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Essays on High-Frequency Asset Pricing

A dissertation submitted in partial satisfaction
of the requirements for the degree
Doctor of Philosophy in Economics

by

Hongxiang Xu

2015

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ABSTRACT OF THE DISSERTATION

Essays on High-Frequency Asset Pricing

by

Hongxiang Xu

Doctor of Philosophy in Economics

University of California, Los Angeles, 2015

Professor Bryan C. Ellickson, Chair

This thesis uses high-frequency data to estimate the stochastic discount factor. The high-frequency data used is sampled at one-second frequency. The fundamental equation of asset pricing is based on the continuous-time no-arbitrage theory. For empirical estimation, I apply the general method of moments to estimate the market price of risk for the risk factors, which consist of exchange-traded funds (ETFs). In Chapter 1, I estimate a one-factor model using the ETF SPY (an SPDR ETF that tracks S&P 500 index) as the risk factor. The estimated risk prices are significant over 2/3 of the sample, and the time series shows plausible patterns of the overall riskiness of the market. An additional factor using IWM (the Russell 2000 ETF that tracks the performance of the small-cap equity market) as the second factor is incorporated into the model in Chapter 2 to arrive at a two-factor model. Adding IWM improves the performance of the model and the estimation precision substantially: the risk price of SPY is almost always significant and the risk price of IWM is significant for about 2/3 of the sample. In Chapter 3 I extend the two-factor model by adding a third factor. Adding a third factor improves the performance of the model to a modest extent, but the large-cap factor SPY followed by the small-cap factor IWM are predominant.

The dissertation of Hongxiang Xu is approved.

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2015

To my wife, for her company and continuous support during my Ph.D study

TABLE OF CONTENTS

1	One-Factor Model	1
1.1	Introduction	2
1.2	Literature	4
1.3	Estimating the Asset-pricing Model	6
1.3.1	Moment Condition	6
1.3.2	Estimation Technique	8
1.4	Data	12
1.4.1	Data	12
1.4.2	Data Cleaning	13
1.4.3	Test Assets	16
1.5	Results	19
1.5.1	Summary	19
1.5.2	The Good Estimates	24
1.6	Conclusion	31
1.7	Appendix: Derivation of the Moment Condition	31
2	Multi-Factor Model: Two-Factor Case	34
2.1	Puzzles	34
2.1.1	Pooling	34
2.1.2	Adding Instrumental Variables	36
2.1.3	Choice of Test Assets	38
2.1.4	Adding Factors	40

2.2	Dividing Up the Day	42
2.2.1	Stacking	42
2.2.2	Estimating Two-factor Models	47
2.3	The Preferred Two-Factor Model	54
2.3.1	The Estimates	55
2.3.2	Variation of Risk Prices Over Time	57
2.4	Conclusion	65
3	Three-Factor Models	67
3.1	Estimating Three-Factor Models	67
3.1.1	Test Assets and the Third Factor	68
3.1.2	Model Setup	69
3.1.3	The Issue of Multicollinearity	70
3.1.4	Using Excess Returns	71
3.1.5	Estimating Three-Factor Models	72
3.2	The Preferred Three-Factor Model	76
3.2.1	Comparison with the Two-Factor Model	76
3.2.2	The Variation of Risk Prices Over Time	78
3.3	Conclusion	90
	References	91

LIST OF FIGURES

1.1	Test Results	21
1.2	Box Plot for $\hat{\theta}$	23
1.3	Time Series Plot of the Significant Estimations	24
1.4	Histogram of $\hat{\theta}$	27
1.5	Time Series of $\hat{\theta}$	28
1.6	Central Tendency and Macro Events	30
2.1	Box Plot with and without the Flash Crash	54
2.2	Distributions of Estimates of θ^1	56
2.3	Time Series of Estimates of θ^1	59
2.4	Time Series of Estimates of θ^2	62
2.5	Estimates of θ^0	63
2.6	The SDF Mimicking Returns	65
3.1	Time Series of Estimates of θ^1 and θ^2	82
3.2	Estimates of θ^0	87
3.3	The SDF Mimicking Returns	89

LIST OF TABLES

1.1	Sample Data Frame of Gross One-second Returns	16
1.2	Individual Stocks	17
1.3	Market ETFs	17
1.4	Sector ETFs	18
1.5	Style ETFs	18
1.6	First 6 Days of GMM Estimates	20
1.7	Significance Test & J-test	21
1.8	Summary Statistics	26
2.1	Pooling Method Results	36
2.2	Estimation Using Instrumental Variables	38
2.3	ETFs as Test Assets	39
2.4	Estimation Using Only ETFs as Test Assets	40
2.5	Candidate Risk Factors	48
2.6	Estimation Results for the One-factor Model (first six days)	49
2.7	Estimation Results for the Two-factor Model (first six days)	50
2.8	One-factor Model using Stacking	52
2.9	One-factor Model using Instrumental Variables	52
2.10	Two-factor Models with Instrumental Variables	52
2.11	Number of Significant Estimates	53
2.12	Summary Statistics for Two-Factor Model	55
2.13	Summary Statistics for Insignificant $\hat{\theta}^2$	57
2.14	Summary Statistics for the Risk Price of SPY	58

2.15	Summary Statistics for the Risk Price of IWM	61
3.1	ETFs as Test Assets	68
3.2	Significant Estimates Across Models	70
3.3	Estimation Results for the Three-factor Model (first six days)	73
3.4	Significant Test and J-test Results	75
3.5	Number of Significant Estimates	75
3.6	Summary Statistics for Three-Factor Model	76
3.7	Summary Statistics for the Risk Prices of SPY and IWM	78
3.8	Summary Statistics for the Risk Price of VWO-SPY	84
3.9	Confidence Bounds	85

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CHAPTER 1

One-Factor Model

Continuous-time asset pricing has a well-developed theoretical background dating back to the early 1900s when Bachelier first used Brownian motion to model stock prices. Modern continuous-time finance started in the 1970s when Black, Scholes and Merton developed a model of option prices for a call option whose underlying asset price process follows a geometric Brownian motion. In the 1980s, Harrison, Kreps and Pliska generalized the theory of arbitrage pricing to semi-martingale processes, the most general class of continuous-time stochastic process for which there exists a well-developed theory of stochastic integration. Delbaen and Schachermayer (2006) completed this theory, proving a version of the fundamental theorem of asset pricing that states: If the stock price process follows a semi-martingale and if there is no free lunch with vanishing risk (a strong form of the no-arbitrage condition), then there exists an equivalent risk neutral measure.

On the other hand, there has been few empirical tests of arbitrage-pricing theory at high frequency. The earliest use of high-frequency data dates back to Epps (1979) who used 10-minute returns to examine the correlation of returns between different stocks. In the late 1990s, as high-frequency data became widely available, many financial econometricians began to examine such data. Most of the attempts focused on using the realized variation to estimate the volatility of geometric Brownian motion, inspired by Merton (1980) who showed that the accuracy of estimating the second moment of the stock return process over a given time period can be improved by subdividing the period into finer sub-intervals.

However, finding that realized variation works poorly at high frequencies, they concluded that observed stock price processes are not semimartingales, but rather the sum of a latent semimartingale process plus a microstructure noise term. They proposed filtering the high-frequency data using various econometric tools or simply avoiding using data with frequencies higher than 15 minutes. This conclusion has been very influential. However, Ellickson et al. (2012) challenge this view. They find that, while stock price volatility is not consistent with geometric Brownian motion, realized variation is consistent with a Heston model of stochastic volatility.

The main focus of this thesis is on asset pricing rather than volatility estimation. This chapter estimates a one-factor model based on the continuous-time no-arbitrage theory. Multi-factor models will be treated in Chapters 2 and 3.

1.1 Introduction

Cochrane (2005) is a good exposition of asset pricing theory. To be specific, consider the following pricing formula, which Cochrane (2005) calls the fundamental equation of asset pricing¹:

$$E [M_{t+h}S_{t+h}^i|\mathcal{F}_t] = S_t^i \quad (1.1)$$

where $[t, t+h]$ is an interval of time, S_t^i and S_{t+h}^i are the prices of asset i sampled at times t and $t+h$, respectively. M_{t+h} is a stochastic discount factor (SDF) over the interval $[t, t+h]$ and $E [M_{t+h}S_{t+h}^i|\mathcal{F}_t]$ is the conditional expectation of the random variable $M_{t+h}S_{t+h}^i$ relative to the information \mathcal{F}_t at time t . In the theory of arbitrage pricing, equation (1.1) is replaced by

$$\tilde{E} [\tilde{S}_{t+h}^i|\mathcal{F}_t] = \tilde{S}_t^i \quad (1.2)$$

¹p36, Cochrane (2005)

where $[t, t + h] \subset [0, 1]$, $\tilde{S}_{t+h}^i = \frac{S_{t+h}^i}{S_{t+h}^0}$ is the price of test asset i relative to a numeraire asset S_{t+h}^0 and \tilde{E} denotes expectation using an equivalent martingale measure \tilde{P} such that \tilde{S}^i is a P -martingale.

In our setting, $[0, 1]$ will represent a trading day where $t = 0$ is the market open and $t = 1$ is the market close. Equation (1.2) is, in turn, equivalent to

$$E \left[M_{t+h} \tilde{S}_{t+h}^i | \mathcal{F}_t \right] = \tilde{S}_t^i \quad (1.3)$$

which I will call the fundamental equation of asset pricing. The key difference of equation (1.3) from equation (1.1) is that equation (1.3) involves **discounted** asset prices, where the asset prices are discounted by a numeraire asset. In the theory of arbitrage pricing, any asset whose prices are positive almost surely can serve as the numeraire asset.

This study incorporates exchange-traded funds (ETFs) into the specification of the stochastic discount factor M_{t+h} , which has some advantages over the use of indices such as the S&P 500 index or the Fama-French “SMB” or “HML” factors. In particular, ETFs are tradable in the market and so their prices can be directly measured. Furthermore, the fact that an ETF is heavily traded can be regarded as evidence that the ETF is regarded by market participants as a measure of systematic risks. The wide range of ETFs available allows us to test various candidates for mimicking the SDF M_{t+h} . For instance, in a one-factor model, one can choose SPY, the most heavily traded ETF, that tracks S&P 500 Index, or, in the spirit of the popular Fama-French three-factor model, one could choose the Russell 2000 small-cap index or choose among various value and growth ETFs.

The structure of this chapter is the following: Section 2 reviews the literature on asset pricing models and empirical estimation techniques. Section 3 briefly derives the model and the moment condition from various assumptions and explains the GMM estimation method. Section 4 explains the data source and the

cleaning process to obtain the data set used in this study. Section 5 covers the estimation results and their interpretation, in particular their link to the market riskiness and macro events. Section 6 concludes.

1.2 Literature

As emphasized by Cochrane (2005), asset pricing models are summarized by the fundamental equation of asset pricing:

$$E [M_{t+h}S_{t+h}|\mathcal{F}_t] = S_t$$

where \mathcal{F}_t is the “information set” at time t ; S_t and S_{t+h} denote the asset prices at time t and $t + h$, respectively; and M_{t+h} is the stochastic discount factor (SDF) of the model at time $t + h$.

The key for the asset pricing model is the specification of the SDF. Cochrane (2005) provides a good summary of the consumption-based asset-pricing literature in discrete time. In consumption-based models, the SDF is considered to be related to consumers’ utility. To be specific, the consumption-based model specifies the SDF as the marginal rate of substitution of the representative consumer who maximizes his or her utility on the consumption stream over time. The parametric form of the SDF depends on the specification of the utility function. Since the original consumption based model does not work well in practice (such as the equity premium puzzle found by Mehra and Prescott (1985)), many economists start to look for different utility functions for the model (for instance, the Epstein-Zin preferences proposed in Epstein and Zin (1989)). Some research works with general equilibrium models by linking consumption to other macro variables including income and production, etc.

Another approach to specify the SDF comes from arbitrage pricing theory.

Delbaen and Schachermayer (2010) and Shreve (2004) have a good summary of the development of the continuous-time arbitrage pricing. Assuming asset prices are semimartingales, they derive the fundamental theorem of asset pricing that the no-arbitrage opportunity condition is equivalent to the existence of the equivalent martingale measure under which the **discounted** asset prices are martingales, which in term implies that

$$E \left[M_{t+h} \tilde{S}_{t+h} | \mathcal{F}_t \right] = \tilde{S}_t$$

where

$$M_{t+h} = \frac{Z_{t+h}}{Z_t}$$

The random variables Z_t and Z_{t+h} are the Radon-Nikodym derivative of the equivalent martingale measure \tilde{P} restricted to the probability spaces $(\Omega, \tilde{F}_t, P_t)$ and $(\Omega, \tilde{F}_{t+h}, P_{t+h})$, respectively. If the market is complete, then \tilde{P} (and hence the Radon-Nikodym process Z) is consequently determined, and $M_{t+h} = \frac{Z_{t+h}}{Z_t}$ is the “return” on the process Z from t to $t+h$. In this study I assume that this return can be replicated by a portfolio of traded assets, in particular exchange traded funds.

Singleton (2006) provides a good review of the empirical methods for the dynamic asset pricing model. According to Singleton (2006), the choice of “preference-based” model or “no-arbitrage” model might also be due to the availability of data, as macro data are usually sampled at lower frequencies such as monthly or quarterly while finance data are sampled at daily frequency or intra-day high-frequencies. Thus, different econometrics has to take into consideration of the property of the underlying data set. In particular, if the data reveal the full information of the distribution of variables in the asset pricing model, then the Maximum Likelihood method can be used to estimate the SDF. On the other hand, if no information of the distribution is known, then the linear projection

method is often applied. More often, partial information is known, such as the moment restrictions of some variables. In this case, one can apply the General Method of Moments (GMM) to estimate the model. In this study, I apply GMM to estimate the SDF of the asset pricing model.

For the use of high-frequency data, most studies focus on volatility estimation, as described in the beginning of this chapter. In terms of asset pricing, there are several papers that study the price dynamics and their implication for algorithm trading using high-frequency data. For instance, Cartea and Jaimungal (2011) models the tick-by-tick dynamics of stock prices using the Hidden Markov Model (HMM) to look for a profitable trading strategy. Bandi and Russell (2005) also includes a discussion of high-frequency estimation of betas. None of them attempts to test the validity of the fundamental equation of asset pricing.

1.3 Estimating the Asset-pricing Model

This section sets up the asset-pricing model used for the estimation and introduces the estimation methodology.

1.3.1 Moment Condition

To start with, let $[0, T]$ denote the length of the “pool”, which is the basic unit of time in our study. Let $(\mathcal{F}_t)_{t \in [0,1]}$ be the filtration. Denote \mathcal{F}_0 to be the initial σ -algebra and \mathcal{F}_T the terminal σ -algebra. In our basic model, $T = 1$, indicating that the pool equals a day.

Assuming that stock prices are semimartingales, one can invoke the fundamental theorem of asset pricing:

Theorem 1.3.1 (Fundamental Theorem of Asset Pricing). *If asset prices are*

semimartingales and if the market satisfies the no free lunch with vanishing risk² condition, then there exists an equivalent martingale measure under which all discounted asset prices are martingales. Furthermore, if the market is complete, then the equivalent martingale measure is unique.

If the market is complete and there is no free lunch with vanishing risk, the fact that \tilde{S}^n is a martingale under the equivalent martingale measure implies that

$$\tilde{S}_t^n = \tilde{E} \left[\tilde{S}_{t+h}^n | \mathcal{F}_t \right] \quad (1.4)$$

for any time $t, t+h$ in the interval $[0, 1]$ where $\tilde{E}[\cdot]$ denotes the conditional expectation under the equivalent martingale measure. To be more specific, we can write (1.4) as the following:

$$Z_t \tilde{S}_t^n = E \left[Z_{t+h} \tilde{S}_{t+h}^n | \mathcal{F}_t \right] \quad (1.5)$$

where Z_1 is the Radon-Nikodym derivative of the equivalent martingale measure (EMM) \tilde{P} with respect to P , $Z_t = E[Z_1 | \mathcal{F}_t]$ for all $t \in [0, 1]$ and E denotes expectation under the true probability measure.

Calling $M_{t+h} = \frac{Z_{t+h}}{Z_t}$ and letting $\tilde{R}_{t+h}^n = \frac{\tilde{S}_{t+h}^n}{\tilde{S}_t^n}$ denote the return of the discounted price of test asset n , equation (1.5) can be written as:

$$E \left[M_{t+h} \tilde{R}_{t+h}^n | \mathcal{F}_t \right] = 1 \quad (1.6)$$

Notice that if we let $n = 0$, i.e., the underlying asset is the numeraire, then we have $E[Z_{t+h} | \mathcal{F}_t] = Z_t$, i.e., $Z = (Z_t)_{t \in [0, 1]}$ is a P-martingale.

²Delbaen and Schachermayer (2006): For a semimartingale S , let $K = \{(H \cdot S)_\infty : H \text{ admissible}, (H \cdot S)_\infty = \lim_{t \rightarrow \infty} (H \cdot S)_t \text{ exists a.s.}\}$ where a strategy is admissible if it is permitted by the market, that is, the wealth is bounded below almost surely, which excludes strategies such as the doubling strategy. Then define $C = \{g \in L^\infty(P) : g \leq f \forall f \in K\}$. S is said to satisfy *no free lunch with vanishing risk* if $\bar{C} \cap L_+^\infty(P) = \{0\}$ such that \bar{C} is the closure of C in the norm topology of $L_+^\infty(P)$.

I assume that market is complete. Then Z_{t+h} is replicable by some portfolio of traded assets. I assume the replicating portfolio can be replicated by a collection of ETFs: i.e., there exists a trading strategy $H_t = (H_t^1, H_t^2, \dots, H_t^d)$ such that:

$$Z_t = \sum_{k=0}^d H_t^k S_t^k \quad (1.7)$$

where H_t is called the stochastic discount factor (SDF) mimicking strategy and S_t^k 's are the prices of some well-traded ETFs. If I further assume that this strategy is simple, i.e., H_t is constant over $[t, t+h]$, then using (1.7), we get the following:

$$M_{t+h} = \frac{\sum_{k=0}^d H_{t+h}^k S_{t+h}^k}{\sum_{k=0}^d H_t^k S_t^k} = \frac{\sum_{k=0}^d H_t^k S_{t+h}^k}{Z_t} = \sum_{k=0}^d \left(\frac{H_t^k S_t^k}{Z_t} \right) \frac{S_{t+h}^k}{S_t^k} = \sum_{k=0}^d \theta_t^k R_{t+h}^k \quad (1.8)$$

where $\theta_t^k = \frac{H_t^k S_t^k}{Z_t}$ is the share of the k th security in the SDF replicating portfolio.

For example, in the case of the one-factor model, i.e., $d = 1$, and assuming θ_t is constant over $[0, 1]$, that is, over the whole day, then combine (1.6) and (1.8), one can get:

$$E \left[(\theta^0 R_{t+h}^0 + \theta^1 R_{t+h}^1) \tilde{R}_{t+h}^i | \mathcal{F}_t \right] = 1, \quad (i = 1, 2, \dots, N) \quad (1.9)$$

Substituting $\theta^0 = 1 - \theta^1$,

$$E \left[(R_{t+h}^0 + \theta^1 R_{t+h}^{10}) \tilde{R}_{t+h}^i | \mathcal{F}_t^i \right] = 1 \quad (1.10)$$

where $R_{t+h}^{10} = R_{t+h}^1 - R_{t+h}^0$.

1.3.2 Estimation Technique

Rearranging equation (1.10) and using the fact that $R_{t+h}^0 \tilde{R}_{t+h}^i = \frac{S_{t+h}^0}{S_t^0} \frac{S_{t+h}^i/S_{t+h}^0}{S_t^i/S_t^0} = \frac{S_{t+h}^i}{S_t^i} = R_{t+h}^i$ and $r_{t+h}^i = R_{t+h}^i - 1$, the net return of asset i over the interval $[t, t+h]$,

we get the following:

$$E \left[r_{t+h}^i + \theta^1 R_{t+h}^{10} \tilde{R}_{t+h}^i | \mathcal{F}_t \right] = 0 \quad (1.11)$$

Applying the law of iterated expectations, we obtain the following unconditional expectation:

$$E \left[\epsilon_{t+h}^i \right] = E \left[r_{t+h}^i + \theta^1 R_{t+h}^{10} \tilde{R}_{t+h}^i \right] = 0 \quad (1.12)$$

where

$$\epsilon_{t+h}^i = r_{t+h}^i + \theta^1 R_{t+h}^{10} \tilde{R}_{t+h}^i. \quad (1.13)$$

We turn to General Method of Moments (GMM) to estimate equation (1.12).

To be more specific, I subdivide each trading day into 2340 10-second intervals, a sample size for each trading day equivalent to 10 years of daily data. For each interval $[t, t + h]$ the gross returns are computed using the prices at t and $t + h$. Because we have 48 test assets and equation (1.12) applies to each of them, we have 48 moment conditions. Since we have only one parameter to estimate and 48 moment conditions, I apply the *multiple-equation GMM with common coefficients* for each trading day to estimate the risk price θ (See Hayashi (2000)).

Hayashi (2000) has a detailed discussion about the assumptions of the multi-equation GMM model, and the notation here follows the one used in Chapter 4 of Hayashi (2000). I assume the model satisfies the usual stationarity and ergodicity conditions. Since there is only one parameter to estimate and 48 moment conditions, the system is over-identified, and we do not require instrumental variables. The orthogonality conditions are:

$$E \left[\epsilon^i \right] = 0 \quad (i = 1, 2, \dots, N) \quad (1.14)$$

We also require that $\{\epsilon_t^i\}_{t \in [0,1]}$ for all $i = 1, 2, \dots, N$ is a martingale difference sequences, that is,

$$E \left[\epsilon_{t+h}^i | \mathcal{F}_t \right] = 0 \quad \forall i = 1, 2, \dots, N \quad (1.15)$$

This is exactly the moment condition of equation (1.11).

Finally, let $z_t^i = R_t^{10} \tilde{R}_t^i$. For the system to be identified, the matrix

$$\Sigma_z = \begin{pmatrix} E[z_t^1] \\ \vdots \\ E[z_t^N] \end{pmatrix} \quad (1.16)$$

needs to have full column rank. Because we have one risk factor and 48 test assets, the system is over-identified. Replace the population moments in (1.16) with sample moments such that

$$S_z = \begin{pmatrix} \frac{1}{T} \sum_{t=1}^T z_t^1 \\ \vdots \\ \frac{1}{T} \sum_{t=1}^T z_t^N \end{pmatrix} \quad (1.17)$$

Also, let

$$S_y = \begin{pmatrix} \frac{1}{T} \sum_{t=1}^T r_t^1 \\ \vdots \\ \frac{1}{T} \sum_{t=1}^T r_t^N \end{pmatrix} \quad (1.18)$$

Then the sample moment condition can be written as

$$g_T(\hat{\theta}) = S_y - S_z \hat{\theta} \quad (1.19)$$

The GMM method minimizes the following quadratic form

$$g_T(\hat{\theta})' \widehat{W} g_T(\hat{\theta}) \quad (1.20)$$

where \widehat{W} is any symmetric and positive definite weighting matrix.

The GMM estimator is

$$\hat{\theta}_T(\widehat{W}) = \left(S_z' \widehat{W} S_z \right)^{-1} \left(S_z' \widehat{W} S_y \right) \quad (1.21)$$

When all the assumptions mentioned above are satisfied, the GMM estimator is consistent, and given $\{\epsilon_t^i\}_{t \in [0,1]} \forall i = 1, 2, \dots, N$ are martingale difference sequences, the estimator is also asymptotically normal.

In the estimation, I utilize the two-step GMM where in the first step the weighting matrix is chosen to be the identity matrix and in the second step, the weighting matrix is the estimated variance-covariance matrix of the residuals from the first step. To be specific, in first step estimation, I obtain

$$\hat{\theta}_{(1)} = \operatorname{argmin} g_T(\hat{\theta})' I g_T(\hat{\theta}) \quad (1.22)$$

where I is the identity matrix. In the second step, I take

$$\widehat{W} = \left(\frac{1}{T} \sum_{t=1}^T g_T(\hat{\theta}_{(1)}) g_T(\hat{\theta}_{(1)})' \right)^{-1} \quad (1.23)$$

The estimator obtained from the two-step estimation is efficient.

I estimate the GMM equation separately for each trading day using the *gmm* package of R. The *gmm* package takes the data matrix as its input and returns the estimation result. The key R codes for the GMM estimation are as following:

```
## GMM method
g = function(theta, x.m){
  gmat = x.m[, 1:48] + theta * x.m[, 49:96]
  return(gmat)
}
res_sdf = gmm(g, x = x.m, t0 = c(0), method = "Brent"),
```



```
lower = -100, upper = 100)
```

where $x = x.m$ is the data matrix. The first 48 columns of $data.m$ form an $T \times N$ matrix of r_t^i and the last 48 columns form an $T \times N$ matrix of $Z_t^i = R_t^{10} \tilde{R}_t^i$. In our settings, over a single day there are $T = 2340$ ten-second intervals and $N = 48$ test assets.

In the GMM estimation, the parameter estimated is $\hat{\theta}$ (called “theta”), the share of SPY in the SDF mimicking portfolio for the given trading day. To obtain the estimation coefficients and statistics, one applies the following R code:

```
summary(res_sdf)$coefficient
```

The J-test result is obtained using:

```
summary(res_sdf)$stest
```

1.4 Data

In this section I discuss the dataset and the test assets used for the estimation and the data-cleaning procedures to treat the raw high-frequency data.

1.4.1 Data

The high frequency data for this study is from the Trade and Quote (TaQ) database on trades³. The test assets used in this paper are the component stocks of the Dow Jones Industrial Average and several well-traded ETFs. In total, there are 48 assets included in the study. In this chapter, a one-factor model is estimated. I use SPY as a proxy for the market, which is a heavily traded ETF

³TaQ also has data on quotes for each stock, which gives for each stock the bid and ask prices associated with each time stamp. We do not use this data for two reasons: first, the amount of data to be handled is too large, and quotes are not necessarily prices at which there was a trade.

that tracks the S&P 500 index. The time frame is from January 1, 2007 to July 31, 2012.

I utilize the procedure developed by Whang (2012) for data downloading and processing⁴. Since the focus of this study is on asset pricing, while Whang (2012) focuses on estimating the volatility using the Heston model, there are some difference in processing the data. First, more ETFs are downloaded for this study to serve as test assets and to construct the SDF mimicking portfolio. Second, I create a data frame of gross returns over each 10-second interval for each test asset. For each day the data frame has 48 columns for each test assets and 2340 rows corresponding to the 2340 ten-second intervals over the trading day. There are 1405 separate data frames, one for each trading day in the sample period.

1.4.2 Data Cleaning

Data cleaning is a standard first-step procedure in the literature, and there is no consensus on the best practice to be applied⁵. This study uses the data cleaning process proposed by Whang (2012).

The original data on the WRDS database are stored in SAS format where each file contains the information for one single trading day. Data sampled at one-second frequency are downloaded from WRDS and are restructured into R data frame format. The reason to use R is for its convenience as the underlying data is irregular where some seconds have no associated transactions at all (called *inactive seconds*), and many have more than one transaction. The purpose of data reduction is to reduce the data set so that each second has only one price for

⁴The original database from WRDS (Wharton Research Data Services) is in SAS format. Whang (2012) developed the Python program to download the Dow Jones stock data and stored them as R data frames. Whang (2012) also designed the data cleaning process by first reducing the data to get the raw median prices, and then filtering the prices to remove extreme outliers to get the prices for the study.

⁵Page 33 of Hautsch (2012) introduces nine different methods for data cleaning that are used in research.

each stock or ETF. Whang (2012) computes the *median share price* of each *active second* (defined to be those seconds with active trades). First, the stock prices for each trading day are sorted in increasing order by time stamp. For each second, prices are ranked from low to high, together with the volume associated with each price. Then the median-share price is computed for the second. The advantage of this method is two fold. First, there is usually a transaction that traded at the median-share price, while other methods such as volume-weighted average usually do not yield a price actually traded. Second, the median share price is a better measure of the “typical” price of a share within a second when there are outliers. Therefore, this method seems preferable to some other alternatives, such as the share-weighted average prices suggested by Hansen and Lunde (2006). The data reduction process significantly reduces the size of the data to store and use relative to the data frame for every transaction.

After reducing the data to get a median prices for each second for which there is a trade, they are further filtered to get the data used for our empirical study. The reasons for filtering the data is because: (1) There are occasional data errors in the data base⁶ that artificially inflate the volatility since mis-recorded prices usually result in large price jumps and consequently consecutive jumps in the realized variation process; (2) Sometimes there are extremely large trades whose prices are so different from adjacent prices that, if included, would result in large realized variation of the price process⁷ and would thus affect the estimation significantly. Many filters are suggested by literature such as Brownlees and Gallo (2006), Barndorff-Nielsen et al. (2008) and Hautsch et al. (2011). Here I apply the method suggested by Whang (2012) that removes prices that increase the realized variation (RV) the most. Whang (2012) detects jumps in the realized

⁶Page 33 of Hautsch (2012) documents one example of Apple trade prices.

⁷Page 24 of Whang (2012) provides an example of SPY trade prices.

variation process and outlier prices using the following influence statistics:

$$\begin{aligned} \text{Influence}_j &:= RV_i^{\mathcal{G}^N} - RV_i^{\mathcal{G}^N \setminus \{X_j\}} \\ &= \begin{cases} (X_{j+1} - X_j)^2 + (X_j - X_{j-1})^2 - (X_{j+1} - X_{j-1})^2 & \text{if } t_0 < t_j < t_N \\ (X_j - X_0)^2 & \text{if } t_j = t_0 \\ (X_N - X_{N-1})^2 & \text{if } t_j = t_N \end{cases} \end{aligned}$$

Here, X_j denotes the j th log price and $RV_i^{\mathcal{G}^N}$ is the realized variation over the block i of the grid \mathcal{G}^N . Therefore, $RV_i^{\mathcal{G}^N} - RV_i^{\mathcal{G}^N \setminus \{X_j\}}$ measures the marginal contribution of the log price X_j to the realized variation of the block that contains X_j . A median-share price is tagged if its influence exceeds 0.2 of the RV of a window of 201 observations centered on X_j , or if its influence exceeds 5% of the total RV for the day. If the price is greater than both the immediately preceding median-share price and the median-share price that immediately follows, I delete the maximum reduced-price of that second (and recompute the median-share price). I also delete the minimum reduced-price of a time stamp if the median price is less than the two immediately adjacent prices. This process is iterated until one obtains a series of median-share prices that do not fail the test. According to Whang (2012), the median number of prices removed per day is much less than 1% of the total transactions.

The data frame obtained using Whang's procedure provides many details of the prices for each one-second time stamp, such as the size of the trade, the number of prices, cumulative variations, and the time stamp of the *inactive seconds*, etc. The focus of this study is the median share price. For each asset, the return over each 10-second interval is computed using the median share prices and arranged into R data frames for estimation. A separate data frame is computed for each trading day with 2340 rows and 48 columns. Each column represents the (gross) returns for one test asset, and each row records the return over one 10-second

time block. Table 1.1 displays a sample of the data frame for the first 10 seconds for one trading day over some of the test assets. It is worth mentioning that most of the “1.000”’s in the table are due to the roundoff of the returns.

Table 1.1: Sample Data Frame of Gross One-second Returns

Date	XLB	XLE	XLF	XLI	XLK	XLP	XLU	XLV	XLY
20080107	1.000	1.000	1.000	1.000	0.997	0.997	0.997	1.000	1.000
20080107	1.002	1.000	0.999	1.000	1.001	1.000	1.000	1.000	1.000
20080107	1.000	1.001	1.000	1.000	1.000	1.000	1.000	1.000	1.000
20080107	1.000	1.002	1.001	1.000	1.000	1.000	1.000	1.000	1.000
20080107	1.000	0.999	1.000	1.000	0.999	1.000	0.999	1.000	1.000
20080107	1.001	1.001	1.003	1.000	1.001	1.000	1.000	1.000	1.000
20080107	1.000	1.000	1.001	1.000	1.000	1.000	1.000	1.000	1.000
20080107	1.000	1.000	1.000	0.999	1.000	1.000	1.000	1.000	1.000
20080107	1.000	0.999	0.999	1.000	1.000	1.000	1.000	1.000	1.000
20080107	0.998	1.000	1.000	0.999	1.000	1.000	1.000	1.000	0.999

1.4.3 Test Assets

For liquidity considerations, the choice of test assets remains in large-cap stocks and heavily-traded ETFs. Table 1.2 to Table 1.5 list the stocks and ETFs chosen as test assets for the estimation of the one-factor model in this chapter.

Table 1.2: Individual Stocks

Ticker	Description	Ticker	Description
AA	Alcoa	AXP	American Express
BA	Boeing	BAC	Bank of America
C	Citigroup	CAT	Caterpillar
CVX	Chevron	DD	DuPont
DIS	Walter Disney	GE	General Electricity
HD	Home Depot	HPQ	Hewlett-Packard
IBM	IBM	INTC	Intel
JNJ	Johnson & Johnson	JPM	JP Morgan
KO	Coca-Cola	MCD	McDonald's
MMM	3M	MRK	Merck & Co.
MSFT	Microsoft	PFE	Pfizer
PG	Procter & Gamble	T	AT&T
UTX	United Technologies Corporation	VZ	Verizon
WMT	Walmart	XOM	Exxon Mobil

For this study, I choose the individual stocks that were components of the Dow Jones Index at January 2007 and were traded during the whole period of our study. In total there are 28 stocks selected.

Table 1.3: Market ETFs

Ticker	Description	Ticker	Description
SPY	S&P 500	DIA	Dow Jones Industrial Average

I choose SPY and DIA as ETFs that track broad market indices: the S&P 500 and the Dow Jones Industrial Average respectively. SPY will also serve as the risk factor.

Table 1.4: Sector ETFs

Ticker	Description	Ticker	Description
IYR	US real estate	XLB	SPDR material
XLE	SPDR energy	XLF	SPDR financial
XLI	SPDR industrial	XLK	SPDR technology
XLP	SPDR consumer staples	XLU	SPDR utility
XLV	SPDR healthcare	XLY	SPDR consumer discretionary

I also include as test assets the nine SPDR sector ETFs, which are heavily traded and represent the performance in different sectors of the economy. In addition, since the period under our study covers the 2008 financial crisis and the burst of the housing market bubble, I include IYR, an ETF that tracks the US real estate market.

Table 1.5: Style ETFs

Ticker	Description	Ticker	Description
IWS	Russell mid-cap value	IWM	Russell 2000
IWO	Russell 2000 growth	IWN	Russell 2000 value
IWB	Russell 1000	IWF	Russell 1000 growth
IWD	Russell 1000 value	IWR	Russell mid-cap

Finally, I include eight ETFs that mimic different investment styles, such as large and small-cap stocks and value and growth stocks. Those styles are popular investment strategies employed by asset managers. In Chapter 2 and 3, they will be used to construct an asset pricing model similar to the Fama-French three-factor model.

1.5 Results

For the main result, I include all the test assets and choose 10-seconds as the frequency for the benchmark estimation. The decision to use 10-second data is a judgment call. I want the frequency to be high to better approximate the continuous-time process. Using 10-second data would lead to 2340 data points for each trading day, approximately the same size as a sample of 10 years of daily returns. On the other hand, I refrain from using one-second data, the highest frequency data possible with this data set. For many test assets, there are many *inactive* seconds during a day that do not have any transaction. For the purpose of estimation, whenever there is one inactive second, the last price of the previous *active* second is used, and therefore a return of “1” is recorded, yet this “interpolated” interval is not informative for the purpose of our study. In addition, for the GMM estimation, using one-second data requires about half an hour to compute the result for one single day and over a week to get the results for the full 1405 trading days under study, which is slow. Thus, 10-second data seems appropriate for the purpose of this study.

1.5.1 Summary

The time period under study is from January 2007 to July 2012, a period of 1405 trading days. This time period includes many macroeconomic events, such as the US financial crisis, Euro crisis and the Fed quantitative easing, etc.

For the one-factor model, I use SPY as the risk factor, a heavily traded ETF that tracks the S&P 500 index. The S&P 500 Index has long been used as a popular proxy for the market portfolio of the US stock market, and SPY is one of the largest and the most actively traded ETFs, and thus is a natural candidate to be used as the risk factor.

For the numeraire, I choose SHY, a bond ETF that tracks the performance

of the US 1-3 year Treasury bond. Although any asset whose price is almost surely positive can serve as the numeraire, SHY is chosen because it tracks the returns of short-term Treasury bonds, which are default-free assets. Therefore, the estimation results of $\hat{\theta}$ can be interpreted naturally as the proportion in the risky asset, and $1 - \hat{\theta}$ in a risk-free asset.

After the estimation computed by the *gmm* package of R, the results are extracted and re-organized into an R data frame for study and analysis using the *summary* command of R, as shown previously. The first few days of the data frame are shown in Table 1.6:

Table 1.6: First 6 Days of GMM Estimates

Date	Theta	S.E.	p.value	J.test
20070103	-0.6494696	0.3612026	7.216504e-02	0.9954959
20070104	-0.4072438	0.6964119	5.586994e-01	0.9994348
20070105	-0.8100496	0.3246761	1.259759e-02	0.9976119
20070108	-0.9506956	0.4825913	4.884055e-02	0.9999648
20070109	-0.8551870	0.7824328	2.744005e-01	0.9999765
20070110	-0.9091853	0.2098295	1.471104e-05	0.9796260

In Table 1.6, the column “Date” lists the trading dates under study. The column “Theta” is the estimated proportion investing in the risky asset in the SDF mimicking portfolio, or $\hat{\theta}$. The column “S.E.” stores the standard errors of the estimates, and the “p.value” column records the p-value of the estimates. The column “significance” marks the estimations with p-value less than 0.05, indicating a 5% significance level. The “J.test” column summarizes the p-value of the J-test of GMM (I call it “Jp-value” hereafter), which tests whether the estimated model is mis-specified.

Table 1.7 summarizes the significance of the daily estimates of the risk price of SPY and the J-test. About 66% of the trading days yield significant estimates of θ at the 5% level. The J-test measures whether the model is mis-specified. The

null hypothesis is that the model is correctly specified. I choose the critical J-test p-value to be 0.1. Table 1.7 suggests that for the trading days with significant estimates, none of them have any evidence of model mis-specification, i.e., with a Jp-value lower than 0.1. Figure 1.1 plots the test results.

Table 1.7: Significance Test & J-test

Level of significance	Number	% significant	% significant pass J-test
0.1	983	70.0%	100%
0.05	925	65.8%	100%
0.01	837	59.6%	100%

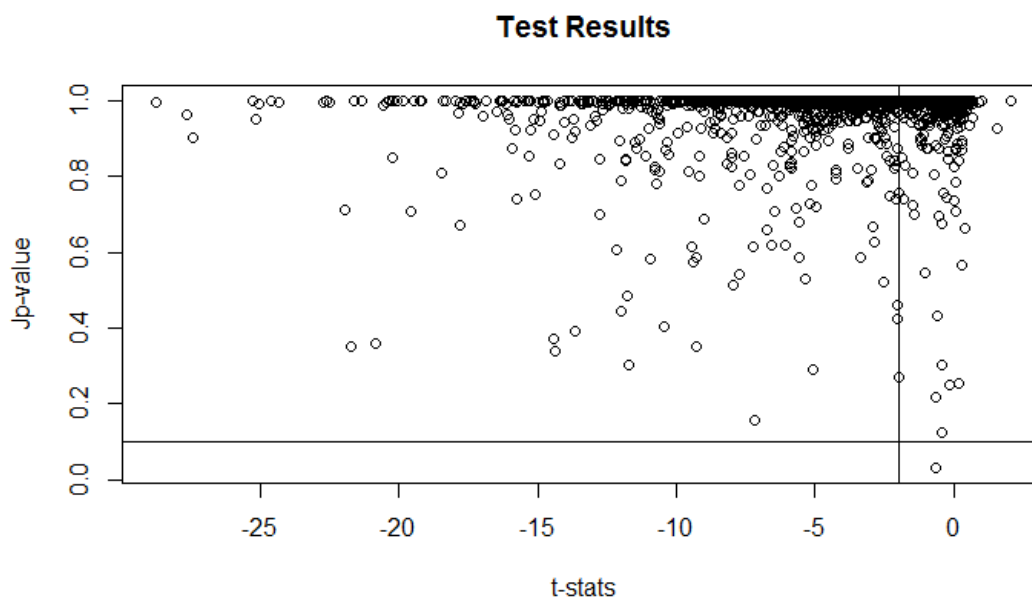


Figure 1.1: Test Results

In Figure 1.1, the x-axis represents the t-statistic of the estimate, and the y-axis represents the J-test significance level. The horizontal and vertical lines divide the plot into four boxes. The top-left box includes estimates that are significant (t-statistics is less than -1.96, corresponding to 5% significance level) and with no evidence of model mis-specification (Jp-value is greater than 0.1). The total number of

estimates in this area is 925, or 65.8% of the whole sample. The top-right box includes estimations that are not significant, though they pass the J-test. There are 473 estimations, 33.7% of the total sample in this box. The single estimate in the bottom-right corners does not pass the model mis-specification test and is not significant, representing 0.07% of the sample. There are no estimates that fall in the bottom-left box.

Focusing on the estimations in the top-left region of Figure 1.1, we observe that most of the estimations yield a Jp-value greater than 0.8 (in fact, many Jp-values are close to 1), strongly suggesting no evidence of model mis-specification. A high proportion of estimates have their t-statistics less than -3. We can conclude the GMM estimation works well on about 2/3 of the trading days.

Figure 1.2 plots the estimation of the θ 's in the top-left region. We notice that on the left box, one estimate (March 13, 2009) is positive and very large in magnitude ($\hat{\theta} = 49.78$, a magnitude 10 times more than the rest), a clear outlier. In the following discussion, I remove this estimate to study the remaining significant estimates in the upper left box of Figure 1.1.

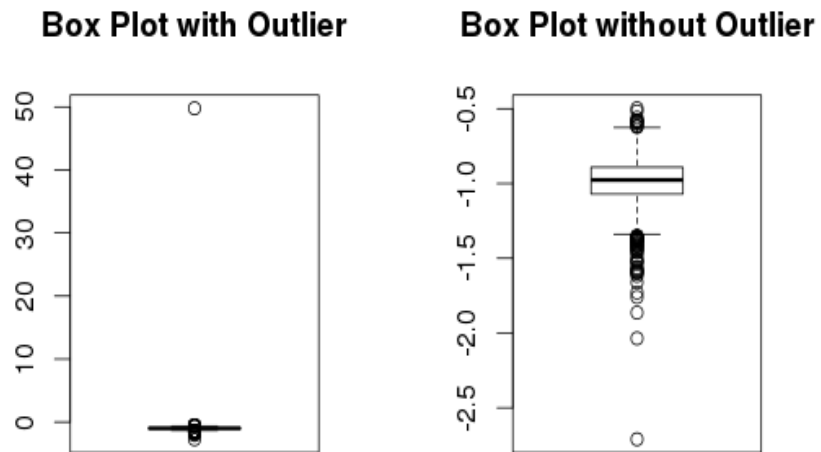


Figure 1.2: Box Plot for $\hat{\theta}$

Figure 1.3 plots the time series of the significant estimations. Recall that the estimation ends at July 31, 2012, leaving a five-month gap at the end of the year 2012.

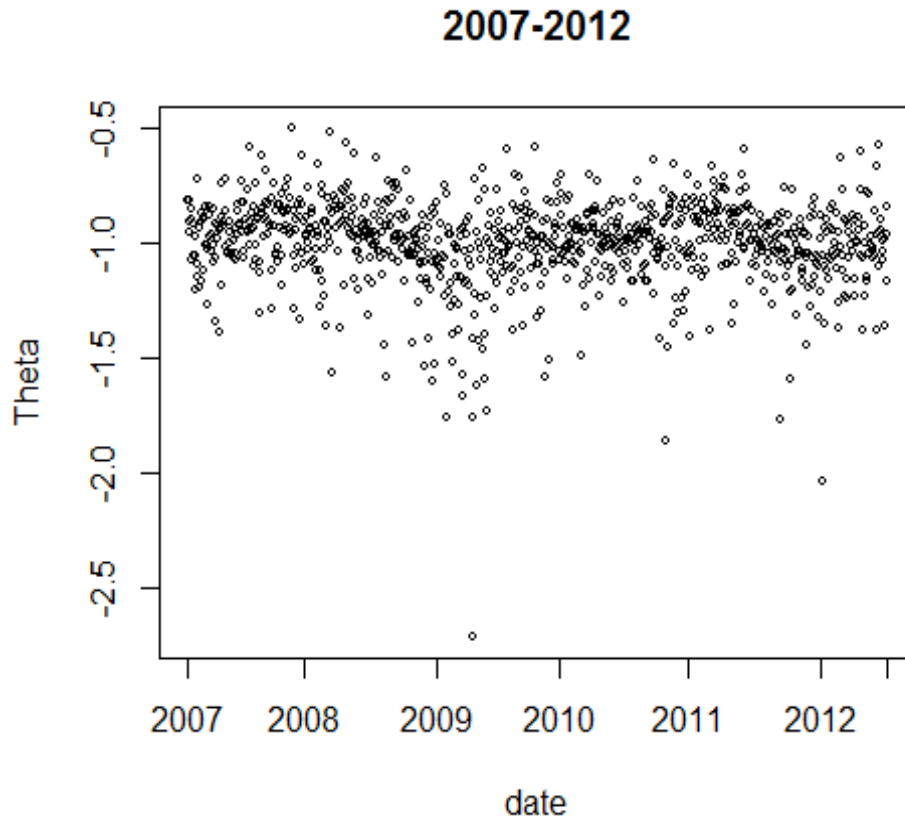


Figure 1.3: Time Series Plot of the Significant Estimations

A quick observation from Figure 1.3 shows that all of the estimated $\hat{\theta}$'s are negative, and most are between -1.2 and -0.8 centering around -1. The estimates are more dispersed with more negative results late in 2008, early in 2009 and late in 2010, with occasionally large, negative estimates such as the one at the beginning of 2012.

1.5.2 The Good Estimates

The rest of this chapter will focus on the study of the significant estimates of the model, which constitute 2/3 of the total sample period.

1.5.2.1 Summary Statistics and Distribution

Table 1.8 provides summary statistics for those days that have significant estimates of the risk price. It is worth mentioning that the year 2012 only includes data from January to July. In Table 1.8, “Mean (-t)” is the mean of the t-statistics of the estimates with the opposite sign. Since all of the good estimates are negative, a negative sign is taken to simplify the presentation. The “Mean precision” is defined to be the mean of the reciprocal of the t-statistics (with sign changed) to measure the error of the estimates.

Several observations can be made from Table 1.8. As discussed previously, θ represents the share of SPY in the SDF mimicking portfolio, and $1 - \theta$ is the share of the numeraire. We observe that $\hat{\theta}$'s, the estimations of the share of SPY in the SDF, are negative. According to the theory, the SDF mimicking portfolio should pay more when SPY has lower return, thus serving as a “hedging” portfolio for the risk factor. Therefore, our estimation results are in accordance with the theory since the negative share of SPY in the SDF mimicking portfolio suggests that the mimicking portfolio actually shorts the SPY. The parameter θ can be viewed as measuring the market riskiness, with a more negative $\hat{\theta}$ indicating a riskier market condition.

Furthermore, the estimates are more negative in 2008, possibly due to the Great Recession. The precision of the estimates improves since 2008, as well.

Table 1.8: Summary Statistics

Year	Statistics	$\hat{\theta}$	Year	Statistics	$\hat{\theta}$
2007	Mean	-0.948	2010	Mean	-0.997
	Median	-0.930		Median	-0.987
	Mean (-t)	5.346		Mean (-t)	6.825
	Mean precision	0.187		Mean precision	0.147
	Min	-1.385		Min	-1.863
	Max	-0.497		Max	-0.639
2008	Mean	-0.979	2011	Mean	-0.986
	Median	-0.961		Median	-0.971
	Mean (-t)	7.864		Mean (-t)	7.982
	Mean precision	0.127		Mean precision	0.125
	Min	-1.601		Min	-1.771
	Max	-0.517		Max	-0.591
2009	Mean	-1.073	2012	Mean	-1.019
	Median	-1.029		Median	-1.010
	Mean (-t)	7.404		Mean (-t)	7.137
	Mean precision	0.135		Mean precision	0.140
	Min	-2.710		Min	-2.036
	Max	-0.580		Max	-0.577

Figure 1.4 shows the distribution of $\hat{\theta}$. For most of the trading days, $\hat{\theta}$ is between -1.2 and -0.8 and is centered on -1. The distribution is negatively skewed as shown from the histogram, the kernel density plot and the box plot, indicating the existence of infrequent but large negative values of $\hat{\theta}$. However, it is worth mentioning that these plots only consider the static property of $\hat{\theta}$, and they implicitly assume that the distribution of $\hat{\theta}$ does not change or shift during the period under study. Given the pattern in Figure 1.3, we might suspect that the distribution of $\hat{\theta}$ might shift according to market conditions, say during the Great Recession and the Euro crisis. The Normal Q-Q plot shows that, except for the tails, the distribution of $\hat{\theta}$ closely resembles a normal distribution.

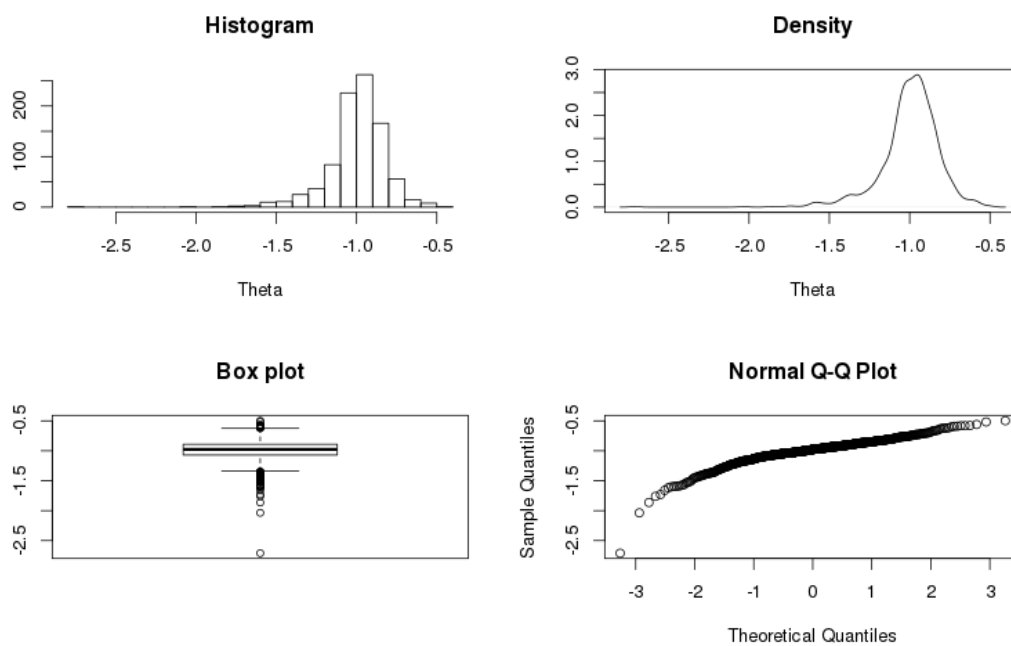
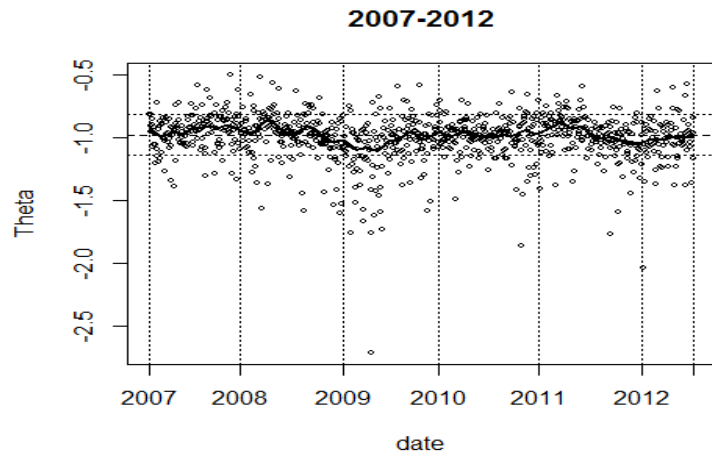


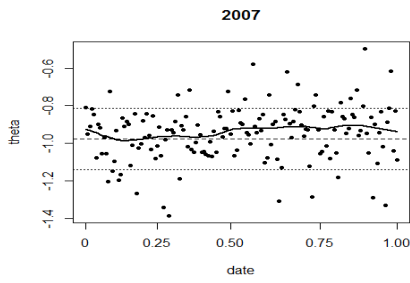
Figure 1.4: Histogram of $\hat{\theta}$

1.5.2.2 Time Series of the Estimation

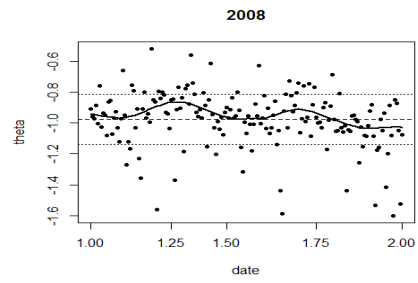
To better make sense of the estimations, we study the behavior of $\hat{\theta}$ over time by plotting the time series of the good estimations for each year under study, as seen in Figure 1.5:



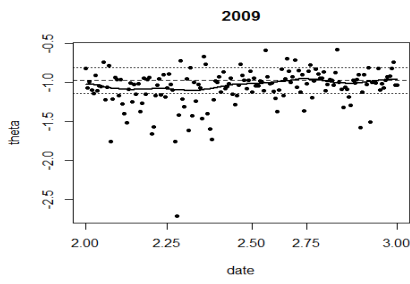
(a) 2007-2012



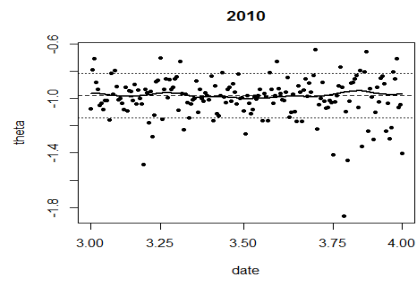
(b) 2007



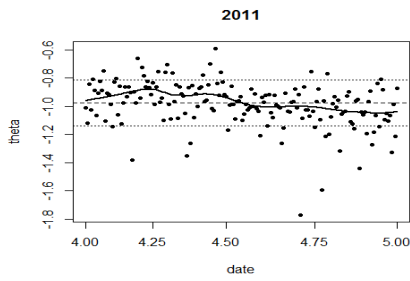
(c) 2008



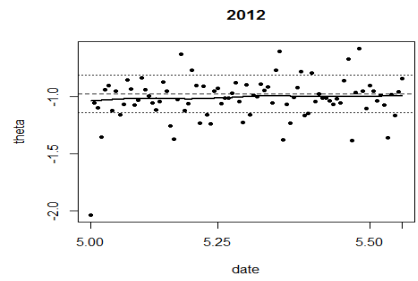
(d) 2009



(e) 2010



(f) 2011



(g) 2012

Figure 1.5: Time Series of $\hat{\theta}$

In Figure 1.5, each year is considered to be a unit of time of length 1. The estimated risk prices are plotted year by year as time series with each dot corresponding to a daily risk price. The dashed line measures the median of the estimates over the entire period, which serves as the reference line of the plots. The median is chosen instead of the mean in order to eliminate the effect of outliers. The dotted lines plot the ± 1 median standard error over the entire period.

For the analysis, I compute the central tendency of the data using the R “lowess” procedure, which performs the locally-weighted polynomial regression (hereafter, the “lowess” method) by fitting polynomials locally to smooth the data. To be more specific, for a given point in a scatter plot (x_i, y_i) , a fitted value of (x_i^f, y_i^f) is the value of a polynomial fit to the data using weighted least squares with a neighborhood of points, called the smooth span. The weight is chosen such that it is large when (x_i, y_i) is close to (x_i^f, y_i^f) and small otherwise. Therefore, the result is less affected by outliers and thus can represent the central tendency of the underlying data. The curve is plotted as the solid line in Figure 1.5.

The “lowess” method requires the choice of smooth span f , which sets the f fraction (thus, $f \times n$ data points, where n is the sample size) of nearest neighborhood used for fitting a local polynomial. The smoother span of the “lowess” function controls the proportion of points that are used to construct the polynomial smoother, and thus a larger number leads to a higher degree of smoothness. On the other hand, a small number would capture more variations of the underlying data. According to Cleveland (1979), $f = 0.5$ is a good starting point, and anywhere between 0.2 and 0.8 is usually chosen. In this study, I choose the smooth span to be 20% of the number of data points in each year so that about 50 days (a little more than two months worth of data) are used to compute the lowess estimates.

Several observations can be made. First, although $\hat{\theta}$ varies from day to day, most of the estimates falls within the dotted lines. Occasionally, there are esti-

mates that falls below the lower band. Most of these estimates are clustered at the end of 2008 and the first half of 2009 (during the financial crisis) and the end of 2011 (during the Euro crisis). Second, the central tendency curve is quite stable as it stays within the band during the whole period. However, some patterns of the central tendency can be observed as it declines during the early 2009. Then it gradually rises until it declines again at the end of 2011.

Figure 1.6 plots the lowest curve and the location of several macroeconomic events⁸ during the period.

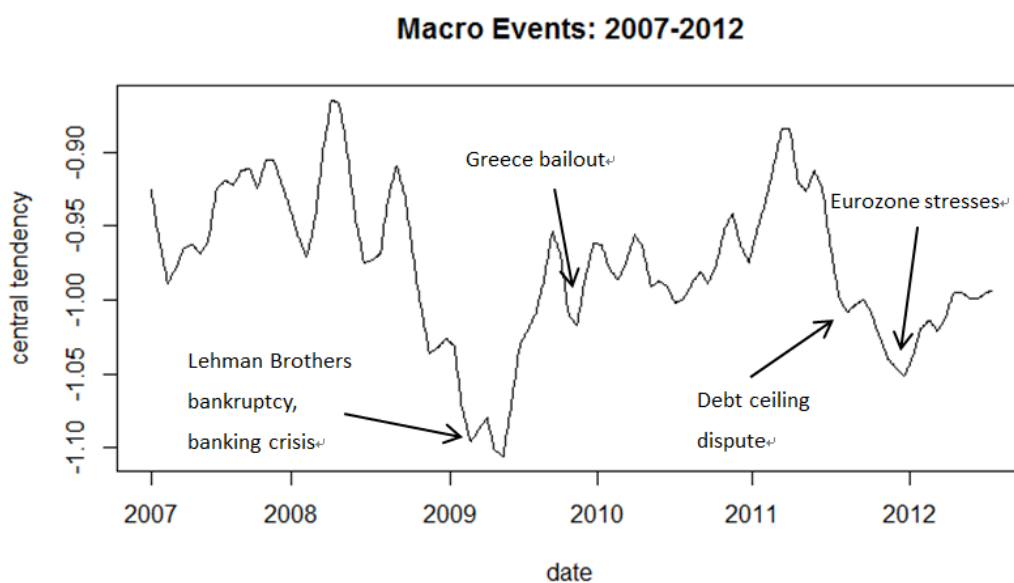


Figure 1.6: Central Tendency and Macro Events

In sum, we are able to obtain reasonable estimation results for our one-factor model of asset pricing. About 2/3 of the trading days have significant estimates of $\hat{\theta}$, and the estimation results make sense as they match the macro events quite well during the sample period. As market becomes more volatile and risky, $\hat{\theta}$ becomes more negative to reflect the need to short more SPY - the risk factor - in order to hedge the risk.

⁸Source: <http://www.policyuncertainty.com/>, which is the study on the Economic Uncertainty Indices of major economies by Scott Baker, Nick Bloom and Steven Davis.

1.6 Conclusion

This chapter uses high frequency data to estimate a one-factor asset-pricing model, assuming that asset prices are semimartingales and there are no arbitrage opportunities in the market. By further assuming that the portfolio weights for the mimicking portfolio are constant over 10 seconds, I obtain a linear specification for the stochastic discount factor. The model performs rather well. I apply the original data set to estimate the stochastic discount factor with minimal data cleaning only, rather than going through the filtering procedure advocated by the microstructure noise literature that claims observed asset prices are semimartingales plus noises. About two thirds of the total sample yield significant estimates of $\hat{\theta}$, the key parameter for our pricing kernel. Our estimates of $\hat{\theta}$ measure the market risk, which seems to correspond to key macro events during the period under study.

1.7 Appendix: Derivation of the Moment Condition

This section provides a rigorous derivation of the moment condition (1.10). The notation is the same as in Section 1.3.1.

I first make the following two assumptions:

Assumption 1.7.1 (Semimartingale). *Asset prices are semimartingales.*

Assumption 1.7.2 (NFLVR). *The market satisfies the no free lunch with vanishing risk condition (NFLVR).*

The notion of semimartingales is a generalization of many familiar stochastic processes, such as the Geometric Brownian Motion and the Ornstein-Uhlenbeck process. When the underlying stochastic process is a semimartingale, the stronger form of no arbitrage condition, the NFLVR, is used to prove the fundamental theorem of asset pricing. These assumptions allow the most general form of the

fundamental theorem of asset pricing without imposing any specific form on the underlying processes under consideration.

To proceed, I introduce the definition of numeraire:

Definition 1.7.1 (Numeraire). *A numeraire is any adapted price process that is strictly positive almost surely.*

Usually, a numeraire is chosen to be the bank account process or any risk-free or, in mathematics, predictable process. However, any adapted process with positive prices almost surely would work, and thus the choice is not unique. The fundamental theorem of asset pricing asserts that discounted asset prices are martingales. For detailed discussion of the numeraire, refer to Section 2.1 of Delbaen and Schachermayer (2005).

One more assumption is needed:

Assumption 1.7.3 (Complete Market). *Market is complete.*

I repeat here the fundamental theorem of asset pricing, which is key in the derivation:

Theorem 1.7.1 (Fundamental Theorem of Asset Pricing). *If asset prices are semimartingales, and if the market satisfies the no free lunch with vanishing risk condition, then there exists an equivalent martingale measure under which all discounted asset prices are martingales. Furthermore, if market is complete, then the equivalent martingale measure is unique.*

Now, consider the $N + 1$ dimensional asset price processes $\{S_t^0, S_t^1, \dots, S_t^N\}_{t=0}^T$ where S_t^0 is the numeraire.

Equation (1.5) follows from the following proposition:

Proposition 1.7.1. *Under assumption 1.7.1 and assumption 1.7.2, we have $Z_t \tilde{S}_t^n = E \left[Z_{t+h} \tilde{S}_{t+h}^n | \mathcal{F}_t \right]$ where $\{Z_t\}_{t=0}^T$ is some martingale.*

Proof. Given $\{\tilde{S}_t^n\}_{t \in [0, T]}$ is a semimartingale, apply the fundamental theorem of asset pricing. There exists a risk neutral measure such that under this measure, $\{\tilde{S}_t^n\}_{t \in [0, T]}$ is a martingale.

Applying Radon-Nikodym theorem, for any $t, t + h \in [0, T]$, we get

$$Z_t \tilde{S}_t^n = E \left[Z_{t+h} \tilde{S}_{t+h}^n | \mathcal{F}_t \right]$$

or

$$E \left[M_{t+h} \tilde{R}_{t+h}^n | \mathcal{F}_t \right] = 1$$

where $M_{t+h} = \frac{Z_{t+h}}{Z_t}$ and $\tilde{R}_{t+h} = \frac{\tilde{S}_{t+h}}{\tilde{S}_t}$. □

CHAPTER 2

Multi-Factor Model: Two-Factor Case

In Chapter 1 I estimated a one-factor asset pricing model using high-frequency data with SPY as the factor. The estimation works well: estimates are significant for over two thirds of the days in our sample period (2007-2012), and the J-tests indicate that the model is well specified. Moreover, all of the significant risk price estimates are negative, which is consistent with our interpretation that the portfolio of the numeraire asset and SPY can mimic the “return” of the Radon-Nikodym derivative of the equivalent martingale measure. This chapter improves the specification using instrumental variables that divide the market day into sub-periods and using two risk factors.

2.1 Puzzles

Although the results of Chapter 1 are surprisingly good, there is clear room for improvement. The risk price is significantly different from zero in only two thirds of the days in our sample, and attempts to allow for factors other than SPY fail decisively. I begin by discussing briefly a number of failed attempts to improve the specification. These failures provide insight into what eventually works.

2.1.1 Pooling

The standard errors for the risk price estimates in Chapter 1 are quite high.

- For 1/3 of the days, the estimates are not significantly different from zero.

- Although a graph of the “lowess” reveals a plausible pattern in the risk price for SPY, the range is contained within two standard errors of the estimates.
- While the mean of the significant estimates is roughly -1, standard errors are about 0.2, which are not as small as one would like.

One possibility is that the 2340 ten-second returns in a day is not a large enough sample. To explore that possibility, I aggregated the date frame used for the estimation over 5-day, 10-day, and 20-day periods. I assume that the estimator of the price of risk, $\hat{\theta}$, is constant over the 5, 10 or 20 days respectively, corresponding to the weekly, half-monthly and monthly estimations. The motivation for pooling is the hypothesis that the large standard errors in the daily estimates reflect sampling error rather than variation of the risk price from day to day.

The following table summarizes the pooled estimates to the daily estimates (the benchmark). Significant estimates are defined to be those whose t-statistics is less than -2, or equivalently, a p-value less than 0.05. (There are no days in which $t > 2$: i.e., significant estimates always have the expected sign.) The significance level for the J-test is chosen to be 10%. Statistics of mean, median, minimum and maximum are computed only for those estimates that pass the J-test and are significant.

Table 2.1: Pooling Method Results

Statistics	Benchmark	5-day	10-day	20-day
No. of estimates	1405	281	140	42
% significant	65.8%	64.8%	59.3%	60.0%
% passes J-test	99.9%	99.6%	100%	100%
Mean	-0.998	-0.988	-1.011	-0.980
Median	-0.976	-0.977	-0.966	-0.967
Mean (-t) - whole sample	4.895	4.647	4.212	4.599
Mean precision - whole sample	0.204	0.215	0.237	0.217
Mean (-t) - significant only	7.124	6.897	6.483	7.444
Mean precision - significant only	0.140	0.145	0.154	0.134
Min	-2.710	-1.690	-1.897	-1.379
Max	-0.497	-0.629	-0.692	-0.535

In Table 2.1, “Mean (-t)” is the mean of the t-statistic of the estimate with the opposite sign. The negative sign is taken to simplify the presentation of the result since all significant estimates are negative. The “Mean precision” is defined to be the mean of the reciprocal of the t-statistics (with sign changed). As Table 2.1 shows, all four estimations yield a similar percentage of estimates that are significant, and the mean, the median and the mean standard error of estimates over 2007-2014 are almost indistinguishable. I conclude that pooling does not help.

2.1.2 Adding Instrumental Variables

The estimates in Chapter 1 are based on the following moment condition:

$$E \left[r_{t+h}^n + \theta^1 R_{t+h}^{10} \tilde{R}_{t+h}^n | \mathcal{F}_t \right] = 0 \quad (2.1)$$

where $R_{t+h}^{10} = R_{t+h}^1 - R_{t+h}^0$. I applied the law of iterated expectations to obtain the unconditional expectation equation:

$$E \left[r_{t+h}^n + \theta^1 R_{t+h}^{10} \tilde{R}_{t+h}^n \right] = 0 \quad (2.2)$$

Equation (2.2) is the moment condition used for the estimation. Since there is one moment condition for each of the 48 test assets and only one parameter to estimate, the system is well over-identified.

To improve the results from Chapter 1, I explored the possibility of adding instrumental variables. As equation (2.1) indicates, valid instrumental variables X_t must be measurable with respect to \mathcal{F}_t , the information set at the beginning of each 10-second interval. Let $\epsilon_{t+h}^n = r_{t+h}^n + \theta^1 R_{t+h}^{10} \tilde{R}_{t+h}^n$ so that equation 2.1 becomes

$$E [\epsilon_{t+h} | \mathcal{F}_t] = 0 \quad (2.3)$$

We have

$$E [X_t \epsilon_{t+h} | \mathcal{F}_t] = X_t E [\epsilon_{t+h} | \mathcal{F}_t] = 0 \quad (2.4)$$

and thus the orthogonality condition is met.

I estimated the model using two different instrumental variables. The first one uses the lagged return of SPY as the instrumental variable, and the other one uses the lagged return of the test asset. Table 2.2 repeats the summary statistics where the significance level is chosen at 5% and the critical J-test p-value is 10%.

Table 2.2: Estimation Using Instrumental Variables

Statistics	Benchmark	Lag SPY	Lag test asset
No. of estimates	1405	1405	1405
% significant	65.8%	24.3%	66.2%
% passes J-test	99.9%	45.8%	99.9%
Mean	-0.998	-1.172	-0.995
Median	-0.976	-1.078	-0.975
Mean (-t) - whole sample	4.895	1.068	4.921
Mean precision - whole sample	0.204	0.936	0.203
Mean (-t) - significant only	7.124	6.453	7.124
Mean precision - significant only	0.140	0.155	0.140
Min	-2.710	-6.166	-2.683
Max	-0.497	-0.037	-0.517

When using the lagged return of SPY as the instrumental variable, the number of significant estimates declines sharply, and less than half of the sample passes the J-test. Using the lagged return of the test assets yields results similar to the benchmark model. While the number of days with significant estimates increases slightly, there is no improvement in the precision of the estimates. Adding instrumental variables does not seem to help.

2.1.3 Choice of Test Assets

By eliminating individual assets and retaining only the ETFs, the estimates can change a lot for some days¹. Using only ETFs is the analog of using portfolios of stocks to replace idiosyncratic risks, a standard practice in the finance literature. I have performed the estimation of the model using ETFs only as dropping individual stocks can avoid the impact of idiosyncratic risks to the estimation. Table 2.3 lists the ETFs used for the estimation.

¹For instance, the estimate for the risk price of SPY on April 13, 2009 in the one-factor model is -2.71 using all individual stocks and ETFs as test assets. However, when removing two individual stocks, BAC and C, the estimate becomes -1.41.

Table 2.3: ETFs as Test Assets

Ticker	Description	Ticker	Description
SPY	S&P 500	VWO	Vanguard foreign stocks
IYR	US real estate	XLB	SPDR material
XLE	SPDR energy	XLF	SPDR financial
XLI	SPDR industrial	XLK	SPDR technology
XLP	SPDR consumer staples	XLU	SPDR utility
XLV	SPDR healthcare	XLY	SPDR consumer discretionary
IWS	Russell mid-cap value	IWM	Russell 2000
IWO	Russell 2000 growth	IWN	Russell 2000 value
IWB	Russell 1000	IWF	Russell 1000 growth
IWD	Russell 1000 value	IWR	Russell mid-cap

Retaining only ETFs leaves a total of 20 test assets, so the model is still over-identified. Table 2.4 presents the summary statistics for estimation of the one-factor model with SPY as the factor and SHY as the numeraire. The percentage of days in which the estimation passes the J-test and the percentage of significant estimates both decline slightly. Although using ETFs gets rid of some of the outliers in the estimates, the summary statistics of the estimates are very close to the results from the benchmark model with no improvement in the number of significant days. Although switching to ETFs only does not improve the results of Chapter 1, getting rid of the individual stocks to reduce idiosyncratic risks seems like a good idea. From now on, estimates will use this more restrictive set of test assets.

Table 2.4: Estimation Using Only ETFs as Test Assets

Statistics	Benchmark	ETFs only
No. of estimates	1405	1405
% significant	65.8%	61.2%
% passes J-test	99.9%	96.8%
Mean	-0.998	-0.998
Median	-0.976	-0.984
Min	-2.710	-2.243
Max	-0.497	-0.591

2.1.4 Adding Factors

I also explored the possibility that adding factors would improve the performance of the model. I used XLF, the ETF that tracks the performance of SPDR financial sector, as a second factor.

The following is the moment condition for the GMM estimation of the two-factor model:

$$E \left[r_{t+h}^n + \theta^1 R_{t+h}^{10} \tilde{R}_{t+h}^n + \theta^2 R_{t+h}^{20} \tilde{R}_{t+h}^n \right] = 0 \quad (2.5)$$

where R_{t+h}^{20} is the excess return of the second factor (here, XLF) relative to the numeraire, and θ^2 is the risk price of the second factor. For notational convenience, write Equation (2.5) as

$$E \left[Y_{t+h}^n + \theta^1 Z_{1,t+h}^n + \theta^2 Z_{2,t+h}^n \right] = 0 \quad (2.6)$$

where $Y_{t+h}^n = r_{t+h}^n$, $Z_{1,t+h}^n = R_{t+h}^{10} \tilde{R}_{t+h}^n$ and $Z_{2,t+h}^n = R_{t+h}^{20} \tilde{R}_{t+h}^n$.

The *gmm* package of R is utilized for the estimation. Below is the core code for the GMM estimation in R:

```
## GMM method
g = function(theta , x = cbind(y , z_1 , z_2)) {
```

```

gmat = y + theta[1] * z_1 + theta[2] * z_2
return(gmat)
}
res_sdf = gmm(g, x = cbind(y, z_1, z_2), t0 = c(0, 0))

```

In the code above, $y = Y_{t+h}^n$, $z_1 = Z_{1,t+h}^n$ and $z_2 = Z_{2,t+h}^n$. x is the data matrix with 2340 rows and $3N$ columns, with the first N columns corresponding to Y_{t+h}^n , the middle N columns to $Z_{1,t+h}^n$, and the last N columns to $Z_{2,t+h}^n$, where N is the number of test assets. $t0 = c(0, 0)$ specifies the initial values for our estimation of the parameter vector $\theta = (\theta^1, \theta^2)$.

In contrast to the one-factor model, the two-factor version of the model works terribly. First, the estimates are very large in magnitude. We expect that the coefficient of SPY to be affected only modestly by adding a second factor for “financial” risk. Second, for almost every day, the standard errors of both parameter estimates are huge, and the estimates are rarely statistically different from zero. This specification completely fails to yield any useful estimates and information about the risk prices with respect to the factors for the sample period under study.

To illustrate, for the first day in the sample January 3, 2007 (which is quite typical), the estimates are $\hat{\theta}^1 = 11.90$, and $\hat{\theta}^2 = -64.76$, which are both large in magnitude, and the standard errors are $SE(\hat{\theta}^1) = 55.10$ and $SE(\hat{\theta}^2) = 82.01$, which are also large enough to render the estimates insignificant. As a comparison, in the one-factor model, the estimate is $\hat{\theta}^1 = -0.66$ and the standard error is $SE(\hat{\theta}^1) = 0.36$, which are much smaller in magnitude, and the estimate is nearly to significant at the 5% level. I tried pooling over three to ten days and adding lagged returns as instrumental variables, but neither method improved the estimation of the two-factor model.

2.2 Dividing Up the Day

Since pooling did not work, I conjectured that the problem is the reverse: we might need to compute moments over intervals shorter than a single day in order to increase the resolution of the model. By using sample moments over the entire day, we may ignore important short-term dynamics within the day.

Let

$$\epsilon_{t+h}^n = r_{t+h}^n + \theta^1 R_{t+h}^{10} \tilde{R}_{t+h}^n + \theta^2 R_{t+h}^{20} \tilde{R}_{t+h}^n$$

The model implies that $\{\epsilon_t^n\}$ is a martingale difference sequence, and this condition has to be satisfied in any 10-second interval over the day.

In Chapter 1, we derived the moment condition that $E[\epsilon_{t+h} | \mathcal{F}_t] = 0$, and as discussed previously, any instrumental variable X_t that is measurable at time t is valid. To restrict the focus within some time interval within the day, we can choose X_t to be an indicator function:

$$X_t = \begin{cases} 1 & \text{for some subset of } \{t = 1, \dots, 2340\} \\ 0 & \text{otherwise} \end{cases} \quad (2.7)$$

where the 1's indicate a subset of 10-second intervals. If the subsets are selected non-randomly, X_t is measurable with respect to \mathcal{F}_t for all $t \in [0, 1]$, and hence is a valid instrumental variable.

In what follows we will use X_t that divide the trading day into 12 non-overlapping intervals.

2.2.1 Stacking

Since the model to be estimated satisfies the GMM model with common coefficients², we can estimate a “stacked” version of the model (see Hayashi (2000)).

²In two-factor case, the common coefficient is a two-tuple vector.

To explain the stacking procedure, we introduce some notation. Let K denote the number of factors in the model. Let the number of test assets be N . In Chapter 1, $N = 48$ with 28 individual stocks and 20 ETFs. In this chapter, the test assets chosen are ETFs only, and thus $N = 20$.³ Let I equal the number of data points we have during a day. Since we are using 10-second returns, $I = 2340$. Let J be the number of sub-intervals of the trading day. In this study $J = 12$. I sub-divide the trading day into 12 disjoint intervals, each covering 32.5 minutes, equivalent to 195 ten-second intervals.

When generalized to K factors, the GMM estimation of the one-factor model in Chapter 1 and the two-factor results described earlier in this chapter included N moment conditions, one for each of the N test assets:

$$\begin{aligned} E [Y_{t+h}^1 + \theta^1 Z_{1,t+h}^1 + \cdots + \theta^k Z_{k,t+h}^1 + \cdots + \theta^K Z_{K,t+h}^1] &= 0 \\ &\vdots \\ E [Y_{t+h}^N + \theta^1 Z_{1,t+h}^N + \cdots + \theta^k Z_{k,t+h}^N + \cdots + \theta^K Z_{K,t+h}^N] &= 0 \end{aligned} \tag{2.8}$$

where $Y_{t+h}^n = r_{t+h}^n$, $Z_{1,t+h}^n = R_{t+h}^{10} \tilde{R}_{t+h}^n$ and $Z_{2,t+h}^n = R_{t+h}^{20} \tilde{R}_{t+h}^n$ for all test assets $n = 1, 2, \dots, N$.

For the stacked model, the moment conditions become:

$$E [\mathbf{Y} + \mathbf{Z}\boldsymbol{\Theta}] = 0 \tag{2.9}$$

where

$$\mathbf{Y} = \begin{pmatrix} Y^1 \\ \vdots \\ Y^n \\ \vdots \\ Y^N \end{pmatrix} \quad \boldsymbol{\Theta} = \begin{pmatrix} \theta^1 \\ \vdots \\ \theta^n \\ \vdots \\ \theta^N \end{pmatrix} \quad \mathbf{Z} = \begin{pmatrix} Z_1^1 & \cdots & Z_k^1 & \cdots & Z_K^1 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ Z_1^n & \cdots & Z_k^n & \cdots & Z_K^n \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ Z_1^N & \cdots & Z_k^N & \cdots & Z_K^N \end{pmatrix}.$$

³The ETF “VWO” was not included in Chapter 1 but is included here

Also,

$$Y^n = \begin{pmatrix} Y_1^n \\ \vdots \\ Y_i^n \\ \vdots \\ Y_I^n \end{pmatrix} \quad Z_k^n = \begin{pmatrix} Z_{k,1}^n \\ \vdots \\ Z_i^n \\ \vdots \\ Z_{k,I}^n \end{pmatrix}$$

for $n = 1, 2, \dots, N$ since there are 2340 observations ($I = 2340$) of 10-second returns for each test asset each day. Thus, \mathbf{Y} is a $2340N \times 1$ matrix and \mathbf{Z} is a $2340N \times K$ matrix.

Let g be the vector of moments. In Chapter 1, $K = 1$, and when stacking the number of moment conditions will be one, so that the model is only exactly identified. One way to make the one-factor model over identified is by adding N instrumental variables X_m , where instrumental variable X_m “selects” the observations for test asset $n = m$. Let \mathbf{X} be the matrix of instrumental variables for the stacked model, then

$$\mathbf{X} = \begin{pmatrix} X_1^1 & \dots & X_m^1 & \dots & X_N^1 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ X_1^n & \dots & X_m^n & \dots & X_N^n \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ X_1^N & \dots & X_m^N & \dots & X_N^N \end{pmatrix}$$

which is a $2340N \times 2340N$ matrix and each column represents one vector of instrumental variables.

$$X_m^n = \begin{cases} \mathbf{1} & \text{if } n = m \\ \mathbf{0} & \text{otherwise} \end{cases}$$

where $\mathbf{1}$ is a column vector of 1's and $\mathbf{0}$ is a column vector of 0's. If $N > K$, the model is over-identified. Essentially, column n of the matrix \mathbf{X} is just an indicator of the observations associated with test asset n .

in Chapter 1, I did not use stacking. Instead the 48 moment conditions (one for each test asset) can be input to the *gmm* program as columns of a matrix

$$\begin{pmatrix} g^1 & g^2 & \dots & g^N \end{pmatrix}$$

where

$$g^n = \begin{pmatrix} g_1^n \\ \vdots \\ g_I^n \end{pmatrix} \quad (n = 1, 2, \dots, N, I = 2340)$$

gives the observations used to compute the sample moment of the n th test asset⁴. With N moment conditions and one parameter to estimate, the model is over-identified and there is no need to specify additional instrumental variables.

However, stacking is very useful in applying over asset pricing models to different intervals within the trading day. The number of instrumental variables is $J = 12$. Provided $J > K$, the model is still over-identified. Let equation (2.9) be the moment condition, and define \mathbf{Y} , Θ and \mathbf{Z} as before. Let \mathbf{X} be the matrix of instrumental variables such that

$$\mathbf{X} = \begin{pmatrix} X_1 & \dots & X_k & \dots & X_K \end{pmatrix} = \begin{pmatrix} X_1^1 & \dots & X_k^1 & \dots & X_K^1 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ X_1^n & \dots & X_k^n & \dots & X_K^n \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ X_1^N & \dots & X_k^N & \dots & X_K^N \end{pmatrix}$$

In this case, we have 12 time intervals over the day and thus 12 instrumental variables. Therefore, \mathbf{X} is a $2340N \times 12$ matrix and

$$X_k^n = \begin{pmatrix} X_{k,1}^n \\ \vdots \\ X_{k,I}^n \end{pmatrix}$$

⁴See the example in Section 3.1 of Chausse (2010).

where

$$X_{k,i}^n = \begin{cases} 1 & \text{if } i \in \{195(k-1) + 1, 195(k-1) + 2, \dots, 195k\} \\ 0 & \text{otherwise} \end{cases}$$

For instance, for $k = 1$,

$$X_{1,i}^n = \begin{cases} 1 & \text{if } i = 1, 2, \dots, 195 \\ 0 & \text{otherwise} \end{cases}$$

and for $k = 2$,

$$X_{2,i}^n = \begin{cases} 1 & \text{if } i = 196, 197, \dots, 390 \\ 0 & \text{otherwise} \end{cases}$$

By choosing X_k as the instrumental variable, we have

$$E[\epsilon_t X_k] = 0$$

over interval k since the moment condition

$$E[\epsilon_t] = 0$$

has to be satisfied at any interval. Although there are $2340N$ data points in each sample moment, the k th moment in effect applies the moment condition only to the k th interval of the day, resulting in an effective number of data points of $\frac{2340N}{12} = 195N$.

For the stacked model, the *gmm* package in R allows us to use the “formula” specification instead of the “vector” specification for the estimation. The following is the R code for the GMM estimation:

```
res_sdf = gmm(Y ~ Z - 1, X)
```

where Y and Z are as defined above and serve as the sample moments and X is the matrix of instrumental variables with $2340N$ rows and 12 columns as defined

previously. The “-1” in the code imposes the requirement that there is no constant term in the specification of the moment condition $g = \mathbf{Y} + \mathbf{Z}\Theta$.

2.2.2 Estimating Two-factor Models

Now we can apply the methodology to estimate a two-factor model. I estimate the two-factor model using only ETFs as test assets to avoid the idiosyncratic risk and excess volatilities incorporated in the individual stocks. The ETFs used are listed in the first section of this chapter. There are 20 test assets (ETFs). SHY is chosen to be the numeraire. Below are the candidate ETFs I consider for the multi-factor model:

- SPY : The SPDR ETF that tracks the S&P 500 index. SPY is the most heavily traded ETF and was shown to play a significant role in explaining the market risk in Chapter 1. SPY is always included as the first factor for the model.
- XLF: The SPDR sector ETF that tracks the financial sector. Since the period under study covers the financial crisis, XLF might be useful in explaining the risk prices during this period.
- XLE: The SPDR sector ETF that tracks the energy sector.
- VWO: The Vanguard ETF that focuses on foreign markets.
- IWM: The heavily-traded ETF that tracks Russell 2000 index to measure the performance of small-cap equities. A two-factor world with SPY and IWM can be regarded as equivalent to a model with SPY and the Fama-French “SMB” (Small Minus Big) factor, where IWM serves as the “S” factor and SPY the “B” factor.
- IWD: The Russell 1000 Value ETF that tracks the performance of large-cap value stocks.

- IWF: The Russell 1000 Growth ETF that tracks the performance of large-cap growth stocks. Although IWD and IWF are not as heavily traded as the previous candidate ETFs, they are good proxies for the sectors of value and growth stocks to mimic the Fama-French “HML” (High Minus Low Price-to-Book ratio) factor.

Table 2.5 lists the average daily volumes (for 20 days in 2015)⁵ for the factors I am considering :

Table 2.5: Candidate Risk Factors

ETFs	Average Daily Volumes (20 days)
SPY	93,768,938
IWM	24,915,324
XLF	24,113,355
XLE	13,528,330
VWO	11,056,910
IWF	1,959,920
IWD	1,595,860

SPY, IWM, XLF, XLE and VWO are usually among the top 30 most heavily traded ETFs, with SPY always the most heavily traded. Note that IWF and IWD are much less heavily traded than the other candidate risk factors.

The daily GMM estimation routine is embedded in a loop that works through the 1405 trading days in our sample. After each daily estimation is completed, the results are assembled into an R data frame⁶. To give a concrete sense of this data frame, Table 2.6 reports the first 6 rows (days) of the data frame for a one-factor model with SPY as the factor. This is the same model explored in Chapter 1 but

⁵Data Source: www.nasdaq.com; Date: April 30, 2015.

⁶Given the big dataset we have of high-frequency data, the GMM estimation takes over five minutes to get the estimation results for each day. However, the estimation procedure is also quite parallel since the estimation for one day is independent of that for other days. I break the whole sample into seven separate parts and run parallel estimations using different R processes. The whole estimation process takes about 1.5 days to finish.

with the methodology and the 12 instrumental variables of this chapter. Table 2.7 reports the corresponding first 6 rows (days) of the data frame for a two-factor model with SPY and IWM as the two factors. SHY is the numeraire asset for both models.

Table 2.6: Estimation Results for the One-factor Model (first six days)

Date	theta1	se1	t1	p1	jpval
20070103	-1.006	0.062	-16.342	4.931e-60	0.005
20070104	-1.067	0.083	-12.812	1.408e-37	0.235
20070105	-1.073	0.085	-12.654	1.062e-36	0.038
20070108	-0.988	0.080	-12.370	3.796e-35	0.810
20070109	-0.958	0.083	-11.542	8.147e-31	0.918
20070110	-0.887	0.109	-8.145	3.796e-16	0.481

Table 2.7: Estimation Results for the Two-factor Model (first six days)

Date	theta1	se1	t1	p1	theta2	se2	t2	p2	jval
20070103	-0.470	0.149	-3.145	1.661e-03	-0.335	0.085	-3.934	8.344e-05	0.124
20070104	-0.664	0.174	-3.812	1.379e-04	-0.220	0.084	-2.620	8.789e-03	0.605
20070105	-0.468	0.214	-2.191	2.847e-02	-0.346	0.118	-2.926	3.431e-03	0.090
20070108	-0.637	0.223	-2.856	4.292e-03	-0.233	0.139	-1.676	9.383e-02	0.930
20070109	-0.832	0.154	-5.420	5.946e-08	-0.092	0.094	-0.976	3.289e-01	0.929
20070110	-0.828	0.163	-5.078	3.814e-07	-0.051	0.106	-0.478	6.328e-01	0.411

In Table 2.6 and 2.7, the “Date” column is the trading date. “theta1” and “theta2” are the estimates $\hat{\theta}^1$ and $\hat{\theta}^2$ for the first and the second factor, respectively. “se1” and “se2” are the standard errors of the estimates; “t1” and “t2” are the t-statistics obtained from the GMM estimation; “p1” and “p2” are the p-values computed from the t-statistics. Lastly, “jpval” is the p-value for the J-test. In Table 2.6, there is only one factor, while in Table 2.7, there are two factors. As in Chapter 1, I consider estimates that pass the J-test to be those whose Jp-value is greater than 0.1. The significance level is chosen at 5% (i.e., the p-value has to be smaller than 0.05).

The results for these first 6 days are fairly typical of the parameters over the entire 1405-day sample period.

- The risk prices of SPY all have a negative sign, consistent with what the theory suggests.
- $\hat{\theta}^1$ is less negative in Table 2.7 than in Table 2.6.
- The risk price for the second factor is also often significantly different from 0.
- The Jp-value from the J-test is usually above the 0.1 threshold, and are often far above. In sum, the estimation works fairly well.

There are only three days (July 3, 2007, July 3, 2008 and November 27, 2009) that the GMM algorithm fails, complaining that the projection matrix is singular. The first two days are the day before the Independence Day and the last one is the day after the Thanksgiving, which are all market holidays and thus have the half-day trading hours that end at 1:00 pm. For these three days, I use 6 instrumental variables instead of 12 to cover the first half of the day.

The following set of tables summarize the results of the significance and the specification tests for three different models. Table 2.8 summarizes the estimation

results based on the same one-factor model in Chapter 1, re-estimated with the “stacking” specification and the old instrumental variables, but using only ETFs as the test assets. The returns are scaled by a multiplicative factor of 234000 so they can be interpreted as the daily equivalent percentage returns. Table 2.9 summarizes the result of the one-factor model using the new set of instrumental variables. Table 2.10 summarizes the results of different versions of the two-factor model. For the two-factor model, a trading day is defined to be significant if at least one of the estimates of $\hat{\theta}^1$ and $\hat{\theta}^2$ is significant. A more detailed discussion of the estimates will be delayed until the next section.

Table 2.8: One-factor Model using Stacking

Factors	Number	% significant	% pass J-test
SPY	1405	64.9%	99.7%

Table 2.9: One-factor Model using Instrumental Variables

Factors	Number	% significant	% pass J-test
SPY	1405	100%	76.4%

Table 2.10: Two-factor Models with Instrumental Variables

Factors	Number	% significant	% pass J-test
SPY & IWM	1404	99.9%	97.5%
SPY & IWD	1205	85.8%	84.1%
SPY & IWF	1404	92.7%	82.1%
SPY & VWO	1405	100%	81.8%
SPY & XLF	1404	99.9%	79.5%
SPY & XLE	1405	100%	78.7%

Comparing Table 2.8 and Table 2.9, there are fewer days that pass the J-test when using instrumental variables. This suggests some problems with the

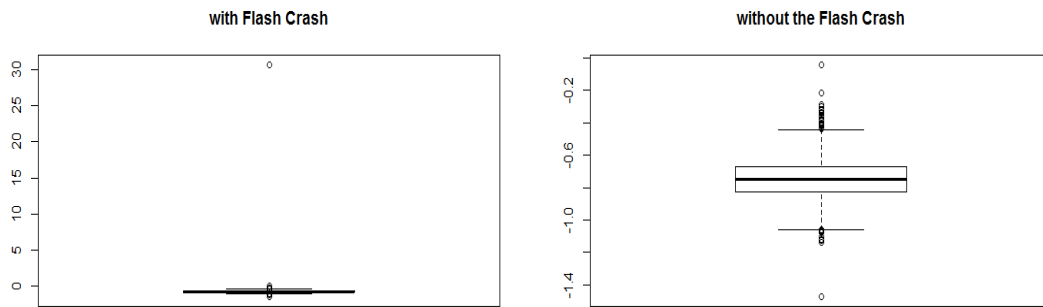
specification, perhaps due to a missing factor. Table 2.10 confirms this suspicion as the percentage that passes the J-test goes up relative to Table 2.9 in all cases. For IWM, the result is much better than Table 2.9 and is almost the same as Table 2.8! The number of trading days with significant estimates increases substantially as compared to the one-factor model in Chapter 1 from 2/3 of the sample to the whole sample in Table 2.10.

Table 2.11 summarizes the number of significant estimates of the risk prices of the first and the second factors, respectively for different choices for the second factor. The risk price of SPY is always significantly different from 0. IWM is not far behind SPY, giving support for the Fama-French “SMB” factor with SPY playing the role of “B” (big- cap stocks) and IWM of “S” (small-cap stocks).

Table 2.11: Number of Significant Estimates

Factors	Significant $\hat{\theta}^1$	% significant	Significant $\hat{\theta}^2$	% significant
SPY & IWM	1396	99.4%	928	66.0%
SPY & VWO	1404	99.9%	287	20.4%
SPY & IWD	1023	72.8%	280	19.9%
SPY & IWF	1192	84.8%	233	16.6%
SPY & XLF	1402	99.8%	165	11.7%
SPY & XLE	1405	100%	128	9.1%

It is also worth mentioning what happens on the day of the Flash Crash, which occurred on May 6, 2010. In only a few minutes the Dow Jones Industrial Average Index plunged by about 9% and recovered within a short period of time. Figure 2.1 shows the box plots of the risk prices of SPY for the two-factor model using SPY and IWM as factors.



(a) SPY & IWM

(b) SPY & IWM

Figure 2.1: Box Plot with and without the Flash Crash

For the box plot, the thick line in the middle of the box represents the median, and the top and the bottom edges are the upper and the lower quantiles, respectively. Those dots outside the upper and the lower bounds of the whiskers are outliers. The impact of the Flash Crash on the scale of the graph on the left compared to that on the right provides graphic evidence of the scale of this event. I remove the estimates made on the day of the Flash Crash from the sample in what follows. After removing this single day, the box plots become very symmetric around the median. As the box plots indicate, including that day would change the scale of our graphs dramatically, rendering the graphs useless.

2.3 The Preferred Two-Factor Model

In this section I focus on the two-factor model with IWM as the second factor. From Table 2.11, it is clear that IWM is the best choice as the second factor: around 2/3 of the days yield significant estimates of $\hat{\theta}^2$, over three times of those of VWO (the second best alternative).

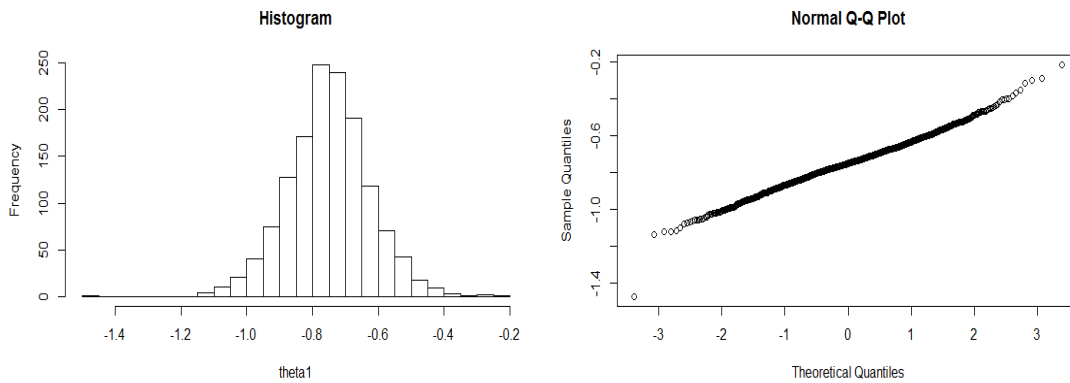
2.3.1 The Estimates

Table 2.12 presents the summary statistics of the estimates of this two-factor model. We compute $\hat{\theta}^0$ using $\hat{\theta}^0 = 1 - \hat{\theta}^1 - \hat{\theta}^2$ (no standard errors or t-statistics are provided for $\hat{\theta}^0$). The statistics of $\hat{\theta}^1$ and $\hat{\theta}^2$ are for the significant estimates of θ^1 and θ^2 , respectively. There are only 8 days where the model yields a significant estimate for θ^2 , but not θ^1 . The statistics of $\hat{\theta}^0$ are computed over the entire sample period (1405 days), regardless of whether $\hat{\theta}^1$ and $\hat{\theta}^2$ are significant or not.

Table 2.12: Summary Statistics for Two-Factor Model

Statistics	$\hat{\theta}^1$	$\hat{\theta}^2$	$\hat{\theta}^0$
Mean	-0.751	-0.248	1.960
Median	-0.750	-0.237	1.963
Mean (-t)	6.137	3.068	
Mean precision	0.163	0.326	
Min	-1.474	-0.658	1.503
Max	-0.215	0.241	2.334

We begin with the analysis of the risk prices for SPY. From Table 2.12, we observe that the estimates of the risk price of SPY are always negative, and $\hat{\theta}^0$ is always positive. This is consistent with the interpretation of the SDF mimicking portfolio, whose return, denoted by R^* , equals $\frac{Z_{t+h}}{Z_t}$ where Z_t and Z_{t+h} are the Radon-Nikodym derivatives for the equivalent martingale measure restricted to \mathcal{F}_t and \mathcal{F}_{t+h} , respectively. Since $R_{t+h}^* = (1 - \theta^0 - \theta^1) R_{t+h}^0 + \theta^1 R_{t+h}^1 + \theta^2 R_{t+h}^2$ in our specification of the SDF, we can interpret the SDF mimicking portfolio as “short SPY” and “long SHY” (the numeraire asset).



(a) SPY & IWM

(b) SPY & IWM

Figure 2.2: Distributions of Estimates of θ^1

Figure 2.2 shows that the distribution of the estimates $\hat{\theta}^1$ are symmetric. The Normal Q-Q plot, ignoring a few outliers at the tails, indicates that the distribution is close to a normal distribution.

The precision ($se(\hat{\theta}^1)/|\hat{\theta}^1|$) of the estimates of θ^1 improves from 20% (0.20) in Chapter 1 to 16% (0.16) in the two-factor model. In Chapter 1, only 2/3 of the days yielded significant estimates of the risk price of SPY. In the two-factor model estimation, the risk price of SPY is almost always significant.

We now turn to the risk prices of IWM. Adding IWM as a second factor can be regarded as a “perturbation” to the original model. Viewed as a “second-order” correction to the one-factor model, the risk price $\hat{\theta}^2$ of IWM could be either positive or negative. The maximum estimate for θ^2 is 0.241, the only case of a positive risk price for the second factor that is statistically significant, which occurred on August 31, 2007. Excluding that day, the maximum estimate is -0.119. Apart from this day, the risk prices of IWM are negative, indicating that the SDF mimicking portfolio is “short IWM” as well as SPY.

Table 2.13 presents the summary statistics of the remaining 1/3 of the insignificant estimates of θ^2 . We observe that the mean t-statistic is slightly larger than one, and about 30% of the estimates are within one standard error around zero.

Table 2.13: Summary Statistics for Insignificant $\hat{\theta}^2$

Statistics	$\hat{\theta}^2$
Mean	-0.142
Median	-0.143
Mean $ t $	1.283
% within one SE	27.7%
Min	-0.413
Max	0.141

2.3.2 Variation of Risk Prices Over Time

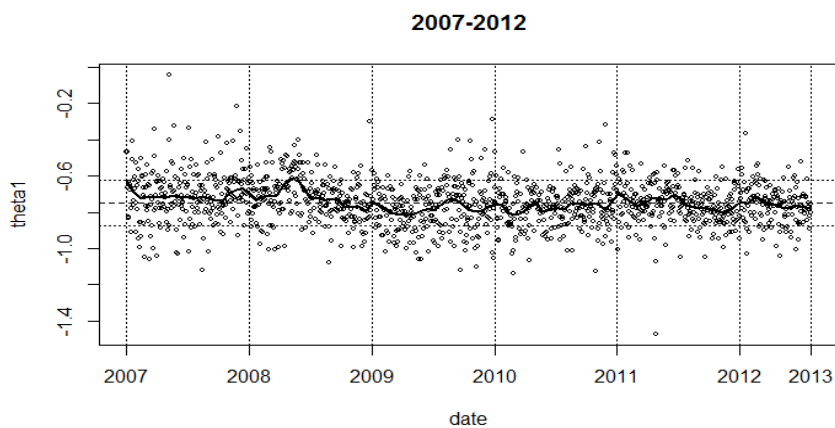
This section studies the behavior of risk prices over time.

Table 2.14 provides summary statistics for those estimates of $\hat{\theta}^1$, the risk price of SPY, by years. The means and medians for $\hat{\theta}^1$ are quite similar across years. The mean estimated risk prices become slightly more negative in 2009 and stay lower than the means and medians in 2007 and 2008, but overall the risk prices are quite stable over the entire period. The precision improves after 2007, is best in the years 2010 and 2011, and decreases in 2012.

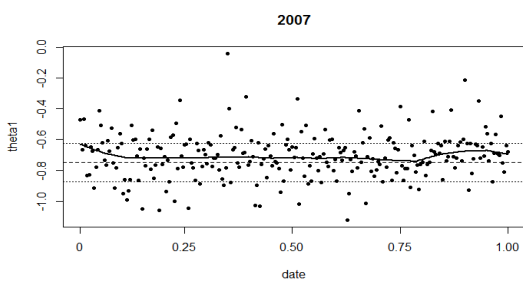
Table 2.14: Summary Statistics for the Risk Price of SPY

Year	Statistics	$\hat{\theta}^1$	Year	Statistics	$\hat{\theta}^1$
2007	Mean	-0.706	2010	Mean	-0.768
	Median	-0.711		Median	-0.772
	Mean (-t)	4.839		Mean (-t)	6.838
	Mean precision	0.207		Mean precision	0.146
	Min	-1.123		Min	-1.138
	Max	-0.041		Max	-0.318
2008	Mean	-0.721	2011	Mean	-0.760
	Median	-0.731		Median	-0.759
	Mean (-t)	6.121		Mean (-t)	6.948
	Mean precision	0.163		Mean precision	0.144
	Min	-1.080		Min	-1.474
	Max	-0.299		Max	-0.471
2009	Mean	-0.780	2012	Mean	-0.758
	Median	-0.778		Median	-0.751
	Mean (-t)	5.752		Mean (-t)	6.259
	Mean precision	0.174		Mean precision	0.160
	Min	-1.118		Min	-1.042
	Max	-0.287		Max	-0.366

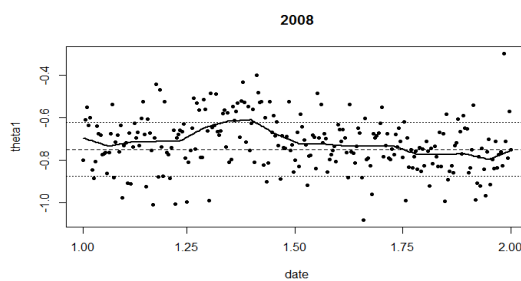
Figure 2.3 displays a plot of the time series of the daily risk price of SPY (recall the sample ends at July 31, 2012).



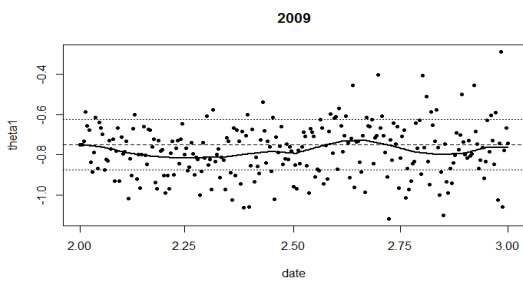
(a) 2007-2012



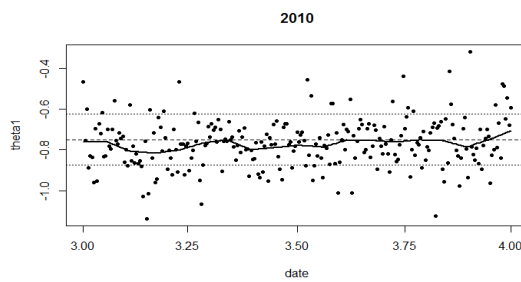
(b) 2007



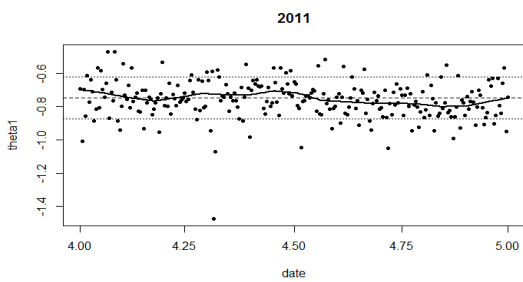
(c) 2008



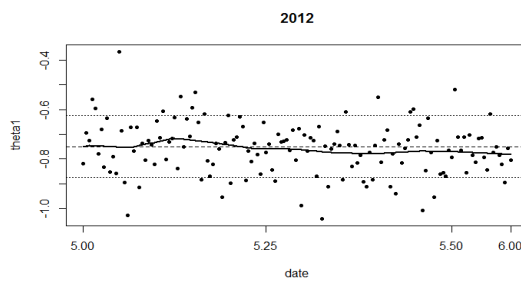
(d) 2009



(e) 2010



(f) 2011



(g) 2012

Figure 2.3: Time Series of Estimates of θ^1

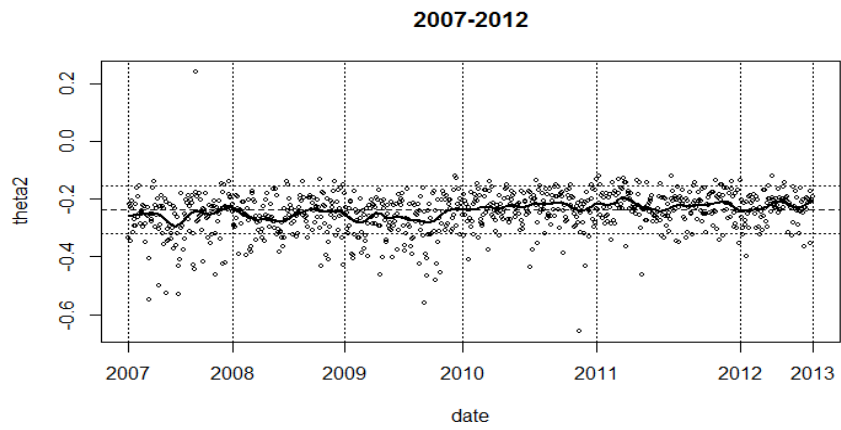
In Figure 2.3, each year is considered to be a unit of time of length 1. The estimated risk prices are plotted year by year as time series with each dot corresponding to a daily risk price. The dashed line measures the median of the estimates over the entire period (in this case equals -0.750), which serves as the reference line of the plots. The median is chosen instead of the mean in order to eliminate the effect of outliers. The dotted lines plot the median standard error over the entire period. The solid lines are obtained from the “lowess” method, which measures the central tendency of the data. Cleveland (1979) suggests that the smooth span be between 0.2 and 0.5. In this study, I choose the smooth span to be 0.2, or 20% of the number of data points in each year so that about 50 days (a little more than two months worth of data) are used to compute the lowess estimates. Most of the estimates fall within -0.8 and -0.6 , suggesting the risk price is rather stable over the whole sample period. The lowess curve rises above the upper band in early 2008, but then starts to decline until mid 2009 and remains below the median until early 2010, which could be attributed to the impact of the financial crisis and the Euro crisis.

We turn next to the risk price of IWM. Table 2.15 provides summary statistics for the significant estimates of the risk price for IWM. Except for one day, the risk prices for IWM are always negative. The mean risk prices are more negative for the years 2007 to 2009, exhibiting more impact from the Great Recession than does SPY.

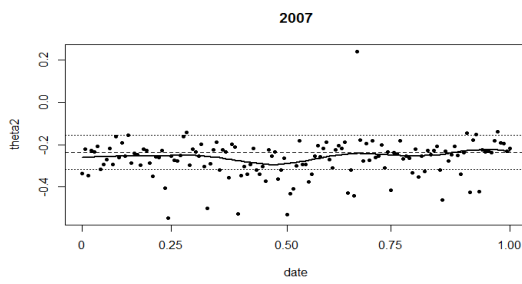
Table 2.15: Summary Statistics for the Risk Price of IWM

Year	Statistics	$\hat{\theta}^2$	Year	Statistics	$\hat{\theta}^2$
2007	Mean	-0.267	2010	Mean	-0.236
	Median	-0.253		Median	-0.227
	Mean (-t)	2.817		Mean (-t)	3.234
	Mean precision	0.355		Mean precision	0.309
	Min	-0.547		Min	-0.658
	Max	0.241		Max	-0.133
2008	Mean	-0.259	2011	Mean	-0.225
	Median	-0.258		Median	-0.220
	Mean (-t)	3.031		Mean (-t)	3.157
	Mean precision	0.330		Mean precision	0.317
	Min	-0.432		Min	-0.464
	Max	-0.132		Max	-0.119
2009	Mean	-0.274	2012	Mean	-0.231
	Median	-0.260		Median	-0.222
	Mean (-t)	2.943		Mean (-t)	3.145
	Mean precision	0.340		Mean precision	0.318
	Min	-0.559		Min	-0.398
	Max	-0.121		Max	-0.120

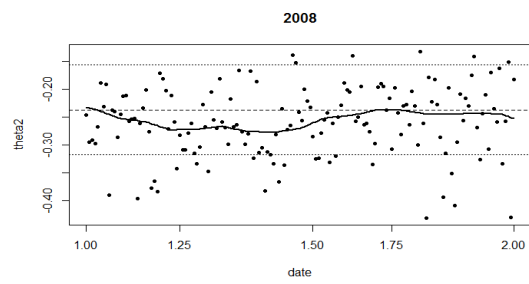
Figure 2.4a gives the plot of the time series of the daily risk price of IWM. The negative estimates for θ^2 indicate that the SDF mimicking portfolio shorts IWM as well. We notice that the risk prices are mostly below the median from 2007 to the end of 2009, and start to rise above the median since 2010. Most of the the estimates are between -0.4 and -0.1 and are reasonably stable over the sample period.



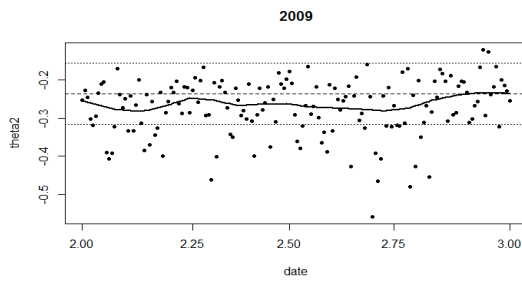
(a) 2007-2012



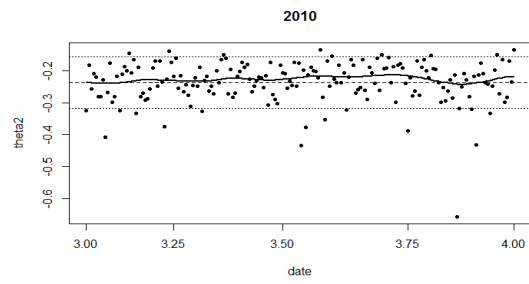
(b) 2007



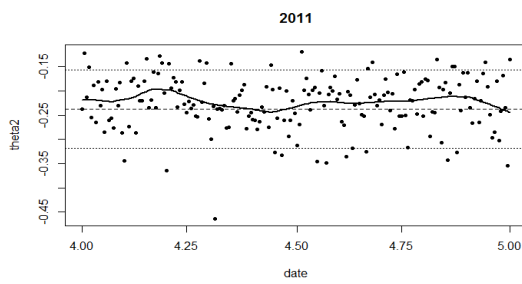
(c) 2008



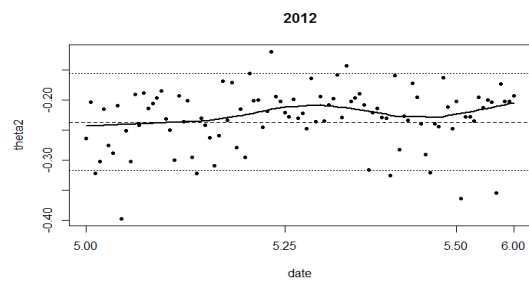
(d) 2009



(e) 2010



(f) 2011



(g) 2012

Figure 2.4: Time Series of Estimates of θ^2

Given the estimates $\hat{\theta}^1$ and $\hat{\theta}^2$, we can compute the estimate $\hat{\theta}^0 = 1 - \hat{\theta}^1 - \hat{\theta}^2$ for the numeraire, which is a measure of the overall risk in the market with more “positive” $\hat{\theta}^0$ being more risky. Figure 2.5 displays the time series plot of $\hat{\theta}^0$ for the whole sample period under study. The dashed line is the median of $\hat{\theta}^0$ for the whole period, and the solid curve shows the central tendency obtained from the lowess method.

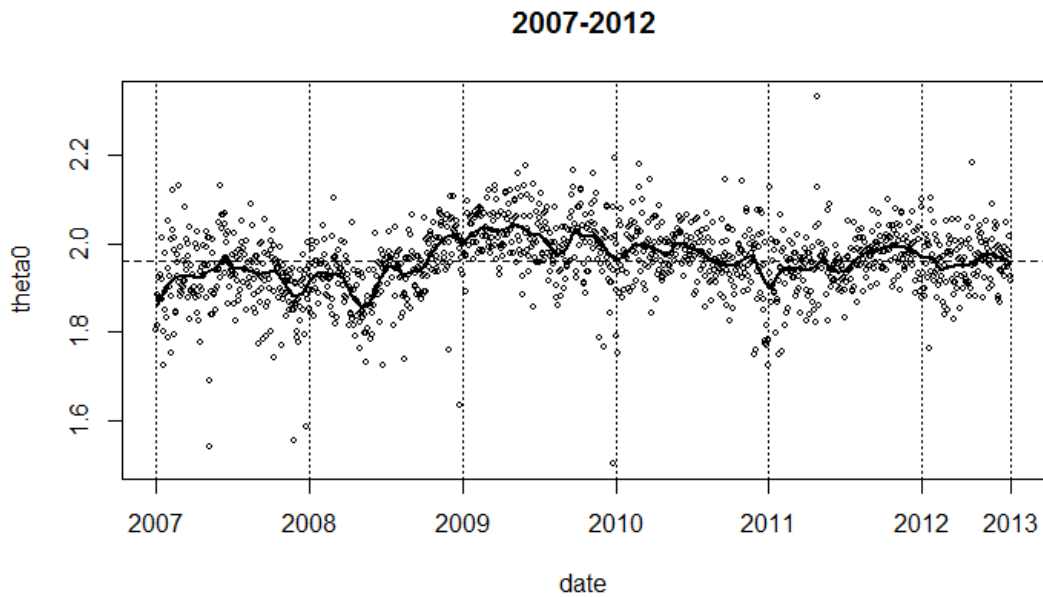


Figure 2.5: Estimates of θ^0

θ^0 represents the share of the numeraire asset in the SDF mimicking portfolio. The SDF mimicking portfolio becomes longer in SHY in late 2008. The long position reaches its peak at around March 2009, and then gradually declines. In other words, the short positions in risky assets (here, SPY and IWM) start to increase since late 2008 to hedge the increasing riskiness of the market caused by the financial crisis, as we would expect.

I also examine the behavior of the SDF mimicking return. With a multi-factor model, we can compute the SDF mimicking return R^* using the estimated $\hat{\theta}^1$ and $\hat{\theta}^2$. To be specific, given there are 2340 10-second intervals in a day, if θ^0 , θ^1 and

θ^2 are constant over the day, then for each 10-second interval, we have

$$R_t^* = (1 - \theta^1 - \theta^2) R_t^0 + \theta^1 R_t^1 + \theta^2 R_t^2, \quad t = 1, 2, \dots, 2340$$

which implies

$$\frac{1}{2340} \sum_{t=1}^{2340} R_t^* = (1 - \theta^1 - \theta^2) \frac{1}{2340} \sum_{t=1}^{2340} R_t^0 + \theta^1 \frac{1}{2340} \sum_{t=1}^{2340} R_t^1 + \theta^2 \frac{1}{2340} \sum_{t=1}^{2340} R_t^2$$

Let

$$\bar{R}^0 = \frac{1}{2340} \sum_{t=1}^{2340} R_t^0, \quad \bar{R}^1 = \frac{1}{2340} \sum_{t=1}^{2340} R_t^1, \quad \bar{R}^2 = \frac{1}{2340} \sum_{t=1}^{2340} R_t^2, \quad \bar{R}^* = \frac{1}{2340} \sum_{t=1}^{2340} R_t^*$$

then we have

$$\bar{R}^* = (1 - \hat{\theta}^1 - \hat{\theta}^2) \bar{R}^0 + \hat{\theta}^1 \bar{R}^1 + \hat{\theta}^2 \bar{R}^2$$

where \bar{R}^0 , \bar{R}^1 and \bar{R}^2 are the average 10-second returns of SHY, SPY and IWM, respectively, over a trading day.

Below is the time series of \bar{R}^* over the whole period. The dashed line is the median of the data and the smooth span of the lowest curve is chosen to be 20% of the number of data points in each year, as explained previously. The returns are scaled by 2340 to get the equivalent daily returns.

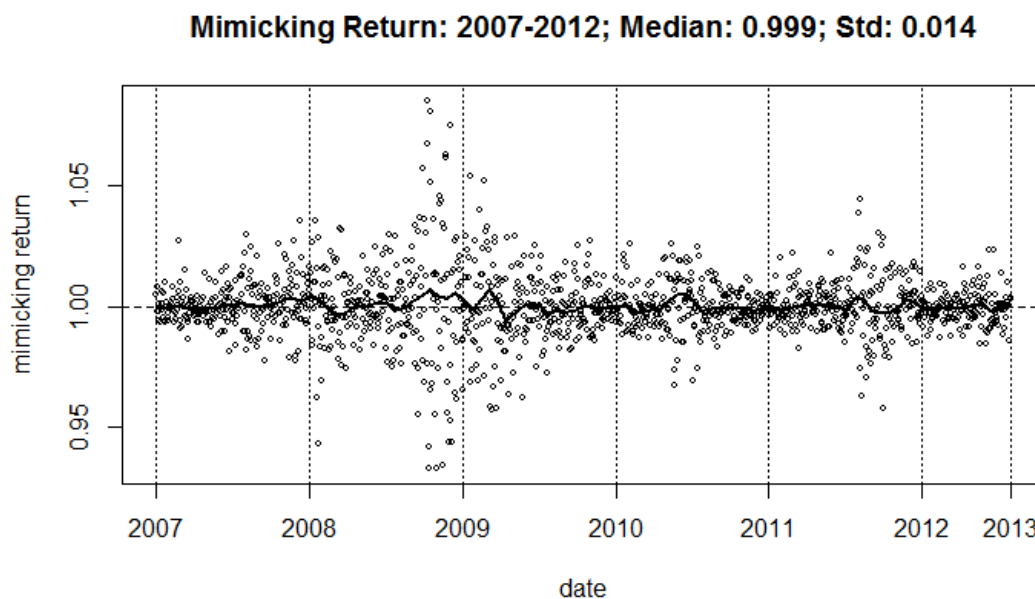


Figure 2.6: The SDF Mimicking Returns

From Figure 2.6, the time series of the SDF mimicking returns is symmetric around the median. It is obviously that the variance of the mimicking returns increases a lot during the financial crisis period (late 2008 to 2009). Also, during the financial crisis in 2008 and 2009 and the Euro crisis in mid 2010, the central tendency clearly deviates from the median. For the rest of the period, it is very close to the median line.

2.4 Conclusion

This chapter improves the estimation of the one-factor model in Chapter 1. Introducing instrumental variables that sub-divide the day into shorter time intervals makes a big difference. The fraction of significant estimates increases from about 2/3 of the sample to essentially every day, and the precision of the estimates increases substantially. The estimation results reinforce the conclusion from Chapter 1 that SPY is an important factor. The identification of IWM as the second

most important factor indicates strongly that multiple risk factors are needed and, in particular, provides support for including an index of small-capitalization stocks as a complement to SPY.

CHAPTER 3

Three-Factor Models

In Chapter 2 I estimated a two-factor model using 10-second returns and instrumental variables that sub-divide the day into 12 non-overlapping time intervals. SPY is the first factor and IWM is the second. The estimation works successfully and substantially improves the results of Chapter 1. SPY is the most important risk factor: its risk price is significant almost every day, and the estimates are always negative, which is consistent with the theory. The second factor, IWM, improves the performance of the model significantly as compared to the one-factor model. The risk price estimates for IWM are significant for about 2/3 of the days, and all estimates are negative.

In this Chapter, I examine the possibility of a third factor to the two-factor model of Chapter 2. The time period is expanded by two years from January 1, 2007 to July 31, 2014, covering a time period of 1886 days. The estimation method is the same as that of Chapter 2 except for the addition of a third factor.

3.1 Estimating Three-Factor Models

Although the two-factor model with SPY and IWM as risk factors works quite well, I want to examine whether adding another factor to the model can further improve the specification of the model. For instance, it is interesting to explore whether adding the Fama-French “HML” factor or a factor that tracks the performance of the foreign market would affect the estimation result. Adding a third

factor can be considered as a further perturbation to the model. Extending the study to the middle of 2014 also provides an “out-of-sample” test of the two-factor model.

3.1.1 Test Assets and the Third Factor

In this chapter I use the same set of test assets as in Chapter 2 to estimate the three-factor model. Specifically I use exchange-traded index funds rather than individual stocks to lessen the impact of idiosyncratic risks. Furthermore, using the same test assets render the results obtained from the three-factor model comparable to those from the two-factor model. Table 3.1 lists the ETFs used for the estimation in this chapter. Twenty test assets are used for the estimation.

Table 3.1: ETFs as Test Assets

Ticker	Description	Ticker	Description
SPY	S&P 500	VWO	Vanguard foreign stocks
IYR	US real estate	XLB	SPDR material
XLE	SPDR energy	XLF	SPDR financial
XLI	SPDR industrial	XLK	SPDR technology
XLP	SPDR consumer staples	XLU	SPDR utility
XLV	SPDR healthcare	XLY	SPDR consumer discretionary
IWS	Russell mid-cap value	IWM	Russell 2000
IWO	Russell 2000 growth	IWN	Russell 2000 value
IWB	Russell 1000	IWF	Russell 1000 growth
IWD	Russell 1000 value	IWR	Russell mid-cap

For the three-factor models in this chapter, I keep SHY as the numeraire and retain SPY and IWM as factors. I consider the following ETFs as candidates for the third factor:

- VWO: A Vanguard ETF that focuses on foreign markets.
- XLF: A SPDR sector ETF that tracks the financial sector.

- IWD: The Russell 1000 Value ETF that tracks the performance of large-cap value stocks.
- IWF: The Russell 1000 Growth ETF that tracks the performance of large-cap growth stocks. IWD and IWF together resemble the “H” and “L” component of the Fama-French “HML” (High Minus Low) factor, where “High” and “Low” refer to the price-to-book ratio.

3.1.2 Model Setup

Recall the moment condition for our asset-pricing model derived from the no-arbitrage theory in Chapter 1:

$$E \left[r_{t+h}^n + \theta^1 R_{t+h}^{10} \tilde{R}_{t+h}^n + \cdots + \theta^k R_{t+h}^{k0} \tilde{R}_{t+h}^n + \cdots + \theta^K R_{t+h}^{K0} \tilde{R}_{t+h}^n \right] = 0 \quad (3.1)$$

where $n = 1, 2, \dots, N$ is the index for the test assets, and $k = 1, 2, \dots, K$ denotes the index for the factors. In our estimation of the three-factor model, we will have $N = 20$ and $K = 3$. R_{t+h}^n and r_{t+h}^n denote the gross and the net returns of the asset n at time $t + h$, respectively. \tilde{R}_{t+h}^n is the return of the asset n discounted by the numeraire, and $R_{t+h}^{k0} = R_{t+h}^k - R_{t+h}^0$, the excess return of the k th factor relative to the numeraire. In Chapter 2, the moment condition for the two-factor model took the form

$$E \left[r_{t+h}^n + \theta^1 R_{t+h}^{10} \tilde{R}_{t+h}^n + \theta^2 R_{t+h}^{20} \tilde{R}_{t+h}^n \right] = 0 \quad (3.2)$$

where R_{t+h}^1 and R_{t+h}^2 are the gross returns of SPY and IWM, respectively. The instrumental variables, the same as in Chapter 2, divide the day into 12 non-overlapping time sub-intervals. The moment condition is applied to all the test assets. The same stacking technique used in Chapter 2 is used for the estimation of the three-factor models in this chapter. Because there are 12 moment conditions

and 3 parameters to estimate, the model is still over-identified.

3.1.3 The Issue of Multicollinearity

For the three-factor model, one can extend equation (3.2) to the three-factor case:

$$E \left[r_{t+h}^n + \theta^1 R_{t+h}^{10} \tilde{R}_{t+h}^n + \theta^2 R_{t+h}^{20} \tilde{R}_{t+h}^n + \theta^3 R_{t+h}^{30} \tilde{R}_{t+h}^n \right] = 0 \quad (3.3)$$

and then the estimation can proceed similarly as in Chapter 2. Table 3.2 compares the number of significant estimates of the risk prices of SPY and IWM in the two-factor from Chapter 2 and a three-factor model using XLF (which tracks an index of financial stocks) as the third factor. The estimation is over the original sample in Chapter 2 (1405 days).

Table 3.2: Significant Estimates Across Models

Model	$\hat{\theta}^1$	$\hat{\theta}^2$	$\hat{\theta}^3$
Two-Factor Model	1396	928	
	99.4%	66.0%	
Three-Factor Model	1327	858	56
	94.4%	61.1%	4.0%

Adding XLF as the third factor to the model decreases the number of significant estimates of the risk price of SPY by 79 days, or 5% of the total sample. The number of significant estimates of the risk price of IWM falls by 70. For the two-factor model multicollinearity was not much an issue because SPY consists of high-capitalization stocks and IWM of low-capitalization stocks. But XLF tracks an index of a subset of the 500 stocks that forms the S&P 500 index, the financial sector. Because it tracks a component of the S&P 500 index, we expect much higher correlation with the ETF SPY that tracks the S&P 500 index. Therefore, there is reason to suspect that multicollinearity is an issue in the estimation when

the third factor is added to the model. The problem is worse with the other factors we are considering as candidates for a third factor.

3.1.4 Using Excess Returns

In order to reduce the multicollinearity problem, I use excess returns of the ETFs, rather than gross returns. To be specific, if XLF or VWO is chosen as the third factor, the following moment condition is used for estimation:

$$E \left[r_{t+h}^n + \theta^1 R_{t+h}^{10} \tilde{R}_{t+h}^n + \theta^2 R_{t+h}^{20} \tilde{R}_{t+h}^n + \theta^3 R_{t+h}^{31} \tilde{R}_{t+h}^n \right] = 0 \quad (3.4)$$

where $R_{t+h}^{31} = R_{t+h}^3 - R_{t+h}^1$ is the excess return of SPY relative to the additional ETF. If XLF is the additional ETF, it is the difference between the return of XLF and the return of SPY, and in the case of VWO, it is the difference between the returns of VWO and SPY.

One can derive the specification using excess returns in the following way. We first consider the following moment condition:

$$E \left[r_{t+h}^n + \hat{\theta}^1 R_{t+h}^{10} \tilde{R}_{t+h}^n + \hat{\theta}^2 R_{t+h}^{20} \tilde{R}_{t+h}^n + \hat{\theta}^3 R_{t+h}^{30} \tilde{R}_{t+h}^n \right] = 0 \quad (3.5)$$

which is the moment condition for the general three-factor model. Equation (3.5) can be written as:

$$E \left[r_{t+h}^n + \left(\hat{\theta}^1 + \hat{\theta}^3 \right) R_{t+h}^{10} \tilde{R}_{t+h}^n + \hat{\theta}^2 R_{t+h}^{20} \tilde{R}_{t+h}^n + \hat{\theta}^3 \left(R_{t+h}^{30} - R_{t+h}^{10} \right) \tilde{R}_{t+h}^n \right] = 0 \quad (3.6)$$

Without loss of generality, by letting $\theta^1 = \hat{\theta}^1 + \hat{\theta}^3$, $\theta^2 = \hat{\theta}^2$ and $\theta^3 = \hat{\theta}^3$, we arrive at equation (3.4). From equation (3.4), the SDF mimicking portfolio can be seen as investing in the numeraire, SPY, IWM, and the excess return of XLF (or VWO) with respect to SPY.

The same moment condition could be applied to IWD and IWF, ETFs that

track the Russell 1000 value and the Russell 1000 growth indices. Instead, I follow the specification of the Fama-French model by using excess return of IWD over IWF as the third factor. We get the following moment condition:

$$E \left[r_{t+h}^n + \theta^1 R_{t+h}^{10} \tilde{R}_{t+h}^n + \theta^2 R_{t+h}^{20} \tilde{R}_{t+h}^n + \theta^3 R_{t+h}^{34} \tilde{R}_{t+h}^n \right] = 0 \quad (3.7)$$

Equation (3.7) can be derived from the following moment condition of the four-factor case:

$$E \left[r_{t+h}^n + \theta^1 R_{t+h}^{10} \tilde{R}_{t+h}^n + \theta^2 R_{t+h}^{20} \tilde{R}_{t+h}^n + \theta^3 R_{t+h}^{30} \tilde{R}_{t+h}^n + \theta^4 R_{t+h}^{40} \tilde{R}_{t+h}^n \right] = 0 \quad (3.8)$$

with the restriction that $\theta^4 = -\theta^3$. From equation (3.7), the SDF mimicking portfolio invests in the numeraire, SPY, IWM, and the excess return of IWD over IWF (known as the “HML” factor in Fama-French model).

3.1.5 Estimating Three-Factor Models

As in the previous chapters, the estimation procedure is estimated day-by-day in a loop over the 1886 days in the sample. The results are assembled into an R data frame. Table 3.3 shows the estimation results for the first six days with the difference of the returns of IWD and IWF as the third factor.

Table 3.3: Estimation Results for the Three-factor Model (first six days)

date	theta1	se1	t1	p1	theta2	se2	t2	p2	theta3	se3	t3	p3	jpval
20070103	-0.46	0.15	-3.11	0.00	-0.30	0.09	-3.31	0.00	-0.14	0.15	-0.91	0.36	0.09
20070104	-0.69	0.18	-3.77	0.00	-0.23	0.08	-2.67	0.01	0.10	0.18	0.56	0.57	0.61
20070105	-0.51	0.23	-2.20	0.03	-0.32	0.13	-2.47	0.01	-0.13	0.27	-0.47	0.64	0.08
20070108	-0.62	0.23	-2.70	0.01	-0.26	0.15	-1.81	0.07	0.16	0.15	1.11	0.27	0.97
20070109	-0.81	0.16	-5.11	0.00	-0.11	0.10	-1.14	0.25	0.11	0.15	0.72	0.47	0.92
20070110	-0.83	0.16	-5.07	0.00	-0.04	0.11	-0.36	0.72	-0.04	0.14	-0.26	0.79	0.35

In Table 3.3, “theta1”, “theta2” and “theta3” are the estimates of the risk prices of the first, the second and the third factor, respectively. “se1”, “se2” and “se3” are the corresponding standard errors; “t1”, “t2” and “t3” are the t-statistics; “p1”, “p2” and “p3” are the p-values. “jpval” is the p-value for the J-test. I consider estimates that pass the J-test to be those whose Jp-value is greater than 0.1. The significance level for parameter estimates is chosen at 5% (i.e., the p-value has to be smaller than 0.05).

The results for the first six days are fairly typical over the entire sample. Similar findings as in Chapter 2 can be concluded for the first two factors:

- The risk prices for SPY and IWM are negative.
- The risk price for SPY is always significant; for most days, the risk price for IWM is significant as well.
- For most days, the estimate passes the J-test with a J-test p-value greater than 0.1.

For the third factor, almost all of the days yield insignificant estimates (for the first six days here, none of the day has significant estimate) for its risk price. In addition, the sign of the estimates can be either positive or negative, although this is not inconsistent with the theory since the perturbation to the model from the third factor incorporates the excess return of two ETFs.

Table 3.4 presents the number of days with significant estimates and the number that passes the J-test across different three-factor models and the two-factor model in Chapter 2. For a trading day to have significant estimates, at least one of the risk prices on that day has to be significant.

Table 3.4: Significant Test and J-test Results

Model	Number	% significant	% pass J-test
Two-Factor (1405 days)	1404	99.9%	97.5%
VWO-SPY	1883	99.8%	98.1%
(First 1405 days)	1402	99.8%	97.8%
IWD-IWF	1883	99.8%	97.6%
(First 1405 days)	1402	99.8%	97.4%
XLF-SPY	1881	99.7%	97.8%
(First 1405 days)	1400	99.6%	97.4%

From Table 3.4, the number and the percentage of significant estimates are quite similar across different models. In particular, all three-factor models yield almost the same number of significant estimates as the two-factor model. The three-factor model with “VWO-SPY” as the third factor has slightly higher percentage of days than the two-factor model that pass the J-test, suggesting an improvement in the model specification when adding the third factor. The other two models has similar percentages as the two-factor model.

Table 3.5 summarizes the number of significant estimates for the three factors separately.

Table 3.5: Number of Significant Estimates

Factors	$\hat{\theta}^1$	% significant	$\hat{\theta}^2$	% significant	$\hat{\theta}^3$	% significant
VWO-SPY	1869	99.1%	1161	61.6%	176	9.3%
IWD-IWF	1866	98.9%	1192	63.2%	69	3.7%
XLF-SPY	1863	98.8%	1168	61.9%	73	3.9%

From Table 3.5, we can see clearly that VWO-SPY is the dominant third factor, accounting for about three times the number of significant estimates of θ^3 than the two alternatives. The percentages of significant estimates of θ^1 and θ^2 are similar across different models, and they are only slightly lower when a third

factor is added than those from the two-factor model of Chapter 2 where 99.4% and 66.0% of the estimates of θ^1 and θ^2 are significant.

3.2 The Preferred Three-Factor Model

For the following analysis, I focus on the three-factor model with VWO-SPY as the third factor. From now on, due to the large impact of the Flash Crash on the scale of the estimates discovered in Chapter 2, I remove the day of the Flash Crash (May 6, 2010) from the sample to further study the estimation results.

3.2.1 Comparison with the Two-Factor Model

Table 3.6 presents the summary statistics of the estimates and their comparison with those from the two-factor model in Chapter 2. We compute $\hat{\theta}^0$ using $\hat{\theta}^0 = 1 - \hat{\theta}^1 - \hat{\theta}^2 - \hat{\theta}^3$ (no standard errors or t-statistics are provided for $\hat{\theta}^0$). Except for $\hat{\theta}^0$ whose summary statistics are over the whole sample period, the summary statistics for the other parameters are for their individual significant estimates only.

Table 3.6: Summary Statistics for Three-Factor Model

Statistics	$\hat{\theta}^1$	$\hat{\theta}^2$	$\hat{\theta}^3$	$\hat{\theta}^0$
Mean	-0.750	-0.243	-0.204	2.032
Median	-0.750	-0.232	-0.199	2.029
Mean $ t $	5.853	2.946	2.374	
Mean precision	0.171	0.340	0.421	
Min	-1.380	-0.638	-0.487	1.303
Max	-0.330	-0.110	0.457	2.545
Mean difference with 2-factor model	0.038	0.026		

In Table 3.6, “Mean $|t|$ ” is the mean of the absolute value of the t-statistics

for each estimated parameter. “Mean precision” is defined as $1/\text{Mean } |t|$, the ratio of the standard error of the estimate to the estimate. “Mean difference with 2-factor model” is the mean of the absolute value of the difference between the estimates from the two-factor model and the three-factor model during the first 1405 days (the sample period covered by the two-factor model).

To begin with, we analyze the risk prices of SPY and IWM in the three-factor model. First, compared to the results from Chapter 2 (Table 2.12), the summary statistics are quite similar for these two risk factors. The precisions of both $\hat{\theta}^1$ and $\hat{\theta}^2$ degrade slightly, but not by much. Second, the risk prices of SPY and IWM are always negative, indicating that, as with the two-factor model, the SDF mimicking portfolio in the three-factor model is still “short” SPY and IWM. Third, the mean differences of the two estimates with those from the two-factor model are small, suggesting that adding the third factor perturbs the model slightly and does not affect the estimates of the risk prices of SPY and IWM a lot.

We now turn to the risk price of the third factor, VWO-SPY. There are only two cases where the estimates are positive and significant: on December 26, 2007 when $\hat{\theta}^3 = 0.204$, and on April 15, 2011 when $\hat{\theta}^3 = 0.457$. Without those two estimates, the maximum estimate is -0.083 , suggesting that the SDF mimicking portfolio is also “short” the excess return of VWO over SPY. The precision is worse than those of the first and the second factor, and for 90% of the days when $\hat{\theta}^3$ is insignificant, the two-factor model would suffice.

For the overall performance of the model, first we notice that the precision of the risk price of SPY is best, followed by that of IWM and then VWO-SPY. This reinforces the conclusion that SPY is the most dominant factor, and IWM is the second most important factor. VWO-SPY, on the other hand, serves as a modest correction to the model. The long position in the “risk-free” asset $\hat{\theta}^0$, is positive over the entire period, indicating that the SDF mimicking portfolio is “long” SHY all the time.

3.2.2 The Variation of Risk Prices Over Time

To further the analysis, we now focus on the time series of the estimates of risk prices.

3.2.2.1 The Risk Prices of SPY and IWM

Table 3.7 shows the summary statistics for the significant estimates of $\hat{\theta}^1$ and $\hat{\theta}^2$ by year. The data set ends at July 31, 2014, leaving a five-month gap at the end of the year 2014.

Table 3.7: Summary Statistics for the Risk Prices of SPY and IWM

Year	Statistics	$\hat{\theta}^1$	$\hat{\theta}^2$	Year	Statistics	$\hat{\theta}^1$	$\hat{\theta}^2$
2007	Mean	-0.713	-0.257	2011	Mean	-0.764	-0.226
	Median	-0.699	-0.246		Median	-0.762	-0.219
	Mean $ t $	4.667	2.763		Mean $ t $	6.624	3.005
	Mean precision	0.214	0.362		Mean precision	0.151	0.333
	Min	-1.084	-0.458		Min	-1.127	-0.389
	Max	-0.358	-0.119		Max	-0.376	-0.120
2008	Mean	-0.723	-0.258	2012	Mean	-0.751	-0.225
	Median	-0.725	-0.254		Median	-0.758	-0.227
	Mean $ t $	5.774	2.875		Mean $ t $	6.237	3.041
	Mean precision	0.173	0.348		Mean precision	0.160	0.329
	Min	-1.238	-0.456		Min	-1.163	-0.368
	Max	-0.330	-0.136		Max	-0.408	-0.127
2009	Mean	-0.781	-0.272	2013	Mean	-0.757	-0.237
	Median	-0.780	-0.259		Median	-0.746	-0.233
	Mean $ t $	5.476	2.817		Mean $ t $	6.067	2.895
	Mean precision	0.183	0.355		Mean precision	0.165	0.345
	Min	-1.209	-0.563		Min	-1.380	0.432
	Max	-0.397	-0.139		Max	-0.427	-0.127
2010	Mean	-0.773	-0.230	2014	Mean	-0.724	-0.219
	Median	-0.776	-0.222		Median	-0.729	-0.206
	Mean $ t $	6.431	3.081		Mean $ t $	5.169	3.039
	Mean precision	0.155	0.325		Mean precision	0.193	0.329
	Min	-1.133	-0.639		Min	-1.004	-0.442
	Max	-0.358	-0.110		Max	-0.378	-0.140

We first focus on the mean and the median of the estimates for the whole period. From Table 3.7, for both $\hat{\theta}^1$ and $\hat{\theta}^2$, their respective mean and median are very close, suggesting the distribution of the estimates is quite symmetric. For $\hat{\theta}^1$, the largest difference is 0.14 (in year 2007). For $\hat{\theta}^2$, the largest difference is 0.13 (in year 2009). Throughout the rest of the years, the difference is almost always smaller than 0.1. The mean and the median estimates of $\hat{\theta}^1$ become more negative in year 2008 and even more negative in 2009. As the financial crisis abated, they become less negative gradually. By 2014 the level is similar to 2008. This pattern illustrates the impact of the financial recession, during which the risk price increased. Overall the risk prices are quite stable over the years. For the risk price of IWM, the overall mean and median estimates are quite similar across the eight-year period except in 2009, which, affected by the financial recession, is more negative.

The minimum and the maximum of the estimated risk prices measure the overall spread of the estimates. For the risk price of SPY, the difference between the minimum (around -1.15) and the maximum (around -0.35) is quite large, amount to about 0.8 over the entire period, suggesting a large spread of the estimates. Also, during the financial crisis both the minimum and the maximum tend to become more negative, reinforcing the findings above about the mean and the median. For the risk price of IWM, the difference between the minimum and the maximum is much smaller, usually around 0.3. In addition, we also observe that the minimum of the estimates becomes substantially more negative in 2009 and 2010 during the financial crisis and the Euro crisis. Correspondingly, the spread also becomes larger during this period.

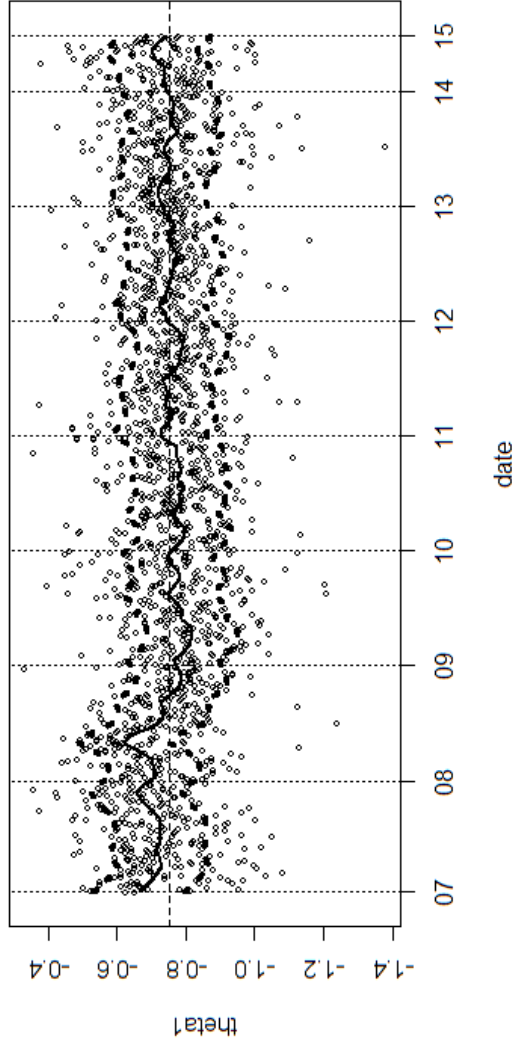
Third, the t-statistics of both $\hat{\theta}^1$ and $\hat{\theta}^2$ are quite high, suggesting the estimates are highly significant. For the risk price of SPY, the mean $|t|$ over the whole sample period is around 5 to 6, corresponding to an average p-value of about 0.002, which is significant even at the 1% level. For the risk price of IWM, the mean $|t|$ over

the whole sample period is about 3, corresponding to an average p-value of 0.011. The estimated risk prices of IWM over the period are also highly significant at 5% level, although not significant at 1% level.

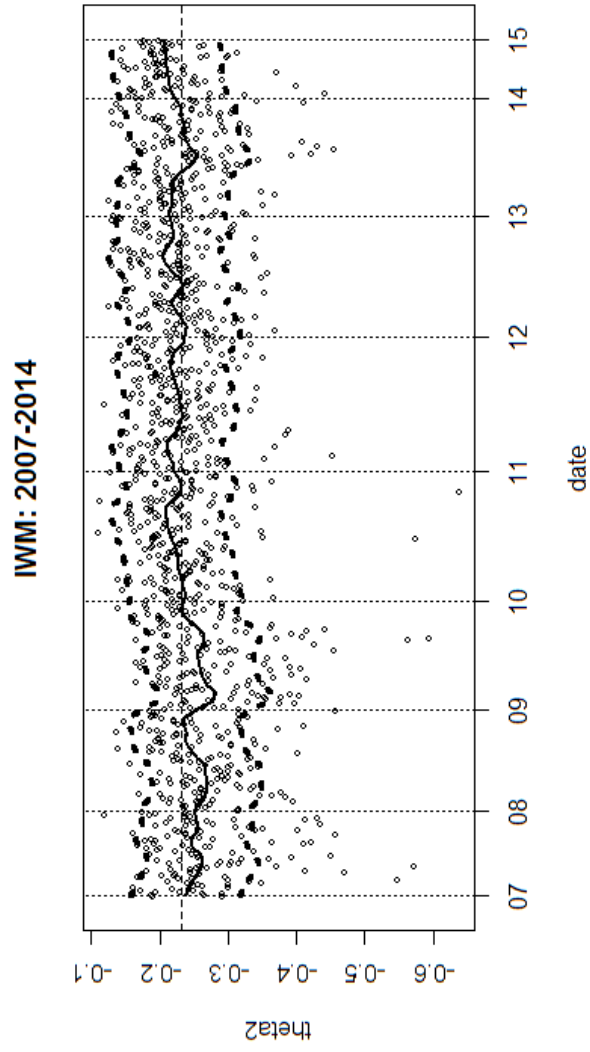
The precision measures the error of the estimates. For $\hat{\theta}^1$, the average precision over the period is about 16%, and for $\hat{\theta}^2$, it is about 34%. For instance, in year 2012 (which is a typical year for the whole sample period), the precision for $\hat{\theta}^1$ is 16%, and thus on average θ^1 is expected to be between $-0.751 \pm (0.16 \times 0.751) = [-0.871, -0.631]$ with a gap of about 0.25 or one and a half standard errors. The precision for $\hat{\theta}^2$ is 33%, and on average θ^2 is between $-0.225 \pm (0.33 \times 0.225) = [-0.299, -0.151]$ with a gap of 0.15. Over the whole sample period, the precision for both $\hat{\theta}^1$ and $\hat{\theta}^2$ are quite stable, and the precision for $\hat{\theta}^1$ is better than that for $\hat{\theta}^2$.

Figure 3.1 plots the time series of the daily risk price of SPY and IWM. Each dot corresponds to a daily risk price. The dashed lines measure the median of the estimates over the whole sample period to serve as a reference line of the plots. The solid lines are computed by the “lowess” method with the same smooth span as in Chapter 2 that uses 20% of the data points in each year to compute the lowess estimates. The lowess lines serve as a measure of the central tendency of the data. The dotted lines are computed as the “lowess” central tendency curve plus/minus one median standard error of the whole period, which measures the spread of the data.

SPY: 2007-2014



(a) Estimates of θ^1



(b) Estimates of θ^2

Figure 3.1: Time Series of Estimates of θ^1 and θ^2

As shown in Figure 3.1, estimates of the risk prices θ^1 and θ^2 vary a lot from day to day. The risk price of SPY starts to decline during the second half of 2008, possibly due to the Great Recession, and becomes stable and close to the median since then. The standard error band contains most of the data, although there are also many data points above or below the band.

The risk price of IWM, on the other hand, starts to drop at mid 2008 and stays below the median until the end of 2009 when it starts to rise above the median. It then stays above the median until the mid 2013 when it slightly declines and rises above the median again immediately. The standard error band also contains most of the data, but occasionally there are data points that fall below the band, in particular during mid 2007 and mid 2009. On the other hand, there are almost no estimates that are above the band. This finding suggests that the distribution of IWM has negative skewness.

3.2.2.2 The Risk Price of VWO-SPY

For around 90% of the days of the sample period, the estimates of the risk price of VWO-SPY is not significant. In fact, for 1092 (57.9%) of the days the estimates are within one standard error of zero, suggesting those estimates are indistinguishable from zero. For the remaining 10% of the days with significant estimates, it is worth mentioning that on those days the risk prices of SPY and IWM are always also significant. This finding reinforces the previous hypothesis that VWO-SPY as the third factor should only be treated as a “third-order correction” to the two-factor model.

Since only about 10% of the days of the sample yield significant estimates of θ^3 , I do not plot the time series of the estimates for the risk price of VWO-SPY, the third factor. Instead, Table 3.8 presents the number of significant estimates of θ^3 and their summary statistics year by year:

Table 3.8: Summary Statistics for the Risk Price of VWO-SPY

Year	Statistics	$\hat{\theta}^3$	Year	Statistics	$\hat{\theta}^3$
2007	Mean	-0.187	2011	Mean	-0.180
	Median	-0.196		Median	-0.231
	Mean $ t $	2.502		Mean $ t $	2.147
	Mean precision	0.400		Mean precision	0.466
	Min	-0.414		Min	-0.361
	Max	0.204		Max	0.457
	Number	36 (20.0%)		Number	15 (8.3%)
2008	Mean	-0.220	2012	Mean	-0.180
	Median	-0.202		Median	-0.181
	Mean $ t $	2.262		Mean $ t $	2.305
	Mean precision	0.442		Mean precision	0.439
	Min	-0.433		Min	-0.333
	Max	-0.116		Max	-0.107
	Number	19 (10.6%)		Number	16 (8.9%)
2009	Mean	-0.266	2013	Mean	-0.160
	Median	-0.243		Median	-0.146
	Mean $ t $	2.417		Mean $ t $	2.359
	Mean precision	0.414		Mean precision	0.424
	Min	-0.487		Min	-0.324
	Max	-0.163		Max	-0.083
	Number	26 (14.4%)		Number	36 (20.0%)
2010	Mean	-0.275	2014	Mean	-0.178
	Median	-0.262		Median	-0.183
	Mean $ t $	2.402		Mean $ t $	2.471
	Mean precision	0.416		Mean precision	0.405
	Min	-0.438		Min	-0.266
	Max	-0.168		Max	-0.096
	Number	18 (10.0%)		Number	9 (5.0%)

From Table 3.8, except for two days as mentioned previously, the risk price of VWO-SPY is always negative. Although there are not enough data points to characterize the distribution of $\hat{\theta}^3$, several observations can be made. First, the difference between the mean and the median of the risk price is usually around 0.1 to 0.2 and can be as high as almost 0.5 in year 2011, which is much larger than the cases for SPY and IWM. Both the mean and the median become more

negative during 2008 to 2010, possibly due to the impact of the Great Recession. Then they starts to become less negative until 2013, when the risk price becomes the least negative over the whole sample period. Correspondingly, the number and the percentage of the significant estimates are higher at year 2009 and 2013, reinforcing the connection of the risk price with macro events. The minimum and the maximum of the estimates also follow the same trend.

Second, the mean t-statistic is much smaller than those of $\hat{\theta}^1$ and $\hat{\theta}^2$. The mean $|t|$ over the whole sample period is around 2.3 to 2.4, corresponding to an average p-value of 0.023. Although significant at 5% level, the estimates are clearly not significant at 1% level. The precision on average is over 40%, which is also much worse than that of the risk prices of SPY and IWM. For instance, in year 2012 (which is a typical year over the sample period), the average $\hat{\theta}^3$ is between $-0.180 \pm 0.439 \times 0.180 = [-0.259, -0.101]$ with a gap of about 0.16. It appears that the third factor is a minor perturbation of the two-factor model.

Table 3.9 summarizes the mean plus and minus one standard error for each estimate each year to show the precision of the estimates. The estimate $\hat{\theta}^0$ is computed as $\hat{\theta}^0 = 1 - \hat{\theta}^1 - \hat{\theta}^2 - \hat{\theta}^3$. For θ^0 , no standard error is computed and only the mean is provided.

Table 3.9: Confidence Bounds

Year	$\hat{\theta}^0$	$\hat{\theta}^1$	$\hat{\theta}^2$	$\hat{\theta}^3$
2007	1.987	$[-0.866, -0.560]$	$[-0.350, -0.164]$	$[-0.262, -0.112]$
2008	2.008	$[-0.848, -0.598]$	$[-0.348, -0.168]$	$[-0.317, -0.123]$
2009	2.106	$[-0.924, -0.638]$	$[-0.369, -0.175]$	$[-0.376, -0.156]$
2010	2.060	$[-0.893, -0.653]$	$[-0.305, -0.155]$	$[-0.389, -0.161]$
2011	2.046	$[-0.879, -0.649]$	$[-0.301, -0.151]$	$[-0.264, -0.096]$
2012	2.017	$[-0.871, -0.631]$	$[-0.299, -0.151]$	$[-0.259, -0.101]$
2013	2.029	$[-0.882, -0.632]$	$[-0.319, -0.155]$	$[-0.228, -0.092]$
2014	1.973	$[-0.864, -0.584]$	$[-0.291, -0.147]$	$[-0.250, -0.106]$

Table 3.9 shows the similar trend as discussed previously. First, $\hat{\theta}^0$, being the share of the numeraire asset in the SDF mimicking portfolio, can be seen as measuring the overall riskiness of the market with more positive estimates of θ^0 being more risky. We see from Table 3.9 that $\hat{\theta}^0$ starts to become more positive since 2008 and continue to rise until 2009, and then it gradually declines. Also, the intervals of $\hat{\theta}^1$, $\hat{\theta}^2$ and $\hat{\theta}^3$ all shift towards more negative during 2008 and 2009, an evidence of the impact of the Great Recession.

Figure 3.2 plots the time series of $\hat{\theta}^0$ for the whole period, with the solid line being its central tendency obtained from the lowess method.

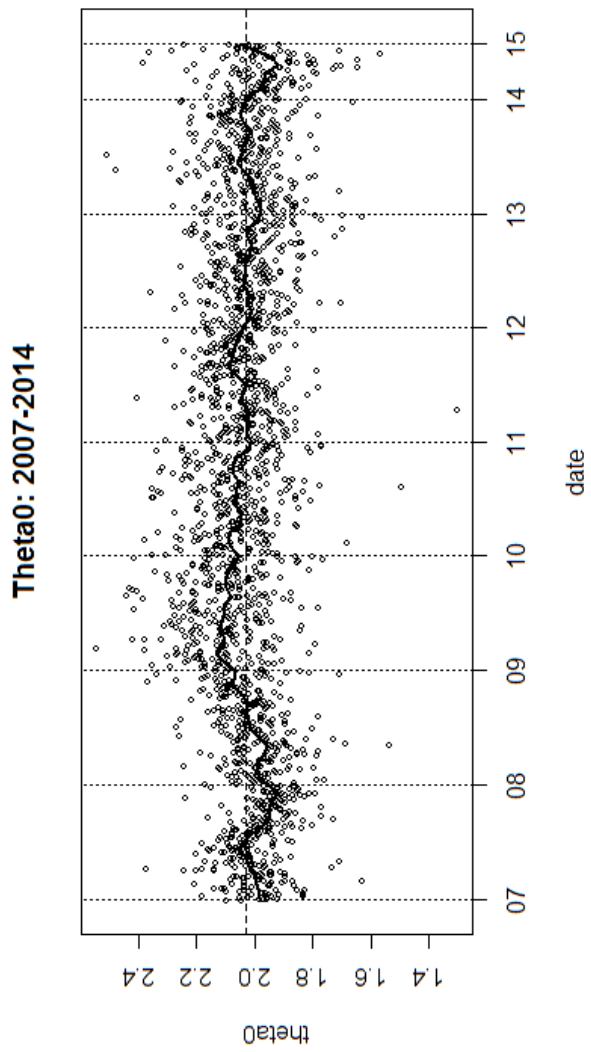


Figure 3.2: Estimates of θ^0

From Figure 3.2, we can also observe that the daily estimate starts to become more positive since 2008 until the early 2009, indicating an overall higher riskiness in the market. It then declines gradually during the rest of the sample period as the economy starts to recovery from the financial crisis.

3.2.2.3 The Mimicking Returns

As in Chapter 2, I compute the SDF mimicking return R^* using

$$R^* = (1 - \hat{\theta}^1 - \hat{\theta}^2) R^0 + \hat{\theta}^1 R^1 + \hat{\theta}^2 R^2 + \hat{\theta}^3 (R^3 - R^1)$$

where R^0 , R^1 , R^2 and R^3 are the average 10-second returns of SHY, SPY, IWM and VWO, respectively, over a trading day.

Figure 3.3 shows the time series of R^* over the whole period. The dashed line is the median of the data and the smooth span of the lowess curve is chosen to be 20% of the number of data points in each year, as explained previously. The returns are scaled by 2340 to get the equivalent daily returns.

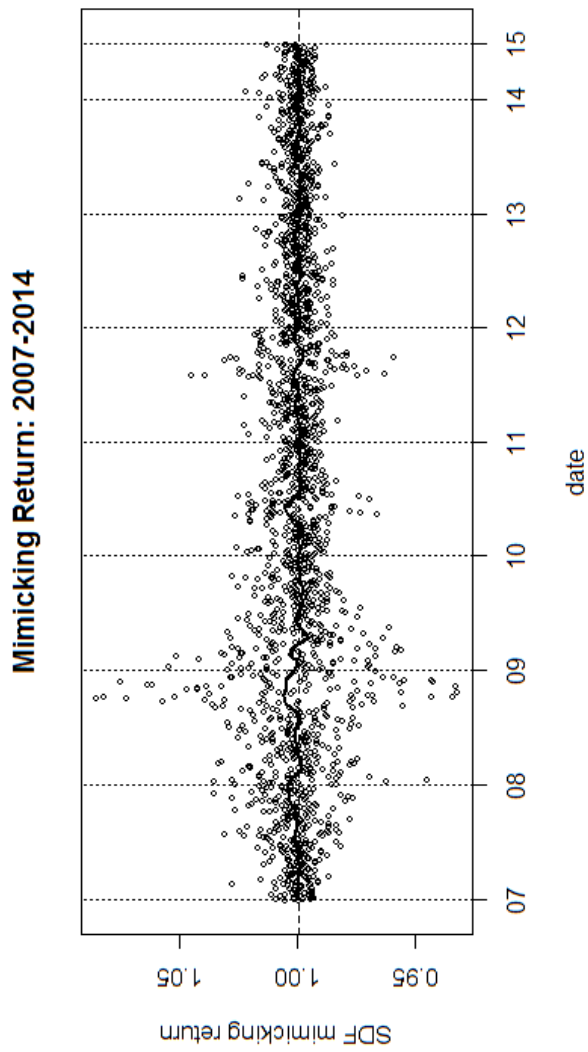


Figure 3.3: The SDF Mimicking Returns

As the two-factor model of Chapter 2, the variance of the SDF mimicking returns becomes much larger during the financial crisis from late 2008 to mid 2009. The central tendency follows close to the median without any obvious trend.

3.3 Conclusion

This chapter extends the two-factor model obtained in Chapter 2 by adding the third factor to the model. The best candidate for a third factor is the excess return of VWO over SPY (VWO-SPY).

Several conclusions can be made from the results. First, the model continues to work well in 2013 and 2014, and the results of Chapter 1 and 2 still hold in this extended period. Second, SPY remains the most important factor, with almost every day yielding a significant estimate of the risk price of SPY. IWM is still the second most important factor with more than 60% of the days having a significant estimate of its risk price. Third, adding VWO-SPY as the third factor has only marginal contribution to the model as only about 10% of the sample yield significant estimates of θ^3 . However, it improves the specification of the model slightly as there are more days (and a higher percentage of the sample) that pass the J-test.

Lastly, one can certainly add more factors, say estimate a four or five-factor model, and the additional factors would be even higher order of corrections to the current model. This will be left for future studies.

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