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A Cognitive Model of Discovering Commutativity

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Abstract

Mathematics is often seen as the uncovering of eternal truths that exist independently of the human mind. However, even if this epistemological view is correct, the mathematics that humans can know can only be the result of cognitive processes. We investigate this ability of the human mind to make mathematical discoveries. More precisely, we present a cognitive model of how the ability to use metaphors and analogies plays a key role in such discoveries. As a proof of concept we present an ACT-R model that uses path-mapping and that is capable of discovering the commutativity property of addition.

Keywords: analogy; metaphor; mathematics; scientific discovery; cognitive modelling.

Mathematical Discoveries

The Cognition of Mathematics

The way in which people construct, evaluate and modify mathematical concepts has received relatively little attention from cognitive science. Likewise, automated mathematical theory formation has so far put little emphasis on cognitively plausible mechanisms. In the *Wheelbarrow* project, we are, therefore, working towards a cognitive theory of mathematical thought in order to substantiate the existing theories and to improve automated theory formation systems.

We build on two streams of research: embodied conceptualisation, which analyses mathematical ideas from a cognitive perspective (Lakoff & Núñez, 2000), and societal conceptualisation based on Lakatos's (1976) philosophical account of the evolution of mathematical ideas. Both argue strongly against the 'romantic' (Lakoff and Núñez) or 'deductivist' (Lakatos) style in which mathematics is presented as an ever-increasing set of universal, absolute, certain truths which exist independently of humans.

Our main interest is how mathematical concepts are formed and modified by the embodied and situated human mind. For instance, Euclid formulated geometric axioms to describe the physical world – the foundations of Euclidean geometry. Euclidean geometry was later modified by rejecting the *parallel postulate* (one of the axioms), and non-Euclidean geometries were formed, along with new sets of concepts. On a less celebrated but equally remarkable level children are able to formulate and modify mathematical rules about their environment such as transitivity or the commutativity in arithmetic. Lakoff and Núñez's theory of embodied mathematics and Lakatos's philosophy of mathematics suggest explanation of how this may work.

The Role of Metaphors

In this paper, we focus on the approach by Lakoff and Núñez (2000). They propose that the human embodied mind brings mathematics into being. That is, human mathematics is grounded in the bodily experience of a physical world, and mathematical entities inherit properties of objects in the world, such as being consistent or stable over time. Via exploration of the physical world we build up mini-domains, which we then map to abstract mathematical domains, allowing us to make inferences in the abstract world by transferring knowledge about the physical world. The main process enabling humans to make this transfer is the ability to use metaphors.

Metaphors and analogies in mathematics have so far been mainly documented by educators (for example, English, 1997; Sfard, 1996). Despite the importance of metaphors and analogies for discovering new concepts in mathematics, historians and philosophers of mathematics, and mathematicians themselves have tended to be silent on the matter, with notable exceptions such as Lakatos (1976, p 9; who recommends embedding a conjecture in a distant body of knowledge, eg a conjecture about solids in the theory of rubber sheets), Polya (1954, p 15–22; who describes and analyses Euler's application of rules for finite equations to infinite equations) and Weil (see Krieger, 2003; who discusses a number of fruitful historical mathematical analogie.s)

Metaphors and Detecting New Scientific Concepts

Lakoff and Núñez (2000) argue that our ordinary conceptual system is fundamentally metaphorical in nature: metaphor makes abstract thought possible, and the development of thought is the process of developing better metaphors. They characterise metaphors as a 'grounded, inference-preserving cross domain mapping' (p 6), thus enabling us to use the inferential structure of one domain to reason about another. They catalogue a large number of mathematical metaphors, thus suggesting how highly abstract mathematical ideas may be discovered and understood, and how they can be traced back to human embodiment.

Lakoff and Núñez show how conceptual metaphors are revealed by everyday language, eg the expression *adding onions to soup*, suggests that *add* can mean physically placing objects in a container. They place great emphasis on the type of domains in a metaphor and distinguish two types of metaphor: *grounding* metaphors, in which one domain is embodied and the other abstract, and *linking* metaphors, in which both domains are abstract. Many linking metaphors in mathematics conceptualise some domain of mathematics in terms of arithmetic, ie arithmetic is used as a source for many mathematical metaphors, including points in space, spaces of any number of dimensions and functions (p 387). Lakoff and Núñez's four different physical domains of Object Collection, Object Construction, Measuring Stick and Motion Along A Path all map to the domain of arithmetic, thus showing a many-to-one mapping between the domains (chapter 3). An example of such many-to-one mappings is *addition*, which is the target of a mapping from putting collections together, putting objects together with other objects to form larger objects, putting physical segments together end-to-end to form longer physical segments and moving a distance away from an origin location.

Examples of one-to-many mappings include the domain of *sets*, which is mapped to *ordered pairs* (p 150), *natural numbers* (p 150) and *naturally continuous space with point locations* (p 263). In these examples different components of the source domain *sets* are identified and mapped, suggesting that the domains are not fully pre-represented but consist in different, interrelated schemata where the most relevant schema is selected or constructed for each new metaphor. This is in line with Hofstadter's (1994) argument that finding a good metaphor relies on extrapolating the currently useful 'gist' from a domain (which depends on the purpose of the metaphor). These ideas go back to William James's (1890) claim that there is no property that is absolutely essential to one thing.

Metaphor and Analogy

Metaphors are similar to analogies. Gentner (1983; see also Gentner & Markman, 1997, p 48) proposes that when comparing two concepts we can distinguish whether there is an analogy, a metaphor, a literal similarity or a mere appearance similarity by looking at the number of relations and properties that the two concepts have in common, see table 1. These notions can, thus, be placed in a two-dimensional space. According to this classification there is no binary distinction between analogy and metaphor but only a difference in degree.

Making these distinctions assumes that source and target domains are represented in terms of objects and predicates. Theories of mapping usually rest on the distinction between attributes, predicates with one argument (to describe object properties) and relations (predicates with more arguments to describe interactions between objects). This is a fuzzy distinction even in natural language, but perhaps even more so in mathematics: for instance, *polyhedron* might be the relationship between the objects *edge*, *face* and *vertex*, an attribute of a specific shape or an object itself.

Lakoff and Núñez do not explicitly make this distinction. However, translating their natural language examples of metaphor, some favour relation over attribute mappings, placing these examples close to analogy. For instance, in the Infinite Sums Are Limits Of Infinite Sequences Of Partial Sums metaphor (p 197), in which limits of infinite sequences are compared to infinite sums, limits of sequences are mapped. Given the similarity between metaphor and analogy, we use a cognitive model of creating analogies: path-mapping.

Table 1: Metaphors and analogies. (Abstraction differs from analogy in that it has only few object attributes in the source as well as the target domain.)

	mapped attributes	mapped relations
literal similarity	many	many
appearance	many	few
analogy	few	many
metaphor	some	some
abstraction	few	many
anomaly	few	few

Modelling Scientific Discovery

Building computational cognitive models of scientific discovery is difficult, because it requires the building of comprehensive models that comprise many different cognitive abilities. Nevertheless, such models have been built with some success, for example by Schunn and Anderson (1998). They conducted experiments and built a model of discovering the fan effect[1] in psychology. However, in order to be able to control their experiments (to allow a statistical analysis of the collected data) they strictly limited the participants' options – something very much unlike real scientific work. Furthermore, their participants were given a preexisting scientific hypothesis. (The hypothesis was set as the goal of the cognitive model.) The task was to confirm or disprove the hypothesis. This is different from the step that we are addressing here, namely the formulation or discovery of a new hypothesis.

Related cognitive models also include algebra learning (Anderson, 2007). However, these are models of learning from instruction not of discovery. Thus, they are models of the effects of increased practise, ie they describe the speedup of solving algebraic equations with increased practise.

A Case Study: Discovering Commutativity

Why Commutativity?

As a first case study we built a computational cognitive model of discovering commutativity for addition, ie the fact that the result of an addition is not affected by the order of the addends ($1+2=2+1$). This seems an appropriate example, because this rather elementary (albeit not trivial) discovery does not require an extraordinary mathematical gift or a particular constellation in mathematical history but has almost certainly been made many times independently and spontaneously. The main cognitive process driving this discovery is analogy (or metaphor). As this process has already been studied in cognitive science it lends itself to cognitive modelling. However, we do not propose that it is the only way in which discoveries

[1]The fan effect is the observation that the time for memory recall increases with the number of relevant facts. Thus, the more relevant knowledge is available, the longer the retrieval of a fact takes. One explanation is that the facts compete to be retrieved, and that resolving this competition takes longer the more facts are available.

are made; insight can take many other forms.

Preconditions

Discovering commutativity requires a number of cognitive abilities and some preexisting knowledge. Because the make or break condition for developing a cognitive model of discovery is to make sure that what the model discovers is not something that the modeller already built in, we will describe the cognitive abilities that can rather safely be assumed to exist when a person (be it a mathematician or a second-grader) discovers commutativity and take this as starting point.

By linking the discovery of certain mathematical facts to using particular metaphors, Lakoff and Núñez (2000) provide us with a useful starting point. They list a number of cognitive abilities required to make a discovery like commutativity.

Subitising and cardinality Psychological experiments with young babies show that there is an innate ability to distinguish object collections of different sizes. This *subitising* ability enables babies (and older humans) to immediately identify the cardinality of small object collections of sizes up to 3 or 4. In the classic experimental setup babies see a display of objects within the subitising range. A screen is placed between the objects and the baby, so that the objects are hidden from view. Then an object is moved behind the screen (added) or appears from behind the screen (subtracted). When the screen is removed again and the resulting number of objects is as expected (say, starting with 2 objects, removing 1 object and revealing 1 object) the babies show no surprise. If the result is unexpected (say, starting with 2 objects, removing 1 object and revealing 2 objects) the babies do show surprise, eg by staring at the objects for a longer period of time or by sucking on a pacifier with an increased frequency. See Lakoff and Núñez (2000, p 18) for a discussion. Even though humans usually have developed an abstract notion of number by the time they are able to discover commutativity, it gives our model a firm footing.

The screen experiments also show that babies have a notion of *cardinality*. In versions of the experiment where, for example, puppets were changed to balls, the babies did respond to differences in the number of objects, but not to differences in the identity of objects (see, Lakoff & Núñez, 2000, p 18).

Arithmetic Is Object Collection Metaphor Lakoff and Núñez (2000) argue that the *Arithmetic Is Object Collection* metaphor (the ability to understand arithmetic operations in terms of manipulating collections of physical objects) is already available to the student when formal education in mathematics begins. They point out that many techniques in teaching arithmetic assume that this metaphor is available. For the case of commutativity this means that the student already knows that the cardinality of an object collection is not affected by the order in which smaller object collections are put together to form it.

Argument Ordering In addition to the knowledge that commutativity holds for object collections the student/model must know that the order of arguments of an operation can affect the result (as it does in removing objects from an object collection). The ordering of arguments is a property of the path-mapping model (see below), where the *roles* that objects fill in relations are explicitly represented.

Mapping Knowledge In addition to having the knowledge the model must also have the ability to create analogies/metaphors that make use of the existing knowledge. For this, we use the path-mapping algorithm.

Symmetry detector We finally assume that the model has a subprocess (a production in our model) that is specialised on detecting the symmetry of roles in path-mapping. Ferguson (1994) provides an account of the role of symmetry in such tasks. This certainly is the strongest assumption, but given the omnipresence of symmetry in the cognition of humans and animals, it seems justified. It ranges from mate selection, where symmetry correlates with mate quality (eg Manning, Scutt, Whitehouse, & Leinster, 1997), to spatial cognition, eg Silverman (2002) makes this point.

Path-Mapping

Overview

The path-mapping model of analogy formation (Salvucci & Anderson, 1998, 2001) is a cognitive model of how an interconnected substructure of a knowledge representation (the source domain) is mapped onto another substructure (the target domain). It is essentially a structure mapping model (Gentner & Markman, 1997) that includes a more detailed account of the cognitive mapping processes. For example, when generating analogies the system could try to map infinitely many possible relations between source and target domain. For an agent acting in the real world, it is computationally too costly to explore all of them, because such agents operate under real time constraints.

Salvucci and Anderson use the ACT-R cognitive architecture, which provides a framework of the invariant or slowly changing parts of cognition, so that these (1) do not need to be implemented again for each model and (2) can provide explanations of parts of the phenomenon by reducing them to already well-established facts about cognition. Put simply, ACT-R is a production system with (1) a model of human declarative memory, (2) a subsymbolic layer and (3) modules for perception and motor control. The problem of the large number of possible relations, for example, is partially solved by ACT-R's declarative memory, which leads to models that consider only a subset of relations – those relations that have a high likelihood of being relevant for the given task.

We decided to use path-mapping, because it already proved its cognitive adequacy for a number of cases (Salvucci & Anderson, 2001). The fact that it is implemented in ACT-R allows us to investigate metaphor and analogy in a wider cognitive context, in particular, how these cognitive abilities work in interaction with the environment (a main aim of our project, on which we do not report here). In line with Gentner's

(1983, p 156) structure mapping, the main feature of path-mapping is that analogies are created by mapping (higher-order) relations.

Reimplementation

Many aspects of the ACT-R architecture have changed since the original path-mapping model was published. (The original model was written in ACT-R 4.0, the current version is ACT-R 6.0.) Because the changes affect many parts of the architecture that are needed for path-mapping, it seemed appropriate to reimplement the model. We just mention three of the most notable changes here. Firstly, there is no goal stack in ACT-R 6.0, which means that all goals that are not currently pursued must be stored in declarative memory. Secondly, there is now a limited set of buffers that contain all temporary information. Each buffer can only hold one chunk (fact), and modules interact by placing chunks in buffers and reading chunks from buffers. The set of all buffers is usually considered ACT-R's working memory. Thirdly, the architecture now has perceptual and motor modules that allow a cognitively adequate interaction with the environment.

An Example

For our case study we used a simple example where the model mapped two additions: $1+2=3$ and $2+1=3$, see figure 1.

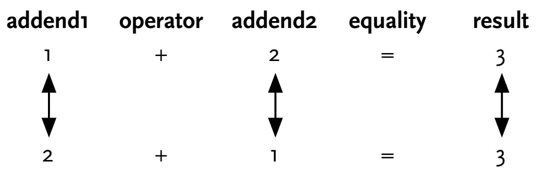

Figure 1: Example of two equations with symmetrical addends.

There are three mappings between these additions. The first mapping, for example, represents the fact that the number 1, which is the first addend in the first addition, is mapped on the number 2, which is the first addend in the second addition.

Knowledge Representation

The knowledge representation used by the path-mapping model for the addition example is graphically shown in figure 2. This representation is parallel to the one given by Salvucci and Anderson (2001). Ovals show objects and relations, boxes show roles. Apart from a name given in bold (and not used by the model), a role consist of five components '**parent**: a pointer to the parent relation, **parent-type**: the semantic type of the parent relation, **slot**: the relation slot that the object fills in the relation, **child**: a pointer to the child object or relation, **child-type**: the semantic type of the child object or relation' (p 75). Because object manipulation is learnt thoroughly, it can be assumed that the commutativity property is an abstraction in the sense of Gentner (1983, p 158–159), ie objects do not have properties.

Simulations

The path-mapping model takes an object or relation from the source domain, for example `add1-one`, and tries to map it on an object in the target domain. It proceeds by retrieving a role that is filled by this object from declarative memory, here: the role labelled `add1-addend1`.[2] The parent relation of this role is `add1-addition`, which is taken as the cue for the next retrieval from declarative memory. This process is iterated until no more role can be retrieved, that is, until the process reaches the root relation. Once that happens the model tries to find roles in the target domain that correspond to the ones it used in the path from the object to the root relation. So, in this case it would map `add1-lhs` onto `add2-lhs` and `add1-addend1` onto `add2-addend1`. The model records both these mappings. (One of them appears as `mapping1` in the top graph of figure 3.) With this, the path-mapping is finished, and the model can try to map the next object. However, our model first attempts to find symmetrical roles, see below.

It should be noted that this implementation of path-mapping does not only work for our addition example but works equally well with Salvucci and Anderson's original example of the solar-system–atom analogy. Thus, our model is further support for Salvucci and Anderson's claim that path-mapping is task independent, ie it is a central process of analogy formation.

Discovering Commutativity

After a path-mapping has been successfully concluded and the mapped path is stored as a chunk in declarative memory, a symmetry detector tries to retrieve a chunk from declarative memory in which the roles were used symmetrically in the source and target domains, see the top of figure 3. In our example, this means the model finds the use of symmetrical roles for the path-mappings of `add1-one/add2-two` for the role `addend1` and `add1-two/add2-one` for the role `addend2`.

After the model found the symmetrical roles it searches the declarative memory for a chunk representing symmetrical roles in another domain (bottom of figure 3). If (1) commutativity holds in this other domain and (2) the knowledge transfer is possible (ie the Arithmetic Is Object Collection metaphor is present) the model 'discovers' that addition is commutative. (See the conclusions on the term *discovery*.)

Unified Theory of Concepts

The cognitively based path-mapping representation is related to a general mathematical characterisation of concepts and relations, the Unified Theory of Concepts (Goguen, 2005), which analyses representations along three axes: the syntactic primitives (called *signature*), the sentences based on a signature, and the (mathematical) structures involved in the semantics. A basic requirement is that of invariance under renaming of syntax. The path-mapping for the atom–solar sys-

[2] In the representation we use here, each object fills only one role. However, nothing depends on this, as can be seen, for example, in Savucci and Anderson's original example of the solar-system–atom analogy. It just means that out case is simpler.

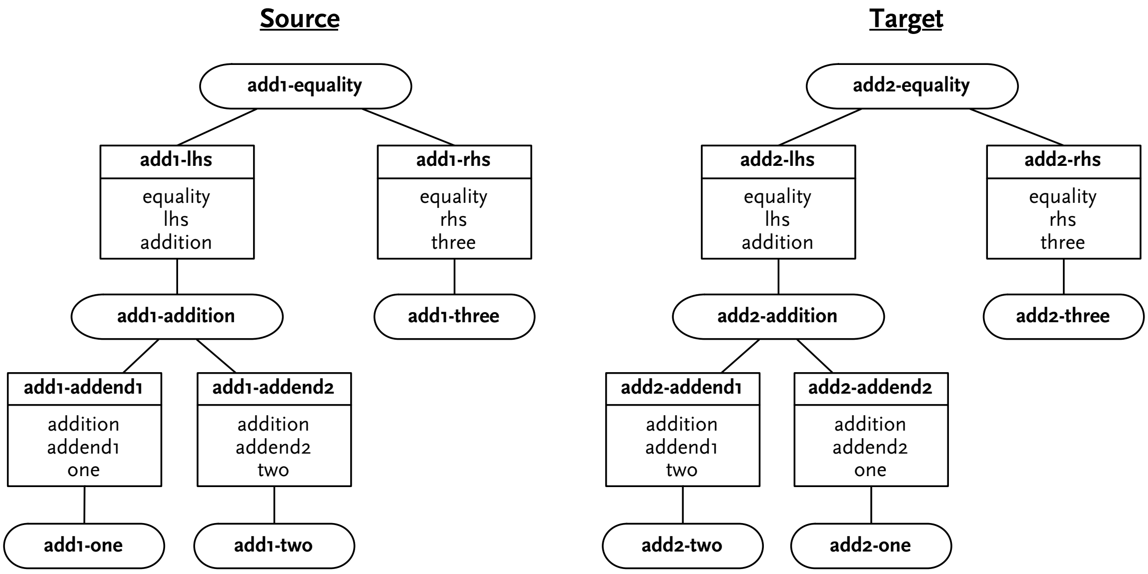

Figure 2: Knowledge representation used for path-mapping in the addition example.

tem analogy is in fact a signature morphism (arity preserving function that changes names), though in general this will be a partial morphism, defined only on a subset of the syntax.

For the arithmetic example, the lexicon is the same. In this case, the path-mapping can be taken as defining an operation on arithmetic equations 'swap arguments to plus', applied uniformly; the fact that this operation preserves arithmetic truth means that it is an *institution morphism* (Goguen, 2006).

These observations suggest that the path-mapping approach may be fruitful over different representation systems, not just the one for which it was originally proposed.

Conclusions

We presented a computational cognitive model of a mathematical discovery. Although the model is still incomplete and limited to a particular case, it provides an important step towards our goal of an embodied, interactive, cognitive model of mathematical discovery.

There still is the question of what a discovery is. In this paper we described the formation of an analogy between a source domain and a target domain. However, more is needed to claim that the model made a discovery. Firstly, the model must be able to distinguish useful discoveries from spurious ones (Boden, 1990, p 32). Otherwise, it will get bogged down by the large number of spurious discoveries. Secondly, after the commutativity property of the object collection domain has been transferred to the arithmetic domain, the discovery *addition is commutative* must be confirmed by a proof or integrated into the target domain as a new axiom.

Analogy/metaphor is not the only way in which scientific concepts can be created, although it seems to be the most common one. For example, we did not consider the processes of concept learning (concept formation) in which observed commonalities between categories of objects or events (shared features) give rise to a new concept, eg commutativity in the object collection domain. In addition to this inductive process, there are certainly also deductive mechanisms of discovery.

Future Research

Our future research will extend the presented model in three main directions. Firstly, we will apply it to more cases of mathematical discoveries to gain a higher level of generality for the model. Again, commutativity is a good example, because in many cases it is known whether it is a property of a mathematical concept, eg rotation, or not, as in the case of subtraction. Secondly, we will embed the model in a task environment. As a first step we will replace the explicit setting of goals for path-mapping attempts by using ACT-R's visual module to read the two equations from a display. Thirdly, we will implement the ability to identify useful discoveries and then formulate a conjecture and attempt to prove it or add the discovery as a new axiom.

Acknowledgments

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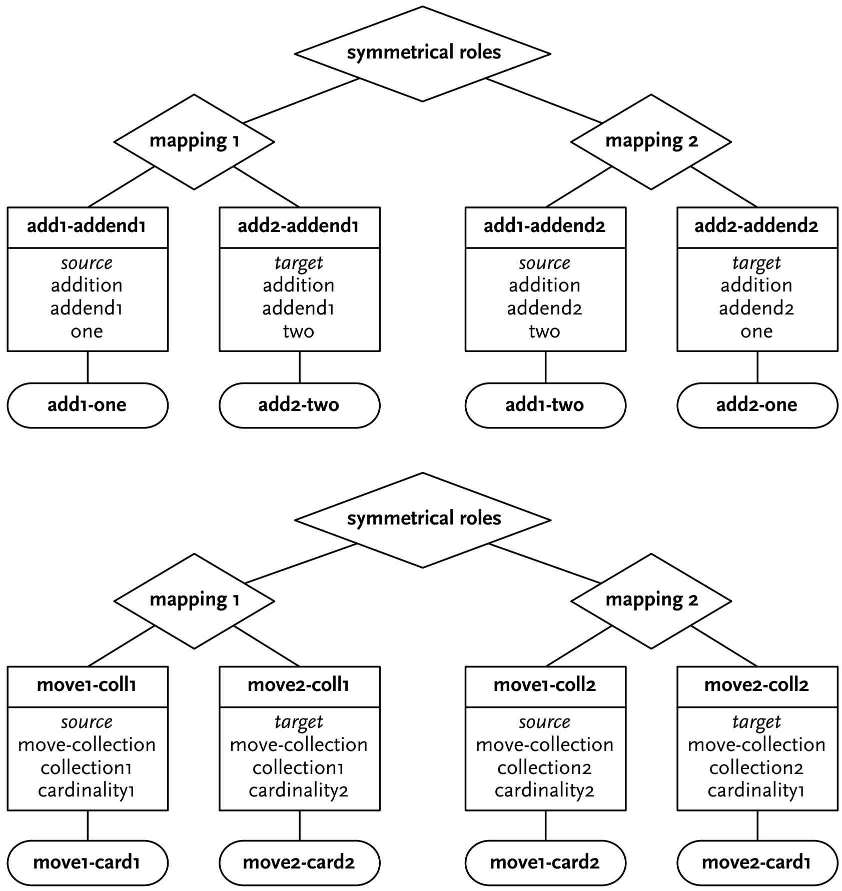

Figure 3: Knowledge representation of symmetrical roles in the example additions and moving collections of objects.

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