \in W_{p-}^-$ and $u([0 : 1]) \in W_{p+}^+$. Restricted to $\{0\} \times \mathcal{B}$ this is the Fredholm section that cuts out $\mathcal{M}^\iota(p_-, p_+; A)$ in (2.17) with $\tilde{J}_H^0 = \tilde{J}$. This determines the Fredholm index as

$$I(p_-, p_+; A) := I^\iota(p_-, p_+; A) + 1 = 2c_1(A) + |p_-| - |p_+| + 1. \quad (2.24)$$

Towards a compactification and polyfold description of these moduli spaces we again – as in §2.4.2, §2.5.1, [40, §8.1] – identify a solution $u : \mathbb{CP}^1 \rightarrow M$ with the map to its graph. Moreover, we again fix marked points $z_0^+ = [1 : 0]$, $z_0^- = [0 : 1]$ to implement evaluation maps to express the conditions $u(z_0^\mp) \in W_{p_\pm}^\pm$. This yields a homeomorphism (in appropriate topologies) between the moduli space (2.23) and the fiber product over $\mathbb{CP}^1 \times M$ with the half-infinite Morse trajectory spaces from §2.3.3,

$$\mathcal{M}(p_-, p_+; A) \cong \mathcal{M}(p_-, M)_{\{z_0^+\} \times_{\text{ev}} \times_{\text{ev}^+}} \mathcal{M}_{\text{SFT}}(A)_{\text{ev}^- \times \{z_0^-\} \times_{\text{ev}}} \mathcal{M}(M, p_+). \quad (2.25)$$

Compared with (2.20) this replaces the Gromov-Witten moduli space in (2.21) with a family of moduli spaces for almost complex structures $\tilde{J}_H^R$ on $\mathbb{CP}^1 \times M$ arising from Y_H^R for $R \in [0, \infty)$,

$$\mathcal{M}_{\text{SFT}}(A) := \bigsqcup_{R \in [0, \infty)} \{v : \mathbb{CP}^1 \rightarrow \mathbb{CP}^1 \times M \mid \bar{\partial}_{\tilde{J}_H^R} v = 0, [v] = [\mathbb{CP}^1] + A\} / \text{Aut}(\mathbb{CP}^1, z_0^-, z_0^+). \quad (2.26)$$

Here, again, we implicitly include the two marked points $z_0^\pm \in \mathbb{CP}^1$. Then, for $R \rightarrow \infty$, the degeneration of the PDE $\bar{\partial}_{\tilde{J}_H^R} v = 0$ is the “neck stretching”⁷ considered more generally in Symplectic Field Theory [13]. The evaluation maps from (2.21) directly generalize to

$$\text{ev}^\pm : \mathcal{M}_{\text{SFT}}(A) \rightarrow \mathbb{CP}^1 \times M, \quad [v] \mapsto v(z_0^\pm). \quad (2.27)$$

Now, as in §2.5.1, each factor in the fiber product (2.25) has natural compactifications – either the compactified Morse trajectory spaces from Theorem 2.3.3, or the compact zero set $\overline{\mathcal{M}}_{\text{SFT}}(A) = \sigma_{\text{SFT}}^{-1}(0)$ of a sc-Fredholm section that we will introduce in §2.5.3. Combined with the construction of fiber products in polyfold theory [15] this will yield compactifications and polyfold descriptions of the moduli spaces (2.23), and the construction of $h : CM \rightarrow CM$ then again proceeds as in §2.4.3. Establishing the algebraic properties in Theorem 2.1.2 relating h with ι and $SSP \circ PSS$ will moreover require an in-depth discussion of the boundary stratification of the polyfold domains $\mathcal{B}_{\text{SFT}}(A)$ of these sections, and “coherent” choices of perturbations in §2.6.

⁷Strictly speaking, $R \in [0, 2]$ parametrizes a family of Gromov-Witten moduli spaces for varying almost complex structure. At $R = 2$, the manifold $S^1 \times M$ with its stable Hamiltonian structure (see §2.4.2) embeds as a stable hypersurface in $\mathbb{CP}^1 \times M$. Then $R \in [2, \infty)$ parametrizes the SFT neck-stretching.

Remark 2.5.3. Gromov-compactifications of the moduli spaces $\mathcal{M}(p_-, p_+; A)$ will result from energy estimates [40, Remark 8.1.7] for solutions of $\bar{\partial}_J u = Y_H^R(u)$,

$$E_R(u) := \frac{1}{2} \int_{\mathbb{CP}^1} |du + X_{H_\beta^R}(u)| \leq \int_{\mathbb{CP}^1} u^* \omega + \|R_{H_\beta^R}\| = \omega([u]) + 2\|H(\theta, \cdot)\|. \quad (2.28)$$

Here $R_{H_\beta^R} \text{dvol}_{\mathbb{CP}^1} = dH_\beta^R + \frac{1}{2} H_\beta^R \wedge H_\beta^R = \beta'_R H \, dr \wedge d\theta$ has uniformly bounded Hofer norm

$$\|R_{H_\beta^R}\| = \int_{\mathbb{CP}^1} (\max R_{H_\beta^R} - \min R_{H_\beta^R}) = \int_0^\infty \int_{S^1} |\beta'_R(r)| \|H(\theta, \cdot)\| \, d\theta \, dr = 2\|H(\theta, \cdot)\|,$$

where $\|H(\theta, \cdot)\| := \max_{x \in M} H(\theta, x) - \min_{x \in M} H(\theta, x)$ and $\beta_R \in \mathcal{C}^\infty((0, \infty), [0, 1])$ is constant except

$$\begin{aligned} \beta_R|_{[e^{-\frac{R}{2}}, e^{1-\frac{R}{2}}]} : r &\mapsto \beta(re^{\frac{R}{2}}) \quad \text{with} \quad \frac{d}{dr} \beta_R \geq 0, & \int_{e^{-\frac{R}{2}}}^{e^{1-\frac{R}{2}}} \left| \frac{d}{dr} \beta_R \right| \, dr &= \beta(e) - \beta(1) = 1, \\ \beta_R|_{[e^{\frac{R}{2}-1}, e^{\frac{R}{2}}]} : r &\mapsto \beta(r^{-1}e^{\frac{R}{2}}) \quad \text{with} \quad \frac{d}{dr} \beta_R \leq 0, & \int_{e^{\frac{R}{2}-1}}^{e^{\frac{R}{2}}} \left| \frac{d}{dr} \beta_R \right| \, dr &= -(\beta(1) - \beta(e)) = 1. \end{aligned}$$

This proves (2.28), and thus establishes energy bounds on the perturbed pseudoholomorphic maps in each of our moduli spaces, where we fix $[u] = A$. Now SFT-compactness [2] asserts that for any $C > 0$ the set of solutions of bounded energy $\bigsqcup_{R \in [0, \infty)} \{u : \mathbb{CP}^1 \rightarrow M \mid \bar{\partial}_J u = Y_H^R(u), E_R(u) \leq C\}$ is compact up to breaking and bubbling. This compactness will be stated rigorously in polyfold terms in Assumption 2.5.5 (ii).

2.5.3 Construction of the morphisms

In this section we construct the Λ -linear maps $\iota : CM \rightarrow CM$ and $h : CM \rightarrow CM$ analogously to §2.4.3 by first obtaining compactifications and polyfold descriptions for the moduli spaces in §2.5.1 and §2.5.2 as in §2.4.2. This construction is motivated by the fiber product descriptions of the moduli spaces in (2.20), (2.25), which couple Morse trajectory spaces from §2.3.3 with moduli spaces of pseudoholomorphic curves in $\mathbb{CP}^1 \times M$ via evaluation maps (2.21), (2.27). Polyfold descriptions of these moduli spaces and their properties are stated in the following Assumption 2.5.5 for reference, with proofs in [33] resp. outlined in [19]. Here we formulate the evaluation map in the context of neck stretching, as explained in the following remark, using a splitting of the sphere as topological space with smooth structures on the complement of the equator

$$\overline{\mathbb{CP}^1_\infty} := \overline{\mathbb{C}^+} \cup_{S^1} \overline{\mathbb{C}^-} \cong \mathbb{C}^+ \sqcup S^1 \sqcup \mathbb{C}^-,$$

using the topologies and smooth structures on $\overline{\mathbb{C}^\pm} = \mathbb{C}^\pm \sqcup S^1 \cong \{z \in \mathbb{C}^\pm \mid |z| \leq 1\}$ from Remark 2.4.2.

Remark 2.5.4. (i) Recall from §2.5.1 that we denote by $\mathcal{B}_{\text{GW}}(A)$ a Gromov-Witten polyfold of curves in class $[\mathbb{CP}^1] + A \in H_2(\mathbb{CP}^1 \times M)$ with 2 marked points. These are determined by $A \in H_2(M)$ as we model graphs of maps $\mathbb{CP}^1 \rightarrow M$, but should not be confused with a

polyfold of curves in M . In particular, $\mathcal{B}_{\text{GW}}(A)$ never contains constant maps and hence is well defined for $A = 0$. The properties of the Gromov-Witten moduli spaces for $\omega(A) \leq 0$ are spelled out abstractly in (ii) below; for the geometric meaning see Remark 2.5.1.

(ii) The SFT polyfolds $\mathcal{B}_{\text{SFT}}(A)$ will similarly describe curves in class $[\mathbb{CP}^1] + A$ in a neck stretching family of targets $(\mathbb{CP}_R^1 \times M)_{R \in [0, \infty]}$ as in [2, §3.4], given by

$$\mathbb{CP}_R^1 := D_+ \sqcup E_R \sqcup D_- / \sim_R \quad \text{with} \quad E_R = \begin{cases} [-R, R] \times S^1 & ; R < \infty, \\ [0, \infty) \times S^1 \sqcup (-\infty, 0] \times S^1 & ; R = \infty. \end{cases}$$

Here we identify the boundaries of the closed unit disks $D_{\pm} = \{z \in \mathbb{C} \mid |z| \leq 1\}$ with the boundary components of the necks E_R via

$$\partial D_{\pm} \in e^{i\theta} \sim_R \begin{cases} (\pm R, e^{\pm i\theta}) & ; R < \infty \\ (0_{\pm}, e^{\pm i\theta}) & ; R = \infty \end{cases} \in \partial E_R,$$

where we denote $0_+ := 0 \in [0, \infty)$ and $0_- := 0 \in (-\infty, 0]$ so that $\partial E_{\infty} = \{0_+\} \times S^1 \sqcup \{0_-\} \times S^1$. To describe convergence and evaluation maps we also embed each $\mathbb{CP}_R^1 \subset \overline{\mathbb{CP}}_{\infty}^1 = \mathbb{C}^+ \sqcup S^1 \sqcup \mathbb{C}^-$ by

$$\begin{aligned} D_+ \sqcup [-R, 0) \times S^1 / \sim_R &\cong D_+ \sqcup [0, \infty) \times S^1 / \sim_{\infty} =: \mathbb{C}^+, \\ D_- \sqcup (0, R] \times S^1 / \sim_R &\cong D_- \sqcup (-\infty, 0] \times S^1 / \sim_{\infty} =: \mathbb{C}^-, \end{aligned}$$

and

$$E_R \supset \{0\} \times S^1 \cong S^1 \subset \overline{\mathbb{CP}}_{\infty}^1.$$

For $R = 0$ this is to be understood as $\mathbb{CP}_0^1 = D_+ \sqcup D_- / \partial D_+ \sim \partial D_-$ with $D_{\pm} \setminus \partial D_{\pm} \cong \mathbb{C}^{\pm}$, and for all $R < \infty$ we view the resulting homeomorphism $\mathbb{CP}_R^1 \cong \overline{\mathbb{CP}}_{\infty}^1 \cong \mathbb{CP}^1$ as identifying the standard marked points $\mathbb{CP}^1 \ni z_0^+ = [1 : 0] \cong 0 \in \mathbb{C}^+$ and $\mathbb{CP}^1 \ni z_0^- = [0 : 1] \cong 0 \in \mathbb{C}^-$. When these embeddings are done via linear shifts $[-R, -1) \cong [0, R-1)$ and $(1, R] \cong (1-R, 0]$ extended by a smooth family of diffeomorphisms $[-1, 0) \cong [R-1, \infty)$ and $(0, 1] \cong (-\infty, 1-R]$, then the pullback of the almost complex structures $\tilde{J}_H^R$ on $\mathbb{CP}_R^1 \times M$ converges for $R \rightarrow \infty$ in $\mathcal{C}_{\text{loc}}^{\infty}((\overline{\mathbb{CP}}_{\infty}^1 \setminus S^1) \times M)$ to the almost complex structures $\tilde{J}_H^+, \tilde{J}_H^-$ on $\mathbb{C}^+ \times M \sqcup \mathbb{C}^- \times M = \overline{\mathbb{CP}}_{\infty}^1 \times M \subset \overline{\mathbb{CP}}_{\infty}^1 \times M$, which are used in the construction of the PSS and SSP moduli spaces in §2.4.2. Moreover, this allows us to extend the evaluation maps from (2.27) to continuous maps $\overline{\text{ev}}^{\pm} : \overline{\mathcal{M}}_{\text{SFT}}(A) \rightarrow \overline{\mathbb{CP}}_{\infty}^1 \times M$ on the compactified SFT moduli space. At $R = \infty$ this involves pseudoholomorphic buildings in $\mathbb{C}^+ \times M \sqcup \mathbb{R} \times S^1 \times M \dots \sqcup \mathbb{R} \times S^1 \times M \sqcup \mathbb{C}^- \times M$, and for any marked point with evaluation into a cylinder $\mathbb{R} \times S^1 \times M$ we project the result to $S^1 \times M \subset \overline{\mathbb{CP}}_{\infty}^1 \times M$ by forgetting the $\mathbb{R}$ -component.

Finally, this formulation with $\overline{\mathbb{CP}}_{\infty}^1 = \overline{\mathbb{C}^+} \cup_{S^1} \overline{\mathbb{C}^-}$ will allow us to compare the evaluation at $R = \infty$ with the product of the evaluations $\overline{\text{ev}}^{\pm} : \overline{\mathcal{M}}_{\text{SFT}}^{\pm}(\gamma; A) \rightarrow \overline{\mathbb{C}^{\pm}} \times M$ constructed

in Remark 2.4.2. While this will be stated rigorously only in Assumption 2.6.3 (iii)(c), note here that we should expect three top boundary strata of an ambient polyfold at $R = \infty$, corresponding to the distribution of marked points on the curves in $\mathbb{C}^+ \times M \sqcup \mathbb{C}^- \times M$. For the fiber product construction, only the boundary components with one marked point in each factor are relevant – in fact only those with marked points near $z_0^+ \cong 0 \in \mathbb{C}^+$ and $z_0^- \cong 0 \in \mathbb{C}^-$. Thus we will work with the open subset $(\overline{\text{ev}}^+)^{-1}(\mathbb{C}^+ \times M) \cap (\overline{\text{ev}}^-)^{-1}(\mathbb{C}^- \times M)$ where the two evaluations for any $R \in [0, \infty]$ are constrained to take values in the open sets given by $\mathbb{C}^\pm \subset \overline{\mathbb{CP}}_\infty^1$.

Assumption 2.5.5. *There is a collection of oriented sc-Fredholm sections of strong polyfold bundles $\sigma_{\text{GW}} : \mathcal{B}_{\text{GW}}(A) \rightarrow \mathcal{E}_{\text{GW}}(A)$ and $\sigma_{\text{SFT}} : \mathcal{B}_{\text{SFT}}(A) \rightarrow \mathcal{E}_{\text{SFT}}(A)$ indexed by $A \in H_2(M)$, sc^∞ maps $\overline{\text{ev}}^\pm : \mathcal{B}_{\text{GW}}(A) \rightarrow \overline{\mathbb{CP}}_\infty^1 \times M$, and continuous maps $\overline{\text{ev}}^\pm : \mathcal{B}_{\text{SFT}}(A) \rightarrow \overline{\mathbb{CP}}_\infty^1 \times M$ with the properties:*

- (i) *The sections have Fredholm indices $\text{ind}(\sigma_{\text{GW}}) = 2c_1(A) + \dim M + 4$ on $\mathcal{B}_{\text{GW}}(A)$ resp. $\text{ind}(\sigma_{\text{SFT}}) = 2c_1(A) + \dim M + 5$ on $\mathcal{B}_{\text{SFT}}(A)$.*
- (ii) *Each zero set $\overline{\mathcal{M}}_{\text{GW}}(A) := \sigma_{\text{GW}}^{-1}(0)$ and $\overline{\mathcal{M}}_{\text{SFT}}(A) := \sigma_{\text{SFT}}^{-1}(0)$ is compact, and given any $C \in \mathbb{R}$ there are only finitely many $A \in H_2(M)$ with nonempty zero set $\overline{\mathcal{M}}_{\text{GW}}(A) \neq \emptyset$ resp. $\overline{\mathcal{M}}_{\text{SFT}}(A) \neq \emptyset$. Moreover, for $\omega(A) \leq 0$ we have $\overline{\mathcal{M}}_{\text{GW}}(A) = \emptyset$ except for $A = 0 \in H_2(M)$ when $\sigma_{\text{GW}}|_{\mathcal{B}_{\text{GW}}(0)} \pitchfork 0$ is in general position with zero set $\overline{\mathcal{M}}_{\text{GW}}(0) \simeq \mathbb{CP}^1 \times \mathbb{CP}^1 \times M$ identified by*

$$\mathcal{B}_{\text{GW}}(0) \supset \sigma_{\text{GW}}^{-1}(0) = \overline{\mathcal{M}}_{\text{GW}}(0) \xrightarrow{\overline{\text{ev}}^+ \times \overline{\text{ev}}^-} \{(z^+, x, z^-, x) \mid z^-, z^+ \in \mathbb{CP}^1, x \in M\}.$$

- (iii) *The polyfolds $\mathcal{B}_{\text{GW}}(A)$ have no boundary, $\partial \mathcal{B}_{\text{GW}}(A) = \emptyset$. For $\mathcal{B}_{\text{SFT}}(A)$ there is a natural inclusion $[0, \infty) \times \mathcal{B}_{\text{GW}}(A) \subset \mathcal{B}_{\text{SFT}}(A)$ that covers the interior $\partial_0 \mathcal{B}_{\text{SFT}}(A) = (0, \infty) \times \mathcal{B}_{\text{GW}}(A)$ and identifies the boundary $\partial \mathcal{B}_{\text{SFT}}(A)$ to consist of the disjoint sets $\{0\} \times \mathcal{B}_{\text{GW}}(A)$ and $\lim_{R \rightarrow \infty} \{R\} \times \mathcal{B}_{\text{GW}}(A)$ of $\mathcal{B}_{\text{SFT}}(A)$. Moreover, this inclusion identifies the section σ_{GW} and evaluation maps $\overline{\text{ev}}^\pm$ with the restricted section $\sigma_{\text{SFT}}|_{\{0\} \times \mathcal{B}_{\text{GW}}(A)}$ and evaluations $\overline{\text{ev}}^\pm|_{\{0\} \times \mathcal{B}_{\text{GW}}(A)}$. (A description of the relevant $R = \infty$ parts of the boundary $\partial \mathcal{B}_{\text{SFT}}(A)$ is given in Assumption 2.6.3.)*

- (iv) *The $\text{Aut}(\mathbb{CP}^1, z_0^-, z_0^+)$ -orbits of smooth maps $v : \mathbb{CP}^1 \rightarrow \mathbb{CP}^1 \times M$ which represent the class $[\mathbb{CP}^1] + A$ form a dense subset $\mathcal{B}_{\text{dense}}(A) \subset \mathcal{B}_{\text{GW}}(A)$. On this subset, the section is given by $\sigma_{\text{GW}}([v]) = [(v, \bar{\partial}_{\bar{J}} v)]$. Moreover, $[0, \infty) \times \mathcal{B}_{\text{dense}}(A) \subset \mathcal{B}_{\text{SFT}}(A)$ is a dense subset that intersects the boundary $\partial \mathcal{B}_{\text{SFT}}(A)$ exactly in $\{0\} \times \mathcal{B}_{\text{dense}}(A)$, and on which the section is given by $\sigma_{\text{SFT}}(R, [v]) = [(v, \bar{\partial}_{\bar{J}_R} v)]$. On these dense subsets, $\overline{\text{ev}}^\pm([v])$ resp. $\overline{\text{ev}}^\pm(R, [v])$ is the evaluation as in (2.27).*

- (v) *The intersection of the zero sets with the dense subsets $\sigma_{\text{GW}}^{-1}(0) \cap \mathcal{B}_{\text{dense}}(A) \cong \mathcal{M}_{\text{GW}}(A)$ and $\sigma_{\text{SFT}}^{-1}(0) \cap [0, \infty) \times \mathcal{B}_{\text{dense}}(A) \cong \mathcal{M}_{\text{SFT}}(A)$ are naturally identified with the Gromov-Witten moduli space (2.21) and SFT moduli space in (2.26).*

(vi) The sections σ_{GW} and σ_{SFT} have tame *sc-Fredholm* representatives in the sense of [15, Def.5.4]. The product of evaluation maps $\overline{\text{ev}}^+ \times \overline{\text{ev}}^- : \mathcal{B}_{\text{GW}}(A) \rightarrow \mathbb{CP}^1 \times M \times \mathbb{CP}^1 \times M$ is σ_{GW} -compatibly submersive in the sense of Definition 2.7.4. On the open subset

$$\mathcal{B}_{\text{SFT}}^{+,-}(A) := (\overline{\text{ev}}^+)^{-1}(\mathbb{C}^+ \times M) \cap (\overline{\text{ev}}^-)^{-1}(\mathbb{C}^- \times M) \subset \mathcal{B}_{\text{SFT}}(A)$$

the evaluation maps $\overline{\text{ev}}^\pm : \mathcal{B}_{\text{SFT}}(A) \rightarrow \overline{\mathbb{CP}}_\infty^1 \times M$ restrict to a σ_{SFT} -compatibly submersive map

$$\text{ev}^+ \times \text{ev}^- : \mathcal{B}_{\text{SFT}}^{+,-}(A) \rightarrow \mathbb{C}^+ \times M \times \mathbb{C}^- \times M. \quad (2.29)$$

On this domain intersected with $\{0\} \times \mathcal{B}_{\text{GW}}(A) \subset \partial_1 \mathcal{B}_{\text{SFT}}(A)$, this map coincides with the Gromov-Witten evaluations $\overline{\text{ev}}^+ \times \overline{\text{ev}}^-$ viewed as maps

$$\text{ev}^+ \times \text{ev}^- : \mathcal{B}_{\text{GW}}^{+,-}(A) \rightarrow \mathbb{C}^+ \times M \times \mathbb{C}^- \times M,$$

where we identify $\mathbb{C}^+ \sqcup \mathbb{C}^- = \mathbb{CP}^1 \setminus S^1$ and restrict to the domain

$$\begin{aligned} \{0\} \times \mathcal{B}_{\text{GW}}^{+,-}(A) &:= (\{0\} \times \mathcal{B}_{\text{GW}}(A)) \cap \mathcal{B}_{\text{SFT}}^{+,-}(A) \\ &= \{0\} \times ((\overline{\text{ev}}^+)^{-1}(\mathbb{C}^+ \times M) \cap (\overline{\text{ev}}^-)^{-1}(\mathbb{C}^- \times M)). \end{aligned}$$

Here the properties (iv),(v) are stated to give an intuitive sense of what spaces we are working with. The polyfold description $\sigma_{\text{GW}} : \mathcal{B}_{\text{GW}}(A) \rightarrow \mathcal{E}_{\text{GW}}(A)$ is developed for the homology classes $[\mathbb{CP}^1] + A \in H_2(\mathbb{CP}^1 \times M)$ in [33], and assumption (v) should follow similarly to the explanation given in [15, Ex.5.1] for the Gromov-Witten polyfolds [33]. Given any such polyfold descriptions of the moduli spaces of pseudoholomorphic curves, we now extend the fiber product descriptions of the moduli spaces $\mathcal{M}^i(p_-, p_+; A) \cong \mathcal{M}(p_-, M)_{\{z_0^+\} \times \text{ev}} \times_{\text{ev}^+} \mathcal{M}_{\text{GW/SFT}}(A)_{\text{ev}^- \times \{z_0^-\} \times \text{ev}} \mathcal{M}(M, p_+)$ in §2.5.1 and §2.5.2 to obtain ambient polyfolds which contain compactifications of the moduli spaces. Towards this we define for each $p_-, p_+ \in \text{Crit}(f)$ and $A \in H_2(M)$ the topological spaces

$$\begin{aligned} \tilde{\mathcal{B}}^i(p_-, p_+; A) &:= \{(\underline{\tau}_-, \underline{v}, \underline{\tau}_+) \in \overline{\mathcal{M}}(p_-, M) \times \mathcal{B}_{\text{GW}}(A) \times \overline{\mathcal{M}}(M, p_+) \mid (z_0^\pm, \text{ev}(\underline{\tau}_\pm)) = \overline{\text{ev}}^\pm(\underline{v})\} \\ &= \{(\underline{\tau}_-, \underline{v}, \underline{\tau}_+) \in \overline{\mathcal{M}}(p_-, M) \times \mathcal{B}_{\text{GW}}^{+,-}(A) \times \overline{\mathcal{M}}(M, p_+) \mid (0, \text{ev}(\underline{\tau}_\pm)) = \text{ev}^\pm(\underline{v})\}, \\ \tilde{\mathcal{B}}(p_-, p_+; A) &:= \{(\underline{\tau}_-, \underline{w}, \underline{\tau}_+) \in \overline{\mathcal{M}}(p_-, M) \times \mathcal{B}_{\text{SFT}}(A) \times \overline{\mathcal{M}}(M, p_+) \mid (z_0^\pm, \text{ev}(\underline{\tau}_\pm)) = \overline{\text{ev}}^\pm(\underline{w})\} \\ &= \{(\underline{\tau}_-, \underline{w}, \underline{\tau}_+) \in \overline{\mathcal{M}}(p_-, M) \times \mathcal{B}_{\text{SFT}}^{+,-}(A) \times \overline{\mathcal{M}}(M, p_+) \mid (0, \text{ev}(\underline{\tau}_\pm)) = \text{ev}^\pm(\underline{w})\}, \end{aligned}$$

where the last equality stems from the identification at the end of Remark 2.5.4 (ii). Then the abstract fiber product constructions in [15] will be used as in Lemma 2.4.5 to obtain the following polyfold description for compactifications of the moduli spaces in §2.5.1 and §2.5.2.

Lemma 2.5.6. *Given any $p_-, p_+ \in \text{Crit}(f)$ and $A \in H_2(M)$, there exist open subsets $\mathcal{B}^i(p_-, p_+; A) \subset \tilde{\mathcal{B}}^i(p_-, p_+; A)_1$ and $\mathcal{B}(p_-, p_+; A) \subset \tilde{\mathcal{B}}(p_-, p_+; A)_1$ which contain the smooth*

levels $\tilde{\mathcal{B}}^\iota(p_-, p_+; A)_\infty$ of the fiber products and inherit natural polyfold structures with smooth level of the interior

$$\begin{aligned}\partial_0 \mathcal{B}^\iota(p_-, p_+; A)_\infty &= \mathcal{M}(p_-, M)_{\{z_0^+\} \times \text{ev} \times \text{ev}^+} \mathcal{B}_{\text{GW}}^{+, -}(A)_\infty_{\text{ev}^- \times \{z_0^-\} \times \text{ev}} \mathcal{M}(M, p_+), \\ \partial_0 \mathcal{B}(p_-, p_+; A)_\infty &= \mathcal{M}(p_-, M)_{\{z_0^+\} \times \text{ev} \times \text{ev}^+} \partial_0 \mathcal{B}_{\text{SFT}}^{+, -}(A)_\infty_{\text{ev}^- \times \{z_0^-\} \times \text{ev}} \mathcal{M}(M, p_+),\end{aligned}$$

and a scale-smooth inclusion

$$\phi_\iota : \mathcal{B}^\iota(p_-, p_+; A) \hookrightarrow \mathcal{B}(p_-, p_+; A), \quad (\underline{\tau}_-, \underline{v}, \underline{\tau}_+) \mapsto (\underline{\tau}_-, 0, \underline{v}, \underline{\tau}_+).$$

Moreover, pullback of the sections and bundles $\sigma_{\text{GW/SFT}} : \mathcal{B}_{\text{GW/SFT}}(A) \rightarrow \mathcal{E}_{\text{GW/SFT}}(A)$ under the projection $\mathcal{B}(p_-, p_+; A) \rightarrow \mathcal{B}_{\text{GW/SFT}}(A)$ induces sc-Fredholm sections of strong polyfold bundles $\sigma_{(p_-, p_+; A)} : \mathcal{B}(p_-, p_+; A) \rightarrow \mathcal{E}(p_-, p_+; A)$ of index $I(p_-, p_+; A)$ as in (2.24) and $\sigma_{(p_-, p_+; A)}^\iota : \mathcal{B}^\iota(p_-, p_+; A) \rightarrow \mathcal{E}^\iota(p_-, p_+; A)$ of index $I^\iota(p_-, p_+; A) = I(p_-, p_+; A) - 1$ as in (2.18). Further, these are related via the inclusion ϕ_ι by natural orientation preserving identification $\sigma_{(p_-, p_+; A)}^\iota \cong \phi_\iota^* \sigma_{(p_-, p_+; A)}$.

The zero sets of these sc-Fredholm sections contain⁸ the moduli spaces from §2.5.1 and §2.5.2,

$$\begin{aligned}\sigma_{(p_-, p_+; A)}^{-1}(0) &= \overline{\mathcal{M}}(p_-, M)_{\{z_0^+\} \times \text{ev} \times \overline{\text{ev}}^+} \sigma_{\text{GW}}^{-1}(0)_{\overline{\text{ev}}^- \times \{z_0^-\} \times \text{ev}} \overline{\mathcal{M}}(M, p_+) \supset \mathcal{M}(p_-, p_+; A), \\ \sigma_{(p_-, p_+; A)}^{\iota^{-1}}(0) &= \overline{\mathcal{M}}(p_-, M)_{\{z_0^+\} \times \text{ev} \times \overline{\text{ev}}^+} \sigma_{\text{SFT}}^{-1}(0)_{\overline{\text{ev}}^- \times \{z_0^-\} \times \text{ev}} \overline{\mathcal{M}}(M, p_+) \supset \mathcal{M}^\iota(p_-, p_+; A).\end{aligned}$$

Finally, each zero set $\sigma_{(p_-, p_+; A)}^{\iota^{-1}}(0)$ is compact, and given any $p_\pm \in \text{Crit}(f)$ and $C \in \mathbb{R}$, there are only finitely many $A \in H_2(M)$ with $\omega(A) \leq C$ and nonempty zero set $\sigma_{(p_-, p_+; A)}^{\iota^{-1}}(0) \neq \emptyset$.

Proof. The inclusion ϕ_ι is sc^∞ since the map $\mathcal{B}_{\text{GW}}(A) \hookrightarrow \mathcal{B}_{\text{SFT}}(A), \underline{v} \mapsto (0, \underline{v})$ is a sc^∞ inclusion by Assumption 2.5.5 (iii). Apart from further relations involving ϕ_ι , the proof is directly analogous to the fiber product construction in Lemma 2.4.5, using Assumption 2.5.5 – in particular the sc^∞ and σ_{SFT} -compatibly submersive evaluation map (2.29) on the open subset $\mathcal{B}_{\text{SFT}}^{+, -}(A) \subset \mathcal{B}_{\text{SFT}}(A)$. This yields polyfold structures on open sets $\mathcal{B}^\iota(p_-, p_+; A) \subset \tilde{\mathcal{B}}^\iota(p_-, p_+; A)_1$ and $\mathcal{B}(p_-, p_+; A) \subset \tilde{\mathcal{B}}(p_-, p_+; A)_1$ as well as the pullback sc-Fredholm sections $\sigma_{(p_-, p_+; A)} = \text{pr}_{\text{SFT}}^* \sigma_{\text{SFT}}$ and $\sigma_{(p_-, p_+; A)}^\iota = \text{pr}_{\text{GW}}^* \sigma_{\text{GW}}$ under the projections $\text{pr}_{\text{GW/SFT}} : \mathcal{B}^\iota(p_-, p_+; A) \rightarrow \mathcal{B}_{\text{GW/SFT}}(A)$. Here we have $\text{pr}_{\text{GW}} = \text{pr}_{\text{SFT}} \circ \phi_\iota$, so the bundle $\mathcal{E}^\iota(p_-, p_+; A) = \text{pr}_{\text{GW}}^* \mathcal{E}_{\text{GW}}(A)$ and section $\sigma_{(p_-, p_+; A)}^\iota = \text{pr}_{\text{GW}}^* \sigma_{\text{GW}}$ are naturally identified with the pullback bundle $\phi_\iota^* \mathcal{E}(p_-, p_+; A) = \text{pr}_{\text{GW}}^* \mathcal{E}_{\text{SFT}}(A)|_{\{0\} \times \mathcal{B}_{\text{GW}}(A)}$ and section $\phi_\iota^* \sigma_{(p_-, p_+; A)} = \text{pr}_{\text{GW}}^* \sigma_{\text{SFT}}|_{\{0\} \times \mathcal{B}_{\text{GW}}(A)}$ using Assumption 2.5.5 (iii). Finally, the index of the induced section $\sigma_{(p_-, p_+; A)}$, and similarly of $\sigma_{(p_-, p_+; A)}^\iota$, is computed by [15, Cor.7.3] as

$$\begin{aligned}\text{ind}(\sigma_{(p_-, p_+; A)}) &= \text{ind}(\sigma_{\text{SFT}}) + \dim \overline{\mathcal{M}}(p_-, M) + \dim \overline{\mathcal{M}}(M, p_+) - 2 \dim(\mathbb{CP}^1 \times M) \\ &= 2c_1(A) + \dim M + 5 + |p_-| + \dim M - |p_+| - 4 - 2 \dim M \\ &= 2c_1(A) + |p_-| - |p_+| + 1 = I(p_-, p_+; A).\end{aligned}$$

□

⁸ As in Remark 2.4.4, this identification is stated for intuition and will ultimately not be used in our proofs.

Given this compactification and polyfold description of the moduli spaces $\mathcal{M}(\alpha) \subset \sigma_\alpha^{-1}(0)$ and $\mathcal{M}^\iota(\alpha) \subset \sigma_\alpha^{\iota^{-1}}(0)$ for all tuples in the indexing set

$$\mathcal{I} := \{ \alpha = (p_-, p_+; A) \mid p_-, p_+ \in \text{Crit}(f), A \in H_2(M) \},$$

we can again apply [34, Theorems 18.2, 18.3, 18.8] to the sc-Fredholm sections σ_α and σ_α^ι and obtain Corollary 2.4.6 verbatim for these collections of moduli spaces. In §2.6 we will moreover make use of the fact that $\sigma_\alpha^\iota = \phi_\iota^* \sigma_\alpha$ arises from restriction of σ_α , so admissible perturbations of σ_α pull back to admissible perturbations of σ_α^ι . For now, we choose perturbations independently and thus as in Definition 2.4.8 obtain perturbation-dependent, and not yet algebraically related, Λ -linear maps.

Definition 2.5.7. *Given admissible sc^+ -multisections $\underline{\kappa} = (\kappa_{(p_-, p_+; A)})_{p_\pm \in \text{Crit}(f), A \in H_2}$ in general position to $(\sigma_{(p_-, p_+; A)})$ and similarly $\underline{\kappa}^\iota = (\kappa_{(p_-, p_+; A)}^\iota)_{p_\pm \in \text{Crit}(f), A \in H_2}$ in general position to $(\sigma_{(p_-, p_+; A)}^\iota)$ as in Corollary 2.4.6, we define the maps $h_{\underline{\kappa}} : CM \rightarrow CM$ and $\iota_{\underline{\kappa}^\iota} : CM \rightarrow CM$ to be the Λ -linear extensions of*

$$\begin{aligned} h_{\underline{\kappa}} \langle p_- \rangle &:= \sum_{\substack{p_+, A \\ I(p_-, p_+; A)=0}} \#Z^{\underline{\kappa}}(p_-, p_+; A) \cdot T^{\omega(A)} \langle p_+ \rangle, \\ \iota_{\underline{\kappa}^\iota} \langle p_- \rangle &:= \sum_{\substack{p_+, A \\ I^\iota(p_-, p_+; A)=0}} \#Z^{\underline{\kappa}^\iota}(p_-, p_+; A) \cdot T^{\omega(A)} \langle p_+ \rangle. \end{aligned}$$

The proof that the coefficients of these maps lie in the Novikov field Λ is verbatim the same as Lemma 2.4.9, based on the compactness properties in Lemma 2.5.6.

Remark 2.5.8. The determination in Corollary 2.4.6 of $\#Z^{\underline{\kappa}}(p_-, p_+; A), \#Z^{\underline{\kappa}^\iota}(p_-, p_+; A) \in \mathbb{Q}$ that is used in Definition 2.5.7 requires an orientation of the sections $\sigma_{(p_-, p_+; A)}$ and $\sigma_{(p_-, p_+; A)}^\iota$. As in Remark 2.4.7 this is determined via the fiber product construction in Lemma 2.5.6 from the orientations of the Morse trajectory spaces in Remark 2.3.5 (i) and the orientations of $\sigma_{\text{GW}}, \sigma_{\text{SFT}}$ given in Assumption 2.5.5. In practice, we will construct the perturbations $\underline{\kappa}, \underline{\kappa}^\iota$ by pullback of perturbations $\underline{\lambda} = (\lambda_A)_{A \in H_2(M)}$ of the SFT-sections σ_{SFT} and their restriction $\underline{\lambda}^\iota$ to $\{0\} \times \mathcal{B}_{\text{GW}}(A) \subset \partial \mathcal{B}_{\text{SFT}}(A)$. So we can specify the orientations of the regularized zero sets by expressing them as transverse fiber products of oriented spaces over $\mathbb{CP}^1 \times M$ or $\mathbb{C}^\pm \times M$,

$$\begin{aligned} Z^{\underline{\kappa}^\iota}(p_-, p_+; A) &= \overline{\mathcal{M}}(p_-, M)_{\text{ev}_0^+ \times \overline{\text{ev}}^+} Z^{\underline{\lambda}^\iota}(A)_{\overline{\text{ev}}^- \times \text{ev}_0^-} \overline{\mathcal{M}}(M, p_+), \\ &= \overline{\mathcal{M}}(p_-, M)_{\text{ev}_0^+ \times \text{ev}^+} (Z^{\underline{\lambda}^\iota}(A) \cap \mathcal{B}_{\text{GW}}^{+, -}(A))_{\text{ev}^- \times \text{ev}_0^-} \overline{\mathcal{M}}(M, p_+), \\ Z^{\underline{\kappa}}(p_-, p_+; A) &= \overline{\mathcal{M}}(p_-, M)_{\text{ev}_0^+ \times \text{ev}^+} (Z^{\underline{\lambda}}(A) \cap \mathcal{B}_{\text{SFT}}^{+, -}(A))_{\text{ev}^- \times \text{ev}_0^-} \overline{\mathcal{M}}(M, p_+), \end{aligned}$$

using $\overline{\text{ev}}^\pm : \mathcal{B}_{\text{GW}}(A) \rightarrow \mathbb{CP}^1 \times M$ resp. $\text{ev}^\pm : \mathcal{B}_{\text{GW/SFT}}^{+, -}(A) \rightarrow \mathbb{C}^\pm \times M$ and the Morse evaluations $\text{ev}_0^\pm : \overline{\mathcal{M}}(\dots) \rightarrow \mathbb{CP}^1 \times M, \underline{\tau} \mapsto (z_0^\pm, \text{ev}(\underline{\tau}))$ resp. $\text{ev}_0^\pm : \overline{\mathcal{M}}(\dots) \rightarrow \mathbb{C}^\pm \times M, \underline{\tau} \mapsto (0, \text{ev}(\underline{\tau}))$.

2.6 Algebraic relations via coherent perturbations

In this section we prove parts (i)–(iii) of Theorem 2.1.2, that is the algebraic properties which relate the maps $PSS : CM \rightarrow CF$, $SSP : CF \rightarrow CM$ constructed in §2.4, and the maps $\iota : CM \rightarrow CM$, $h : CM \rightarrow CM$ constructed in §2.5. More precisely, we will make so-called “coherent” choices of perturbations in §2.6.2, §2.6.3, and §2.6.4 which guarantee that (i) ι is a chain map, (ii) ι is a A -module isomorphism, and (iii) h is a chain homotopy between the composition $SSP \circ PSS$ and ι .

2.6.1 Coherent polyfold descriptions of moduli spaces

The general approach to obtaining not just counts as discussed in §2.4.2 but well-defined algebraic structures from moduli spaces of pseudoholomorphic curves is to replace them by compact manifolds with boundary and corners (or generalizations thereof which still carry ‘relative virtual fundamental classes’) in such a manner that their boundary strata are given by Cartesian products of each other. In the context of polyfold theory, this requires a description of the compactified moduli spaces $\overline{\mathcal{M}}(\alpha) = \sigma_\alpha^{-1}(0)$ as zero sets of a “coherent collection” of sc-Fredholm sections $(\sigma_\alpha : \mathcal{B}(\alpha) \rightarrow \mathcal{E}(\alpha))_{\alpha \in \mathcal{I}}$ of strong polyfold bundles. Here “coherence” indicates a well organized identification of the boundaries $\partial \mathcal{B}(\alpha)$ with unions of Cartesian products of other polyfolds in the collection $\mathcal{I}$, which is compatible with the bundles and sections.

As a first example, the moduli spaces $\mathcal{M}^\iota(p_-, p_+; A)$ in §2.5.1 which yield the map $\iota : CM \rightarrow CM$ are given polyfold descriptions $\sigma_{(p_-, p_+; A)}^\iota : \mathcal{B}^\iota(p_-, p_+; A) \rightarrow \phi_\iota^* \mathcal{E}(p_-, p_+; A)$ in Lemma 2.5.6 that arise as fiber products with polyfolds $\mathcal{B}_{\text{GW}}(A)$ without boundary. Thus their coherence properties stated below follow from properties of the fiber product in [15] and the boundary stratification of the Morse trajectory spaces in Theorem 2.3.3. We state this result to illustrate the notion of coherence. The full technical statement – on the level of ep-groupoids and including compatibility with bundles and sections – can be found in the second bullet point of Lemma 2.6.4.

Lemma 2.6.1. *For any $p_\pm \in \text{Crit}(f)$ and $A \in H_2(M)$ the smooth level of the first boundary stratum of the fiber product $\mathcal{B}^\iota(p_-, p_+; A)$ in Lemma 2.5.6 is naturally identified with*

$$\partial_1 \mathcal{B}^\iota(p_-, p_+; A)_\infty \cong \bigcup_{q \in \text{Crit}(f)} \mathcal{M}(p_-, q) \times \partial_0 \mathcal{B}^\iota(q, p_+; A)_\infty \sqcup \bigcup_{q \in \text{Crit}(f)} \partial_0 \mathcal{B}^\iota(p_-, q; A)_\infty \times \mathcal{M}(q, p_+).$$

Proof. By the fiber product construction [15, Cor.7.3] of $\mathcal{B}^\iota(p_-, p_+; A)$ in Lemma 2.5.6, the degeneracy index satisfies $d_{\mathcal{B}^\iota(p_-, p_+; A)}(\underline{\tau}^-, \underline{v}, \underline{\tau}^+) = d_{\overline{\mathcal{M}}(p_-, M)}(\underline{\tau}^-) + d_{\mathcal{B}_{\text{GW}}(A)}(\underline{v}) + d_{\overline{\mathcal{M}}(M, p_+)}(\underline{\tau}^+)$, and the smooth level given by the fiber product of smooth levels, that is, $\mathcal{B}^\iota(p_-, p_+; A)_\infty = \overline{\mathcal{M}}(p_-, M)_{\{z_0^-\} \times_{\text{ev}} \times_{\text{ev}} \mathcal{B}_{\text{GW}}^{+, -}(A)_\infty} \times_{\{z_0^+\} \times_{\text{ev}}} \overline{\mathcal{M}}(M, p_+)$. The polyfold $\mathcal{B}_{\text{GW}}(A)$ and its open subset $\mathcal{B}_{\text{GW}}^{+, -}(A)$ are boundaryless by Assumption 2.5.5 (iii), which means $d_{\mathcal{B}_{\text{GW}}(A)} = d_{\mathcal{B}_{\text{GW}}^{+, -}(A)} \equiv 0$. Hence we have $d_{\mathcal{B}^\iota(p_-, p_+; A)}(\underline{\tau}^-, \underline{v}, \underline{\tau}^+) = 1$ if and only if $\underline{\tau}^- \in \partial_1 \overline{\mathcal{M}}(p_-, M)$ and

$\tau^+ \in \partial_0 \overline{\mathcal{M}}(M, p_+)$ or the other way around. These two cases are disjoint but analogous, so it remains to show that the first case consists of points in the union $\bigcup_{q \in \text{Crit}(f)} \mathcal{M}(p_-, q) \times \partial_0 \mathcal{B}^u(q, p_+; A)_\infty$. For that purpose, we recall the boundary identification $\partial_1 \overline{\mathcal{M}}(p_-, M) = \bigcup_{q \in \text{Crit}(f)} \mathcal{M}(p_-, q) \times \mathcal{M}(q, M)$ in Theorem 2.3.3, which is compatible with the evaluation $\text{ev} : \mathcal{M}(p_-, q) \times \mathcal{M}(q, M) \rightarrow M, (\tau_1, \tau_2) \mapsto \text{ev}(\tau_2)$ by construction, and thus

$$\begin{aligned} \partial_1 \overline{\mathcal{M}}(p_-, M) & \big|_{\{z_0^-\} \times \text{ev}^- \times \text{ev}^-} \mathcal{B}_{\text{GW}}^{+, -}(A)_\infty \big|_{\text{ev}^+ \times \{z_0^+\} \times \text{ev}} \partial_0 \overline{\mathcal{M}}(M, p_+) \\ &= \left(\bigcup_{q \in \text{Crit}(f)} \mathcal{M}(p_-, q) \times \mathcal{M}(q, M) \right) \big|_{\{z_0^-\} \times \text{ev}^- \times \text{ev}^-} \mathcal{B}_{\text{GW}}^{+, -}(A)_\infty \big|_{\text{ev}^+ \times \{z_0^+\} \times \text{ev}} \mathcal{M}(M, p_+) \\ &= \bigcup_{q \in \text{Crit}(f)} \mathcal{M}(p_-, q) \times \left(\mathcal{M}(q, M) \big|_{\{z_0^-\} \times \text{ev}^- \times \text{ev}^-} \mathcal{B}_{\text{GW}}^{+, -}(A)_\infty \big|_{\text{ev}^+ \times \{z_0^+\} \times \text{ev}} \mathcal{M}(M, p_+) \right) \\ &= \bigcup_{q \in \text{Crit}(f)} \mathcal{M}(p_-, q) \times \partial_0 \mathcal{B}^u(q, p_+; A)_\infty \end{aligned}$$

Here we also used the identification of the interior smooth level in Lemma 2.5.6. $\square$

Next, the polyfold description in Lemma 2.5.6 for the moduli spaces $\mathcal{M}(p_-, p_+; A)$ in §2.5.2, which yield the map $h : CM \rightarrow CM$, are obtained as fiber products of the Morse trajectory spaces with polyfold descriptions $\sigma_{\text{SFT}} : \mathcal{B}_{\text{SFT}}(A) \rightarrow \mathcal{E}_{\text{SFT}}(A)$ of SFT moduli spaces given in [19, 18]. We will state as assumption only those parts of their coherence properties that are relevant to our argument in §2.6.4 for the chain homotopy $\iota - SSP \circ PSS = d \circ h + h \circ d$. Here the contributions to $d \circ h + h \circ d$ will arise from boundary strata of the Morse trajectory spaces, whereas $\iota - SSP \circ PSS$ arises from the following identification of the boundary of the polyfold $\mathcal{B}_{\text{SFT}}^{+, -}(A)$, which is given as open subset of $\mathcal{B}_{\text{SFT}}(A)$ in Assumption 2.5.5 (vi).⁹

Remark 2.6.2. In the following we will use the word “face” loosely for Cartesian products of polyfolds such as $\mathcal{F} = \mathcal{B}_{\text{SFT}}^+(\gamma; A_+) \times \mathcal{B}_{\text{SFT}}^-(\gamma; A_-)$ and their immersions into the boundary of another polyfold such as $\partial \mathcal{B}_{\text{SFT}}^{+, -}(A)$. We also refer to the image of the immersion $\mathcal{F} \hookrightarrow \partial \mathcal{B}_{\text{SFT}}^{+, -}(A)$ as a face of $\mathcal{B}_{\text{SFT}}^{+, -}(A)$. Compared with the formal definition of faces in [34, Definitions 2.21, 11.1, 16.13], ours are disjoint unions of faces and carry more structure, as we will specify.

Assumption 2.6.3. *The collection of oriented sc-Fredholm sections of strong polyfold bundles $\sigma_{\text{SFT}}^\pm : \mathcal{B}_{\text{SFT}}^\pm(\gamma; A) \rightarrow \mathcal{E}_{\text{SFT}}(\gamma; A)$, $\sigma_{\text{GW}} : \mathcal{B}_{\text{GW}}(A) \rightarrow \mathcal{E}_{\text{GW}}(A)$, $\sigma_{\text{SFT}} : \mathcal{B}_{\text{SFT}}(A) \rightarrow \mathcal{E}_{\text{SFT}}(A)$ for $\gamma \in \mathcal{P}(H)$ and $A \in H_2(M)$ together with the evaluation maps $\overline{\text{ev}}^\pm : \mathcal{B}_{\text{SFT}}^\pm(\gamma; A) \rightarrow \mathbb{C}^\pm \times M$, $\overline{\text{ev}}^\pm : \mathcal{B}_{\text{GW}}(A) \rightarrow \mathbb{CP}^1 \times M$, $\overline{\text{ev}}^\pm : \mathcal{B}_{\text{SFT}}(A) \rightarrow \overline{\mathbb{CP}^1} \times M$, and their sc^∞ restrictions on open subsets, $\text{ev}^\pm : \mathcal{B}_{\text{SFT}}^{\pm, \mathbb{C}}(\gamma; A) \rightarrow \mathbb{C}^\pm \times M$, $\text{ev}^\pm : \mathcal{B}_{\text{GW/SFT}}^{+, -}(A) \rightarrow \mathbb{C}^\pm \times M$ from Assumptions 2.4.3, 2.5.5 has the following coherence properties.*

(i) *For each $\gamma \in \mathcal{P}(H)$ and $A_-, A_+ \in H_2(M)$ such that $A_- + A_+ = A$, there is a sc^∞ immersion*

$$l_{\gamma, A_\pm} : \mathcal{B}_{\text{SFT}}^+(\gamma; A_+) \times \mathcal{B}_{\text{SFT}}^-(\gamma; A_-) \rightarrow \partial \mathcal{B}_{\text{SFT}}(A)$$

⁹See also the end of Remark 2.5.4 (ii) for the motivation of $\mathcal{B}_{\text{SFT}}^{+, -}(A)$ as open subset that intersects the boundary strata $\lim_{R \rightarrow \infty} \{R\} \times \mathcal{B}_{\text{GW}}(A) \subset \partial \mathcal{B}_{\text{SFT}}(A)$ in the buildings which have one marked point in each of the components mapping to $\mathbb{C}^\pm \times M$, and no marked points mapping to intermediate cylinders $\mathbb{R} \times S^1 \times M$.

whose restriction to the interior $\partial_0 \mathcal{B}_{\text{SFT}}^+(\gamma; A_+) \times \partial_0 \mathcal{B}_{\text{SFT}}^-(\gamma; A_-)$ is an embedding into the boundary of the open subset $\mathcal{B}_{\text{SFT}}^{+,-}(A) \subset \mathcal{B}_{\text{SFT}}(A)$. They map into the limit set $\lim_{R \rightarrow \infty} \{R\} \times \mathcal{B}_{\text{GW}}(A)$ from Assumption 2.5.5(iii), so cover most of the boundary¹⁰

$$\partial \mathcal{B}_{\text{SFT}}(A) \supset \{0\} \times \mathcal{B}_{\text{GW}}(A) \sqcup \bigcup_{\substack{\gamma \in \mathcal{P}(H) \\ A_- + A_+ = A}} l_{\gamma, A_{\pm}} (\mathcal{B}_{\text{SFT}}^+(\gamma; A_+) \times \mathcal{B}_{\text{SFT}}^-(\gamma; A_-)).$$

(ii) The union of the images $l_{\gamma, A_{\pm}} (\mathcal{B}_{\text{SFT}}^{+, \mathbb{C}}(\gamma; A_+) \times \mathcal{B}_{\text{SFT}}^{-, \mathbb{C}}(\gamma; A_-)) \subset \partial \mathcal{B}_{\text{SFT}}^{+,-}(A)$ for all admissible choices of $\gamma, A_{\pm}$ is the intersection of $\mathcal{B}_{\text{SFT}}^{+,-}(A)$ with $\lim_{R \rightarrow \infty} \{R\} \times \mathcal{B}_{\text{GW}}(A) \subset \partial \mathcal{B}_{\text{SFT}}(A)$, i.e.

$$\begin{aligned} \partial \mathcal{B}_{\text{SFT}}^{+,-}(A) &= \{0\} \times \mathcal{B}_{\text{GW}}^{+,-}(A) \sqcup \partial^{R=\infty} \mathcal{B}_{\text{SFT}}^{+,-}(A), \\ \text{where } \partial^{R=\infty} \mathcal{B}_{\text{SFT}}^{+,-}(A) &= \bigcup_{\substack{\gamma \in \mathcal{P}(H) \\ A_- + A_+ = A}} l_{\gamma, A_{\pm}} (\mathcal{B}_{\text{SFT}}^{+, \mathbb{C}}(\gamma; A_+) \times \mathcal{B}_{\text{SFT}}^{-, \mathbb{C}}(\gamma; A_-)). \end{aligned}$$

When restricted to the interiors, this yields a disjoint cover of the top boundary stratum,

$$\partial_1 \mathcal{B}_{\text{SFT}}^{+,-}(A) = \{0\} \times \mathcal{B}_{\text{GW}}^{+,-}(A) \sqcup \bigsqcup_{\substack{\gamma \in \mathcal{P}(H) \\ A_- + A_+ = A}} l_{\gamma, A_{\pm}} (\partial_0 \mathcal{B}_{\text{SFT}}^+(\gamma; A_+) \times \partial_0 \mathcal{B}_{\text{SFT}}^-(\gamma; A_-)).$$

(iii) The immersions $l_{\gamma, A_{\pm}}$ are compatible with the evaluation maps, bundles, and sections – as required for the construction [18] of coherent perturbations for SFT, that is:

- (a) The boundary restriction of the evaluation maps $\overline{\text{ev}}^{\pm}|_{\{0\} \times \mathcal{B}_{\text{GW}}(A) \subset \partial \mathcal{B}_{\text{SFT}}(A)}$ coincides with the Gromov-Witten evaluation maps $\overline{\text{ev}}^{\pm} : \mathcal{B}_{\text{GW}}(A) \rightarrow \overline{\mathbb{CP}}_{\infty}^1$, and the same holds for their sc^{∞} restriction $\text{ev}^+ \times \text{ev}^-|_{\{0\} \times \mathcal{B}_{\text{GW}}^{+,-}(A) \subset \partial \mathcal{B}_{\text{SFT}}^{+,-}(A)} = \text{ev}^+ \times \text{ev}^- : \mathcal{B}_{\text{GW}}^{+,-}(A) \rightarrow \mathbb{C}^+ \times M \times \mathbb{C}^- \times M$ with values in $\mathbb{C}^{\pm} \subset \overline{\mathbb{CP}}_{\infty}^1 = \mathbb{C}^+ \sqcup S^1 \sqcup \mathbb{C}^-$. The restriction of $\overline{\text{ev}}^{\pm} : \mathcal{B}_{\text{SFT}}(A) \rightarrow \overline{\mathbb{CP}}_{\infty}^1$ to each boundary face $\text{im } l_{\gamma, A_{\pm}} \subset \partial \mathcal{B}_{\text{SFT}}(A)$ takes values in $\overline{\mathbb{C}}^{\pm} \subset \overline{\mathbb{CP}}_{\infty}^1$, and its pullback under $l_{\gamma, A_{\pm}}$ coincides with $\overline{\text{ev}}^{\pm} : \mathcal{B}_{\text{SFT}}^{\pm}(\gamma; A_{\pm}) \rightarrow \overline{\mathbb{C}}^{\pm} \times M$. Moreover, pullback of the restricted sc^{∞} evaluations $\text{ev}^+ \times \text{ev}^- : \mathcal{B}_{\text{SFT}}^{+,-}(A) \rightarrow \mathbb{C}^+ \times M \times \mathbb{C}^- \times M$ under $l_{\gamma, A_{\pm}}$ coincides with $\text{ev}^+ \times \text{ev}^- : \mathcal{B}_{\text{SFT}}^{+, \mathbb{C}}(\gamma; A_+) \times \mathcal{B}_{\text{SFT}}^{-, \mathbb{C}}(\gamma; A_-) \rightarrow \mathbb{C}^+ \times M \times \mathbb{C}^- \times M$.
- (b) The restriction of σ_{SFT} to $\mathcal{F} = \{0\} \times \mathcal{B}_{\text{GW}}(A) \subset \partial \mathcal{B}_{\text{SFT}}(A)$ equals to σ_{GW} via a natural identification $\mathcal{E}_{\text{SFT}}(A)|_{\mathcal{F}} \cong \mathcal{E}_{\text{GW}}(A)$. This identification reverses the orientation of sections.
- (c) The restriction of σ_{SFT} to each face $\mathcal{F} = \mathcal{B}_{\text{SFT}}^+(\gamma; A_+) \times \mathcal{B}_{\text{SFT}}^-(\gamma; A_-) \subset \partial \mathcal{B}_{\text{SFT}}(A)$ is related by pullback to $\sigma_{\text{SFT}}^+ \times \sigma_{\text{SFT}}^- = \sigma_{\text{SFT}} \circ l_{\gamma, A_{\pm}}$ via a naturally defined identification $l_{\gamma, A_{\pm}}^* \mathcal{E}_{\text{SFT}}(A) \cong \mathcal{E}_{\text{SFT}}^+(\gamma; A_+) \times \mathcal{E}_{\text{SFT}}^-(\gamma; A_-)$. This identification preserves the orientation of sections.

¹⁰ The extra boundary faces of $\mathcal{B}_{\text{SFT}}(A)$ arise from both marked points mapping to the same component in the $R \rightarrow \infty$ neck stretching limit. These will not be relevant to our construction of coherent perturbations.

2.6.2 Coherent perturbations for chain map identity

In this section we prove Theorem 2.1.2 (i), that is we construct $\iota_{\underline{\kappa}^\iota}$ in Definition 2.5.7 as a chain map on the Morse complex (2.5) with differential $d : CM \rightarrow CM$ given by (2.4). This requires the following construction of the perturbations $\underline{\kappa}^\iota$ that is coherent in the sense that it is compatible with the boundary identifications of the polyfolds $\mathcal{B}^\iota(p_-, p_+; A)$ in Lemma 2.6.1. Here we will indicate smooth levels by adding ∞ as superscript – denoting e.g. $\mathcal{X}_{p_-, p_+; A}^{\iota, \infty}$ as the smooth level of an ep-groupoid representing $\mathcal{B}^\iota(p_-, p_+; A)_\infty$.

Lemma 2.6.4. *There is a choice of $(\kappa_\alpha^\iota)_{\alpha \in \mathcal{I}}$ in Corollary 2.4.6 for $\mathcal{I} = \{(p_-, p_+; A) \mid p_\pm \in \text{Crit}(f), A \in H_2(M)\}$ that is coherent w.r.t. the identifications in Lemma 2.6.1 in the following sense.*

- Each $\kappa_\alpha^\iota : \mathcal{W}_\alpha^\iota \rightarrow \mathbb{Q}^+$ for $\alpha \in \mathcal{I}$ is an admissible sc^+ -multisection of a strong bundle $P_\alpha : \mathcal{W}_\alpha^\iota \rightarrow \mathcal{X}_\alpha^\iota$ that is in general position to a sc -Fredholm section functor $S_\alpha^\iota : \mathcal{X}_\alpha^\iota \rightarrow \mathcal{W}_\alpha^\iota$ which represents $\sigma_\alpha^\iota|_{\mathcal{V}_\alpha}$ on an open neighborhood $\mathcal{V}_\alpha \subset \mathcal{B}^\iota(\alpha)$ of the zero set $\sigma_\alpha^{-1}(0)$.
- The identification of top boundary strata in Lemma 2.6.1 holds for the representing ep-groupoids,

$$\partial_1 \mathcal{X}_{p_-, p_+; A}^{\iota, \infty} \cong \bigcup_{q \in \text{Crit}(f)} \mathcal{M}(p_-, q) \times \partial_0 \mathcal{X}_{q, p_+; A}^{\iota, \infty} \quad \sqcup \quad \bigcup_{q \in \text{Crit}(f)} \partial_0 \mathcal{X}_{p_-, q; A}^{\iota, \infty} \times \mathcal{M}(q, p_+),$$

and the oriented section functors $S_\alpha^\iota : \mathcal{X}_\alpha^\iota \rightarrow \mathcal{W}_\alpha^\iota$ are compatible with these identifications in the sense that the restriction of $S_{p_-, p_+; A}^\iota$ to any face $\mathcal{F}_{(p_-, q_-), \alpha'}^\infty := \mathcal{M}(p_-, q_-) \times \partial_0 \mathcal{X}_{\alpha'}^{\iota, \infty} \subset \partial_1 \mathcal{X}_{p_-, p_+; A}^{\iota, \infty}$ resp. $\mathcal{F}_{\alpha', (q_+, p_+)}^\infty := \partial_0 \mathcal{X}_{\alpha'}^{\iota, \infty} \times \mathcal{M}(q_+, p_+) \subset \partial_1 \mathcal{X}_{p_-, p_+; A}^{\iota, \infty}$ for another $\alpha' \in \mathcal{I}$ coincides on the smooth level with the pullback $S_\alpha^\iota|_{\mathcal{F}^\infty} = \text{pr}_{\mathcal{F}}^* S_{\alpha'}^\iota|_{\mathcal{F}^\infty}$ of $S_{\alpha'}^\iota$ via the projection $\text{pr}_{\mathcal{F}} : \mathcal{F} = \mathcal{F}_{(p_-, q_-), \alpha'} := \mathcal{M}(p_-, q_-) \times \partial_0 \mathcal{X}_{\alpha'}^\iota \rightarrow \mathcal{X}_{\alpha'}^\iota$ resp. $\text{pr}_{\mathcal{F}} : \mathcal{F} = \mathcal{F}_{\alpha', (q_+, p_+)} := \partial_0 \mathcal{X}_{\alpha'}^\iota \times \mathcal{M}(q_+, p_+) \rightarrow \mathcal{X}_{\alpha'}^\iota$.

- Each restriction $\kappa_\alpha^\iota|_{P_\alpha^{-1}(\mathcal{F}^\infty)}$ to a face $\mathcal{F}^\infty = \mathcal{F}_{(p_-, q_-), \alpha'}^\infty$ resp. $\mathcal{F}^\infty = \mathcal{F}_{\alpha', (q_+, p_+)}^\infty$ is given by pullback $\kappa_\alpha^\iota|_{P_\alpha^{-1}(\mathcal{F}^\infty)} = \kappa_{\alpha'}^\iota \circ \text{pr}_{\mathcal{F}}^*$ via the identification $P_\alpha^{-1}(\mathcal{F}^\infty) \cong \text{pr}_{\mathcal{F}}^* \mathcal{W}_{\alpha'}^\iota|_{\partial_0 \mathcal{X}_{\alpha'}^{\iota, \infty}}$ and natural map $\text{pr}_{\mathcal{F}}^* : \text{pr}_{\mathcal{F}}^* \mathcal{W}_{\alpha'}^\iota \rightarrow \mathcal{W}_\alpha^\iota$.

For any such choice of $\underline{\kappa}^\iota = (\kappa_\alpha^\iota)_{\alpha \in \mathcal{I}}$, the resulting map $\iota_{\underline{\kappa}^\iota} : CM \rightarrow CM$ in Definition 2.5.7 satisfies $\iota_{\underline{\kappa}^\iota} \circ d + d \circ \iota_{\underline{\kappa}^\iota} = 0$. By setting $\iota\langle p \rangle := (-1)^{|p|} \iota_{\underline{\kappa}^\iota}\langle p \rangle$ we then obtain a chain map $\iota : C_* M \rightarrow C_* M$, that is $\iota \circ d = d \circ \iota$.

Proof. We will first assume the claimed coherence and discuss the algebraic consequences up to signs, then construct the coherent data, and finally use this construction to compute the orientations.

Construction of chain map: Assuming $\iota_{\underline{\kappa}^\iota} \circ d + d \circ \iota_{\underline{\kappa}^\iota} = 0$, recall that d decreases the degree on the Morse complex (2.6) by 1. Thus $\iota : C_* M \rightarrow C_* M$ defined as above satisfies, for any $q \in \text{Crit}(f)$, $(\iota \circ d - d \circ \iota)\langle q \rangle = (-1)^{|q|-1} \iota_{\underline{\kappa}^\iota}(d\langle q \rangle) - d((-1)^{|q|} \iota_{\underline{\kappa}^\iota}\langle q \rangle) = (-1)^{|q|-1} (\iota_{\underline{\kappa}^\iota} \circ d + d \circ \iota_{\underline{\kappa}^\iota})\langle q \rangle = 0$. By A -linearity this proves $\iota \circ d = d \circ \iota$ on $C_* M$.

Proof of identity: To prove $\iota_{\kappa^\iota} \circ d + d \circ \iota_{\kappa^\iota} = 0$ note that both ι_{κ^ι} and d are Λ -linear, so the claimed identity is equivalent to the collection of identities $(\iota_{\kappa^\iota} \circ d)\langle p_- \rangle + (d \circ \iota_{\kappa^\iota})\langle p_- \rangle = 0$ for all generators $p_- \in \text{Crit}(f)$. That is we wish to verify that the following sum vanishes:

$$\sum_{\substack{q, p_+, A \\ I^\iota(q, p_+, A) = 0 \\ |q| = |p_-| - 1}} \# \mathcal{M}(p_-, q) \cdot \# Z^{\kappa^\iota}(q, p_+, A) \cdot T^{\omega(A)} \langle p_+ \rangle + \sum_{\substack{q, p_+, A \\ I^\iota(p_-, q, A) = 0 \\ |p_+| = |q| - 1}} \# Z^{\kappa^\iota}(p_-, q, A) \cdot \# \mathcal{M}(q, p_+) \cdot T^{\omega(A)} \langle p_+ \rangle.$$

Here, by the index formula (2.18), both sides can be written as sums over $p_+ \in \text{Crit}(f)$ and $A \in H_2(M)$ for which $I^\iota(p_-, p_+, A) = 1$. Then it suffices to prove for any such pair $\alpha = (p_-, p_+, A)$ with $I^\iota(\alpha) = 1$

$$\sum_{|q|=|p_-|-1} \# \mathcal{M}(p_-, q) \cdot \# Z^{\kappa^\iota}(q, p_+, A) + \sum_{|q|=|p_+|+1} \# Z^{\kappa^\iota}(p_-, q, A) \cdot \# \mathcal{M}(q, p_+) = 0. \quad (2.30)$$

This identity will follow by applying Corollary 2.4.6 (v) to the sc^+ -multisection $\kappa_\alpha : \mathcal{W}_\alpha^\iota \rightarrow \mathbb{Q}^+$. Its perturbed zero set is a weighted branched 1-dimensional orbifold $Z^{\kappa^\iota}(\alpha)$, whose boundary is given by the intersection with the smooth level¹¹ of the top boundary stratum $\partial_1 \mathcal{B}^\iota(\alpha) \cap \mathcal{V}_\alpha = |\partial_1 \mathcal{X}_\alpha^\iota|$. By coherence (and with orientations discussed below) this boundary is

$$\begin{aligned} \partial Z^{\kappa^\iota}(\alpha) &= Z^{\kappa^\iota}(\alpha) \cap |\partial_1 \mathcal{X}_\alpha^\iota| \\ &= \bigcup_{q \in \text{Crit}(f)} Z^{\kappa^\iota}(\alpha) \cap (\mathcal{M}(p_-, q) \times |\partial_0 \mathcal{X}_{q, p_+, A}^\iota|) \\ &\quad \sqcup \bigcup_{q \in \text{Crit}(f)} Z^{\kappa^\iota}(\alpha) \cap (|\partial_0 \mathcal{X}_{p_-, q, A}^\iota| \times \mathcal{M}(q, p_+)) \\ &= \bigcup_{q \in \text{Crit}(f)} \mathcal{M}(p_-, q) \times (Z^{\kappa^\iota}(q, p_+, A) \cap |\partial_0 \mathcal{X}_{q, p_+, A}^\iota|) \\ &\quad \sqcup \bigcup_{q \in \text{Crit}(f)} (Z^{\kappa^\iota}(p_-, q, A) \cap |\partial_0 \mathcal{X}_{p_-, q, A}^\iota|) \times \mathcal{M}(q, p_+), \\ &= \bigcup_{|q|=|p_-|-1} \mathcal{M}(p_-, q) \times Z^{\kappa^\iota}(q, p_+, A) \quad \sqcup \quad \bigcup_{|q|=|p_+|+1} Z^{\kappa^\iota}(p_-, q, A) \times \mathcal{M}(q, p_+). \end{aligned}$$

Here the first summand of the third identification on the level of object spaces,

$$\begin{aligned} &\{([\tau], x) \in \mathcal{M}(p_-, q) \times \partial_0 X_{q, p_+, A} \subset \partial_1 X_\alpha \mid \kappa_\alpha(S_\alpha^\iota([\tau], x)) > 0\} \\ &\cong \{([\tau], x) \in \mathcal{M}(p_-, q) \times \partial_0 X_{q, p_+, A} \mid \kappa_{q, p_+, A}(S_{q, p_+, A}^\iota(x)) > 0\} \\ &= \mathcal{M}(p_-, q) \times \{x \in \partial_0 X_{q, p_+, A} \mid \kappa_{q, p_+, A}(S_{q, p_+, A}^\iota(x)) > 0\}, \end{aligned}$$

follows if we assume coherence of sections and multisections on the faces $\mathcal{F}_{(p_-, q), \alpha'} \subset \partial_1 \mathcal{X}_\alpha^\iota$,

$$\kappa_\alpha(S_\alpha^\iota([\tau], x)) = \kappa_\alpha(S_{q, p_+, A}^\iota(x)) = \kappa_{q, p_+, A}(S_{q, p_+, A}^\iota(x)).$$

The second summand is identified similarly by assuming coherence on the faces $\mathcal{F}_{\alpha', (q_-, p_+)} \subset \partial_1 \mathcal{X}_\alpha^\iota$.

¹¹ Here and in the following we suppress indications of the smooth level, as the perturbed zero sets automatically lie in the smooth level; see Remark 2.7.3.

Finally, the fourth identification in $\partial Z^{\kappa'}(\alpha)$ for $\alpha = (p_-, p_+; A)$ with $I^\iota(\alpha) = 1$ follows from index and regularity considerations as follows. Corollary 2.4.6 (iii),(iv) guarantees that the perturbed solution spaces $Z^{\kappa'}(\alpha')$ are nonempty only for Fredholm index $I^\iota(\alpha') \geq 0$, and for $I^\iota(\alpha') = 0$ are contained in the interior, $Z^{\kappa'}(\alpha') \subset \partial_0 \mathcal{B}(\alpha')$. The Morse trajectory spaces $\mathcal{M}(p_-, q)$ resp. $\mathcal{M}(q, p_+)$ are nonempty only for $|p_-| - |q| \geq 1$ resp. $|q| - |p_+| \geq 1$, so the perturbed solution spaces in the Cartesian products have Fredholm index (2.18)

$$I^\iota(q, p_+; A) = 2c_1(A) + |q| - |p_+| = I^\iota(p_-, p_+; A) + |q| - |p_-| = 1 + |q| - |p_-| \leq 0,$$

and analogously $I^\iota(p_-, q; A) = I^\iota(p_-, p_+; A) + |p_+| - |q| \leq 0$. By the above regularity of the perturbed solution spaces this implies that the unions on the left hand side of the fourth identification are over $|q| = |p_-| - 1$ resp. $|q| = |p_+| + 1$ as in (2.30), and for these critical points we have the inclusions $Z^{\kappa'}(q, p_+; A) \subset \partial_0 \mathcal{B}(q, p_+; A)$ and $Z^{\kappa'}(p_-, q; A) \subset \partial_0 \mathcal{B}(p_-, q; A)$ that verify the equality.

This finishes the identification of the boundary $\partial Z^{\kappa'}(\alpha)$. Now Corollary 2.4.6 (v) asserts that the sum of weights over this boundary is zero – when counted with signs that are induced by the orientation of $Z^{\kappa'}(\alpha)$. So in order to prove the identity (2.30) we need to compare the boundary orientation of $\partial Z^{\kappa'}(\alpha)$ with the orientations on the faces. We will compute the relevant signs in (2.31) below, after first making coherent choices of representatives $S_\alpha^\iota : \mathcal{X}_\alpha^\iota \rightarrow \mathcal{W}_\alpha^\iota$ of the oriented sections σ_α^ι , and constructing coherent sc^+ -multisections $\kappa_\alpha^\iota : \mathcal{W}_\alpha^\iota \rightarrow \mathbb{Q}^+$ for $\alpha \in \mathcal{I}$.

Coherent ep-groupoids, sections, and perturbations: Recall that the fiber product construction in Lemma 2.5.6 defines each bundle $\mathcal{W}_\alpha^\iota = \text{pr}_\alpha^* \mathcal{W}_A^{\text{GW}}$ for $\alpha = (p_-, p_+; A) \in \mathcal{I}$ as the pullback of a strong bundle $\mathcal{W}_A^{\text{GW}} \rightarrow \mathcal{X}_A^{\text{GW}}$ under a projection of ep-groupoids – with abbreviated notation $\text{ev}_0^\pm := \{z_0^\pm\} \times \text{ev} : \overline{\mathcal{M}}(\dots) \rightarrow \mathbb{CP}^1 \times M$ –

$$\text{pr}_{p_-, p_+; A} : \mathcal{X}_{p_-, p_+; A}^\iota = \overline{\mathcal{M}}(p_-, M) \underset{\text{ev}_0^-}{\times_{\overline{\text{ev}}^-}} \mathcal{X}_A^{\text{GW}} \underset{\overline{\text{ev}}^+}{\times_{\text{ev}_0^+}} \overline{\mathcal{M}}(M, p_+) \longrightarrow \mathcal{X}_A^{\text{GW}}.$$

Moreover, the section $S_\alpha^\iota = S_A^{\text{GW}} \circ \text{pr}_\alpha$ is induced by the section $S_A^{\text{GW}} : \mathcal{X}_A^{\text{GW}} \rightarrow \mathcal{W}_A^{\text{GW}}$ which cuts out the Gromov-Witten moduli space $\overline{\mathcal{M}}_{\text{GW}}(A) = |(S_A^{\text{GW}})^{-1}(0)|$. Then the identification of the top boundary stratum proceeds exactly as the proof of Lemma 2.6.1. Coherence of the bundles and sections follows from coherence of the projections $\text{pr}_\alpha : \mathcal{X}_\alpha^\iota \rightarrow \mathcal{X}_A^{\text{GW}}$ in the sense that $\text{pr}_\alpha|_{\mathcal{F}^\infty} = \text{pr}_{\alpha'} \circ \text{pr}_{\mathcal{F}}$ for all smooth levels of faces $\mathcal{F} \supset \mathcal{F}^\infty \subset \partial_1 \mathcal{X}_\alpha^\iota$ and their projections $\text{pr}_{\mathcal{F}} : \mathcal{F} = \mathcal{F}_{(p_-, q_-), \alpha'} \rightarrow \mathcal{X}_{\alpha'}^\iota$ resp. $\text{pr}_{\mathcal{F}} : \mathcal{F} = \mathcal{F}_{\alpha', (q_-, p_+)} \rightarrow \mathcal{X}_{\alpha'}^\iota$. For example, the face $\mathcal{F} = \mathcal{F}_{(p_-, q_-), (q_-, p_+; A)}$ with $\mathcal{F}^\infty \subset \partial_1 \mathcal{X}_{p_-, p_+; A}^\iota$ identifies

$$\begin{aligned} ([\tau], (\tau_-, [\underline{v}], \tau_+)) &\in \mathcal{F}_{(p_-, q_-), (q_-, p_+; A)}^\infty = \mathcal{M}(p_-, q_-) \times \partial_0 \mathcal{X}_{q_-, p_+; A}^{\iota, \infty} \\ &= \mathcal{M}(p_-, q_-) \times \mathcal{M}(q_-, M) \underset{\text{ev}_0^-}{\times_{\overline{\text{ev}}^-}} \mathcal{X}_A^{\text{GW}, \infty} \underset{\overline{\text{ev}}^+}{\times_{\text{ev}_0^+}} \mathcal{M}(M, p_+) \end{aligned}$$

with

$$(([\tau], \tau_-), [\underline{v}], \tau_+) \in \overline{\mathcal{M}}(p_-, M)_1 \underset{\text{ev}_0^-}{\times_{\overline{\text{ev}}^-}} \mathcal{X}_A^{\text{GW}, \infty} \underset{\overline{\text{ev}}^+}{\times_{\text{ev}_0^+}} \mathcal{M}(M, p_+) \subset \partial_1 \mathcal{X}_{p_-, p_+; A}^{\iota, \infty},$$

and $\text{pr}_{p_-, p_+; A}(([\tau], \tau_-), [\underline{v}], \tau_+) = [\underline{v}] \in \mathcal{X}_A^{\text{GW}}$ coincides with $(\text{pr}_{q_-, p_+; A} \circ \text{pr}_{\mathcal{F}})([\tau], (\tau_-, [\underline{v}], \tau_+)) = \text{pr}_{q_-, p_+; A}(\tau_-, [\underline{v}], \tau_+) = [\underline{v}] \in \mathcal{X}_A^{\text{GW}}$. Now any choice of sc^+ -multisections $(\lambda_A^{\text{GW}} : \mathcal{W}_A^{\text{GW}} \rightarrow \mathbb{Q}^+)_{A \in H_2(M)}$ induces a coherent collection of sc^+ -multisections $(\kappa_\alpha^\ell := \lambda_A^{\text{GW}} \circ \text{pr}_\alpha^* : \text{pr}_\alpha^* \mathcal{W}_A^{\text{GW}} \rightarrow \mathbb{Q}^+)_{\alpha \in \mathcal{I}}$ by composition with the natural maps $\text{pr}_\alpha^* : \text{pr}_\alpha^* \mathcal{W}_A^{\text{GW}} \rightarrow \mathcal{W}_A^{\text{GW}}$ covering $\text{pr}_\alpha : \mathcal{X}_\alpha^\ell \rightarrow \mathcal{X}_A^{\text{GW}}$. Indeed, $\text{pr}_\alpha|_{\mathcal{F}} = \text{pr}_{\alpha'} \circ \text{pr}_{\mathcal{F}}$ lifts to $\text{pr}_\alpha^*|_{P_\alpha^{-1}(\mathcal{F}^\infty)} = \text{pr}_{\alpha'}^* \circ \text{pr}_{\mathcal{F}}^*$ so that

$$\kappa_\alpha^\ell|_{P_\alpha^{-1}(\mathcal{F}^\infty)} = \lambda_A^{\text{GW}} \circ \text{pr}_\alpha^*|_{P_\alpha^{-1}(\mathcal{F}^\infty)} = \lambda_A^{\text{GW}} \circ \text{pr}_{\alpha'}^* \circ \text{pr}_{\mathcal{F}}^* = \kappa_{\alpha'}^\ell \circ \text{pr}_{\mathcal{F}}^*.$$

Construction of admissible Gromov-Witten perturbations: It remains to choose the sc^+ -multisections $(\lambda_A^{\text{GW}} : \mathcal{W}_A^{\text{GW}} \rightarrow \mathbb{Q}^+)_{A \in H_2(M)}$ so that the induced coherent collection $\underline{\kappa}^\ell = (\lambda_A^{\text{GW}} \circ \text{pr}_\alpha^*)_{\alpha \in \mathcal{I}}$ is admissible and in general position. To do so, for each $A \in H_2(M)$ we apply Theorem 2.7.9 to the sc -Fredholm section functor $S_A^{\text{GW}} : \mathcal{X}_A^{\text{GW}} \rightarrow \mathcal{W}_A^{\text{GW}}$, the sc^∞ submersion $\bar{\text{ev}}^- \times \bar{\text{ev}}^+ : \mathcal{X}_A^{\text{GW}} \rightarrow \mathbb{CP}^1 \times M \times \mathbb{CP}^1 \times M$, and the collection of Cartesian products of stable and unstable manifolds $\{z_0^-\} \times W_{p_-}^- \times \{z_0^+\} \times W_{p_+}^+$ for all pairs of critical points $p_-, p_+ \in \text{Crit}(f)$.

After fixing a pair controlling compactness $(N_A, \mathcal{U}_A)$ for each $A \in H_2(M)$, Theorem 2.7.9 yields $(N_A, \mathcal{U}_A)$ -admissible sc^+ -multisections $\lambda_A^{\text{GW}} : \mathcal{W}_A^{\text{GW}} \rightarrow \mathbb{Q}^+$ in general position to S_A^{GW} for each $A \in H_2(M)$. Moreover, they can be chosen such that restriction of evaluations to the perturbed zero set $\bar{\text{ev}}^- \times \bar{\text{ev}}^+ : Z^{\lambda_A^{\text{GW}}} \rightarrow \mathbb{CP}^1 \times M \times \mathbb{CP}^1 \times M$ is transverse to all of the products of unstable and stable submanifolds $\{z_0^-\} \times W_{p_-}^- \times \{z_0^+\} \times W_{p_+}^+$ for $p_-, p_+ \in \text{Crit}(f)$. Note that these embedded submanifolds cover the images of all evaluation maps on the compactified Morse trajectory spaces $\text{ev}_0^- \times \text{ev}_0^+ : \overline{\mathcal{M}}(p_-, M) \times \overline{\mathcal{M}}(M, p_+) \rightarrow \mathbb{CP}^1 \times M \times \mathbb{CP}^1 \times M$, by construction of the evaluations $\text{ev} : \overline{\mathcal{M}}(\dots) \rightarrow M$ in (2.8), which determine $\text{ev}_0^\pm(\underline{\tau}) = (z_0^\pm, \text{ev}(\underline{\tau}))$. Thus we obtain transverse fiber products $\overline{\mathcal{M}}(p_-, M)_{\text{ev}_0^- \times \bar{\text{ev}}^-} Z^{\lambda_A^{\text{GW}}} \times_{\text{ev}_0^+ \times \bar{\text{ev}}^+} \overline{\mathcal{M}}(M, p_+)$ for every $\alpha \in \mathcal{I}$. This translates into the pullbacks $\kappa_\alpha^\ell = \lambda_A^{\text{GW}} \circ \text{pr}_\alpha^*$ being in general position to the pullback sections S_α^ℓ for $\alpha \in \mathcal{I}$. Moreover, κ_α^ℓ is admissible with respect to a pullback of $(N_A, \mathcal{U}_A)$, so the perturbed zero set is a compact weighted branched orbifold for each $\alpha = (p_-, p_+; A)$,

$$\left| (S_\alpha^\ell + \kappa_\alpha^\ell)^{-1}(0) \right| = Z^{\kappa^\ell}(\alpha) = \overline{\mathcal{M}}(p_-, M)_{\text{ev}_0^- \times \bar{\text{ev}}^-} Z^{\lambda_A^{\text{GW}}} \times_{\text{ev}_0^+ \times \bar{\text{ev}}^+} \overline{\mathcal{M}}(M, p_+).$$

This finishes the construction of coherent perturbations.

Computation of orientations: To prove the identity (2.30) it remains to compute the effect of the orientations in Remark 2.5.8 on the algebraic identity in Corollary 2.4.6 (v) that arises from the boundary $\partial Z^{\kappa^\ell}(\alpha)$ of the 1-dimensional weighted branched orbifolds arising from regularization of the moduli spaces with index $I^\ell(\alpha) = I^\ell(p_-, p_+; A) = 1$. Here $Z^{\lambda_A^{\text{GW}}}$ is of even dimension and has no boundary since the Gromov-Witten polyfolds in Assumption 2.5.5 have no boundary, and the index of σ_{GW} is even. For the Morse trajectory spaces, the boundary strata are determined in Theorem 2.3.3, with relevant orientations computed in Remark 2.3.5. Thus for $I^\ell(\alpha) = |p_-| - |p_+| + 2c_1(A) = 1$ we can compute orientations – at the level of well defined finite dimensional tangent spaces at a solution; in

whose neighborhood the evaluation maps are guaranteed to be scale-smooth –

$$\begin{aligned}
\partial Z^{\kappa^t}(\alpha) &= \partial_1 \overline{\mathcal{M}}(p_-, M)_{\text{ev} \times \text{ev}} Z^{\lambda_A^{\text{GW}}}{}_{\text{ev} \times \text{ev}} \partial_0 \overline{\mathcal{M}}(M, p_+) \\
&\sqcup (-1)^{\dim \overline{\mathcal{M}}(p_-, M)} \partial_0 \overline{\mathcal{M}}(p_-, M)_{\text{ev} \times \text{ev}} Z^{\lambda_A^{\text{GW}}}{}_{\text{ev} \times \text{ev}} \partial_1 \overline{\mathcal{M}}(M, p_+) \\
&= \left(\bigsqcup_{q \in \text{Crit} f} \mathcal{M}(p_-, q) \times \mathcal{M}(q, M) \right)_{\text{ev} \times \text{ev}} Z^{\lambda_A^{\text{GW}}}{}_{\text{ev} \times \text{ev}} \mathcal{M}(M, p_+) \\
&\sqcup (-1)^{|p_-| + |p_+| + 1} \mathcal{M}(p_-, M)_{\text{ev} \times \text{ev}} Z^{\lambda_A^{\text{GW}}}{}_{\text{ev} \times \text{ev}} \left(\bigsqcup_{q \in \text{Crit} f} \mathcal{M}(M, q) \times \mathcal{M}(q, p_+) \right) \\
&= \bigsqcup_{q \in \text{Crit} f} \mathcal{M}(p_-, q) \times Z^{\kappa^t}(q, p_+; A) \sqcup \bigsqcup_{q \in \text{Crit} f} Z^{\kappa^t}(p_-, q; A) \times \mathcal{M}(q, p_+).
\end{aligned} \tag{2.31}$$

Here the signs in the first equality arise from the ambient Cartesian product $\partial(\overline{\mathcal{M}}_- \times Z \times \overline{\mathcal{M}}_+) \subset (-1)^{\dim(\overline{\mathcal{M}}_- \times Z)} \overline{\mathcal{M}}_- \times Z \times \overline{\mathcal{M}}_+$; in the second equality we used Remark 2.3.5; and in the final equality we use $|p_-| + |p_+| + 1 \equiv I^t(\alpha) = 1 \equiv 0$ modulo 2. This finishes the computation of the oriented boundaries $\partial Z^{\kappa^t}(\alpha)$ for $I^t(\alpha) = 1$ that proves (2.30) and thus yields a chain map. $\square$

2.6.3 Admissible perturbations for isomorphism property

In this section we prove Theorem 2.1.2 (ii), i.e. construct $\iota = (-1)^* \iota_{\kappa^t} : C_* M \rightarrow C_* M$ in Definition 2.5.7 and Lemma 2.6.4 as a Λ -module isomorphism on the chain complex $CM = CM_\Lambda$ over the Novikov field as in (2.5). This requires a construction of the perturbations κ^t that preserves the properties of the zero sets in Remark 2.5.1 for nonpositive symplectic area $\omega(A) \leq 0$.

Lemma 2.6.5. *The coherent collection of sc^+ -multisections κ^t in Lemma 2.6.4 can be chosen such that $\#Z^{\kappa^t}(p_-, p_+; A) = 0$ for $A \in H_2(M) \setminus \{0\}$ with $\omega(A) \leq 0$, or for $A = 0$ and $p_- \neq p_+$, and $\#Z^{\kappa^t}(p, p; 0) \neq 0$. As a consequence, $\iota = (-1)^* \iota_{\kappa^t} : CM_\Lambda \rightarrow CM_\Lambda$ is a Λ -module isomorphism.*

Proof. The sc^+ -multisections κ^t in Lemma 2.6.4 are obtained from a choice of family of sc^+ -multisections $(\kappa_A : \mathcal{W}_A^\iota \rightarrow \mathbb{Q}^+)_{A \in H_2(M)}$ that are in general position to sc -Fredholm sections $S_A : \mathcal{X}_A^{\text{GW}} \rightarrow \mathcal{W}_A$ which cut out the Gromov-Witten moduli space $\overline{\mathcal{M}}_{\text{GW}}(A) = |S_A^{-1}(0)|$, and such that moreover the evaluation maps restricted to the perturbed zero sets, $\text{ev}^- \times \text{ev}^+ : Z(\kappa_A) \rightarrow \mathbb{CP}^1 \times M \times \mathbb{CP}^1 \times M$ are transverse to the unstable and stable manifolds $\{z_0^-\} \times W_{p_-}^- \times \{z_0^+\} \times W_{p_+}^+ \subset \mathbb{CP}^1 \times M \times \mathbb{CP}^1 \times M$ for any pair of critical points $p_-, p_+ \in \text{Crit}(f)$.

We will first consider $\alpha = (p_-, p_+; A) \in \mathcal{I}$ for nontrivial homology classes $A \in H_2(M) \setminus \{0\}$ with nonpositive symplectic area $\omega(A) \leq 0$. Recall from Remark 2.5.1 that these moduli spaces are empty $|S_A^{-1}(0)| = \emptyset$, so as in Corollary 2.4.6 we can choose empty neighborhoods $\emptyset = |\mathcal{U}_A| \subset |\mathcal{X}_A^{\text{GW}}|$ to control compactness. Then the perturbed zero set $Z(\kappa_A) = |\{x \in X_A \mid \kappa_A(S_A(x)) > 0\}| \subset |\mathcal{U}_A|$ is forced to be empty, i.e. $\kappa_A \circ S_A \equiv 0$. This is an allowed choice in Lemma 2.6.4 since evaluation maps from an empty set are trivially transverse to

any submanifold. This choice induces for any $p_{\pm} \in \text{Crit}(f)$ in $\alpha = (p_-, p_+; A)$ an induced sc^+ -multisection $\kappa_{\alpha}^{\iota} = \kappa_A \circ \text{pr}_{\alpha}^* : \mathcal{W}_{\alpha}^{\iota} \rightarrow \mathbb{Q}^+$. Its perturbed zero set is

$$Z^{\kappa_{\alpha}^{\iota}}(\alpha) = |\{(\underline{\tau}^-, x, \underline{\tau}^+) \in X_{\alpha} \mid \kappa_{\alpha}^{\iota}(S_{\alpha}^{\iota}(\underline{\tau}^-, x, \underline{\tau}^+)) > 0\}| = \emptyset$$

since the coherence in Lemma 2.6.4 implies $\kappa_{\alpha}^{\iota} \circ S_{\alpha}^{\iota} = \kappa_A \circ \text{pr}_{\alpha}^* \circ S_{\alpha}^{\iota} = \kappa_A \circ S_A \circ \text{pr}_{\alpha} \equiv 0$, or more concretely $\kappa_{(p_-, p_+; A)}^{\iota}(S_{p_-, p_+; A}^{\iota}(\underline{\tau}^-, x, \underline{\tau}^+)) = \kappa_A(S_A(x)) = 0$. Thus we have ensured vanishing counts $\#Z^{\kappa^{\iota}}(p_-, p_+; A) = 0$ for $A \in H_2(M) \setminus \{0\}$ with $\omega(A) \leq 0$ whenever $I^{\iota}(p_-, p_+; A) = 0$.

Next we consider $A = 0 \in H_2(M)$ and recall from Remark 2.5.1 and Assumption 2.5.5 (ii) that the Gromov-Witten moduli space $\overline{\mathcal{M}}_{\text{GW}}(0) = Z(\kappa_0)$ is already compact and transversely cut out. Thus the trivial sc^+ -multisection $\kappa_0 : \mathcal{W}_0 \rightarrow \mathbb{Q}^+$, given by $\kappa_0(0_x) = 1$ on zero vectors $0_x \in (\mathcal{W}_0)_x$ and $\kappa_0|_{(\mathcal{W}_0)_x \setminus \{0_x\}} \equiv 0$, is an admissible sc^+ -multisection in general position to $S_0 : \mathcal{X}_0^{\text{GW}} \rightarrow \mathcal{W}_0$. Recall moreover that the evaluation maps on the unperturbed zero set are

$$\overline{\text{ev}}^- \times \overline{\text{ev}}^+ : Z(\kappa_0) \simeq \mathbb{CP}^1 \times \mathbb{CP}^1 \times M \rightarrow \mathbb{CP}^1 \times M \times \mathbb{CP}^1 \times M, \quad (z^-, z^+, x) \mapsto (z^-, x, z^+, x).$$

In the $\mathbb{CP}^1$ -factors this is submersive so transverse to the fixed points $(z_0^-, z_0^+) \in \mathbb{CP}^1 \times \mathbb{CP}^1$. In the M -factors this is the diagonal map, which is transverse to the unstable and stable manifolds $W_{p_-}^- \times W_{p_+}^+ \subset M \times M$ for any pair $p_-, p_+ \in \text{Crit}(f)$ by the Morse-Smale condition on the metric on M chosen in §2.3. Thus the trivial multisection κ_0 is in fact an allowed choice in Lemma 2.6.4. Now with this choice, the tuples $(p_-, p_+; 0) \in \mathcal{I}$ for which we need to compute

$$\begin{aligned} \#Z^{\kappa^{\iota}}(p_-, p_+; 0) &= \# |\{(\underline{\tau}^-, [v], \underline{\tau}^+) \in \overline{\mathcal{M}}(p_-, M) \times Z(\kappa_0) \times \overline{\mathcal{M}}(M, p_+) \\ &\quad \text{such that } (z_0^{\pm}, \text{ev}(\underline{\tau}^{\pm})) = \overline{\text{ev}}^{\pm}([v])\}| \\ &\cong \# |\{(\underline{\tau}^-, \underline{\tau}^+) \in \overline{\mathcal{M}}(p_-, M) \times \overline{\mathcal{M}}(M, p_+) \mid \text{ev}(\underline{\tau}^-) = \text{ev}(\underline{\tau}^+)\}| \end{aligned}$$

are those with $0 = I^{\iota}(p_-, p_+; 0) = 2c_1(0) + |p_-| - |p_+|$, i.e. $|p_-| = |p_+|$. These are the fiber products identified in Remark 2.3.5 (ii) as either empty or a one point set,

$$\overline{\mathcal{M}}(p_-, M)_{\text{ev} \times \text{ev}} \overline{\mathcal{M}}(M, p_+) = \begin{cases} \emptyset & ; p_- \neq p_+, \\ (\tau^- \equiv p_-, \tau^+ \equiv p_+) & ; p_- = p_+. \end{cases}$$

Thus we have counts $\#Z^{\kappa^{\iota}}(p_-, p_+; 0) = 0$ for $p_- \neq p_+$ and $\#Z^{\kappa^{\iota}}(p, p; 0) \neq 0$ for each $p \in \text{Crit}(f)$.

Finally, we will use these computations of $\#Z^{\kappa^{\iota}}(p_-, p_+; A)$ for $\omega(A) \leq 0$ to prove that the resulting map $\iota := (-1)^* \iota_{\kappa^{\iota}} : CM_{\Lambda} \rightarrow CM_{\Lambda}$ is a Λ -module isomorphism. For that purpose we choose an arbitrary total order of the critical points $\text{Crit}(f) = \{p_1, \dots, p_{\ell}\}$ and for $i, j \in \{1, \dots, \ell\}$ denote the coefficients of $\iota(\langle p_j \rangle) = \sum_{i=1}^{\ell} \lambda^{ij} \langle p_i \rangle$ by $\lambda^{ij} \in \Lambda$. We claim that the $(\ell \times \ell)$ -matrix with entries $\lambda^{ij} = \sum_{r \in \Gamma} \lambda_r^{ij} T^r$ satisfies the conditions of Lemma 2.2.1. To check this recall that we have by construction in Definition 2.5.7 and change of signs in Lemma 2.6.4

$$\lambda_r^{ij} = \sum_{\substack{A \in H_2(M), \omega(A)=r \\ I^{\iota}(p_j, p_i; A)=0}} (-1)^{|p_j|} \#Z^{\kappa^{\iota}}(p_j, p_i; A).$$

For $r < 0$ we obtain $\lambda_r^{ij} = 0$ since each coefficient $\#Z^{\kappa'}(p_j, p_i; A) = 0$ vanishes for $\omega(A) = r < 0$. For $r = 0$ and $i \neq j$ we also have $\lambda_0^{ij} = 0$ since $\#Z^{\kappa'}(p_j, p_i; A) = 0$ also holds for $\omega(A) = 0$ and $p_j \neq p_i$. Finally, for $r = 0$ and $i = j$ we use $\#Z^{\kappa'}(p_j, p_i; A) = 0$ for $A \neq 0$ with $\omega(A) = 0$ to compute $\lambda_0^{ii} = \#Z^{\kappa'}(p_j, p_i; 0) \neq 0$. This confirms that Lemma 2.2.1 applies, and thus $\iota \cong (\lambda_{ij})_{1 \leq i, j \leq \ell}$ is invertible. This finishes the proof. $\square$

2.6.4 Coherent perturbations for chain homotopy

In this section we prove Theorem 2.1.2 (iii) by constructing $h_{\underline{\kappa}} : CM \rightarrow CM$ in Definition 2.5.7 as a chain homotopy between $SSP_{\underline{\kappa}^+} \circ PSS_{\underline{\kappa}^-}$ and $\iota_{\underline{\kappa}^\iota}$ from Definitions 2.4.8, 2.5.7, with appropriate sign adjustments as in Lemma 2.6.4. This requires a coherent construction of perturbations $\underline{\kappa}, \underline{\kappa}^\iota, \underline{\kappa}^-, \underline{\kappa}^+$ over the indexing sets

$$\begin{aligned} \mathcal{I} = \mathcal{I}^\iota &:= \{ \alpha = (p_-, p_+, A) \mid p_-, p_+ \in \text{Crit}(f), A \in H_2(M) \}, \\ \mathcal{I}^+ &:= \{ \alpha = (p, \gamma, A) \mid p \in \text{Crit}(f), \gamma \in \mathcal{P}(H), A \in H_2(M) \}, \\ \mathcal{I}^- &:= \{ \alpha = (\gamma, p, A) \mid p \in \text{Crit}(f), \gamma \in \mathcal{P}(H), A \in H_2(M) \}. \end{aligned}$$

Here we will use notation from Lemma 2.7.7 for Cartesian products of multisections.

Lemma 2.6.6. *There is a choice of $\underline{\kappa}^+ = (\kappa_\alpha^+)_{\alpha \in \mathcal{I}^+}$, $\underline{\kappa}^- = (\kappa_\alpha^-)_{\alpha \in \mathcal{I}^-}$, $\underline{\kappa}^\iota = (\kappa_\alpha^\iota)_{\alpha \in \mathcal{I}^\iota}$, $\underline{\kappa} = (\kappa_\alpha)_{\alpha \in \mathcal{I}}$ in Definitions 2.4.8, 2.5.7 that is coherent in the following sense.*

- (i) *Each $\kappa_\alpha^\iota : \mathcal{W}_\alpha^\iota \rightarrow \mathbb{Q}^+$ for $\alpha \in \mathcal{I}^+ \sqcup \mathcal{I}^- \sqcup \mathcal{I}^\iota \sqcup \mathcal{I}$ is an admissible sc^+ -multisection of a strong bundle $P_\alpha^\iota : \mathcal{W}_\alpha^\iota \rightarrow \mathcal{X}_\alpha^\iota$ that is in general position to a sc -Fredholm section functor $S_\alpha^\iota : \mathcal{X}_\alpha^\iota \rightarrow \mathcal{W}_\alpha^\iota$ which represents $\sigma_\alpha^\iota|_{\mathcal{V}_\alpha^\iota}$ on an open neighborhood $\mathcal{V}_\alpha^\iota \subset \mathcal{B}^\iota(\alpha)$ of the zero set $\sigma_\alpha^{\iota-1}(0)$. The tuple $\underline{\kappa}^\iota = (\kappa_\alpha^\iota)_{\alpha \in \mathcal{I}^\iota}$ satisfies the conclusions of Lemma 2.6.4 and 2.6.5.*
- (ii) *The smooth level of the first boundary stratum of $\mathcal{X}_{p_-, p_+, A}$ for every $(p_-, p_+, A) \in \mathcal{I}$ is naturally identified – on the level of object spaces, and compatible with morphisms – with*

$$\begin{aligned} \partial_1 \mathcal{X}_{p_-, p_+, A}^\infty &\cong \partial_0 \mathcal{X}_{p_-, p_+, A}^{\iota, \infty} \sqcup \bigcup_{\gamma \in \mathcal{P}(H), A = A_- + A_+} \partial_0 \mathcal{X}_{p_-, \gamma, A_+}^{+, \infty} \times \partial_0 \mathcal{X}_{\gamma, p_+, A_-}^{-, \infty} \\ &\sqcup \bigcup_{q \in \text{Crit}(f)} \mathcal{M}(p_-, q) \times \partial_0 \mathcal{X}_{q, p_+, A}^\infty \sqcup \bigcup_{q \in \text{Crit}(f)} \partial_0 \mathcal{X}_{p_-, q, A}^\infty \times \mathcal{M}(q, p_+), \end{aligned} \tag{2.32}$$

and the oriented section functors S_α^ι are compatible with these identifications in the sense that the restriction of $S_{p_-, p_+, A}$ to any of these faces $\mathcal{F}^\infty \subset \partial_1 \mathcal{X}_{p_-, p_+, A}^\infty$ is given by pullback $S_{p_-, p_+, A}|_{\mathcal{F}^\infty} = \text{pr}_{\mathcal{F}}^* S_{\mathcal{F}}$ of another sc -Fredholm section of a strong bundle over an ep -groupoid $S_{\mathcal{F}} : \mathcal{X}_{\mathcal{F}} \rightarrow \mathcal{W}_{\mathcal{F}}$ given by $S_{q, p_+, A}$, $S_{p_-, q, A}$, $S_{p_-, p_+, A}^\iota$, resp.

$$S_{\mathcal{F}} = S_{p_-, \gamma, A_+}^+ \times S_{\gamma, p_+, A_-}^- : \mathcal{X}_{p_-, \gamma, A_+}^+ \times \mathcal{X}_{\gamma, p_+, A_-}^- \rightarrow \mathcal{W}_{p_-, \gamma, A_+}^+ \times \mathcal{W}_{\gamma, p_+, A_-}^-$$

via the projection $\text{pr}_{\mathcal{F}} : \mathcal{F} \rightarrow \mathcal{X}_{\mathcal{F}}$ given by the natural maps

$$\begin{aligned} \mathcal{M}(p_-, q) \times \partial_0 \mathcal{X}_{q, p_+, A} &\rightarrow \mathcal{X}_{q, p_+, A}, & \partial_0 \mathcal{X}_{p_-, p_+, A}^{\iota} &\rightarrow \mathcal{X}_{p_-, p_+, A}^{\iota}, \\ \partial_0 \mathcal{X}_{p_-, q, A} \times \mathcal{M}(q, p_+) &\rightarrow \mathcal{X}_{p_-, q, A}, & \partial_0 \mathcal{X}_{p_-, \gamma, A_+}^+ \times \partial_0 \mathcal{X}_{\gamma, p_+, A_-}^- &\rightarrow \mathcal{X}_{p_-, \gamma, A_+}^+ \times \mathcal{X}_{\gamma, p_+, A_-}^-. \end{aligned}$$

(iii) Each restriction $\kappa_{\alpha}|_{P_{\alpha}^{-1}(\mathcal{F}^{\infty})}$ for $\alpha = (p_-, p_+, A) \in \mathcal{I}$ to one of the faces $\mathcal{F}^{\infty} \subset \partial_1 \mathcal{X}_{\alpha}$ is given via the identification $P_{\alpha}^{-1}(\mathcal{F}^{\infty}) \cong \text{pr}_{\mathcal{F}}^* \mathcal{W}_{\mathcal{F}}|_{\partial_0 \mathcal{X}_{\mathcal{F}}}$ and natural map $\text{pr}_{\mathcal{F}}^* : \text{pr}_{\mathcal{F}}^* \mathcal{W}_{\mathcal{F}} \rightarrow \mathcal{W}_{\mathcal{F}}$ by

$$\kappa_{\alpha}|_{P_{\alpha}^{-1}(\mathcal{F}^{\infty})} = \begin{cases} \kappa_{q, p_+, A} \circ \text{pr}_{\mathcal{F}}^* & \text{for } \mathcal{F} = \mathcal{M}(p_-, q) \times \partial_0 \mathcal{X}_{q, p_+, A}, \\ \kappa_{p_-, q, A} \circ \text{pr}_{\mathcal{F}}^* & \text{for } \mathcal{F} = \partial_0 \mathcal{X}_{p_-, q, A} \times \mathcal{M}(q, p_+), \\ \kappa_{p_-, p_+, A}^{\iota} \circ \text{pr}_{\mathcal{F}}^* & \text{for } \mathcal{F} = \partial_0 \mathcal{X}_{p_-, p_+, A}^{\iota}, \\ (\kappa_{p_-, \gamma, A_+}^+ \cdot \kappa_{\gamma, p_+, A_-}^-) \circ \text{pr}_{\mathcal{F}}^* & \text{for } \mathcal{F} = \partial_0 \mathcal{X}_{p_-, \gamma, A_+}^+ \times \partial_0 \mathcal{X}_{\gamma, p_+, A_-}^-. \end{cases}$$

For any such choice of $\kappa^{\iota} = (\kappa_{\alpha}^{\iota})_{\alpha \in \mathcal{I}}$, the resulting maps $PSS_{\kappa^+}, SSP_{\kappa^-}, \iota_{\kappa^{\iota}}, h_{\kappa}$ in Definitions 2.4.8, 2.5.7 satisfy $(-1)^{|p|} \iota_{\kappa^{\iota}} \langle p \rangle = (-1)^{|p|} SSP_{\kappa^-} (PSS_{\kappa^+} \langle p \rangle) + h_{\kappa}(\text{d} \langle p \rangle) + \text{d}(h_{\kappa} \langle p \rangle)$, where d is the Morse differential from §2.3. By setting $\iota \langle p \rangle := (-1)^{|p|} \iota_{\kappa^{\iota}} \langle p \rangle$ as done in Lemma 2.6.4, $PSS \langle p \rangle := (-1)^{|p|} PSS_{\kappa^+} \langle p \rangle$, $SSP := SSP_{\kappa^-}$, and $h := h_{\kappa}$, we then obtain a chain homotopy between ι and $SSP \circ PSS$, that is $\iota - SSP \circ PSS = \text{d} \circ h + h \circ \text{d}$.

Proof. This proof is similar to Lemma 2.6.4, with more complicated combinatorics of the boundary faces due to the boundary of $\mathcal{B}_{\text{SFT}}$ described in Assumption 2.6.3, and presented in different order: We will first make the coherent constructions and then deduce the algebraic consequences.

Coherent ep-groupoids and sections: To construct coherent representatives $S_{\alpha}^{\cdots} : \mathcal{X}_{\alpha}^{\cdots} \rightarrow \mathcal{W}_{\alpha}^{\cdots}$ for $\alpha \in \mathcal{I}^+ \sqcup \mathcal{I}^- \sqcup \mathcal{I}^{\iota} \sqcup \mathcal{I}$ as claimed in (ii) recall that the fiber product construction in Lemma 2.5.6 defines each bundle $\mathcal{W}_{\alpha} = \text{pr}_{\alpha}^* \mathcal{W}_A^{\text{SFT}}$ for $\alpha = (p_-, p_+, A) \in \mathcal{I}$ as the pullback of a strong bundle $P_A : \mathcal{W}_A^{\text{SFT}} \rightarrow \mathcal{X}_A^{\text{SFT}}$ under the natural projection of ep-groupoids

$$\text{pr}_{p_-, p_+, A} : \mathcal{X}_{p_-, p_+, A} = \overline{\mathcal{M}}(p_-, M) \underset{\text{ev}_0^+}{\times} \underset{\text{ev}^+}{\times} \mathcal{X}_A^{\text{SFT}} \underset{\text{ev}^-}{\times} \underset{\text{ev}_0^-}{\times} \overline{\mathcal{M}}(M, p_+) \longrightarrow \mathcal{X}_A^{\text{SFT}}.$$

Here $\text{ev}_0^{\pm} : \overline{\mathcal{M}}(\dots) \rightarrow \mathbb{C}^{\pm} \times M, \tau \mapsto (0, \text{ev}(\tau))$ arise from Morse evaluation (2.8). The ep-groupoid $\mathcal{X}_A^{\text{SFT}} \subset \tilde{\mathcal{X}}_A^{\text{SFT}}$ is a full subcategory – determined by the open subset $\mathcal{B}_{\text{SFT}}^{+, -}(A) = (\text{ev}^+)^{-1}(\mathbb{C}^+ \times M) \cap (\text{ev}^-)^{-1}(\mathbb{C}^- \times M) \subset \mathcal{B}_{\text{SFT}}(A)$ – of an ep-groupoid $\tilde{\mathcal{X}}_A^{\text{SFT}}$ from Assumption 2.5.5 that represents $\mathcal{B}_{\text{SFT}}(A)$ and thus contains the compactified SFT neck stretching moduli space $\overline{\mathcal{M}}_{\text{SFT}}(A) = |(S_A^{\text{SFT}})^{-1}(0)|$ as zero set of a sc-Fredholm section $S_A^{\text{SFT}} : \tilde{\mathcal{X}}_A^{\text{SFT}} \rightarrow \tilde{\mathcal{W}}_A^{\text{SFT}}$. We will work with both groupoids: Multisection perturbations are constructed over $\tilde{\mathcal{X}}_A^{\text{SFT}}$ since we need a compact zero set to specify the admissibility that guarantees preservation of compactness under perturbations – both for S_A^{SFT} and its fiber product restrictions S_{α} . On the other hand, $|\tilde{\mathcal{X}}_A^{\text{SFT}}| = \mathcal{B}_{\text{SFT}}(A)$ has more complicated boundary than $\mathcal{B}_{\text{SFT}}^{+, -}(A)$ – due to

the distribution of marked points into building levels – and does not support a sc^∞ evaluation map. Thus we discuss coherence only over subgroupoids $\mathcal{X}_A^{\text{SFT}} \subset \tilde{\mathcal{X}}_A^{\text{SFT}}$ with the boundary stratification of $\mathcal{B}_{\text{SFT}}^{\pm,-}(A)$, and which support sc^∞ functors $\text{ev}^\pm : \mathcal{X}_A^{\text{SFT}} \rightarrow \mathbb{C}^\pm \times M$ representing the evaluation maps (2.29). Here we may even use subgroupoids $\mathcal{X}_A^{\text{SFT}}$ representing a smaller open subset $(\overline{\text{ev}}^+)^{-1}(\Delta_r^+ \times M) \cap (\overline{\text{ev}}^-)^{-1}(\Delta_r^- \times M) \subset \mathcal{B}_{\text{SFT}}(A)$ of preimages of the disks $\Delta_r^\pm := \{z \in \mathbb{C}^\pm \mid |z| < r\} \subset \mathbb{C}^\pm$, which contain the standard marked points $z_0^\pm \cong 0 \in \mathbb{C}^\pm$.¹² The polyfold structure on the fiber products $\mathcal{X}_\alpha$ in Lemma 2.5.6 is independent of the choice of open neighborhood in $\mathcal{B}_{\text{SFT}}(A)$ of the subset satisfying the fiber product condition. After obtaining the subgroupoid $\mathcal{X}_A^{\text{SFT}} \subset \tilde{\mathcal{X}}_A^{\text{SFT}}$ from such an open subset, we obtain the bundle $\mathcal{W}_A^{\text{SFT}} = \tilde{\mathcal{W}}_A^{\text{SFT}}|_{\mathcal{X}_A^{\text{SFT}}}$ and section $S_A^{\text{SFT}}|_{\mathcal{X}_A^{\text{SFT}}} : \mathcal{X}_A^{\text{SFT}} \rightarrow \mathcal{W}_A^{\text{SFT}}$ by restriction. Finally, each section $S_\alpha = S_A^{\text{SFT}} \circ \text{pr}_\alpha$ is induced by the above projection $\text{pr}_\alpha : \mathcal{X}_\alpha \rightarrow \mathcal{X}_A^{\text{SFT}} \subset \tilde{\mathcal{X}}_A^{\text{SFT}}$.

Next, restriction to the boundary faces given in Assumption 2.6.3 (i) induces representatives $S_A^{\text{GW}} : \tilde{\mathcal{X}}_A^{\text{GW}} \rightarrow \tilde{\mathcal{W}}_A^{\text{GW}}$ resp. $S_{\gamma, A_\pm}^\pm : \tilde{\mathcal{X}}_{\gamma, A_\pm}^\pm \rightarrow \tilde{\mathcal{W}}_{\gamma, A_\pm}^\pm$ of the sections $\sigma_{\text{GW}} : \mathcal{B}_{\text{GW}}(A) \rightarrow \mathcal{E}_{\text{GW}}(A)$ resp. $\sigma_{\text{SFT}} : \mathcal{B}_{\text{SFT}}^\pm(\gamma; A_\pm) \rightarrow \mathcal{E}_{\text{SFT}}^\pm(\gamma; A_\pm)$ from Assumption 2.4.3 resp. 2.5.5. Moreover, the boundary of the open subset $(\overline{\text{ev}}^+ \times \overline{\text{ev}}^-)^{-1}(\Delta_r^+ \times M \times \Delta_r^- \times M)$ for $0 < r \leq \infty$ (with $\Delta_\infty^\pm := \mathbb{C}^\pm$) yields subgroupoids $\mathcal{X}_A^{\text{GW}} \subset \tilde{\mathcal{X}}_A^{\text{GW}}$ representing $(\overline{\text{ev}}^+ \times \overline{\text{ev}}^-)^{-1}(\Delta_r^+ \times M \times \Delta_r^- \times M) \subset \mathcal{B}_{\text{GW}}(A)$ resp. $\mathcal{X}_{\gamma, A_\pm}^\pm \subset \tilde{\mathcal{X}}_{\gamma, A_\pm}^\pm$ representing $(\overline{\text{ev}}^\pm)^{-1}(\Delta_r^\pm \times M) \subset \mathcal{B}_{\text{SFT}}(A)$, along with restricted sections $S_A^{\text{GW}} : \mathcal{X}_A^{\text{GW}} \rightarrow \mathcal{W}_A^{\text{GW}} = \tilde{\mathcal{W}}_A^{\text{GW}}|_{\mathcal{X}_A^{\text{GW}}}$ resp. $S_{\gamma, A_\pm}^\pm : \mathcal{X}_{\gamma, A_\pm}^\pm \rightarrow \mathcal{W}_{\gamma, A_\pm}^\pm = \tilde{\mathcal{W}}_{\gamma, A_\pm}^\pm|_{\mathcal{X}_{\gamma, A_\pm}^\pm}$. Then the evaluation maps restrict to sc^∞ functors $\text{ev}^\pm : \mathcal{X}_A^{\text{GW}} \rightarrow \Delta_r^\pm \times M$ resp. $\text{ev}^\pm : \mathcal{X}_{\gamma, A}^\pm \rightarrow \Delta_r^\pm \times M$, which yield – again independent of $r > 0$ – the fiber product construction of $\mathcal{B}^\pm(\alpha)$ in Lemma 2.4.5, and of $\mathcal{B}^i(\alpha)$ in Lemma 2.5.6.

Now the identification of the top boundary strata $\partial_1 \mathcal{X}_{p_-, p_+, A}^\infty$ will proceed similar to the proof of Lemma 2.6.1 with $\mathcal{B}_{\text{GW}}(A)$ replaced by $\mathcal{B}_{\text{SFT}}^{\pm,-}(A)$, apart from the fact that the SFT polyfold has boundary. This boundary is identified in Assumption 2.6.3 (ii) as

$$\partial_1 \mathcal{X}_A^{\text{SFT}} \cong \mathcal{X}_A^{\text{GW}} \sqcup \bigsqcup_{\substack{\gamma \in \mathcal{P}(H) \\ A_- + A_+ = A}} \partial_0 \mathcal{X}_{\gamma, A_+}^+ \times \partial_0 \mathcal{X}_{\gamma, A_-}^-. \quad (2.33)$$

By the fiber product construction [15, Cor.7.3] of $\mathcal{B}(p_-, p_+; A)$ in Lemma 2.5.6, the degeneracy index satisfies $d_{\mathcal{B}(p_-, p_+; A)}(\underline{\tau}_-, \underline{u}, \underline{t}_+) = d_{\mathcal{M}(p_-, M)}(\underline{\tau}_-) + d_{\mathcal{B}_{\text{SFT}}(A)}(\underline{u}) + d_{\mathcal{M}(M, p_+)}(\underline{t}_+)$. Hence we have $d_{\mathcal{B}(p_-, p_+; A)}(\underline{\tau}_-, \underline{u}, \underline{t}_+) = 1$ if and only if the degeneracy index of exactly one of the three arguments $\underline{\tau}_-, \underline{u}, \underline{t}_+$ is 1 and the other two are 0. This identifies $|\partial_1 \mathcal{X}_{p_-, p_+, A}| = \partial_1 \mathcal{B}(p_-, p_+; A)$ as in the first line of the displayed equation below. Then the subsequent identifications result by comparing the resulting expressions with the interiors in Lemma 2.4.5, 2.5.6. We obtain an identification that throughout is to be interpreted on the smooth level

¹²These disks should not be confused with the closed disks $D_\pm$ in the construction of $\mathbb{CP}_R^1$, as e.g. $\Delta^+ \subset \mathbb{C}^+ \cong (D_+ \sqcup [-R, 0) \times S^1) / \sim_R$ is a precompact subset of the first hemisphere in $\mathbb{CP}_R^1 \cong \mathbb{C}^+ \cup S^1 \cup \mathbb{C}^-$ for any $R \geq 0$.

(as fiber product constructions drop some non-smooth points)

$$\begin{aligned}
\partial_1 \mathcal{X}_{p_-, p_+, A} &\cong \partial_0 \overline{\mathcal{M}}(p_-, M)_{\text{ev}_0^+ \times \text{ev}^+} \partial_1 \mathcal{X}_A^{\text{SFT}}_{\text{ev}^- \times \text{ev}_0^-} \partial_0 \overline{\mathcal{M}}(M, p_+) \\
&\sqcup \partial_1 \overline{\mathcal{M}}(p_-, M)_{\text{ev}_0^+ \times \text{ev}^+} \partial_0 \mathcal{X}_A^{\text{SFT}}_{\text{ev}^- \times \text{ev}_0^-} \partial_0 \overline{\mathcal{M}}(M, p_+) \\
&\sqcup \partial_0 \overline{\mathcal{M}}(p_-, M)_{\text{ev}_0^+ \times \text{ev}^+} \partial_0 \mathcal{X}_A^{\text{SFT}}_{\text{ev}^- \times \text{ev}_0^-} \partial_1 \overline{\mathcal{M}}(M, p_+) \\
&= \mathcal{M}(p_-, M)_{\text{ev}_0^+ \times \text{ev}^+} \mathcal{X}_A^{\text{GW}}_{\text{ev}^- \times \text{ev}_0^-} \mathcal{M}(M, p_+) \\
&\sqcup \bigcup_{\substack{\gamma \in \mathcal{P}(H) \\ A_- + A_+ = A}} \mathcal{M}(p_-, M)_{\text{ev}_0^+ \times \text{ev}^+} \partial_0 \mathcal{X}_{\gamma, A_+}^+ \times \partial_0 \mathcal{X}_{\gamma, A_-}^-_{\text{ev}^- \times \text{ev}_0^-} \mathcal{M}(M, p_+) \\
&\sqcup \bigcup_{q \in \text{Crit}(f)} \mathcal{M}(p_-, q) \times \mathcal{M}(q, M)_{\text{ev}_0^+ \times \text{ev}^+} \partial_0 \mathcal{X}_A^{\text{SFT}}_{\text{ev}^- \times \text{ev}_0^-} \partial_0 \overline{\mathcal{M}}(M, p_+) \\
&\sqcup \bigcup_{q \in \text{Crit}(f)} \mathcal{M}(p_-, M)_{\text{ev}_0^+ \times \text{ev}^+} \partial_0 \mathcal{X}_A^{\text{SFT}}_{\text{ev}^- \times \text{ev}_0^-} \mathcal{M}(M, q) \times \mathcal{M}(q, p_+) \\
&= \partial_0 \mathcal{X}_{p_-, p_+, A}^\iota \sqcup \bigcup_{\substack{\gamma \in \mathcal{P}(H) \\ A_- + A_+ = A}} \partial_0 \mathcal{X}_{p_-, \gamma, A_+}^+ \times \partial_0 \mathcal{X}_{\gamma, p_+, A_-}^- \\
&\sqcup \bigcup_{q \in \text{Crit}(f)} \mathcal{M}(p_-, q) \times \partial_0 \mathcal{X}_{q, p_+, A} \sqcup \bigcup_{q \in \text{Crit}(f)} \partial_0 \mathcal{X}_{p_-, q, A} \times \mathcal{M}(q, p_+).
\end{aligned}$$

Here we also used the identification of evaluation maps in Assumption 2.6.3 (iii)(a). Then compatibility in (ii) of the oriented section functors S_α° with the identification of these (smooth levels of) faces $\mathcal{F}^\infty \subset \partial_1 \mathcal{X}_{p_-, p_+, A}^\infty$ follows from compatibility of $\text{pr}_{p_-, p_+, A} : \mathcal{X}_{p_-, p_+, A} \rightarrow \mathcal{X}_A^{\text{SFT}}$ with the projections $\text{pr}_\alpha^\pm : \mathcal{X}_\alpha^\pm \rightarrow \mathcal{X}_{\gamma, A_\pm}^\pm$ for $\alpha \in \mathcal{I}^\pm$ used in Lemma 2.4.5 and $\text{pr}_\alpha^\iota : \mathcal{X}_\alpha^\iota \rightarrow \mathcal{X}_A^{\text{GW}}$ used in Lemma 2.5.6. More precisely, $S_{p_-, p_+, A}|_{\mathcal{F}^\infty} = \text{pr}_{\mathcal{F}}^* S_{\mathcal{F}}$ follows from compatibility of the sections in Assumption 2.6.3 (iii) and

$$\text{pr}_{p_-, p_+, A}|_{\mathcal{F}^\infty} = \begin{cases} \text{pr}_{p_-, p_+, A}^\iota \circ \text{pr}_{\mathcal{F}} & \text{for } \mathcal{F} = \partial_0 \mathcal{X}_{p_-, p_+, A}^\iota, \\ (\text{pr}_{p_-, \gamma, A_-}^+ \times \text{pr}_{\gamma, p_+, A_+}^-) \circ \text{pr}_{\mathcal{F}} & \text{for } \mathcal{F} = \partial_0 \mathcal{X}_{p_-, \gamma, A_+}^+ \times \partial_0 \mathcal{X}_{\gamma, p_+, A_-}^-, \\ \text{pr}_{q, p_+, A} \circ \text{pr}_{\mathcal{F}} & \text{for } \mathcal{F} = \mathcal{M}(p_-, q) \times \partial_0 \mathcal{X}_{q, p_+, A}, \\ \text{pr}_{p_-, q, A} \circ \text{pr}_{\mathcal{F}} & \text{for } \mathcal{F} = \partial_0 \mathcal{X}_{p_-, q, A} \times \mathcal{M}(q, p_+). \end{cases}$$

Construction of coherent perturbations: We construct admissible sc^+ -multisections $\kappa_\alpha^\circ : \mathcal{W}_\alpha^\circ \rightarrow \mathbb{Q}^+$ for $\alpha \in \mathcal{I}^+ \cup \mathcal{I}^- \cup \mathcal{I}^\iota \cup \mathcal{I}$ as claimed in (i), i.e. in general position to the respective sections $S_\alpha^\circ : \mathcal{X}_\alpha^\circ \rightarrow \mathcal{W}_\alpha^\circ$, while also coherent as claimed in (iii). The existence of such coherent transverse perturbations will ultimately be guaranteed by an abstract perturbation theorem for coherent systems of sc -Fredholm sections. Since we have to deal with nontrivial isotropy and the SFT perturbation package [18] has only been outlined in [19], we give a detailed construction of the perturbations for our purposes. We proceed as in Lemma 2.6.4 and construct them all as pullbacks $\kappa_\alpha^\circ := \lambda_A^\circ \circ (\text{pr}_\alpha^\circ)^*$ of a collection of sc^+ -multisections on the SFT resp. Gromov-Witten polyfold bundles – without Morse trajectories –

$$\underline{\lambda} = \begin{pmatrix} (\lambda_{\gamma, A}^+ : \widetilde{\mathcal{W}}_{\gamma, A}^+ \rightarrow \mathbb{Q}^+)_{\gamma \in \mathcal{P}(H), A \in H_2(M)} & (\lambda_A^{\text{GW}} : \widetilde{\mathcal{W}}_A^{\text{GW}} \rightarrow \mathbb{Q}^+)_{A \in H_2(M)} \\ (\lambda_{\gamma, A}^- : \widetilde{\mathcal{W}}_{\gamma, A}^- \rightarrow \mathbb{Q}^+)_{\gamma \in \mathcal{P}(H), A \in H_2(M)} & (\lambda_A^{\text{SFT}} : \widetilde{\mathcal{W}}_A^{\text{SFT}} \rightarrow \mathbb{Q}^+)_{A \in H_2(M)} \end{pmatrix}.$$

For this to induce a coherent collection of sc^+ -multisections as required in (iii),

$$\begin{aligned} (\kappa_{p,\gamma,A}^+ &:= \lambda_{\gamma,A}^+ \circ (\text{pr}_{p,\gamma,A}^+)^*)_{(p,\gamma,A) \in \mathcal{I}^+}, & (\kappa_{p-,p+,A}^\ell &:= \lambda_A^{\text{GW}} \circ (\text{pr}_{p-,p+,A}^\ell)^*)_{(p-,p+,A) \in \mathcal{I}^\ell}, \\ (\kappa_{\gamma,p,A}^- &:= \lambda_{\gamma,A}^- \circ (\text{pr}_{\gamma,p,A}^-)^*)_{(\gamma,p,A) \in \mathcal{I}^-}, & (\kappa_{p-,p+,A} &:= \lambda_A^{\text{SFT}} \circ (\text{pr}_{p-,p+,A})^*)_{(p-,p+,A) \in \mathcal{I}}, \end{aligned}$$

it suffices to pick $\underline{\lambda}$ compatible with respect to the faces of the SFT neck stretching polyfolds $\mathcal{X}_A^{\text{SFT}}$ in (2.33). More precisely, using the natural identifications of bundles from Assumption 2.6.3 (iii), we will construct $\underline{\lambda}$ coherent in the sense that – for some choice of $r > 0$ in the construction of $|\mathcal{X}_{\gamma,A}^\pm| = (\bar{\text{ev}}^\pm)^{-1}(\Delta_r^\pm \times M) \subset \mathcal{B}_{\text{SFT}}^\pm(\gamma; A)$ and $\mathcal{W}_{\gamma,A}^\pm = \widetilde{\mathcal{W}}_{\gamma,A}^\pm|_{\mathcal{X}_{\gamma,A}^\pm}$ – we have

$$\lambda_A^{\text{SFT}}(w) = \lambda_A^{\text{GW}}(w) \quad \forall w \in \widetilde{\mathcal{W}}_A^{\text{GW}}, \quad (2.34)$$

$$\lambda_A^{\text{SFT}}((l_{\gamma,A_\pm})_*(w^+, w^-)) = \lambda_{\gamma,A_+}^+(w^+) \cdot \lambda_{\gamma,A_-}^-(w^-) \quad \forall (w^+, w^-) \in \mathcal{W}_{\gamma,A_+}^+ \times \mathcal{W}_{\gamma,A_-}^-. \quad (2.35)$$

So to finish this proof it remains to choose the sc^+ -multisections $\underline{\lambda}$ so that each induced sc^+ -multisection in the induced coherent collection for $(\kappa_\alpha^\cdots)_{\alpha \in \mathcal{I}^+ \cup \mathcal{I}^- \cup \mathcal{I}^\ell \cup \mathcal{I}}$ is admissible and in general position, while also satisfying the coherence requirements (2.34), (2.35) and the requirements on $\underline{\kappa}^\ell$ in the proofs of Lemma 2.6.4 and 2.6.5. The construction of coherent perturbations for the SFT polyfolds outlined in [19] proceeds by first choosing coherent compactness controlling data, i.e. pairs $(N, \mathcal{U})$ of auxiliary norms on all the bundles and saturated neighborhoods of the compact zero sets in all the ep-groupoids $\widetilde{\mathcal{X}}_{\gamma,A}^\pm, \widetilde{\mathcal{X}}_A^{\text{GW}}, \widetilde{\mathcal{X}}_A^{\text{SFT}}$ (c.f. Definition 2.7.5), which are compatible with the immersions to boundary faces in (2.33). Then it constructs the perturbations λ_A^{GW} as in Lemma 2.6.5 and also $\lambda_{\gamma,A_\pm}^\pm$ to be in general position, admissible w.r.t. the coherent data $(2N, \mathcal{U})$, and coherent in the sense that continuous extension of (2.34)–(2.35) induces a well defined multisection $\lambda_A^\partial : \widetilde{\mathcal{W}}_A^{\text{SFT}}|_{\partial \mathcal{X}_A^{\text{SFT}}} \rightarrow \mathbb{Q}^+$. Here coherence of the perturbations on the intersection of faces (see Remark 2.6.2) is required to guarantee existence of scale-smooth extensions of λ_A^∂ to multisections $\lambda_A^{\text{SFT}} : \widetilde{\mathcal{W}}_A^{\text{SFT}} \rightarrow \mathbb{Q}^+$. Coherence of the compactness controlling pairs guarantees that the multisection λ_A^∂ over $\partial \mathcal{X}_A^{\text{SFT}} \subset \widetilde{\mathcal{X}}_A^{\text{SFT}}$ satisfies the auxiliary norm bounds $N(\lambda_A^\partial) \leq \frac{1}{2}$ and support requirements that guarantee compactness for extensions λ_A^{SFT} of λ_A^∂ with $N(\lambda_A^{\text{SFT}}) \leq 1$ and appropriate support requirements. Moreover, we may choose each of the extensions λ_A^{SFT} using Theorem 2.7.9 to ensure – as in Lemma 2.6.4 – that the induced multisections $\kappa_\alpha^\cdots$ are in general position as well. The latter will automatically be admissible with respect to pullback of the pair controlling compactness. In more detail (but without specifying the auxiliary norm bounds) the inductive construction of perturbations in [18] – simplified to the subset of SFT moduli spaces considered here – proceeds as follows:

Construction of λ_A^{GW} and $\underline{\kappa}^\ell$: Since the Gromov-Witten ep-groupoids $\widetilde{\mathcal{X}}_A^{\text{GW}}$ are boundaryless by Assumption 2.5.5 (iii), the sc^+ -multisections λ_A^{GW} can be chosen independently of all other multisections. So we construct λ_A^{GW} as in the proofs of Lemma 2.6.4 and 2.6.5, to ensure that the conclusions in these lemmas hold, as required by (i). This prescribes (2.34) on the boundary face $\widetilde{\mathcal{X}}_A^{\text{GW}} \subset \partial \widetilde{\mathcal{X}}_A^{\text{SFT}}$.

Moreover, recall that λ_A^{GW} is obtained by applying Theorem 2.7.9 to the sc-Fredholm section functors S_A^{GW} , the sc^∞ submersion $\overline{\text{ev}}^+ \times \overline{\text{ev}}^- : \tilde{\mathcal{X}}_A^{\text{GW}} \rightarrow \mathbb{CP}^1 \times M \times \mathbb{CP}^1 \times M$, and the collection of Cartesian products of stable and unstable manifolds $\{z_0^+\} \times W_{p_-}^- \times \{z_0^-\} \times W_{p_+}^+$. As in the proof of Lemma 2.6.4 this ensures that the pullbacks $\underline{\kappa}^\iota = (\kappa_\alpha^\iota = \lambda_A^{\text{GW}} \circ (\text{pr}_\alpha^\iota)^*)_{\alpha \in \mathcal{I}^\iota}$ are in general position. Moreover, these pullbacks are admissible w.r.t. the pairs controlling compactness on $\mathcal{W}_\alpha^\iota \rightarrow \mathcal{X}_\alpha^\iota$ that result by pullback from the coherent compactness controlling pair on $\tilde{\mathcal{W}}_A^{\text{GW}} \rightarrow \tilde{\mathcal{X}}_A^{\text{GW}}$, which is constructed in a preliminary step by [19, Lecture 13].

Coherence for $\lambda_{\gamma,A}^\pm$: The next step is to construct sc^+ -multisections $\lambda_{\gamma,A}^\pm : \tilde{\mathcal{W}}_{\gamma,A}^\pm \rightarrow \mathbb{Q}^+$ over the SFT ep-groupoids $\tilde{\mathcal{X}}_{\gamma,A}^\pm$ of planes with limit orbit $\gamma \in \mathcal{P}(H)$ from Assumption 2.4.3, which then induce the perturbations $\underline{\kappa}^\pm$ for the PSS/SSP moduli spaces. These constructions are independent of the choice of λ_A^{GW} since the corresponding boundary faces of $\tilde{\mathcal{X}}_A^{\text{SFT}}$ do not intersect by Assumption 2.6.3 (ii). However, to enable the subsequent construction of λ_A^{SFT} as extension of the boundary values prescribed in (2.34) and (2.35), we need to make sure that each sc^+ -multisection $(\lambda_{\gamma,A_+}^+ \cdot \lambda_{\gamma,A_-}^-) \circ (l_{\gamma,A_\pm})_*^{-1}$ is well defined on the (open subset of) face $\mathcal{F}_{\gamma,A_\pm} := l_{\gamma,A_\pm}(\mathcal{X}_{\gamma,A_+}^+ \times \mathcal{X}_{\gamma,A_-}^-) \subset \partial \tilde{\mathcal{X}}_A^{\text{SFT}}$ and coincides with the other sc^+ -multisections $(\lambda_{\gamma',A'_+}^+ \cdot \lambda_{\gamma',A'_-}^-) \circ (l_{\gamma',A'_\pm})_*^{-1}$ on their intersection $\mathcal{F}_{\gamma,A_\pm} \cap \mathcal{F}_{\gamma',A'_\pm}$. Then this yields a well defined sc^+ -multisection on $\bigcup \mathcal{F}_{\gamma,A_\pm} = \partial \tilde{\mathcal{X}}_A^{\text{SFT}} \subset \partial \tilde{\mathcal{X}}_A^{\text{SFT}}$. To describe these intersections we note that [19] constructs the ep-groupoids $\tilde{\mathcal{X}}_{\gamma,A_\pm}^\pm$ with coherent boundaries – involving ep-groupoids $(\mathcal{X}_{\gamma^\pm, \gamma^\pm, B}^{\text{Fl}})_{\gamma^\pm \in \mathcal{P}(H), B \in H_2(M)}$ which contain the moduli spaces of Floer trajectories between periodic orbits $\gamma^\pm$, as well as further ep-groupoids for Floer trajectories carrying a marked point. We will avoid dealing with the latter by specifying values $r < \infty$ when pulling back perturbations from the ep-groupoids $\mathcal{X}_{\gamma,A}^\pm \subset \tilde{\mathcal{X}}_{\gamma,A}^\pm$ given by $|\mathcal{X}_{\gamma,A}^\pm| = (\overline{\text{ev}}^\pm)^{-1}(\Delta_r^\pm \times M) \subset \mathcal{B}_{\text{SFT}}^\pm(\gamma; A)$, as this will prevent the appearance of marked Floer trajectories even in the closure. For any fixed value $0 < r \leq \infty$, the j -th boundary stratum is given by j Floer trajectories breaking off,

$$\begin{aligned} \partial_j \mathcal{X}_{\gamma,A}^+ &= \bigsqcup_{\substack{\gamma^0, \dots, \gamma^j \in \mathcal{P}(H) \\ A_+ + B_1 + \dots + B_j = A}} \partial_0 \mathcal{X}_{\gamma^0, A_+}^+ \times \partial_0 \mathcal{X}_{\gamma^0, \gamma^1, B_1}^{\text{Fl}} \times \dots \times \partial_0 \mathcal{X}_{\gamma^{j-1}, \gamma^j, B_j}^{\text{Fl}}, \\ \partial_{k-j} \mathcal{X}_{\gamma,A}^- &= \bigsqcup_{\substack{\gamma = \gamma^j, \dots, \gamma^k \in \mathcal{P}(H) \\ B_{j+1} + \dots + B_k + A_- = A}} \partial_0 \mathcal{X}_{\gamma^j, \gamma^{j+1}, B_{j+1}}^{\text{Fl}} \times \dots \times \partial_0 \mathcal{X}_{\gamma^{k-1}, \gamma^k, B_k}^{\text{Fl}} \times \partial_0 \mathcal{X}_{\gamma^k, A_-}^-. \end{aligned} \quad (2.36)$$

Now, for example, $\partial_0 \mathcal{X}_{\gamma^0, A_+}^+ \times \partial_0 \mathcal{X}_{\gamma^0, \gamma^1, B}^{\text{Fl}} \times \partial_0 \mathcal{X}_{\gamma^1, A_-}^-$ is both a subset of $\partial_0 \mathcal{X}_{\gamma^0, A_+}^+ \times \partial_1 \mathcal{X}_{\gamma^0, A_- + B}^- \subset \partial(\mathcal{X}_{\gamma^0, A_+}^+ \times \mathcal{X}_{\gamma^0, A_- + B}^-)$ and of $\partial_1 \mathcal{X}_{\gamma^1, A_+ + B}^+ \times \partial_0 \mathcal{X}_{\gamma^1, A_-}^- \subset \partial(\mathcal{X}_{\gamma^1, A_+ + B}^+ \times \mathcal{X}_{\gamma^1, A_-}^-)$, and the embeddings $l_{\gamma^0, A_\pm^0}$ and $l_{\gamma^1, A_\pm^1}$ for the two splittings $A_+ + (A_- + B) = A_+^0 + A_-^0 = A = A_+^1 + A_-^1 = (A_+ + B) + A_-$ coincide under this identification. Generally, the boundary of the Floer ep-groupoids is given by broken trajectories, and this yields a disjoint cover of $\partial^{R=\infty} \mathcal{X}_A^{\text{SFT}} \subset \partial \mathcal{X}_A^{\text{SFT}}$,

$$\partial^{R=\infty} \mathcal{X}_A^{\text{SFT}} = \bigsqcup_{\substack{\gamma^0, \dots, \gamma^k \in \mathcal{P}(H) \\ A_+ + B_1 + \dots + B_k + A_- = A}} l_{\gamma, A_\pm, B}(\partial_0 \mathcal{X}_{\gamma^0, A_+}^+ \times \partial_0 \mathcal{X}_{\gamma^0, \gamma^1, B_1}^{\text{Fl}} \times \dots \times \partial_0 \mathcal{X}_{\gamma^{k-1}, \gamma^k, B_k}^{\text{Fl}} \times \partial_0 \mathcal{X}_{\gamma^k, A_-}^-)$$

in which the embeddings $l_{\underline{\gamma}, A_{\pm}, \underline{B}}$ coincide with each of the embeddings $l_{\gamma^j, A_{\pm}^j}$ for $0 \leq j \leq k$ and $A_+^j = A_+ + \sum_{i \leq j} B_i$, $A_-^j = A_- + \sum_{i > j} B_i$ – when restricted to the subsets

$$\partial_0 \mathcal{X}_{\gamma^0, A_+}^+ \times \partial_0 \mathcal{X}_{\gamma^0, \gamma^1, B_1}^{\text{Fl}} \times \dots \times \partial_0 \mathcal{X}_{\gamma^{k-1}, \gamma^k, B_k}^{\text{Fl}} \times \partial_0 \mathcal{X}_{\gamma^k, A_-}^- \subset \partial_j \mathcal{X}_{\gamma^j, A_+^j}^+ \times \partial_{k-j} \mathcal{X}_{\gamma^j, A_-^j}^-.$$

Now on these subsets we require coherence $\lambda_{\gamma^j, A_+^j}^+ \cdot \lambda_{\gamma^j, A_-^j}^- = \lambda_{\gamma^{j'}, A_+^{j'}}^+ \cdot \lambda_{\gamma^{j'}, A_-^{j'}}^-$ for all $0 \leq j \neq j' \leq k$, as this is equivalent to (2.35) being well defined on $\text{im } l_{\underline{\gamma}, A_{\pm}, \underline{B}} = \bigcap_{j=0}^k \mathcal{F}_{\gamma^j, A_{\pm}^j}$. This will be achieved by constructing the sc^+ -multisections $(\lambda_{\gamma, A_{\pm}}^{\pm})$ to have product structure on the boundary – where the bundles $P_{\gamma, A} : \mathcal{W}_{\gamma, A}^{\pm} \rightarrow \mathcal{X}_{\gamma, A}^{\pm}$ are restricted to various faces of $\partial \mathcal{X}_{\gamma, A}^{\pm}$ –

$$\begin{aligned} \lambda_{\gamma^j, A_+^j}^+ \big|_{P_{\gamma^j, A_+^j}^{-1}(\mathcal{X}_{\gamma^0, A_+}^+ \times \mathcal{X}_{\gamma^0, \gamma^1, B_1}^{\text{Fl}} \times \dots \times \mathcal{X}_{\gamma^{j-1}, \gamma^j, B_j}^{\text{Fl}})} &= \lambda_{\gamma^0, A_+}^+ \cdot \lambda_{\gamma^0, \gamma^1, B_1}^{\text{Fl}} \cdot \dots \cdot \lambda_{\gamma^{j-1}, \gamma^j, B_j}^{\text{Fl}}, \\ \lambda_{\gamma^j, A_-^j}^- \big|_{P_{\gamma^j, A_-^j}^{-1}(\mathcal{X}_{\gamma^j, \gamma^{j+1}, B_{j+1}}^{\text{Fl}} \times \dots \times \mathcal{X}_{\gamma^{k-1}, \gamma^k, B_k}^{\text{Fl}} \times \mathcal{X}_{\gamma^k, A_-}^-)} &= \lambda_{\gamma^j, \gamma^{j+1}, B_{j+1}}^{\text{Fl}} \cdot \dots \cdot \lambda_{\gamma^{k-1}, \gamma^k, B_k}^{\text{Fl}} \cdot \lambda_{\gamma^k, A_-}^-, \end{aligned} \quad (2.37)$$

for a collection of sc^+ -multisections $\lambda_{\gamma^-, \gamma^+, B}^{\text{Fl}} : \mathcal{W}_{\gamma^-, \gamma^+, B}^{\text{Fl}} \rightarrow \mathbb{Q}$ over the Floer ep-groupoids $\mathcal{X}_{\gamma^-, \gamma^+, B}^{\text{Fl}}$. While this guarantees coherence on each overlap of embeddings $\text{im } l_{\underline{\gamma}, A_{\pm}, \underline{B}} \subset \mathcal{F}_{\gamma^j, A_{\pm}^j} \cap \mathcal{F}_{\gamma^{j'}, A_{\pm}^{j'}}$,

$$\lambda_{\gamma^j, A_+^j}^+ \cdot \lambda_{\gamma^j, A_-^j}^- = \lambda_{\gamma^0, A_+}^+ \cdot \lambda_{\gamma^0, \gamma^1, B_1}^{\text{Fl}} \cdot \dots \cdot \lambda_{\gamma^{k-1}, \gamma^k, B_k}^{\text{Fl}} \cdot \lambda_{\gamma^k, A_-}^- = \lambda_{\gamma^{j'}, A_+^{j'}}^+ \cdot \lambda_{\gamma^{j'}, A_-^{j'}}^-,$$

we are now faced with the challenge of satisfying the coherence conditions in (2.37). These conditions uniquely determine the boundary restrictions $\lambda_{\gamma, A_{\pm}}^{\pm} \big|_{P_{\gamma, A_{\pm}}^{-1}(\partial \mathcal{X}_{\gamma, A_{\pm}}^{\pm})}$ via the identification of the boundaries with Cartesian products of interiors in (2.36). Thus (2.36) on Cartesian products involving boundary strata poses coherence conditions on the choice of $\lambda_{\beta}^{\text{Fl}}$ for $\beta \in \mathcal{I}^{\text{Fl}} := \mathcal{P}(H) \times \mathcal{P}(H) \times H_2(M)$.

Construction of $\lambda_{\gamma^-, \gamma^+, B}^{\text{Fl}}$: To achieve the coherence in (2.37), [19] first constructs the sc^+ -multisections $(\lambda_{\beta}^{\text{Fl}})_{\beta \in \mathcal{I}^{\text{Fl}}}$ by iteration over the maximal degeneracy index $k_{\beta} := \max\{k \in \mathbb{N}_0 \mid (S_{\beta}^{\text{Fl}})^{-1}(0) \cap \partial_k \mathcal{X}_{\beta}^{\text{Fl}} \neq \emptyset\}$ of unperturbed solutions (which is finite by Gromov compactness): For $k_{\beta} = -\infty$ the section S_{β}^{Fl} has no zeros so is already transverse, so that $\lambda_{\beta}^{\text{Fl}}$ can be chosen as the trivial perturbation. (The trivial multivalued section functor $\lambda : \mathcal{W} \rightarrow \mathbb{Q}^+$ is given by $\lambda(0) = 1$ and $\lambda(w \neq 0) = 0$.) For $k_{\beta} = 0$ the section S_{β}^{Fl} has all zeros in the interior, so that $\lambda_{\beta}^{\text{Fl}}$ can be chosen admissible and trivial on the boundary – by applying Corollary 2.4.6 (i) with a neighborhood of the unperturbed zero set in the interior, $|(S_{\beta}^{\text{Fl}})^{-1}(0)| \subset \mathcal{V}_{\beta} \subset |\partial_0 \mathcal{X}_{\beta}^{\text{Fl}}|$. For $\beta = (\gamma^-, \gamma^+, B) \in \mathcal{I}^{\text{Fl}}$ with $k_{\beta} \geq 1$ the restriction $\lambda_{\beta}^{\text{Fl}} \big|_{P_{\beta}^{-1}(\partial \mathcal{X}_{\beta}^{\text{Fl}})}$ to the boundary $\partial \mathcal{X}_{\beta}^{\text{Fl}} = \bigcup_{\gamma^- = \gamma^0, \gamma^1, \dots, \gamma^{k-1}, \gamma^k = \gamma^+, B = B_1 + \dots + B_k} \partial_0 \mathcal{X}_{\gamma^0, \gamma^1, B_1}^{\text{Fl}} \times \dots \times \partial_0 \mathcal{X}_{\gamma^{k-1}, \gamma^k, B_k}^{\text{Fl}}$ is prescribed by the previous iteration steps $\lambda_{\beta}^{\text{Fl}} \big|_{P_{\beta}^{-1}(\mathcal{X}_{\gamma^0, \gamma^1, B_1}^{\text{Fl}} \times \dots \times \mathcal{X}_{\gamma^{k-1}, \gamma^k, B_k}^{\text{Fl}})} := \lambda_{\gamma^0, \gamma^1, B_1}^{\text{Fl}} \cdot \dots \cdot \lambda_{\gamma^{k-1}, \gamma^k, B_k}^{\text{Fl}}$ on all boundary faces that contain unperturbed solutions in their closure. Indeed, existence of

a solution in $\mathcal{X}_{\gamma^0, \gamma^1, B_1}^{\text{Fl}} \times \dots \times \mathcal{X}_{\gamma^{k-1}, \gamma^k, B_k}^{\text{Fl}}$ implies $k_{\gamma^{i-1}, \gamma^i, B_i} \geq 0$ for $i = 1, \dots, k$, and the Cartesian product of solutions of maximal degeneracy yields $1 + k_{\gamma^0, \gamma^1, B_1} + \dots + k_{\gamma^{k-1}, \gamma^k, B_k} \leq k_\beta$. Thus these prescriptions are made for $0 \leq k_{\gamma^{i-1}, \gamma^i, B_i} \leq k_\beta - 1$, and on boundary faces with no solutions in their closure we prescribe the trivial perturbation throughout. This yields a well defined sc^+ -multisection $\lambda_\beta^{\text{Fl}}|_{P_\beta^{-1}(\partial\mathcal{X}_\beta^{\text{Fl}})}$ by coherence in the prior iteration steps, so that $\lambda_\beta^{\text{Fl}}$ can be constructed by applying the extension result [34, Thm.15.5] which provides general position and admissibility with respect to a pair controlling compactness that extends the pair which was chosen on the boundary in prior iteration steps.

Construction of $\lambda_{\gamma, A}^\pm$ and $\kappa^\pm$: With the Floer perturbations in place, [19] next constructs the collections of sc^+ -multisections $(\lambda_{\gamma, A}^\pm)_{\gamma \in \mathcal{P}(H), A \in H_2(M)}$ to satisfy (2.37) by iteration over degeneracy $k_{\gamma, A} := \max\{k \in \mathbb{N}_0 \mid (S_{\gamma, A}^\pm)^{-1}(0) \cap \partial_k \tilde{\mathcal{X}}_{\gamma, A}^\pm \neq \emptyset\}$. For $k_{\gamma, A} = -\infty$ one takes $\lambda_{\gamma, A}^\pm$ to be trivial. For $k_{\gamma, A} = 0$ one applies Theorem 2.7.9 to the sc -Fredholm section functor $S_{\gamma, A}^\pm : \tilde{\mathcal{X}}_{\gamma, A}^\pm \rightarrow \tilde{\mathcal{W}}_{\gamma, A}^\pm$, the map $\bar{\text{ev}}^\pm : \tilde{\mathcal{X}}_{\gamma, A}^\pm \rightarrow \mathbb{C}^\pm \times M$, and the collection of stable resp. unstable manifolds $\{0\} \times W_p^\pm$ for all critical points $p \in \text{Crit}(f)$. These satisfy the assumptions as the zero set $|(S_{\gamma, A}^\pm)^{-1}(0)|$ is compact and the preimages $(\bar{\text{ev}}^\pm)^{-1}(\{0\} \times W_p^\pm)$ lie within the open subset $\mathcal{X}_{\gamma, A}^\pm \subset \tilde{\mathcal{X}}_{\gamma, A}^\pm$ on which $\bar{\text{ev}}^\pm$ restricts to a sc^∞ submersion $\text{ev}^\pm : \mathcal{X}_{\gamma, A}^\pm \rightarrow \mathbb{C}^\pm \times M$. We can moreover prescribe $\lambda_{\gamma, A}^\pm|_{P_{\gamma, A}^{-1}(\partial\tilde{\mathcal{X}}_{\gamma, A}^\pm)}$ to be trivial, since in the absence of solutions the trivial perturbation is in general position. Then Theorem 2.7.9 provides $\lambda_{\gamma, A}^\pm$ that is supported in the interior and transverse to each submanifold $\{0\} \times W_p^\pm$ in the sense that these submanifolds are transverse to the evaluation from the perturbed zero set

$$\text{ev}^\pm : |\{x \in \mathcal{X}_{\gamma, A}^\pm \mid \lambda_{\gamma, A}^\pm(S_{\gamma, A}^\pm(x)) > 0\}| \rightarrow \mathbb{C}^\pm \times M. \quad (2.38)$$

Now suppose that admissible $\lambda_{\gamma', A'}^\pm$ in general position have been constructed for $k_{\gamma', A'} \leq k \in \mathbb{N}_0$, and satisfy both the transversality in (2.38) and the coherence condition (2.37) over the ep-groupoids $|\mathcal{X}_{\gamma', A'}^\pm| = (\bar{\text{ev}}^\pm)^{-1}(\Delta_{r_k}^\pm \times M)$ with $r_k := 2 + 2^{-k}$. Then for $k_{\gamma, A} = k + 1$ we will construct $\lambda_{\gamma, A}^\pm$ to satisfy (2.37) over $(\bar{\text{ev}}^\pm)^{-1}(\Delta_{r_{k+1}}^\pm \times M)$ by first noting that the previous iteration – and requiring triviality on boundary faces without solutions – determines a well defined sc^+ -multisection $\lambda_{\gamma, A}^\pm|_{P_{\gamma, A}^{-1}(\partial\mathcal{X}_{\gamma, A}^\pm)}$ over the $r = r_k$ boundary $\partial\mathcal{X}_{\gamma, A}^\pm \simeq \bigcup_{\gamma', A=A_\pm+B} \partial_0 \mathcal{X}_{\gamma', A_\pm}^\pm \times \mathcal{X}_{\gamma', \gamma, B}^{\text{Fl}}$. For faces (w.r.t. $\partial\mathcal{X}_{\gamma, A}^\pm$) with solutions it is given by $\lambda_{\gamma, A}^\pm|_{P_{\gamma, A}^{-1}(\mathcal{X}_{\gamma', A_\pm}^\pm \times \mathcal{X}_{\gamma', \gamma, B}^{\text{Fl}})} = \lambda_{\gamma', A_\pm}^\pm \times \lambda_{\gamma', \gamma, B}^{\text{Fl}}$ where $k_{\gamma, A} \geq 1 + k_{\gamma', A_\pm} + k_{\gamma', \gamma, B}$. This is well defined at $(x^\pm, \underline{x}, \underline{x}') \in \partial_0 \mathcal{X}_{\gamma', A_\pm}^\pm \times \mathcal{X}_{\gamma', \gamma'', B'}^{\text{Fl}} \times \mathcal{X}_{\gamma'', \gamma, B-B'}^{\text{Fl}}$, which appears both as $(x^\pm, (\underline{x}, \underline{x}')) \in \partial_0 \mathcal{X}_{\gamma', A_\pm}^\pm \times \partial\mathcal{X}_{\gamma', \gamma, B}^{\text{Fl}}$ and $((x^\pm, \underline{x}), \underline{x}') \in \partial\mathcal{X}_{\gamma'', A_\pm+B'}^\pm \times \mathcal{X}_{\gamma'', \gamma, B-B'}^{\text{Fl}}$, by the coherence of the Floer multisections and the prior iteration: For vectors in the respective fibers $(w^\pm, w, w') \in P_{\gamma', A_\pm}^{-1}(x^\pm) \times P_{\gamma', \gamma'', B'}^{-1}(\underline{x}) \times P_{\gamma'', \gamma, B-B'}^{-1}(\underline{x}')$ we have

$$\begin{aligned} \lambda_{\gamma', A_\pm}^\pm(w^\pm) \cdot \lambda_{\gamma', \gamma, B}^{\text{Fl}}(w, w') &= \lambda_{\gamma', A_\pm}^\pm(w^\pm) \cdot \lambda_{\gamma', \gamma'', B'}^{\text{Fl}}(w) \cdot \lambda_{\gamma'', \gamma, B-B'}^{\text{Fl}}(w') \\ &= \lambda_{\gamma'', A_\pm+B'}^\pm(w^\pm, w) \cdot \lambda_{\gamma'', \gamma, B-B'}^{\text{Fl}}(w'). \end{aligned}$$

Moreover, $\text{ev}^\pm : |\{x \in \partial\mathcal{X}_{\gamma, A}^\pm \mid \lambda_{\gamma, A}^\pm(S_{\gamma, A}^\pm(x)) > 0\}| \rightarrow \mathbb{C}^\pm \times M$ is transverse to the submanifolds $\{0\} \times W_p^\pm$. However, this defines an admissible sc^+ -multisection in general position

only over the open subset of the boundary $\partial\mathcal{X}_{\gamma,A}^\pm = (\text{ev}^\pm)^{-1}(\Delta_{r_k}^\pm \times M) \cap \partial\tilde{\mathcal{X}}_{\gamma,A}^\pm$. We multiply the given data by a scale-smooth cutoff function – guaranteed by the existence of partitions of unity for the open cover $|\tilde{\mathcal{X}}_{\gamma,A}^\pm| = (\bar{\text{ev}}^\pm)^{-1}(\Delta_{r_k}^\pm \times M) \cup (\bar{\text{ev}}^\pm)^{-1}((\bar{\mathbb{C}}^\pm \setminus \Delta_{r_{k+\frac{1}{2}}}^\pm) \times M)$; see

Remark 2.7.6 – to obtain an admissible sc^+ -multisection $\lambda_{\gamma,A}^\partial : \widetilde{\mathcal{W}}_{\gamma,A}^\pm|_{\partial\tilde{\mathcal{X}}_{\gamma,A}^\pm} \rightarrow \mathbb{Q}^+$ which coincides with the prescribed data – thus in general position and with evaluation transverse to each $\{0\} \times W_p^\pm$ – over the closed subset $(\text{ev}^\pm)^{-1}(\overline{\Delta_{r_{k+1}}}^\pm \times M) \cap \partial\tilde{\mathcal{X}}_{\gamma,A}^\pm$. Then $\lambda_{\gamma,A}^\pm : \widetilde{\mathcal{W}}_{\gamma,A}^\pm \rightarrow \mathbb{Q}^+$ is constructed with these given boundary values using Theorem 2.7.9 to achieve not just general position but also transversality as in (2.38). By admissibility of the prior iteration and coherence of the pairs controlling compactness, $\lambda_{\gamma,A}^\pm$ can moreover be chosen admissible.

As required in the coherence discussion, this determines right hand sides of (2.35) which agree on overlaps of different immersions $l_{\gamma,A_\pm}(\mathcal{X}_{\gamma,A_+}^+ \times \mathcal{X}_{\gamma,A_-}^-)$ for $r = 2$. Thus it constructs a well defined sc^+ -multisection on $\partial^{R=\infty}\mathcal{X}_A^{\text{SFT}} = \bigcup l_{\gamma,A_\pm}(\mathcal{X}_{\gamma,A_+}^+ \times \mathcal{X}_{\gamma,A_-}^-) \subset \partial\tilde{\mathcal{X}}_A^{\text{SFT}}$ that is admissible and has evaluations transverse to the submanifolds $\{0\} \times W_{p_-}^- \times \{0\} \times W_{p_+}^+$ for all pairs $p_-, p_+ \in \text{Crit}(f)$.

Moreover, for $\alpha \in \mathcal{I}^\pm$ we obtain a pair controlling compactness by pullback of the coherent pairs constructed in [19] on the bundles $\mathcal{W}_{\gamma,A}^\pm$. Then the pullback multisections $\underline{\kappa}^\pm = (\kappa_\alpha^\pm = \lambda_{\gamma,A}^\pm \circ (\text{pr}_\alpha^\pm)^*)_{\alpha \in \mathcal{I}^\pm}$ are sc^+ , admissible w.r.t. the pullback pair, and in general position by the arguments in the proof of Lemma 2.6.4.

Construction of λ_A^{SFT} and $\underline{\kappa}$: The above constructions determine the right hand sides in the coherence requirements $\lambda_A^{\text{SFT}}|_{P_A^{-1}(\tilde{\mathcal{X}}_A^{\text{GW}})} = \lambda_A^{\text{GW}}$ over $\tilde{\mathcal{X}}_A^{\text{GW}} \subset \tilde{\mathcal{X}}_A^{\text{SFT}}$ in (2.34), as well as $\lambda_A^{\text{SFT}}|_{P_A^{-1}(\mathcal{F}_{\gamma,A_\pm}(2))} = (\lambda_{\gamma,A_+}^+ \cdot \lambda_{\gamma,A_-}^-) \circ (l_{\gamma,A_\pm})_*^{-1}$ on $\bigcup_{\gamma \in \mathcal{P}(H), A_- + A_+ = A} \mathcal{F}_{\gamma,A_\pm}(2) \subset \partial\tilde{\mathcal{X}}_A^{\text{SFT}}$ in (2.35), where we denote by $\mathcal{F}_{\gamma,A_\pm}(r) := l_{\gamma,A_\pm}(\mathcal{X}_{\gamma,A_+}^+ \times \mathcal{X}_{\gamma,A_-}^-) \subset \partial\tilde{\mathcal{X}}_A^{\text{SFT}}$ the image of the immersion $l_{\gamma,A_\pm}$ on the ep-groupoids representing $|\mathcal{X}_{\gamma,A_\pm}^\pm| = (\bar{\text{ev}}^\pm)^{-1}(\Delta_r^\pm \times M) \subset \mathcal{B}_{\text{SFT}}^\pm(\gamma; A)$. By admissibility in the prior steps and existence of scale-smooth partitions of unity (see Remark 2.7.6) these induce for every $A \in H_2(M)$ an admissible sc^+ -multisection $\lambda_A^\partial : \widetilde{\mathcal{W}}_A^\pm|_{\partial\tilde{\mathcal{X}}_A} \rightarrow \mathbb{Q}^+$ which coincides with the prescribed data over $\tilde{\mathcal{X}}_A^{\text{GW}} \sqcup \bigcup_{\gamma,A_\pm} \overline{\mathcal{F}_{\gamma,A_\pm}(1)} \subset \partial\tilde{\mathcal{X}}_A^{\text{SFT}}$. Thus on this closed subset we have general position and transversality of the evaluation map

$$\text{ev}^+ \times \text{ev}^- : |\{x \in \partial\mathcal{X}_A^{\text{SFT}} \mid \lambda_A^\partial(S_A^{\text{SFT}}(x)) > 0\}| \rightarrow \mathbb{C}^+ \times M \times \mathbb{C}^- \times M \quad (2.39)$$

to $\{0\} \times W_{p_-}^- \times \{0\} \times W_{p_+}^+$ for any pair of critical points $p_-, p_+ \in \text{Crit}(f)$. Then the admissible sc^+ -multisection $\lambda_A^{\text{SFT}} : \widetilde{\mathcal{W}}_A^{\text{SFT}} \rightarrow \mathbb{Q}^+$ is constructed with these given boundary values – and auxiliary norm and support prescribed by the coherent pairs controlling compactness – using Theorem 2.7.9 to achieve general position on all of $\tilde{\mathcal{X}}_A^{\text{SFT}}$ and extend transversality of the evaluation $\text{ev}^+ \times \text{ev}^-$ to $\{0\} \times W_{p_-}^- \times \{0\} \times W_{p_+}^+$ to the entire perturbed zero set $|\{x \in \mathcal{X}_A^{\text{SFT}} \mid \lambda_A^{\text{SFT}}(S_A^{\text{SFT}}(x)) > 0\}|$, where $|\mathcal{X}_A^{\text{SFT}}| = (\bar{\text{ev}}^+)^{-1}(\Delta_1^+ \times M) \cap (\bar{\text{ev}}^-)^{-1}(\Delta_1^- \times M) \subset \mathcal{B}_{\text{SFT}}(A)$.

As in the proof of Lemma 2.6.4, the transversality of the evaluation maps implies that the pullbacks $\underline{\kappa} = (\kappa_\alpha = \lambda_A \circ (\text{pr}_\alpha^t)^*)_{\alpha \in \mathcal{I}}$ are in general position. They are also admissible with respect to the pullback of pairs controlling compactness. This finishes the construction of the sc^+ -multisections claimed in (i) with the boundary restrictions required in (iii).

Proof of identity: By Λ -linearity of all maps involved, it suffices to fix two generators $p_-, p_+ \in \text{Crit}(f)$ of CM and check that $\iota_{\kappa^\iota} \langle p_- \rangle$ and $(SSP_{\kappa^-} \circ PSS_{\kappa^+}) \langle p_- \rangle + (-1)^{|p_-|} (d \circ h_{\kappa^-}) \langle p_- \rangle + (-1)^{|p_-|} (h_{\kappa^-} \circ d) \langle p_- \rangle$ have the same coefficient in Λ on $\langle p_+ \rangle$. That is, we claim

$$\begin{aligned} \sum_{\substack{A \in H_2(M) \\ I^\iota(p_-, p_+; A) = 0}} \#Z^{\kappa^\iota}(p_-, p_+; A) \cdot T^{\omega(A)} = \\ \sum_{\substack{\gamma \in \mathcal{P}(H), A_-, A_+ \in H_2(M) \\ I(p_-, \gamma; A_+) = I(\gamma, p_+; A_-) = 0}} \#Z^{\kappa^+}(p_-, \gamma; A_+) \#Z^{\kappa^-}(\gamma, p_+; A_-) \cdot T^{\omega(A_-) + \omega(A_+)} \\ + (-1)^{|p_-|} \sum_{\substack{q \in \text{Crit}(f), A \in H_2(M) \\ I(p_-, q; A) = |q| - |p_+| - 1 = 0}} \#Z^{\kappa}(p_-, q; A) \# \mathcal{M}(q, p_+) \cdot T^{\omega(A)} \\ + (-1)^{|p_-|} \sum_{\substack{q \in \text{Crit}(f), A \in H_2(M) \\ |p_-| - |q| - 1 = I(q, p_+; A) = 0}} \# \mathcal{M}(p_-, q) \# Z^{\kappa}(q, p_+; A) \cdot T^{\omega(A)}. \end{aligned}$$

Here the sums on the right hand side are over counts of pairs of moduli spaces of index 0. From §2.3 we have $\mathcal{M}(q, p_+) = \emptyset$ for $|q| - |p_+| - 1 < 0$ and $\mathcal{M}(p_-, q) = \emptyset$ for $|p_-| - |q| - 1 < 0$, and general position of the sc^+ -multisections $\kappa^\bullet$ as in Corollary 2.4.6 (iii) implies $Z^{\kappa^\bullet}(\dots) = \emptyset$ for $I(\dots) < 0$. Thus the right hand side can be rewritten as sum over pairs of moduli spaces with indices summing to zero, and by (2.9), (2.18), (2.24) this is moreover equivalent to

$$\begin{aligned} 0 &= I(p_-, \gamma; A_+) + I(\gamma, p_+; A_-) = I^\iota(p_-, p_+; A_- + A_+) = I(p_-, p_+; A_- + A_+) - 1, \\ 0 &= I(p_-, q; A) + |q| - |p_+| - 1 = I(p_-, p_+; A) - 1, \\ 0 &= |p_-| - |q| - 1 + I(q, p_+; A) = I(p_-, p_+; A) - 1. \end{aligned}$$

So all sums can be rewritten with the index condition $I(p_-, p_+; A) = 1$ for $A = A_- + A_+ \in H_2(M)$, and since the symplectic area is additive $\omega(A_-) + \omega(A_+) = \omega(A_- + A_+)$, it suffices to show the following identity for each $\alpha = (p_-, p_+; A) \in \mathcal{I}$ with $I(p_-, p_+; A) = 1$,

$$\begin{aligned} (-1)^{|p_-|} \#Z^{\kappa^\iota}(p_-, p_+; A) &= (-1)^{|p_-|} \sum_{\substack{\gamma \in \mathcal{P}(H) \\ A_- + A_+ = A}} \#Z^{\kappa^+}(p_-, \gamma; A_+) \#Z^{\kappa^-}(\gamma, p_+; A_-) \\ &+ \sum_{q \in \text{Crit}(f)} \#Z^{\kappa}(p_-, q; A) \# \mathcal{M}(q, p_+) + \sum_{q \in \text{Crit}(f)} \# \mathcal{M}(p_-, q) \# Z^{\kappa}(q, p_+; A). \end{aligned} \tag{2.40}$$

This identity will follow from Corollary 2.4.6 (v) applied to the weighted branched 1-dimensional orbifold $Z^{\kappa}(\alpha)$ that arises from an admissible sc^+ -multisection $\kappa_\alpha : \mathcal{W}_\alpha \rightarrow \mathbb{Q}^+$. The boundary $\partial Z^{\kappa}(\alpha)$ is given by the intersection with the top boundary stratum $\partial_1 \mathcal{B}(\alpha) \cap \mathcal{V}_\alpha = |\partial_1 \mathcal{X}_\alpha|$, and will be determined here, with orientations computed in (2.41)

below.

$$\begin{aligned}
 \partial Z^\kappa(\alpha) &= Z^\kappa(\alpha) \cap |\partial_1 \mathcal{X}_\alpha| \\
 &= Z^\kappa(\alpha) \cap |\partial_0 \mathcal{X}_{p_-, p_+; A}^\iota| \sqcup \bigcup_{\substack{\gamma \in \mathcal{P}(H) \\ A_- + A_+ = A}} Z^\kappa(\alpha) \cap |\partial_0 \mathcal{X}_{p_-, \gamma; A_+}^+ \times \partial_0 \mathcal{X}_{\gamma, p_+; A_-}^-| \\
 &\sqcup \bigcup_{q \in \text{Crit}(f)} Z^\kappa(\alpha) \cap (\mathcal{M}(p_-, q) \times |\partial_0 \mathcal{X}_{q, p_+; A}|) \\
 &\sqcup \bigcup_{q \in \text{Crit}(f)} Z^\kappa(\alpha) \cap (|\partial_0 \mathcal{X}_{p_-, q; A}| \times \mathcal{M}(q, p_+)) \\
 &= Z^{\kappa^\iota}(p_-, p_+; A) \sqcup \bigcup_{\gamma \in \mathcal{P}(H), A = A_- + A_+} Z^{\kappa^+}(p_-, \gamma; A_+) \times Z^{\kappa^-}(\gamma, p_+; A_-) \\
 &\sqcup \bigcup_{q \in \text{Crit}(f)} \mathcal{M}(p_-, q) \times Z^\kappa(q, p_+; A) \sqcup \bigcup_{q \in \text{Crit}(f)} Z^\kappa(p_-, q; A) \times \mathcal{M}(q, p_+).
 \end{aligned}$$

Here the second identity uses coherence of the ep-groupoid as in (2.32). The third identity follows from the coherence of sections S_α° and sc^+ multisections κ_α° as stated in (ii), (iii), together with the fact from Corollary 2.4.6 (iv) that perturbed zero sets $Z^{\kappa^\circ}(\alpha) \subset |\partial_0 \mathcal{X}_\alpha^\circ|$ are contained in the interior of the polyfolds when the Fredholm index is 0. For the second summand we moreover use Lemma 2.7.7 which ensures that each restriction $\kappa_\alpha|_{P_\alpha^{-1}(\mathcal{F})}$ to a face $\mathcal{F} = \partial_0 \mathcal{X}_{p_-, \gamma; A_+}^+ \times \partial_0 \mathcal{X}_{\gamma, p_+; A_-}^- \subset \partial_1 \mathcal{X}_{p_-, p_+; A}$, given by $\kappa_{p_-, \gamma; A_+}^+ \cdot \kappa_{\gamma, p_+; A_-}^-$, is in general position to the section $S_{p_-, \gamma; A_+}^+ \times S_{\gamma, p_+; A_-}^-$. Then its perturbed zero set $Z^{\kappa^\iota}(p_-, p_+; A) \cap |\mathcal{F}|$ is contained in the interior $|\partial_0 \mathcal{X}_{p_-, \gamma; A_+}^+ \times \mathcal{X}_{\gamma, p_+; A_-}^-| = |\partial_0 \mathcal{X}_{p_-, \gamma; A_+}^+ \times \partial_0 \mathcal{X}_{\gamma, p_+; A_-}^-|$ as the complement of the pairs of points (x^+, x^-) with

$$\begin{aligned}
 0 &= \kappa_{p_-, p_+; A}(S_{p_-, p_+; A}(x^+, x^-)) = (\kappa_{p_-, \gamma; A_+}^+ \cdot \kappa_{\gamma, p_+; A_-}^-)((S_{p_-, \gamma; A_+}^+ \times S_{\gamma, p_+; A_-}^-)(x^+, x^-)) \\
 &= \kappa_{p_-, \gamma; A_+}^+(S_{p_-, \gamma; A_+}^+(x^+)) \cdot \kappa_{\gamma, p_+; A_-}^-(S_{\gamma, p_+; A_-}^-(x^-)).
 \end{aligned}$$

Since a product in $\mathbb{Q}^+ = \mathbb{Q} \cap [0, \infty)$ is nonzero exactly when both factors are nonzero, this identifies the objects of the perturbed zero set of $\kappa_{p_-, p_+; A}$ with the product of perturbed zero objects for $\kappa^\pm$, that is, $\{(x^+, x^-) \in \mathcal{F} \mid \kappa_{p_-, p_+; A}(S_{p_-, p_+; A}(x^+, x^-)) > 0\} = \{x^+ \in \mathcal{X}_{p_-, \gamma; A_+}^+ \mid \kappa_{p_-, \gamma; A_+}^+(S_{p_-, \gamma; A_+}^+(x^+)) > 0\} \times \{x^- \in \mathcal{X}_{\gamma, p_+; A_-}^- \mid \kappa_{\gamma, p_+; A_-}^-(S_{\gamma, p_+; A_-}^-(x^-)) > 0\}$. And the realization of this set is precisely $Z^{\kappa^+}(p_-, \gamma; A_+) \times Z^{\kappa^-}(\gamma, p_+; A_-)$, as claimed above.

Computation of orientations: To prove the identity (2.40) it remains to compute the effect of the orientations in Remark 2.5.8 on the algebraic identity in Corollary 2.4.6 (v) that arises from the boundary $\partial Z^\kappa(\alpha)$ of the 1-dimensional weighted branched orbifolds arising from regularization of the moduli spaces with index $I(\alpha) = I(p_-, p_+; A) = 1$. Here $Z^{\lambda_A^{\text{SFT}}}$ is of odd dimension with oriented boundary determined by the orientation relations in Assumption 2.6.3 (iii)(b) and (c) as

$$\partial_1 Z^{\lambda_A^{\text{SFT}}} = Z^{\lambda_A^{\text{SFT}}} \cap \partial_1 \mathcal{B}_{\text{SFT}}(A) = (-1)Z^{\lambda_A^{\text{GW}}} \sqcup \bigsqcup_{\substack{\gamma \in \mathcal{P}(H) \\ A_- + A_+ = A}} Z^{\lambda_{\gamma, A_+}^+} \times Z^{\lambda_{\gamma, A_-}^-}.$$

Moreover, the index of σ_{SFT} is $I(\alpha) = |p_-| - |p_+| + 2c_1(A) + 1 = 1$, so we compute orientations in close analogy to (2.31) – while also giving an alternative identification of the boundary components –

$$\begin{aligned} \partial Z^\kappa(\alpha) &= \partial_1 \overline{\mathcal{M}}(p_-, M)_{\text{ev} \times \text{ev}} Z_A^{\lambda \text{SFT}}_{\text{ev} \times \text{ev}} \partial_0 \overline{\mathcal{M}}(M, p_+) \\ &\sqcup (-1)^{\dim \overline{\mathcal{M}}(p_-, M)} \partial_0 \overline{\mathcal{M}}(p_-, M)_{\text{ev} \times \text{ev}} \partial_1 Z_A^{\lambda \text{SFT}}_{\text{ev} \times \text{ev}} \partial_0 \overline{\mathcal{M}}(M, p_+) \\ &\sqcup (-1)^{\dim \overline{\mathcal{M}}(p_-, M) + 1} \partial_0 \overline{\mathcal{M}}(p_-, M)_{\text{ev} \times \text{ev}} \partial_0 Z_A^{\lambda \text{SFT}}_{\text{ev} \times \text{ev}} \partial_1 \overline{\mathcal{M}}(M, p_+) \\ &= \left(\bigsqcup_{q \in \text{Crit} f} \mathcal{M}(p_-, q) \times \mathcal{M}(q, M) \right)_{\text{ev} \times \text{ev}} Z_A^{\lambda \text{SFT}}_{\text{ev} \times \text{ev}} \mathcal{M}(M, p_+) \end{aligned} \quad (2.41)$$

$$\begin{aligned} &\sqcup (-1)^{|p_-| + |p_+|} \mathcal{M}(p_-, M)_{\text{ev} \times \text{ev}} Z_A^{\lambda \text{SFT}}_{\text{ev} \times \text{ev}} \left(\bigsqcup_{q \in \text{Crit} f} \mathcal{M}(M, q) \times \mathcal{M}(q, p_+) \right) \\ &\sqcup (-1)^{|p_-|} \mathcal{M}(p_-, M)_{\text{ev} \times \text{ev}} \end{aligned} \quad (2.42)$$

$$\begin{aligned} &\left(\bigsqcup_{\gamma \in \mathcal{P}(H), A = A_- + A_+} Z^{\lambda^+_{\gamma, A_+}} \times Z^{\lambda^-_{\gamma, A_-}} \right)_{\text{ev} \times \text{ev}} \mathcal{M}(M, p_+) \\ &\sqcup (-1)^{|p_-| + 1} \mathcal{M}(p_-, M)_{\text{ev} \times \text{ev}} Z_A^{\lambda \text{GW}}_{\text{ev} \times \text{ev}} \mathcal{M}(M, p_+) \\ &= \bigsqcup_{q \in \text{Crit} f} \mathcal{M}(p_-, q) \times Z^\kappa(q, p_+; A) \sqcup \bigsqcup_{q \in \text{Crit} f} Z^\kappa(p_-, q; A) \times \mathcal{M}(q, p_+) \\ &\sqcup (-1)^{|p_-|} \bigsqcup_{\gamma \in \mathcal{P}(H), A = A_- + A_+} Z^{\kappa^+}(p_-, \gamma; A_+) \times Z^{\kappa^-}(\gamma, p_+; A_-) \\ &\sqcup (-1)^{|p_-| + 1} Z^{\kappa'}(p_-, p_+; A). \end{aligned}$$

This computation should be understood in a neighborhood of a solution, so in particular with scale-smooth evaluation maps to $\mathbb{C}^\pm \times M$. Based on this, Corollary 2.4.6 (v) implies – as claimed –

$$\begin{aligned} 0 &= h_\kappa(d\langle p_- \rangle) + d(h_\kappa\langle p_- \rangle) + (-1)^{|p_-|} SSP_{\kappa^-}(PSS_{\kappa^+}\langle p_- \rangle) - (-1)^{|p_-|} \iota_{\kappa'}\langle p_- \rangle \\ &= (h \circ d + d \circ h + SSP \circ PSS - \iota)\langle p_- \rangle. \end{aligned}$$

□

2.7 Appendix: Summary of Polyfold Theory

This section gives an overview of the main notions of polyfold theory that are used in this paper. The following language is used to describe settings with trivial isotropy.¹³

Remark 2.7.1. (i) An **M-polyfold without boundary** is analogous to the notion of a Banach manifold: While the latter are locally homeomorphic to open subsets of a Banach space, an M-polyfold is locally homeomorphic to the image $\mathcal{O} = \text{im } \rho$ of a retract $\rho : U \rightarrow U$ of an open subset $U \subset E$ of a Banach space E . While ρ is generally not classically differentiable, it is required to be scale-smooth (sc^∞) with respect to a scale structure on E , which is indicated by $\mathbb{E}$.

¹³Trivial isotropy would be guaranteed in our settings by an almost complex structure J for which there are no nonconstant J -holomorphic spheres.

(i') An **M-polyfold**, as defined in [34, Def.2.8], is a paracompact Hausdorff space X together with an atlas of charts $\phi_\iota : U_\iota \rightarrow \mathcal{O}_\iota \subset [0, \infty)^{s_\iota} \times \mathbb{E}^{\iota}$ (i.e. homeomorphisms between open sets $U_\iota \subset X$ and sc-retracts $\mathcal{O}_\iota$ such that $\cup_\iota U_\iota = X$), whose transition maps are sc-smooth.

For $k \in \mathbb{N}_0$ the k -th boundary stratum $\partial_k X$ is the set of all $x \in X$ of degeneracy index $d(x) = k$ given¹⁴ by the number of components equal to 0 for the point in a chart $\phi_\iota(x) \in [0, \infty)^{s_\iota} \times \mathbb{E}^{\iota}$. In particular, $\partial_0 X$ is the interior of X .

(ii) A **strong bundle** over an M-polyfold X , as defined in [34, Def.2.26], is a sc-smooth surjection $P : W \rightarrow X$ with linear structures on each fiber $W_x = P^{-1}(x)$ for $x \in X$, and an equivalence class of compatible strong bundle charts, which in particular encode a sc-smooth subbundle $W \supset W^1 \rightarrow X$ whose fiber inclusions $W_x^1 \hookrightarrow W_x$ are compact and dense.

(iii) The notion of **sc-Fredholm** for a scale smooth section $S : X \rightarrow W$ of a strong bundle in [34, Def.3.8] encodes elliptic regularity and a nonlinear contraction property [34, Def.3.6,3.7]. The latter is a stronger condition than the classical notion of linearizations being Fredholm operators, and is crucial to ensure an implicit function theorem; see [17].

A more detailed survey of these trivial isotropy notions can be found in [14]. Then the generalization to nontrivial isotropy is directly analogous to the notion of smooth sections of orbi-bundles, in which orbifolds are realizations of étale proper groupoids [42].

Remark 2.7.2. A sc-Fredholm section $\sigma : \mathcal{B} \rightarrow \mathcal{E}$ of a strong polyfold bundle as introduced in [34, Def.16.16,16.40] is a map between topological spaces together with an equivalence class of sc-Fredholm section functors $s : \mathcal{X} \rightarrow \mathcal{W}$ of strong bundles $\mathcal{W}$ over ep-groupoids $\mathcal{X}$, whose realization $|s| : |\mathcal{X}| \rightarrow |\mathcal{W}|$ together with homeomorphisms $|\mathcal{X}| := \text{Obj}_{\mathcal{X}}/\text{Mor}_{\mathcal{X}} \cong \mathcal{B}$ and $|\mathcal{W}| \cong \mathcal{E}$ induces σ . To summarize these notions we use conventions of [34] in denoting object and morphism spaces as $\text{Obj}_{\mathcal{X}} = X$ and $\text{Mor}_{\mathcal{X}} = \mathbf{X}$. These will be equipped with M-polyfold structures, so that the k -th boundary stratum of a polyfold $\mathcal{B} \cong |\mathcal{X}|$ is given as $\partial_k \mathcal{B} \cong \partial_k X/\mathbf{X} \subset |\mathcal{X}|$ for all $k \in \mathbb{N}_0$.

(i) An **ep-groupoid** as in [34, Def.7.3] is a groupoid $\mathcal{X} = (X, \mathbf{X})$ equipped with M-polyfold structures on the object and morphism sets such that all structure maps are local sc-diffeomorphisms and every $x \in X$ has a neighborhood $V(x)$ such that the mapping $t : s^{-1}(\text{cl}_X(V(x))) \rightarrow X$ is proper. As in [34, §7.4] we require that the realization $|\mathcal{X}|$ is paracompact and thus metrizable.

(ii) A **strong bundle** as in [34, Def.8.4] over the ep-groupoid $\mathcal{X}$ is a pair (P, μ) of a strong bundle $P : W \rightarrow X$ and a strong bundle map $\mu : \mathbf{X}_s \times_P W \rightarrow W$ so that P lifts to a functor $\mathcal{P} : \mathcal{W} \rightarrow \mathcal{X}$ from an ep-groupoid $\mathcal{W} = (W, \mathbf{W})$ induced by (P, μ) . Then $\mathcal{P}$ restricts to a

¹⁴ The degeneracy index $d(x) \in \mathbb{N}_0$ in [34, Def.2.13,Thm.2.3] is a priori independent of the choice of chart ϕ_ι only for points in a dense subset $X_\infty \subset X$ specified in Remark 2.7.3. With that $d(x) := \max\{\limsup d(x_i) \mid X_\infty \ni x_i \rightarrow x\}$ is well defined for all $x \in X$ and can also be computed in any fixed chart.

functor $\mathcal{W}^1 \rightarrow \mathcal{X}$ on the full subcategory whose object space is the sc-smooth subbundle $W^1 \subset W$.

- (iii) A **sc-Fredholm section functor** of the strong bundle $\mathcal{P} : \mathcal{W} \rightarrow \mathcal{X}$ as in [34, Def.8.7] is a functor $S : \mathcal{X} \rightarrow \mathcal{W}$ that is sc-smooth on object and morphism spaces, satisfies $\mathcal{P} \circ S = \text{id}_{\mathcal{X}}$, and such that $S : X \rightarrow W$ is sc-Fredholm on the M-polyfold X .

Now a polyfold description of a compact moduli space $\overline{\mathcal{M}}$ is a sc-Fredholm section $\sigma : \mathcal{B} \rightarrow \mathcal{E}$ of a strong polyfold bundle with zero set $\sigma^{-1}(0) \cong \overline{\mathcal{M}}$. The polyfold descriptions used in this paper are obtained as fiber products of existing polyfolds and sc-Fredholm sections over them. This requires a technical shift in levels described in the following remark, and a notion of submersion below.

Remark 2.7.3. Polyfolds carry a level structure $\mathcal{B}_{\infty} \subset \dots \subset \mathcal{B}_1 \subset \mathcal{B}_0 = \mathcal{B}$ as follows: For any M-polyfold X , in particular the object space of the ep-groupoid representing $\mathcal{B} = |\mathcal{X}|$, a sequence of dense subsets $X_{\infty} \subset \dots \subset X_1 \subset X_0 = X$ is induced by the scale structures $\mathbb{E}^{\iota} = (E_m^{\iota})_{m \in \mathbb{N}_0}$ of the charts, that is $X_m = \bigcup_{\iota} \phi_{\iota}^{-1}(\mathcal{O}_{\iota} \cap \mathbb{R}^{s_{\iota}} \times E_m^{\iota})$. Then $\mathcal{B}_m := X_m / \text{Mor}_{\mathcal{X}}$ is well defined since morphisms of $\mathcal{X}$ – locally represented by scale-diffeomorphisms – preserve the levels on $\text{Obj}_{\mathcal{X}} = X$.

The restriction $\sigma|_{\mathcal{B}_m}$ of a sc-Fredholm section $\sigma : \mathcal{B} \rightarrow \mathcal{E}$ is again sc-Fredholm with values in $\mathcal{E}_m$, and the choice of such a shift in levels is irrelevant for applications since the zero set $\sigma^{-1}(0) \subset \mathcal{B}_{\infty}$ – as well as the perturbed zero set for any admissible perturbation – is always contained in the so-called “smooth part” that is densely contained in each level $\mathcal{B}_{\infty} \subset \mathcal{B}_m$.

For a finite dimensional manifold or orbifold M – such as the Morse trajectory spaces in §2.3.3 – viewed as polyfold, the level structure is trivial $M_{\infty} = \dots = M_1 = M_0 = M$.

Definition 2.7.4. [15, Def.5.8] A sc^{∞} functor $f : \mathcal{X} \rightarrow M$ from an ep-groupoid $\mathcal{X} = (X, \mathbf{X})$ to a finite dimensional manifold M is a **submersion** if for all $x \in X_{\infty}$ the tangent map $D_x f : T_x^R X \rightarrow T_{f(x)} M$ is surjective, where $T_x^R X$ is the reduced tangent space [34, Def.2.15].

Consider in addition a sc-Fredholm section functor $S : \mathcal{X} \rightarrow \mathcal{W}$. Then the sc^{∞} functor $f : \mathcal{X} \rightarrow M$ is **S -compatibly submersive** if for all $x \in X_{\infty}$ there exists a sc-complement $L \subset T_x^R X$ of $\ker(D_x f) \cap T_x^R X$ and a tame sc-Fredholm chart for S at x [15, Def.5.4] in which the change of coordinates $\psi : \mathcal{O} \rightarrow [0, \infty)^s \times \mathbb{R}^{k-s} \times \mathbb{W}$ that puts S in basic germ form – which by tameness has the form $\psi(v, e) = (v, \overline{\psi}(e))$ for $(v, e) \in \mathcal{O} \subset [0, \infty)^s \times \mathbb{E}$ and a linear sc-isomorphism $\overline{\psi}$ – moreover satisfies $\overline{\psi}(L) \subset \{0\}^{k-s} \times \mathbb{W}$, where the chart identifies $L \subset T_x^R X \cong T_0^R \mathcal{O} = \{0\} \times \mathbb{E}$.

More generally, given a smooth submanifold $N \subset M$, the sc^{∞} functor f is **transverse** to N if for all $x \in f^{-1}(N) \cap X_{\infty}$ we have $D_x f(T_x^R X) + T_{f(x)} N = T_{f(x)} M$, and f is **S -compatibly transverse** to N if there exists a sc-complement L of $(D_x f)^{-1}(T_{f(x)}(N)) \cap T_x^R X$ satisfying the above condition.

The purpose of giving a moduli space a polyfold description is to utilize the perturbation theory for sc-Fredholm sections over polyfolds, which allows to “regularize” the moduli space by associating to it a well defined cobordism class of weighted branched orbifolds. (For a

technical statement see Corollary 2.4.6 and the references therein.) Since the ambient space $|\mathcal{X}|$ is almost never locally compact, this requires “admissible perturbations” of the section to preserve compactness of the zero set. This admissibility is determined by the following data introduced in [34, Def.12.2,15.4].

Definition 2.7.5. *A saturated open subset $\mathcal{U} \subset \mathcal{X}$ of an ep-groupoid $\mathcal{X} = (X, \mathbf{X})$ is an open subset $\mathcal{U} \subset X$ with $\pi^{-1}(\pi(\mathcal{U})) = \mathcal{U}$, where $\pi : X \rightarrow |\mathcal{X}| = X/\mathbf{X}$ is the projection to the realization.*

A pair controlling compactness for a sc-Fredholm section $S : \mathcal{X} \rightarrow \mathcal{W}$ of a strong bundle $P : \mathcal{W} \rightarrow \mathcal{X}$ consists of an auxiliary norm $N : \mathcal{W}[1] \rightarrow [0, \infty)$ (see [34, Def.12.2]) and a saturated open subset $\mathcal{U} \subset \mathcal{X}$ that contains the zero set $S^{-1}(0) \subset \mathcal{U}$, such that $|\{x \in \mathcal{U} \mid N(S(x)) \leq 1\}| \subset |\mathcal{X}|$ has compact closure.

Given such a pair, a section $s : \mathcal{X} \rightarrow \mathcal{W}$ is $(N, \mathcal{U})$ -admissible if $N(s(x)) \leq 1$ and $\text{supp } s \subset \mathcal{U}$.

The construction of perturbations moreover requires scale-smooth partitions of unity, which will be guaranteed by the following standing assumptions.

Remark 2.7.6. Throughout this paper we assume that the realizations $|\mathcal{X}|$ of ep-groupoids are paracompact, and the Banach spaces E in all M-polyfold charts are Hilbert spaces. This guarantees the existence of scale-smooth partitions of unity by [34, §5.5, §7.5.2]. In order to guarantee the same on every level $\mathcal{B}_m$ as discussed in Remark 2.7.3, we moreover assume that each scale structure $\mathbb{E} = (E_m)_{m \in \mathbb{N}_0}$ consists of Hilbert spaces E_m . These assumptions hold in applications, such as the ones cited [33, 19]. Then paracompactness and thus existence of scale-smooth partitions of unity on every level is guaranteed by [34, Prop.7.12].

When discussing coherence of perturbations of a system of sc-Fredholm sections, the boundaries are described in terms of Cartesian products of polyfolds, bundles, and sections. So we will make use of Cartesian products of multivalued perturbations as follows, to obtain multisections over the boundary as summarized in the subsequent remark.

Lemma 2.7.7. *Let $S_1 : \mathcal{X}_1 \rightarrow \mathcal{W}_1$ and $S_2 : \mathcal{X}_2 \rightarrow \mathcal{W}_2$ be sc-Fredholm section of strong bundles $P_i : \mathcal{W}_i \rightarrow \mathcal{X}_i$ over ep-groupoids. Then the Cartesian product $\mathcal{X}_1 \times \mathcal{X}_2$ is naturally an ep-groupoid and $(S_1 \times S_2) : \mathcal{X}_1 \times \mathcal{X}_2 \rightarrow \mathcal{W}_1 \times \mathcal{W}_2$ is a sc-Fredholm section of the strong bundle $P_1 \times P_2$.*

Moreover, if $\lambda_i : \mathcal{W}_i \rightarrow \mathbb{Q}^+$ are sc^+ -multisections for $i = 1, 2$, then there is a well defined sc^+ -multisection $\lambda_1 \cdot \lambda_2 : \mathcal{W}_1 \times \mathcal{W}_2 \rightarrow \mathbb{Q}^+$ given by $(\lambda_1 \cdot \lambda_2)(w_1, w_2) = \lambda_1(w_1) \cdot \lambda_2(w_2)$. If, for $i = 1, 2$, the sections λ_i are $(N_i, \mathcal{U}_i)$ -admissible for some fixed pair controlling compactness as in Definition 2.7.5, then $\lambda_1 \cdot \lambda_2$ is $(\max(N_1, N_2), \mathcal{U}_1 \times \mathcal{U}_2)$ -admissible. Finally, if λ_i is in general position to S_i for $i = 1, 2$ then $\lambda_1 \cdot \lambda_2$ is in general position to $S_1 \times S_2$.

Proof. A detailed treatment of sc-Fredholmness of the product section $S_1 \times S_2$ can be found in [15, Lemma 7.2]. The remaining statements follow easily from the definitions in [34] (as do the statements in the first paragraph).

Recall in particular from [34, Def.13.4] that a sc^+ -multisection on a strong bundle $P : \mathcal{W} \rightarrow \mathcal{X}$ is a functor $\lambda : \mathcal{W} \rightarrow \mathbb{Q}^+$ that is locally of the form $\lambda(w) = \sum_{\{j \mid w=p_j(P(w))\}} q_j$, represented by sc^+ -sections $p_1, \dots, p_k : \mathcal{V} \rightarrow P^{-1}(\mathcal{V})$ (i.e. sc^∞ sections of $\mathcal{W}^1$; see [34, Def.2.27]) and weights $q_1, \dots, q_k \in \mathbb{Q} \cap [0, \infty)$ with $\sum_j q_j = 1$. Then for local sections p_j^i and weights q_j^i representing λ_i for $i = 1, 2$, the multisection $\lambda_1 \cdot \lambda_2$ is locally represented by the sections $(p_j^1, p_{j'}^2)$ with weights $q_j^1 q_{j'}^2$, and all admissibility and general position arguments are made at the level of these local sections.

In particular, the $(N_i, \mathcal{U}_i)$ -admissibility can be phrased as the existence of local representations by sections with $N_i(p_j^i(x)) \leq 1$ and $Z(S_i, p_j^i) := \{x \in \mathcal{V}_i \mid \exists t \in [-1, 1] : S_i(x) = t p_j^i(x)\} \subset \mathcal{U}_i$. Then $(\max(N_1, N_2), \mathcal{U}_1 \times \mathcal{U}_2)$ -admissibility uses the observation $\{(x_1, x_2) \mid \exists t \in [-1, 1] : (S_1, S_2)(x_1, x_2) = t(p_j^1(x_1), p_{j'}^2(x_2))\} \subset Z(S_1, p_j^1) \times Z(S_2, p_{j'}^2) \subset \mathcal{U}_1 \times \mathcal{U}_2$. $\square$

Remark 2.7.8. Let $P : \mathcal{W} \rightarrow \mathcal{X}$ be a strong bundle over a tame ep-groupoid $\mathcal{X} = (X, \mathbf{X})$. Then for every $x \in X_\infty$ there is a chart $\phi : U_x \rightarrow \mathcal{O}$ from a locally uniformizing¹⁵ neighborhood $U_x \subset X$ of x to a sc -retract $\mathcal{O} \subset [0, \infty)^n \times \mathbb{E}$, with $\phi(x) = 0$ lying in the intersection of the n local faces $\mathcal{F}_k := \phi^{-1}(\{(\underline{v}, e) \in [0, \infty)^n \times \mathbb{E} \mid v_k = 0\})$ which cover the boundary $\partial X \cap U_x = \bigcup_{k=1}^n \mathcal{F}_k$.

Now a **sc^+ -multisection over the boundary** is a functor $\lambda^\partial : P^{-1}(\partial \mathcal{X}) \rightarrow \mathbb{Q}^+$ whose restriction $\lambda^\partial|_{P^{-1}(\mathcal{F}_k)}$ to each local face is a sc^+ -multisection of the strong bundle $P^{-1}(\mathcal{F}_k) \rightarrow \mathcal{F}_k$. In the presence of a sc -Fredholm section $S : \mathcal{X} \rightarrow \mathcal{W}$, such a sc^+ -multisection is **in general position over the boundary** if for each intersection of faces $\mathcal{F}_K := \bigcap_{k \in K} \mathcal{F}_k \subset \partial X$ the restriction of the perturbed multi-section $\lambda^\partial \circ S|_{\mathcal{F}_K} : P^{-1}(\mathcal{F}_K) \rightarrow \mathbb{Q}^+$ has surjective linearizations at all solutions. If, moreover, $(N, \mathcal{U})$ is a pair controlling compactness, then λ^∂ is $(N, \mathcal{U})$ -admissible if each restriction $\lambda^\partial|_{P^{-1}(\mathcal{F}_k)}$ is admissible w.r.t. the pair $(N|_{P^{-1}(\mathcal{F}_k)}, \mathcal{U} \cap \mathcal{F}_k)$.

In our applications, as described in Assumption 2.6.3, the local faces $\mathcal{F}_k$ are images of open subsets of global face immersions $l_{\mathcal{F}} : \mathcal{F} \rightarrow \partial \mathcal{X}$, where each $\mathcal{F}$ is a Cartesian product of two polyfolds, and the restriction to the interior $l_{\mathcal{F}}|_{\partial_0 \mathcal{F}}$ is an embedding into the top boundary stratum $\partial_1 \mathcal{X}$. The bundles over each face are naturally identified with the pullbacks $l_{\mathcal{F}}^* \mathcal{W}$, and then the pushforwards of sc^+ -multisections $\lambda_{\mathcal{F}} : l_{\mathcal{F}}^* \mathcal{W} \rightarrow \mathbb{Q}^+$ form a sc^+ -multisection over the boundary $\lambda^\partial : P^{-1}(\bigcup \text{im } \lambda_{\mathcal{F}}) \rightarrow \mathbb{Q}^+$ if they agree on overlaps and self-intersections of the immersions $l_{\mathcal{F}}$, at the boundary $\partial \mathcal{F}$ of the faces. In this setting, general position of λ^∂ is equivalent to general position of the multisections $\lambda_{\mathcal{F}}$.

The following perturbation theorem allows us to refine the construction of coherent perturbations in [19] for the SFT moduli spaces such that moreover the evaluation maps from the perturbed solution sets are transverse to the unstable and stable manifolds in the symplectic manifold. This is a generalization of the polyfold perturbation theorem over ep-groupoids and the extension of transverse perturbations from the boundary [34, Theorems 15.4, 15.5]

¹⁵ A neighborhood $U_x \subset X$ forms a local uniformizer as in [34, Def.7.9] if the morphisms between points in U_x are given by a local action of the isotropy group G_x .

(with norm bound given by $h \equiv 1$ for simplicity). Another version of this – with the submanifolds representing cycles whose Gromov-Witten invariants are then obtained as counts – also appears in [46][47]. We are working under the assumptions made in this section – e.g. paracompactness – without further mention. The limitation to finitely many submanifolds in the extension result seems to be of technical nature and possibly a removable hypothesis.

Theorem 2.7.9. *Suppose $S : \mathcal{X} \rightarrow \mathcal{W}$ is a sc -Fredholm section functor of a strong bundle $P : \mathcal{W} \rightarrow \mathcal{X}$ over a tame ep -groupoid $\mathcal{X}$ with compact solution set $|S^{-1}(0)| \subset |\mathcal{X}|$, and let $(N, \mathcal{U})$ be a pair controlling compactness. Moreover, let $e : \mathcal{X} \rightarrow M$ be a sc^0 -map to a finite dimensional manifold M which has a sc^∞ submersive restriction $e|_{\mathcal{V}} : \mathcal{V} \rightarrow M$ on a saturated open set $\mathcal{V} \subset \mathcal{X}$.*

Then, for any countable collection of smooth submanifolds $(C_i \subset M)_{i \in I}$ with the property $e^{-1}(\bigcup_{i \in I} C_i) \subset \mathcal{V}$, there exists an $(N, \mathcal{U})$ -admissible sc^+ -multisection $\lambda : \mathcal{W} \rightarrow \mathbb{Q}^+$ so that (S, λ) is in general position (see [34, Definition 15.6]) and the restriction $e|_{Z^\lambda} : Z^\lambda \rightarrow M$ to the perturbed zero set $Z^\lambda = |\{x \in X \mid \lambda(S(x)) > 0\}|$ is in general position¹⁶ to the submanifolds C_i for all $i \in I$.

Moreover, suppose I is finite and $\lambda^\partial : P^{-1}(\partial\mathcal{X}) \rightarrow \mathbb{Q}^+$ for some $0 < \alpha < 1$ is an $(\frac{1}{\alpha}N, \mathcal{U})$ -admissible structurable sc^+ -multisection in general position over the boundary such that the restriction $e|_{Z^\partial} : Z^\partial \rightarrow M$ to the perturbed zero set in the boundary $Z^\partial := |\{x \in \partial\mathcal{X} \mid \lambda^\partial(S(x)) > 0\}|$ is in general position¹⁷ to the submanifolds C_i for all $i \in I$. Then λ above can be chosen with $\lambda|_{P^{-1}(\partial\mathcal{X})} = \lambda^\partial$.

Proof. Our proof follows the perturbation procedure of [34, Theorem 15.4], which proves the special case when there is no condition on a map $e : \mathcal{X} \rightarrow M$, i.e. when $M = \{pt\}$ and $C_i = \{pt\}$. To obtain the desired transversality of e to the submanifolds $C_i \subset M$ we will go through the proof and indicate adjustments in three steps: A local stabilization construction, which adds a finite dimensional parameter space to cover the cokernels near a point $x \in S^{-1}(0)$; a local-to-global argument which combines the local constructions into a global stabilization which covers the cokernels near $S^{-1}(0)$; and a global Sard argument which shows that regular values yield transverse perturbations. Within these arguments we need to consider restrictions to any intersection of faces to ensure general position to the boundary, use submersivity of e to achieve transversality to the C_i , and work with multisections due to isotropy. The statement with prescribed boundary values λ^∂ generalizes the extension result [34, Theorem 15.5], which hinges on the fact that general position over the boundary persists in an open neighborhood – something that is generally guaranteed only for finitely many transversality conditions; see the end of this proof. The first step in any construction of perturbations is the existence of local stabilizations which cover the cokernels, as follows.

¹⁶ General position to C_i requires transversality to C_i of each restriction $e|_{Z^\lambda \cap \mathcal{F}_K}$ to the perturbed solution set within an intersection of local faces $\mathcal{F}_K = \bigcap_{k \in K} \mathcal{F}_k$ as defined in Remark 2.7.8, including for $\mathcal{F}_\emptyset := Z^\lambda$.

¹⁷ This requires general position of each restriction $e|_{Z^\lambda \cap \mathcal{F}_k}$ to a local face $\mathcal{F}_k \subset \partial\mathcal{X}$ as defined in Remark 2.7.8.

Local stabilization constructions: For every zero $x \in S^{-1}(0)$ of the unperturbed sc-Fredholm section we construct a finite dimensional parameter space $\mathbb{R}^l$ for $l = l_x \in \mathbb{N}_0$ and sc^+ -multisection

$$\tilde{A}^x : \mathbb{R}^l \times \mathcal{W} \rightarrow \mathbb{Q}^+, \quad (t, w) \mapsto A_t^x(w)$$

such that A_0^x is the trivial multisection, i.e. $A_0^x(0) = 1$, $A_0^x(w) = 0$ for $w \in \mathcal{W}_x \setminus \{0\}$. This multisection $\tilde{A}^x$ is viewed as local perturbation near $(0, x)$ of a sc-Fredholm section functor $\tilde{S}^x$ of a bundle $\tilde{P}^x$,

$$\begin{aligned} \tilde{S}^x : \mathbb{R}^l \times \mathcal{X} &\rightarrow \mathbb{R}^l \times \mathcal{W} & \tilde{P}^x : \mathbb{R}^l \times \mathcal{W} &\rightarrow \mathbb{R}^l \times \mathcal{X} \\ (t, y) &\mapsto (t, S(y)) & (t, w) &\mapsto (t, P(w)). \end{aligned}$$

It is constructed in [34] to be structurable in the sense of [34, Def.13.17], in general position in the sense that the linearization $T_{(\tilde{S}^x, \tilde{A}^x)}(0, x) : T_0\mathbb{R}^l \times T_x^R X \rightarrow W_x$ is surjective¹⁸ and admissible in the sense that the domain support of $\tilde{A}^x$ is contained in $\mathcal{U}$ and the auxiliary norm is bounded linearly, $N(\Lambda)(t, y) \leq c_x |t|$ for some constant c_x . In case $x \in \mathcal{V} \cap S^{-1}(0)$ we refine this construction to require surjectivity of the restrictions

$$T_{(\tilde{S}^x, \tilde{A}^x)}(0, x)|_{T_0\mathbb{R}^l \times K_x} : T_0\mathbb{R}^l \times K_x \rightarrow W_x, \quad (2.43)$$

where $K_x := \ker(D_x e|_{T_x^R X}) \subset T_x^R X$ is the kernel of the linearization $D_x e : T_x^R X \rightarrow T_{e(x)} M$ restricted to the reduced tangent space. For that purpose note that e is sc^∞ near x by assumption, so has a well defined linearization, and since its codomain is finite dimensional, its kernel has finite codimension. Moreover $\text{im } D_x S \subset W_x$ has finite codimension by the sc-Fredholm property of S , and the reduced tangent space $T_x^R X \subset T_x X$ has finite codimension by the definition of M-polyfolds with corners. Thus we can find finitely many vectors $w^1, \dots, w^l \in W_x$ which together with $D_x S(K_x)$ span W_x . These vectors are extended to sc^+ -sections of the form $p^j(t, y) = \sum t_j w^j(y)$, multiplied with sc^∞ cutoff functions of sufficiently small support, and pulled back by local isotropy actions to construct the functor $\tilde{A}^x$ as in [34, Thm.15.4]. We claim that this yields the following local properties with respect to the sc^∞ functor

$$\tilde{e}^x : \mathbb{R}^l \times \mathcal{V} \rightarrow M, \quad (t, y) \mapsto e(y).$$

Local stabilization properties: *There exists $\epsilon_x > 0$ and a locally uniformizing neighborhood $Q(x) \subset X$ of x whose closure is contained in $\mathcal{U}$, such that*

$$\Theta^x : \{t \in \mathbb{R}^l \mid |t| < \epsilon_x\} \times Q(x) \rightarrow \mathbb{Q}^+, \quad (t, y) \mapsto A_t^x(S(y)) = \tilde{A}^x(\tilde{S}^x(t, y)) \quad (2.44)$$

is a tame ep^+ -subgroupoid, and for $(t, y) \in \text{supp } \Theta^x = \{(t, y) \mid \Theta^x(t, y) > 0\} \subset \mathbb{R}^l \times X$ the reduced linearizations $T_{(\tilde{S}^x, \tilde{A}^x)}^R(t, y) := T_{(\tilde{S}^x, \tilde{A}^x)}(t, y)|_{T_t\mathbb{R}^l \times T_y^R X}$ are surjective. Moreover, if

¹⁸ This is shorthand for $\tilde{S}^x + p_j$ having surjective linearization for every section p_j in a local representation of $\tilde{A}^x$ with $\tilde{S}^x(0, x) = 0 = p_j(0, x)$, and restricted to the reduced tangent space $T_x^R X$.

$x \in \mathcal{V}$ then we may choose $Q(x) \subset \mathcal{V}$ such that for all $(t, y) \in \text{supp } \Theta^x$ we have surjections¹⁹

$$D_{(t,y)}\tilde{e}^x|_{N_{t,y}^x} : N_{t,y}^x := \ker T_{(\tilde{S}^x, \tilde{A}^x)}^R(t, y) \rightarrow T_{e(y)}M.$$

In particular, the realization $|\text{supp } \Theta^x|$ is a weighted branched orbifold and $\tilde{e}^x$ induces a submersion $|\text{supp } \Theta^x| \rightarrow M$ in the sense of Definition 2.7.4. Moreover, for all $y \in S^{-1}(0) \cap U_x$ we have $(0, y) \in \text{supp } \Theta^x$ so that the reduced linearizations $T_{(\tilde{S}^x, \tilde{A}^x)}^R(0, y)$ and the restriction to their kernel $D_{(0,y)}\tilde{e}^x|_{N_{0,y}^x}$ are surjective. These properties persist for $y \in S^{-1}(0)$ with $|y| \in |Q(x)|$.

The structure of $\text{supp } \Theta^x$ and surjectivity of linearizations $T_{(\tilde{S}^x, \tilde{A}^x)}^R$ follows from the local implicit function theorem [34, Theorems 15.2, 15.3]. Then the kernels $N_{t,y}^x = \ker T_{(\tilde{S}^x, \tilde{A}^x)}^R(t, y)$ represent the reduced tangent spaces at $|(t, y)|$ to the weighted branched orbifold $|\text{supp } \Theta^x|$. Surjectivity of $D_{(0,x)}\tilde{e}^x|_{N_{0,x}^x}$ holds since $D_{(0,x)}\tilde{e}^x$ is surjective by assumption, and the preimage of any given vector in $T_{e(x)}M$ can be adjusted by vectors in $\ker D_{(0,x)}\tilde{e}^x$ to lie in $N_{0,x}^x = \ker T_{(\tilde{S}^x, \tilde{A}^x)}^R(0, x)$, because $T_{(\tilde{S}^x, \tilde{A}^x)}^R(t, y)|_{\ker D_{(0,x)}\tilde{e}^x}$ is surjective by (2.43). Then $\tilde{e}^x$ restricts to a map $|\text{supp } \Theta^x| \rightarrow M$ that is classically smooth on each (finite dimensional) branch of $\text{supp } \Theta^x$, and thus surjectivity of $D_{(t,y)}\tilde{e}^x|_{N_{t,y}^x}$ is an open condition along each branch. Since $\text{supp } \Theta^x$ is locally compact – in particular with finitely many branches near x – we can then choose ϵ_x and $Q(x)$ sufficiently small to guarantee that each $D_{(t,y)}\tilde{e}^x|_{N_{t,y}^x}$ is surjective. This proves submersivity in the sense of Definition 2.7.4.

From local to global stabilization: In this portion of the proof, we proceed almost verbatim to the corresponding portion of [34, Thm.15.4], with extra considerations to deduce submersivity of (2.47). By assumption, $|S^{-1}(0)|$ is compact and $|e| : |\mathcal{X}| \rightarrow M$ is continuous. Then $|S^{-1}(0)| \cap |e^{-1}(C)|$ is compact since $C := \bigcup_{i \in I} (C_i) \subset M$ is closed. We moreover have the identity $|S^{-1}(0) \cap e^{-1}(C)| = |S^{-1}(0)| \cap |e^{-1}(C)|$ since both sets are saturated. Thus we have an open covering $(|Q(x)|)_{x \in S^{-1}(0) \cap e^{-1}(C)}$ by the open subsets chosen above, and can pick finitely many points $x_1, \dots, x_r \in S^{-1}(0) \cap e^{-1}(C)$ to obtain a finite open cover $|S^{-1}(0) \cap e^{-1}(C)| \subset \bigcup_{i=1}^r |Q(x_i)|$. Then $|S^{-1}(0)| \setminus \bigcup_{i=1}^r |Q(x_i)|$ is compact, with open cover by $(|Q(x)|)_{x \in S^{-1}(0)}$, so we may pick further $x_{r+1}, \dots, x_k \in S^{-1}(0)$ to obtain the covers

$$\begin{aligned} |S^{-1}(0)| &\subset \bigcup_{i=1}^k |Q(x_i)|, & |S^{-1}(0) \cap e^{-1}(C)| &\subset \bigcup_{i=1}^r |Q(x_i)|, \\ S^{-1}(0) &\subset \tilde{Q} := \pi^{-1}(\bigcup_{i=1}^k |Q(x_i)|) \subset \mathcal{U}. \end{aligned} \quad (2.45)$$

For each $x = x_i$ we constructed above a family of sc^+ -multisections $(\Lambda_t^{x_i} : \mathcal{W} \rightarrow \mathbb{Q}^+)_{t \in \mathbb{R}^{l_{x_i}}}$. These are summed up, using [34, Def.13.11], to a sc^+ -multisection

$$\tilde{\Lambda} : \mathbb{R}^{\tilde{l}} \times \mathcal{W} \rightarrow \mathbb{Q}^+, \quad (t = (t_1, \dots, t_k), w) \mapsto \Lambda_t(w) := (\Lambda_{t_1}^{x_1} \oplus \dots \oplus \Lambda_{t_k}^{x_k})(w)$$

¹⁹ As before, this is shorthand for surjectivity on each reduced tangent space $\ker D_{(t,y)}(\tilde{S}^x + p_j)|_{T_t \mathbb{R}^l \times T_y^R X}$.

for $\tilde{l} := l_{x_1} + \dots + l_{x_k}$. Here each $\Lambda_t : \mathcal{W} \rightarrow \mathbb{Q}^+$ for $t \in \mathbb{R}^{\tilde{l}}$ is a structurable sc^+ -multisection by [34, Prop.13.3]. We view the multisection $\tilde{\Lambda}$ as global perturbation of a sc -Fredholm section functor $\tilde{S}$ of a bundle $\tilde{P}$,

$$\begin{aligned} \tilde{S} : \mathbb{R}^{\tilde{l}} \times X &\rightarrow \mathbb{R}^{\tilde{l}} \times \mathcal{W} =: \tilde{\mathcal{W}} & \tilde{P} : \mathbb{R}^{\tilde{l}} \times \mathcal{W} &\rightarrow \mathbb{R}^{\tilde{l}} \times \mathcal{X} \\ (t, y) &\mapsto (t, S(y)) & (t, w) &\mapsto (t, P(w)), \end{aligned}$$

and claim that $e : \mathcal{X} \rightarrow M$ induces a submersion on its perturbed solution set in the following sense.

Global stabilization properties: *There exists $\epsilon_0 > 0$ such that for every $0 < \epsilon < \epsilon_0$*

$$\tilde{\Theta} : \{t \in \mathbb{R}^{\tilde{l}} \mid |t| < \epsilon\} \times \mathcal{X} \rightarrow \mathbb{Q}^+, \quad (t, y) \mapsto \Lambda_t(S(y)) = \tilde{\Lambda}(\tilde{S}(t, y)) \quad (2.46)$$

is a tame ep^+ -subgroupoid with surjective reduced linearizations $T_{(\tilde{S}, \tilde{\Lambda})}^R(t, y)$ for all $(t, y) \in \text{supp } \tilde{\Theta}$. In particular, the realization $|\text{supp } \tilde{\Theta}|$ is a weighted branched orbifold. Moreover, there is a neighborhood $\mathcal{V}' \subset \mathcal{X}$ of $S^{-1}(0) \cap e^{-1}(C)$ such that

$$\tilde{e}|_{\text{supp } \tilde{\Theta}} : \text{supp } \tilde{\Theta} \rightarrow M, \quad (t, y) \mapsto e(y) \quad (2.47)$$

satisfies $(\tilde{e}|_{\text{supp } \tilde{\Theta}})^{-1}(C) \subset \mathbb{R}^{\tilde{l}} \times \mathcal{V}'$, and its restriction to $\text{supp } \tilde{\Theta} \cap (\mathbb{R}^{\tilde{l}} \times \mathcal{V}')$ is classically smooth and submersive as in Definition 2.7.4.

Note that the auxiliary norm N on $\mathcal{W}$ pulls back to an auxiliary norm $\tilde{N}$ on $\tilde{\mathcal{W}}$, and compactness of $\tilde{S}$ is controlled in the sense that for any compact subset $K \subset \mathbb{R}^{\tilde{l}}$ we have compactness of

$$|\{(t, x) \in K \times \mathcal{U} \mid \tilde{N}(\tilde{S}(t, x)) \leq 1\}| = K \times |\{(x \in \mathcal{U} \mid N(S(x)) \leq 1\}| \subset \mathbb{R}^{\tilde{l}} \times |\mathcal{X}|. \quad (2.48)$$

Next, the restriction of $\tilde{\Lambda}$ to each $\mathbb{R}^{l_{x_i}} \times X \hookrightarrow \mathbb{R}^{\tilde{l}} \times X$ is the local perturbation $\tilde{\Lambda}^{x_i}$ of $\tilde{S}^{x_i}$, since we identify $\mathbb{R}^{l_{x_i}} \cong \{(t_1, \dots, t_k) \in \mathbb{R}^{\tilde{l}} \mid t_j = 0 \ \forall j \neq i\}$ and each $\Lambda_0^{x_j}$ is trivial. In particular, Λ_0 is the trivial multisection, with $N(\Lambda_0) = 0$. Moreover, we have an estimate $N(\Lambda_t) \leq c|t|$ that results from the linear estimates on each $\Lambda_t^{x_i}$. Now for $\epsilon_0 \leq \frac{1}{c}$ we can deduce compactness of the stabilized solution set as closed subset of (2.48),

$$\tilde{Z} := |\{(t, x) \in \mathbb{R}^{\tilde{l}} \times X \mid |t| \leq \epsilon_0, \Lambda_t(S(x)) > 0\}|. \quad (2.49)$$

The next step is to argue that (2.49) is smooth in a neighborhood of $\tilde{Z} \cap (\{0\} \times |\mathcal{X}|) = \{0\} \times |S^{-1}(0)|$. Recall here that $\tilde{Q} = \pi^{-1}(\bigcup_{i=1}^k |Q(x_i)|) \subset X$ is an open neighborhood of $S^{-1}(0)$. So for any $x \in \tilde{Q}$ we can use the local properties of some $\tilde{\Lambda}^{x_i}$ with $|x| \in |Q(x_i)|$ to deduce surjectivity of $T_{(\tilde{S}, \tilde{\Lambda})}^R(0, x)$. Then the local implicit function theorems [34, Thms 15.2, 15.3, Rmk.15.2] yield an open neighborhood $U(0, x) = \{|t| < \epsilon'_x\} \times U(x) \subset \mathbb{R}^{\tilde{l}} \times X$ of $(0, x)$ for some $0 < \epsilon'_x < \epsilon_0$, and hence a saturated neighborhood $\tilde{U}(0, x) := \{|t| < \epsilon'_x\} \times \pi^{-1}(|U(x)|) \subset \mathbb{R}^{\tilde{l}} \times \mathcal{X}$

such that $\tilde{\Theta}|_{\tilde{U}(0,x)} = \tilde{A} \circ \tilde{S}|_{\tilde{U}(0,x)}$ is a tame branched ep^+ -subgroupoid of $\tilde{U}(0,x)$. As a consequence, the orbit space of the support $|\text{supp } \tilde{\Theta}|_{\tilde{U}(0,x)}$ is a weighted branched orbifold with boundary and corners.

For $x \in S^{-1}(0) \setminus e^{-1}(C)$ we can moreover choose $U(x) \cap e^{-1}(C) = \emptyset$, since $|e^{-1}(C)| \subset |\mathcal{X}|$ is closed. For $x \in S^{-1}(0) \cap e^{-1}(C) \subset \mathcal{V}$ the covering (2.45) guarantees $|x| \in |Q(x_i)|$ for some $1 \leq i \leq r$ with $Q(x_i) \subset \mathcal{V}$ and we choose $U(x) \subset Q(x_i)$. This guarantees that the restriction of $\tilde{e} : \mathbb{R}^{\tilde{l}} \times X \rightarrow M, (t, y) \mapsto e(y)$ to $\tilde{U}(0, x)$ is sc^∞ , and surjectivity of $D_{(0,x)} \tilde{e}^{x_i}|_{\ker T_{(\tilde{S}^{x_i}, \tilde{A}^{x_i})}^R(0,x)}$ implies surjectivity of $D_{(0,x)} \tilde{e}|_{N_{0,x}} : N_{0,x} \rightarrow T_{e(x)}M$ on $N_{0,x} := \ker T_{(\tilde{S}, \tilde{A})}^R(0, x)$. Here $N_{0,x}$ represents the reduced tangent space at $|(0, x)|$ to the weighted branched orbifold $|\text{supp } \tilde{\Theta}|_{\tilde{U}(0,x)}$. Now $\tilde{e}|_{\text{supp } \tilde{\Theta} \cap \tilde{U}(0,x)} : \text{supp } \tilde{\Theta}|_{\tilde{U}(0,x)} \rightarrow M$ is classically smooth since it is a restriction of an sc^∞ map to finite dimensions, and we have shown it to be submersive at $(0, x)$. Hence, by openness of submersivity along each corner stratum, and local compactness of $\text{supp } \tilde{\Theta}|_{\tilde{U}(0,x)} \subset \tilde{Z}$ it follows that $\tilde{U}(0, x) \subset \mathbb{R}^{\tilde{l}} \times \mathcal{V}$ can be chosen sufficiently small to ensure that $\tilde{e}|_{\text{supp } \tilde{\Theta} \cap \tilde{U}(0,x)}$ is submersive as in Definition 2.7.4.

Now compactness of $|S^{-1}(0) \cap e^{-1}(C)|$ and $|S^{-1}(0)|$ again allows us to find finite covers

$$|S^{-1}(0)| \subset \bigcup_{i=1}^{k'} |U(x'_i)|, \quad |S^{-1}(0) \cap e^{-1}(C)| \subset \bigcup_{i=1}^{r'} |U(x'_i)|$$

with $x'_i \in S^{-1}(0) \cap e^{-1}(C)$ for $i = 1, \dots, r'$ and $U(x'_i) \cap e^{-1}(C) = \emptyset$ for $r' < i \leq k'$. Then we have $\epsilon := \min\{\epsilon'_{x'_1}, \dots, \epsilon'_{x'_{k'}}\} > 0$, an open cover $S^{-1}(0) \subset \mathcal{A} := \pi^{-1}(\bigcup_{i=1}^{k'} |U(x'_i)|)$, and the functor $\{t \in \mathbb{R}^{\tilde{l}} \mid |t| < \epsilon\} \times \mathcal{A} \rightarrow \mathbb{Q}^+, (t, y) \mapsto \Lambda_t(S(y))$ is a tame branched ep^+ -subgroupoid, since it is the restriction of $\tilde{\Theta} = \tilde{A} \circ \tilde{S}$ to an open subset of $\bigcup_{i=1}^{k'} \tilde{U}(0, x'_i)$. Moreover, we claim that for a possibly smaller $0 < \epsilon < \epsilon_0$ we have

$$(t, y) \in \{|t| < \epsilon\} \times X, \quad \tilde{\Theta}(t, y) > 0 \quad \implies \quad y \in \mathcal{A}. \quad (2.50)$$

By contradiction, consider a sequence $\mathbb{R}^{\tilde{l}} \ni t_n \rightarrow 0, y_n \in X$ with $\tilde{\Theta}(t_n, y_n) > 0$ but $y_n \in X \setminus \mathcal{A}$. Then compactness of (2.49) guarantees a convergent subsequence $|(t_n, y_n)| \rightarrow |(0, y_\infty)| \in \tilde{Z}$, and since $\tilde{Z} \cap \{0\} \times |\mathcal{X}| = \{0\} \times |\text{supp } \Lambda_0 \circ S| = \{0\} \times |S^{-1}(0)|$ this contradicts the fact that $|y_n| \in |\mathcal{X}| \setminus |\mathcal{A}|$, where $|\mathcal{A}| = \bigcup_{i=1}^{k'} |U(x'_i)| \subset |\mathcal{X}|$ is an open neighborhood of $|S^{-1}(0)|$. Thus we have shown (2.50) and can deduce that $\tilde{\Theta} = \tilde{A} \circ \tilde{S} : \{t \in \mathbb{R}^{\tilde{l}} \mid |t| < \epsilon\} \times \mathcal{X} \rightarrow \mathbb{Q}^+$ is a tame branched ep^+ -subgroupoid with $\text{supp } \tilde{\Theta} \subset \mathbb{R}^{\tilde{l}} \times \mathcal{A}$, and thus $|\text{supp } \tilde{\Theta}| \subset \mathbb{R}^{\tilde{l}} \times \bigcup_{i=1}^{k'} |U(x'_i)|$ is a weighted branched orbifold with boundary and corners, as claimed.

Moreover, from the properties of $\tilde{e}|_{\text{supp } \tilde{\Theta} \cap \tilde{U}(0,x'_i)}$ for $i = 1, \dots, r'$ we know that the restriction of $\tilde{e}$ to $\text{supp } \tilde{\Theta} \cap (\mathbb{R}^{\tilde{l}} \times \mathcal{V}')$ for $\mathcal{V}' := \pi^{-1}(\bigcup_{i=1}^{r'} U(x'_i)) \subset \mathcal{V}$ is classically smooth and submersive. Here we have $e^{-1}(C) \cap \mathcal{A} \subset \mathcal{V}'$ since $U(x_i)$ for $i > r'$ was chosen disjoint from $e^{-1}(C)$, and hence we have

$$(\tilde{e}|_{\text{supp } \tilde{\Theta}})^{-1}(C) = \text{supp } \tilde{\Theta} \cap (\mathbb{R}^{\tilde{l}} \times e^{-1}(C)) \subset \mathbb{R}^{\tilde{l}} \times (e^{-1}(C) \cap \mathcal{A}) \subset \mathbb{R}^{\tilde{l}} \times \mathcal{V}',$$

and thus $\tilde{e}|_{\text{supp } \tilde{\Theta}} : \text{supp } \tilde{\Theta} \rightarrow M$ is classically smooth and submersive (in the sense of Definition 2.7.4) in the open neighborhood $\text{supp } \tilde{\Theta} \cap (\mathbb{R}^{\tilde{I}} \times \mathcal{V}')$ of $(\tilde{e}|_{\text{supp } \tilde{\Theta}})^{-1}(C_i) \subset \text{supp } \tilde{\Theta}$ for all $i \in I$.

Global transversality from regular values: As we continue to follow the proof of [34, Thm.15.4], we replace each application of the Sard theorem by countably many Sard arguments to obtain general position to the countably many submanifolds $C_i \subset M$ for $i \in I$. For that purpose we will consider various restrictions of the projection

$$\text{supp } \tilde{\Theta} = \{(t, y) \in \mathbb{R}^{\tilde{I}} \times \mathcal{X} \mid |t| < \epsilon, \Lambda_t(S(y)) > 0\} \rightarrow \mathbb{R}^{\tilde{I}}, \quad (t, y) \mapsto t.$$

The global properties of $\tilde{\Theta}$ imply that every $(t_0, y_0) \in \text{supp } \tilde{\Theta}$ has a saturated open neighborhood $\tilde{U}(t_0, y_0) = \{t \in \mathbb{R}^{\tilde{I}} \mid |t - t_0| < \delta\} \times \pi^{-1}(|U(y_0)|) \subset \mathbb{R}^{\tilde{I}} \times \mathcal{X}$ satisfying the following:

- $U(y_0) \subset X$ admits the natural action of the isotropy group G_{y_0} ; see [34, Thm.7.1], satisfies the properness property [34, Def.7.17], and has $d_{\mathcal{X}}(y_0)$ local faces $\mathcal{F}_1^{y_0}, \dots, \mathcal{F}_{d_{\mathcal{X}}(y_0)}^{y_0}$ which contain y_0 ; see [34, Def.2.21, Prop.2.14].
- The branched ep^+ -subgroupoid $\text{supp } \tilde{\Theta} \cap \tilde{U}(t_0, y_0)$ has a local branching structure

$$\tilde{\Theta}(t, y) = \Lambda_t(S(y)) = \frac{1}{|J|} \cdot |\{j \in J \mid (t, y) \in M_j^{t_0, y_0}\}|,$$

given by finitely many properly embedded submanifolds $M_j^{t_0, y_0} \subset \tilde{U}(t_0, y_0)$ with boundary and corners, which intersect any intersection of local faces in a manifold with boundary and corners.

- On each branch $M_j^{t_0, y_0}$, the reduced linearizations $T_{(\tilde{S}, \tilde{\Lambda})}^R(t, y)$ are surjective for all $(t, y) \in M_j^{t_0, y_0}$, and the restriction of $\tilde{e}|_{\text{supp } \tilde{\Theta}}$ is a submersion $M_j^{t_0, y_0} \cap (\mathbb{R}^{\tilde{I}} \times \mathcal{V}') \rightarrow M$ in general position to the boundary in the sense of Definition 2.7.4. That is, $D_{(t, y)} \tilde{e}|_{N_{t, y}} : N_{t, y} \rightarrow T_{e(y)} M$ is surjective on $N_{t, y} := \ker T_{(\tilde{S}, \tilde{\Lambda})}^R(t, y)$ for all $(t, y) \in M_j^{t_0, y_0} \cap (\mathbb{R}^{\tilde{I}} \times \mathcal{V}')$.

There is a countable cover $\text{supp } \tilde{\Theta} \subset \bigcup_{\beta \in \mathbb{Z}} \tilde{U}(t_{\beta}, y_{\beta})$ indexed by $(t_{\beta}, y_{\beta})_{\beta \in \mathbb{Z}} \subset \text{supp } \tilde{\Theta}$, since $\mathbb{R}^{\tilde{I}} \times X$ – and hence its subspace $\text{supp } \tilde{\Theta}$ – is second-countable, and every open cover of a second-countable space has a countable subcover. Moreover, for any given $\beta \in \mathbb{Z}$ there are finitely many choices $\tilde{\mathcal{F}}_K := \{|t - t_0| < \delta\} \times \bigcap_{k \in K} \mathcal{F}_k^{y_{\beta}} \subset \tilde{U}(t_{\beta}, y_{\beta})$ of intersections of finitely many local faces $K \subset \{1, \dots, d_{\mathcal{X}}(y_{\beta})\}$, with $\tilde{\mathcal{F}}_{\emptyset} := \tilde{U}(t_{\beta}, y_{\beta})$. Finally, for each $\beta \in \mathbb{Z}$ and intersection of faces $\tilde{\mathcal{F}}_K$, there are finitely many smooth manifolds $\tilde{\mathcal{F}}_K \cap M_j^{t_{\beta}, y_{\beta}}$ indexed by $j \in J_{\beta}$. For each of these countably many choices, Sard's theorem asserts that $\tilde{\mathcal{F}}_K \cap M_j^{t_{\beta}, y_{\beta}} \rightarrow \mathbb{R}^{\tilde{I}}, (t, y) \mapsto t$ has an open and dense subset $R_{K, j}^{\beta} \subset \mathbb{R}^{\tilde{I}}$ of regular values. Then, since $\mathbb{R}^{\tilde{I}}$ is a Baire space, the set of common regular values $\mathcal{R}_0 := \bigcap_{\beta \in \mathbb{Z}} \bigcap_{K, j} R_{K, j}^{\beta} \subset \mathbb{R}^{\tilde{I}}$ is still dense. For any $t_0 \in \mathcal{R}_0$, the sc^+ -multisection $\Lambda_{t_0} : \mathcal{W} \rightarrow \mathbb{Q}^+$ is in general position by the usual linear algebra for each restriction of the linearized operators to intersections of faces: Consider $(t_0, x_0) \in \tilde{\mathcal{F}}_K \cap M_j^{t_{\beta}, y_{\beta}} \subset \text{supp } \tilde{\Theta}$ and a local section $S + p^j$ in the representation of

$\tilde{\Theta} = \tilde{A} \circ \tilde{S}$ with $M_j^{t_\beta, y_\beta} \subset (S + p^j)^{-1}(0)$. The surjective differential along this intersection of faces can be written as $D_{t_0, x_0}(S + p^j)|_{\tilde{\mathcal{F}}_K} = D \oplus L$, where L is a bounded operator (arising from differentiating p^j in the direction of $\mathbb{R}^{\tilde{l}}$) and D is the reduced linearization – on the intersection of faces $\mathcal{F}_K := \bigcap_{k \in K} \mathcal{F}_k^{y_\beta} \subset U(y_\beta) \subset X$ – of the section $S + p^j(t_0, \cdot)$ that is a part of the representation of $\Lambda_{t_0} \circ S$. Then regularity of t_0 implies surjectivity of the projection $\Pi : \ker(D \oplus L) \rightarrow \mathbb{R}^{\tilde{l}}$, which in turn is equivalent to surjectivity of D ; see e.g. [40, Lemma A.3.6].

Moreover, each Λ_t for $|t| < \epsilon$ is $(N, \mathcal{U})$ -admissible, thus any sufficiently small regular $t_0 \in \mathcal{R}_0$ yields an admissible sc^+ -multisection $\lambda := \Lambda_{t_0}$ in general position as in [34, Thm.15.4]. To prove our theorem, we have to moreover choose $t_0 \in \mathcal{R}_0$ so that the restriction $e|_{Z^\lambda} : Z^\lambda \rightarrow M$ to the solution set $Z^\lambda = |\text{supp } \lambda \circ S|$ is in general position to $C_i \subset M$ for all $i \in I$. For that purpose we consider the countably many projections

$$\tilde{e}^{-1}(C_i) \cap \tilde{\mathcal{F}}_K \cap M_j^{t_\beta, y_\beta} \rightarrow \mathbb{R}^{\tilde{l}}, \quad (t, x) \mapsto t \quad (2.51)$$

for any $i \in I$, index $\beta \in \mathbb{Z}$ of the countable cover, intersection of local faces $\tilde{\mathcal{F}}_K$, and smooth branch $M_j^{t_\beta, y_\beta} \subset \text{supp } \tilde{\Theta} \cap \tilde{U}(t_\beta, y_\beta)$. Here we have $\tilde{e}^{-1}(C_i) \cap M_j^{t_\beta, y_\beta} \subset (\tilde{e}|_{\text{supp } \tilde{\Theta}})^{-1}(C_i)$, so that the restriction $\tilde{e}|_{\tilde{\mathcal{F}}_K \cap M_j^{t_\beta, y_\beta}} : \tilde{\mathcal{F}}_K \cap M_j^{t_\beta, y_\beta} \rightarrow M$ is smooth and submersive in a neighborhood of $\tilde{e}^{-1}(C_i)$. In particular, it is transverse to C_i so that there is a natural smooth structure on $\tilde{e}^{-1}(C_i) \cap \tilde{\mathcal{F}}_K \cap M_j^{t_\beta, y_\beta}$. Thus we can apply the Sard theorem to each (2.51) to find open and dense subsets $T_{K,j}^{i,\beta} \subset \mathbb{R}^{\tilde{l}}$ of regular values, and a dense set of common regular values $\mathcal{T}_0 := \bigcap_{\beta \in \mathbb{Z}} \bigcap_{K,j} R_{K,j}^\beta \cap \bigcap_i T_{K,j}^{i,\beta} \subset \mathbb{R}^{\tilde{l}}$. Note that $\mathcal{T}_0 \subset \mathcal{R}_0$, so sufficiently small $t_0 \in \mathcal{T}_0$ yield admissible sc^+ -multisections $\lambda := \Lambda_{t_0}$ in general position. Moreover, general position of $e|_{Z^\lambda} : Z^\lambda \rightarrow M$ to C_i at $x \in Z^\lambda \cap e^{-1}(C_i)$ means that the linearizations of $e|_{\mathcal{F}_K \cap Z^\lambda}$ map onto $T_{e(x)}M / T_{e(x)}C_i$ for each intersection of local faces $\mathcal{F}_K \subset U(y_\beta) \subset X$ that contains x . Here the tangent spaces of $\mathcal{F}_K \cap Z^\lambda$ at x are given by those of $\tilde{\mathcal{F}}_K \cap M_j^{t_\beta, y_\beta} \cap (\{t_0\} \times X)$ for each branch with $(t_0, x) \in M_j^{t_\beta, y_\beta} \subset \text{supp } \tilde{\Theta}$, so we need to ensure surjectivity of $D_{(t_0, x)}\tilde{e} : \ker \Pi \rightarrow T_{e(x)}M / T_{e(x)}C_i$ on the kernel of the projection $\Pi : T_{(t_0, x)}(\tilde{\mathcal{F}}_K \cap M_j^{t_\beta, y_\beta}) \rightarrow \mathbb{R}^{\tilde{l}}$. Here $D_{(t_0, x)}\tilde{e} : T_{(t_0, x)}(\tilde{\mathcal{F}}_K \cap M_j^{t_\beta, y_\beta}) \rightarrow T_{e(x)}M$ is surjective (since $\tilde{e}|_{\text{supp } \tilde{\Theta}}$ is submersive), and regularity $t_0 \in T_{K,j}^{i,\beta}$ means that we have $\Pi(D_{(t_0, x)}\tilde{e})^{-1}(T_{e(x)}C_i) = \mathbb{R}^{\tilde{l}}$, so for any $Y \in T_{e(x)}M$ we find $(T, X) \in T_{(t_0, x)}(\tilde{\mathcal{F}}_K \cap M_j^{t_\beta, y_\beta})$ with $D_{(t_0, x)}\tilde{e}(T, X) = Y$ and $(T, X') \in (D_{(t_0, x)}\tilde{e})^{-1}(T_{e(x)}C_i)$, so that $(0, X - X') \in \ker \Pi$ proves the required surjectivity $D_{(t_0, x)}\tilde{e}(0, X - X') = Y - D_{(t_0, x)}\tilde{e}(T, X') = [Y] \in T_{e(x)}M / T_{e(x)}C_i$. Thus this choice of sufficiently small $t_0 \in \mathcal{T}_0$ also guarantees general position of $e|_{Z^\lambda}$ to each of the countably many submanifolds C_i , which finishes the proof of the theorem when no boundary values are prescribed.

Regular extension: To prove the last paragraph of the theorem we consider a given $(\frac{1}{\alpha}N, \mathcal{U})$ -admissible structurable sc^+ -multisection $\lambda^\partial : P^{-1}(\partial\mathcal{X}) \rightarrow \mathbb{Q}^+$ that is in general position over the boundary, and with $e|_{Z^\partial} : Z^\partial = \text{supp } \lambda^\partial \circ S|_{\partial\mathcal{X}} \rightarrow M$ in general position to finitely many submanifolds C_i . Then we will adjust the above construction of $\lambda : \mathcal{W} \rightarrow \mathbb{Q}^+$

to also satisfy $\lambda|_{P^{-1}(\partial\mathcal{X})} = \lambda^\partial$, by following the proof of the transversal extension theorem over ep-groupoids [34, Thm.15.5].

Since λ^∂ is supported in $\mathcal{U} \cap \partial\mathcal{X}$ with $N(\lambda^\partial)(x) < \alpha$ for all $x \in \partial\mathcal{X}$ we can find a continuous functor $h : \mathcal{X} \rightarrow [0, 1)$ supported in $\mathcal{U}$ with $N(\lambda^\partial)(x) < h(x) < \frac{1}{2}N(\lambda^\partial)(x) + \frac{1}{2}$ for all $x \in \partial\mathcal{X}$. Then [34, Thm.14.2] yields a sc^+ -multisection $\Lambda' : \mathcal{W} \rightarrow \mathbb{Q}^+$ with $\Lambda'|_{P^{-1}(\partial\mathcal{X})} = \lambda^\partial$, domain support in $\mathcal{U}$, and $N(\Lambda')(x) \leq h(x) \leq \frac{\alpha+1}{2}$ for all $x \in \mathcal{X}$. This guarantees compactness of $|\text{supp } \Lambda' \circ S| \subset |\mathcal{X}|$ and regularity of $|\text{supp } \Lambda' \circ S| \cap |\partial\mathcal{X}| = |\text{supp } \lambda^\partial \circ S|_{\partial\mathcal{X}}$. To obtain regularity in the interior we construct $\lambda = \Lambda' \oplus \Lambda_t$ by the above arguments with $S^{-1}(0)$ replaced by $\mathcal{S}' := \text{supp } \Lambda' \circ S \subset \mathcal{X}$, noting that $|\mathcal{S}'| \subset |\mathcal{X}|$ is also compact. To achieve general position to the C_i we need further adjustments.

Local constructions relative to boundary values: For interior points $x \in \mathcal{S}' \cap \partial_0\mathcal{X}$ we construct $\tilde{\Lambda}^x : \mathbb{R}^l \times \mathcal{W} \rightarrow \mathbb{Q}^+$ with domain support in the interior $\mathbb{R}^l \times (\partial_0\mathcal{X} \cap \mathcal{U})$ to cover the cokernels of $T_{(\tilde{S}^x, \tilde{\Lambda}^x)}^R$ for the stabilized multisection $\tilde{\Lambda}' : \mathbb{R}^l \times \mathcal{W} \rightarrow \mathbb{Q}^+$, $(t, w) \mapsto \Lambda'(w)$. For $x \in \mathcal{S}' \cap \partial\mathcal{X}$ we need no stabilization by a $\mathbb{R}^l$ factor (i.e. take $l = 0$) due to the general position of λ^∂ at x . However, we only obtain general position to the C_i , rather than submersivity in the following claim.

Local properties relative to boundary: *For each $x \in \mathcal{S}'$ there exists $l_x \in \mathbb{N}_0$ – with $l_x = 0$ for $x \in \mathcal{S}' \cap \partial\mathcal{X}$ – and a locally uniformizing neighborhood $Q(x) \subset X$ of x whose closure is contained in $\mathcal{U}$, such that for some $\epsilon_x > 0$ we have a tame ep^+ -subgroupoid $\Theta^x : \{t \in \mathbb{R}^l \mid |t| < \epsilon_x\} \times Q(x) \rightarrow \mathbb{Q}^+$, $(t, y) \mapsto (\Lambda' \oplus \Lambda_t^x)(S(y))$ with surjective reduced linearizations, and thus a weighted branched orbifold $|\text{supp } \Theta^x|$. Moreover, if $x \in \mathcal{S}' \cap \mathcal{V}$ then $\tilde{e}^x$ induces a smooth map $|\text{supp } \Theta^x| \rightarrow M$, which is in general position to C_i for each $i \in I$.*

The structure of Θ^x is established in [34, Thm.15.5.], and the general position to each C_i for $x \in \partial_0\mathcal{X}$ follows from submersivity. To achieve general position to the C_i for $x \in \partial\mathcal{X}$, recall that $C = \overline{\cup_{i \in I} (C_i)} \subset M$ is closed, so for $x \notin e^{-1}(C)$ we can choose $Q(x)$ disjoint from $e^{-1}(C)$ so that general position to the $C_i \subset C$ is automatic. For $x \in e^{-1}(C) \subset \mathcal{V}$ we have $e : \text{supp } \Theta^x \cap \partial\mathcal{X} = \text{supp } \lambda^\partial \circ S|_{\partial\mathcal{X}} \rightarrow M$ in general position to each C_i by assumption on λ^∂ . Moreover, we choose $Q(x) \subset \mathcal{V}$ so that $e : Q(x) \cap \text{supp } \Theta^x \rightarrow M$ is smooth, and thus general position to each C_i extends to a neighborhood $Q_i \subset \mathcal{X}$ of x . Then $Q' := \bigcap_{i \in I} Q_i$ is a neighborhood of x since I is finite, and we can replace $Q(x)$ by a uniformizing neighborhood in Q' to achieve general position to all C_i .

From local to global relative to boundary: This portion of the proof is started by picking a finite cover $|\mathcal{S}'| \cap |\partial\mathcal{X}| = |\text{supp } \lambda^\partial \circ S|_{\partial\mathcal{X}} = \bigcup_{i=-k_\partial}^0 |Q(x_i)| \subset |\mathcal{X}|$ by the above neighborhoods for $x_i \in \mathcal{S}' \cap \partial\mathcal{X}$. Next we cover $|\mathcal{S}'| \setminus \bigcup_{i=-k_\partial}^0 |Q(x_i)| \subset \bigcup_{i=1}^k |Q(x_i)|$ with neighborhoods of interior points $x_i \in \mathcal{S}' \cap \partial_0\mathcal{X}$ whose associated multisections Λ^{x_i} are supported in the interior, $\text{dom-supp } \Lambda^{x_i} \subset \mathbb{R}^{l_x} \cap \partial_0\mathcal{X}$. Then we define $\tilde{\Lambda} : \mathbb{R}^l \times \mathcal{W} \rightarrow \mathbb{Q}^+$ by $\tilde{\Lambda}(t, w) := \Lambda_t(w) := (\Lambda' \oplus \Lambda_{t_1}^{x_1} \oplus \cdots \oplus \Lambda_{t_k}^{x_k})(w)$. This multisection is constructed so that $\Lambda_0 = \Lambda'$ and $\Lambda_t|_{P^{-1}(\partial\mathcal{X})} = \lambda^\partial$ for any $t \in \mathbb{R}^l$. Moreover, the estimate $N(\Lambda_t) \leq N(\Lambda') + c|t| \leq \frac{1+\alpha}{2} + c|t|$ allows us to guarantee admissibility $N(\Lambda_t) \leq 1$ by choosing $|t| \leq \frac{1-\alpha}{2c}$. Then compactness of $\tilde{Z}$ in (2.49) follows as above, and its smoothness is established using a covering

$|S^{-1}(0)| \subset \bigcup_{i=-k_\partial}^{k'} |U(x'_i)|$ where $|U(x'_i)|$ for $i \leq 0$ arise from $x'_i \in \mathcal{S}' \cap \partial\mathcal{X}$ and cover a neighborhood of $|\partial\mathcal{X}|$. Moreover, $U(x'_i) \subset \mathbb{R}^{\tilde{l}} \times Q(x'_i)$ can be chosen as in the prior proof of the local properties such that $\tilde{e}|_{\text{supp } \tilde{\Theta}} : U(x'_i) \rightarrow M$ is in general position to C_i for each $i \in I$. This establishes the following.

Global stabilization properties with fixed boundary values: *There exists $\epsilon_0 > 0$ such that $\tilde{\Theta} := \tilde{A} \circ \tilde{S} : \{|t| < \epsilon\} \times \mathcal{X} \rightarrow \mathbb{Q}^+$ is a tame ep^+ -subgroupoid with surjective reduced linearizations for every $0 < \epsilon < \epsilon_0$. In particular, $|\text{supp } \tilde{\Theta}|$ is a weighted branched orbifold. Moreover, there is a neighborhood $\mathcal{V}' \subset \mathcal{X}$ of $S^{-1}(0) \cap e^{-1}(C)$ such that $\tilde{e}|_{\text{supp } \tilde{\Theta}} : \text{supp } \tilde{\Theta} \rightarrow M$ satisfies $(\tilde{e}|_{\text{supp } \tilde{\Theta}})^{-1}(C) \subset \mathbb{R}^{\tilde{l}} \times \mathcal{V}'$, and its restriction to $\text{supp } \tilde{\Theta} \cap (\mathbb{R}^{\tilde{l}} \times \mathcal{V}')$ is classically smooth and in general position to each C_i .*

Global transversality relative to boundary: In this final step we use the fact that Λ_t is $(N, \mathcal{U})$ -admissible for $|t| \leq \frac{1-\alpha}{2c}$ and choose a common regular value of countably many projections as before. The only difference to the proof above is that the restriction of $\tilde{e}|_{\text{supp } \tilde{\Theta}}$ to a branch $M_j^{t_0, y_0} \cap (\mathbb{R}^{\tilde{l}} \times \mathcal{V}') \rightarrow M$ is not necessarily submersive but still in general position to each of the C_i , that is $D_{(t,y)} \tilde{e}|_{N_{t,y}} : N_{t,y} \rightarrow T_{e(y)} M / T_{e(y)} C_i$ is surjective for each $i \in I$. When considering the projections (2.51), this suffices to obtain smooth structures on $\tilde{e}^{-1}(C_i) \cap \tilde{\mathcal{F}}_K \cap M_j^{t_\beta, y_\beta}$ for each branch and intersection of faces $\tilde{\mathcal{F}}_K$. Then general position of $e|_{Z^\lambda} : Z^\lambda \rightarrow M$ to C_i at $x \in Z^\lambda \cap e^{-1}(C_i)$ for $\lambda = \Lambda_{t_0}$ with a regular value $t_0 \in \mathbb{R}^l$ again requires surjectivity of $D_{(t_0, x)} \tilde{e} : \ker \Pi \rightarrow T_{e(x)} M / T_{e(x)} C_i$ on the kernel of the projection $\Pi : T_{(t_0, x)}(\tilde{\mathcal{F}}_K \cap M_j^{t_\beta, y_\beta}) \rightarrow \mathbb{R}^{\tilde{l}}$. To see that $[Y] \in T_{e(x)} M / T_{e(x)} C_i$ is in the image we use the above surjectivity of $D_{(t_0, x)} \tilde{e}|_{N_{t_0, x}}$ to find $(T, X) \in T_{(t_0, x)}(\tilde{\mathcal{F}}_K \cap M_j^{t_\beta, y_\beta})$ with $D_{(t_0, x)} \tilde{e}(T, X) \in [Y]$. Then regularity of t_0 yields $(T, X') \in (D_{(t_0, x)} \tilde{e})^{-1}(T_{e(x)} C_i)$, so that $(0, X - X') \in \ker \Pi$ solves $[D_{(t_0, x)} \tilde{e}(0, X - X')] = [Y - D_{(t_0, x)} \tilde{e}(T, X')] = [Y] \in T_{e(x)} M / T_{e(x)} C_i$. This finishes the proof with prescribed boundary values. $\square$

Chapter 3

Counterexamples in Scale Calculus

This chapter is a reproduction of the first three sections of the paper [17], co-authored with Zhengyi Zhou and Katrin Wehrheim.

Abstract:

We construct counterexamples to classical calculus facts such as the Inverse and Implicit Function Theorems in Scale Calculus – a generalization of Multivariable Calculus to infinite dimensional vector spaces in which the reparameterization maps relevant to Symplectic Geometry are smooth. Scale Calculus is a cornerstone of Polyfold Theory, which was introduced by Hofer-Wysocki-Zehnder as a broadly applicable tool for regularizing moduli spaces of pseudoholomorphic curves. We show how the novel nonlinear scale-Fredholm notion in Polyfold Theory overcomes the lack of Implicit Function Theorems, by formally establishing an often implicitly used fact: The differentials of basic germs – the local models for scale-Fredholm maps – vary continuously in the space of bounded operators when the base point changes. This justifies the high technical complexity in the foundations of Polyfold Theory.

3.1 From Calculus to Scale Calculus

The Inverse and Implicit Function Theorems are core facts in Calculus for functions of one or several variables (i.e. maps $f : \mathbb{R}^m \rightarrow \mathbb{R}^n$). They also hold in all previously known contexts – e.g. on Banach spaces¹ and on manifolds² – in which the classical chain rule holds.

Chain Rule: *If two maps $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ are differentiable, then their composition $g \circ f : X \rightarrow Z, x \mapsto g(f(x))$ is differentiable. Its differential at $x \in X$ is given by composition of the differentials, $d(g \circ f)(x) = dg(f(x)) \circ df(x)$.*

Here and throughout we use the Fréchet notion of differentiability; see e.g. [35, §16.2]. When X, Y, Z are normed vector spaces, it guarantees that each differential $df(x) : X \rightarrow Y$ at a given point x is a linear map. In single variable Calculus for $X = Y = \mathbb{R}$ this map is multiplication $r \mapsto f'(x)r$ by the classical derivative $f'(x) \in \mathbb{R}$. More generally, when X, Y are manifolds, then the differential $df(x) : T_x X \rightarrow T_{f(x)} Y$ is a linear map between tangent spaces; see e.g. [36, Prop.3.6]. The chain rule, e.g. [35, §15.2], is used for example to prove a formula relating the differentials of a function and its inverse as follows: Since $s^{-1} \circ s = \text{id}$, we have $ds^{-1}(s(x)) \circ ds(x) = d(s^{-1} \circ s)(x) = d\text{id}(x) = \text{id}$, where id denotes the identity map on X (and its tangent space $T_x X$), and thus the differential of s^{-1} at $s(x)$ is inverse to the differential of s at x . This is a key ingredient for the following classical result; see e.g. [35, §17.3].

¹A Banach space is a vector space with a norm $X \rightarrow [0, \infty), x \mapsto \|x\|$ that induces a complete topology. The spaces $X = \mathbb{R}^n$ with any norm are Banach spaces, but the term usually denotes infinite dimensional Banach spaces such as the space of square integrable functions $L^2(\mathbb{R}) = \{f : \mathbb{R} \rightarrow \mathbb{R} \mid \|f\|_{L^2} := \int |f(x)|^2 dx < \infty\}$.

²A manifold is a topological space X that can locally be described in terms of coordinates in $\mathbb{R}^n$. More formally, X is also required to be second countable and Hausdorff, and the charts (local homeomorphisms to $\mathbb{R}^n$) are required to be smoothly compatible, which in particular implies that the dimension n is fixed on connected components of X . For an introduction to manifolds see e.g. [36].

Inverse Function Theorem: *Let $s : X \rightarrow Y$ be a continuously differentiable map whose differential $ds(x_0) : X \rightarrow Y$ at some $x_0 \in X$ is an isomorphism (i.e. has a continuous inverse). Then there exists a neighborhood $U \subset X$ of x_0 such that the map $s : U \rightarrow s(U)$ is invertible with open image $s(U) \subset Y$, and the inverse $s^{-1} : s(U) \rightarrow U$ is continuously differentiable with differential $ds^{-1}(s(x)) = ds(x)^{-1}$.*

Similarly, the chain rule is used to compute the implicit function $y : X \rightarrow Y$ that parameterizes the locus defined by a function $h(x, y) = 0$, as follows (for simplicity) in case $X = Y = \mathbb{R}$: Since $h(x, y(x)) = 0$, we have $\partial_x h + \partial_y h \cdot y'(x) = 0$ and thus $y'(x) = -\partial_x h / \partial_y h$. Note that this requires the partial derivative $\partial_y h$ to be nonzero (or more generally invertible as map $T_y Y \rightarrow T_{h(x,y)} Z$), and this in fact is also a sufficient condition for the local existence of the implicit function $y : X \rightarrow Y$, by the following classical result; see e.g. [35, §17.4].

Implicit Function Theorem: *Let $h : X \times Y \rightarrow Z$ be a continuously differentiable map whose partial differential $\partial_Y h(x_0, y_0)$ is an isomorphism. Then there exist neighborhoods $U \subset X$ of x_0 and $V \subset Y$ of y_0 and a differentiable map $y : U \rightarrow V$ whose graph parameterizes the local zero set; that is, $h^{-1}(0) \cap (U \times V) = \{(x, y(x)) \mid x \in U\}$.*

This result is critical for Differential Geometry, which studies “smooth geometric shapes,” i.e. manifolds, by describing them locally in terms of implicit functions. For example, the circle $S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$ has the structure of a 1-dimensional manifold because it can be covered by the four (smoothly compatible) charts arising from applying the Implicit Function Theorem to $h(x, y) = x^2 + y^2 - 1$,

$$S^1 = \{(x, \pm\sqrt{1-x^2}) \mid -1 < x < 1\} \cup \{(\pm\sqrt{1-y^2}, y) \mid -1 < y < 1\}.$$

In classical Calculus and Differential Geometry one can also study the zero sets of more general functions such as $h(x, y) = x^2 + y^2$, which do not meet the transversality condition of dh being surjective. (Such transversality is equivalent, up to change of coordinates, to a partial differential being an isomorphism.) *Singular zero sets are regularized by perturbing the function to achieve transversality.* The result is a well-defined cobordism class of manifolds of the expected dimension. In our example, $h : \mathbb{R}^2 \rightarrow \mathbb{R}$ imposes one condition on two variables, so is expected to have 1-dimensional zero set. While the unperturbed zero set $h^{-1}(0) = \{(0, 0)\}$ consists of a single point, its perturbations $(h - \epsilon)^{-1}(0) = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = \epsilon\}$ are either circles (for $\epsilon > 0$) or empty sets (for $\epsilon < 0$). These perturbed zero sets are all cobordant. In more elementary terms, the integral of a conservative vector field along $(h - \epsilon)^{-1}(0)$ is independent of ϵ (in fact zero).

Scale Calculus was recently developed by Hofer-Wysocki-Zehnder [34][32] as the cornerstone of Polyfold Theory, which provides an analogous perturbation theory for functions whose zero sets are the moduli spaces of pseudoholomorphic curves studied in Symplectic Geometry. It satisfies a Chain Rule [34, §1], and with the appropriate scale-Fredholm notion it satisfies an Implicit Function Theorem. But we show in §3.2 that, for general scale-differentiable (or even scale-smooth) functions, no reasonable version of the Inverse or Implicit Function Theorems can be true. This does not affect the validity of Polyfold

Theory (as we make more explicit in §3.3), but it justifies novel extra conditions in the scale-Fredholm notion, and explains the level of technical difficulties in the polyfold approach to overcoming the foundational challenges in regularizing moduli spaces. The latter have been discussed at length, e.g. in [41], and are not the topic of this paper – apart from one such challenge having motivated the development of Scale Calculus. The following remark gives a brief introduction to Scale Calculus from this point of view; for a more in-depth motivation see [14, §2.2]. The Scale Calculus notions require more analysis proficiency than the calculus level discussion so far. The basic claims and constructions in the rest of this paper should be accessible at the advanced calculus level – when taking for granted the existence of a Scale Calculus in which the constructed maps are “smooth.” The proofs are formulated at the undergraduate analysis level such as in [35] as much as possible, but require some standard graduate topology and analysis such as compactness considerations and Hölder and Sobolev estimates. Use of prior results in Scale Calculus is labeled.

Remark 3.1.1. Scale Calculus works with a sequence $\mathbb{E} = (E_i)_{i \in \mathbb{N}_0}$ of Banach spaces with natural embeddings $E_{i+1} \hookrightarrow E_i$. This is motivated by the reparameterization map $\tau : \mathbb{R} \times \{f : S^1 \rightarrow \mathbb{R}\} \rightarrow \{f : S^1 \rightarrow \mathbb{R}\}, (s, f) \mapsto f(s + \cdot)$ given by viewing the circle as the quotient $S^1 = \mathbb{R}/\mathbb{Z}$. Its two-dimensional analogues appear crucially in the description of moduli spaces in symplectic geometry. While τ is not classically differentiable in any known norm on an infinite dimensional vector space of functions $\{f : S^1 \rightarrow \mathbb{R}\}$, it is Fréchet differentiable as a map $\tau : \mathcal{C}^{i+1}(S^1) \rightarrow \mathcal{C}^i(S^1)$. Note here the shift in differentiability between the spaces $\mathcal{C}^i(S^1) = \{f : S^1 \rightarrow \mathbb{R} \mid f, f', \dots, f^{(i)} \text{ continuous}\}$. This notion of “shifted differentiability” reproduces classical Multivariable Calculus by viewing $\mathbb{R}^n$ as the constant sequence $\mathbb{E} = (E_i = \mathbb{R}^n)_{i \in \mathbb{N}_0}$.

Hofer-Wysocki-Zehnder [34] generalized this notion to infinite dimensions while preserving the chain rule by requiring extra conditions both in the definition of differentiability and on the scale structure $(E_i)_{i \in \mathbb{N}_0}$ as follows: A scale-Banach space is given by sequences of compact³ embeddings $E_{i+1} \hookrightarrow E_i$, whose intersection yields a vector space $E_\infty := \bigcap_{i \in \mathbb{N}_0} E_i$ that is dense in each E_i . Then a function $\tau : \mathbb{E} \rightarrow \mathbb{F}$ is scale-continuous if it is continuous as map $\tau : E_i \rightarrow F_i$ for all $i \in \mathbb{N}_0 = \{0, 1, 2, \dots\}$. Further, the notion of scale-differentiability requires classical differentiability of $\tau : E_{i+1} \rightarrow F_i$ together with a well-defined differential $d\tau(e) : E_i \rightarrow F_i$ for $e \in E_{i+1}$ and continuity of the maps $E_{i+1} \times E_i \rightarrow F_i, (e, X) \mapsto d\tau(e)X$ for $i \in \mathbb{N}_0$; see [34, Definitions 1.1, 1.9]. The latter can be phrased as scale-continuity of the tangent map $T\tau : T\mathbb{E} \rightarrow T\mathbb{F}, (e, X) \mapsto (\tau(e), d\tau(e)X)$, where the shift is encoded in the notion of tangent space $T\mathbb{E} = (E_{i+1} \times E_i)_{i \in \mathbb{N}_0}$.

With these Scale Calculus notions, it holds that the above reparameterization map τ is scale-differentiable and in fact scale-smooth (i.e. all its iterated tangent maps $T^k\tau$ for $k \in \mathbb{N}$ are scale-differentiable) when specifying $\{f : S^1 \rightarrow \mathbb{R}\}$ as the scale-Banach space of functions $(\mathcal{C}^i(S^1))_{i \in \mathbb{N}_0}$; see [14, §2.2]. Here the smooth functions form a dense subspace $E_\infty = \mathcal{C}^\infty(S^1)$ of each Banach space $E_i = \mathcal{C}^i(S^1)$ in the scale structure. The Banach space $E_0 = L^2(\mathbb{R})$

³ Compactness of embeddings means that any bounded sequence in E_{i+1} has a convergent subsequence in E_i . When E_i is infinite dimensional, this requires nontrivial embeddings $E_{i+1} \subsetneq E_i$.

and scale structure $E_i = H^{i, \delta_i}(\mathbb{R})$ that we work with in §3.2 are somewhat more complicated since we require inner products and wish to work with a space of functions $f : \mathbb{R} \rightarrow \mathbb{R}$ whose domain is noncompact. However, the above example is a good proxy for nonexperts since smooth functions with compact support $C_0^\infty(\mathbb{R})$ are dense in E_∞ and thus in each E_i . $\square$

To regularize moduli spaces of pseudoholomorphic curves despite an absence of Inverse and Implicit Function Theorems, Hofer-Wysocki-Zehnder [33] show that they are in fact the zero set of scale-Fredholm maps – a special class of scale-differentiable functions, with the Implicit Function Theorem essentially built into the definition. This is in stark contrast to classical Fredholm theory – which establishes, e.g., the Implicit Function Theorem as stated above for continuously differentiable maps between Banach spaces $X \times Y$ and Z , when the factor X is finite dimensional. These assumptions are equivalent (after change of coordinates and splitting) to the (generally nonlinear) function $h : X \times Y \rightarrow Z$ being transverse (i.e. surjective differential) and Fredholm in the classical sense: At every $(x, y) \in h^{-1}(0)$ the differential $dh(x, y)$ is a (linear) Fredholm operator; that is, its kernel and cokernel are finite dimensional. *Thus our results demonstrate that the highly nontrivial variation of the nonlinear Fredholm notion in Scale Calculus [34, Definition 3.7] is in fact necessary to obtain the desired perturbation theory [34, Theorems 3.4, 5.5].* This scale-Fredholm notion requires a contraction property – after change of coordinates and splitting off finite dimensions in domain and target – and we illuminate this definition in §3.3 by showing that the contraction property implies a continuity of the differentials. This is crucial to various proofs of [34] but only implicitly stated. Unfortunately, this continuity holds only in specific coordinates since changes of coordinates in Scale Calculus generally do not preserve continuity of the differential – another deviation from classical calculus facts. However, our results are sufficient to deduce persistence of transversality in neighborhoods of a transverse zero in Corollary 3.3.3 for general scale-Fredholm maps. This further illuminates why the Inverse and Implicit Function Theorems – while false for general scale-smooth maps – actually do hold for scale-Fredholm maps.

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3.2 Counterexamples to Inverse and Implicit Function Theorems

A full polyfold analogue of the Inverse Function Theorem would require replacing (open subsets of) Banach spaces by sc-retracts as defined in [34, Definition 2.2]. Somewhat simplified,

a sc-retract $R = \text{im } \rho$ is the image of a continuous map $\rho : E \rightarrow E$ on a Banach space E satisfying $\rho \circ \rho = \rho$, where E is equipped with a scale-structure with respect to which ρ is sc-smooth.⁴ As it turns out, the first nontrivial example of a sc-smooth retraction from [32, Lemma 1.23] provides the analytic basis for all of the counterexamples in this section. To construct it explicitly (and fit our later needs), fix a smooth function $\beta : \mathbb{R} \rightarrow [0, \infty)$ with support in $[-1, 1]$ and L^2 -norm $\int_{-\infty}^{\infty} \beta(x)^2 dx = 1$, denote its shifts by $\beta_t := \beta(e^{1/t} + \cdot)$ for $t > 0$, and denote the $L^2(\mathbb{R})$ -inner product by $\langle f, g \rangle := \int_{-\infty}^{\infty} f(x)g(x)dx$. Then [32, Lemma 1.23] gives $E_0 = \mathbb{R} \times L^2(\mathbb{R})$ a scale structure $\mathbb{E} = (E_i)_{i \in \mathbb{N}_0}$ in which the following map is scale-smooth:

$$\rho : \mathbb{R} \times L^2(\mathbb{R}) \rightarrow \mathbb{R} \times L^2(\mathbb{R}), \quad \rho(t, f) := \begin{cases} (t, \langle f, \beta_t \rangle \beta_t) & \text{for } t > 0; \\ (t, 0) & \text{for } t \leq 0. \end{cases}$$

This map is moreover a retraction in the sense that $\rho \circ \rho = \rho$, and the corresponding sc-retract is

$$R := \text{im } \rho = \{(t, 0) \mid t \leq 0\} \cup \{(t, s\beta_t) \mid t > 0, s \in \mathbb{R}\} \subset \mathbb{R} \times L^2(\mathbb{R}), \quad (3.1)$$

with topology induced by its inclusion in $\mathbb{R} \times L^2(\mathbb{R})$. The tangent spaces to this retract are defined as $T_{(t,f)}R = \text{im } d\rho(t, f)$, which are 1-dimensional for $t \leq 0$ and 2-dimensional for $t > 0$, as follows for $f = 0$ from the computation of the differential⁵

$$d\rho(t, 0) : (T, F) \mapsto \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} \rho(t + \epsilon T, \epsilon F) = \begin{cases} (T, \langle F, \beta_t \rangle \beta_t) & \text{for } t > 0; \\ (T, 0) & \text{for } t \leq 0. \end{cases} \quad (3.2)$$

While ρ is not classically differentiable (see Remark 3.2.7), the above map is the differential of ρ in scale calculus. And from here we quickly obtain a first counterexample to the Inverse Function Theorem, in which the map is not invertible since it is not even locally surjective.

Lemma 3.2.1. *There exists a sc-smooth map $s : O \rightarrow R$ between sc-retracts O, R , whose differential $ds(0) : T_0O \rightarrow T_{s(0)}R$ is a sc-isomorphism, but $s(O) \subset R$ contains no neighborhood of $s(0)$.*

Proof. The identity map $\mathbb{R} \rightarrow \mathbb{R}$ is a sc-smooth retraction with corresponding sc-retract $O := \mathbb{R}$. Then the map $s : O \rightarrow R, t \mapsto (t, 0)$ is sc-smooth (as defined in [34, Definition 2.4]) since $\mathbb{R} \rightarrow \mathbb{R} \times L^2(\mathbb{R}), t \mapsto (t, 0)$ is linear and thus sc-smooth. Its differential at $0 \in O$ is the map $ds(0) : T \mapsto (T, 0)$ from $T_0O = \mathbb{R}$ to $T_{(0,0)}R = \text{im } d\rho(0, 0) = \mathbb{R} \times \{0\}$ from (3.2). While this differential is an isomorphism, the image $s(O) \subset R$ does not contain any element of the line $(t, t\beta_t) \in O$ for $t > 0$, which for $t \rightarrow 0$ converges to $s(0) = (0, 0)$ as $\|t\beta_t\|_{L^2} = t$. $\square$

Next, we show that the complications are not caused by the retracts, but by the differences between classical and scale differentiability.

⁴Here and throughout we usually abbreviate ‘scale’ with ‘sc.’

⁵In the case $t = 0$ this computation is based on the convergence $\int F\beta_t \rightarrow 0$ as $t \searrow 0$ for any fixed $F \in L^2(\mathbb{R})$.

Lemma 3.2.2. *There exists a sc-smooth map $s : \mathbb{E} \rightarrow \mathbb{E}$ on a sc-Banach space $\mathbb{E}$, whose differential $ds(0) : \mathbb{E} \rightarrow \mathbb{E}$ is a sc-isomorphism, but $s(\mathbb{E}) \subset \mathbb{E}$ contains no neighborhood of $s(0)$.*

Proof. After giving $\mathbb{E} = \mathbb{R} \times L^2(\mathbb{R})$ a sc-Banach space structure as in [32, Lemma 1.23], we obtain a sc-smooth map

$$s : \mathbb{R} \times L^2(\mathbb{R}) \rightarrow \mathbb{R} \times L^2(\mathbb{R}), \quad (t, f) \mapsto (2t, f) - \rho(t, f) = \begin{cases} (t, f - \langle f, \beta_t \rangle \beta_t) & \text{for } t > 0; \\ (t, f) & \text{for } t \leq 0. \end{cases}$$

Its differential $ds(0, 0) : (T, F) \mapsto (2T, F) - d\rho(0, 0)(T, F) = (T, F)$ is the identity, hence an isomorphism, but the image of s does not contain the line $(t, t\beta_t) \xrightarrow{t \rightarrow 0} (0, 0)$ for $t > 0$ since $f \mapsto f - \langle f, \beta_t \rangle \beta_t$ is projection to the orthogonal complement of $\mathbb{R}\beta_t \subset L^2(\mathbb{R})$. $\square$

In fact, local invertibility is unclear even if the differentials are sc-isomorphisms on an open set.

Question: Given a sc-smooth map $s : \mathbb{E} \rightarrow \mathbb{F}$, whose differential $ds(e) : \mathbb{E} \rightarrow \mathbb{F}$ is a sc-isomorphism for every $e \in \mathbb{E}$, is s (locally) bijective?

We suspect that the answer may in fact be ‘no’ as we have the following example with discontinuous inverse.

Lemma 3.2.3. *There exists a sc-smooth map $\tilde{s} : \mathbb{F} \rightarrow \mathbb{F}$, whose differential $d\tilde{s}(e) : \mathbb{F} \rightarrow \mathbb{F}$ is a sc-isomorphism for every $e \in \mathbb{F}$, but whose inverse $\tilde{s}^{-1} : F_i \rightarrow F_0$ is not continuous on any scale $i \in \mathbb{N}_0$.*

Proof. We modify the construction of Lemma 3.2.2 by adding a new $\mathbb{R}$ component. The map $\tilde{s} : \mathbb{R} \times \mathbb{R} \times L^2(\mathbb{R}) \rightarrow \mathbb{R} \times \mathbb{R} \times L^2(\mathbb{R})$ is defined by

$$\tilde{s} : (t, y, f) \mapsto \begin{cases} (t, y + \phi(t)\langle f, \beta_t \rangle, f - \langle f, \beta_t \rangle \beta_t + y\phi(t)\beta_t) & \text{for } t > 0; \\ (t, y, f) & \text{for } t \leq 0. \end{cases} \quad (3.3)$$

where $\phi(t) = 0$ for $t \leq 0$ and $\phi(t) = e^{-e^{1/t^2}}$ for $t > 0$. We will show that this choice of $\phi \in \mathcal{C}^\infty(\mathbb{R})$ guarantees classical smoothness of

$$g_0 : \mathbb{R} \rightarrow H^{k, \delta}(\mathbb{R}), \quad t \mapsto \phi(t)\beta_t \quad \text{for } k \geq 0, \delta \geq 0. \quad (3.4)$$

Here the weighted Sobolev space $H^{k, \delta}(\mathbb{R})$ is the completion of the smooth compactly supported functions $\mathcal{C}_0^\infty(\mathbb{R})$ with respect to the norm $\|f\|_{H^{k, \delta}} = \sum_{i=0}^k \|e^{\delta|x|} f^{(i)}(x)\|_{L^2}$. Smoothness of (3.4) then implies classical smoothness of $\mathbb{R} \times L^2(\mathbb{R}) \rightarrow \mathbb{R}$, $(t, f) \mapsto \phi(t)\langle f, \beta_t \rangle = \langle f, g_0(t) \rangle$ and thus, together with Lemma 3.2.2, proves sc-smoothness of (3.3) – using a scale structure $E_i = H^{i, \delta_i}(\mathbb{R})$ for $\delta_{i+1} > \delta_i \geq 0$ on $E_0 = L^2(\mathbb{R})$. To show smoothness of (3.4) we express it in the general form $g(t) = \psi(t)\phi(t)\gamma_t$ with $\gamma_t := \gamma(e^{1/t} + \cdot)$

for $\gamma = \beta$ and $\psi \equiv 1$. Any map of this form with $\psi \in \mathcal{C}^\infty((0, \infty))$ and compactly supported $\gamma = \frac{d^k}{dt^k}\beta \in \mathcal{C}^\infty(\mathbb{R})$ has derivative zero for $t \leq 0$, and for $t > 0$ we have $\frac{d}{dt}g(t) = \psi'(t)\phi(t)\gamma_t + \psi(t)\phi'(t)\gamma_t - \psi(t)\phi(t)\frac{1}{t^2}e^{1/t}\gamma'_t$. So $\frac{d}{dt}g = g_1 + g_2 + g_3$ is the sum of three functions of the same form, with $\psi_1(t) = \psi'(t)$, $\psi_2(t) = \frac{2}{t^3}e^{1/t^2}\psi(t)$, $\psi_3(t) = \frac{1}{t^2}e^{1/t}\psi(t)$, $\gamma_1 = \gamma_2 = \gamma$, and $\gamma_3 = \gamma'$. Thus to prove continuity of all derivatives of g it suffices to prove $\|w_\delta\psi(t)\phi(t)\gamma_t\|_{L^2} = \psi(t)\phi(t)\|w_\delta\gamma_t\|_{L^2} \rightarrow 0$ for $t \searrow 0$ with weight function $w_\delta(x) = e^{\delta|x|}$ and any function ψ obtained from $\psi_0(t) = 1$ in finitely many steps of multiplying with $\frac{2}{t^3}e^{1/t^2}$ or $\frac{1}{t^2}e^{1/t}$, or taking the t -derivative. This yields a convex combination of functions of the form $\psi_{\ell,m,n}(t) = \frac{1}{t^\ell}e^{m/t^2}e^{n/t}$ for $\ell, m, n \in \mathbb{N}$. Since $\gamma = \frac{d^k}{dt^k}\beta$ is supported in $[-1, 1]$ we can estimate

$$\|w_\delta\gamma_t\|_{L^2}^2 = \int_{-\infty}^{\infty} |e^{\delta|y-e^{1/t}}\gamma(y)|^2 dy \leq \|\gamma\|_\infty e^{\delta(e^{1/t}+1)} \leq Ce^{\delta e^{1/t}}.$$

Then change of variables $x = \frac{1}{t^2} \rightarrow \infty$ yields the desired convergence

$$\begin{aligned} \lim_{t \searrow 0} \psi_{\ell,m,n}(t)\phi(t)\|w_\delta\gamma_t\|_{L^2} &\leq \lim_{t \searrow 0} \frac{1}{t^\ell} e^{\delta e^{1/t} + m/t^2 + n/t - e^{1/t^2}} \\ &= \left(\lim_{x \rightarrow \infty} x^{\frac{\ell}{2}} e^{-\frac{1}{2}e^x} \right) \cdot e^{\lim_{x \rightarrow \infty} (\delta e^{\sqrt{x}} + mx + n\sqrt{x} - \frac{1}{2}e^x)} = 0. \end{aligned}$$

To prove that the differentials $d\tilde{s}(t, y, f)$ are sc-isomorphisms for all $(t, y, f) \in \mathbb{R}^2 \times L^2(\mathbb{R})$, first note that the differential is the identity for $t \leq 0$. Next, for fixed $t > 0$ and splitting off the first $\mathbb{R}$ -factor, the map $s_t := \text{pr}_{\mathbb{R} \times L^2(\mathbb{R})} \circ \tilde{s}(t, \cdot, \cdot) : \mathbb{R} \times L^2(\mathbb{R}) \rightarrow \mathbb{R} \times L^2(\mathbb{R})$ is linear with inverse

$$s_t^{-1}(y, f) = \left(\frac{\langle f, \beta_t \rangle}{\phi(t)}, f - \langle f, \beta_t \rangle \beta_t + \frac{y\phi(t) - \langle f, \beta_t \rangle}{\phi(t)^2} \beta_t \right).$$

Now the full differential $d\tilde{s}(t, y, f) : (T, Y, F) \mapsto (T, T \frac{d}{dt} s_t(y, f) + s_t(Y, F))$ for $t > 0$ has inverse $(T', Y', F') \mapsto (T', s_t^{-1}((Y', F') - T' \frac{d}{dt} s_t(y, f)))$. This shows that in fact $d\tilde{s}(t, y, f)$ is a sc-isomorphism for any fixed $(t, y, f) \in \mathbb{R}^2 \times L^2(\mathbb{R})$, since β_t is smooth with compact support, so that the bounded linear operators $d\tilde{s}(t, y, f)$ and $d\tilde{s}(t, y, f)^{-1}$ on $\mathbb{R}^2 \times L^2(\mathbb{R})$ restrict to bounded linear operators on the scales $\mathbb{R}^2 \times H^{i, \delta_i}$. On the other hand, the inverse of the nonlinear map $\tilde{s}$,

$$\tilde{s}^{-1} : (t, y, f) \mapsto \begin{cases} (t, \frac{\langle f, \beta_t \rangle}{\phi(t)}, f - \langle f, \beta_t \rangle \beta_t + \frac{y\phi(t) - \langle f, \beta_t \rangle}{\phi(t)^2} \beta_t) & \text{for } t > 0; \\ (t, y, f) & \text{for } t \leq 0. \end{cases}$$

is not even continuous as a map $\mathbb{R}^2 \times H^{i, \delta_i}(\mathbb{R}) \rightarrow \mathbb{R}^2 \times L^2(\mathbb{R})$. To see this, pick $f \in H^{i, \delta_i}(\mathbb{R})$ such that $f(x) = e^{-\delta_i|x|}x^{-2}$ for $|x| > 1$. Then the second component of $\tilde{s}^{-1}(t, 0, f)$ for $0 < t \leq 1$ satisfies an estimate

$$\text{pr}_{\mathbb{R}_y}(\tilde{s}^{-1}(t, 0, f)) = \frac{\langle f, \beta_t \rangle}{\phi(t)} \geq \frac{e^{-\delta_i(e^{1/t}+1)}(e^{1/t}+1)^{-2}}{\phi(t)} = \frac{1}{4}e^{-2\delta_i e^{1/t} - 2/t + e^{1/t^2}} \xrightarrow{t \rightarrow 0} \infty,$$

so does not extend continuously to $\text{pr}_{\mathbb{R}_y}(\tilde{s}^{-1}(0, 0, f)) = 0$. $\square$

Remark 3.2.4. Lemma 3.2.3 also provides a counterexample to the Implicit Function Theorem and its classical consequence that zero sets of smooth Fredholm maps with surjective linearization are smooth manifolds, as follows.

Let $\hat{s} : \mathbb{R} \times \mathbb{R} \times L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$ denote the projection of (3.3) to $L^2(\mathbb{R})$. Then $d\hat{s}$ is surjective everywhere but the zero set $\hat{s}^{-1}(0)$ is

$$\{(t, y, 0) \mid t \leq 0, y \in \mathbb{R}\} \cup \{(t, 0, v\beta_t) \mid t > 0, v \in \mathbb{R}\}.$$

This subset of $\mathbb{R}^2 \times L^2(\mathbb{R})$ is not a topological manifold, as it admits no manifold chart at $(0, 0, 0)$.

This can be seen by failure of local compactness of $\hat{s}^{-1}(0)$ as follows: Given any $\epsilon > 0$, the intersection $\hat{s}^{-1}(0) \cap B_\epsilon$ with the open ϵ -ball in $\mathbb{R}^2 \times L^2(\mathbb{R})$ centered at $(0, 0, 0)$ contains the sequence $e_n = (\frac{1}{n}, 0, \frac{\epsilon}{2}\beta_{1/n})$ for $n > \frac{2}{\sqrt{3}}\epsilon$, which has no convergent subsequence in $\mathbb{R}^2 \times L^2(\mathbb{R})$ since $\|\beta_{1/n} - \beta_{1/m}\|_{L^2(\mathbb{R})} = 2$ for $m \gg n$. $\square$

Next, we obtain an even sharper contrast to the classical Implicit Function Theorem by constructing a nonlinear sc-smooth map with surjective Fredholm linearizations but a branched 1-dimensional zero set.

Theorem 3.2.5. *There exists a sc-smooth map $h : \mathbb{R} \times \mathbb{E} \rightarrow \mathbb{E}, (t, e) \mapsto h_t(e)$ on a Banach space $\mathbb{E}$, whose partial differential $dh_0 : \mathbb{E} \rightarrow \mathbb{E}$ is a sc-isomorphism, but whose zero set branches at $(0, 0)$ in the sense that $h^{-1}(0) = \{(t, 0), (t, z(t))\}$ with a sc-smooth function $z : \mathbb{R} \rightarrow \mathbb{E}$ such that $z(t) = 0$ for $t \leq 0$ and $z(t) \neq 0$ for $t > 0$. In fact, h is transverse to 0 in the sense that $dh(t, e)$ is surjective for all $(t, e) \in \mathbb{R} \times E$, and $dh_t(e)$ is surjective whenever $h_t(e) = 0$.*

Proof. We modify the construction of Lemma 3.2.2 by dropping the first component in the codomain and adjusting the second to

$$h : \mathbb{R} \times L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R}), \quad (t, f) \mapsto h_t(f) := \begin{cases} f - \phi(t, \langle f, \beta_t \rangle) \beta_t & \text{for } t > 0; \\ f & \text{for } t \leq 0; \end{cases} \quad (3.5)$$

for a smooth function $\phi : [0, \infty) \times \mathbb{R} \rightarrow \mathbb{R}, (t, x) \mapsto \phi_t(x)$. The previous example is reproduced by $\phi(t, x) = x$, but for the present result we multiply this function with a t -dependent smooth function to obtain e.g. $\phi(t, x) = x(1 - e^{-1/t^2} + x)$. To prove sc-smoothness of h – using the same sc-structure on $\mathbb{E} = L^2(\mathbb{R})$ as before – we may subtract the identity on $L^2(\mathbb{R})$ and consider the map

$$\mathbb{R} \times \mathbb{E} \rightarrow \mathbb{E}, \quad (t, f) \mapsto h(t, f) - f = \psi(t, \langle f, \beta_t \rangle) \cdot \Phi(t, f).$$

Here $\Phi(t, f) \mapsto \langle f, \beta_t \rangle \beta_t$ for $t > 0$ extends sc-smoothly to $\Phi(t, f) = 0$ for $t \leq 0$ by [32, Lemma 1.23], and $\psi : \mathbb{R}^2 \rightarrow \mathbb{R}$ is some smooth function such as $(t, x) \mapsto 1 - e^{-1/t^2} + x$. So by the product and chain rules in scale calculus [34, §1] it remains to prove sc-smoothness

of the function $\Psi : \mathbb{R} \times \mathbb{E} \rightarrow \mathbb{R}$ given by $\Psi(t, f) = \langle f, \beta_t \rangle$ for $t > 0$ and $\Psi(t, f) = 0$ for $t \leq 0$. For $t \neq 0$ this map is smooth and thus sc-smooth. At $(0, f_0) \in \mathbb{R} \times L^2(\mathbb{R})$ it is sc^0 because both terms in

$$|\Psi(t, f) - \Psi(0, f_0)| = |\langle f, \beta_t \rangle| \leq \|\beta\|_{C^0} \|f - f_0\|_{L^2} + |\langle f_0, \beta_t \rangle|$$

converge to 0 as $(t, f) \rightarrow (0, f_0)$. Scale differentiability is only required at $(0, f_0) \in \mathbb{R} \times H^{1, \delta_1}(\mathbb{R})$ with $\delta_1 > 0$, where we estimate for $t > 0$

$$|\Psi(t, f) - \Psi(0, f_0)| = |\langle f, \beta_t \rangle| \leq \|f\|_{H^{0, \delta_1}} \cdot \left(\int e^{-2\delta_1 |s - e^{1/t}|} \beta(s) ds \right)^{\frac{1}{2}} \leq C \|f\|_{H^{0, \delta_1}} e^{-\delta_1 e^{1/t}}.$$

This shows differentiability with trivial differential $D\Psi(0, f_0) \equiv 0$ because $\lim_{t \rightarrow 0} t^{-1} e^{-\delta_1 e^{1/t}} = 0$. Continuity of the differential then boils down to continuity of $(t, f) \mapsto -t^{-2} e^{1/t} \langle f, \beta'(\cdot + e^{1/t}) \rangle$ at $t = 0$, and further differentiability uses analogous estimates with β replaced by its (still smooth and compactly supported) derivatives. The required limit is seen to be $\lim_{t \rightarrow 0} t^{-k} e^{\ell/t} e^{-\delta_1 e^{1/t}} = 0$ for $k, \ell \in \mathbb{N}$, which holds since for $x = \frac{1}{t} \rightarrow \infty$ we know that $e^{\delta_1 e^x}$ grows faster than $x^k e^{\ell x}$.

This proves sc-smoothness of $h : \mathbb{R} \times \mathbb{E} \rightarrow \mathbb{E}$. Next, its partial differentials are $dh_t = \text{id}$ for $t \leq 0$ but for $t > 0$ we compute

$$dh_t(f) : F \mapsto F - \phi'_t(\langle f, \beta_t \rangle) \langle F, \beta_t \rangle \beta_t.$$

Whenever $c := \phi'_t(\langle f, \beta_t \rangle) \neq 1$ this is a sc-isomorphism on $\mathbb{E}$ with inverse $G \mapsto G - \frac{c}{c-1} \langle G, \beta_t \rangle \beta_t$, but for $c = 1$ it is the projection to the orthogonal complement of $\mathbb{R}\beta_t$ with 1-dimensional kernel and cokernel. To find the zero set, we know $h_t^{-1}(0) = \{0\}$ for $t \leq 0$ and compute for $t > 0$

$$h_t(f) = 0 \quad \Leftrightarrow \quad f = \phi_t(\langle f, \beta_t \rangle) \beta_t \quad \Leftrightarrow \quad f = x \beta_t, \quad x = \phi_t(x) \quad \Leftrightarrow \quad f \in \{0, e^{-e^{1/t^2}} \beta_t\}$$

since for our specific choice of the function ϕ we have

$$x = x(1 - e^{-e^{1/t^2}} + x) \quad \Leftrightarrow \quad x = 0 \quad \text{or} \quad 1 = 1 - e^{-e^{1/t^2}} + x.$$

This proves the first part of the theorem with $z(t) = e^{-e^{1/t^2}} \beta_t$ for $t > 0$, which extends to a sc-smooth path $z : \mathbb{R} \rightarrow \mathbb{E}$ by $z(t) = 0$ for $t \leq 0$ by classical smoothness of (3.4).

To check transversality of h and prove the final remark, we compute $\phi'_t(x) = 1 - e^{-e^{1/t^2}} + 2x$ so that $\phi'_t(x) = 1 \Leftrightarrow x = \frac{1}{2} e^{-e^{1/t^2}}$, and thus surjectivity of $dh_t(f)$ fails exactly for $t > 0$ on the hyperplane $\langle f, \beta_t \rangle = \frac{1}{2} e^{-e^{1/t^2}} = \langle \frac{1}{2} z(t), \beta_t \rangle$. This is the hyperplane through the mid-point $\frac{1}{2} e^{-e^{1/t^2}} \beta_t$ on the line segment between the two zeros $0, z(t) = e^{-e^{1/t^2}} \beta_t$, and orthogonal to the line $\mathbb{R}\beta_t$ through them, so the hyperplane does not intersect the zero set, as claimed. Moreover, although the differential $dh(t, f)$ is generally defined only at $(t, f) \in \mathbb{R} \times E_1$, our

particular choice of function allows us to compute at any $f \in L^2(\mathbb{R})$ and obtain a prospective differential $dh(t, f) : (T, F) \mapsto F$ for $t \leq 0$ and for $t > 0$ with $x_t := \langle f, \beta_t \rangle$

$$dh(t, f) : (T, F) \mapsto dh_t(f)F - T((\partial_t \phi_t)(x_t)\beta_t + \phi'_t(x_t)\langle f, \partial_t \beta_t \rangle \beta_t + \phi_t(x_t)\partial_t \beta_t).$$

To see that this map $\mathbb{R} \times L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$ is surjective, we consider an element $G \in L^2(\mathbb{R})$ in the orthogonal complement to its image and aim to show that it must be zero. From the established properties of dh_t , the only case that remains to be considered is $t > 0$, $f = x_t \beta_t$, $x_t = \frac{1}{2}e^{-e^{1/t^2}}$, and $G \in \mathbb{R}\beta_t$. In that case we use the identity $2\langle \beta_t, \partial_t \beta_t \rangle = \partial_t \|\beta_t\|^2 = 0$ and compute $(\partial_t \phi_t)(x) = -\frac{2}{t^3}e^{1/t^2}e^{-e^{1/t^2}}x$ to obtain

$$\begin{aligned} \langle dh(t, x_t \beta_t)(1, 0), G \rangle &= (\partial_t \phi_t)(x_t)\langle \beta_t, G \rangle + \phi'_t(x_t)\langle x_t \beta_t, \partial_t \beta_t \rangle \langle \beta_t, G \rangle + \phi_t(x_t)\langle \partial_t \beta_t, G \rangle \\ &= -\frac{2}{t^3}e^{1/t^2-e^{1/t^2}} \cdot \frac{1}{2}e^{-e^{1/t^2}} \langle \beta_t, G \rangle = -\frac{1}{t^3}e^{1/t^2-2e^{1/t^2}} \langle \beta_t, G \rangle. \end{aligned}$$

This implies $\langle \beta_t, G \rangle = 0$ and thus $G = 0$, finishing the proof of surjectivity of $dh(t, f)$. $\square$

To better understand the origin of these differences between classical and scale versions of calculus, note that the proofs of the Implicit and Inverse Function Theorems rely on surjectivity (and hence invertibility) of the differential persisting in a neighborhood as follows.

Openness of Transversality: *Let $s : E \rightarrow F$ be a continuously differentiable Fredholm map between two Banach spaces E, F whose differential $ds(0)$ is surjective. Then there is a neighborhood $U \subset E$ of 0 such that $ds(p)$ is surjective for all $p \in U$.*

The examples of Lemmas 3.2.2, 3.2.3, and Theorem 3.2.5 also disprove the scale calculus version of this classical fact. In contrast with Corollary 3.3.3 this also shows that these examples are not sc-Fredholm in the sense of [34, Definition 3.7].

Remark 3.2.6. *Lemma 3.2.2 constructs s on $\mathbb{E} = \mathbb{R} \times L^2(\mathbb{R})$ so that $ds(t, f)$ is a sc-isomorphism for $t \leq 0$, but for $t > 0$ has 1-dimensional kernel and cokernel. Lemma 3.2.3 constructs $\tilde{s}$ on $\mathbb{F} = \mathbb{R} \times \mathbb{R} \times L^2(\mathbb{R})$ so that $\text{pr}_{\mathbb{E}} \circ \tilde{s}|_{\mathbb{E}} \simeq s$ on $\mathbb{E} \simeq \mathbb{R} \times \{0\} \times L^2(\mathbb{R})$ recovers s of Lemma 3.2.2.*

Indeed, the sc-smooth map $s : \mathbb{R} \times L^2(\mathbb{R}) \rightarrow \mathbb{R} \times L^2(\mathbb{R})$ of Lemma 3.2.2 has differential $ds(t, f) = \text{id}$ for $t \leq 0$ but for $t > 0$ we compute

$$ds(t, f) : (T, F) \mapsto (T, F - \langle F, \beta_t \rangle \beta_t + T \frac{e^{1/t}}{t^2} (\langle f, \beta'_t \rangle \beta_t + \langle f, \beta_t \rangle \beta'_t)).$$

For $f = 0$ the second component simplifies to the projection $F \mapsto F - \langle F, \beta_t \rangle \beta_t$ to the orthogonal complement of $\mathbb{R}\beta_t$. Thus, $ds(t, 0)$ is still a sc-Fredholm operator but has kernel $\{0\} \times \mathbb{R}\beta_t$ and cokernel $\mathbb{F}/\text{im } ds(t, 0) \simeq \mathbb{R}\beta_t$. For $f \neq 0$ a brief computation shows the same. The claim on Lemma 3.2.3 follows by setting $y = 0$ in (3.3) and dropping the second component.

Theorem 3.2.5, as established in the proof, constructs h so that the partial differential $dh_t(e)$ is a sc-isomorphism whenever $t \leq 0$ or $t > 0$ and $\langle e - \frac{1}{2}z(t), \beta_t \rangle \neq 0$. However, for

$t > 0$ on the hyperplane $\langle e - \frac{1}{2}z(t), \beta_t \rangle = 0$ through $\frac{1}{2}z(t)$ orthogonal to $\mathbb{R}\beta_t$ the differential $dh_t(e)$ has 1-dimensional kernel and cokernel.

In fact, this failure of fiber-wise transversality of the sc-smooth family of maps $h_t : \mathbb{E} \rightarrow \mathbb{E}$ along some path $t \mapsto x_t\beta_t$ with limit $0 \mapsto 0$ (in our case $x_t\beta_t = \frac{1}{2}z(t)$) is a universal effect for any choice of the function ϕ_t in the construction (3.5) with branching zero set. Indeed, with $\psi_t(x) := x - \phi_t(x)$ we have $h_t^{-1}(0) = \{x | \psi_t(x) = 0\}\beta_t$ and transversality fails at $\{x | \psi'_t(x) = 0\}\beta_t$. So, by the mean value theorem, there is fiber-restricted transversality failure between any two solutions on the line $\mathbb{R}\beta_t$.

While the total differential dh is surjective everywhere in this example, it remains an open question whether there is a scale calculus counterexample to the implicit function theorem in which all partial differentials dh_t are surjective in a neighborhood of a branching point. $\square$

Remark 3.2.6 shows that Openness of Transversality does not hold for general sc-smooth maps. On the other hand, Corollary 3.3.3 below proves Openness of Transversality for sc-Fredholm maps – based on continuity of the differential as an operator in specific coordinates established in Proposition 3.3.2. The difference between continuity of the differential required by sc-smoothness and continuity as an operator is illuminated in Remark 3.2.7.

Remark 3.2.7. The map $h : \mathbb{R} \times L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$, $(t, f) \mapsto f - \phi_t(\langle f, \beta_t \rangle)\beta_t$ with $\phi_t \equiv 0$ for $t \leq 0$, which appears in all counterexamples above for some choice of ϕ_t , has a continuous tangent map

$$Th : \mathbb{R} \times L^2(\mathbb{R}) \times \mathbb{R} \times L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R}) \times L^2(\mathbb{R}), \quad (t, f, T, F) \mapsto (h(t, f), dh(t, f)(T, F))$$

but the differential is discontinuous as a map to the Banach space of bounded operators

$$dh : \mathbb{R} \times L^2(\mathbb{R}) \rightarrow \mathcal{L}(\mathbb{R} \times L^2(\mathbb{R}), L^2(\mathbb{R})), \quad (t, f) \mapsto dh(t, f).$$

Explicitly, we can see that the difference of differentials,

$$dh(t, 0) - dh(0, 0) : (T, F) \mapsto \begin{cases} -\phi'_t(0)\langle F, \beta_t \rangle\beta_t & \text{for } t > 0; \\ 0 & \text{for } t \leq 0; \end{cases}$$

converges pointwise to $(0, 0)$ as $t \rightarrow 0$ since $\langle F, \beta_t \rangle \rightarrow 0$ for any fixed $F \in L^2(\mathbb{R})$. However, the operator norm in $\mathcal{L}(\mathbb{R} \times L^2(\mathbb{R}), L^2(\mathbb{R}))$ is bounded below by $\|dh(0, 0) - dh(t, 0)\| \geq \|\phi'_t(0)\langle \beta_t, \beta_t \rangle\beta_t\|_{L^2} = |\phi'_t(0)| \geq 1$ for every $t > 0$ and both $\phi_t(x) = x$ and $\phi_t(x) = e^t x$. Here we used $F = \beta_t$ with $\|\beta_t\|_{L^2} = 1$. The higher operator norms in the scale structure (whose specifics we do not discuss) are bounded analogously, $\|dh(0, 0) - dh(t, 0)\|_{\mathcal{L}(\mathbb{R} \times H^{i, \delta}, H^{i, \delta})} \geq \|\beta_t\|_{H^{i, \delta}}^{-1} \|\phi'_t(0)\langle \beta_t, \beta_t \rangle\beta_t\|_{H^{i, \delta}} = |\phi'_t(0)| \frac{\|\beta_t\|_{H^{i, \delta}}}{\|\beta_t\|_{H^{i, \delta}}} \geq 1$. In comparison with Proposition 3.3.2 this shows that h is not equivalent to a basic germ at $(0, 0)$ since otherwise Proposition 3.3.2 would imply continuity of the differential as operator on level $i \geq 1$ for variations of the base point in $\mathbb{R} \times \{0\}$, which lies in the ∞ -level of the sc-structure on $\mathbb{R} \times L^2(\mathbb{R})$.

On the other hand, scale smoothness of h requires continuity of the differential only in $\mathcal{L}(\mathbb{R} \times H^{1,\delta}(\mathbb{R}), L^2(\mathbb{R}))$, where the Sobolov space

$$H^{1,\delta}(\mathbb{R}) = \{F : \mathbb{R} \rightarrow \mathbb{R} \mid e^{\delta|x|}F(x), e^{\delta|x|}F'(x) \in L^2\}$$

carries a weight $\delta > 0$. In that operator norm we have convergence $\|dh(0,0) - dh(t,0)\| \leq \sup_{\|F\|_{H^{1,\delta}}=1} |\phi'_t(0)| \|e^{\delta|x|}F(x)\|_{L^2} \|e^{-\delta|x|}\beta_t(x)\|_{L^2} \leq e^{-\delta(e^{1/t}-1)} |\phi'_t(0)| \rightarrow 0$ as $t \searrow 0$. $\square$

3.3 Continuity of differential for basic germs

The examples in §3.2 demonstrate that sc-smoothness and Fredholm linearizations are insufficient for an Inverse or Implicit Function Theorem. Instead, recall from [34, Definitions 3.4–3.7] that sc-Fredholm sections in polyfold theory are required to be locally equivalent to a *basic germ*. Here a section can be thought of (locally, and after a notion of filling) as map $s : \mathbb{E} \rightarrow \mathbb{F}$ between sc-Banach spaces, with the admissible changes of coordinates being governed by the bundle structure, which is specified for experts in a footnote.

Definition 3.3.1. *A sc-smooth map $s : \mathbb{E} \rightarrow \mathbb{F}$ is sc-Fredholm at $e_0 \in E_\infty$ if it is regularizing⁶ and there is an admissible change of coordinates that brings s into the form of a basic germ at 0.⁷ Such a basic germ is a sc-smooth map of the form*

$$f : \mathbb{R}^k \times \mathbb{W} \supset V \rightarrow \mathbb{R}^N \times \mathbb{W}, \quad (c, w) \mapsto (a(c, w), w - B(c, w)), \quad (3.6)$$

where the sc-smooth map $B : V \rightarrow \mathbb{W}$ is a contraction on all levels of $\mathbb{W}$, in the sense that for any $i \in \mathbb{N}_0$ and $\epsilon > 0$ there exists $\delta > 0$ such that for $c \in \mathbb{R}^k$ and $w_1, w_2 \in W_i$ with $|c|, |w_1|_i, |w_2|_i < \delta$ we have the contraction property

$$\|B(c, w_1) - B(c, w_2)\|_i \leq \epsilon \|w_1 - w_2\|_i. \quad (3.7)$$

Recall here that the sc-space $\mathbb{W} = (W_i)_{i \in \mathbb{N}_0}$ consists of Banach spaces W_i with norm $\|\cdot\|_i$ and compact embeddings $W_i \subset W_j$ for $i > j$ such that $W_\infty := \bigcap_{i \in \mathbb{N}_0} W_i$ is dense in each W_i .

The purpose of this section is to illuminate this nonlinear sc-Fredholm property by proving a continuity property of the differentials of a basic germ, which is implicit in various proofs of [34], and does not hold for general sc-smooth maps. Recall from Remark 3.1.1 that general sc-smooth maps $s : \mathbb{E} \rightarrow \mathbb{F}$ restrict to continuously differentiable maps $E_{i+1} \rightarrow F_i$ and the differential is continuous as map $E_{i+1} \times E_i \rightarrow F_i, (e, X) \mapsto ds(e)X$. This can also be

⁶The regularizing property requires $s^{-1}(F_i) \subset E_i$ for each $i \in \mathbb{N}$. This plays a minor but still necessary role in the proof of the Implicit Function Theorem of scale calculus.

⁷Admissible changes of coordinates are given by a sc⁺-section $U \rightarrow U \triangleleft \mathbb{F}, e \mapsto (e, s_0(e))$ with $s_0(e_0) = s(e_0)$ on a neighborhood $U \subset \mathbb{E}$ of e_0 and a strong bundle isomorphism $U \triangleleft \mathbb{F} \rightarrow V \triangleleft (\mathbb{R}^N \times \mathbb{W}), (e, f) \mapsto (\psi(e), \Psi_e f)$ covering a sc-diffeomorphism $\psi : U \rightarrow V \subset \mathbb{R}^k \times \mathbb{W}$ with $\psi(e_0) = 0$. The result of this change of coordinates applied to a map $s : \mathbb{E} \rightarrow \mathbb{F}$ is the map $f : V \rightarrow \mathbb{R}^N \times \mathbb{W}, v \mapsto \Psi_{\psi^{-1}(v)}(s(\psi^{-1}(v)) - s_0(\psi^{-1}(v)))$.

phrased as the differential forming a map $ds : E_{i+1} \rightarrow \mathcal{L}(E_i, F_i)$; that is, the differential at any given base point $e \in E_{i+1}$ is an element of the vector space $\mathcal{L}(E_i, F_i)$, which is defined to consist of bounded (i.e. continuous) linear operators such as $ds(e) : E_i \rightarrow F_i$. However, the differential as map that takes the base point e to the linear operator $ds(e)$ may not be continuous in the operator norm on the vector space $\mathcal{L}(E_i, F_i)$; see [34, Remark 1.1]. That is – as in the previous examples of §3.2 by Remark 3.2.7 – we cannot generally guarantee $\|ds(e+h) - ds(e)\| = \sup_{\|X\|_{E_i}=1} \|ds(e+h)X - ds(e)X\|_{F_i} \rightarrow 0$ as $\|h\|_{E_{i+1}} \rightarrow 0$. However, the following proposition establishes this type of continuity at $e = 0$ if $s = f$ is a basic germ.

Proposition 3.3.2. *Let f be a basic germ as in (3.6). Then for every $i \in \mathbb{N}$ the differential*

$$df : \mathbb{R}^k \times W_{i+1} \rightarrow \mathcal{L}(\mathbb{R}^k \times W_i, \mathbb{R}^N \times W_i), \quad (c, w) \mapsto df(c, w)$$

is continuous at $(0, 0)$ with respect to the operator norm on $\mathcal{L}(\dots)$. In fact, the partial differential in the directions of $\mathbb{W}$,

$$d_{\mathbb{W}}f : \mathbb{R}^k \times W_{i+1} \rightarrow \mathcal{L}(W_i, \mathbb{R}^N \times W_i), \quad (c, w) \mapsto df(c, \cdot)|_w$$

is continuous at $(0, 0)$ with respect to the W_i -topology on W_{i+1} .

Proof. First note that $\text{pr}_{\mathbb{R}^N} \circ f = a : \mathbb{R}^k \times \mathbb{W} \rightarrow \mathbb{R}^N$ is a sc-smooth map with finite dimensional codomain $\mathbb{F} = \mathbb{R}^N$, so for any $i \geq 1$ it restricts to a continuously differentiable map $a : \mathbb{R}^k \times W_i \rightarrow \mathbb{R}^N$ by [34, Proposition 1.7] and triviality of the sc-structure $F_i = \mathbb{R}^N$ from [34, p.4]. Therefore $\text{pr}_{\mathbb{R}^N} \circ df = da : \mathbb{R}^k \times W_i \rightarrow \mathcal{L}(\mathbb{R}^k \times W_i, \mathbb{R}^N)$ is continuous at $(0, 0)$ for $i \geq 1$. Now the composition of this map with the inclusion $W_{i+1} \rightarrow W_i$ yields continuity of $\text{pr}_{\mathbb{R}^N} \circ df = da : \mathbb{R}^k \times W_{i+1} \rightarrow \mathcal{L}(\mathbb{R}^k \times W_i, \mathbb{R}^N)$ at $(0, 0)$ for $i \geq 1$ with respect to both the W_{i+1} -topology and the W_i -topology on W_{i+1} .

Next, the linear map $(c, w) \mapsto w$ in the second component of f has differential $\text{pr}_{\mathbb{W}}$, which restricts to the bounded projections $\mathbb{R}^k \times W_i \rightarrow W_i$ and does not vary with the base point. Thus, the crucial step for this proof is to show continuity of dB at $(0, 0)$. Sc-differentiability of $B : \mathbb{R}^k \times \mathbb{W} \rightarrow \mathbb{W}$, by [34, Proposition 1.5] can be split up into existence of partial differentials $d_{\mathbb{R}^k}B(c, w) : \mathbb{R}^k \rightarrow W_0$ and $d_{\mathbb{W}}B(c, w) : W_0 \rightarrow W_0$ for $(c, w) \in \mathbb{R}^k \times W_1$, which for $w \in W_{i+1}$ restrict to bounded operators in $\mathcal{L}(\mathbb{R}^k, W_i)$ resp. $\mathcal{L}(W_i, W_i)$, such that the shifted difference quotients converge,

$$\lim_{\|(d, h)\|_{\mathbb{R}^k \times W_{i+1}} \rightarrow 0} \frac{\|B(c+d, w+h) - B(c, w) - d_{\mathbb{R}^k}B(c, w)d - d_{\mathbb{W}}B(c, w)h\|_{W_i}}{\|(d, h)\|_{\mathbb{R}^k \times W_{i+1}}} = 0,$$

and $(c, w, d) \mapsto d_{\mathbb{R}^k}B(c, w)d$ restricts to continuous maps $\mathbb{R}^k \times W_{i+1} \times \mathbb{R}^k \rightarrow W_i$, as well as $(c, w, h) \mapsto d_{\mathbb{W}}B(c, w)h$ restricts to continuous maps $\mathbb{R}^k \times W_{i+1} \times W_i \rightarrow W_i$ for every $i \geq 0$. For the first component of the differential, $d_{\mathbb{R}^k}B$, the vector-wise continuity implies continuity of $(c, w) \mapsto d_{\mathbb{R}^k}B(c, w)$ in the operator topology $\mathbb{R}^k \times W_{i+1} \rightarrow \mathcal{L}(\mathbb{R}^k, W_i)$ since the domain $\mathbb{R}^k$ of the bounded operators is finite dimensional. To show the continuity of $(c, w) \mapsto d_{\mathbb{W}}B(c, w)$ in the operator topology $\mathbb{R}^k \times W_{i+1} \rightarrow \mathcal{L}(W_i, W_i)$ at $(0, 0)$, recall that, given $\epsilon > 0$, the

contraction property (3.7) provides $\delta > 0$ so that $\|B(c, w_1) - B(c, w_2)\|_i < \epsilon \|w_1 - w_2\|_i$ whenever $|c|, \|w_1\|_i, \|w_2\|_i < \delta$. We claim that this implies $\|d_{\mathbb{W}}B(c, w)\|_{\mathcal{L}(W_i, W_i)} \leq 2\epsilon$ for $w \in \mathbb{W}_\infty$ with $\|w\|_i < \delta$. Indeed, assume by contradiction $\|d_{\mathbb{W}}B(c, w)h\|_i > 2\epsilon \|h\|_i$ for some $h \in W_i$. Since W_{i+1} is dense in W_i and $d_{\mathbb{W}}B(c, w)$ is continuous, we can find a nearby $h \in W_{i+1}$ that satisfies the same inequality. Then for $t > 0$ sufficiently small such that $\|w + th\|_i, \|w\|_i < \delta$ we can bound the shifted difference quotient

$$\begin{aligned} \frac{\|B(c, w + th) - B(c, w) - d_{\mathbb{W}}B(c, w)th\|_i}{\|th\|_{i+1}} &\geq \frac{t\|d_{\mathbb{W}}B(c, w)h\|_i - \|B(c, w + th) - B(c, w)\|_i}{\|th\|_{i+1}} \\ &\geq \frac{2t\epsilon\|h\|_i - \epsilon\|w + th - w\|_i}{\|th\|_{i+1}} = \frac{\epsilon\|h\|_i}{\|h\|_{i+1}} > 0. \end{aligned}$$

This contradicts the above condition of sc-differentiability for $d = 0$ and $t \rightarrow 0$. Thus, given any $\epsilon > 0$ we found $\delta > 0$ so that $\|d_{\mathbb{W}}B(c, w)\|_{\mathcal{L}(W_i, W_i)} \leq 2\epsilon$ for $w \in \mathbb{W}_{i+1}$ with $\|w\|_i < \delta$. Therefore $d_{\mathbb{W}}B$ is continuous at $(0, 0)$ not just in the natural topology on W_{i+1} but even in the coarser topology induced by the embedding $W_{i+1} \subset W_i$. The same is true for $\text{pr}_{\mathbb{R}^N} \circ df$ with $i \geq 1$, which proves the claimed continuity of $d_{\mathbb{W}}f$. For $d_{\mathbb{R}^k}B$, the scale differentiability yields continuity only in the topology of W_{i+1} , so the overall differential df is continuous at $(0, 0)$ in the W_{i+1} -topology. $\square$

Unfortunately, Proposition 3.3.2 does not prove continuity of the differential as operator for general sc-Fredholm maps, since a change of coordinates by a nonlinear sc-diffeomorphism of the domain does not generally preserve continuity of the differential. In applications, we do expect sc-Fredholm maps such as the Cauchy-Riemann operator in [33] to have continuous differentials, as the changes of coordinates in practice are linear – arising from splitting off kernel and cokernel of linearized operators. However, we deduce from Proposition 3.3.2 that any property which (i) follows from continuity of the differential in the operator norm, and (ii) is preserved under admissible changes of coordinates, must also hold for sc-Fredholm maps. This proves the following scale calculus analogues of “Openness of Transversality” and “Openness of isomorphic differentials.” Here we also note the full polyfold theoretic version of this result in the language of [34].

Corollary 3.3.3. *Let $s : \mathbb{E} \rightarrow \mathbb{F}$ be sc-Fredholm in the sense of Definition 3.3.1 at every $e_0 \in E_\infty$. Then for any $i \in \mathbb{N}$ the following subsets of E_∞ are open with respect to the E_{i+1} -topology,*

$$\{e \in E_\infty \mid ds(e)(E_i) = F_i\}, \quad \{e \in E_\infty \mid ds(e) : E_i \rightarrow F_i \text{ is a sc-isomorphism}\}.$$

Let $\sigma : \mathcal{X} \rightarrow \mathcal{Y}$ be a sc-Fredholm section of a strong bundle $P : \mathcal{Y} \rightarrow \mathcal{X}$. Then, given any local trivialization $P^{-1}(\mathcal{U}) \simeq K \subset \mathbb{E} \triangleleft \mathbb{F}$ over an open subset $\mathcal{U} \subset \mathcal{X}$, the following subsets of $\mathcal{U}_\infty = \mathcal{U} \cap \mathcal{X}_\infty$ are open with respect to the $\mathcal{X}_{i+1}$ -topology for any $i \in \mathbb{N}$,

$$\{x \in \mathcal{U}_\infty \mid D\sigma(x)(T_x\mathcal{X}_i) = (\mathcal{Y}_x)_i\}, \quad \{x \in \mathcal{U}_\infty \mid D\sigma(x) : T_x\mathcal{X}_i \rightarrow (\mathcal{Y}_x)_i \text{ is a sc-isomorphism}\}.$$

Here $(\mathcal{Y}_x)_i$ is the i -th scale of the fiber $\mathcal{Y}_x := P^{-1}(x)$, and the linearizations $D\sigma(x)$ are determined by the choice of local trivialization.

Chapter 4

Persistent homology of the sum metric

This chapter is a reproduction of the paper [6], co-authored with Gunnar Carlsson.

Abstract:

Given finite metric spaces (X, d_X) and (Y, d_Y) , we investigate the persistent homology $PH_*(X \times Y)$ of the Cartesian product $X \times Y$ equipped with the sum metric $d_X + d_Y$. Interpreting persistent homology as a module over a polynomial ring, one might expect the usual Künneth short exact sequence to hold. We prove that it holds for PH_0 and PH_1 , and we illustrate with the Hamming cube $\{0, 1\}^k$ that it fails for PH_n , $n \geq 2$. For $n = 2$, the prediction for $PH_2(X \times Y)$ from the expected Künneth short exact sequence has a natural surjection onto $PH_2(X \times Y)$. We compute the nontrivial kernel of this surjection for the splitting of Hamming cubes $\{0, 1\}^k = \{0, 1\}^{k-1} \times \{0, 1\}$. For all $n \geq 0$, the interleaving distance between the prediction for $PH_n(X \times Y)$ and the true persistent homology is bounded above by the minimum of the diameters of X and Y . As preliminary results of independent interest, we establish an algebraic Künneth formula for simplicial modules over the ring $\kappa[\mathbb{R}_+]$ of polynomials with coefficients in a field κ and exponents in $\mathbb{R}_+ = [0, \infty)$, as well as a Künneth formula for the persistent homology of $\mathbb{R}_+$ -filtered simplicial sets – both of these Künneth formulas hold in all homological dimensions $n \geq 0$.

4.1 Introduction

Topological Data Analysis (TDA) aims to understand the topology of an ambient space from a finite sample; see [5] for an overview of the field. We assume that the topology of the ambient space is induced by a metric d , making the finite sample X into a finite metric space (X, d) . In applications, the ambient space is often embedded in $\mathbb{R}^n$ and various metrics d on $\mathbb{R}^n$ can be useful.

A central tool in TDA is the *persistent homology* $PH_*(X)$ of the Vietoris-Rips complex associated to a finite metric space (X, d) . Persistent homology provides approximations of the homology of the ambient space from which X was sampled, and it does so at all scales $t \geq 0$. The scale parameter t takes all values $t \in \mathbb{R}_+ = [0, \infty)$, and for each fixed value it upper bounds the distances in X that are ‘seen at scale t .’

In this paper, given two finite metric spaces (X, d_X) and (Y, d_Y) , we equip the Cartesian product $X \times Y$ with the sum metric $d_X + d_Y$. We are interested in computing $PH_*(X \times Y)$ in terms of $PH_*(X)$ and $PH_*(Y)$. A Künneth formula of this type would allow us to compute persistent homology of interesting spaces, such as the Hamming cube I^k for $k \geq 1$, $I = \{0, 1\}$ with the Hamming distance; see Example 4.1.6 and Section 4.4.5 for a discussion of the Hamming cube. In [25], the authors derive a Künneth formula for the maximum metric $d_{X \times Y} := \max\{d_X, d_Y\}$ on $X \times Y$ which holds in all homological dimensions.

Our main result (Theorem 4.1.1) is that the familiar Künneth short exact sequence holds for the sum metric $d_{X \times Y} := d_X + d_Y$ in homological dimensions 0 and 1, and in dimension 2 it computes a module that admits a surjection onto $PH_2(X \times Y)$; see Theorem 4.4.5 for a more refined statement and a proof. Example 4.1.6 below shows that the short exact sequence fails in homological dimension $n \geq 2$. We discuss the underlying reason for this

failure below (both in this introduction and in the beginning of Section 4.4). Here we view persistent homology $PH_*(X)$ as a module over a ring $\kappa[\mathbb{R}_+]$ of polynomials in a single variable with exponents in $\mathbb{R}_+ = [0, \infty)$ and coefficients in a field κ ; see Section 4.2 for a study of $\kappa[\mathbb{R}_+]$ -modules, and in particular see Remark 4.2.1 for a discussion of their relationship with persistence vector spaces.

Theorem 4.1.1. *Consider finite metric spaces (X, d_X) and (Y, d_Y) and the product space equipped with the sum metric $(X \times Y, d_X + d_Y)$. Then for $n = 0, 1$ there is a short exact sequence*

$$\begin{aligned} 0 \rightarrow \bigoplus_{i+j=n} PH_i(X) \otimes_{\kappa[\mathbb{R}_+]} PH_j(Y) &\rightarrow PH_n(X \times Y) \\ &\rightarrow \bigoplus_{i+j=n-1} Tor_1(PH_i(X), PH_j(Y)) \rightarrow 0 \end{aligned}$$

which is natural with respect to distance non-increasing maps $(X, d_X) \rightarrow (X', d_{X'})$ and $(Y, d_Y) \rightarrow (Y', d_{Y'})$. This sequence splits, but not naturally.

Moreover, in dimension $n = 2$, the direct sum of the terms in the left position (tensor product terms) and the right position (Tor terms) admits a canonically defined surjection onto $PH_2(X \times Y)$. $\square$

Remark 4.1.2. *Theorem 4.1.1 and all other results about metric spaces (X, d) in this paper hold for a more general class of spaces: Precisely, it is required that the pairing $d : X \times X \rightarrow \mathbb{R}_+$ is symmetric $d(x, y) = d(y, x)$, but d is not required to be reflexive $d(x, y) = 0 \iff x = y$ or satisfy the triangle inequality $d(x, z) \leq d(x, y) + d(y, z)$.*

For all $n \geq 0$, denote the prediction for $PH_n(X \times Y)$ from the short exact sequence in Theorem 4.1.1 by $PH_n(X, Y)$. That is, $PH_n(X, Y)$ is isomorphic to the direct sum of the tensor product terms on the left and the Tor terms on the right. The following result bounds the interleaving distance between the homology $PH_n(X \times Y)$ and the prediction $PH_n(X, Y)$ from above by the minimum of the diameters of X and Y ; see Theorem 4.4.9 for the proof. We discuss the interleaving distance in Section 4.4.4. The idea to look for a bound on this distance was suggested by Leonid Polterovich.

Theorem 4.1.3. *The interleaving distance between $PH_*(X, Y)$ and $PH_*(X \times Y)$ is less than or equal to the minimum of the diameters*

$$\min(\text{diameter}(X), \text{diameter}(Y)),$$

where the diameter is the maximum distance

$$\text{diameter}(X) = \max\{d_X(x_0, x_1) \mid x_0, x_1 \in X\}.$$

$\square$

We now outline the structure of the paper, including preliminary Künneth type results used in the proof of Theorem 4.1.1 which are of independent interest. These other Künneth formulas hold in all homological dimensions $n \geq 0$. We also summarize why Theorem 4.1.1 holds, and why it fails in homological dimensions $n \geq 2$.

In Section 4.2 we study finitely presented $\kappa[\mathbb{R}_+]$ -modules and simplicial modules (Definition 4.2.7). Given a simplicial $\kappa[\mathbb{R}_+]$ -module M , there is an associated chain complex $C_*(M)$; see Definition 4.2.8. The main result (Theorem 4.1.4) proved in Section 4.2.4 is an algebraic Künneth formula for the homology $H_*(C_*(M \otimes_{\kappa[\mathbb{R}_+]} N))$ for free finitely generated simplicial $\kappa[\mathbb{R}_+]$ -modules M and N . See Theorem 4.2.10 for the proof. We remark that, if $\kappa[\mathbb{R}_+]$ were a principal ideal domain (PID), then this theorem would be immediate from the usual Künneth short exact sequence for simplicial modules over a PID. However, unlike the usual polynomial ring $\kappa[\mathbb{Z}_+]$ with exponents in $\mathbb{Z}_+ = \{0, 1, 2, \dots\}$, the ring $\kappa[\mathbb{R}_+]$ is not a PID. Related Künneth formulas have been proven in [45] and [3].

Theorem 4.1.4. (Algebraic Persistent Künneth Theorem) *Let M, N be $\mathbb{R}_+$ -graded free finitely generated simplicial $\kappa[\mathbb{R}_+]$ -modules. Then for $n \geq 0$ there is a short exact sequence*

$$\begin{aligned} 0 \rightarrow \bigoplus_{i+j=n} H_i(C_*(M)) \otimes_{\kappa[\mathbb{R}_+]} H_j(C_*(N)) &\rightarrow H_n(C_*(M \otimes_{\kappa[\mathbb{R}_+]} N)) \\ &\rightarrow \bigoplus_{i+j=n-1} \text{Tor}_1(H_i(C_*(M)), H_j(C_*(N))) \rightarrow 0 \end{aligned}$$

which is natural in both M and N . Moreover, the sequence splits, but not naturally. $\square$

Building on this algebraic Künneth formula over $\kappa[\mathbb{R}_+]$, in Section 4.3 we establish a Künneth formula (Theorem 4.1.5) for persistent homology of $\mathbb{R}_+$ -filtered simplicial sets $(\mathbf{X}, l)$ (Definitions 4.3.1, 4.3.3). These are simplicial sets $\mathbf{X}$ (see Section 4.5 for a review of simplicial sets) together with a map $l : \mathbf{X}_n \rightarrow \mathbb{R}_+$ on the n -simplices $\mathbf{X}_n$ of $\mathbf{X}$ for each $n \geq 0$ such that l is non-increasing along the structure maps (face and degeneracy) in the simplicial set. For any $t \in \mathbb{R}_+$, the collection of simplices $\sigma \in \mathbf{X}$ such that $l(\sigma) \leq t$ forms a simplicial set $\mathbf{X}^{(t)}$, and if $t \leq t'$ then $\mathbf{X}^{(t)} \subseteq \mathbf{X}^{(t')}$. The persistent homology $PH_*(\mathbf{X}, l)$ is a $\mathbb{R}_+$ -graded $\kappa[\mathbb{R}_+]$ -module with homogeneous degree- t part equal to the homology of the simplicial set $\mathbf{X}^{(t)}$, i.e. $PH_*^{(t)}(\mathbf{X}, l) = H_*(\mathbf{X}^{(t)}; \kappa)$.

The product of filtered simplicial sets $(\mathbf{X}, l_X)$ and $(\mathbf{Y}, l_Y)$ is the pair $(\mathbf{X} \times \mathbf{Y}, l_X + l_Y)$ where $\mathbf{X} \times \mathbf{Y}$ is the usual product of simplicial sets and $l_X + l_Y$ is the pointwise sum of the filtration functions. In the notation of the above paragraph, this corresponds to the filtration $(\mathbf{X} \times \mathbf{Y})^{(t)} = \bigcup_{t_X + t_Y \leq t} \mathbf{X}^{(t_X)} \times \mathbf{Y}^{(t_Y)}$ for $t \in \mathbb{R}_+$. The following Künneth theorem follows from Theorem 4.1.4; see Section 4.3 for the proof. In [25], the authors derive Künneth formulas for the persistent homology of the product $\mathcal{X} \times \mathcal{Y}$ of $\mathbb{Z}_+$ -filtered topological spaces $\mathcal{X} = (\mathcal{X}^{(0)} \subset \mathcal{X}^{(1)} \subset \mathcal{X}^{(2)} \subset \dots)$ for various choices of filtration on $\mathcal{X} \times \mathcal{Y}$, including $(\mathcal{X} \times \mathcal{Y})^{(t)} = \bigcup_{t_X + t_Y \leq t} \mathcal{X}^{(t_X)} \times \mathcal{Y}^{(t_Y)}$ for $t \in \mathbb{Z}_+$.

Theorem 4.1.5. (Künneth Theorem for $\mathbb{R}_+$ -filtered simplicial sets) *Given $\mathbb{R}_+$ -filtered simplicial sets $(\mathbf{X}, l_X)$ and $(\mathbf{Y}, l_Y)$ such that $\mathbf{X}_n$ and $\mathbf{Y}_n$ are finite sets for all $n \geq 0$, then for $n \geq 0$ there is a short exact sequence*

$$\begin{aligned} 0 \rightarrow \bigoplus_{i+j=n} PH_i(\mathbf{X}, l_X) \otimes_{\kappa[\mathbb{R}_+]} PH_j(\mathbf{Y}, l_Y) &\rightarrow PH_n(\mathbf{X} \times \mathbf{Y}, l_X + l_Y) \\ &\rightarrow \bigoplus_{i+j=n-1} Tor_1(PH_i(\mathbf{X}, l_X), PH_j(\mathbf{Y}, l_Y)) \rightarrow 0 \end{aligned}$$

which is natural with respect to maps $(\mathbf{X}, l_X) \rightarrow (\mathbf{X}', l_{X'})$ and $(\mathbf{Y}, l_Y) \rightarrow (\mathbf{Y}', l_{Y'})$. Moreover, the sequence splits, but not naturally. $\square$

In Section 4.4 we study the persistent homology $PH_*(X)$ of metric spaces (X, d_X) . We define $PH_*(X)$ to be the persistent homology $PH_*(\mathbf{X}, l_X)$ of a $\mathbb{R}_+$ -filtered simplicial set $(\mathbf{X}, l_X)$ associated to the metric space. The filtration function l_X is defined using the metric d_X (see (4.10)). The homogeneous degree- t part $PH_*^{(t)}(X)$ is equal to homology $H_*(\mathbf{X}^{(t)}; \kappa)$, which in turn is isomorphic to homology of the time- t Vietoris-Rips simplicial complex $V(X, t)$ since $V(X, t)$ and $\mathbf{X}^{(t)}$ are homotopy equivalent; see Remark 4.4.2 for Vietoris-Rips $V(X, t)$. That is, we have $PH_*^{(t)}(X) \cong H_*(V(X, t); \kappa)$ for all $t \in \mathbb{R}_+$.

The product of two metric spaces $X \times Y$ equipped with the sum metric $d_X + d_Y$ has associated $\mathbb{R}_+$ -filtered simplicial set $(\mathbf{X} \times \mathbf{Y}, l_{X \times Y})$, where the filtration function $l_{X \times Y}$ is defined using the sum metric $d_X + d_Y$. The underlying reason for the failure of Theorem 4.1.1 in dimensions $n \geq 2$ is that the filtrations functions satisfy the inequality (see (4.15))

$$l_{X \times Y} \leq l_X + l_Y,$$

but equality does not necessarily hold. The inequality provides a graded module homomorphism

$$PH_n(\mathbf{X} \times \mathbf{Y}, l_X + l_Y) \rightarrow PH_n(\mathbf{X} \times \mathbf{Y}, l_{X \times Y}) = PH_n(X \times Y),$$

but since equality does not hold, it is not necessarily an isomorphism. If it were an isomorphism, then Theorem 4.1.1 would follow immediately from Theorem 4.1.5 in all dimensions $n \geq 0$. We show in Section 4.4.2 that this homomorphism fits into a long exact sequence, and in Section 4.4.3 we show that the correction term in the long exact sequence vanishes in homological dimensions $n = 0, 1, 2$. Hence the homomorphism is an isomorphism for $n = 0, 1$ and a surjection for $n = 2$. Theorem 4.1.1 then follows from Theorem 4.1.5.

We remark that, in the notation of Theorem 4.1.3, we have

$$PH_n(X, Y) = PH_n(\mathbf{X} \times \mathbf{Y}, l_X + l_Y)$$

by definition; see (4.11) and Proposition 4.4.3.

In Section 4.4.5 and Example 4.1.6, we investigate $PH_*(I^k)$ where $I^k = \{0, 1\}^k$ is the Hamming cube with the Hamming metric. We use the splitting $I^k = I^{k-1} \times I$ of the Hamming metric to apply Theorem 4.1.1 inductively to compute $PH_n(I^k)$ for $n = 0, 1, 2$, and for all $k \geq 1$. This involves a technical computation of the nontrivial kernel of the surjection $PH_2(I \times I^{k-1}, l_I + l_{I^{k-1}}) \rightarrow PH_2(I^k)$; see Propositions 4.4.12, 4.4.13.

Example 4.1.6. Consider the metric space $I = \{0, 1\}$ with distance 1 between its two points. The k -dimensional **Hamming Cube** is the k -fold Cartesian product I^k consisting of k -tuples of zeros and ones equipped with the Hamming metric: The distance between two k -tuples in I^k is the number of coordinates in which they are not equal. The Hamming metric is equal to the sum metric on I^k given by summing the metric on each of the k factors of I , which is also equal to the sum metric on the two factor splitting $I^k = I^{k-1} \times I$. So, from a full Künneth formula for the sum metric on a product $X \times Y$, one hopes to inductively compute $PH_n(I^k)$ for all n and k .

In Section 4.4.5 we prove that $PH_2(I^k) = 0$ for all k . This shows that Theorem 4.1.1 does not hold in dimension $n = 2$. Indeed, for $X = I$ and $Y = I^2$, the product is $X \times Y = I^3$, and the short exact sequence in Theorem 4.1.1 for $n = 2$ would predict that $PH_2(I^3)$ has 1 bar; see Section 4.2.2 for a review of bars and how to compute tensor product and Tor of bars.

The table below displays the number of bars in $PH_n(I^k)$ for low values of n and k . This data shows that Theorem 4.1.1 does not hold in higher dimensions $n > 2$. Indeed, again for $X = I$ and $Y = I^2$ with product $X \times Y = I^3$, the short exact sequence in Theorem 4.1.1 for $n = 3$ would predict that $PH_3(I^3) = 0$, but in fact $PH_3(I^3)$ has 1 bar.

Number of bars in $PH_n(I^k)$							
	$k = 1$	$k = 2$	$k = 3$	$k = 4$	$k = 5$	$k = 6$	$k = 7$
$PH_0(I^k)$	2	4	8	16	32	64	128
$PH_1(I^k)$	0	1	5	17	49	129	321
$PH_2(I^k)$	0	0	0	0	0	0	0
$PH_3(I^k)$	0	0	1	9	49	209	769
$PH_4(I^k)$	0	0	0	0	1	11	71
$PH_5(I^k)$	0	0	0	0	0	—	—
$PH_6(I^k)$	0	0	0	0	0	—	—
$PH_7(I^k)$	0	0	0	1	10	—	—

We discuss the Hamming cube further in Section 4.4.5. □

4.2 $\kappa[\mathbb{R}_+]$ -modules

The ring $\kappa[\mathbb{R}_+]$ is the monoid ring of $\mathbb{R}_+ = [0, \infty)$ over a field κ . Concretely, it is the polynomial ring in a single variable T with exponents in $\mathbb{R}_+$ and coefficients in κ . There is a natural $\mathbb{R}_+$ -grading on $\kappa[\mathbb{R}_+]$ where the homogeneous degree $t \in \mathbb{R}_+$ elements are the monomials with exponent t . The usual polynomial ring in a single variable is the monoid ring $\kappa[\mathbb{Z}_+]$ over the nonnegative integers $\mathbb{Z}_+ = \{0, 1, 2, \dots\}$, i.e., the exponents on the variable are in $\mathbb{Z}_+$.

We are interested in the ring $\kappa[\mathbb{R}_+]$ because the persistent homology $PH_*(X)$ of a metric space (X, d) is a $\kappa[\mathbb{R}_+]$ -module. Indeed, $PH_*(X)$ is often viewed as a persistence vector space – or in other words a functor from the poset $\mathbb{R}_+$ to the category of κ -vector spaces – and a persistence vector space is equivalent to a $\mathbb{R}_+$ -graded $\kappa[\mathbb{R}_+]$ -module; see Remark 4.2.1. For the purposes of this paper, it is more convenient to work with $\kappa[\mathbb{R}_+]$ -modules.

Our main result in this section is an algebraic Künneth formula (Theorem 4.2.10) for simplicial modules over $\kappa[\mathbb{R}_+]$ (Section 4.2.3). Theorem 4.2.10 would be immediate from the usual Künneth short exact sequence for simplicial modules over a principal ideal domain – but $\kappa[\mathbb{R}_+]$ is not a principal ideal domain, unlike the usual polynomial ring $\kappa[\mathbb{Z}_+]$ with integer exponents. Indeed, $\kappa[\mathbb{R}_+]$ is not even Noetherian: Consider the ascending chain of principle ideals $(T) \subset (T^{1/2}) \subset (T^{1/3}) \subset \dots$. We resolve these issues in Section 4.2.1: The ring $\kappa[\mathbb{R}_+]$ has the crucial property that the kernel and the image of a homomorphism between free finitely generated modules are both free and finitely generated; see Lemma 4.2.4. This property of $\kappa[\mathbb{R}_+]$ is enough to establish Theorem 4.2.10.

In Section 4.2.2, we discuss the barcode classification of finitely presented persistence vector spaces in the language of finitely presented $\kappa[\mathbb{R}_+]$ -modules (Proposition 4.2.5), and we compute the tensor product and Tor of bars in Proposition 4.2.6. This allows for efficient computation with the algebraic Künneth theorem (Theorem 4.2.10).

Remark 4.2.1. (Persistence vector spaces and Artin-Rees) *The Artin-Rees construction provides an equivalence between finitely generated $\mathbb{R}_+$ -graded $\kappa[\mathbb{R}_+]$ -modules and finitely generated persistence vector spaces over κ . We describe this equivalence in this remark. See [56, Theorem 3.1] for a discussion over $\mathbb{Z}_+$ instead of $\mathbb{R}_+$.*

Recall the definition of a persistence vector space $V = \{V_t\}_{t \in \mathbb{R}_+}$ over κ from [5, Def. 3.3]. This is a functor $V : \mathbb{R}_+ \rightarrow \text{Vec}_\kappa$, where here $\mathbb{R}_+$ denotes the partially ordered set $\mathbb{R}_+$ viewed as a category and Vec_κ is the category of vector spaces over κ . Let $P\text{Vec}_\kappa$ denote the category of persistence vector spaces over κ .

The Artin-Rees construction provides an equivalence of categories

$$A : P\text{Vec}_\kappa \rightarrow \kappa[\mathbb{R}_+]\text{-Mod},$$

where $\kappa[\mathbb{R}_+]\text{-Mod}$ is the category of graded $\kappa[\mathbb{R}_+]$ -modules. Explicitly, $A(V)$ is the κ -vector space $\bigoplus_{t \in \mathbb{R}_+} V_t$. The $\kappa[\mathbb{R}_+]$ -module structure on $A(V)$ is given as follows. Let T denote the variable in $\kappa[\mathbb{R}_+]$, and for $t \leq t'$ let $l_{t,t'} : V_t \rightarrow V_{t'}$ denote the corresponding map in V . Then for a homogeneous element $v \in V_t$ we define $T^a \cdot v = l_{t,t+a}(v) \in V_{t+a}$ for $a \in \mathbb{R}_+$. This gives $A(V)$ the structure of a $\mathbb{R}_+$ -graded $\kappa[\mathbb{R}_+]$ -module.

The inverse functor $A^{-1} : \kappa[\mathbb{R}_+]\text{-Mod} \rightarrow P\text{Vec}_\kappa$ is as follows. For $M \in \kappa[\mathbb{R}_+]\text{-Mod}$ and $t \in \mathbb{R}_+$, the κ -vector space $A^{-1}(M)_t$ is the homogeneous degree t part of M . For $t \leq t'$ the linear map $l_{t,t'} : A^{-1}(M)_t \rightarrow A^{-1}(M)_{t'}$ is multiplication by $T^{t'-t}$.

4.2.1 Finiteness

Over the ring $\kappa[\mathbb{R}_+]$, submodules of finitely generated free modules are not necessarily free: Indeed, the ideal in $\kappa[\mathbb{R}_+]$ consisting of all nonconstant polynomials is not finitely generated because it contains $T^{1/n}$ for all $n \geq 1$ but does not contain constants. This is in stark contrast to the polynomial ring with integer exponents $\kappa[\mathbb{Z}_+]$ which is a principal ideal domain and hence does have the property that submodules of finitely generated free modules are always free.

To prove the algebraic Künneth formula over $\kappa[\mathbb{R}_+]$ (Theorem 4.2.10) and to describe homology using bars via the classification theorem for finitely presented $\kappa[\mathbb{R}_+]$ -modules (Proposition 4.2.5), it is essential that, given a homomorphism of finitely generated graded free $\kappa[\mathbb{R}_+]$ -modules, the kernel and the image are free. This is the content of Lemma 4.2.4.

Remark 4.2.2. *The following results, mostly in this subsection, are a rephrasing of results in [5, Sec. 3.4] from the language of persistence vector spaces to the language of $\kappa[\mathbb{R}_+]$ -modules (see Remark 4.2.1 for a discussion of the equivalence between these notions): Lemma 4.2.3 corresponds to [5, Prop. 3.7]. Lemma 4.2.4 corresponds to the essential step in the proof of [5, Prop. 3.12]. And in Proposition 4.2.5, we reinterpret the classification of finitely presented persistence vector spaces [5, Prop. 3.12, 3.13] as a classification of finitely presented $\mathbb{R}_+$ -graded $\kappa[\mathbb{R}_+]$ -modules.*

To prepare for Lemma 4.2.4, we first understand the matrix of a $\mathbb{R}_+$ -graded $\kappa[\mathbb{R}_+]$ -module homomorphism with respect to a basis.

Lemma 4.2.3. *Consider free and finitely generated $\mathbb{R}_+$ -graded $\kappa[\mathbb{R}_+]$ -modules P and Q with bases $\{p_1, \dots, p_n\}$ and $\{q_1, \dots, q_m\}$, respectively, such that each p_j and q_i is homogeneous.*

Then, given a $\mathbb{R}_+$ -graded $\kappa[\mathbb{R}_+]$ -module homomorphism $\varphi : P \rightarrow Q$, its matrix A^φ with respect to the given bases has homogeneous entries satisfying

$$\deg(A_{i,j}^\varphi) = \deg(p_j) - \deg(q_i) \geq 0 \text{ or } A_{i,j}^\varphi = 0 \text{ for all } i, j. \quad (4.1)$$

Conversely, any $(m \times n)$ -matrix A with homogeneous entries in $\kappa[\mathbb{R}_+]$ satisfying (4.1) determines a $\mathbb{R}_+$ -graded $\kappa[\mathbb{R}_+]$ -module homomorphism $\varphi^A : P \rightarrow Q$ by $\varphi^A(p_j) = \sum_{i=1}^m A_{i,j} q_i$.

Moreover, the correspondences $\varphi \mapsto A^\varphi$ and $A \mapsto \varphi^A$ are inverses of each other.

Proof. Consider a $\mathbb{R}_+$ -graded $\kappa[\mathbb{R}_+]$ -module homomorphism $\varphi : P \rightarrow Q$. Then, the entries of the matrix A^φ are defined by $\varphi(p_j) = \sum_{i=1}^m A_{i,j}^\varphi q_i$. Since φ is $\mathbb{R}_+$ -graded, the image $\varphi(p_j)$ is homogeneous of degree $\deg(p_j)$. Since the q_i form a basis of Q , this implies that either $A_{i,j}^\varphi q_i$ is 0 or it is homogeneous of degree $\deg(p_j)$ for all i , which implies that either $A_{i,j}^\varphi = 0$ or $\deg(A_{i,j}^\varphi) = \deg(p_j) - \deg(q_i)$ for all i , as claimed. The converse statement is clear. $\square$

Lemma 4.2.4. *Suppose $\varphi : P \rightarrow Q$ is a homomorphism of $\mathbb{R}_+$ -graded free¹ finitely generated $\kappa[\mathbb{R}_+]$ -modules.*

¹Graded free means that there exists a basis of homogeneous elements.

Then, there exist bases $\{p_1, \dots, p_n\}$ of P and $\{q_1, \dots, q_m\}$ of Q consisting of homogeneous elements such that the matrix of φ with respect to these bases has the property that every row and every column has at most one nonzero entry.

In particular, the kernel and the image of φ are graded free and finitely generated.

Proof. Since P and Q are graded free and finitely generated, there exists a basis $\{p_1, \dots, p_n\}$ of P and a basis $\{q_1, \dots, q_m\}$ of Q , both consisting of homogeneous elements. Let A be the matrix of φ with respect to these bases. We must show that there exist invertible matrices B and C satisfying (4.1) such that the matrix BAC has at most one nonzero entry in each row and column; precisely, (4.1) requires that B satisfies $\deg(B_{i,j}) = \deg(q_j) - \deg(q_i) \geq 0$ or $B_{i,j} = 0$ for all $1 \leq i, j \leq m$ and that C satisfies the same condition with respect to the p_i .

Let q_{i_0} be the basis element such that $\deg(q_{i_0})$ is maximal among the degrees of the q_i such that the i -th row of A is nonzero, and let p_{j_0} be the basis element such that $\deg(p_{j_0})$ is minimal among the degrees of the p_j such that $A_{i_0,j} \neq 0$. By Lemma 4.2.3, we have

$$A_{i_0,j_0} = l_0 \cdot T^{\deg(p_{j_0}) - \deg(q_{i_0})}$$

for some $l_0 \in \kappa$. For any $j \neq j_0$ such that $A_{i_0,j} \neq 0$, we have $A_{i_0,j} = l \cdot T^{\deg(p_j) - \deg(q_{i_0})}$ for some $l \in \kappa$. There is a column operation given by a matrix C with $C_{j_0,j} = -(l/l_0) \cdot T^{\deg(p_j) - \deg(p_{j_0})}$, ones on the diagonal, and zeros elsewhere, and we have $(AC)_{i_0,j} = 0$. Applying these column operations for all $j \neq j_0$ such that $A_{i_0,j} \neq 0$ arrives at a matrix with i_0 -th row equal to 0 except for the (i_0, j_0) -entry. Rename C to be the product of all these column operations, and set $A' = AC$. Now, for any $i \neq i_0$ such that $A'_{i,j_0} \neq 0$, we have $A'_{i,j_0} = l \cdot T^{\deg(p_{j_0}) - \deg(q_i)}$ for some $l \in \kappa$. There is a row operation given by a matrix B with $B_{i,i_0} = -(l/l_0) \cdot T^{\deg(q_{i_0}) - \deg(q_i)}$, ones on the diagonal, and zeros elsewhere, and we have $(BA')_{i,j_0} = 0$. Applying these row operations for all $i \neq i_0$ such that $A'_{i,j_0} \neq 0$ arrives at a matrix with j_0 -th column equal to 0 except for the (i_0, j_0) -entry. Rename B to be the product of all these row operations. Then the matrix BAC has the property that the i_0 -th row and the j_0 -th column are zero except for the (i_0, j_0) -entry.

We may now iterate this process to produce the desired result. Indeed, relabel BAC by A , let q_{i_1} denote the basis element such that $\deg(q_{i_1})$ is maximal among the degrees of the $q_i \neq q_{i_0}$ such that the i -th row of A is nonzero, and let p_{j_1} denote the basis element such that $\deg(p_{j_1})$ is minimal among the degrees of the $p_j \neq p_{j_0}$ such that $A_{i_1,j} \neq 0$. Then the row and column operations as described above that zero out all entries in the i_1 -row and j_1 -column (except for the (i_1, j_1) -entry) do not affect the i_0 -row and the j_0 -column. Repeating this process completes the construction. $\square$

4.2.2 Bars

A common notation for finitely presented $\kappa[\mathbb{R}_+]$ -modules uses bars (a, b) for $0 \leq a \leq b \leq \infty$, where (a, b) is shorthand for the module generated by a single generator σ of degree a such

that $T^{b-a} \cdot \sigma = 0$; precisely,

$$(a, b) = \Gamma^a \kappa[\mathbb{R}_+] / T^{b-a},$$

where Γ^a denotes a degree shift by a . We allow $b = \infty$, in which case

$$(a, \infty) = \Gamma^a \kappa[\mathbb{R}_+].$$

The following classification result is equivalent to the classification of finitely presented persistence vector spaces in [5, Prop. 3.12, 3.13]; see Remark 4.2.2. The essential step in the setting of $\kappa[\mathbb{R}_+]$ -modules is Lemma 4.2.4.

Proposition 4.2.5. *Every $\mathbb{R}_+$ -graded finitely presented $\kappa[\mathbb{R}_+]$ -module M is isomorphic to a direct sum of the form*

$$M \cong \bigoplus_{i=1}^n (a_i, b_i),$$

where $0 \leq a_i < b_i \leq \infty$. Moreover, this decomposition is unique, i.e. if M is isomorphic to another such direct sum then the number n of bars is the same and the set of pairs (a_i, b_i) that occur, with multiplicities, is the same. $\square$

Due to the above classification, for many computations it suffices to understand how single bars interact with each other. In this paper we use the following computations of the tensor product and Tor of bars; equivalent computations appear in [45][3].

Proposition 4.2.6. *Consider the bars (a, b) and (c, d) . Then there are isomorphisms of $\mathbb{R}_+$ -graded $\kappa[\mathbb{R}_+]$ -modules*

$$(a, b) \otimes_{\kappa[\mathbb{R}_+]} (c, d) \cong (a + c, \min\{a + d, b + c\}) \quad (4.2)$$

and

$$Tor_1((a, b), (c, d)) \cong (\max\{a + d, b + c\}, b + d). \quad (4.3)$$

Moreover, given $\mathbb{R}_+$ -graded finitely presented $\kappa[\mathbb{R}_+]$ -modules M, N , we have $Tor_n(M, N) = 0$ for $n \geq 2$.

Proof. We compute $(a, b) \otimes_{\kappa[\mathbb{R}_+]} (c, d) = \Gamma^a \kappa[\mathbb{R}_+] / T^{b-a} \otimes \Gamma^c \kappa[\mathbb{R}_+] / T^{d-c} \cong \Gamma^{a+c} \kappa[\mathbb{R}_+] / \langle T^{b-a}, T^{d-c} \rangle = \Gamma^{a+c} \kappa[\mathbb{R}_+] / T^{\min\{b-a, d-c\}} = (a + c, a + c + \min\{b - a, d - c\}) = (a + c, \min\{a + d, b + c\})$, as claimed.

Next, we compute (4.3). A projective resolution of (a, b) is given by the exact sequence

$$0 \rightarrow (b, \infty) \rightarrow (a, \infty) \rightarrow (a, b) \rightarrow 0. \quad (4.4)$$

Removing the (a, b) term and tensoring with (c, d) yields the complex

$$0 \rightarrow (b + c, b + d) \rightarrow (a + c, a + d) \rightarrow 0.$$

The homology at the $(b + c, b + d)$ term is the kernel of the map to $(a + c, a + d)$. If $b + c \geq a + d$ then the map is 0 and hence the homology is $(b + c, b + d)$. If $b + c \leq a + d$ then the kernel is $(a + d, b + d)$. So, in both cases, the homology is $(\max\{a + d, b + c\}, b + d)$, proving (4.3).

To prove the final statement of the proposition, first apply Proposition 4.2.5 to both M and N to write them as direct sums of bars. Then the statement follows since the projective resolution (4.4) is zero to the left of the (b, ∞) term. $\square$

4.2.3 Simplicial $\kappa[\mathbb{R}_+]$ -modules

Let $\kappa[\mathbb{R}_+]\text{-Mod}$ denote the category with objects the $\mathbb{R}_+$ -graded $\kappa[\mathbb{R}_+]$ -modules and maps the graded module homomorphisms.

Definition 4.2.7. A $\mathbb{R}_+$ -graded simplicial $\kappa[\mathbb{R}_+]$ -module M is a simplicial object in $\kappa[\mathbb{R}_+]\text{-Mod}$, i.e. a contravariant functor $M : \underline{\Delta} \rightarrow \kappa[\mathbb{R}_+]\text{-Mod}$, where $\underline{\Delta}$ is the simplex category (see Section 4.5).

Concretely, M is a collection of $\mathbb{R}_+$ -graded $\kappa[\mathbb{R}_+]$ -modules $M_n = M([n])$ for $n \geq 0$ together with $\mathbb{R}_+$ -graded $\kappa[\mathbb{R}_+]$ -module maps $M_\varphi : M_n \rightarrow M_m$ for every map $\varphi : [m] \rightarrow [n]$ in $\underline{\Delta}$ that together satisfy the usual functorial properties. We call M_n the n -simplices of M . The images $M_{d_i^\Delta}$ and $M_{s_i^\Delta}$ of the coface maps d_i^Δ and the codegeneracy maps s_i^Δ are called face maps and degeneracy maps, respectively, and for convenience we often denote them by d_i and s_i .

There is a natural chain complex associated to a simplicial module.

Definition 4.2.8. The alternating face maps complex $C_*(M)$ associated to a graded simplicial $\kappa[\mathbb{R}_+]$ -module M is a chain complex of $\mathbb{R}_+$ -graded $\kappa[\mathbb{R}_+]$ -modules which in level $n \geq 0$ is the module

$$C_n(M) = M_n$$

and where the differential is given by the alternating sum of the face maps, i.e.

$$\begin{aligned} d : C_{n+1}(M) &\rightarrow C_n(M) \\ x &\mapsto \sum_{i=0}^{n+1} (-1)^i d_i(x). \end{aligned}$$

4.2.4 The Algebraic Persistent Künneth Theorem

For graded simplicial $\kappa[\mathbb{R}_+]$ -modules M and N (Definition 4.2.7), the tensor product $M \otimes N$ is defined level-wise, i.e.

$$(M \otimes N)_n := M_n \otimes_{\kappa[\mathbb{R}_+]} N_n$$

for $n \geq 0$. For a morphism $\varphi : [m] \rightarrow [n]$ in $\underline{\Delta}$, the corresponding map in $M \otimes N$ is given by

$$\begin{aligned} (M \otimes N)_\varphi : M_n \otimes N_n &\rightarrow M_m \otimes N_m \\ x \otimes y &\mapsto M_\varphi(x) \otimes N_\varphi(y). \end{aligned}$$

We establish in Theorem 4.2.10 a Künneth formula for the homology of $C_*(M \otimes N)$ in terms of $C_*(M)$ and $C_*(N)$, where $C_*(-)$ is the alternating face maps complex (Definition 4.2.8).

In level $n \geq 0$, we have

$$C_n(M \otimes N) = C_n(M) \otimes_{\kappa[\mathbb{R}_+]} C_n(N).$$

There is another chain complex of $\kappa[\mathbb{R}_+]$ -modules associated to the pair M, N , given by the usual tensor product of chain complexes $C_*(M) \otimes C_*(N)$, which in level n is the $\kappa[\mathbb{R}_+]$ -module

$$(C_*(M) \otimes C_*(N))_n = \bigoplus_{i+j=n} C_i(M) \otimes_{\kappa[\mathbb{R}_+]} C_j(N).$$

The Eilenberg-Zilber theorem (see [11], [54, Thm. 8.5.1]) says that there is a natural isomorphism on homology

$$H_*(C_*(M) \otimes C_*(N)) \cong H_*(C_*(M \otimes N)). \quad (4.5)$$

Definition 4.2.9. *A simplicial module M is **graded free** (resp. **finitely generated**) if, for all $n \geq 0$, the module M_n is graded free (resp. finitely generated).*

We now prove Theorem 4.1.4 from the introduction.

Theorem 4.2.10. (Algebraic Persistent Künneth Theorem) *Let M, N be $\mathbb{R}_+$ -graded free finitely generated simplicial $\kappa[\mathbb{R}_+]$ -modules. Then for $n \geq 0$ there is a short exact sequence*

$$\begin{aligned} 0 \rightarrow \bigoplus_{i+j=n} H_i(C_*(M)) \otimes_{\kappa[\mathbb{R}_+]} H_j(C_*(N)) &\rightarrow H_n(C_*(M \otimes_{\kappa[\mathbb{R}_+]} N)) \\ &\rightarrow \bigoplus_{i+j=n-1} \text{Tor}_1(H_i(C_*(M)), H_j(C_*(N))) \rightarrow 0 \end{aligned}$$

which is natural in both M and N . Moreover, the sequence splits, but not naturally.

Proof. For $n \geq 0$, the submodules $B_n(M)$ and $Z_n(M)$ of $C_n(M) = M_n$ consisting of boundaries and cycles, respectively, are graded free by Lemma 4.2.4, and similarly for the boundaries and cycles in $C_n(N)$. Hence, although $\kappa[\mathbb{R}_+]$ is not a principal ideal domain, the proof of the Künneth short exact sequence for the homology of the tensor product of chain complexes $C_*(M) \otimes C_*(N)$ still goes through; see for example the proof of [26, Theorem 3B.5]. That is, for $n \geq 0$ there is a natural short exact sequence

$$\begin{aligned} 0 \rightarrow \bigoplus_{i+j=n} H_i(C_*(M)) \otimes_{\kappa[\mathbb{R}_+]} H_j(C_*(N)) &\rightarrow H_n(C_*(M) \otimes C_*(N)) \\ &\rightarrow \bigoplus_{i+j=n-1} \text{Tor}_1(H_i(C_*(M)), H_j(C_*(N))) \rightarrow 0 \end{aligned}$$

which splits (but the splitting is not natural). The theorem then follows from the natural Eilenberg-Zilber isomorphism (4.5). $\square$

We can compute $H_n(C_*(M \otimes_{\kappa[\mathbb{R}_+]} N))$ using the Künneth short exact sequence in Theorem 4.2.10 together with the classification result Proposition 4.2.5 and the calculations of tensor product and Tor of bars in Proposition 4.2.6.

4.3 $\mathbb{R}_+$ -filtered simplicial sets

We define a notion of a simplicial set filtered by $\mathbb{R}_+ = [0, \infty)$ (see Section 4.5 for a review of simplicial sets). Its homology is a $\mathbb{R}_+$ -graded module over the ring $\kappa[\mathbb{R}_+]$, whose theory we discussed in Section 4.2. In Section 4.4 we associate a $\mathbb{R}_+$ -filtered simplicial set to a metric space (X, d) and we define persistent homology of the metric space $PH_*(X)$ to be the homology of the filtered simplicial set. This is a rephrasing of the Vietoris-Rips construction; see Remark 4.4.2.

In this section we prove the Künneth theorem for $\mathbb{R}_+$ -filtered simplicial sets (Theorem 4.1.5), which follows easily from the algebraic Künneth theorem for simplicial $\kappa[\mathbb{R}_+]$ -modules (Theorem 4.2.10).

Definition 4.3.1. A $\mathbb{R}_+$ -filtered simplicial set $(\mathbf{X}, l)$ is a simplicial set $\mathbf{X}$ together with maps $l : \mathbf{X}_n \rightarrow \mathbb{R}_+$ for all $n \geq 0$ satisfying $l(\mathbf{X}_\varphi(\sigma)) \leq l(\sigma)$ for all simplices $\sigma \in \mathbf{X}_n$ and all morphisms $\varphi : [m] \rightarrow [n]$ in $\underline{\Delta}$.

Remark 4.3.2. A natural notion of a $\mathbb{R}_+$ -filtered simplicial set would be a collection of simplicial sets $\{\mathbf{X}^{(t)}\}_{t \in \mathbb{R}_+}$ and inclusion maps $\mathbf{X}^{(t)} \hookrightarrow \mathbf{X}^{(t')}$ for $t \leq t'$. Given a $\mathbb{R}_+$ -filtered simplicial set $(\mathbf{X}, l)$ as in Definition 4.3.1, for any $t \in \mathbb{R}_+$, the collection of simplices $\sigma \in \mathbf{X}_*$ such that $l(\sigma) \leq t$ forms a simplicial set $\mathbf{X}^{(t)}$, and if $t \leq t'$ then there is an inclusion $\mathbf{X}^{(t)} \hookrightarrow \mathbf{X}^{(t')}$.

The persistent homology $PH_*(\mathbf{X}, l)$ (see Definition 4.3.3) is a $\mathbb{R}_+$ -graded $\kappa[\mathbb{R}_+]$ -module with homogeneous degree- t part equal to the homology of the simplicial set $\mathbf{X}^{(t)}$, i.e. we have $PH_*^{(t)}(\mathbf{X}, l) = H_*(\mathbf{X}^{(t)}; \kappa)$.

Let $\mathbb{R}_+$ -sSet denote the category² with objects the $\mathbb{R}_+$ -filtered simplicial sets $(\mathbf{X}, l)$ and the morphisms $f : (\mathbf{X}, l) \rightarrow (\mathbf{X}', l')$ given by the simplicial maps $\mathbf{X} \rightarrow \mathbf{X}'$ satisfying $l'(f(\sigma)) \leq l(\sigma)$ for all $\sigma \in \mathbf{X}$.

Let $\kappa[\mathbb{R}_+]$ -sMod denote the category of $\mathbb{R}_+$ -graded simplicial $\kappa[\mathbb{R}_+]$ -modules (Definition 4.2.7).

We now define a functor

$$F : \mathbb{R}_+\text{-sSet} \rightarrow \kappa[\mathbb{R}_+]\text{-sMod} \quad (4.6)$$

and then define the persistent homology of $(\mathbf{X}, l)$ (Definition 4.3.3) to be the homology of $F(\mathbf{X}, l)$. Let the n -simplices of $F(\mathbf{X}, l)$ be the free module

$$F_n(\mathbf{X}, l) := \bigoplus_{\sigma \in \mathbf{X}_n} \Gamma^{l(\sigma)} \kappa[\mathbb{R}_+] \cdot \langle \sigma \rangle, \quad (4.7)$$

²Alternatively, we may define $\mathbb{R}_+$ -sSet as follows. Consider the category of pairs (A, l) of sets A and maps $l : A \rightarrow \mathbb{R}_+$, where the morphisms $f : (A, l) \rightarrow (A', l')$ are the set maps $f : A \rightarrow A'$ such that $l'(f(a)) \leq l(a)$ for all $a \in A$. Such a pair (A, l) is called a $\mathbb{R}_+$ -filtered set, as defined in [5, Sec. 3.4], and we denote the category of these pairs by $\mathbb{R}_+\text{-Set}$. Then a $\mathbb{R}_+$ -filtered simplicial set is a simplicial object in $\mathbb{R}_+\text{-Set}$, i.e. a contravariant functor $\underline{\Delta} \rightarrow \mathbb{R}_+\text{-Set}$, and $\mathbb{R}_+\text{-sSet}$ is the category of these functors.

where the notation Γ^a means a degree shift of $a \in \mathbb{R}_+$. So, F is essentially the free $\kappa[\mathbb{R}_+]$ -module functor, with additional degree shifts of the generators by the filtration function l . For a morphism $\varphi : [m] \rightarrow [n]$ in $\underline{\Delta}$, we define the corresponding morphism in $F(\mathbf{X}, l)$ on generators by

$$\begin{aligned} F(\mathbf{X}, l)_\varphi : F_n(\mathbf{X}, l) &\rightarrow F_m(\mathbf{X}, l) \\ \langle \sigma \rangle &\mapsto T^{l(\sigma) - l(\mathbf{X}_\varphi(\sigma))} \langle \mathbf{X}_\varphi(\sigma) \rangle. \end{aligned}$$

Extending $\kappa[\mathbb{R}_+]$ -linearly defines $F(\mathbf{X}, l)_\varphi$ on all of $F_n(\mathbf{X}, l)$. Note that $F(\mathbf{X}, l)_\varphi$ is a graded map due to the degree shifts in (4.7). In particular, the face maps on $F(\mathbf{X}, l)$ are given on generators by

$$d_i(\langle \sigma \rangle) := T^{l(\sigma) - l(d_i(\sigma))} \langle d_i(\sigma) \rangle,$$

where on the left we are writing $d_i = F(\mathbf{X}, l)_{d_i^\Delta}$ and on the right we are writing $d_i = \mathbf{X}_{d_i^\Delta}$.

For a map $f : (\mathbf{X}, l) \rightarrow (\mathbf{X}', l')$ in $\mathbb{R}_+$ -sSet, we define $F(f)$ on generators by

$$\begin{aligned} F(f) : F_n(\mathbf{X}, l) &\rightarrow F_n(\mathbf{X}', l') \\ \langle \sigma \rangle &\mapsto T^{l(\sigma) - l'(f(\sigma))} \langle f(\sigma) \rangle. \end{aligned}$$

Note that the property $l(\sigma) - l'(f(\sigma)) \geq 0$, which holds by definition of a map $f \in \mathbb{R}_+$ -sSet, is crucial in the definition of $F(f)$; indeed, the factor $T^{l(\sigma) - l'(f(\sigma))}$ is necessary to make $F(f)$ a $\mathbb{R}_+$ -graded map due to the degree shifts in (4.7).

Definition 4.3.3. Given $(\mathbf{X}, l) \in \mathbb{R}_+$ -sSet, its **persistent chain complex** $PC_*(\mathbf{X}, l)$ is the alternating face maps complex (Definition 4.2.8) of the module $F(\mathbf{X}, l)$,

$$PC_*(\mathbf{X}, l) := C_*(F(\mathbf{X}, l)),$$

and its **persistent homology** is the homology $PH_*(\mathbf{X}, l) := H_*(PC_*(\mathbf{X}, l))$. For $t \in \mathbb{R}_+$, the **time- t persistent homology** $PH_*^{(t)}(\mathbf{X}, l)$ is the homogeneous degree t part of $PH_*(\mathbf{X}, l)$.

Consider $\mathbb{R}_+$ -filtered simplicial sets $(\mathbf{X}, l_X)$ and $(\mathbf{Y}, l_Y)$. We define their Cartesian product by

$$(\mathbf{X}, l_X) \times (\mathbf{Y}, l_Y) := (\mathbf{X} \times \mathbf{Y}, l_X + l_Y),$$

where the simplicial set $\mathbf{X} \times \mathbf{Y}$ has n -simplices $(\mathbf{X} \times \mathbf{Y})_n = \mathbf{X}_n \times \mathbf{Y}_n$ and structure maps $(\mathbf{X} \times \mathbf{Y})_\varphi = \mathbf{X}_\varphi \times \mathbf{Y}_\varphi$ for $\varphi \in \underline{\Delta}$, and where $(l_X + l_Y)(\sigma, \tau) = l_X(\sigma) + l_Y(\tau)$.

We now establish the Künneth formula (Theorem 4.1.5) that computes persistent homology of the Cartesian product $PH_*(\mathbf{X} \times \mathbf{Y}, l_X + l_Y)$ when $\mathbf{X}_n$ and $\mathbf{Y}_n$ are finite sets for $n \geq 0$.

Proof of Theorem 4.1.5. Recall the functor $F : \mathbb{R}_+$ -sSet $\rightarrow \kappa[\mathbb{R}_+]$ -sMod from (4.6). There is a canonical isomorphism

$$F(\mathbf{X} \times \mathbf{Y}, l_X + l_Y) \cong F(\mathbf{X}, l_X) \otimes_{\kappa[\mathbb{R}_+]} F(\mathbf{Y}, l_Y).$$

Then the theorem follows from the definition of persistent homology (Definition 4.3.3) and the Künneth formula in $\kappa[\mathbb{R}_+]$ -sMod (Theorem 4.2.10). $\square$

4.4 Metric spaces

The main purpose of this paper is to investigate the persistent homology of the sum metric $d_X + d_Y$ on the product $X \times Y$ of metric spaces (X, d_X) and (Y, d_Y) .

In Section 4.4.1, we define persistent homology $PH_*(X)$ of a metric space (X, d_X) as the persistent homology $PH_*(\mathbf{X}, l_X)$ of a $\mathbb{R}_+$ -filtered simplicial set $(\mathbf{X}, l_X)$ (Definition 4.3.1, 4.3.3) associated to (X, d_X) . See Remark 4.4.2 for a discussion of the equivalence of this definition to the usual Vietoris-Rips construction.

The Künneth theorem for $\mathbb{R}_+$ -filtered simplicial sets (Theorem 4.1.5) computes $PH_*(\mathbf{X} \times \mathbf{Y}, l_X + l_Y)$, however the filtration function $l_{X \times Y}$ associated to the sum metric $d_X + d_Y$ is not equal to $l_X + l_Y$; it only satisfies the inequality (see (4.15))

$$l_{X \times Y} \leq l_X + l_Y.$$

We discuss this in Section 4.4.2. The inequality provides a graded module homomorphism

$$PH_*(X, Y) := PH_*(\mathbf{X} \times \mathbf{Y}, l_X + l_Y) \rightarrow PH_*(X \times Y) \quad (4.8)$$

which fits into a long exact sequence (Proposition 4.4.4). The third term in the triple that forms the long exact sequence is denoted $\overline{PH}_*(X, Y)$.

In Section 4.4.3 we prove the vanishing $\overline{PH}_n(X, Y) = 0$ in dimensions $n = 0, 1, 2$. This implies that the mapping (4.8) is an isomorphism in dimensions 0, 1 and a surjection in dimension 2. Then from the Künneth Theorem 4.1.5 for $PH_n(\mathbf{X} \times \mathbf{Y}, l_X + l_Y)$, we obtain our Künneth results for $PH_n(X \times Y)$ in low dimensions $n = 0, 1, 2$ (Theorem 4.4.5).

In Section 4.4.4 we bound the interleaving distance between the computable module $PH_*(X, Y)$ and the homology $PH_*(X \times Y)$ that we are interested in by the minimum of the diameters of X and Y ; see Theorem 4.4.9.

In Section 4.4.5, we use Theorem 4.4.5 to compute PH_n of the Hamming cube $\{0, 1\}^k$ for $n = 0, 1, 2$ and for all $k \geq 1$. This involves a technical computation of the nontrivial kernel of the surjection (4.8) in dimension 2 for the example $X = \{0, 1\}$ and $Y = \{0, 1\}^{k-1}$.

4.4.1 Persistent homology

We associate a $\mathbb{R}_+$ -filtered simplicial set (Definition 4.3.1) to every metric space, and we define the persistent homology of the metric space (Definition 4.4.1) to be the persistent homology of the associated filtered simplicial set (Definition 4.3.3). This is a simplicial set version of the usual Vietoris-Rips construction; see Remark 4.4.2 for further discussion.

Let Met denote the category with metric spaces (X, d_X) as objects and with morphisms the maps $f : (X, d_X) \rightarrow (Y, d_Y)$ that are distance non-increasing, i.e.

$$d_Y(f(a), f(b)) \leq d_X(a, b) \text{ for all } a, b \in X. \quad (4.9)$$

Recall from Section 4.3 the category $\mathbb{R}_+$ -sSet of $\mathbb{R}_+$ -filtered simplicial sets.

We now describe a functor

$$\begin{aligned} Met &\rightarrow \mathbb{R}_+\text{-sSet} \\ (X, d_X) &\mapsto (\mathbf{X}, l_X) \end{aligned}$$

and define persistent homology of the metric space (X, d_X) (Definition 4.4.1) to be the persistent homology of $(\mathbf{X}, l_X)$.

Let $(X, d_X) \in Met$. There is a simplicial set $\mathbf{X}$ with n -simplices

$$\mathbf{X}_n = \{(x_0, \dots, x_n) \mid x_i \in X \text{ for all } 0 \leq i \leq n\}$$

consisting of all $(n+1)$ -tuples of points in X . For a morphism $\varphi : [m] \rightarrow [n]$ in $\underline{\Delta}$, the map $\mathbf{X}_\varphi : \mathbf{X}_n \rightarrow \mathbf{X}_m$ is defined on a n -simplex $\sigma = (x_0, \dots, x_n) \in \mathbf{X}_n$ by

$$\mathbf{X}_\varphi(x_0, \dots, x_n) = (x_{\varphi(0)}, \dots, x_{\varphi(m)}).$$

There is also a *max-length* map

$$\begin{aligned} l_X : \mathbf{X}_n &\rightarrow \mathbb{R}_+ \\ (x_0, \dots, x_n) &\mapsto \max\{d_X(x_i, x_j) \mid 0 \leq i, j \leq n\} \end{aligned} \tag{4.10}$$

given by the maximum of the pairwise distances between points in a simplex. We observe $l_X(\mathbf{X}_\varphi(\sigma)) \leq l_X(\sigma)$ holds since the points in $\mathbf{X}_\varphi(\sigma)$ are a subset of the points in σ . Hence, the pair $(\mathbf{X}, l_X)$ is a $\mathbb{R}_+$ -filtered simplicial set. This defines the functor on objects $(X, d_X) \mapsto (\mathbf{X}, l_X)$. For a morphism $f : (X, d_X) \rightarrow (Y, d_Y)$ in Met there is an obvious induced map on simplicial sets $\mathbf{f} : \mathbf{X} \rightarrow \mathbf{Y}$ that satisfies $l_Y(\mathbf{f}(\sigma)) \leq l_X(\sigma)$ by the distance non-increasing property (4.9) of f . So, $\mathbf{f}$ is indeed a morphism in $\mathbb{R}_+\text{-sSet}$. This completes the definition of the functor $Met \rightarrow \mathbb{R}_+\text{-sSet}$.

We now define the persistent homology of a metric space (X, d_X) .

Definition 4.4.1. *Given a metric space (X, d_X) , its **persistent chain complex** $PC_*(X)$ is the persistent chain complex of the associated $\mathbb{R}_+$ -filtered simplicial set $(\mathbf{X}, l_X)$ (see Definition 4.3.3), i.e.*

$$PC_*(X) := PC_*(\mathbf{X}, l_X).$$

*The **persistent homology** of (X, d_X) is the homology $PH_*(X) := PH_*(\mathbf{X}, l_X)$ of this chain complex.*

*For $t \in \mathbb{R}_+$, the **time- t persistent homology** $PH_*^{(t)}(X)$ is the homogeneous degree t part of $PH_*(X)$.*

Remark 4.4.2. (Vietoris-Rips) *The classical definition of persistent homology of a metric space (X, d_X) uses the Vietoris-Rips construction, which we now recall. For $t \in \mathbb{R}_+$, the time- t Vietoris-Rips complex of X is the abstract simplicial complex $V(X, t)$ with n -simplices given by all subsets $\{x_0, \dots, x_n\} \subset X$ satisfying $d_X(x_i, x_j) \leq t$ for all $0 \leq i, j \leq n$. For $s \leq t$*

there is an inclusion $V(X, s) \hookrightarrow V(X, t)$ which induces a map on homology $H_*(V(X, s)) \rightarrow H_*(V(X, t))$. The homology of Vietoris-Rips at all times $t \in \mathbb{R}_+$ together with the induced maps of the inclusions for $s \leq t$ forms the persistent homology of X .

Letting $(\mathbf{X}, l_X)$ denote the $\mathbb{R}_+$ -filtered simplicial set associated to (X, d_X) , for all $t \in \mathbb{R}_+$ there is a simplicial set

$$\mathbf{X}^{(t)} := \{\text{simplices } \sigma \text{ in } \mathbf{X} \mid l_X(\sigma) \leq t\}.$$

There is a homotopy equivalence $V(X, t) \rightarrow \mathbf{X}^{(t)}$. The homogeneous degree- t persistent homology of (X, d_X) is the homology of $\mathbf{X}^{(t)}$. In conclusion, for all $t \in \mathbb{R}_+$ there are isomorphisms of κ -vector spaces

$$PH_*^{(t)}(X) \cong H_*(\mathbf{X}^{(t)}) \cong H_*(V(X, t))$$

which are natural in t , i.e. the induced map $H_*(V(X, s)) \rightarrow H_*(V(X, t))$ for $s \leq t$ is identified with the map $PH_*^{(s)}(X) \rightarrow PH_*^{(t)}(X)$ given by multiplication by $T^{t-s} \in \kappa[\mathbb{R}_+]$.

4.4.2 A long exact sequence for the sum metric $d_X + d_Y$

Consider metric spaces X and Y with metrics d_X and d_Y . Ultimately, we are interested in a Künneth formula for $PH_*(X \times Y)$ in terms of $PH_*(X)$ and $PH_*(Y)$, where the product space $X \times Y$ is equipped with the sum metric $d_{X \times Y} = d_X + d_Y$.

In this section, we describe a long exact sequence in Proposition 4.4.4 that relates the persistent homology of the product metric space $(X \times Y, d_X + d_Y)$ and the persistent homology

$$PH_*(X, Y) := PH_*(\mathbf{X} \times \mathbf{Y}, l_X + l_Y) \quad (4.11)$$

of the product filtered simplicial set $\mathbf{X} \times \mathbf{Y}$ with filtration function $l_X + l_Y$, where l_X and l_Y are the max-length maps on the simplicial sets $\mathbf{X}$ and $\mathbf{Y}$, respectively; see Section 4.4.1. This is useful because the Künneth theorem for filtered simplicial sets (Theorem 4.1.5) provides the following formula for $PH_*(X, Y)$ in terms of $PH_*(X)$ and $PH_*(Y)$.

Proposition 4.4.3. *Given finite metric spaces (X, d_X) and (Y, d_Y) , for $n \geq 0$ there is a short exact sequence*

$$\begin{aligned} 0 \rightarrow \bigoplus_{i+j=n} PH_i(X) \otimes_{\kappa[\mathbb{R}_+]} PH_j(Y) &\rightarrow PH_n(X, Y) \\ &\rightarrow \bigoplus_{i+j=n-1} Tor_1(PH_i(X), PH_j(Y)) \rightarrow 0 \end{aligned} \quad (4.12)$$

which is natural with respect to distance non-increasing maps $(X, d_X) \rightarrow (X', d_{X'})$ and $(Y, d_Y) \rightarrow (Y', d_{Y'})$. Moreover, the sequence splits, but not naturally.

Proof. By definition of persistent homology of a metric space (Definition 4.4.1) we have $PH_*(X) = PH_*(\mathbf{X}, l_X)$ and $PH_*(Y) = PH_*(\mathbf{Y}, l_Y)$. Hence the proposition follows immediately from Theorem 4.1.5 and the definition $PH_*(X, Y) := PH_*(\mathbf{X} \times \mathbf{Y}, l_X + l_Y)$. $\square$

We now describe the long exact sequence which relates $PH_*(X \times Y)$ to $PH_*(X, Y)$. By definition, we have the chain complexes

$$\begin{aligned} PC_n(X, Y) &:= PC_n(\mathbf{X} \times \mathbf{Y}, l_X + l_Y) \\ &= \bigoplus_{(\sigma, \tau) \in \mathbf{X}_n \times \mathbf{Y}_n} \Gamma^{l_X(\sigma) + l_Y(\tau)} \kappa[\mathbb{R}_+] \cdot \langle \sigma, \tau \rangle \end{aligned} \quad (4.13)$$

and

$$PC_n(X \times Y) = \bigoplus_{(\sigma, \tau) \in \mathbf{X}_n \times \mathbf{Y}_n} \Gamma^{l_{X \times Y}(\sigma, \tau)} \kappa[\mathbb{R}_+] \cdot \langle \sigma, \tau \rangle. \quad (4.14)$$

Given simplices $\sigma = (x_0, \dots, x_n) \in \mathbf{X}_n$ and $\tau = (y_0, \dots, y_n) \in \mathbf{Y}_n$, we claim that the max-length $l_{X \times Y}(\sigma, \tau)$ of the n -simplex (σ, τ) in $X \times Y$ must be less than or equal to the sum $l_X(\sigma) + l_Y(\tau)$. Indeed, for some $0 \leq i \leq j \leq n$, the max-length of (σ, τ) is equal to $l_{X \times Y}(\sigma, \tau) = d_{X \times Y}((x_i, y_i), (x_j, y_j)) = d_X(x_i, x_j) + d_Y(y_i, y_j)$. Then since $d_X(x_i, x_j) \leq l_X(\sigma)$ and $d_Y(y_i, y_j) \leq l_Y(\tau)$, the claim follows:

$$l_{X \times Y}(\sigma, \tau) \leq l_X(\sigma) + l_Y(\tau) \text{ for all } (\sigma, \tau) \in \mathbf{X}_n \times \mathbf{Y}_n. \quad (4.15)$$

So, due to the degree shifts Γ in (4.13) and (4.14), for $n \geq 0$ there is a $\mathbb{R}_+$ -graded $\kappa[\mathbb{R}_+]$ -linear embedding

$$\begin{aligned} \iota_n : PC_n(X, Y) &\hookrightarrow PC_n(X \times Y) \\ \langle \sigma, \tau \rangle &\mapsto T^{l_X(\sigma) + l_Y(\tau) - l_{X \times Y}(\sigma, \tau)} \cdot \langle \sigma, \tau \rangle. \end{aligned} \quad (4.16)$$

Together these form a chain map $\iota : PC_*(X, Y) \hookrightarrow PC_*(X \times Y)$ and hence induce maps on homology

$$\iota_{n*} : PH_n(X, Y) \rightarrow PH_n(X \times Y). \quad (4.17)$$

Consider the relative chain complex

$$\begin{aligned} \overline{PC}_n(X, Y) &:= PC_n(X \times Y) / \iota_n(PC_n(X, Y)) \\ &= \bigoplus_{(\sigma, \tau) \in \mathbf{X}_n \times \mathbf{Y}_n} \Gamma^{l_{X \times Y}(\sigma, \tau)} \kappa[\mathbb{R}_+] / (T^{l_X(\sigma) + l_Y(\tau) - l_{X \times Y}(\sigma, \tau)} \cdot \langle \sigma, \tau \rangle) \end{aligned} \quad (4.18)$$

with homology $\overline{PH}_*(X, Y)$.

Let $q_n : PC_n(X \times Y) \rightarrow \overline{PC}_n(X, Y)$ denote the quotient map implicit in the definition (4.18) and q_{n*} the induced map on homology. There is a connecting homomorphism $\delta_n : \overline{PH}_n(X, Y) \rightarrow PH_{n-1}(X, Y)$ defined in the usual way, which we now recall. Consider a chain $\alpha \in PC_n(X \times Y)$ that represents a cycle in $\overline{PC}_n(X, Y)$ and hence a homology class $[\alpha] \in \overline{PH}_n(X, Y)$. This means that $d(\alpha) \in \iota_{n-1}(PC_{n-1}(X, Y))$. Then define $\delta_n([\alpha]) := [\iota_{n-1}^{-1}(d(\alpha))] \in PH_{n-1}(X, Y)$. It is routine to check that δ_n is independent of the choice of representative α of the class $[\alpha]$, and that it fits into the following long exact sequence. Moreover, this sequence is natural in the arguments X and Y . The result is the following.

Proposition 4.4.4. *For metric spaces (X, d_X) and (Y, d_Y) , there is a long exact sequence*

$$\dots \xrightarrow{\delta_{n+1}} PH_n(X, Y) \xrightarrow{\iota_{n*}} PH_n(X \times Y) \xrightarrow{q_{n*}} \overline{PH}_n(X, Y) \xrightarrow{\delta_n} \dots$$

which is natural with respect to distance non-increasing maps $(X, d_X) \rightarrow (X', d_{X'})$ and $(Y, d_Y) \rightarrow (Y', d_{Y'})$. $\square$

4.4.3 The Künneth Theorem for $d_X + d_Y$ in low dimensions

The purpose of this section is to prove our main Theorem 4.4.5, which is a stronger version of Theorem 4.1.1 from the introduction.

Theorem 4.4.5. *Consider metric spaces (X, d_X) and (Y, d_Y) and the product space equipped with the sum metric $(X \times Y, d_X + d_Y)$.*

Then, the map $\iota_{n} : PH_n(X, Y) \rightarrow PH_n(X \times Y)$ (see (4.17)) is an isomorphism for $n = 0, 1$, and is a surjection for $n = 2$.*

In particular, by Proposition 4.4.3, if X and Y are finite sets, then for $n = 0, 1$ there is a short exact sequence

$$\begin{aligned} 0 \rightarrow \bigoplus_{i+j=n} PH_i(X) \otimes_{\kappa[\mathbb{R}_+]} PH_j(Y) &\rightarrow PH_n(X \times Y) \\ &\rightarrow \bigoplus_{i+j=n-1} Tor_1(PH_i(X), PH_j(Y)) \rightarrow 0 \end{aligned} \quad (4.19)$$

which is natural with respect to distance non-increasing maps $(X, d_X) \rightarrow (X', d_{X'})$ and $(Y, d_Y) \rightarrow (Y', d_{Y'})$. Moreover, this sequence splits, but not naturally. $\square$

Proof. In Lemma 4.4.6 below we prove that $\overline{PH}_n(X, Y) = 0$ for $n = 0, 1, 2$. Hence it follows from the long exact sequence in Proposition 4.4.4 that ι_{n*} is an isomorphism for $n = 0, 1$, and a surjection for $n = 2$, as claimed. The short exact sequence for $n = 0, 1$ follows by replacing $PH_n(X, Y)$ with $PH_n(X \times Y)$ in the short exact sequence (4.12) from Proposition 4.4.3. $\square$

The following lemma is used in the proof of Theorem 4.4.5 above.

Lemma 4.4.6. *For $n = 0, 1, 2$, the homology of the chain complex (4.18) vanishes, that is, $\overline{PH}_n(X, Y) = 0$.*

In particular, the map $\iota_{n} : PH_n(X, Y) \rightarrow PH_n(X \times Y)$ (see (4.17)) is an isomorphism for $n = 0, 1$, and is a surjection for $n = 2$.*

Proof. In dimensions $n = 0$ and $n = 1$, we claim that every simplex $(\sigma, \tau) \in \mathbf{X}_n \times \mathbf{Y}_n$ has the property $l_X(\sigma) + l_Y(\tau) = l_{X \times Y}(\sigma, \tau)$. In dimension 0 this is because the max-length of any 0-simplex is zero. In dimension 1 this is because the max-length is the length of the unique edge in the simplex. Hence, by the definition (4.18) of the relative chain complex, we have $\overline{PC}_n(X, Y) = 0$ and hence $\overline{PH}_n(X, Y) = 0$ for $n = 0, 1$.

We now consider dimension $n = 2$. Since $\overline{PC}_1(X, Y) = 0$, we have

$$\overline{PH}_2(X, Y) = \overline{PC}_2(X, Y) / \text{im}(\bar{d} : \overline{PC}_3(X, Y) \rightarrow \overline{PC}_2(X, Y)).$$

So, we must show that the differential

$$\bar{d} : \overline{PC}_3(X, Y) \rightarrow \overline{PC}_2(X, Y)$$

is surjective. Let $(\sigma, \tau) \in \mathbf{X}_2 \times \mathbf{Y}_2$ and consider the corresponding generator $[\langle \sigma, \tau \rangle] \in \overline{PC}_2(X, Y)$. To show that $[\langle \sigma, \tau \rangle]$ is in the image of $\bar{d}$ we must find some $\alpha \in PC_3(X \times Y)$ such that $d(\alpha) = \langle \sigma, \tau \rangle + r$ for some $r \in \iota_2(PC_2(X, Y))$. This is possible by Lemma 4.4.7 below. Hence $\bar{d}$ is surjective and so $\overline{PH}_2(X, Y) = 0$.

The final statement of the lemma about ι_{n*} for $n = 0, 1, 2$ is an immediate consequence of exactness of the long exact sequence in Proposition 4.4.4 and the vanishing $\overline{PH}_n(X, Y) = 0$ for $n = 0, 1, 2$. $\square$

The following lemma is used in the proof of Lemma 4.4.6 above.

Lemma 4.4.7. *Given a 2-simplex $(\sigma, \tau) \in \mathbf{X}_2 \times \mathbf{Y}_2$, there exists an element $\alpha \in PC_3(X \times Y)$ such that $d(\alpha) = \langle \sigma, \tau \rangle + r$ for some $r \in \iota_2(PC_2(X, Y))$.*

Proof. Say $\sigma = (x_0, x_1, x_2)$ and $\tau = (y_0, y_1, y_2)$. Up to permutations of the indices and switching the roles of x and y , there are four possible cases:

- (i) $x_0 = x_1 = x_2$,
- (ii) $x_0 = x_1 \neq x_2$ and $y_1 = y_2$,
- (iii) $x_0 = x_1 \neq x_2$ and $y_1 \neq y_2$,
- (iv) the x_i are all distinct.

Notice that if $\langle \sigma, \tau \rangle \in \iota_2(PC_2(X, Y))$, then $\alpha := 0$ and $r := -\langle \sigma, \tau \rangle$ satisfies the required conditions. We claim that Case (i) and Case (ii) are of this form.

Indeed, in Case (i) we have $l_X(\sigma) = 0$, and moreover $l_X(\sigma) + l_Y(\tau) = l_Y(\tau) = l_{X \times Y}(\sigma, \tau)$. Hence by definition (4.16) of ι_2 , we have $\langle \sigma, \tau \rangle = \iota_2(\langle \sigma, \tau \rangle) \in \iota_2(PC_2(X, Y))$, as claimed.

Similarly, in Case (ii), observe that $l_{X \times Y}(\sigma, \tau) = d_{X \times Y}((x_0, y_0), (x_2, y_2)) = d_X(x_0, x_2) + d_Y(y_0, y_2) = l_X(\sigma) + l_Y(\tau)$, and so $\langle \sigma, \tau \rangle = \iota_2(\langle \sigma, \tau \rangle) \in \iota_2(PC_2(X, Y))$.

We now consider Case (iii). Consider the 3-simplices

$$\begin{aligned} \sigma' &:= (x_0, x_0, x_1, x_2) \in \mathbf{X}_3, \\ \tau' &:= (y_2, y_0, y_1, y_2) \in \mathbf{Y}_3. \end{aligned}$$

We claim that $\alpha := \langle \sigma', \tau' \rangle \in PC_3(X \times Y)$ satisfies the required conditions. The faces of the 3-simplex $(\sigma', \tau') \in \mathbf{X}_3 \times \mathbf{Y}_3$ are

$$\begin{aligned} f_0 &:= d_0(\sigma', \tau') = (\sigma, \tau), \\ f_1 &:= d_1(\sigma', \tau') = \{(x_0, y_2), (x_1, y_1), (x_2, y_2)\}, \\ f_2 &:= d_2(\sigma', \tau') = \{(x_0, y_2), (x_0, y_0), (x_2, y_2)\}, \\ f_3 &:= d_3(\sigma', \tau') = \{(x_0, y_2), (x_0, y_0), (x_1, y_1)\}. \end{aligned}$$

From the assumption $x_0 = x_1$, it follows that $l_{X \times Y}(\sigma, \tau) = l_{X \times Y}(\sigma', \tau')$. Hence by definition of the differential d on $PC_*(X \times Y)$ we have

$$d(\langle \sigma', \tau' \rangle) = \langle \sigma, \tau \rangle + \sum_{i=1}^3 (-1)^i \cdot T^{l_{X \times Y}(\sigma, \tau) - l_{X \times Y}(f_i)} \cdot \langle f_i \rangle. \quad (4.20)$$

Observe that, since $x_0 = x_1$, up to a permutation of the vertices Case (ii) applies to the simplices f_1 and f_2 and Case (i) applies to the simplex f_3 . Hence for $i = 1, 2, 3$, we have shown above that $\langle f_i \rangle = \iota_2(\langle f_i \rangle)$. Hence the summation term in (4.20) is an element of $i_2(PC_2(X, Y))$, i.e. $d(\langle \sigma', \tau' \rangle) = \langle \sigma, \tau \rangle + r$ where $r \in i_2(PC_2(X, Y))$. This completes the proof of Case (iii).

We now consider Case (iv). By definition of the max-length $l_{X \times Y}(\sigma, \tau)$ and the sum metric $d_{X \times Y} = d_X + d_Y$, we have in particular

$$\begin{aligned} d_X(x_0, x_2) + d_Y(y_0, y_2) &= d_{X \times Y}((x_0, y_0), (x_2, y_2)) \leq l_{X \times Y}(\sigma, \tau), \\ d_X(x_1, x_2) + d_Y(y_1, y_2) &= d_{X \times Y}((x_1, y_1), (x_2, y_2)) \leq l_{X \times Y}(\sigma, \tau). \end{aligned}$$

It follows that either

$$d(x_0, x_2) + d(y_1, y_2) \leq l_{X \times Y}(\sigma, \tau) \quad (4.21)$$

or

$$d(x_1, x_2) + d(y_0, y_2) \leq l_{X \times Y}(\sigma, \tau)$$

holds. Without loss of generality, assume (4.21) holds.

Consider the 3-simplices

$$\begin{aligned} \sigma' &:= (x_0, x_0, x_1, x_2) \in \mathbf{X}_3, \\ \tau' &:= (y_1, y_0, y_1, y_2) \in \mathbf{Y}_3. \end{aligned}$$

The faces of the 3-simplex $(\sigma', \tau') \in \mathbf{X}_3 \times \mathbf{Y}_3$ are

$$\begin{aligned} f_0 &:= d_0(\sigma', \tau') = (\sigma, \tau), \\ f_1 &:= d_1(\sigma', \tau') = \{(x_0, y_1), (x_1, y_1), (x_2, y_2)\}, \\ f_2 &:= d_2(\sigma', \tau') = \{(x_0, y_1), (x_0, y_0), (x_2, y_2)\}, \\ f_3 &:= d_3(\sigma', \tau') = \{(x_0, y_1), (x_0, y_0), (x_1, y_1)\}. \end{aligned}$$

By (4.21), an examination of all edge lengths shows that

$$l_{X \times Y}(\sigma', \tau') = l_{X \times Y}(\sigma, \tau)$$

holds. Hence by definition of the differential d on $PC_*(X \times Y)$ we have

$$d(\langle \sigma', \tau' \rangle) = \langle \sigma, \tau \rangle + \sum_{i=1}^3 (-1)^i \cdot T^{l_{X \times Y}(\sigma, \tau) - l_{X \times Y}(f_i)} \cdot \langle f_i \rangle.$$

Observe that, up to a permutation of the vertices and interchanging the roles of x and y , the simplices f_1, f_2 and f_3 are all of type $(i), (ii)$, or (iii) . So, the lemma says that for $i = 1, 2, 3$, there exists $\alpha_i \in PC_3(X \times Y)$ such that $d(\alpha_i) = \langle f_i \rangle + r_i$ for some $r_i \in \iota_2(PC_2(X, Y))$. We claim that

$$\alpha := \langle \sigma', \tau' \rangle + \sum_{i=1}^3 (-1)^{i+1} \cdot T^{l_{X \times Y}(\sigma, \tau) - l_{X \times Y}(f_i)} \cdot \alpha_i$$

satisfies the required conditions; indeed,

$$\begin{aligned} d(\alpha) &= \langle \sigma, \tau \rangle + \sum_{i=1}^3 \left[(-1)^i \cdot T^{l_{X \times Y}(\sigma, \tau) - l_{X \times Y}(f_i)} \cdot \langle f_i \rangle \right. \\ &\quad \left. + (-1)^{i+1} \cdot T^{l_{X \times Y}(\sigma, \tau) - l_{X \times Y}(f_i)} \cdot d(\alpha_i) \right] \\ &= \langle \sigma, \tau \rangle + \sum_{i=1}^3 (-1)^i \cdot T^{l_{X \times Y}(\sigma, \tau) - l_{X \times Y}(f_i)} \cdot (\langle f_i \rangle - d(\alpha_i)) \\ &= \langle \sigma, \tau \rangle + \sum_{i=1}^3 (-1)^i \cdot T^{l_{X \times Y}(\sigma, \tau) - l_{X \times Y}(f_i)} \cdot (-r_i). \end{aligned}$$

This completes the proof of Case (iv) , and hence the lemma. $\square$

4.4.4 Interleaving distance

We bound the interleaving distance between the module $PH_*(X, Y)$ defined in (4.11), which we can compute using the split short exact sequence in Proposition 4.4.3, and the homology $PH_*(X \times Y)$ of the metric space $(X \times Y, d_X + d_Y)$ that we would like to compute; see Theorem 4.4.9. Leonid Polterovich suggested we look for a bound of this type.

Here we view $\kappa[\mathbb{R}_+]$ -modules as persistence vector spaces ($t \mapsto PH_*^{(t)}(X, Y)$) and ($t \mapsto PH_*^{(t)}(X \times Y)$); see Remark 4.2.1. The interleaving distance is a quantitative measure of how far away the computable module $PH_*(X, Y)$ is from the true persistent homology $PH_*(X \times Y)$ of the product $(X \times Y, d_X + d_Y)$. The long exact sequence in Proposition 4.4.4 can be viewed as an algebraic measure of the difference between these modules.

The general definition of interleaving distance between persistence vector spaces is as follows. The interleaving distance was introduced in [7], and has since become a standard measure of distance between persistence vector spaces.

Definition 4.4.8. Consider persistence vector spaces $V = \{V_t\}_{t \in \mathbb{R}_+}$ and $W = \{W_t\}_{t \in \mathbb{R}_+}$ with structure maps $l_{t,t'} : V_t \rightarrow V_{t'}$ and $s_{t,t'} : W_t \rightarrow W_{t'}$, respectively, for $t \leq t'$.

For $\delta \geq 0$, we say that V and W are δ -interleaved if there exist linear maps

$$\begin{aligned} F_t : V_t &\rightarrow W_{t+\delta}, \\ G_t : W_t &\rightarrow V_{t+\delta} \end{aligned}$$

such that, for all $t \leq t'$, we have

$$l_{t+\delta,t'+\delta} \circ G_t = G_{t'} \circ s_{t,t'}, \quad s_{t+\delta,t'+\delta} \circ F_t = F_{t'} \circ l_{t,t'},$$

and

$$G_{t+\delta} \circ F_t = l_{t,t+2\delta}, \quad F_{t+\delta} \circ G_t = s_{t,t+2\delta}.$$

The **interleaving distance** between V and W is the infimum over all $\delta \geq 0$ such that V and W are δ -interleaved.

Theorem 4.4.9. The interleaving distance between the persistence vector spaces

$$(t \mapsto PH_*^{(t)}(X, Y)) \text{ and } (t \mapsto PH_*^{(t)}(X \times Y))$$

is less than or equal to the minimum of the diameters

$$\min(\text{diameter}(X), \text{diameter}(Y)),$$

where the diameter is the maximum distance

$$\text{diameter}(X) = \max\{d_X(x_0, x_1) \mid x_0, x_1 \in X\}.$$

Proof. Let $\delta := \min(\text{diameter}(X), \text{diameter}(Y))$ denote the desired bound on the interleaving distance. We must show that the persistence vector spaces $(t \mapsto PH_*^{(t)}(X, Y))$ and $(t \mapsto PH_*^{(t)}(X \times Y))$ are δ -interleaved. Denote the structure maps on these persistence vector spaces by $\bar{l}_{t,t'} : PH_*^{(t)}(X, Y) \rightarrow PH_*^{(t')}(X, Y)$ and $\bar{s}_{t,t'} : PH_*^{(t)}(X \times Y) \rightarrow PH_*^{(t')}(X \times Y)$ for $t \leq t'$. Note that the structure maps $\bar{l}_{t,t'}$ and $\bar{s}_{t,t'}$ on homology are induced by the inclusions of chain complexes $l_{t,t'} : PC_*^{(t)}(X, Y) \rightarrow PC_*^{(t')}(X, Y)$ and $s_{t,t'} : PC_*^{(t)}(X \times Y) \rightarrow PC_*^{(t')}(X \times Y)$.

We claim it suffices to show that the persistence vector spaces $(t \mapsto PC_*^{(t)}(X, Y))$ and $(t \mapsto PC_*^{(t)}(X \times Y))$ are δ -interleaved with respect to chain maps

$$\begin{aligned} F^{(t)} : PC_*^{(t)}(X, Y) &\rightarrow PC_*^{(t+\delta)}(X \times Y) \\ G^{(t)} : PC_*^{(t)}(X \times Y) &\rightarrow PC_*^{(t+\delta)}(X, Y). \end{aligned}$$

(The structure maps on these persistence vector spaces are the inclusions $l_{t,t'}$ and $s_{t,t'}$.) Indeed, since $F^{(t)}$ and $G^{(t)}$ are chain maps, they induce maps on homology $\bar{F}^{(t)} : PH_*^{(t)}(X, Y) \rightarrow$

$PH_*^{(t+\delta)}(X \times Y)$ and $\overline{G}^{(t)} : PH_*^{(t)}(X \times Y) \rightarrow PH_*^{(t+\delta)}(X, Y)$, and so the required properties of a δ -interleaving

$$\begin{aligned} \overline{G}^{(t+\delta)} \circ \overline{F}^{(t)} &= \bar{l}_{t,t+2\delta}, & \overline{F}^{(t+\delta)} \circ \overline{G}^{(t)} &= \bar{s}_{t,t+2\delta}, \\ \bar{l}_{t+\delta,t'+\delta} \circ \overline{G}^{(t)} &= \overline{G}^{(t')} \circ \bar{s}_{t,t'}, & \bar{s}_{t+\delta,t'+\delta} \circ \overline{F}^{(t)} &= \overline{F}^{(t')} \circ \bar{l}_{t,t'}, \end{aligned}$$

follow from the corresponding properties on the chain level.

We now define the maps $F^{(t)}$ and $G^{(t)}$ and show that they form a δ -interleaving on the chain level. By definition we have

$$\begin{aligned} PC_*^{(t)}(X, Y) &= \bigoplus_{l_X(\sigma) + l_Y(\tau) \leq t} \kappa \cdot \langle \sigma, \tau \rangle, \\ PC_*^{(t)}(X \times Y) &= \bigoplus_{l_{X \times Y}(\sigma, \tau) \leq t} \kappa \cdot \langle \sigma, \tau \rangle. \end{aligned}$$

Since we have $l_{X \times Y} \leq l_X + l_Y$ (see (4.15)), it is immediate that $l_{X \times Y} \leq l_X + l_Y + \delta$ holds, and so we can define $F^{(t)}$ be the natural inclusion that takes generators to generators $F^{(t)}(\langle \sigma, \tau \rangle) = \langle \sigma, \tau \rangle$.

Towards defining $G^{(t)}$, we first establish the bound

$$l_{X \times Y}(\sigma, \tau) \geq \max(l_X(\sigma), l_Y(\tau)).$$

Given simplices $\sigma = (x_0, \dots, x_n) \in \mathbf{X}_n$ and $\tau = (y_0, \dots, y_n) \in \mathbf{Y}_n$, there are integers $i, j \in \{0, \dots, n\}$ such that $l_X(\sigma) = d_X(x_i, x_j)$, and similarly there are i', j' such that $l_Y(\tau) = d_X(y_{i'}, y_{j'})$, by definition (4.10) of the filtration functions l_X and l_Y . Hence by definition (4.10) of $l_{X \times Y}$ and the sum metric $d_{X \times Y} = d_X + d_Y$, we have

$$\begin{aligned} l_{X \times Y}(\sigma, \tau) &= \max\{d_{X \times Y}((x_a, y_a), (x_b, y_b)) \mid 0 \leq a, b \leq n\} \\ &= \max\{d_X(x_a, x_b) + d_Y(y_a, y_b) \mid 0 \leq a, b \leq n\} \\ &\geq \max\{d_X(x_i, x_j), d_X(y_{i'}, y_{j'})\} \\ &= \max(l_X(\sigma), l_Y(\tau)), \end{aligned}$$

as claimed. From this, we derive the bound

$$\begin{aligned} l_X(\sigma) + l_Y(\tau) - l_{X \times Y}(\sigma, \tau) &\leq l_X(\sigma) + l_Y(\tau) - \max(l_X(\sigma), l_Y(\tau)) \\ &= \min(l_X(\sigma), l_Y(\tau)) \\ &\leq \delta. \end{aligned}$$

Hence we have $l_X + l_Y \leq l_{X \times Y} + \delta$, and so we can define $G^{(t)}$ to be the natural inclusion $G^{(t)}(\langle \sigma, \tau \rangle) = \langle \sigma, \tau \rangle$.

It is immediate that the required properties of a δ -interleaving hold, since the structure maps $l_{t,t'}$ and $s_{t,t'}$ as well as the interleaving maps $G^{(t)}$ and $F^{(t)}$ are all the natural inclusions of chain complexes defined on generators by $\langle \sigma, \tau \rangle \mapsto \langle \sigma, \tau \rangle$. This completes the proof. $\square$

4.4.5 Example: The Hamming Cube I^k

Let $I = \{0, 1\}$ denote the metric space consisting of 2 points at distance 1. Then for any integer $k \geq 1$, the Cartesian product I^k with the sum metric is the *Hamming k -cube*, consisting of all k -tuples of zeros and ones with distance given by the number of coordinates in which tuples differ.

The purpose of this section is to make the following computation using Theorem 4.4.5. In particular, we carry out a technical computation of the nontrivial kernel of the surjection $\iota_{2*} : PH_2(X, Y) \rightarrow PH_2(X \times Y)$ in the case $X = I$ and $Y = I^{k-1}$, $k \geq 3$.

Proposition 4.4.10. *For $k \geq 1$, the Hamming cube $I^k = \{0, 1\}^k$ satisfies the following:*

$$\begin{aligned} PH_0(I^k) &\cong (0, 1)^{2^k-1} \oplus (0, \infty), \\ PH_1(I^k) &\cong (1, 2)^{k \cdot 2^{k-1} - (2^k - 1)}, \\ PH_2(I^k) &= 0. \end{aligned}$$

Proof. See Proposition 4.4.12 and Proposition 4.4.13. □

We introduce the following useful terminology for a standard coordinate embedding $I^r \hookrightarrow I^k$.

Definition 4.4.11. *Given $0 \leq r \leq k$, a point $\xi \in I^{k-r}$, and any choice of r coordinates $1 \leq i_1 < i_2 < \dots < i_r \leq k$ of I^k , there is an isometric embedding*

$$\varphi : I^r \rightarrow I^k,$$

where for $x \in I^r$ the r chosen coordinates of $\varphi(x)$ are equal to x , i.e.

$$x = (x_1, \dots, x_r) = (\varphi(x)_{i_1}, \dots, \varphi(x)_{i_r}),$$

*and the other $k - r$ coordinates of $\varphi(x)$ are given by ξ . We call these maps φ the **coordinate inclusions of I^r into I^k** .*

For $r \leq k$, let $C(r, k)$ denote the set of coordinate inclusions $I^r \rightarrow I^k$.

Proposition 4.4.12. *For $k \geq 1$, we have*

$$\begin{aligned} PH_0(I^k) &\cong (0, 1)^{2^k-1} \oplus (0, \infty), \\ PH_1(I^k) &\cong (1, 2)^{k \cdot 2^{k-1} - (2^k - 1)}. \end{aligned}$$

Moreover, $PH_1(I^k)$ is generated by the collection of images of the induced maps $PH_1(I^2) \rightarrow PH_1(I^k)$ of the coordinate inclusions $I^2 \rightarrow I^k$. Precisely, we have

$$PH_1(I^k) = \langle \bigcup_{\varphi \in C(2, k)} \varphi_*(PH_1(I^2)) \rangle.$$

Proof. Recall from Remark 4.4.2 that for $t \in \mathbb{R}_+$ there is an isomorphism $PH_*^{(t)}(I^k) \cong H_*(V(I^k, t))$, where $PH_*^{(t)}(I^k)$ is the homogeneous degree t persistent homology and $V(I^k, t)$ is the Vietoris-Rips complex of I^k at time t .

A simple calculation of $PH_0(I^k)$ follows from the observation that $V(I^k, 0)$ is a simplicial complex consisting of 2^k points and no higher dimensional simplices, and $V(I^k, 1)$ is connected. Alternatively, $PH_0(I^k)$ can be computed inductively using the Künneth Theorem 4.4.5 for metric spaces applied to $I^k = I \times I^{k-1}$, together with the observation $PH_0(I) \cong (0, 1) \oplus (0, \infty)$ and the formulas for tensor product and *Tor* of bars in Proposition 4.2.6.

We now compute $PH_1(I^k)$. The claimed formula holds for $k = 1$ since $PH_1(I) = 0$. We now prove the formula for $k > 1$, assuming inductively that it holds for $k - 1$. Indeed, by Theorem 4.4.5 applied to $X = I$ and $Y = I^{k-1}$ together with the formulas for tensor product and *Tor* of bars in Proposition 4.2.6, we compute

$$\begin{aligned} PH_1(I^k) &\cong (PH_0(I) \otimes PH_1(I^{k-1})) \oplus Tor_1(PH_0(I), PH_0(I^{k-1})) \\ &\cong (1, 2)^{(k-1) \cdot 2^{k-1} - (2^k - 2)} \oplus (1, 2)^{2^{k-1} - 1} \\ &= (1, 2)^{k \cdot 2^{k-1} - (2^k - 1)}, \end{aligned}$$

as claimed.

It remains to prove that $PH_1(I^k)$ is generated by the images of the induced maps of the coordinate inclusions $I^2 \rightarrow I^k$. From the computed formula for $PH_1(I^k)$, we see that it is generated by homogeneous degree 1 elements. Hence it suffices to show that $PH_1^{(1)}(I^k)$ is generated by the images of the induced maps $PH_1^{(1)}(I^2) \rightarrow PH_1^{(1)}(I^k)$. This is the same as showing that $H_*(V(I^k, 1))$ is generated by the images of the induced maps $H_*(V(I^2, 1)) \rightarrow H_*(V(I^k, 1))$.

The simplicial complex $V(I^k, 1)$ is easily describable. The 0-simplices are all the points in I^k . The 1-simplices are all pairs $\{x, y\}$ such that $d_{I^k}(x, y) = 1$. There is a canonical identification $I^k \cong \mathbb{F}_2^k$ where $\mathbb{F}_2 = \{0, 1\}$ is the finite field with 2 elements. Under this identification, the set of 1-simplices in $V(I^k, 1)$ is given by all pairs $\{x, x + e_i\}$ for $x \in I^k$ and $1 \leq i \leq k$, where $e_i \in \mathbb{F}_2^k$ is the i -th standard basis vector. We claim that there are no higher dimensional simplices in $V(I^k, 1)$. Indeed, suppose for contradiction that $\{x, x + e_i, y_1, \dots, y_n\} \subset I^k$ is a simplex in $V(I^k, 1)$ for some $n \geq 1$. Then $d_{I^k}(y_1, x) = 1$ implies $y_1 = x + e_j$ for some standard basis vector e_j . If $j = i$ then $y_1 = x + e_i$, which is not allowed, and if $j \neq i$ then $d_{I^k}(y_1, x + e_i) = 2$, which is not allowed in $V(I^k, 1)$.

We identify the Vietoris-Rips complexes with their geometric realizations. The complex $V(I^2, 1)$ is homeomorphic to S^1 . So, to show that $H_1(V(I^k, 1))$ is generated by the induced maps $H_1(V(I^2, 1)) \rightarrow H_1(V(I^k, 1))$ of the coordinate inclusions $I^2 \rightarrow I^k$, it suffices to show that if we glue a disk D^2 along the induced map on Vietoris-Rips $\varphi_V : \partial D^2 \cong V(I^2, 1) \rightarrow V(I^k, 1)$ for every $\varphi \in C(2, k)$, then every loop in $V(I^k, 1)$ is nullhomotopic in this new space.

Precisely, we form the space

$$X(k) := V(I^k, 1) \cup \bigcup_{\varphi \in C(2, k)} D^2,$$

where for each $\varphi \in C(2, k)$ we identify $\partial D^2 \cong V(I^2, 1)$ with its image $\varphi_V(V(I^2, 1)) \subset V(I^k, 1)$.

It suffices to show that every loop in $V(I^k, 1)$ is nullhomotopic in $X(k)$. This clearly holds for $k = 1$ since $X(k) = V(I^k, 1) = [0, 1]$ is contractible.

We proceed by induction. Let $k > 1$ and assume that every loop in $V(I^{k-1}, 1)$ is nullhomotopic in $X(k-1)$.

Observe from the above description of the 1-simplices in $V(I^k, 1)$ that we have an inclusion

$$\begin{aligned} V(I^k, 1) &= (\{0\} \times V(I^{k-1}, 1)) \cup ([0, 1] \times I^{k-1}) \cup (\{1\} \times V(I^{k-1}, 1)) \\ &\subset [0, 1] \times V(I^{k-1}, 1). \end{aligned}$$

Moreover, there is an inclusion

$$[0, 1] \times V(I^{k-1}, 1) \hookrightarrow X(k)$$

defined in the following way. Every 1-simplex σ in $V(I^{k-1}, 1)$ has boundary points $\{x, x + e_i\}$ for some $x \in I^{k-1}$ and standard basis vector $e_i \in I^{k-1}$, and so together with the first coordinate of $I^k = I \times I^{k-1}$ the simplex σ determines a coordinate inclusion $\varphi_\sigma : I^2 \rightarrow I^k$, where the 2 chosen coordinates of I^k are the first one and the coordinate corresponding to $e_i \in I^{k-1}$, and the $\xi \in I^{k-2}$ is given by x in the other coordinates. Then the inclusion $[0, 1] \times V(I^{k-1}, 1) \hookrightarrow X(k)$ is defined by identifying $[0, 1] \times \sigma$ with the copy of D^2 corresponding to φ_σ .

Consider a loop $S^1 \rightarrow V(I^k, 1)$. This loop is homotopic in $[0, 1] \times V(I^{k-1}, 1)$ to a loop in $\{0\} \times V(I^{k-1}, 1)$. After gluing on the disks corresponding to the coordinate inclusions $I^2 \rightarrow I^k$ that land in $\{0\} \times I^{k-1}$, our loop is in the space $\{0\} \times X(k-1)$, where it is nullhomotopic by the induction hypothesis. This completes the proof. $\square$

Proposition 4.4.13. *For $k \geq 1$, we have $PH_2(I^k) = 0$.*

Proof. By direct computation, one can check that the result holds for $k \leq 3$. We now proceed by induction. Let $k > 3$ and assume $PH_2(I^{k-1}) = 0$. By Theorem 4.4.5, the map $\iota_{2*} : PH_2(I, I^{k-1}) \rightarrow PH_2(I^k)$ is surjective, so it suffices to show that $\iota_{2*} = 0$.

Let $\varphi : I^2 \rightarrow I^{k-1}$ be any coordinate inclusion. Note that $PH_n(I) = 0$ for $n > 0$. Then by naturality of the short exact sequence (4.12) and the long exact sequence from Proposition 4.4.4, we have a commutative diagram

$$\begin{array}{ccccc} PH_2(I^3) & \xleftarrow{\iota_{2*}} & PH_2(I, I^2) & \xrightarrow{\alpha} & Tor_1(PH_0(I), PH_1(I^2)) \\ \downarrow & & \downarrow & & \downarrow id_{I*} \otimes \varphi_* \\ PH_2(I^k) & \xleftarrow{\iota_{2*}} & PH_2(I, I^{k-1}) & \xrightarrow{\alpha} & Tor_1(PH_0(I), PH_1(I^{k-1})). \end{array}$$

Moreover, since $PH_n(I) = 0$ for $n > 0$ and since by induction we have $PH_2(I^{k-1}) = 0$, it follows from (4.12) that both maps labeled α are isomorphisms.

Since $PH_0(I) \cong (0, 1) \oplus (0, \infty)$, by Proposition 4.2.6 we have

$$Tor_1(PH_0(I), PH_1(I^{k-1})) \cong Tor_1((0, 1), PH_1(I^{k-1})).$$

A free resolution of the bar $(0, 1)$ is given by

$$0 \rightarrow (1, \infty) \rightarrow (0, \infty) \rightarrow (0, 1) \rightarrow 0.$$

Now consider the bar $(1, 2)$. Dropping the $(0, 1)$ term from the free resolution and tensoring with $(1, 2)$ yields (by Proposition 4.2.6) the sequence

$$0 \rightarrow (2, 3) \rightarrow (1, 2) \rightarrow 0.$$

Observe in particular that the map $(2, 3) \rightarrow (1, 2)$ is necessarily zero. Hence, since $PH_1(I^{k-1})$ is a direct sum of bars $(1, 2)$ by Proposition 4.4.12, we have

$$Tor_1((0, 1), PH_1(I^{k-1})) = (1, \infty) \otimes PH_1(I^{k-1}).$$

Hence we have shown that we have an isomorphism

$$Tor_1(PH_0(I), PH_1(I^{k-1})) \cong (1, \infty) \otimes PH_1(I^{k-1}).$$

Under this isomorphism, the map $id_{I*} \otimes \varphi_*$ in the commutative diagram above corresponds to the map $id_{(1, \infty)} \otimes \varphi_*$ in the following commutative diagram, where we have also used $PH_2(I^3) = 0$.

$$\begin{array}{ccccc} 0 & \longleftarrow & PH_2(I, I^2) & \xrightarrow{\cong} & (1, \infty) \otimes PH_1(I^2) \\ \downarrow & & \downarrow & & \downarrow id_{(1, \infty)} \otimes \varphi_* \\ PH_2(I^k) & \xleftarrow{\iota_{2*}} & PH_2(I, I^{k-1}) & \xrightarrow{\cong} & (1, \infty) \otimes PH_1(I^{k-1}). \end{array}$$

Now, there is a generating set of $PH_2(I, I^{k-1})$ corresponding under the isomorphism with $(1, \infty) \otimes PH_1(I^{k-1})$ to the generating set of $PH_1(I^{k-1})$ described in Proposition 4.4.12 (tensored with the generator of $(1, \infty)$), which consists of the images of the induced maps $\varphi_* : PH_1(I^2) \rightarrow PH_1(I^{k-1})$ of the coordinate inclusions $\varphi : I^2 \rightarrow I^{k-1}$. For any such coordinate inclusion φ , commutativity of the above diagram shows that the image of $(1, \infty) \otimes PH_1(I^2)$ in $PH_2(I^k)$ is 0. Hence $\iota_{2*} = 0$. $\square$

4.5 Appendix: simplicial sets

We review basic notions about simplicial sets.

Let $\underline{\Delta}$ denote the *simplex category*. It has objects $[n] = \{0, \dots, n\}$ for $n \geq 0$ and morphisms the weakly order preserving maps, i.e. $\varphi : [m] \rightarrow [n]$ is a morphism in $\underline{\Delta}$ if it satisfies $\varphi(0) \leq \varphi(1) \leq \dots \leq \varphi(m)$.

A *simplicial set* is a contravariant functor $\mathbf{X} : \underline{\Delta} \rightarrow \text{Set}$. Concretely, $\mathbf{X}$ consists of a set $\mathbf{X}_n$ for each $n \geq 0$ together with a set map $\mathbf{X}_\varphi : \mathbf{X}_n \rightarrow \mathbf{X}_m$ for each morphism $\varphi : [m] \rightarrow [n]$ in $\underline{\Delta}$ such that the collection of these maps satisfy the usual functorial properties.

The category of simplicial sets $s\text{Set}$ has simplicial sets as objects and natural transformations as morphisms. The morphisms are called *simplicial maps*.

There are special morphisms in $\underline{\Delta}$ called the *coface maps*

$$\begin{aligned} d_i^\Delta : [n] &\rightarrow [n+1] \\ j &\mapsto j \text{ if } j < i, \\ j &\mapsto j+1 \text{ if } j \geq i \end{aligned}$$

for $0 \leq i \leq n+1$ and the *codegeneracy maps*

$$\begin{aligned} s_i^\Delta : [n] &\rightarrow [n-1] \\ j &\mapsto j \text{ if } j \leq i, \\ j &\mapsto j-1 \text{ if } j > i. \end{aligned}$$

for $0 \leq i \leq n-1$. These generate the morphisms in $\underline{\Delta}$ in the sense that any morphism can be written as the composition of coface and codegeneracy maps.

Given $\mathbf{X} \in s\text{Set}$, the maps $\mathbf{X}_{d_i^\Delta}$ are called *face maps* and the maps $\mathbf{X}_{s_i^\Delta}$ are called *degeneracy maps*. Often we denote $\mathbf{X}_{d_i^\Delta}$ by d_i and $\mathbf{X}_{s_i^\Delta}$ by s_i when the simplicial set $\mathbf{X}$ is implicit.

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