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Impact of Frame Dragging on the Kepler Frequency of Relativistic Stars[†]

N. K. Glendenning and F. Weber

Abstract

It has long been known that in general relativity the centrifugal force on an element in a rotating star involves the frequency of the star *relative* to the frequency at which the local inertial frame is dragged by the rotation. Intuitively, one would expect that this would increase the critical frequency at which rotation disrupts the star. Our analysis shows the opposite to be true.

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N. K. Glendenning and F. Weber

1 Introduction

A particle in stable circular orbit at the equator of a static star has a frequency in general relativity that is precisely equal to the classical one [1],

$$\Omega^2 = M/R^3 \ . \tag{1}$$

In classical mechanics this expresses the balance of gravitational and centrifugal forces. Here M and R are the gravitational mass and radius of the star (orbit), and Ω is the uniform angular velocity of the star. In classical mechanics the same expression holds for a rotating, axially symmetric star, though it is understood that the equilibrium shape of the star is no longer a sphere and that R denotes the equatorial radius. For a rotating star in general relativity, the situation is drastically altered, as is well known. Among the important effects is the phenomenon of dragging of local inertial frames by the rotating star [2, 3, 4, 5]. Mach's critical attention to the concept of inertial forces no doubt played an important role in ultimately focussing attention on the effects of rotating matter. Thirring appears to have been the first to realize that in Einstein's theory, a rotating mass shell drags the local inertial frames [2]. The effect was studied in greater generality by Brill and Cohen [5]. Shortly thereafter, Hartle incorporated the effect into his calculation of the equilibrium configurations of rotating stars [6]. He notes that the centrifugal force acting on a fluid element of the star is governed by the rate of rotation of the star, assumed to be uniform, *relative* to the *local* inertial frames, which are dragged by the stars rotation, in the same direction. The frequency with which the local inertial frames are dragged is largest at the center of the star, never exceeds the frequency of the star itself, and goes to zero at great distance from the star. It is this problem that we revisit in this paper, because the effect of frame dragging on the Kepler frequency, relative to the classical result, is counter to the intuition that one would bring from classical mechanics. Inasmuch as the centrifugal force acting on a fluid element is governed by $\Omega - \omega(r)$, where the angular velocity of the local inertial frame is denoted by $\omega(r)$, it is natural to expect that the dragging of the local inertial frames will allow a larger rotational frequency, Ω, before the centrifugal force will disrupt the star than if there were no dragging. According to what has been said, one expects that the role of dragging alone is to *increase* the Kepler frequency over the value given by (1). This turns out to be incorrect. Of course there are other factors that effect the Kepler frequency of a relativistic star but they are not at issue, and have been analyzed elsewhere [7]. Our analytic discussion progresses in two stages, with an improvement in the metric at the second stage.

2 Analytic treatment

While the classical result (1) holds for a particle in orbit around a *static* star also in general relativity, it is easy to understand why it cannot hold, for several reasons, for a *rotating* star in general relativity. The radially dependent dragging of local inertial frames must perforce effect the actual distribution of matter in the *rotating* star and hence the metric of spacetime is altered by the star, that is by the particular distribution of matter, determined by the condition of equilibrium or balance of forces. In classical mechanics space and time are assumed to be absolute. In general relativity the metric functions are dynamically determined by the distribution of mass, which itself responds to the metric. It should not be surprising therefore that the expression for the Kepler frequency does not resemble the classical one. Instead it is (cf. Appendix A)

$$\Omega_{\rm K} \;=\; \omega + \frac{\omega'}{2\psi'} + e^{\nu-\psi}\left[\frac{\nu'}{\psi'} + \left(\frac{\omega'}{2\psi'}e^{\psi-\nu}\right)^2\right]^{1/2} . \qquad (2)$$

The primes denote derivatives with respect to Schwarzschild radial coordinate r, and all functions on the right are evaluated at the star's equator. More than this, they depend also on $\Omega_{\rm K}$, so that the above is *not* an equation for $\Omega_{\rm K}$, but a transcendental relationship which the solution of the equations of stellar structure must satisfy if it is rotating at its Kepler frequency. The frame dragging frequency, $\omega(r)$, satisfies a particular boundary condition at the equator of the star that has been written before and, along with the above transcendental equation is derived in Appendix C.

2.1 Restriction to static metric

To obtain an analytic solution to the problem, we shall, in a first step, take the metric which corresponds to that of a static star, i.e. the Schwarzschild metric. This will provide a first orientation. Corrections to this metric will be considered in the next section. Thus at the equator we take

$$e^{2\nu} = 1 - \frac{2M}{R} , \tag{3}$$

$$e^{2\psi} = R^2 , \tag{4}$$

where for our approximate solution to Eq. (2) we take M to be the mass of the rotating star and R its equatorial radius. (The second of these equations looks strange, but we follow an old precedent so as not to introduce confusion [8, 9, 10].) Combined with the condition that outside the star, $\omega(r)$ must obey (cf. Appendix C),

$$\omega(r) = \frac{2I}{r^3}\,\Omega , \qquad r > R \tag{5}$$

we are able to write an approximate solution to the transcendental equation for Ω_K, namely

$$\begin{aligned} \Omega_K^2 &= \left(1 + \frac{\omega(R)}{\Omega_K} - 2\left(\frac{\omega(R)}{\Omega_K}\right)^2\right)^{-1} \frac{M}{R^3}, \\ &= \left(1 + \frac{2I}{R^3} - 2\left(\frac{2I}{R^3}\right)^2\right)^{-1} \frac{M}{R^3}. \end{aligned} \tag{6}$$

This approximate result has a very interesting structure, for it shows the classical result modified by a prefactor. The prefactor leads to a *reduction* in the relativistic Kepler frequency when $\omega(R)/\Omega_K < 1/2$ or equivalently $4I/R^3 < 1$. There is no apparent reason why this limit *must* be obeyed, even if in practice it is (cf. Ref. [7, 10, 11]). Therefore we proceed to an improved metric.

2.2 Monopole corrected static metric

Here, we carry the analytic investigation one step further by taking monopole corrections to the Schwarzschild metric into account [6, 12] (see Appendix B). In this case Eq. (3) reads

$$e^{2\nu} = 1 - \frac{2M}{R} + \frac{2J^2}{R^4} , \tag{7}$$

while Eq. (4) remains unchanged. Here $J \equiv I\Omega$ is the angular momentum. From Eq. (2) one finds for the Kepler frequency

$$\begin{aligned} \Omega_K^2 &= \left(1 + \frac{\omega(R)}{\Omega_K} - \left(\frac{\omega(R)}{\Omega_K}\right)^2\right)^{-1} \frac{M}{R^3}, \\ &= \left(1 + \frac{2I}{R^3} - \left(\frac{2I}{R^3}\right)^2\right)^{-1} \frac{M}{R^3}, \end{aligned} \tag{8}$$

The prefactor in Eq. (8) *always* leads to a *reduction* of the Kepler frequency below its classical value because $\omega(R)/\Omega_{\rm K} < 1$. The dragging frequency cannot exceed the star frequency [6]. This universal limit is what the improved metric has bought.

It is interesting that the above result constitutes also a derivation of Eq. (1) for a particle in stable orbit around a static relativistic star since in that case $\omega(r) \equiv 0$.

It may be of some interest that Eq. (5) places a limit involving the moment of inertia and radius of a star,

$$\frac{2I}{R^3} < 1 \tag{9}$$

3 Summary

In this work we showed that the dragging of local inertial frames caused by the rotation of any massive star, *reduces* its Kepler (mass shedding) frequency relative to that of a static star, contrary to the intuitive expectation that naturally follows from the fact that the centrifugal force on fluid elements of the star are determined by the frequency of the star *relative* to the local inertial frames which are dragged in the direction of the star's rotation.

This counter-intuitive behavior can be understood mathematically as following from the fact that Eq. (2) is not a formula for Ω_K, but a transcendental equation, in which all quantities on the right depend also on Ω_K and on $\omega(r)$. Thus to say that the centrifugal effect on a fluid element of the star at r depends on $\Omega_K - \omega(r)$, while true, does *not* inform us that there is a *reduction* in the centrifugal effect with corresponding increase in the Kepler frequency.

Appendix

A Kepler frequency in general relativity

We are interested in models of compact stars that are uniformly rotating, axi-symmetric fluid configurations. Therefore, the spacetime is stationary and axi-symmetric, which corresponds to respectively time translation and rotational symmetry. The line element can be written as [13, 10, 14]

$$ds^2 = - e^{2\nu(r,\theta;\Omega)}dt^2 + e^{2\psi(r,\theta;\Omega)}\,[d\phi - \omega(r,\theta;\Omega)\,dt]^2 + e^{2\mu(r,\theta;\Omega)}\,d\theta^2 + e^{2\lambda(r,\theta;\Omega)}dr^2 \,. \qquad (10)$$

As a consequence of the underlying symmetries, the metric functions ν, ψ, μ, and λ are independent of t and ϕ. The function $\omega(r,\theta;\Omega)$ denotes the angular velocity of the local inertial frames (dragging of the local inertial frames). As indicated, it depends on the radial coordinate r and the azimuthal coordinate θ, and is proportional to the star's rotational velocity Ω.

The frequency Ω is assumed to be constant throughout the star's fluid. The frequency $\bar{\omega}(r,\theta;\Omega) \equiv \Omega - \omega(r,\theta;\Omega)$, which is the star's rotational frequency relative to the frequency of the local inertial frames, is the one on which the centrifugal force acting on the mass elements of the rotating star's fluid depends [6]. It is this frequency relative to which the fluid inside the star moves.

From Eq. (10) one finds for a material particle rotating at the star's surface (constant r and θ coordinates)

$$1 = e^{2\nu}\left(\frac{dt}{d\tau}\right)^2 - e^{2\psi}\left(\frac{d\phi}{d\tau} - \omega\,\frac{dt}{d\tau}\right)^2 . \qquad (11)$$

For the purpose of brevity, the arguments of the functions here and in the following are omitted. From $u^\phi = \Omega u^t$, where $u^\phi \equiv d\phi/d\tau$ and $u^t \equiv dt/d\tau$, one obtains $d\phi/d\tau = \Omega\, dt/d\tau$. Thus the time-component of the particle's four-velocity is given by

$$\frac{dt}{d\tau} = \frac{e^{-\nu}}{\sqrt{1 - V^2}} \,, \qquad (12)$$

where

$$V \equiv e^{\psi-\nu}\,\bar{\omega} \,. \qquad (13)$$

denotes the particle's orbital velocity ($u^r = u^\theta = 0$). Equation (13) serves to express the star's rotational frequency in terms of V and the frame dragging frequency,

$$\Omega = e^{\nu-\psi}V + \omega \,, \qquad (14)$$

which, in other words, is the expression for the rotational frequency of a massive particle rotating in a stable orbit of constant radial distance, i.e., $r = R_{\rm eq}$ and $\theta = \pi/2$, from the star's origin. For its evaluation, knowledge of V is necessary. The relevant mathematical expression for V will be derived now. Since the particle path is a circular orbit, we can determine V simply as the extremal of $ds^2(t, r, \phi)$, i.e., $ds^2/dr = 0$. From Eq. (10) one obtains

$$\psi' \, e^{2\nu} \, V^2 \; - \; \omega' \, e^{\psi+\nu} \, V \; - \; \nu' \, e^{2\nu} \; = \; 0 \; , \tag{15}$$

where according to Eq. (13), $d\phi - \omega \, dt = (\Omega - \omega) \, dt = V \, e^{\nu-\psi} \, dt$. Equation (15) constitutes a quadratic equation in the equatorial velocity V. Its solutions are

$$V_{+,-} \; = \; \frac{\omega'}{2\,\psi'} \; e^{\psi - \nu} \; \pm \; \sqrt{\frac{\nu'}{\psi'} \; + \; \left(\frac{\omega'}{2\,\psi'} \; e^{\psi - \nu} \right)^2} \; . \tag{16}$$

The solution, V_+ corresponds to co-rotation which is the desired one in connection with the stability of the star to mass shedding. The other solution corresponds to a counter-rotating satellite at its Kepler frequency.

In summary, Eqs. (14) and (16) are to be solved simultaneously in combination with the stellar structure equations by means of a self-consistent iteration procedure in order to find the general relativistic Kepler frequency of a rotating star model of given central density [10, 11].

B Monopole correction to the metric of a static star

For our purpose we recall [6, 12] only the metric functions ν and ψ occurring in Eq. (10). These are given by

$$e^{2\,\nu(r,\theta;\Omega)} \; = \; e^{2\,\Phi(r)} \, [1 + 2 \, (h_0(r;\Omega) + h_2(r;\Omega) \, P_2(\cos\theta))] \, , \tag{17}$$

$$e^{2\,\psi(r,\theta;\Omega)} \; = \; r^2 \, \sin^2\!\theta \, [1 + 2 \, (v_2(r;\Omega) - h_2(r;\Omega)) \, P_2(\cos\theta)] \, , \tag{18}$$

where

$$e^{2\Phi(r)} \; = \; (1 - \frac{2M}{r}) \; , \qquad r \geq R \; . \tag{19}$$

The functions h_l, m_l, $(l = 0, 2)$ and v_2 of Eqs. (17)-(18) stand for the monopole and quadrupole perturbation functions, and the quantity P_2 is the second order Legendre polynomial, $P_2(x) = (3x^2 - 1)/2$. In the non-rotating limit, the perturbation functions vanish identically, and the metric functions reduce to those of a static star.

The monopole function h_0 is given by

$$e^{2\Phi(r)}\, h_0(r) \;=\; -\frac{\Delta M}{r} + \frac{J^2}{r^4} \;, \qquad r \geq R \;. \tag{20}$$

Neglecting the quadrupole perturbation functions in Eqs. (17) and (18), one obtains for the metric functions at the star's equator

$$e^{2\nu(R)} \;=\; 1 - \frac{2M}{R} + \frac{2J^2}{R^4} \;, \tag{21}$$

$$e^{2\psi(R)} \;=\; R^2 \;. \tag{22}$$

The quantity ΔM in Eq. (20) denotes the mass increase of a rotating star caused by rotation, J ($\equiv I\Omega$) refers to the star's angular momentum.

C Frame dragging frequency at the equator of a rotating star

We derive the expression for the frequency of the local inertial frames, ω, at the equator of a rotating star, which rotates with frequency Ω. The result is accurate to order $O(J/r^4)$ [13], where J denotes the star's angular momentum (cf. Appendix B). We begin by deriving an expression for the moment of inertia of a stationary rotating, axi-symmetric, relativistic star in equilibrium. Under these restrictions, the expression for the moment of inertia is given by [15]

$$I(\mathcal{A},\Omega) \;\equiv\; \frac{1}{\Omega}\int_{\mathcal{A}} dr d\theta d\phi\, \mathcal{T}_3^{\,0}\, \sqrt{-g} \;. \tag{23}$$

In the above equations, $\mathcal{A}$ denotes an axially symmetric region in the interior of a body where all matter is rotating with the same angular velocity Ω. The quantity g refers to the determinant of the metric tensor. For the metric of Eq. (10) one finds (cf. Ref. [16] for details)

$$\sqrt{-g} \;=\; e^{\lambda+\mu+\nu+\psi} \;, \tag{24}$$

$$\mathcal{T}_3^{\,0} \;=\; -\,(\epsilon + P)\,\bar{\omega}(r,\Omega)\, e^{\psi(r,\Omega)} \,\Big[\, e^{\nu(r,\Omega)} \;-\; \bar{\omega}(r,\Omega)^2\, e^{\psi(r,\Omega)} \,\Big]^{-1} \;. \tag{25}$$

The expression for the moment of inertia of Eq. (23) then leads to

$$I \;=\; 4\,\pi \int_0^{\pi/2} d\theta \sin\theta \int_0^{R(\theta)} dr\, r\, \frac{\sqrt{e^{\lambda(r,\Omega)}\; e^{\mu(r,\Omega)}\; e^{\nu(r,\Omega)}}\,[\epsilon + P(\epsilon)]}{e^{\nu(r,\Omega)-\psi(r,\Omega)} \;-\; \bar{\omega}(r,\Omega)^2}\; \frac{\bar{\omega}(r,\Omega)}{\Omega} \;, \tag{26}$$

which reads in the case of a rotationally non-deformed star

$$J \;\equiv\; I\,\Omega = \frac{8\,\pi}{3}\int_0^R dr\, r^4\, \frac{\epsilon + P(\epsilon)}{\sqrt{1 - 2\,m(r)\,/\,r}}\, \bar{\omega}(r,\Omega)\, e^{-\Phi(r)} \;. \tag{27}$$

The quantity J denotes the star's angular momentum. From the field equation $\mathcal{R}_3^0 = 8\pi \mathcal{T}_3^0$ one obtains a differential equation for $\bar{\omega}$ [6],

$$\frac{d}{dr}\left(r^4\, j(r)\, \frac{d\bar{\omega}(r)}{dr}\right) + 4\, r^3\, \frac{dj(r)}{dr}\, \bar{\omega}(r) = 0\,, \qquad r \leq R_{\rm eq}\,, \tag{28}$$

where

$$j(r) = e^{-\Phi(r)}\, \sqrt{1 - \frac{2m(r)}{r}}\,. \tag{29}$$

From Eq. (29) it follows that

$$\frac{dj}{dr} = -4\, \pi\, r\, (\epsilon + P)\, e^{-\Phi} \,/\, \sqrt{1 - 2m/r}\,, \tag{30}$$

which is used to find

$$r^4\, j\, \frac{d\bar{\omega}}{dr} = 6\, J\,, \qquad r = R_{\rm eq} \tag{31}$$

from Eq. (28). Here use of Eq. (27) has been made. For $r \geq R_{\rm eq}$ one has $j \equiv 1$, and one obtains from Eq. (28)

$$\bar{\omega} = -\frac{A}{r^3} + B\,. \tag{32}$$

Since $\bar{\omega} \longrightarrow \Omega$ for $r \longrightarrow \infty$ (frame dragging vanishes at infinity) one gets $B = \Omega$. To determine the constant A in Eq. (32), we compute $d\bar{\omega}/dr$ from Eq. (32) and make use of Eq. (31), evaluated at $r = R_{\rm eq}$, leading to $A = 2J$. Thus, the angular velocity of the dragged inertial frames at the star's equator is given by

$$\omega = \frac{2\, I}{R_{\rm eq}^3}\, \Omega\,. \tag{33}$$

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