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Correspondence Between the Spectrum of Jacobi Operator and Dominated Splitting of its Cocycle Map

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by

Kateryna V. Alkorn

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To my family.

ABSTRACT OF THE DISSERTATION

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by

Kateryna V. Alkorn

Doctor of Philosophy, Graduate Program in Mathematics
University of California, Riverside, June 2021
Dr. Zhenghe Zhang, Chairperson

In this paper we show a version of Johnson's theorem for $\mathrm{M}(2,\mathbb{C})$-sequences. In particular, motivated by the work of [18] and [10], we show that one can identify the spectrum of Jacobi operator $J_{a,b} : \ell^2(\mathbb{Z},\mathbb{R}) \to \ell^2(\mathbb{Z},\mathbb{R})$ by those energies $E \in \mathbb{R}$, whose cocycle map $B^E : \mathbb{Z} \to \mathrm{M}(2,\mathbb{R})$ does not admit dominated splitting, i.e.

$$\sigma(J_{a,b}) = \{E \in \mathbb{R} : B^E \notin \mathcal{DS}\}.$$

This result extends a well-known Johnson's theorem for Schrödinger operators, which identifies the spectrum of Schrödinger operator by all energy values $E \in \mathbb{R}$, such that the cocycle map $A^E : \mathbb{Z} \to \mathrm{SL}(2,\mathbb{R})$ is not uniformly hyperbolic. Jacobi operator is a natural extension to Schrödinger operator, and dominated splitting generalizes the notion of uniform hyperbolicity. The biggest difference lies in the fact that for Schrödinger operator, the cocycle map takes value in $\mathrm{SL}(2,\mathbb{R})$, whereas for Jacobi operator, the cocycle map takes value in $\mathrm{M}(2,\mathbb{R})$. In particular, it is allowed to be singular, i.e. have zero determinant. This is one of the main challenges of this paper that we had to overcome. In this paper we also discuss

the importance of Jacobi operators, develop definition of dominated splitting for $M(2, \mathbb{C})$-sequences, show stability theorem and finish this paper with the discussion of possibility of extending this result to dynamically defined Jacobi operators.

Contents

Chapter 1

Introduction

In this paper we study Jacobi operators defined by $\ell^\infty(\mathbb{Z}, \mathbb{R})$ sequences. We show that the spectrum of such operators is characterized by those energies, whose cocycle map does not admit dominated splitting. This paper is self contained and is intended for both graduate students, and experienced researchers as it contains original material.

We start this paper with the section called "Background". First we introduce the notation we are using throughout the paper. Then we review some important results from spectral theory that are relevant to our research. We finish this section with an extensive discussion of Schrödinger operators and Johnson's theorem for Schrödinger operators.

Section "Jacobi Operator and Dominated Splitting" is the summary of this paper. We start by discussing Jacobi operators and their history. Then we talk about the existing version of Johnson's theorem for Jacobi operators and the notion of dominated splitting. We discuss the deficiency of the existing version of Johnson's theorem for Jacobi operators and talk about the need to find a more general version of Johnson's theorem for $\mathrm{M}(2, \mathbb{R})$-

sequences. We finish this section by stating results that will be proven in the later sections, in particular the stability theorem and Johnson's theorem for $\mathrm{M}(2,\mathbb{R})$-sequences.

In the section called "Stability of Dominated Splitting" we prove that our definition of dominated splitting for $\mathrm{M}(2,\mathbb{C})$-sequences is stable under $\|\cdot\|_\infty$-perturbation. To do that we first prove that the existence of an invariant cone field implies domination for a $\mathrm{M}(2,\mathbb{C})$-sequence. Then we use that fact together with multiple lemmas to prove the stability theorem.

Sections "Dominated Splitting and the Invertability of the Operator" and "Dominated Splitting Away From the Spectrum" contain the full prove of Johnson's theorem for $\mathrm{M}(2,\mathbb{R})$-sequences. We prove that the resolvent set of Jacobi operator is characterized by those energies, whose cocycle map admits dominated splitting. In each section we show one direction of the if and only if statement of the theorem.

Finally, in section "Conclusions", we summarize our results and talk about the possible extension of our results to the dynamically defined Jacobi operators. In particular we discuss the strategy of proving the dynamical version of Johnson's theorem for Jacobi operators when the base dynamics is topologically transitive.

Chapter 2

Background

2.1 Notation and terminology

For the most part throughout the paper we try to be consistent with the standard notation and terminology. Because it is a self-contained paper, we include it here for reader's reference.

We denote $\mathbb{N}$ to be the set of natural numbers, $\mathbb{Z}$ to be the set on integers, $\mathbb{R}$ to be the set of real numbers and $\mathbb{C}$ to be the field of complex numbers. For any set $\mathrm{I} \subset \mathbb{Z}$ and any set K we denote by $\ell(\mathrm{I}, K)$ the space of K-valued sequences $(\phi_n)_{n\in\mathrm{I}}$. When it is clear from the context, we might omit K and just write $\ell(\mathrm{I}, K) = \ell(\mathrm{I})$. For any $n_1, n_2 \in \mathbb{Z}$, $n_1 < n_2$ we defined $\ell(n_1, n_2) = \ell(\{n \in \mathbb{Z} | n_1 < n < n_2\})$, $\ell(n_2, \infty) = \ell(\{n \in \mathbb{Z} | n > n_2\})$ and $\ell(-\infty, n_1) = \ell(\{n \in \mathbb{Z} | n < n_1\})$.

For $0 < p < \infty$, the introduce the space of sequences called $\ell^p(\mathbb{Z})$, defined as

$$\ell^p(\mathbb{Z}) = \big\{\phi \in \ell(\mathbb{Z}) \big| \sum_{n\in\mathbb{Z}} |\phi_n|^p < \infty\big\}.$$

When $p = \infty$, $\ell^\infty(\mathbb{Z})$ is defined to be the space of bounded sequences, i.e.

$$\ell^\infty(\mathbb{Z}) = \{\phi \in \ell(\mathbb{Z}) \big| \sup_{n\in\mathbb{Z}} |\phi_n| < \infty\}.$$

We introduce the norms

$$\|\phi\|_p = \left(\sum_{n\in\mathbb{Z}} |\phi_n|^p\right)^{1/p}, \quad 1 \leq p < \infty$$

$$\|\phi\|_\infty = \sup_{n\in\mathbb{Z}} |\phi_n|$$

These norms make the $\ell^p(\mathbb{Z})$, $1 \leq p \leq \infty$ a Banach space. When it is clear from the context, we might omit the subscript in the norm and just write $\|\cdot\|$. Throughout this paper, we will most often use the $\ell^2(\mathbb{Z})$ space, which is the space of square-summable sequences, i.e.

$$\ell^2(\mathbb{Z}) = \{\phi \in \ell(\mathbb{Z}) \big| \sum_{n\in\mathbb{Z}} |\phi_n|^2 < \infty\}$$

with the $\|\cdot\|_2$ norm,

$$\|\phi\|_2 = \left(\sum_{n\in\mathbb{Z}} |\phi_n|^2\right)^{1/2}.$$

In addition, $\ell^2(\mathbb{Z})$ is a Hilbert space with inner product and norm defined as

$$< f, g >= \sum_{n\in\mathbb{Z}} < f_n, g_n >$$

$$\|f\| = \|f\|_2 = \sqrt{< f, f >}, \quad f, g \in \ell^2(\mathbb{Z}).$$

We let $\mathrm{M}(2, \mathbb{R})$ be the associated algebra of real 2×2 matrices. We define general group $\mathrm{GL}(2, \mathbb{R})$ to be the group of real 2×2 matrices whose determinant is not zero, i.e.

$$\mathrm{GL}(2, \mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{R}, ad - bc \neq 0 \right\}$$

and special linear group $\mathrm{SL}(2,\mathbb{R})$ to be the group of real 2×2 matrices whose determinant equals 1, i.e.

$$\mathrm{SL}(2,\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a,b,c,d \in \mathbb{R}, ad-bc=1 \right\}$$

Consistent with functional analysis notation, we define $L : X \to Y$ to be a bounded linear operator between topological vector spaces X and Y if it maps bounded subsets of X to bounded subsets of Y. We set $\|L\|$ to be the usual operator norm, i.e.

$$\|L\| := \inf\{\alpha > 0 : \|Lx\|_Y \le \alpha \|x\|_X, x \in X\}$$

L is a bounded linear operator if and only if $\|L\| < \infty$.

We define $\mathbb{RP}^n$ to be the quotient space of $\mathbb{R}^{n+1}\setminus\{0\}$ under the equivalence relation $x \sim \lambda x$ for all real numbers $\lambda \neq 0$. In particular, we will be dealing with $\mathbb{RP}^1 = \mathbb{R}/(\pi\mathbb{Z})$ which is a real projective line, and $\mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$ which is the one-dimensional complex projective space. We mainly use the following projection map from $\mathbb{C}^2 \setminus \{\vec{0}\} \to \mathbb{CP}^1$:

$$\pi : \mathbb{C}^2 \setminus \{\vec{0}\} \to \mathbb{CP}^1 \text{ where } \pi_2 \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \frac{z_2}{z_1} \tag{2.1}$$

Through this projection, each one-dimensional space in $\mathbb{C}^2$ can be identified with a point in $\mathbb{CP}^1$. Hence, we may view a point $z \in \mathbb{CP}^1$ as a one-dimension space of $\mathbb{C}^2$. For instance, by $\vec{v} \in z$, we mean $\vec{v}$ is a vector in the one-dimensional space z. In particular, we let $\vec{z}$ denotes a unit vector in z. We shall mainly use the following metric on $\mathbb{CP}^1$:

$$d(z,z') = \begin{cases} \frac{2|z-z'|}{\sqrt{(1+|z|^2)(1+|z'|^2)}}, & z,\ z' \in \mathbb{C} \\ \frac{2}{\sqrt{1+|z|^2}}, & z' = \infty. \end{cases} \tag{2.2}$$

Note $d(z,z') \le 2|z-z'|$ for all $z, z' \in \mathbb{C}$. Let $\vec{v}, \vec{v}' \in \mathbb{C}^2$ be two nonzero vectors. We let $(\vec{v}, \vec{v}') \in \mathrm{M}(2,\mathbb{C})$ denotes the matrix whose column vectors are $\vec{v}$ and $\vec{v}'$. Then a direct computation shows that

$$d(\pi(\vec{v}), \pi(\vec{v}')) = \frac{|\det(\vec{v}, \vec{v}')|}{\|\vec{v}\| \cdot \|\vec{v}'\|} \tag{2.3}$$

In particular, if $\vec{v}, \vec{v}'$ are two unit vectors, then

$$d(\pi(\vec{v}), \pi(\vec{v}')) = |\det(\vec{v}, \vec{v}')|. \tag{2.4}$$

Since one-dimensional space can be identified by the points in $\mathbb{CP}^1$, abusing the notation slightly, for two one-dimensional subspaces V, W of $\mathbb{C}^2$, we set

$$d(E, F) := d(\pi(\vec{v}), \pi(\vec{w})),$$

where $\vec{v} \in V$ and $\vec{w} \in W$ are nonzero vectors.

Throughout this paper, we also make the following assumptions: C, c will be universal constants, where C is large and c is small; $\|\cdot\|_\infty$ always denotes the usual supremum norm in various scenarios; there is a $M > 0$ so that $\|A\| < M$ for all $A \in \mathrm{M}(2, \mathbb{C})$ that appear in the remaining part of the paper.

2.2 Basics of Spectral theory

Spectral theory is an important branch of mathematics that extends the study of the eigenvalues and eigenvectors of a single matrix to a much broader set of matrices. The spectrum of bounded linear operator on a finite-dimensional vector space is the set of eigenvalues, however the spectrum of an operator on a infinite-dimensional vector space can include additional elements and might or might not include eigenvalues. In this paper we work primarily with bounded, self-adjoint linear operators on Hilbert space.

Definition 1 *Let H be a bounded linear operator on a Hilbert space $\mathcal{H}$. We say that H^* is the adjoint of H, if for all $x, y \in \mathcal{H}$*

$$< Hx, y > = < x, H^* y > .$$

We say that H is self-adjoint, if $H^* = H$.

Definition 2 *Let H be a bounded linear operator on a Hilbert space $\mathcal{H}$. The spectrum $\sigma(H)$, of the operator H is defined as:*

$$\sigma(H) = \{E \in \mathbb{C} : H - E \textit{ does not have a bounded inverse}\}.$$

The resolvent $\rho(H)$, of the operator H is defined as:

$$\rho(H) = \mathbb{C} - \sigma(H).$$

The spectrum of linear operator can be decomposed into the point spectrum, which consists of the set of eigenvalues, and continuous spectrum, which consists of all the other $E \in \mathbb{C}$ such that $H - E$ does not have abounded inverse. Many bounded, linear operators are associated with the solutions to various differential equations. If we understand what is in the spectrum of the operator, we can find the directions along which the operator acts by "stretching" or "flipping" the vectors. Therefore understanding what is in the spectrum of the operator can provide us with better understanding of the operator itself.

The purpose of this paper is to find the correspondence between the spectrum of Jacobi operator and the dynamics of its cocycle map. We provide some standard results from the spectral theory that we will be using throughout this paper. We include them here for completeness but omit the proofs as they can easily be found in the textbooks on spectral theory.

Result 1 *Let H be a bounded, linear operator on a Hilbert space $\mathcal{H}$. Then the spectrum $\sigma(H)$ is a nonempty compact subset of $\mathbb{C}$.*

Result 2 *Let H be a bounded, linear, self-adjoint operator on a Hilbert space $\mathcal{H}$. Then the spectrum $\sigma(H)$ of the operator H is real, i.e.*

$$\sigma(H) = \{E \in \mathbb{R} : H - E \textit{ does not have a bounded inverse}\}.$$

2.3 Schrödinger operator and uniform hyperbolicity

In this section we discuss the background on which my research was built. My work started as an effort to generalize a result known as Johnson's theorem. Johnson's theorem first appeared in 1986 in Russel Johnson's paper [9] and was stated for dynamically defined Schrödinger operators. The theorem identifies the spectrum of Schrödinger operator by the energies whose cocycles are not uniformly hyperbolic. Johnson's work was inspired by Sacker and Sell [12] and Selgrade [13]. Since then it was widely used to study spectral and quantum dynamical properties of discrete one-dimensional Schrödinger operators [8], [4], [18].

To fully understand Johnson's theorem and the implications it has, we need some more definitions.

Let Ω be a compact metric space, $T : \Omega \to \Omega$ - a homeomorphism and $f : \Omega \to \mathbb{R}$ - a continuous function. $H_\omega : \ell^2(\mathbb{Z}, \mathbb{R}) \to \ell^2(\mathbb{Z}, \mathbb{R})$ is called Schrödinger operator and is given by

$$(H_\omega \psi)_n = \psi_{n+1} + \psi_{n-1} + f(T^n\omega)\psi_n, \quad \psi = (\psi_n)_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z}, \mathbb{R}), \omega \in \Omega.$$

Schrödinger cocycle map $A^{E-f} : \Omega \to \mathrm{SL}(2, \mathbb{R})$ is defined as

$$A^{E-f}(\omega) = \begin{pmatrix} E - f(\omega) & -1 \\ 1 & 0 \end{pmatrix},$$

where $E \in \mathbb{R}$ and $\|A\|_\infty < M$. Then we define the dynamical system in the following way:

$$(T, A^{E-f}) : \Omega \times \mathbb{R}^2 \to \Omega \times \mathbb{R}^2 \quad (T, A^{E-f})(\omega, \vec{v}) = (T(\omega), A^{E-f}(\omega)\vec{v}).$$

The iteration of the map is $(T^n, A_n^{E-f}) = (T, A^{E-f})^n$, where the cocycle map is defined as

$$A_n^{E-f}(\omega) = \begin{cases} A^{E-f}(T^{n-1}\omega) \cdots A^{E-f}(\omega), & n \geq 1, \\ I_2, & n = 0, \\ A^{E-f}(T^n\omega)^{-1} \cdots A^{E-f}(T^{-1}\omega)^{-1}, & n \leq -1. \end{cases}$$

Next we need the notion of uniform hyperbolicity for dynamical systems.

Definition 3 *We say that (T, A) is uniformly hyperbolic if there exists two maps $u, s : \Omega \to \mathbb{RP}^1$ such that*

1. *u, s are (T, A) -invariant, meaning that for all $\omega \in \Omega$,*

$$A(\omega) \cdot u(\omega) = u[T(\omega)] \text{ and } A(\omega) \cdot s(\omega) = s[T(\omega)].$$

2. *There exists $C > 0, \lambda > 1$ such that $\|A_{-n}(\omega)\vec{v}\|, \|A_n(\omega)\vec{w}\| \leq C\lambda^{-n}$ for all $n \geq 1$, all $\omega \in \Omega$, and all unit vectors $\vec{v} \in u(\omega), \vec{w} \in s(\omega)$.*

Then we have the original Johnson's theorem [9, Theorem 3.1]:

Theorem 1 *Let (Ω, T) be topologically transitive, i.e. $\overline{\mathrm{Orb}(\omega_0)} = \Omega$ for some $\omega_0 \in \Omega$. Then (T, A^E) is uniformly hyperbolic if and only if $E \in \rho(H_{\omega_0})$.*

This is a powerful and well known result. One of the variations of the theorem appeared in [18, Theorem 1], when the author noticed that Johnson's theorem does not necessarily need to involve a base dynamics (T, Ω).

Indeed, we can define Schrödinger operator $H_v : \ell^2(\mathbb{Z}, \mathbb{R}) \to \ell^2(\mathbb{Z}, \mathbb{R})$ as follows

$$(H_v\psi)_n = \psi_{n+1} + \psi_{n-1} + v(n)\psi_n, \ \psi = (\psi_n)_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z}, \mathbb{R}),$$

where sequence $v : \mathbb{Z} \to \mathbb{R}$, which is called the potential, describes the medium and $\psi \in \ell^2(\mathbb{Z}, \mathbb{R})$, which is called the state, describes the wave function. We assume that $\|v\|_\infty < M$. Schrödinger cocycle map $A^E : \mathbb{Z} \to \mathrm{SL}(2, \mathbb{R})$ is defined as

$$A^E(j) = \begin{pmatrix} E - v(j) & -1 \\ 1 & 0 \end{pmatrix}.$$

Notice that $\psi \in \mathbb{C}^{\mathbb{Z}}$ is the solution to the spectral equation

$$H_v\psi = E\psi,$$

if and only if

$$A^E(j)\begin{pmatrix} \psi_j \\ \psi_{j-1} \end{pmatrix} = \begin{pmatrix} \psi_{j+1} \\ \psi_j \end{pmatrix}, \ j \in \mathbb{Z}.$$

The cocycle iteration is defined as

$$A_n^E(j) = \begin{cases} A^E(j+n-1) \cdots A^E(j), & n \geq 1, \\ I_2, & n = 0, \\ A^E(j+n)^{-1} \cdots A^E(j-1)^{-1}, & n \leq -1. \end{cases}$$

We can also define uniform hyperbolicity of $\mathrm{SL}(2, \mathbb{R})$-cocycle for a sequence of $\mathrm{SL}(2, \mathbb{R})$-matrices in the following way:

Definition 4 *We say that bounded $A : \mathbb{Z} \to \mathrm{SL}(2, \mathbb{R})$ is* uniformly hyperbolic ($\mathcal{UH}$) *if there are two maps*

$$u,\ s : \mathbb{Z} \to \mathbb{RP}^1$$

such that u, s are A–invariant in the sense that for all $j \in \mathbb{Z}$, it holds that

$$A(j) \cdot u(j) = u(j+1) \text{ and } A(j) \cdot s(j) = s(j+1).$$

there exists $C > 0, \lambda > 1$ such that $\|A_{-n}(j)\vec{v}\|, \|A_n(j)\vec{w}\| \leq C\lambda^{-n}$ for all $n \geq 1$, all $j \in \mathbb{Z}$, and all unit vectors $\vec{v} \in u(j), \vec{w} \in s(j)$.

Then the following theorem is from [18, Theorem 1]:

Theorem 2 *$A^E : \mathbb{Z} \to \mathrm{SL}(2, \mathbb{R})$ is uniformly hyperbolic if and only if $E \in \rho(H_v)$.*

Moreover, in [18] the author proved that uniform hyperbolicity of sequences is equivalent to the uniform hyperbolicity of dynamically defined cocycles if the base dynamics is topologically transitive. Indeed, if we start with a uniformly hyperbolic cocycle (T, A^E) over a base dynamics (T, Ω), then $A^E : \mathbb{Z} \to \mathrm{SL}(2, \mathbb{R})$ where $A^E(n) = A^E(T^n\omega)$ is a $\mathrm{SL}(2, \mathbb{R})$-uniformly hyperbolic sequence. Notice that no restrictions are imposed on the base dynamics in this case. However, without further restrictions on base dynamics, for a continuous cocycle map $A : \Omega \to \mathrm{SL}(2, \mathbb{R})$, uniform hyperbolicity of sequences does not immediately imply uniform hyperbolicity of cocycle (T, A). The reason is that uniformity of some constants might be lost. However, if we assume base dynamics to be topologically transitive, uniform hyperbolicity is equivalent to a uniform exponential growth condition which can easily be passed uniformly from a dense orbit to the whole space.

So in some sense, it might be more intuitive to start with Johnson's theorem for $\mathrm{SL}(2,\mathbb{R})$-sequences and then deduce the Johnson's theorem for dynamically defined Schrödinger operator by requiring the underlying space to be topologically transitive.

Our goal is to extend this result to a more general setting. The question that we ask is what if instead of Schrödinger operator we work with a more general operator that might not be in $\mathrm{SL}(2,\mathbb{R})$, how can we extend the notion of uniform hyperbolicity for such operators? Would Johnson's theorem still hold in that setting? These questions greatly influenced my research and they were the building blocks of the analysis of relationship between the spectrum of Jacobi operator and the properties that its cocycle map possesses. But to answer those questions we first need to explore what are the Jacobi operators and what is the importance of studying them.

Chapter 3

Jacobi Operator and Dominated Splitting

Operator $J_{a,b} : \ell^2(\mathbb{Z}, \mathbb{R}) \to \ell^2(\mathbb{Z}, \mathbb{R})$ is called Jacobi operator and is given by

$$(J_{a,b}\psi)_n = a_n\psi_{n+1} + a_{n-1}\psi_{n-1} + b_n\psi_n, \tag{3.1}$$

where $\psi = (\psi_n)_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z}, \mathbb{R})$, $a = (a_n)_{n\in\mathbb{Z}} \in \ell^\infty(\mathbb{Z}, \mathbb{R})$, and $b = (b_n)_{n\in\mathbb{Z}} \in \ell^\infty(\mathbb{Z}, \mathbb{R})$.

Jacobi operator is associated with the following symmetric tri-diagonal matrix

$$J_{a,b}(j) = \begin{pmatrix} \ddots & \ddots & \ddots & & & \\ a_{j+n-2} & b_{j+n-1} & a_{j+n-1} & & & \\ & a_{j+n-1} & b_{j+n} & a_{j+n} & & \\ & & a_{j+n} & b_{j+n+1} & a_{j+n+1} & \\ & & \ddots & \ddots & \ddots & \end{pmatrix}$$

The operator is named after Carl Gustav Jacob Jacobi, a German mathematician who was one of the first to study infinite symmetric tri-diagonal matrices. In 1845 Jacobi wrote a paper where he introduced the new iterative way of solving symmetric tri-diagonal matrices that arise in the method of least squares [7]. He used the rotations to increase the diagonal dominance of such linear systems which he then solved by what we today call a point Jacobi method. This might have been one of the first starting points in the study of the symmetric tri-diagonal matrices, or the Jacobi matrices.

Further advancement in the theory of tri-diagonal symmetric matrices came from Lanczos, who in the early 1950s introduced the methods for solving linear systems of equations called the conjugate gradient and Lanczos algorithms [6]. The idea behind the Lanczos algorithm is the successive transformation of the given matrix into tri-diagonal form and then computing the approximations to eigenvalues. The use of commuters and the increased computer memories made it possible to use these methods to solve problems which could not be solved in any other way. Further refinements of algorithms even allowed solving some nonlinear systems of equations and unconstrained and constrained optimization problems.

In the more recent years there has been many more developments in the study of Jacobi operators. Jacobi operators are used in the study of completely integrable nonlinear lattices, for example Toda lattice, which is a model for a one dimensional crystal is solid state physics, and its modified counterpart, the Kac-van Moerbeke lattice [16]. Jacobi operators are widely used and play an important role in the inverse spectrum theory [15].

Notice that the Jacobi operator is a natural extension to one-dimensional discrete Schrödinger operator, which we encountered in the previous section. If we set $a_n = 1$ for all $n \in \mathbb{Z}$, then we get $J_{a,b} = H_v$, where $v_n = b_n$.

In this paper we study the relationship between the spectrum of Jacobi operator and the dynamics of its cocycle map.

We want to point out that usually in the definition of Jacobi operators, the real -valued sequence $a = (a_n)_{n\in\mathbb{Z}}$ is set to satisfy $a_n > 0$ or $a_n \in \mathbb{R} - \{0\}$ for all $n \in \mathbb{N}$. By allowing $a_n = 0$ for some $n \in \mathbb{N}$, we greatly complicate the study of the operator, since in this case the associated cocycle map is singular.

Jacobi operator is a bounded, self-adjoint, linear operator. Hence, the spectrum $\sigma(J_{a,b})$ of the operator $J_{a,b}$ which is defined as

$$\sigma(J_{a,b}) := \{E \in \mathbb{R} : J_{a,b} - E \textit{ does not have bounded inverse}\}$$

is a nonempty compact subset of $\mathbb{R}$. Resolvent, $\rho(J_{a,b})$, of the operator $J_{a,b}$ is then:

$$\rho(J_{a,b}) = \mathbb{C} - \sigma(J_{a,b}).$$

A key part of the spectral analysis of the operator (3.1) is to understand the asymptotic behaviors of solutions of the spectral equation

$$J_{a,b}\psi = E\psi, \tag{3.2}$$

where $E \in \mathbb{C}$ is the *energy parameter*. A direct computation shows that if $\psi \in \mathbb{C}^{\mathbb{Z}}$ solves equation (3.2), then

$$B^E(j)\begin{pmatrix}\psi_j\\ \psi_{j-1}\end{pmatrix} = a_j\begin{pmatrix}\psi_{j+1}\\ \psi_j\end{pmatrix},\ j \in \mathbb{Z}, \tag{3.3}$$

where $B^E : \mathbb{Z} \to \mathrm{M}(2,\mathbb{R})$ is called the *Jacobi cocycle map* and is defined as

$$B^E(j) = \begin{pmatrix} E - b_j & -a_{j-1} \\ a_j & 0 \end{pmatrix}. \tag{3.4}$$

The cocycle iteration is defined as

$$B_n^E(j) = \begin{cases} B^E(j+n-1)\cdots B^E(j), & n \geq 1, \\ I_2, & n = 0, \end{cases} \tag{3.5}$$

where I_2 is the identity matrix. If B is invertible, then

$$B_n^E(j) = B^E(j+n)^{-1} \cdots B^E(j-1)^{-1}, \ n \leq -1.$$

By (3.3),

$$B_n^E(j)\begin{pmatrix} \psi_j \\ \psi_{j-1} \end{pmatrix} = \left(\prod_{i=j}^{j+n-1} a_i\right)\begin{pmatrix} \psi_{j+n} \\ \psi_{j+n-1} \end{pmatrix} \text{ for all } j, n \in \mathbb{Z}. \tag{3.6}$$

Through this relation, the spectral analysis of the operator (3.1) may then be turned into the study of the dynamics of the cocycle iterations (3.5). Notice that if $|a_j| > c$ for all $j \in \mathbb{Z}$, then

$$A^E(j) = \frac{1}{a_j} B^E(j),$$

where $A^E(j)$ is the transfer matrix. In this case we can follow the same logic as in [18] to get Johnson's theorem for such sequences. So the interesting case happens when we let $B^E(j)$ take values in $\mathrm{M}(2, \mathbb{R})$, in particular when we allow determinant to be close to or equal zero. For this general case the notion of uniform hyperbolicity is not sufficient. Just like Jacobi operators are the generalization of Schrödinger operators, we now need to generalize the notion of uniform hyperbolicity. C. Marx [10] first noted that in such scenario the right choice should be cocycles admitting dominated splitting, which is a natural generalization of uniformly hyperbolic cocycles to those allowing $\mathrm{M}(2, \mathbb{R})$-values.

Let (T, B) be a dynamical system, where $T : \Omega \to \Omega$ is a homeomorphism on a compact metric space Ω and $B : \Omega \to \mathrm{M}(2, \mathbb{C})$ be a continuous map.

Definition 5 *We say that (T, B) admits dominated splitting if there exist two maps*

$$u, s : \Omega \to \mathbb{CP}^1,$$

such that

1. *u, s are (T, B) -invariant, meaning that for all $\omega \in \Omega$,*

$$B(\omega) \cdot u(\omega) \subseteq u[T(\omega)] \text{ and } A(\omega) \cdot s(\omega) \subseteq s[T(\omega)].$$

2. *There exists $N > 0, \lambda > 1$ such that*

$$\|B_N(\omega)\vec{v}\| > \lambda \|B_N(\omega)\vec{w}\|$$

for all $\omega \in \Omega$, and all unit vectors $\vec{v} \in u(\omega), \vec{w} \in s(\omega)$.

3. *There exists $\delta > 0$, such that $d(u(\omega), s(\omega)) > \delta$ for all $\omega \in \Omega$ uniformly.*

Moreover, let $a \in C(\Omega, \mathbb{C})$ and $b \in C(\Omega, \mathbb{R})$, i.e. a and b are continuous functions on Ω. Then for each $\omega \in \Omega$, we may define Jacobi operator as follows

$$(J_\omega \psi)(n) = \overline{a(T^{n-1}\omega)}\psi(n-1) + a(T^n\omega)\psi(n+1) + b(T^n\omega)\psi(n). \tag{3.7}$$

Let B^E be the Jacobi cocycle map associated with the energy $E \in \mathbb{C}$. We say T is minimal if $\overline{\mathrm{Orb}(\omega)} = \Omega$ for all ω. We say T is strictly ergodic if T is uniquely ergodic and minimal. Note that when T is minimal, then $\rho(J_\omega)$ is independent of $\omega \in \Omega$. This is a direct consequence of the following proposition [18]:

Proposition 1 *If $\overline{\mathrm{Orb}(\omega_0)} = \Omega$, then $\sigma(J_\omega) \subset \sigma(J_{\omega_0})$ for all $\omega \in \Omega$.*

The proposition above was stated for Schrödinger operators, but the proof works for Jacobi operators as well.

Now, Marx [10] showed the following version of Johnson's theorem for Jacobi operators:

Theorem 3 *Let $T : \Omega \to \Omega$ be strictly ergodic. Assume $B^E \in C^0(\Omega, \mathrm{M}(2, \mathbb{C}))$ and $\int_\Omega \big|\log|\det(B^E(\omega))|\big| d\mu < \infty$ for all $E \in \mathbb{C}$. Assume that the map $\omega \mapsto J_\omega$ is continuous with respect to the operator norm on J_ω. Then $E \in \rho(J_\omega)$ if and only if (T, B^E) admits dominated splitting.*

The original version of Marx's theorem was stated for quasiperiodic Jacobi cocycles whose base dynamics are minimal translations on the d-dimensional torus. Such base dynamics satisfy all the conditions stated in Theorem 3. Even with recent improvements, the above theorem is very restrictive. Motivated by results in [18], we believe that many of the constraints used in Theorem 3 are not necessary and that Johnson's theorem should be true for Jacobi operators as long as the base space is topologically transitive.

We also believe that there should be a version of Johnson's theorem for the Jacobi operators defined by sequences $a, b \in \ell^\infty(\mathbb{Z}, \mathbb{R})$. That is the main goal of this paper: to establish thorough version of Johnson's theorem for $\mathrm{M}(2, \mathbb{R})$-sequences. After this is done, we can try to lift this result to dynamically defined cocycle maps.

In the case of $\mathrm{SL}(2, \mathbb{R})$ matrices, the definition of uniform hyperbolicity for cocycles defined over base dynamics and for $\mathrm{SL}(2, \mathbb{R})$-sequences are almost identical, see [18]. However, though the definition of dominated splitting for $\mathrm{M}(2, \mathbb{C})$ cocycles over base dynamics exists, it's not immediate clear what the one for $\mathrm{M}(2, \mathbb{C})$-sequences should be. When cresting a definition, we wanted it to satisfy the following properties:

1. It should generalize the definition of uniformly hyperbolic $\mathrm{SL}(2,\mathbb{R})$-cocycles.

2. If a dynamically defined cocycle (T, B) admits dominated splitting, then

$$B(T^{(\cdot)}\omega) : \mathbb{Z} \to \mathrm{M}(2,\mathbb{C})$$

   should be a sequence that admits dominated splitting for each $\omega \in \Omega$.

3. It should be stable under $\|\cdot\|_\infty$-perturbation.

4. It should be the right definition to show Johnson's theorem for the Jacobi operator $J_{a,b}$ with any choice of $a, b \in \ell^\infty(\mathbb{Z},\mathbb{R})$.

Notice that if we can obtain such definition, then condition (2) will automatically guarantee that we can lift dominated splitting from cocycles defined dynamically, to cocycles defined by sequences. So if one hopes to establish thorough version of Johnson's theorem for Jacobi operators, one would need to only show that the the dominated splitting can be lifted from $\mathrm{M}(2,\mathbb{C})$-sequences to $\mathrm{M}(2,\mathbb{C})$-cocycles. Such definition is presented below.

Definition 6 *We say that bounded $B : \mathbb{Z} \to \mathrm{M}(2,\mathbb{C})$ admits* dominated splitting *($\mathcal{DS}$) if for each $j \in \mathbb{Z}$, there are one-dimensional spaces*

$$E^u(j) \oplus E^s(j) = \mathbb{C}^2$$

with the following properties.

1. *E^u, E^s are B–invariant in the sense that for all $j \in \mathbb{Z}$, it holds that*

$$B(j)[E^u(j)] \subseteq E^u(j+1) \text{ and } B(j)[E^s(j)] \subseteq E^s(j+1).$$

2. *There exists $N \in \mathbb{Z}_+$, $\lambda > 1$ such that*

$$\|B_N(j)\vec{u}(j)\| > \lambda \|B_N(j)\vec{s}(j)\|$$

for all $j \in \mathbb{Z}$, and all unit vectors $\vec{u}(j) \in E^u(j), \vec{s}(j) \in E^s(j)$.

3. *There exists $\delta > 0$, such that $d(E^u(j), E^s(j)) > \delta$ for all $j \in \mathbb{Z}$ uniformly.*

4. *For all $n \in \mathbb{Z}_+$, there exists $c = c(n) > 0$, such that $\|B_n(j)\| \geq c$ for all $j \in \mathbb{Z}$.*

Here we say the space E^u dominates the space E^s. Throughout this paper, without loss of generality, we set $\lambda = 2$ in condition (2) in Definition 6. From now on, $B \in \mathcal{DS}$ means B admits dominated splitting. Conditions (1)-(3) of Definition 6 are equivalent to conditions (1)-(3) of Definition 5 for dynamically defined $\mathrm{M}(2,\mathbb{C})$-cocycles that admit dominated splitting. However, when it comes to $\mathrm{M}(2,\mathbb{C})$-sequences, conditions (1)-(3) do not imply condition (4) which is necessary to guarantee the stability of dominated splitting.

Notice that our definition clearly generalizes the definition of uniformly hyperbolic $\mathrm{SL}(2,\mathbb{R})$-cocycles. Moreover, if a dynamically defined cocycle (T, B) admits dominated splitting, then $B(T^{(\cdot)}\omega) : \mathbb{Z} \to \mathrm{M}(2,\mathbb{C})$ satisfies Definition 6 for each $\omega \in \Omega$. Hence, our work is divided into two parts. First we need to show that indeed, the Definition 6 is stable under $\|\cdot\|_\infty$-perturbation. This is done by proving the following theorem:

Theorem 4 *Let $B : \mathbb{Z} \to \mathrm{M}(2,\mathbb{C})$ be such that $B \in \mathcal{DS}$. There exists $\varepsilon > 0$ such that if $\widetilde{B} : \mathbb{Z} \to \mathrm{M}(2,\mathbb{C})$ satisfies $\|\widetilde{B} - B\|_\infty < \varepsilon$, then $\widetilde{B} \in \mathcal{DS}$.*

Then we show Johnson's theorem for $\mathrm{M}(2,\mathbb{R})-$sequences.

Theorem 5 *Let $B^E : \mathbb{Z} \to \mathrm{M}(2,\mathbb{R})$ be the Jacobi cocycle for the Jacobi operator $J_{a,b}$. Then*

$$\sigma(J_{a,b}) = \{E \in \mathbb{R} : B^E \notin \mathcal{DS}\}.$$

Chapter 4

Stability of Dominated Splitting

As mentioned earlier, stability of dominated splitting is essential in order for Definition 6 to be meaningful. If we can not show stability, the definition would be insufficient. The key step to show Theorem 4 is the following theorem:

Theorem 6 *Let* $A : \mathbb{Z} \to \mathrm{M}(2, \mathbb{C})$ *be that* $\inf_{j \in \mathbb{Z}} \|A(j)\| > c > 0$. *If there exist* $\alpha > \alpha' > 0$ *such that* $A(j) \cdot (\mathbb{D}_\alpha) \subset \mathbb{D}_{\alpha'}$ *for all* $j \in \mathbb{Z}$*, then* $A \in \mathcal{DS}$.

Theorem 6 basically says that the existence of an invariant cone field implies domination for a $\mathrm{M}(2, \mathbb{C})$-sequence. In this section we will now prove Theorems 4 and 6.

Let $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{M}(2, \mathbb{C})$ be a nonzero matrix. Under the projection π as in (2.1), there is an induced projectivized map of A acting on projective space $(\mathbb{CP}^1) \setminus \{\alpha\}$, where α is the eigenspace of the 0 eigenvalue of A, if such exists. We denoted the induced map by $A \cdot z$. Then a direct computation shows that

$$A \cdot z : \mathbb{CP}^1 \to \mathbb{CP}^1,\ A \cdot z = \frac{c + dz}{a + bz}. \tag{4.1}$$

Recall $\vec{z}$ denotes a unit vector in the one-dimensional space $z \in \mathbb{CP}^1$. By (2.3) and (2.4), we have

$$\begin{aligned} d(A \cdot z, A \cdot z') &= d(A \cdot \pi(\vec{z}), A \cdot \pi(\vec{z}')) \\ &= d(\pi(A\vec{z}), \pi(A\vec{z}')) \\ &= \frac{|\det(A\vec{z}, A\vec{z}')|}{\|A\vec{z}\| \cdot \|A\vec{z}'\|} \\ &= \frac{|\det(A)|}{\|A\vec{z}\| \cdot \|A\vec{z}'\|} |\det(\vec{z}, \vec{z}')| \\ &= \frac{|\det(A)|}{\|A\vec{z}\| \cdot \|A\vec{z}'\|} |d(z, z')| \end{aligned} \tag{4.2}$$

In particular, if $|\det(A)| > \delta > 0$, then it holds that

$$\inf_{\|\vec{v}\|=1} \|A\vec{v}\| = \frac{|\det(A)|}{\|A\|} > \frac{\delta}{M}$$

which in turn implies that

$$\frac{\delta}{M^2} d(z, z') \leq d(A \cdot z, A \cdot z') \leq \frac{M^4}{\delta^2} d(z, z') \text{ for all } z, z' \in \mathbb{CP}^1. \tag{4.3}$$

Sometimes, we may need to use the another projection

$$\bar{\pi} : \mathbb{C}^2 \setminus \{\vec{0}\} \to \mathbb{CP}^1 \text{ where } \pi_2 \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \frac{z_1}{z_2}.$$

Then the projectivized map of A under $\bar{\pi}$, denoted by $\bar{A}$, becomes the well-known Möbius transformation:

$$\bar{A}(z) = \frac{az + b}{cz + d}. \tag{4.4}$$

All the computation above still hold true for $\bar{A}$. In fact, $d(\bar{A}(z), \bar{A}(z')) = d(A \cdot z, A \cdot z')$.

4.1 Invariant cone field and dominated splitting

Lemma 7 *Let* $f : \mathbb{D}_\alpha \to \mathbb{D}_{\alpha'}$ *be a holomorphic function for some* $0 < \alpha' < \alpha$. *Then there exists a* $0 < \rho = \rho(\alpha, \alpha') < 1$ *such that it holds for all* $z_1, z_2 \in \mathbb{D}_\alpha$

$$\left|\frac{f(z_2) - f(z_1)}{\alpha^2 - \overline{f(z_1)}f(z_2)}\right| < \rho \left|\frac{z_2 - z_1}{\alpha^2 - \overline{z_1}z_2}\right|.$$

Proof. Let $\mathbb{D} = \mathbb{D}_1$. Define $g : \mathbb{D} \to \mathbb{D}$ to be

$$g(z) := \frac{1}{\alpha'} f(\alpha z) \tag{4.5}$$

which is a holomorphic function on the unit disc. By Schwarz–Pick theorem (see [5]), for all $z_1, z_2 \in \mathbb{D}$,

$$\left|\frac{g(z_2) - g(z_1)}{1 - \overline{g(z_1)}g(z_2)}\right| \leq \left|\frac{z_2 - z_1}{1 - \overline{z_1}z_2}\right|.$$

Then, from (4.5), for all $z \in \mathbb{D}_\alpha$

$$f(z) = \alpha' g\left(\frac{z}{\alpha}\right).$$

Hence, for all $z_1, z_2 \in \mathbb{D}_\alpha$, we have

$$\begin{aligned}
\left|\frac{f(z_2) - f(z_1)}{\alpha'^2 - \overline{f(z_1)}f(z_2)}\right| &= \left|\frac{\alpha' g\left(\frac{z_1}{\alpha}\right) - \alpha' g\left(\frac{z_2}{\alpha}\right)}{\alpha'^2 - \overline{\alpha' g\left(\frac{z_1}{\alpha}\right)}\alpha' g\left(\frac{z_2}{\alpha}\right)}\right| \\
&= \frac{1}{\alpha'}\left|\frac{g\left(\frac{z_1}{\alpha}\right) - g\left(\frac{z_2}{\alpha}\right)}{1 - \overline{g\left(\frac{z_1}{\alpha}\right)}g\left(\frac{z_2}{\alpha}\right)}\right| \\
&\leq \frac{1}{\alpha'}\left|\frac{\frac{z_2}{\alpha} - \frac{z_1}{\alpha}}{1 - \frac{\overline{z_1}}{\alpha}\frac{z_2}{\alpha}}\right| \\
&= \frac{\alpha}{\alpha'}\left|\frac{z_2 - z_1}{\alpha^2 - \overline{z_1}z_2}\right|
\end{aligned}$$

We can rewrite

$$\begin{aligned}
\left|\frac{f(z_2) - f(z_1)}{\alpha'^2 - \overline{f(z_1)}f(z_2)}\right| &= \frac{\alpha^2}{\alpha'^2}\left|\frac{f(z_2) - f(z_1)}{\alpha^2 - (\frac{\alpha}{\alpha'})^2\overline{f(z_1)}f(z_2)}\right| \\
&= \frac{\alpha^2}{\alpha'^2}\left|\frac{f(z_2) - f(z_1)}{\alpha^2 - \overline{f(z_1)}f(z_2)}\right|\left|\frac{\alpha^2 - \overline{f(z_1)}f(z_2)}{\alpha^2 - (\frac{\alpha}{\alpha'})^2\overline{f(z_1)}f(z_2)}\right|
\end{aligned}$$

Hence, combining the last two formulas, we get

$$\left|\frac{f(z_2)-f(z_1)}{\alpha^2-\overline{f(z_1)}f(z_2)}\right| = \left|\frac{\alpha'^2-\overline{f(z_1)}f(z_2)}{\alpha^2-\overline{f(z_1)}f(z_2)}\right| \cdot \left|\frac{f(z_2)-f(z_1)}{\alpha'^2-\overline{f(z_1)}f(z_2)}\right| \tag{4.6}$$
$$\leq \frac{\alpha}{\alpha'}\left|\frac{\alpha'^2-\overline{f(z_1)}f(z_2)}{\alpha^2-\overline{f(z_1)}f(z_2)}\right| \cdot \left|\frac{z_2-z_1}{\alpha^2-\overline{z_1}z_2}\right|.$$

All that is left to show is that

$$\frac{\alpha}{\alpha'}\left|\frac{\alpha'^2-\overline{f(z_1)}f(z_2)}{\alpha^2-\overline{f(z_1)}f(z_2)}\right| < \rho < 1 \text{ for all } z_1, z_2 \in \mathbb{D}_\alpha. \tag{4.7}$$

The estimate above can be reduced to finding an upper bound of the following function

$$h : \mathbb{D}_a \to \mathbb{R} \text{ where } \ h(z) = \left|\frac{a-z}{b-z}\right| \text{ and } 0 < a < b.$$

Set $z = te^{i\theta}$ for some $0 \leq t < a$. Then,

$$\begin{aligned} h(te^{i\theta}) &= \left|\frac{a-te^{i\theta}}{b-te^{i\theta}}\right| \\ &= \sqrt{\frac{(a-t\cos(\theta))^2+(t\sin(\theta))^2}{(b-t\cos(\theta))^2+(t\sin(\theta))^2}} \\ &= \left(\frac{a^2+t^2-2at\cos\theta}{b^2+t^2-2bt\cos\theta}\right)^{\frac{1}{2}}. \end{aligned}$$

e, $\max_{\theta\in[0,2\pi)} h(te^{i\theta}) = h(te^{i\pi}) = \frac{a+t}{b+t}$ which is monotone increasing in $t \in [0,\infty)$. Hence we obtain

$$\sup_{z\in\mathbb{D}_a} |h(z)| \leq \frac{2a}{a+b}.$$

Thus, we have for the left hand side of (4.7) that

$$\frac{\alpha}{\alpha'}\left|\frac{\alpha'^2-\overline{f(z_1)}f(z_2)}{\alpha^2-\overline{f(z_1)}f(z_2)}\right| \leq \frac{\alpha}{\alpha'}\frac{2\alpha'^2}{\alpha^2+\alpha'^2} = \frac{2\frac{\alpha}{\alpha'}}{1+(\frac{\alpha}{\alpha'})^2} < 1 \text{ for all } z_1, z_2 \in \mathbb{D}_\alpha.$$

In other words, we can set $\rho = \rho(\alpha,\alpha') = \frac{2\frac{\alpha}{\alpha'}}{1+(\frac{\alpha}{\alpha'})^2} < 1$. ■

Now we prove one of the main results in this section, Theorem 6.

Proof of Theorem 6. Replacing α by a smaller number if necessary, we may assume that for all $j \in \mathbb{Z}$

$$A(j) \cdot (\overline{\mathbb{D}}_\alpha) \subset \overline{A(j) \cdot (\mathbb{D}_\alpha)} \subset \mathbb{D}_{\alpha'}.$$

By Lemma 7, there exists a $0 < \rho < 1$ such that for all $j \in \mathbb{Z}$ and all $z_1, z_2 \in \mathbb{D}_\alpha$ that

$$\left| \frac{A(j) \cdot z_2 - A(j) \cdot z_1}{\alpha^2 - \overline{A(j) \cdot z_1} A(j) \cdot z_2} \right| < \rho \left| \frac{z_2 - z_1}{\alpha^2 - \overline{z_1} z_2} \right|.$$

Thus we have for all $n \in \mathbb{Z}_+$, all $j \in \mathbb{Z}$, and all $z_1, z_2 \in \overline{\mathbb{D}}_\alpha$ that

$$\left| \frac{A_n(j) \cdot z_2 - A_n(j) \cdot z_1}{\alpha^2 - \overline{A_n(j) \cdot z_1} A_n(j) \cdot z_2} \right| < \rho^{n-1} \left| \frac{A(j) \cdot z_2 - A(j) \cdot z_1}{\alpha^2 - \overline{A(j) \cdot z_1} A(j) \cdot z_2} \right|$$

which in turn implies that

$$|A_n(j) \cdot z_2 - A_n(j) \cdot z_1| < \frac{2\alpha^2 \alpha'^2}{\alpha^2 - \alpha'^2} \rho^{n-1} \text{ for all } z_1, z_2 \in \mathbb{D}_\alpha \tag{4.8}$$

Hence, the sequence of sets

$$A_n(j-n) \cdot (\overline{\mathbb{D}}_\alpha), \; n \geq 1$$

is a sequence of nested compact sets whose diameter goes to 0 as n goes to ∞. Hence,

$$\bigcap_{n \geq 1} A_n(j-n) \cdot (\overline{\mathbb{D}}_\alpha)$$

is a single point. We define

$$E^u(j) := \bigcap_{n \geq 1} A_n(j-n) \cdot (\overline{\mathbb{D}}_\alpha).$$

We claim that E^u is A- invariant. Indeed, we have for all $j \in \mathbb{Z}$,

$$\begin{aligned}
A(j) \cdot E^u(j) &= A(j) \cdot \left(\bigcap_{n \geq 1} A_n(j-n) \cdot (\overline{\mathbb{D}}_\alpha) \right) \\
&\subset \bigcap_{n \geq 1} A(j) \cdot \left[A_n(j-n) \cdot (\overline{\mathbb{D}}_\alpha) \right] \\
&= \bigcap_{n \geq 1} A_n(j+1-n) \cdot \left[(A(j-n)\overline{\mathbb{D}}_\alpha) \right] \\
&\subset \bigcap_{n \geq 1} A_n(j+1-n) \cdot (\overline{\mathbb{D}}_\alpha) \\
&= E^u(j+1).
\end{aligned}$$

Moreover, it's clear that

$$E^u(j) \in \mathbb{D}_{\alpha'} \text{ for all } j \in \mathbb{Z}. \tag{4.9}$$

Next, we compute $E^s(j)$ for all $j \in \mathbb{Z}$. First, suppose $\det A \neq 0$. Then the fact that $A \cdot (\mathbb{D}_\alpha) \subset \mathbb{D}_{\alpha'}$ implies that

$$A^{-1} \cdot (\mathbb{D}_{\alpha'}^{\complement}) \subset \mathbb{D}_{\alpha}^{\complement}.$$

Let $g : \mathbb{CP}^1 \to \mathbb{CP}^1$ be the diffeomorphism such that $g(z) = \frac{1}{z}$. Then it's clear that $g = g^{-1}$ and

$$g \circ A \cdot = \bar{A} \circ g$$

Also, it's clear that $g(\mathbb{D}_r^{\complement}) = \overline{\mathbb{D}}_{\frac{1}{r}}$. Thus if $\det(A(j)) \neq 0$, then the above estimates imply that

$$\overline{A(j)^{-1}} \cdot (\overline{\mathbb{D}}_{\frac{1}{\alpha'}}) = \overline{A(j)^{-1}} \cdot g(\mathbb{D}_{\alpha'}^{\complement}) = g \circ A(j)^{-1} \cdot (\mathbb{D}_{\alpha'}^{\complement}) \subset g(\mathbb{D}_{\alpha}^{\complement}) = \overline{\mathbb{D}}_{\frac{1}{\alpha}}$$

Thus we have for all $\frac{1}{\alpha'} > r > \frac{1}{\alpha}$

$$\overline{A(j)^{-1}} \left(\overline{\mathbb{D}}_{\frac{1}{\alpha'}} \right) \subset \overline{\mathbb{D}}_r.$$

Now we fix an arbitrary $j_0 \in \mathbb{Z}$. To find $E^s(j_0)$, we consider two different cases:

Case 1. Assume $\det A(j) \neq 0$ for all $j \geq j_0$.

Then, similar to the argument where we obtained E^u, we have that

$$\left\{\overline{A_{-n}(j_0+n)} \cdot (\overline{\mathbb{D}}_{\frac{1}{\alpha'}})\right\}_{n\geq 1} = \left\{\overline{A_n(j_0)^{-1}} \cdot (\overline{\mathbb{D}}_{\frac{1}{\alpha'}})\right\}_{n\geq 1}$$

is a nested sequence of compact sets whose diameters tend to zero as $n \to \infty$. Thus we can define

$$E^s(j_0) := g\left[\bigcap_{n\geq 1} \overline{A_{-n}(j_0+n)} \cdot (\overline{\mathbb{D}}_{\frac{1}{\alpha'}})\right].$$

Note that $E^s(j)$ can be defined in the same way for all $j \geq j_0$ in this case. Now similar to the E^u, E^s is A-invariant for all $j \geq j_0$,

$$\begin{aligned}
A^{-1}(j+1) \cdot E^s(j+1) &= A^{-1}(j+1) \cdot g\left[\bigcap_{n\geq 1} \overline{A_{-n}(j+1+n)} \cdot (\overline{\mathbb{D}}_{\frac{1}{\alpha'}})\right] \\
&= g \cdot \overline{A^{-1}(j+1)}\left[\bigcap_{n\geq 1} \overline{A_{-n}(j+1+n)} \cdot (\overline{\mathbb{D}}_{\frac{1}{\alpha'}})\right] \\
&= g \cdot \left[\bigcap_{n\geq 1} \overline{A_{-n}(j+n)} \cdot [A^{-1}(j+n)(\overline{\mathbb{D}}_{\frac{1}{\alpha'}})]\right] \\
&\supset g \cdot \left[\bigcap_{n\geq 1} \overline{A_{-n}(j+n)} \cdot (\overline{\mathbb{D}}_{\frac{1}{\alpha'}})\right] \\
&= E^s(j).
\end{aligned}$$

Hence, for all $j \geq j_0$

$$A(j) \cdot E^s(j) = E^s(j+1).$$

Moreover, $E^s(j) \in g\left(\bigcap_{r>\frac{1}{\alpha}} \mathbb{D}_r\right) = \mathbb{D}_{\alpha}^{\complement}$.

Case 2. Assume $\det A(j) = 0$ for some $j \geq j_0$.

Then we set $j' = \min\{j \geq j_0 : \det A(j) = 0\}$ and define

$$E^s(j_0) := \ker(A_{j'-j_0+1}(j_0)).$$

We know that $\mathbb{D}_\alpha$ is contained in the domain of $A_n(j)\cdot$ for all $j \in \mathbb{Z}$ and all $n \geq 1$. Thus $A_n(j)$ cannot be a zero matrix which implies that $E^s(j_0)$ is an one-dimensional subspace of $\mathbb{C}^2$. Moreover, since $A_{j'-j_0}(j_0)$ cannot be defined at $\pi(E^s(j_0))$, $E^s(j_0)$ cannot be in the domain of $A_{j'-j_0}(j_0)$. In particular, it holds that

$$E^s(j_0) \in \mathbb{D}_\alpha^{\complement}.$$

To show that E^s is A- invariant, we have to further consider two subcases. First, we assume $j' = j_0$. Then

$$E^s(j_0) = \ker A(j_0) \text{ and } A(j_0)[E^s(j_0)] = \{\vec{0}\} \subset E^s(j_0+1)$$

no matter what is $E^s(j_0+1)$. Second, $j' > j_0$. Then $j' \geq j_0+1$ which implies j_0+1 is in this second case as well. Thus, by our definition, we have

$$E^s(j_0+1) = \ker(A_{j'-j_0}(j_0+1)).$$

Moreover, we have $\det A(j_0) \neq 0$ in this case. Thus we must have

$$\begin{aligned} E^s(j_0) &= \ker\left[A_{j'-j_0+1}(j_0)\right] \\ &= \ker\left[A_{j'-j_0}(j_0+1)A(j_0)\right] \\ &= A(j_0)^{-1}[\ker(A_{j'-j_0}(j_0+1))] \\ &= A(j_0)^{-1}[E^s(j_0+1)] \end{aligned}$$

or equivalently

$$A(j_0)[E^s(j)] = E^s(j_0+1).$$

Combining cases 1 and 2, we have defined $E^s(j)$ for all $j \in \mathbb{Z}$, showed its invariance, and obtained $E^s(j) \in \mathbb{D}_\alpha^\complement$. Thus we have

$$d(E^s(j), E^u(j)) \geq \inf\{d(z, z') : z \in \mathbb{D}_\alpha^\complement, z' \in \mathbb{D}_{\alpha'}\}$$

It's straightforward calculation that for all $z \in \mathbb{D}_\alpha^\complement$ and all $z' \in \mathbb{D}_{\alpha'}$ that

$$\begin{aligned} d(z, z') &= \frac{2|z - z'|}{\sqrt{(1 + |z|^2)(1 + |z'|^2)}} \\ &\geq \frac{2(|z| - \alpha')}{\sqrt{(1 + |z|^2)(1 + \alpha'^2)}} \\ &= \frac{2(1 - \frac{\alpha'}{|z|})}{\sqrt{(1 + |z|^{-2})(1 + \alpha'^2)}} \\ &\geq \frac{2(1 - \frac{\alpha'}{\alpha})}{\sqrt{(1 + \alpha^{-2})(1 + \alpha'^2)}} \\ &= \frac{2(\alpha - \alpha')}{\sqrt{(1 + \alpha^2)(1 + \alpha'^2)}}. \end{aligned} \tag{4.10}$$

Let $\delta = \frac{2(\alpha-\alpha')}{\sqrt{(1+\alpha^2)(1+\alpha'^2)}} > 0$. Thus, we have

$$\inf_{j\in\mathbb{Z}} d(E^s(j), E^u(j)) > \delta. \tag{4.11}$$

So far we have shown conditions (1) and (3) from Definition 6. Now we need to show that E^u dominates E^s, i.e. condition (2) from Definition 6.

Let $\vec{u}(j) \in E^u(j)$ and $\vec{s}(j) \in E^s(j)$ be unit vectors. We set $D(j) = \big(\vec{u}(j),\ \vec{s}(j)\big) \in \mathrm{M}(2, \mathbb{C})$. In other words, $\vec{u}(j)$ and $\vec{s}(j)$ are the column vectors of $D(j)$. By (2.4) and (4.11), we have

$$\inf_{j\in\mathbb{Z}} |\det D(j)| = \inf_{j\in\mathbb{Z}} d(E^u(j), E^s(j)) > \delta. \tag{4.12}$$

Clearly, $D(j)$ are invertible for all $j \in \mathbb{Z}$ and there are c, C, depending on δ, such that

$$c < \|D(j)^{\pm 1}\| < C \text{ for all } j \in \mathbb{Z}. \tag{4.13}$$

Together with $|\det(D(j))| \leq \|D(j)\|^2 \leq 1$, (4.13) implies

$$1 \leq |\det D(j)^{-1}| < C \text{ for all } j \in \mathbb{Z}. \tag{4.14}$$

Thus, we may apply (4.3) to $D(j)^{-1}$ and obtain for some c, C, depending on δ only, such that

$$cd(z, z') < d(D(j)^{-1} \cdot z, D(j)^{-1} \cdot z') < Cd(z, z') \text{ for all } j \in \mathbb{Z} \text{ and all } z, z' \in \mathbb{CP}^1. \tag{4.15}$$

We define

$$\Lambda(j) := D(j+1)^{-1} A(j) D(j).$$

Since $\Lambda(j)$ leaves one dimensional spaces correspond to both 0 and $\infty \in \mathbb{CP}^1$ invariant, it must be diagonal. In other words, there are $\lambda_j^+, \lambda_j^- \in \mathbb{C}$ for all $j \in \mathbb{Z}$ such that

$$\Lambda(j) = \begin{pmatrix} \lambda_j^+ & 0 \\ 0 & \lambda_j^- \end{pmatrix}.$$

Change c and C if necessary, (4.13) together with the fact $c < \|A(j)\| < M$ for all $j \in \mathbb{Z}$ imply that

$$c < \|\Lambda(j)\| < C \text{ for all } j \in \mathbb{Z}. \tag{4.16}$$

Set $\mathcal{F}(j) = D(j)^{-1}\mathbb{D}_\alpha$. The computation (4.10) shows that $\inf_{j \in \mathbb{Z}} d(E^u(j), \partial \mathbb{D}_\alpha) > \delta$ where $\partial \mathbb{D}_\alpha$ is the boundary of $\mathbb{D}_\alpha$. Thus (4.15) implies that

$$\inf_{j \in \mathbb{Z}} d(0, \partial \mathcal{F}(j)) = \inf_{j \in \mathbb{Z}} d\big(D(j)^{-1} \cdot E^u(j), D(j)^{-1} \cdot \partial \mathbb{D}_\alpha\big) > c\delta > 0 \tag{4.17}$$

In particular, there is a $r' > 0$, depending on δ only, such that

$$\mathbb{D}_{r'} \subset \mathcal{F}(j) \text{ for all } j \in \mathbb{Z}.$$

Let $\operatorname{diam}(S)$ denotes the diameter of a set $S \subset \mathbb{CP}^1$ under the metric d. By (4.8), the fact $d(z, z') < |z - z'|$ for all $z, z' \in \mathbb{C}$, and (4.15), we obtain

$$\begin{aligned}
\operatorname{diam}\big(\Lambda_n(j)\mathcal{F}(j)\big) &= \operatorname{diam}\big(D^{-1}(j)A_n(j) \cdot \mathbb{D}_\alpha\big) \\
&\leq C\operatorname{diam}\big(A_n(j) \cdot \mathbb{D}_\alpha\big) \\
&\leq C'\rho^{n-1}
\end{aligned}$$

where $0 < \rho < 1$. Note by our construction, $E^u(j) \in A_n(j-n) \cdot D_\alpha$ for all $n \geq 1$ and all $j \in \mathbb{Z}$ which implies that $0 \in \Lambda_n(j)\mathcal{F}(j)$ for all $n \geq 1$ and $j \in \mathbb{Z}$. Thus the estimate above shows that $\Lambda_n(j)\mathcal{F}(j)$ uniformly converges to 0 as $n \to \infty$. In particular, there exists a $N \geq 1$ such that for all $j \in \mathbb{Z}$

$$\Lambda_N(j) \cdot \big(\mathbb{D}_{r'}\big) \subset \mathbb{D}_{\frac{r'}{2}}$$

where

$$\Lambda_N(j) = \begin{pmatrix} \prod_{k=j}^{j+N-1} \lambda_k^+ & 0 \\ 0 & \prod_{k=j}^{j+N-1} \lambda_k^- \end{pmatrix}$$

Hence, we must have for all $j \in \mathbb{Z}$ that

$$\left| \prod_{k=j}^{j+N-1} \lambda_k^+ \right| > 2 \left| \prod_{k=j}^{j+N-1} \lambda_k^- \right|.$$

Since $A_N(j)D(j) = D(j+N)\Lambda_N(j)$ where $D(j) = (\vec{u}(j), \vec{s}(j))$, the estimate above implies that

$$\|A_N(j)\vec{u}(j)\| > 2\|A_N(j)\vec{s}(j)\| \text{ for all } j \in \mathbb{Z},$$

where $\vec{u}(j)$ and $\vec{s}(j)$ are unit vectors in $E^u(j)$ and $E^s(j)$, respectively. Thus we've shown that E^u dominates E^s.

To complete the proof that $A \in \mathcal{DS}$, the only thing left is to show for all $n \geq 1$, there is a $c = c(n) \geq 0$ such that $\|A_n(j)\| > c(n)$ for all $j \in \mathbb{Z}$, i.e. condition (3) from Definition 6.

Since A is conjugate to Λ via D which satisfies (4.13), we only need to show such condition holds true for Λ. Since Λ is diagonal, it's then sufficient to show that there is $c > 0$ so that

$$\inf_{j \in \mathbb{Z}} |\lambda_j^+| > c.$$

Suppose it's not true. Then for any $\varepsilon > 0$, there exists a $j \in \mathbb{Z}$ such that $|\lambda_j^+| < \varepsilon$. By (4.16), we must have $|\lambda_j^-| > c$. Then, we must have for all $t > 0$ that

$$\mathbb{D}_{\frac{c}{\varepsilon}t} \subset \Lambda(j) \cdot \mathbb{D}_t$$

By (4.15) and the simple fact that $d(z, z') < |z - z'| < C(r)d(z, z')$ for all $z, z' \in \mathbb{D}_r$, there are $0 < \alpha_1 < \alpha_2$ such that $\mathbb{D}_{\alpha_1} \subset \mathcal{F}(j) = D(j)^{-1}\mathbb{D}_\alpha \subset \mathbb{D}_{\alpha_2}$ for all $j \in \mathbb{Z}$. Hence, we obtain

$$\mathbb{D}_{\frac{c}{\varepsilon}\alpha_1} \subset \Lambda(j) \cdot \mathbb{D}_{\alpha_1} \subset \Lambda(j) \cdot \mathcal{F}(j) \subset \mathcal{F}(j+1) \subset \mathbb{D}_{\alpha_2}$$

which is clearly not true when ε is sufficiently small. This completes the proof. ■

4.2 Stability theorem

Lemma 8 *Let $B \in \ell^\infty(\mathbb{Z}, \mathrm{M}(2, \mathbb{C}))$. Define $B^{(m,N)}(j) := B_N(jN + m)$. Then, $B \in \mathcal{DS}$ if and only if there exists $N \in \mathbb{Z}_+$, such that $B^{(m,N)} \in \mathcal{DS}$ for all $m = 0, 1, 2, ..., N - 1$.*

Proof. We start by showing the *only if* part of the lemma. Let $B \in \mathcal{DS}$. Fix $N \in \mathbb{N}$ such that it satisfies condition (2) in definition 6. For simplicity, we define $A^{(m)} : \mathbb{Z} \to \mathrm{M}(2, \mathbb{C})$

as

$$A^{(m)}(j) = B^{(m,N)}(j) = B_N(jN+m).$$

By condition (3) of Definition 6, it's clear that for each $k \in \mathbb{Z}_+$ there is a $c(k) > 0$ such that $\|A_k^{(m)}(j)\| \geq c(k) > 0$ for all $j \in \mathbb{Z}$ and for all $m = 0, 1, 2, ..., N-1$.

For every $m = 0, 1, 2, ..., N-1$, define $E_m^u(j) := E^u(jN+m)$ and $E_m^s(j) := E^s(jN+m)$. Clearly E_m^s and E_m^u are $A^{(m)}$-invariant and they satisfy the conditions (4) of Definition 6. Finally, with such choices of E_m^s and E_m^u, by our definition of $A^{(m)}$, it's clear that $A^{(m)}$ satisfies condition (2) of Definition 6 at step one. With these subspaces $A^{(m)}$ satisfies conditions (1)-(4) from the definition of $\mathcal{DS}$. Hence, $A^{(m)} = B^{(m,N)} \in \mathcal{DS}$.

Now we show the *if* part of the lemma. Assume that $B^{(m,N)}(j) \in \mathcal{DS}$ for some $N \in \mathbb{Z}_+$ and for all $m = 0, 1, 2, ..., N-1$. Again, we define $A^{(m)} : \mathbb{Z} \to \mathrm{M}(2, \mathbb{C})$ as

$$A^{(m)}(j) = B^{(m,N)}(j) = B_N(jN+m).$$

Then, for every $0 \leq m < N$, $A^{(m)} \in \mathcal{DS}$. In particular, we define $E_m^u(j)$ and $E_m^s(j)$ to be the invariant subspaces of $A^{(m)}$. Let $\vec{s}^{(m)}(j)$ and $\vec{u}^{(m)}(j)$ be unit vectors in $E_m^s(j)$ and $E_m^u(j)$, respectively. We may choose a $\delta > 0$ and a $K \in \mathbb{Z}_+$ so that for all $0 \leq m < N$ and all $j \in \mathbb{Z}$, it holds that

$$\|E_m^s(j) - E_m^u(j)\|_{\mathbb{CP}^1} > \delta \tag{4.18}$$

$$\|A_K^{(m)}(j)\vec{u}^{(m)}(j)\| > 2\|A_K^{(m)}(j)\vec{s}^{(m)}(j)\| \tag{4.19}$$

For every $m \in \{0, 1, 2, ..., N-1\}$, we can define a pair of invariant directions $E^{u,m}$ and $E^{s,m}$ for $B(j)$ as follows. Fix any $j \in \mathbb{Z}$, we have $j = kN + m + \ell$ for some $k \in \mathbb{Z}$ and some $0 \leq \ell < N$. We define $E^{u,m}(j)$ to be

$$E^{u,m}(j) := B_\ell(j - \ell) E^u_m(k)$$

which is always a one-dimensional subspace of $\mathbb{C}^2$. Otherwise, we will have $A^m(k)E^u_m(k) = \{\vec{0}\}$ which contradicts the condition (2) of Definition 6. We define $E^{s,m}(j)$ to be the one-dimensional subspace of $\mathbb{C}^2$ so that

$$B_{N-\ell}(j)E^{s,m}(j) \subset E^s_m(k+1).$$

That is, if $B_{N-\ell}(j)$ is invertible, then $E^{s,m}(j) = B_{N-\ell}(j)^{-1}E^s_m(k+1)$; otherwise, $E^{s,m}(j)$ is the eigenspace of $B_{N-\ell}(j)$ for the eigenvalue 0.

Then $E^{u,m}(j)$ and $E^{s,m}(j)$ are B-invariant for each $0 \leq m < N$. Moreover, $E^{s,m}(j) \neq E^{u,m}(j)$ for all $j \in \mathbb{Z}$. We now have N pairs of invariant subspaces for $B(j)$. We claim that the N choices must be the same. To this end, we first note that it holds that for all $k \in \mathbb{Z}$ that

$$E^{u,m}(kN + m) = E^u_m(k) \text{ and } E^{s,m}(kN + m) = E^s_m(k).$$

We define $\vec{s}_m(j) \in E^{s,m}(j)$ and $\vec{u}_m(j) \in E^{u,m}(j)$ to be unit vectors. Fix any $j \in \mathbb{Z}$, we may assume $j = kN + m$ for some $k \in \mathbb{Z}$ and some $0 \leq m < N$. It then suffices to show that for all $0 \leq \ell < N$, $\ell \neq m$, we have

$$E^{s,\ell}(j) = E^{s,m}(j) \text{ and } E^{u,\ell}(j) = E^{u,m}(j).$$

We assume that $\ell > m$ since the proof of the case $\ell < m$ is completely analogous. We write $\vec{s}_\ell(j) = a\vec{u}_m(j) + b\vec{s}_m(j)$ and $\vec{u}_\ell(j) = c\vec{u}_m(j) + d\vec{s}_m(j)$ for some $a, b, c, d \in \mathbb{C}$. It's clear that $\vec{u}_m(j) \in E^u_m(k)$ and $\vec{s}_m(j) \in E^s_m(k)$. Recall that E^u_m dominates E^s_m under $A^{(m)} = B_N(N(\cdot)+m)$. Thus for all $\lambda > 0$, we have for some large $n_1 = pN \in \mathbb{Z}_+$ that

$$\|B_{n_1}(j)\vec{u}_m(j)\| > \lambda \|B_{n_1}(j)\vec{s}_m(j)\|. \tag{4.20}$$

Note we have $B_{n_1}(j)\vec{u}_m(j) \in E^u_m(p+k)$ and $B_{n_1}(j)\vec{s}_m(j) \in E^s_m(p+k)$.

On the other hand, $B_{\ell-m}(j)\vec{s}_\ell(j) \in E^s_\ell(k)$ and $B_{\ell-m}(j)\vec{u}_\ell(j) \in E^u_\ell(k)$. Thus for the same $\lambda > 0$ above and by choosing p appropriately, we can find a $n_2 = n_1 + \ell - m$ such that

$$\|B_{n_2}(j)\vec{s}_\ell(j)\| < \frac{1}{\lambda}\|B_{n_2}(j)\vec{u}_\ell(j)\| \tag{4.21}$$

For simplicity, we set $D = B_{\ell-m}(j+n_1)$. (4.21) implies that

$$\begin{aligned}
|a|\|DB_{n_1}(j)\vec{u}_m(j)\| - |b|\|DB_{n_1}(j)\vec{s}_m(j)\| &\le \|DB_{n_1}(j)\vec{s}_\ell(j)\| \\
&= \|B_{n_2}(j)\vec{s}_\ell(j)\| \\
&< \frac{1}{\lambda}\|B_{n_2}(j)\vec{u}_\ell(j)\| \\
&= \frac{1}{\lambda}\|DB_{n_1}(j)\vec{u}_\ell(j)\| \\
&\le \frac{|c|}{\lambda}\|DB_{n_1}(j)\vec{u}_m(j)\| + \frac{|d|}{\lambda}\|DB_{n_1}(j)\vec{s}_m(j)\|
\end{aligned}$$

If $a \neq 0$, for large λ, the estimate above and (4.20) imply that

$$\begin{aligned}
\|DB_{n_1}(j)\vec{u}_m(j)\| &< \frac{2(|b|+|d|)}{|a|}\|DB_{n_1}(j)\vec{s}_m(j)\| \\
&\le \frac{2(|b|+|d|)\|D\|}{|a|}\|B_{n_1}(j)\vec{s}_m(j)\| \\
&\le \frac{2(|b|+|d|)\|D\|}{\lambda|a|}\|B_{n_1}(j)\vec{u}_m(j)\|
\end{aligned}$$

which in turn implies for some $C_1 > 0$, the choice of which is independent of λ, that

$$\|B_{\ell-m}(j+n_1)\vec{u}^{(m)}(k+p)\| \le \frac{C_1}{\lambda} \tag{4.22}$$

where $\vec{u}^{(m)}(k+p)$ is a unit vector in $E_m^u(p+k)$. Note that (4.22) implies for some $C_2 > 0$, the choice of which is independent of λ, that

$$\|B_{KN}((k+p)N+m)\vec{u}^{(m)}(k+p)\| \leq \frac{C_2}{\lambda}$$

which together with (4.19) imply that

$$\|B_{KN}((k+p)N+m)\vec{s}^{(m)}(k+p)\| \leq \frac{C_2}{2\lambda}$$

where $\vec{s}^{(m)}(k+p)$ is an unit vector in $E_m^s(k+p)$. Note (4.18) implies that the angle between $\vec{u}^{(m)}(k+p)$ and $\vec{u}^{(m)}(k+p)$ is at least δ away from 0 and π. Thus, the two estimates above clearly imply that

$$\|B_{KN}((k+p)N+m)\vec{w}\| \leq \frac{C_3}{\lambda}$$

for all unit vectors $\vec{w} \in \mathbb{R}^2$. Thus, we have

$$\|B_{KN}((k+p)N+m)\| = \|A_K^{(m)}(k+p)\| < \frac{C_3}{\lambda}$$

where the choice of C_3 is again independent of λ. Since we can find such p for all $\lambda > 0$, it contradicts with condition (3) of Definition 6 for $A^{(m)}$. Thus we must have $a = 0$ which implies that

$$\vec{s}_\ell(j) = e^{it}\vec{s}_m(j) \text{ for some } t \in \mathbb{R}.$$

Use the same strategy of proof, we can show that $d = 0$ which implies that

$$\vec{u}_\ell(j) = e^{it}\vec{u}_m(j) \text{ for some } t \in \mathbb{R}.$$

Indeed, by B- invariance of E_ℓ^u, we have $B_n(j-n)\vec{u}_\ell(j-n)$ is linearly dependent with $\vec{u}_\ell(j)$. If $d \neq 0$, then we must have that $\vec{u}_\ell(j-n) = a_n\vec{u}_m(j-n) + b_n\vec{s}_m(j-n)$ where $b_n \neq 0$

and $B_n(j-n)s_m(j-n) \neq 0$ for all $n > 0$. This implies that $B_n(j-n)$ is invertible for all $n > 1$ which in turn implies that $B(j-n)$ is invertible for all $n > 0$. Hence we can get for $K \in \mathbb{Z}_+$, for all $0 \leq m < N$ and all $j \in \mathbb{Z}$,

$$\|A_{-K}^{(m)}(j)\vec{s}^{(m)}(j)\| > 2\|A_{-K}^{(m)}(j)\vec{u}^{(m)}(j)\| \tag{4.23}$$

Then we have

$$\|B_{-n_1}(j)\vec{s}_m(j)\| > \lambda\|B_{-n_1}(j)\vec{u}_m(j)\|. \tag{4.24}$$

and

$$\|B_{-n_2}(j)\vec{u}_\ell(j)\| < \frac{1}{\lambda}\|B_{-n_2}(j)\vec{s}_\ell(j)\| \tag{4.25}$$

where $\lambda > 0$, $n_1 = pN \in \mathbb{Z}_+$ and $n_2 = n_1 + \ell - m$.

We set $P = B_{m-\ell}(j-n_1)$. (4.25) implies that

$$\begin{aligned}
|d|\|PB_{-n_1}(j)\vec{s}_m(j)\| - |c|\|PB_{-n_1}(j)\vec{u}_m(j)\| \leq& \|PB_{-n_1}(j)\vec{u}_\ell(j)\| \\
=& \|B_{-n_2}(j)\vec{u}_\ell(j)\| \\
<& \frac{1}{\lambda}\|B_{-n_2}(j)\vec{s}_\ell(j)\| \\
=& \frac{1}{\lambda}\|PB_{-n_1}(j)\vec{s}_\ell(j)\| \\
\leq& \frac{|a|}{\lambda}\|PB_{-n_1}(j)\vec{u}_m(j)\| + \frac{|b|}{\lambda}\|PB_{-n_1}(j)\vec{s}_m(j)\|
\end{aligned}$$

If $d \neq 0$, for large λ, the estimate above and (4.24) imply that

$$\begin{aligned}
\|PB_{-n_1}(j)\vec{s}_m(j)\| &< \frac{2(|c|+|a|)}{|d|}\|PB_{-n_1}(j)\vec{u}_m(j)\| \\
&\leq \frac{2(|c|+|a|)\|P\|}{|d|}\|B_{-n_1}(j)\vec{u}_m(j)\| \\
&\leq \frac{2(|c|+|a|)\|P\|}{\lambda|d|}\|B_{-n_1}(j)\vec{s}_m(j)\|
\end{aligned}$$

which in turn implies for some $\widetilde{C}_1 > 0$, the choice of which is independent of λ, that

$$\|B_{m-\ell}(j-n_1)\vec{s}^{(m)}(k-p)\| \leq \frac{\widetilde{C}_1}{\lambda} \tag{4.26}$$

where $\vec{s}^{(m)}(k-p)$ is a unit vector in $E^s_m(k-p)$. Then (4.26) implies for some $\widetilde{C}_2 > 0$, the choice of which is independent of λ, that

$$\|B_{-KN}((k-p)N+m)\vec{s}^{(m)}(k-p)\| \leq \frac{\widetilde{C}_2}{\lambda}$$

which together with and (4.23) imply that

$$\|B_{-KN}((k-p)N+m)\vec{u}^{(m)}(k-p)\| \leq \frac{\widetilde{C}_2}{2\lambda}$$

where $\vec{u}^{(m)}(k-p)$ is an unit vector in $E^u_m(k-p)$. Again, (4.18) implies that the angle between $\vec{u}^{(m)}(k-p)$ and $\vec{s}^{(m)}(k-p)$ is at least δ away from 0 and π. Thus, the two estimates above imply

$$\|B_{-KN}((k-p)N+m)\vec{w}\| \leq \frac{\widetilde{C}_3}{\lambda}$$

for all unit vectors $\vec{w} \in \mathbb{R}^2$. Thus, we have

$$\|B_{-KN}((k-p)N+m)\| = \|A^{(m)}_{-K}(k-p)\| < \frac{\widetilde{C}_3}{\lambda}$$

where the choice of $\widetilde{C}_3$ is again independent of λ. Since we can find such p for all $\lambda > 0$, we get a contradiction. Thus we must have $d = 0$ which implies that

$$\vec{u}_\ell(j) = e^{it}\vec{u}_m(j) \text{ for some } t \in \mathbb{R}.$$

Now, we just need to define E^u and E^s of B to be $E^{s,m}$ and $E^{u,m}$ for any $0 \leq m < N$, respectively. They satisfy condition (1) of Definition 6 since they are B-invariant. On the other hand, for any $j \in \mathbb{Z}$, we have $j = kN + m$ for some $k \in \mathbb{Z}$ and $0 \leq m < N$.

Thus $E^s(j) = E^s_m(k)$ and $E^u(j) = E^u_m(k)$ which implies the following two things. First, by (4.18), we have $\|E^s(j) - E^u(j)\| > \delta$ for all $j \in \mathbb{Z}$. In other words, E^s and E^u satisfy condition (3) of Definition 6. Second, by (4.19), it holds for all $j \in \mathbb{Z}$ that

$$\|B_{KN}(j)\vec{u}(j)\| > 2\|B_{KN}(j)\vec{s}(j)\|$$

which is the condition (2) of Definition 6. Finally, submultiplicativity of the norm together with condition (4) of Definition 6 for $A^{(m)}$ imply that for all $n \in \mathbb{Z}_+$, there is a $c(n) > 0$ such that $\|B_n(j)\| \geq c(n) > 0$ for all $j \in \mathbb{Z}$. In other words, B satisfies condition (3) of Definition 6. Hence, $B \in \mathcal{DS}$ with the corresponding invariant spaces to be E^u and E^s. ∎

Lemma 9 *Let* $\Lambda = \begin{pmatrix} \lambda^+ & 0 \\ 0 & \lambda^- \end{pmatrix} \in \mathrm{M}(2, \mathbb{C})$ *be such that* $|\lambda^+| > \gamma > 0$ *and* $|\lambda^+| > 2|\lambda^-|$. *Fix any* $r > 0$. *Then there exist* $c = c(\gamma, r) > 0$ *and* $C = C(\gamma, r) > 0$ *so that if* $\widetilde{\Lambda} : \mathbb{Z} \to \mathrm{M}(2, \mathbb{R})$ *satisfies* $\|\widetilde{\Lambda} - \Lambda\| \leq c$, *then*

$$\sup_{z \in \mathbb{D}_r} |\Lambda \cdot z - \widetilde{\Lambda} \cdot z| < C\|\widetilde{\Lambda} - \Lambda\|. \tag{4.27}$$

Proof. Let $\tilde{\Lambda} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and let $\delta = \|\widetilde{\Lambda} - \Lambda\|$. Then, $|\lambda^+ - a| \leq \delta$, $|\lambda^- - d| \leq \delta$, $|b| \leq \delta$ and $|c| \leq \delta$ which in turn imply that

$$|c + dz - \lambda^- z| \leq |c| + |z||d - \lambda^-| \leq \delta(1 + |z|) \text{ and}$$

$$|a + bz - \lambda^+| \leq |a - \lambda^+| + |b||z| \leq \delta(1 + |z|).$$

In particular, we have

$$|a + bz| \geq \lambda^+ - \delta(1 + |z|) \geq \gamma - \delta(1 + r) \geq \frac{\gamma}{2}$$

provided $\delta < \frac{\gamma}{2(1+r)}$. Thus, for such δ, we have for all $z \in \mathbb{D}_r$ that

$$\begin{aligned}
|\Lambda \cdot z - \widetilde{\Lambda} \cdot z| &\leq \left| \frac{\lambda^- z}{\lambda^+} - \frac{c + dz}{a + bz} \right| \\
&\leq \left| \frac{\lambda^- z}{\lambda^+} - \frac{\lambda^- z}{a + bz} \right| + \left| \frac{\lambda^- z}{a + bz} - \frac{c + dz}{a + bz} \right| \\
&= \left| \frac{\lambda^- z}{\lambda^+ (a + bz)} \right| |a + bz - \lambda^+| + \left| \frac{1}{a + bz} \right| |\lambda^- z - (c + dz)| \\
&\leq \frac{1}{|a + bz|} \delta (1 + |z|) \left(\left| \frac{\lambda_j^-}{\lambda_j^+} z \right| + 1 \right) \\
&\leq \frac{1}{\gamma} (1 + r)(2 + r) \delta \\
&= \frac{1}{\gamma} (1 + r)(2 + r) \|\Lambda - \widetilde{\Lambda}\|.
\end{aligned}$$

In other words, (4.27) holds true with $c = \frac{\gamma}{2(1+r)}$ and $C = \frac{1}{\gamma}(1+r)(2+r)$. ■

We are now ready to show that our definition of the dominated splitting for $\mathrm{M}(2, \mathbb{C})$-sequences is a stable property under the $\| \cdot \|_\infty$- perturbation.

Proof of Theorem 4. Let $B \in \mathcal{DS}$. Let E^u and E^s be its invariant spaces. By condition (2) of Definition 6, there exists $N \in \mathbb{Z}_+$ such that $\|B_N(j)\vec{u}(j)\| > 2\|B_N(j)\vec{s}(j)\|$ for all $j \in \mathbb{Z}$, and all unit vectors $\vec{u}(j) \in E^u(j), \vec{s}(j) \in E^s(j)$. By Lemma 8, we may assume that $N = 1$ i.e. E^u dominates E^s at step 1. We define $\Lambda : \mathbb{Z} \to \mathrm{M}(2, \mathbb{C})$ to be

$$\Lambda(j) := D(j+1)^{-1} B(j) D(j) \tag{4.28}$$

where $D(j)$ is defined as

$$D(j) = \big(\vec{u}(j),\ \vec{s}(j)\big)$$

and

$$\Lambda(j) = \begin{pmatrix} \lambda_j^+ & 0 \\ 0 & \lambda_j^- \end{pmatrix},$$

where

$$\left|\lambda_j^+\right| > 2\left|\lambda_j^-\right| \text{ for all } j \in \mathbb{Z}.$$

By condition (4) of Definition 6,

$$\inf_{j\in\mathbb{Z}} d(E^u(j), E^s(j)) > \delta > 0$$

which implies

$$\inf_{j\in\mathbb{Z}} |\det D(j)| = \inf_{j\in\mathbb{Z}} d(E^u(j), E^s(j)) > \delta > 0.$$

Thus, there exist c, C, depending on δ only, such that

$$c < \|D(j)^{\pm 1}\| < C \text{ for all } j \in \mathbb{Z} \tag{4.29}$$

By condition (3) of Definition 6, $c < \|B(j)\| < M$ for all $j \in \mathbb{Z}$. Hence, we must have

$$c < \|\Lambda(j)\| < C \text{ for all } j \in \mathbb{Z}. \tag{4.30}$$

Since $\left|\lambda_j^+\right| > 2|\lambda_j^-|$ for all $j \in \mathbb{Z}$, there must exist a $\gamma > 0$ such that $|\lambda_j^+| > \gamma$ for all $j \in \mathbb{Z}$. Moreover, for any $\alpha > 0$, we have

$$\Lambda(j) \cdot (\mathbb{D}_\alpha) \subset \mathbb{D}_{\frac{\alpha}{2}} \text{ for all } j \in \mathbb{Z}. \tag{4.31}$$

For $\widetilde{B} : \mathbb{Z} \to \mathrm{M}(2, \mathbb{C})$, we define

$$\widetilde{\Lambda}(j) := D(j+1)^{-1}\widetilde{B}(j)D(j). \tag{4.32}$$

Then by (4.29), we have

$$\|\widetilde{\Lambda}(j) - \Lambda(j)\| < C\|\widetilde{B} - B\|.$$

Hence, if $\|\widetilde{B} - B\|$ small, we have the following two properties. First, by (4.30), we have

$$\inf_{j\in\mathbb{Z}} \|\widetilde{\Lambda}(j)\| > c. \tag{4.33}$$

Second, by (4.27) of Lemma 9, we have

$$\sup_{z\in\mathbb{D}_\alpha} |\Lambda(j)\cdot z - \widetilde{\Lambda}(j)\cdot z| < C\|\widetilde{\Lambda}(j) - \Lambda(j)\| < C^2\|\widetilde{B}(j) - B(j)\|.$$

Thus for all $\frac{\alpha}{2} < \alpha' < \alpha$, there exists a $\varepsilon = \varepsilon(\delta, \alpha, \alpha') > 0$ such that if $\|\widetilde{B} - B\| < \varepsilon$, then

$$\widetilde{\Lambda}(j)\cdot\mathbb{D}_\alpha \subset \mathbb{D}_{\alpha'} \text{ for all } j \in \mathbb{Z}. \tag{4.34}$$

Now, by Theorem 6, $\widetilde{\Lambda} \in \mathcal{DS}$. Since $\widetilde{\Lambda}$ is conjugate to $\widetilde{B}$ via D, if $\tilde{E}^u$ and $\tilde{E}^s$ are the invariant spaces of $\widetilde{\Lambda}$, then $\bar{E}^u(j) = D(j)\cdot E^u(j)$ and $\bar{E}^s(j) = D(j)\cdot E^s(j)$ are the invariant spaces of $\widetilde{B}$. For all $j \in \mathbb{Z}$,

$$\widetilde{B}(j)\bar{E}^u(j) = D(j+1)\widetilde{\Lambda}(j)E^u(j) \subset D(j+1)E^u(j+1) = \bar{E}^u(j+1)$$

Similarly, $\widetilde{B}(j)\bar{E}^s(j) \subset \bar{E}^s(j+1)$. Hence, we have invariance. Let $N > 0$ be such that for all $j \in \mathbb{Z}$, and all unit vectors $\vec{u}(j) \in E^u(j), \vec{s}(j) \in E^s(j)$.

$$\|\widetilde{\Lambda}_N(j)\vec{u}(j)\| > \lambda\|\widetilde{\Lambda}_N(j)\vec{s}(j)\|$$

Consider unit vectors $\vec{v}(j) \in \bar{E}^u(j), \vec{w}(j) \in \bar{E}^s(j)$. Then

$$\begin{aligned}
\|\widetilde{B}_N(j)\vec{v}(j)\| &= \left\|D(j+N)\widetilde{\Lambda}_N(j)D^{-1}(j)\frac{D(j)\vec{u}(j)}{|D(j)\vec{u}(j)|}\right\| \\
&> \frac{\lambda}{|D(j)\vec{u}(j)|}\left\|D(j+N)\widetilde{\Lambda}_N(j)\vec{s}(j)\right\| \\
&= \frac{\lambda}{|D(j)\vec{u}(j)|}\left\|\widetilde{B}_N(j)D(j)\vec{s}(j)\right\| \\
&= \frac{\lambda|D(j)\vec{s}(j)|}{|D(j)\vec{u}(j)|}\left\|\widetilde{B}_N(j)\vec{w}(j)\right\| \\
&= c\lambda\left\|\widetilde{B}_N(j)\vec{w}(j)\right\|
\end{aligned}$$

where the last equality comes from (4.29). Condition (3) of Definition 6 for $\widetilde{B}$ follows by applying (4.3) to $D(j)$. Moreover, (4.29) implies that $\|\widetilde{B}_n(j)\| \geq c > 0$ for all $j \in \mathbb{Z}$. Hence, $\widetilde{B} \in \mathcal{DS}$. ■

Chapter 5

Dominated Splitting and the Invertibility of the Operator

The goal of this section is to show that

$$\{E \in \mathbb{C} : B^E \in \mathcal{DS}\ \} \subset \rho(J_{a,b}),$$

which is equivalent to

$$\sigma(J_{a,b}) \subset \{E \in \mathbb{C} : B^E \notin \mathcal{DS}\ \}.$$

Now that we have shown that dominated splitting is a stable property for $\mathrm{M}(2, \mathbb{R})$-sequences, we go back to our setting. Recall that the cocycle of the Jacobi operator is given by

$$B^E(j) = \begin{pmatrix} E - b_j & -a_{j-1} \\ a_j & 0 \end{pmatrix},$$

and if $\psi \in \mathbb{C}^{\mathbb{Z}}$ solves spectral equation (3.2), then

$$B^E(j) \begin{pmatrix} \psi_j \\ \psi_{j-1} \end{pmatrix} = a_j \begin{pmatrix} \psi_{j+1} \\ \psi_j \end{pmatrix}, \ j \in \mathbb{Z}.$$

We proceed by showing the following lemma.

Lemma 10 *Suppose $E \in \mathbb{C}$ is such that $B^E \in \mathcal{DS}$ with E^u, E^s being its invariant directions. Then there exists a $\lambda_0 > 1$ so that the following holds true.*

1. *Suppose $a_j \neq 0$ for all $j \in \mathbb{Z}$. Then, for all $j \in \mathbb{Z}$, there exists $k = k(j) > 0$, such that for all unit vectors $\vec{u}(j) \in E^u(j)$*

$$\|A_n^E \vec{u}(j)\| \geq k\lambda_0^n \|\vec{u}(j)\| \text{ for all } n \geq 1,$$

and for all unit vectors $\vec{s}(j) \in E^s(j)$

$$\|A_{-n}^E \vec{s}(j)\| \geq k\lambda_0^n \|\vec{s}(j)\| \text{ for all } n \geq 1.$$

2. *Suppose there exists $j_{\max} \in \mathbb{Z}$ such that $a_j \neq 0$ for all $j > j_{\max}$. Then, for all $j \geq j_{\max} + 1$, there exists $k = k(j) > 0$, such that for all unit vectors $\vec{u}(j) \in E^u(j)$*

$$\|A_n^E(j)\vec{u}(j)\| \geq k\lambda_0^n \text{ for all } n \geq 1. \tag{5.1}$$

3. *Suppose there exists $j_{\min} \in \mathbb{Z}$ such that $a_j \neq 0$ for all $j < j_{\min}$. Then, for all $j \leq j_{\min}$, there exists $k = k(j) > 0$, such that for all unit vectors $\vec{s}(j) \in E^s(j)$*

$$\|A_{-n}^E(j)\vec{s}(j)\| \geq k\lambda_0^n \text{ for all } n \geq 1. \tag{5.2}$$

Proof. By conditions (1) and (2) of Definition 6, $B^E \in \mathcal{DS}$ guarantees that there exist $c > 0$ such that

$$\|B_n^E(j)\vec{u}(j)\| > c\lambda^n \|B_n^E(j)\vec{s}(j)\| \tag{5.3}$$

for all $j \in \mathbb{Z}$, and all unit vectors $\vec{u}(j) \in E^u(j), \vec{s}(j) \in E^s(j)$, where $\lambda > 1$. By condition (3) of Definition 6, there exists $\delta > 0$, such that $\left|\det\begin{pmatrix} \vec{u}(j), \vec{s}(j) \end{pmatrix}\right| = d(E^u(j), E^s(j)) > \delta > 0$

for all $j \in \mathbb{Z}$. Hence, we have

$$\begin{aligned}
\left|\delta \prod_{i=j}^{j+n-1} a_i a_{i-1}\right| &\leq |\det(B_n^E(j)\left(\vec{u}(j), \vec{s}(j)\right))| \\
&= |\det\left(B_n^E(j)\vec{u}(j), B_n^E(j)\vec{s}(j)\right)| \\
&= \|B_n^E(j)\vec{u}(j)\| \cdot \|B_n^E(j)\vec{s}(j)\| \cdot |\det(\vec{u}(j+n), \vec{s}(j+n))| \\
&= \|B_n^E(j)\vec{u}(j)\| \cdot \|B_n^E(j)\vec{s}(j)\| \cdot |d\big(E^u(j+n), E^s(j+n)\big)| \\
&\leq \delta\|B_n^E(j)\vec{u}(j)\|\|B_n^E(j)\vec{s}(j)\|.
\end{aligned}$$

By (5.3), there exists some $\tilde{c} > 0$, independent of j, such that

$$\left|\widetilde{c} \prod_{i=j}^{j+n-1} a_i a_{i-1}\right| \leq \|B_n^E(j)\vec{u}(j)\|^2 \lambda^{-n} \text{ for all } j \in \mathbb{Z} \text{ and all } n \geq 1. \tag{5.4}$$

If $a_j \neq 0$ for all $j \in \mathbb{Z}$, then we can define

$$A^E(j) = \frac{1}{a_j} B^E(j). \tag{5.5}$$

Then, from (5.4),

$$\|A_n^E(j)\vec{u}(j)\|^2 \geq \widetilde{c}\lambda^n \left|\frac{a_{j-1}}{a_{j+n-1}}\right| \geq \widetilde{c_1}\lambda^n |a_{j-1}| \text{ for all } n \geq 1 \tag{5.6}$$

where the last inequality comes form the fact that $(a_j)_{j\in\mathbb{Z}}$ is a bounded sequence. Similarly, $B^E \in \mathcal{DS}$ guarantees that there exist $c > 0$, the choice of which is independent of j, such that

$$\|B_{-n}^E(j)\vec{s}(j)\| > c\lambda^n \|B_{-n}^E(j)\vec{u}(j)\|$$

for all unit vectors $\vec{u}(j) \in E^u(j), \vec{s}(j) \in E^s(j)$. Following the same logic as above, we get that

$$\left|\frac{\widetilde{c}}{\prod_{i=j-n}^{j-1} a_i a_{i-1}}\right| \leq \|B_{-n}^E(j)\vec{u}(j)\|\|B_{-n}^E(j)\vec{s}(j)\| \leq \|B_{-n}^E(j)\vec{s}(j)\|^2 \lambda^{-n}.$$

By (5.5), it is clear that $a_j(B^E)^{-1}(j) = (A^E)^{-1}(j)$ for all $j \in \mathbb{Z}$, and so for all $n \geq 1$

$$\|A^E_{-n}(j)\vec{s}(j)\|^2 \geq \widetilde{c}\lambda^n \left|\frac{a_{j-1}}{a_{j-n}}\right| \geq \widetilde{c_1}\lambda^n |a_{j-1}|. \tag{5.7}$$

Estimates 5.6 and 5.7 imply case (1) of this lemma, when $\lambda_0 = \sqrt{\lambda}$.

If $a_j \neq 0$ for all $j > j_{\max}$, then (5.6) holds for all $j > j_{\max} + 1$. It then can be extended to $j_{\max} + 1$ by the fact that

$$A^E(j_{\max} + 1)E^u(j_{\max} + 1) = E^u(j_{\max} + 2).$$

It clearly implies (5.1) with $\lambda_0 = \sqrt{\lambda}$ when $j \geq j_{\max} + 1$.

If $a_j \neq 0$ for all $j < j_{\min}$, then $B^E_{-n}(j)$ is well-defined for all $j \leq j_{\min}$ and for all $n \geq 1$. Thus, (5.7) holds with $\lambda_0 = \sqrt{\lambda}$ when $j \leq j_{\min}$. ■

For $j_2 > j_1$, we define $J_{[j_1,j_2)}$ to be the finite restriction of $J_{a,b}$ to the subspaces $\ell^2[j_1, j_2)$ with Dirichlet boundary conditions. That is $j_{[j_1,j_1+1)} = b_{j_1}$ and for $j_2 > j_1 + 1$

$$(J_{[j_1,j_2)}\phi)_j = \begin{cases} a_{j_1}\phi_{j_1+1} + b_{j_1}\phi_{j_1}, & j = j_1, \\ (J_{a,b}\phi)_j, & j_1 < j < j_2 - 1 \text{ if } j_2 > j_1 + 2, \\ a_{j_2-2}\phi_{j_2-2} + b_{j_2-1}\phi_{j_2-1}, & j = j_2 - 1. \end{cases} \tag{5.8}$$

We also denote restrictions of $J_{a,b}$ on half lines $[j_0, +\infty)$ and $(-\infty, j_0)$ with Dirichlet boundary condition as $J_{[j_0,+)}$ and $J_{(-,j_0)}$, respectively. In other words, we have

$$(J_{[j_0,+)}\phi)_j = \begin{cases} a_{j_0}\phi_{j_0+1} + b_{j_0}\phi_{j_0}, & j = j_0, \\ (J_{a,b}\phi)_j, & j > j_0, \end{cases} \tag{5.9}$$

$$(J_{(-,j_0)}\phi)_j = \begin{cases} a_{j_0-2}\phi_{j_0-2} + b_{j_0-1}\phi_{j_0-1}, & j = j_0 - 1, \\ (J_{a,b}\phi)_j, & j < j_0 - 1. \end{cases} \tag{5.10}$$

We define $p_N(j,E)=\det(E-J_{[j,j+N)})$ for $N\geq 1$, $p_0(j,E)=1$, and $p_{-1}(j,E)=0$. Then it's a standard result that the following is true for all $j\in\mathbb{Z}$ and all $N\geq 1$:

$$B_N^E(j)=\begin{pmatrix} p_N(j,E) & -a_{j-1}p_{N-1}(j+1,E) \\ a_{j+N-1}p_{N-1}(j,E) & -a_{j-1}a_{j+N-1}p_{N-2}(j+1,E)\end{pmatrix}$$

Lemma 11 *Suppose $E\in\mathbb{C}$ is such that $B^E\in\mathcal{DS}$ with E^u, E^s being its invariant directions. Then we have to following cases:*

1. *If there exists $j_{\max}\in\mathbb{Z}$ such that $a_{\max}=0$ and $a_j\neq 0$ for all $j>j_{\max}$, then the solution space of $J_{(j_{\max},+)}\phi=E\phi$ is one-dimensional.*

2. *If there exists $j_{\min}\in\mathbb{Z}$ such that $a_{\min}=0$ and $a_j\neq 0$ for all $j<j_{\min}$, then the solution space of $J_{(-,j_{\min}]}\phi=E\phi$ is one-dimensional.*

3. *If $a_j\neq 0$ for all $j\in\mathbb{Z}$, then the solution space of $J_{a,b}\psi=E\psi$ is two dimensional.*

4. *If there exist $j_1<j_2$ such that $a_{j_1}=a_{j_2}=0$, then the solution space of $J_{(j_1,j_2]}\phi=E\phi$ is trivial.*

Moreover, any nontrivial solutions grow exponentially fast along some subsequence that goes to ∞ or along a subsequence that goes to $-\infty$.

Proof. Without loss of generality, we may focus on case (1) since case (2) can be done completely similarly. Consider the solution space $\{\phi:\mathbb{Z}_{j>j_{\max}}\to\mathbb{C}:J_{(j_{\max},+)}\phi=E\phi\}$. By (5.9), the solution is uniquely determined by $\phi_{j_{\max}}$ which implies the space is one-dimensional. Moreover, by (5.5) and (5.9), for all $j\geq j_{\max}+2$,

$$\begin{pmatrix}\phi_{j+1}\\ \phi_j\end{pmatrix}=A^E_{j-j_{\max}-1}(j_{\max}+1)\begin{pmatrix}\phi_{j_{\max}+1}\\ \phi_{j_{\max}}\end{pmatrix}.$$

Note in the equation above, the choice of $\phi_{j_{\max}}$ is not relevant as it will be cancelled by $a_{j_{\max}} = 0$. In particular, if we define ϕ^u to be a solution generated by some $\binom{\phi^u_{j_{\max}+1}}{\phi^u_{j_{\max}}} \in E^u(j_{\max}+1)$, then by Lemma 10, we will have for all $j \geq j_{\max}+1$

$$\left\| \begin{pmatrix} \phi^u_{j+1} \\ \phi^u_j \end{pmatrix} \right\| = \left\| \begin{pmatrix} \phi^u_{j_{\max}+1} \\ \phi^u_{j_{\max}} \end{pmatrix} \right\| \|A^E_{j-j_{\max}-1}(j_{\max}+1)\vec{u}(j)\| \geq \tilde{k}\lambda^{j-j_{\max}}. \tag{5.11}$$

Then there exists a monotonically increasing sequence $\{n_l\}_{l\in\mathbb{N}}$, such that $n_l > j_{\max}$ and $|\phi^u(n_l)| > \tilde{k}\lambda^{n_l-j_{\max}}$. In particular, it's a nontrivial solution which must form a basis of the solution space. Since the solution space of $J_{(j_{\max},+)}\phi = E\phi$ is one dimensional, all nontrivial solutions are some ϕ^u generated by a nonzero vector in $E^u(j_{\max}+1)$ and it holds that for some $\tilde{k} = \tilde{k}(\phi^u) > 0$,

$$|\phi^u(n_l)| > \tilde{k}\lambda^{n_l-j_{\max}} \text{ for all } l \geq 1. \tag{5.12}$$

Similarly for case (2), any nontrivial solution ϕ^s of $J_{(-,j_{\min}]}\phi = E\phi$ is generated by a vector in $E^s(j_{\min})$. Hence, by case (2) of Lemma 10 and by applying the same logic as above, there exists a monotonically decreasing sequence $\{n_t\}_{t\in\mathbb{N}}$, such that $n_t < j_{\min}$ such that

$$|\phi^s(n_t)| > \tilde{k}\lambda^{j_{\min}-n_t} \text{ for all } t \geq 1. \tag{5.13}$$

To consider case (3), we fix any j_0. Then all solutions are of the form $\psi = \alpha\phi^u + \beta\phi^s$ where ϕ^u is generated by a vector in $E^u(j_0)$ and ϕ^s is generated by a vector in $E^s(j_0)$. Since $a_j \neq 0$ for all $j \in \mathbb{Z}$, the same proof as in the first two cases yield that $\phi^u(n_l)$ grows exponentially fast as $n_l \to \infty$ and $\phi^s(n_t)$ grows exponentially fast as $n_t \to -\infty$ where $\{n_l\}$ and $\{n_t\}$ are some sequences. Hence, if ψ is nontrivial, it must grow exponentially fast at least along one of the sequences $\{n_l\}$ and $\{n_t\}$.

Now, we consider case (4), i.e. the finite restrictions $J_{(j_1,j_2]}$. We want to show

$$\dim\{\phi^f(n), j_1 < n \le j_2 : (J_{(j_1,j_2]} - E)\phi^f(n) = 0\} = 0.$$

If not, then $(J_{(j_1,j_2]} - E)\phi^f = 0$ for some nonzero vector ϕ^f. Set $N = j_2 - j_1$. Then we have $p_N(j_1+1, E) = \det(E - J_{(j_{k+1},j_k]}) = 0$ since $E \in \sigma(J_{(j_1,j_2]})$. Recall $a_{j_1} = a_{j_2} = 0$. Thus, we have

$$B_N^E(j_1+1) = \begin{pmatrix} p_N(j_1+1,E) & -a_{j_1}p_{N-1}(j_1+2,E) \\ a_{j_2}p_{N-1}(j_1+1,E) & -a_{j_1}a_{j_2}p_{N-2}(j_1+2,E) \end{pmatrix}$$

is a zero matrix, which contradicts condition (4) of Definition 6. Hence, the solutions space of finite restriction matrices is trivial. ■

Let $\mathcal{E}_g(J_{a,b})$ be the set of generalized eigenvalues of $J_{a,b}$, i.e. all $E \subset \mathbb{C}$ that admit a nontrivial polynomially bounded solution of $J_{a,b}\psi = E\psi$. It is a well know fact (see Sch'nol-Berezanskii's theorem in [2]) that

$$\overline{\mathcal{E}_g(J_{a,b})} = \sigma(J_{a,b}).$$

We are now ready to prove that

$$\{E : B^E \in \mathcal{DS}\} \subset \rho(J_{a,b}).$$

Let $E \in \mathbb{C}$ be such that $B^E \in \mathcal{DS}$. We want to show that $E \notin \overline{\mathcal{E}_g(J_{a,b})}$. First we show that $E \notin \mathcal{E}_g(J_{a,b})$, i.e. all the nontrivial solutions of the spectral equation $J_{a,b}\psi = E\psi$ are not polynomially bounded, and then we show that E in not a limit point of $\mathcal{E}_g(J_{a,b})$. Define

$$j_{\max} := \sup\{j \in \mathbb{Z} : a_j = 0\} \text{ and } j_{\min} := \inf\{j \in \mathbb{Z} : a_j = 0\},$$

if they exist. Note we allow $j_{\max}$ to be ∞ and $j_{\min}$ to be $-\infty$. We consider different cases.

Case 1. $a_j \neq 0$ for all $j \in \mathbb{Z}$.

Then Lemma 11 directly implies that all nontrivial solutions of $J_{a,b}\psi = E\psi$ are not polynomially bounded.

Case 2. $-\infty < j_{\min} \le j_{\max} < \infty$.

If $j_{\min} = j_{\max}$, then we can decompose $J_{a,b}$ as the following direct sum

$$J_{a,b} = J_{(-,j_{\min}]} \oplus J_{(j_{\max},+)}.$$

If $j_{\min} < j_{\max}$, then we have

$$J_{a,b} = J_{(-,j_{\min}]} \oplus J_{(j_{\min},j_{\max}]} \oplus J_{(j_{\max},+)}.$$

By cases (1), (2), and (4) of Lemma 11, in both cases, the solution space is spanned by

$$\{(\phi^s, 0, 0, \dots), (\dots, 0, 0, \phi^u)\}$$

where ϕ^s and ϕ^u are any nontrivial solutions of $J_{(-,j_{\min}]}\phi = E\phi$ and $J_{(j_{\max},+)}\phi = E\phi$, respectively. Hence, by (5.12) and (5.13) and similar to Case 1, all nontrivial solutions of the eigenvalue equation in both cases must grow exponentially fast at least along some subsequences that go to ∞ or $-\infty$.

Case 3. Either $j_{\min} = -\infty$ and $j_{\max} < \infty$; or $j_{\min} > -\infty$ and $j_{\max}$.

In the first case, there exists a strictly monotone decreasing sequence of integers $\{j_k\}_{k\ge 1}$ such that $j_1 = j_{\max}$ and $a_{j_k} = 0$ for all $k \ge 1$. Hence we may decompose the operator $J_{a,b}$ as

$$J_{a,b} = \ldots \oplus J_{(j_{k+1},j_k]} \oplus \cdots \oplus J_{(j_2,j_1]} \oplus J_{(j_1,+)}$$

Then by case (1) and (4) of Lemma 11, the solution space of $J_{a,b}\phi = E\phi$ is spanned by $(\dots, 0, 0, \phi^u)$, where ϕ^u is any nontrivial solution of $J_{(j_{\max},+)}\phi = E\phi$. By (5.12), all nontrivial solutions of $J_{a,b}\phi = E\phi$ grow exponentially fast along $\{n_l\}$ where $n_l \to \infty$ as $l \to \infty$.

In the second case, we may decompose the operator $J_{a,b}$ as

$$J_{a,b} = J_{(-,j_1)} \oplus J_{(j_1,j_2]} \oplus \cdots \oplus J_{(j_k,j_{k+1}]} \oplus \cdots$$

where $j_1 = j_{\min}$ and $a_{j_k} = 0$ for all $k \geq 1$. Hence, the solution space of $J_{a,b}\phi = E\phi$ is spanned by $(\phi^s, 0, 0 \ldots)$, where ϕ^s is any nontrivial solution of $J_{(-,j_{\min}]}\phi = E\phi$. By (5.13), all nontrivial solutions of $J_{a,b}\phi = E\phi$ grow exponentially fast along $\{n_t\}$ where $n_t \to -\infty$ as $t \to \infty$.

Case 4. $j_{\min} = -\infty$ and $j_{\max} = \infty$.

Then there exists a subsequence $(j_k)_{k\in\mathbb{Z}}$ such that $j_k \to \pm\infty$ as $k \to \pm\infty$ and $a_{j_k} = 0$ for all $k \in \mathbb{Z}$. Hence, we may decompose $J_{a,b}$ as

$$J_{a,b} = \bigoplus_{k=-\infty}^{\infty} J_{(j_k,j_{k+1}]}.$$

By case (4) of Lemma 11, all solutions of $J_{a,b}\phi = E\phi$ are trivial.

Therefore, $E \notin \mathcal{E}_g(J_{a,b})$. If E is a limit point of $\mathcal{E}_g(J_{a,b})$, then for sufficiently small $\epsilon > 0$, there exists E' such that $|E - E'| < \epsilon$, $E' \in \mathcal{E}_g(J_{a,b})$, and, by Theorem **??**, $B^{E'} \in \mathcal{DS}$. Then, applying the same argument as above to $B^{E'}$, we get that all nontrivial solutions of $J_{a,b}\psi = E'\psi$ are not polynomially bounded which implies that $E' \notin \mathcal{E}_g(J_{a,b})$, and so we get a contradiction. Hence, $E \in \rho(J_{a,b})$.

Chapter 6

Dominated Splitting Away From the Spectrum

The goal of this section is to show that

$$\rho(J_{a,b}) \subset \{E \in \mathbb{C} : B^E \in \mathcal{DS}\},$$

which is equivalent to

$$\{E \in \mathbb{C} : B^E \notin \mathcal{DS}\} \subset \sigma(J_{a,b}).$$

6.1 Uniform low bound of the norm of cocycle iterations

We start by assuming that $E \in \rho(J_{a,b})$ and show condition (4) of Definition 6. Notice that our ultimate goal is to show that $B^E \in \mathcal{DS}$, which means we need to find the invariant directions of B^E that satisfy all the conditions of Definition 6. However, condition (4) of Definition 6 does not depend on the existence of directions, it simply provides the existence of a uniform low bound of cocycle iterations when $E \in \rho(J_{a,b})$. For this section we use complicated iterative inductive argument, so it is best to have a good visual picture in mind. Recall that

$$E - J_{a,b}(j) = \begin{pmatrix} \ddots & \ddots & \ddots & & & & \\ -a_{j-1} & E-b_j & -a_j & 0 & & & \\ 0 & -a_j & E-b_{j+1} & -a_{j+1} & 0 & & \\ & \ddots & \ddots & \ddots & & & \\ & 0 & -a_{j+n-2} & E-b_{j+n-1} & -a_{j+n-1} & 0 & \\ & & 0 & -a_{j+n-1} & E-b_{j+n} & -a_{j+n} & \\ & & & \ddots & \ddots & \ddots & \end{pmatrix}$$

Let $J_N(j) = J_{[j,j+N-1]}$ be the restriction of $J_{a,b}$ on $[j, j+N-1]$. Recall that $p_N(j,E) = \det(E - J_N(j))$ and

$$B_N^E(j) = \begin{pmatrix} p_N(j,E) & -a_{j-1}p_{N-1}(j+1,E) \\ a_{j+N-1}p_{N-1}(j,E) & -a_{j-1}a_{j+N-1}p_{N-2}(j+1,E) \end{pmatrix}.$$

Let $\mathrm{diam}(S)$ be the diameter of a subset $S \subset \mathbb{C}$. We start with the following lemma.

Lemma 12 *Let $E \in \rho(J_{a,b})$. For all $\epsilon > 0$, there exists delta $= \delta(\epsilon) > 0$ such that if $|p_N(j,E)| \le \delta$ for some $N \in \mathbb{N}$, and some $j \in \mathbb{Z}$, then there exists $E_0 \in \sigma(J_N(j))$, such that $|E - E_0| < \epsilon$.*

Proof. To prove this lemma we instead show the following: *Fix $N \in \mathbb{Z}_+$. For all $\epsilon > 0$, there exists $\delta = \delta(\epsilon) > 0$ such that for all $j \in \mathbb{Z}$ and for all connected components $\mathcal{K}$ of $I_N(j,\delta)$,*

$$\mathrm{diam}(\mathcal{K}) < \epsilon,$$

where $I_N(j,\delta) = \{E \in \mathbb{C} : |p_N(j,E)| \leq \delta\}$*. Moreover, there is a zero of* $p_N(j,E)$ *in* $\mathcal{K}$*.*

The result above implies the lemma. So we focus on the proof of this result.

To show that each connected component of $I_N(j,\delta)$ contains a zero of $p_N(j,E)$, we just need to show that $|p_N(j,E)|$ has no positive local minimum or a negative local maximum. Suppose $|p_N(j,E)|$ has a positive local minimum at E_0. Then $p_N(j,E) : B_r(E_0) = \{E : |E - E_0| < r\} \to \mathbb{C}$ has no zeros for small $r > 0$. Hence, the holomorphic function $\frac{1}{p_N(j,E)} : B_r(E_0) \to \mathbb{C}$ attains its maximum at an interior point E_0. This contradicts the maximum principle since $p_N(j,E)$ is nonconstant for all $j \in \mathbb{Z}$ and all $N \in \mathbb{Z}_+$. Similarly, $|p_N(j,E)|$ can not have a negative local maximum. Therefore, each connected component of $I_N(j,\delta)$ must contain a zero of $p_N(j,E)$.

Now we show that for all $\epsilon > 0$, there exists $\delta = \delta(\epsilon) > 0$ such that for all $j \in \mathbb{Z}$ and for all connected component $\mathcal{K}$ of $I_N(j,\delta)$,

$$\operatorname{diam}(\mathcal{K}) < \epsilon.$$

We need to use the following version of Markov's inequality [14]: For any connected compact set $\mathcal{K} \subset \mathbb{C}$ with $\operatorname{diam}(\mathcal{K}) > \eta > 0$, there exist positive constants M, α, depending only on η, such that it holds for all polynomials $p(z)$ and all $r \in \mathbb{N}$ that

$$\left\| \frac{d^r p_n}{dz^r} \right\|_{\mathcal{K}} \leq M (\deg p)^{r\alpha} \|p_n\|_{\mathcal{K}}, \tag{6.1}$$

where $\|\cdot\|_{\mathcal{K}}$ denotes the supremum norm on $\mathcal{K}$ and $\deg p$ is the degree of p.

We will argue by contradiction. Suppose that there exists $\epsilon_0 > 0$ such that for all $\delta > 0$, there exists $j \in \mathbb{Z}$ and a connected component $\mathcal{K}$ of $I_N(j,\delta)$ such that

$$\operatorname{diam}(\mathcal{K}) \geq \varepsilon_0.$$

Applying (6.1) to $p_N(j,E)$ on $\mathcal{K}$, we obtain

$$\left\| \frac{d^N p_N(j,E)}{dE^N} \right\|_{\mathcal{K}} \le MN^{N\alpha} \|p_N(j,E)\|_{\mathcal{K}} \le MN^{N\alpha}\delta.$$

However, for all $j \in \mathbb{Z}$, $p_N(j,E)$ is the N-th degree polynomial with a leading coefficient 1, which implies that

$$\left\| \frac{d^N p_N(j,E)}{dE^N} \right\|_{\mathcal{K}} \equiv N!$$

Since N is fixed and M and s depend only on ε_0, choosing $\delta < \frac{N!}{MN^{N\alpha}}$, we get

$$N! \le MN^{N\alpha}\delta < N!$$

which causes a contradiction. ■

First, note that since $J_{a,b}$ is a self-adjoint operator, then the Weyl's criterion [11] says:

$$E \in \sigma(J_{a,b}) \text{ if and only if for all } \epsilon > 0, \|(J_{a,b} - E)u\| < \epsilon \text{ for some unit vector } u. \tag{6.2}$$

Moreover, for each $z \in \rho(H)$, we have

$$\|(J_{a,b} - zI)^{-1}\|^{-1} = d(z, \sigma(J_{a,b})). \tag{6.3}$$

Now we will show condition (4) of Definition 6.

Lemma 13 *Let $E \in \rho(J_{a,b})$. Then for all $n \in \mathbb{Z}_+$, there exists $c = c(n) > 0$, such that*

$$\inf_{j \in \mathbb{Z}} \|B_n^E(j)\| > c.$$

Proof. We will prove this lemma by induction. First, we set $n = 1$. We need to show that there exists $c > 0$ such that

$$\inf_{j \in \mathbb{Z}} \|B^E(j)\| > c.$$

We proceed with the proof by contradiction. Suppose $\inf_{j\in\mathbb{Z}} \|B^E(j)\| = 0$. Then, for all $\epsilon > 0$, there exists $j_0 \in \mathbb{Z}$ such that $\|B^E(j_0)\| < \epsilon$, i.e.

$$\left\| \begin{pmatrix} E - b_{j_0} & -a_{j_0-1} \\ a_{j_0} & 0 \end{pmatrix} \right\| < \epsilon \implies \begin{cases} |E - b_{j_0}| < \epsilon \\ |a_{j_0-1}| < \epsilon \\ |a_{j_0}| < \epsilon \end{cases}$$

Now, let's consider

$$(J_{a,b} - E)\delta_{j_0}.$$

Clearly,

$$(J_{a,b} - E)\delta_{j_0}(n) = a_n\delta_{j_0}(n+1) + a_{n-1}\delta_{j_0}(n-1) + (b_n - E)\delta_{j_0}(n)$$

which is 0 for all $n \neq j_0 - 1, j_0, j_0 + 1$. Thus, we only need to consider the cases when $n = j_0 - 1$, j_0, $j_0 + 1$. It is easy to see that

$$(J_{a,b} - E)\delta_{j_0}(n) = \begin{cases} a_{j_0-1}, & \text{if } n = j_0 - 1 \\ b_{j_0} - E, & \text{if } n = j_0 \\ a_{j_0}, & \text{if } n = j_0 + 1 \end{cases}$$

Hence,

$$\|(J_{a,b} - E)\delta_{j_0}\|_{\ell^2} < 3\epsilon$$

Since $\epsilon > 0$ is arbitrary and because $\|\delta_0\|_{\ell^2} = 1$, by (6.2), $E \in \sigma(J_{a,b})$. We arrive at contradiction, since we assumed that $E \in \rho(J_{a,b})$. Hence, $\inf_{j\in\mathbb{Z}} \|B^E(j)\| > c$ for some $c > 0$.

Now assume that $\inf_{j\in\mathbb{Z}} \|B_n^E(j)\| > c$ for all $1 \leq n < N$. We want to show that $\inf_{j\in\mathbb{Z}} \|B_n^E(j)\| > c$ for $n = N$. We will again argue by contradiction.

Assume that $\inf_{j\in\mathbb{Z}} \|B_N^E(j)\| = 0$. Recall that $|E - b_j| < M$ and $|a_j| < M$ for all $j \in \mathbb{Z}$. Let $\epsilon > 0$. By Lemma 12, there exists $\delta > 0$ such that if $|p_N(j, E)| < \delta$, then there exists $E_0 \in \sigma(J_N(j))$ such that $|E - E_0| < \epsilon$. Set $0 < \gamma < \max\{\frac{\epsilon^2}{2}, \delta, \frac{1}{M^2}\}$. Then, for all such $\gamma > 0$, there exists $j_0 \in \mathbb{Z}$ such that $\|B_N^E(j_0)\| < \gamma^{3N}$. Without loss of generality we may assume that $j_0 = 0$. Then

$$\left\| \begin{pmatrix} p_N(0,E) & -a_{-1}p_{N-1}(1,E) \\ a_{N-1}p_{N-1}(0,E) & -a_{-1}a_{N-1}p_{N-2}(1,E) \end{pmatrix} \right\| < \gamma^{3N} \implies \begin{cases} |p_N(0,E)| < \gamma^{3N} \\ |-a_{-1}p_{N-1}(1,E)| < \gamma^{3N} \\ |a_{N-1}p_{N-1}(0,E)| < \gamma^{3N} \\ |-a_{-1}a_{N-1}p_{N-2}(1,E)| < \gamma^{3N} \end{cases}$$

There are three possible cases that describe the above system of inequalities and we will show that all of those cases cause contradiction.

Case 1. $|p_N(0,E)| < \gamma^{3N}$, $|a_{N-1}| < \gamma$, $|a_{-1}| < \gamma$.

By Lemma 12, there exists $E_0 \in \sigma(J_N(0))$, such that $|E-E_0| < \epsilon$. Choose a unit eigenvector $\vec{u} = (u_0, \ldots, u_{N-1})$ of $J_N(0)$ for the eigenvalue E_0. We define $\phi = (\ldots, 0, 0, \vec{u}, 0, 0, \ldots)$ which is a unit vector in $\ell^2(\mathbb{Z})$. Then,

$$(J_{a,b} - E)\phi = (J_{a,b} - E_0)\phi - (E - E_0)\phi,$$

where $\|(E - E_0)\phi\| < \epsilon$. Moreover, we have

$$(J_{a,b} - E_0)\phi(n) = a_n\phi(n+1) + a_{n-1}\phi(n-1) + (b_n - E_0)\phi(n),$$

which together with (5.8) implies that

$$(J_{a,b} - E_0)\phi(n) = \begin{cases} 0, & 0 \le n \le N-1,\ n \ge N+1,\ \text{or } n \le -2 \\ a_{-1}\phi(0), & n = -1 \\ a_{N-1}\phi(N-1), & n = N, \end{cases}$$

Since $|\phi(0)| = |u_0| \le 1$, $|\phi(N-1)| = |u_{N-1}| \le 1$ and since $|a_{N-1}| < \gamma$, $|a_{-1}| < \gamma$, we get that $\|(J_{a,b} - E_0)\phi\| < 2\gamma < 2\epsilon$. Hence, for the unit vector $\phi \in \ell^2(\mathbb{Z})$,

$$\|(J_{a,b} - E)\phi\| = \|(J_{a,b} - E_0)\phi - (E - E_0)\phi\| < 3\epsilon.$$

By (6.2), $E \in \sigma(J_{a,b})$, which is a contradiction.

Case 2. $|p_N(0, E)| < \gamma^{3N}$. Moreover, either $|a_{N-1}| < \gamma$ and $|a_{-1}| \ge \gamma$; or $|a_{N-1}| \ge \gamma$ and $|a_{-1}| < \gamma$.

Without loss of generality, we focus on the case $|a_{N-1}| < \gamma$ and $|a_{-1}| \ge \gamma$. In this case we have

$$\gamma|p_{N-1}(1, E)| \le |a_{-1}p_{N-1}(1, E)| < \gamma^{3N}$$

which implies $|p_{N-1}(1, E)| < \gamma^{3N-1}$.

If $|a_0| < \sqrt{2\gamma} < \epsilon$, then together with

$$|p_{N-1}(1, E)| < \gamma^{3N-1} < \delta \text{ and } |a_{N-1}| < \gamma < \epsilon$$

we apply the same argument as in case 1 to the operator $J_{N-1}(1)$ and get a unit vector $\phi \in \ell^2(\mathbb{Z})$ such that

$$\|(J_{a,b} - E)\phi\| < 3\epsilon,$$

which again contradicts the assumption that $E \in \rho(J_{a,b})$.

If $|a_{j_0}| \geq \sqrt{2\gamma}$, then since

$$|p_N(0,E)| = |(E-b_0)p_{N-1}(1,E) - a_0^2 p_{N-2}(2,E)| < \gamma^{3N},$$

we must have that

$$|a_0^2 p_{N-2}(2,E)| < \gamma^{3N-2} + \gamma^{3N} < 2\gamma^{3N-2},$$

which in turn implies

$$|p_{N-2}(2,E)| < \gamma^{3N-3}.$$

Repeating this procedure, we either at some step obtain $\|(J_{a,b}-E)\phi\| < 3\epsilon$ for some unit vector $\phi \in \ell^2(\mathbb{Z})$, which would cause a contradiction, or we eventually get

$$|p_2(N-2,E)| = |(E-b_{N-2})(E-b_{N-1}) - a_{N-2}^2| < \gamma^9$$

and

$$|p_1(N-1,E)| = |E-b_{N-1}| < \gamma^6.$$

So we have that $|E-b_{N-1}| < \gamma^6 < \epsilon$, $|a_{N-1}| < \gamma < \epsilon$ and $|a_{N-2}| < \sqrt{2\gamma^5} < \epsilon$, which again yields

$$\|(J_{a,b}-E)\phi\| < 3\epsilon,$$

for some unit vector $\phi \in \ell^2(\mathbb{Z})$, causing a contradiction.

Case 3. $|p_N(0,E)| < \gamma^{3N}$, $|a_{N-1}| \geq \gamma$, $|a_{-1}| \geq \gamma$.

In this case we must have $|p_{N-1}(1,E)| < \gamma^{3N-1}$, $|p_{N-1}(0,E)| < \gamma^{3N-1}$ and $|p_{N-2}(1,E)| < \gamma^{3N-2}$. Also,

$$|p_N(0,E)| = |(E-b_0)p_{N-1}(1,E) - a_0^2 p_{N-2}(2,E)| < \gamma^{3N}$$

$$|p_{N-1}(0,E)| = |(E-b_0)p_{N-2}(1,E) - a_0^2 p_{N-3}(2,E)| < \gamma^{3N-1}$$

So we must have that

$$|a_0^2 p_{N-2}(2,E)| < 2\gamma^{3N-2}$$

$$|a_0^2 p_{N-3}(2,E)| < 2\gamma^{3N-3}$$

If $|a_0| > \sqrt{2\gamma}$, then $|p_{N-2}(2,E)| < \gamma^{3N-3}$ and $|p_{N-3}(2,E)| < \gamma^{3N-4}$. Now,

$$B_{N-1}^E(1) = \begin{pmatrix} p_{N-1}(1,E) & -a_0 p_{N-2}(2,E) \\ a_{N-1} p_{N-2}(1,E) & -a_0 a_{N-1} p_{N-3}(2,E) \end{pmatrix}$$

We have that $|p_{N-1}(1,E)| < \gamma^{3N-1}$, $|-a_0 p_{N-2}(2,E)| < \gamma^{3N-4}$, $|a_{N-1} p_{N-2}(1,E)| < \gamma^{3N-3}$ and $|-a_0 a_{N-1} p_{N-3}(2,E)| < \gamma^{3N-5}$. Hence, combining all the estimates above, we obtain

$$\|B_{N-1}^E(1)\| < 5\gamma^{3N-5} < 5\gamma < 5\epsilon,$$

which contradicts our induction assumption if we choose $\epsilon < \frac{5}{c}$. Thus we must have $|a_0| < \sqrt{2\gamma} < \epsilon$. By the same argument as above, we see that $|a_{N-2}| < \sqrt{2\gamma} < \epsilon$. But now we have $|a_0| < \epsilon$, $|a_{N-2}| < \epsilon$ and $|p_{N-2}(1,E)| < \gamma^{3N-2} < \delta$, which again, by the same logic as in Case 1, implies that $E \notin \rho(J_{a,b})$, causing a contradiction.

Combining all the possible cases, we see that $\inf_{j\in\mathbb{Z}} \|B_N^E(j)\| > c$ for some $c = c(N) > 0$ concluding our proof. ■

6.2 Resolvent set and dominated splitting

We are now ready to prove the other direction of Theorem 5, mainly

$$\rho(J_{a,b}) \subset \{E \in \mathbb{C} : B^E \in \mathcal{DS}\ \}.$$

Fix $E \in \rho(J_{a,b})$. First, we follow the same approach as in [18] and perform a Combes-Thomas type of estimates [3] concerning the exponential decay of the Green's function.

Define M_β to be the multiplication operator $(M_\beta\psi)(n) = e^{\beta n}\psi(n)$. Without loss of generality, we may assume that $|\beta| \leq 1$. Then,

$$\begin{aligned}
(M_{-\beta}(J_{a,b}-E)M_\beta\phi)(n) &= e^{-\beta n}(J_{a,b}-E)(e^{\beta n}\phi(n)) \\
&= e^{-\beta n}(a_{n-1}e^{\beta(n-1)}\phi(n-1) + a_n e^{\beta(n+1)}\phi(n+1) + (b_n - E)e^{\beta n}\phi(n)) \\
&= a_{n-1}e^{-\beta}\phi(n-1) + a_n e^{\beta}\phi(n+1) + (b_n - E)\phi(n) \\
&= (J_{a,b}-E)\phi(n) + a_n(e^{\beta}-1)\phi(n+1) + a_{n-1}(e^{-\beta}-1)\phi(n-1) \\
&= (J_{a,b}-E)\phi(n) + a_n(e^{\beta}-1)(T\phi(n)) + a_{n-1}(e^{-\beta}-1)(T^{-1}\phi(n)),
\end{aligned}$$

where T is the left shift operator, i.e. $(T\phi)(n) = \phi(n+1)$. Hence,

$$M_{-\beta}(J_{a,b}-E)M_\beta = J_{a,b} - E + a_n(e^{\beta}-1)T + a_{n-1}(e^{-\beta}-1)T^{-1} = J_{a,b} - E + B,$$

The operator B is bounded on $\ell^2(\mathbb{Z})$ and

$$\|B\| \leq |a_n||(e^{\beta}-1)| + |a_{n-1}||(e^{-\beta}-1)| \leq C|\beta|,$$

for some $C = C(M) > 0$. Clearly,

$$\|(J_{a,b}-E)^{-1}B\| \leq \frac{1}{2}$$

if $\beta \leq \|(J_{a,b}-E)^{-1}\|^{-1}/(2C)$. Then

$$M_{-\beta}(J_{a,b}-E)M_\beta = J_{a,b} - E + B = (J_{a,b}-E)[I + (J_{a,b}-E)^{-1}B]$$

is invertible. Moreover

$$(M_{-\beta}(J_{a,b}-E)M_\beta)^{-1} = M_{-\beta}(J_{a,b}-E)^{-1}M_\beta = [I + (J_{a,b}-E)^{-1}B]^{-1}(J_{a,b}-E)^{-1},$$

which implies

$$\|M_{-\beta}(J_{a,b}-E)^{-1}M_\beta\| \leq 2\|(J_{a,b}-E)^{-1}\| := K.$$

Hence, it holds for all $p, q \in \mathbb{Z}$ that

$$\begin{aligned}|\langle \delta_p, M_{-\beta}(J_{a,b}-E)^{-1}M_\beta \delta_q\rangle| &= |\langle M_{-\beta}\delta_p, (J_{a,b}-E)^{-1}M_\beta \delta_q\rangle| \\ &= |(J_{a,b}-E)^{-1}(p,q)|e^{-\beta(p-q)} \\ &\le K\end{aligned}$$

which, when choosing the sign of β appropriately, gives exponential decay of the Green's Function:

$$|(J_{a,b}-E)^{-1}(p,q)| \le Ke^{-\beta|p-q|}. \tag{6.4}$$

Now we are ready to construct the two invariant directions of B^E. After we construct such directions, we will show that they satisfy conditions (1)-(3) of Definition 6, which together with Lemma 13 would imply that $B^E \in \mathcal{DS}$.

Let $g_j(n) = (J_{a,b}-E)^{-1}(n,j)$. Then $(g_j(n))_{n\in\mathbb{Z}}$ is the unique solution of the equation

$$(J_{a,b}-E)g_j = \delta_j, \tag{6.5}$$

where δ_j is the vector that $\delta_j(m) = 1$ if $m = j$ and 0 otherwise. By (6.4), it holds that

$$|g_j(n)| < Ke^{-u|n-j|}, \text{ for all } n, j \in \mathbb{Z}. \tag{6.6}$$

For each $j \in \mathbb{Z}$, we define $\vec{v}(j)$ and $\vec{w}(j) \in \mathbb{R}^2$ so that

$$\vec{v}(j) = \begin{pmatrix} g_{j-1}(j) \\ g_{j-1}(j-1)\end{pmatrix} \text{ and } \vec{w}(j) = \begin{pmatrix} g_j(j) \\ g_j(j-1)\end{pmatrix}.$$

Note that $\|\vec{v}(j)\| \le 2K$ and $\|\vec{w}(j)\| \le 2K$ for all $j \in \mathbb{Z}$. Recall

$$B^E(j) = \begin{pmatrix} E-b_j & -a_{j-1} \\ a_j & 0\end{pmatrix}$$

By (6.5), it holds for each $j \in \mathbb{Z}$ that

$$B^E(j-1)\begin{pmatrix} g_{j-1}(j-1) \\ g_{j-1}(j-2) \end{pmatrix} = \begin{pmatrix} a_{j-1}g_{j-1}(j) - 1 \\ a_{j-1}g_{j-1}(j-1) \end{pmatrix}. \tag{6.7}$$

Then we have

$$(E - b_{j-1})g_{j-1}(j-1) - a_{j-2}g_{j-1}(j-2) = a_{j-1}g_{j-1}(j) - 1$$

$$a_{j-2}g_{j-1}(j-2) = (1 - a_{j-1}g_{j-1}(j) + (E - b_{j-1})g_{j-1}(j-1))$$

Since sequences a_n and b_n are bounded, there exists some M, N such that $|a_n| < M$ and $|E - b_n| < N$. Then, there is a constant $C(M, N)$, such that either

$$C^{-1} \leq \|\vec{v}(j)\| = \left\| \begin{pmatrix} g_{j-1}(j) \\ g_{j-1}(j-1) \end{pmatrix} \right\| \leq 2K,$$

or $C^{-1} < |g_{j-1}(j-2)|$ which in turn implies that

$$C^{-1} \leq \|\vec{v}(j-1)\| = \left\| \begin{pmatrix} g_{j-2}(j-1) \\ g_{j-2}(j-2) \end{pmatrix} \right\| \leq 2K.$$

Similarly,

$$a_j g_j(j+1) = (1 - a_{j-1}g_j(j-1) + (E - b_j)g_j(j))$$

Same argument yields similar estimates for $\|\vec{w}(j)\|$ or $\|\vec{w}(j+1)\|$. Now for each $j \in \mathbb{Z}$, we define $\vec{s}(j)$ so that

$$\vec{s}(j) = \begin{cases} \vec{v}(j), & \text{if } \|\vec{v}(j)\| > C^{-1} \\ \begin{pmatrix} g_{j-2}(j) \\ g_{j-2}(j-1) \end{pmatrix}, & \text{otherwise.} \end{cases}$$

Similarly, we define $\vec{u}(j)$ so that

$$\vec{u}(j) = \begin{cases} \vec{w}(j), & \text{if } \|\vec{w}(j)\| > C^{-1} \\ \begin{pmatrix} g_{j+1}(j) \\ g_{j+1}(j-1) \end{pmatrix}, & \text{otherwise.} \end{cases}$$

Thus for all $j \in \mathbb{Z}$, we have

$$\|\vec{s}(j)\| > C^{-1} \text{ and } \|\vec{u}(j)\| > C^{-1}. \tag{6.8}$$

In particular, $\vec{s}(j) \neq \binom{0}{0}$ and $\vec{u}(j) \neq \binom{0}{0}$. Thus for each $j \in \mathbb{Z}$, we may define $E^s(j)$, $E^u(j)$ to be two one-dimensional subspaces of $\mathbb{C}^2$ such that $E^s(j) = \text{span}\{\vec{s}(j)\}$ and $E^u(j) = \text{span}\{\vec{u}(j)\}$.

So far we have constructed subspaces $E^s(j)$, $E^u(j)$. In Lemmas 14, 15 and 16 we will show conditions (1), (2) and (3) of Definition 6 respectively.

Lemma 14 *E^u, E^s are B^E-invariant in the sense that for all $j \in \mathbb{Z}$, it holds that*

$$B^E(j) \cdot E^u(j) = E^u(j+1) \textit{ and } B^E(j) \cdot E^s(j) \subseteq E^s(j+1).$$

Proof. Let $G(E; n, m) = (J_{a,b} - E)^{-1}(n, m)$ be the Green's function of $J_{a,b}$ at E. Then, by (6.5),

$$g_j(n) = G(E; j, n) =< \delta_j, (J_{a,b} - E)^{-1}\delta_n > .$$

By construction,

$$(J_{a,b} - E)g_j(n) = \delta_j(n).$$

Hence, we have the following representation of $(J_{a,b} - E)^{-1}$:

$$
\begin{array}{c} \\ \\ j-2 \\ j-1 \\ j \\ j+1 \\ j+2 \\ \\ \end{array}
\begin{array}{c}
\begin{array}{ccccc} j-2 & j-1 & j & j+1 & j+2 \end{array} \\
\left(\begin{array}{ccccccc}
\ddots & & & & & & \\
& g_{j-2}(j-2) & g_{j-1}(j-2) & g_j(j-2) & g_{j+1}(j-2) & g_{j+2}(j-2) & \\
& g_{j-2}(j-1) & g_{j-1}(j-1) & g_j(j-1) & g_{j+1}(j-1) & g_{j+2}(j-1) & \\
& g_{j-2}(j) & g_{j-1}(j) & g_j(j) & g_{j+1}(j) & g_{j+2}(j) & \\
& g_{j-2}(j+1) & g_{j-1}(j+1) & g_j(j+1) & g_{j+1}(j+1) & g_{j+2}(j+1) & \\
& g_{j-2}(j+2) & g_{j-1}(j+2) & g_j(j+2) & g_{j+1}(j+2) & g_{j+2}(j+2) & \\
& & & & & & \ddots
\end{array}\right)
\end{array}
$$

Case 1. Assume $a_j \neq 0$ for all $j \in \mathbb{Z}$.

If we take $\vec{s}(j)$ as out starting point, we can get a solution $\phi_j^s(n)$ of $J_{a,b}\phi = E\phi$ that coincides with $g_j(n)$ for $n \geq j$. Hence, $\phi_j^s(n) \in \ell^2(\mathbb{Z}_{\geq j})$. Similarly, we can obtain a solution $\phi_j^u(n) \in \ell^2(\mathbb{Z}_{\leq j})$ to $J_{a,b}\phi = E\phi$ for the case $n \leq j$. So, for all $j \in \mathbb{Z}$ we have

$$g_j(n) = \begin{cases} \phi_j^s(n), & n \geq j. \\ \phi_j^u(n), & n \leq j, \end{cases}$$

In particular, we obtain that ϕ^s and ϕ^u must be linear independent. Otherwise, they both decay exponentially fast as $n \to \pm\infty$ which implies they are eigenvectors. Then, $E \in \sigma(J_{a,b})$ which contradicts with our assumption that $E \in \rho(J_{a,b})$.

Recall that if $\phi \in \mathbb{C}^{\mathbb{Z}}$ solves equation $J_{a,b}\phi = E\phi$, then

$$B^E(j)\begin{pmatrix} \phi(j) \\ \phi(j-1) \end{pmatrix} = a_j \begin{pmatrix} \phi(j+1) \\ \phi(j) \end{pmatrix}, \ j \in \mathbb{Z}.$$

The solution ϕ to the spectral equation $J_{a,b}\phi = E\phi$ is uniquely determined by $\phi(j)$, $\phi(j+1)$ at consecutive points $j, j+1$. Hence the solution space of $J_{a,b}\phi = E\phi$ is two-dimensional.

We denote by $\ell^2(\mathbb{Z}_\pm)$ all the two-sided infinite sequences who is square summable on $\mathbb{Z}_\pm$, respectively. Since the solution space of $J_{a,b}\phi = E\phi$ is two-dimensional, it's clear that all solution in $\ell^2(\mathbb{Z}_\pm)$ are linear dependent with $\phi^{s(u)}$, respectively. Therefore $\phi_j^{s(u)}$ and $\phi_k^{s(u)}$ are linearly dependent for all $j \neq k \in \mathbb{Z}$, respectively. Hence

$$A^E(j)\vec{s}(j) = A^E(j)\begin{pmatrix} g_p(j) \\ g_p(j-1) \end{pmatrix} = A^E(j)\begin{pmatrix} \phi_p^s(j) \\ \phi_p^s(j-1) \end{pmatrix} = \begin{pmatrix} \phi_p^s(j+1) \\ \phi_p^s(j) \end{pmatrix}$$

must be linearly dependent with $\binom{\phi_q^s(j+1)}{\phi_q^s(j)}$, where $p = j-1$ or $j-2$ and $q = j-1$ or j. In other words, we have $B^E(j)\cdot\vec{s}(j) \subseteq E^s(j+1)$. Similar, we obtain $B^E(j-1)\vec{u}(j-1) \subseteq E^u(j)$ for all $j \in \mathbb{Z}$. Hence, for all $j \in \mathbb{Z}$,

$$B^E(j)E^s(j) = E^s(j+1) \text{ and } B^E(j-1)E^u(j-1) = E^u(j).$$

Case 2. Assume $a_j = 0$ for some $j \in \mathbb{Z}$ and $a_n \neq 0$ for all $n \neq j$.

In this case we can decompose $J_{a,b}$ into the direct sum

$$J_{a,b} = J_{a,b(-,j+1)} \oplus J_{a,b(j,+)}$$

on $\ell^2(-\infty, j] \oplus \ell^2(j, +\infty)$. Then, if ϕ^u solves $J_{a,b(j,+)}\phi^u = E\phi^u$, we have

$$\begin{pmatrix} b_{j+1}-E & a_{j+1} & 0 & \dots \\ a_{j+1} & b_{j+2}-E & a_{j+2} & \dots \\ 0 & a_{j+2} & b_{j+3}-E & a_{j+3} \\ \vdots & \vdots & a_{j+3} & \ddots \end{pmatrix}\begin{pmatrix} \phi^u(j+1) \\ \phi^u(j+2) \\ \phi^u(j+3) \\ \vdots \end{pmatrix} = 0$$

As discussed in Lemma 11 we see that ϕ^u is uniquely determined by $\phi^u(j+1)$ at some point $j+1$. Hence, the solutions space is one-dimensional, i.e.

$$\dim\{\phi^u(n), n > j : (J_{a,b(j,+)} - E)\phi^u(n) = 0\} = 1.$$

Now we consider $g_m(n)$ for all $m > j$. Note $(J_{a,b} - E)g_m = \delta_m$. In particular, we have

$$(J_{a,b} - E)g_m(n) = \begin{cases} 0, & j \le n < m \\ 1, & n = m \end{cases}$$

Such $\phi^u(n)$ coincides with $g_m(n)$ for all $j < n \le m$, since

$$\begin{pmatrix} b_{j+1} - E & a_{j+1} & 0 & \dots \\ a_{j+1} & b_{j+2} - E & a_{j+2} & \dots \\ 0 & a_{j+2} & b_{j+3} - E & a_{j+3} \\ \vdots & \vdots & a_{j+3} & \ddots \end{pmatrix} \begin{pmatrix} g_m(j+1) \\ g_m(j+2) \\ g_m(j+3) \\ \vdots \end{pmatrix} = 0$$

up until the point when $j < m = n$ in $g_m(n)$, since then $(J_{a,b(j,+)} - E)g_m(m) = 1$. Hence $g_m(n) = \phi^u_m(n)$, $j < n \le m$.

Now, let $\phi^+ : \mathbb{Z}_{\ge j+1} \to \mathbb{R}$ be such that

1. $a_{n-1}\phi^+(n-1) + a_n\phi^+(n+1) + (E - b_n)\phi^+(n) = 0$ for all $n \ge j+2$,
2. $\phi^+ \in \ell^2(\mathbb{Z}_+)$, and
3. $a_{j+1}\phi^+(j+2) + (E - b_{j+1})\phi^+(j+1) \neq 0$.

Note the existence of such ϕ^+ that satisfies conditions (1) and (2) above is guaranteed by the restriction of g_j on $\mathbb{Z}_{\ge j+1}$, since $\phi^+_{j+1}(n) = g_{j+1}(n)$ for all $n \ge j+1$ and $g_{j+1}(n) \in \ell^2(\mathbb{Z})$. On the other hand, all such vectors are uniquely determined by $\binom{\phi^+(j+2)}{\phi^+(j+1)}$ since for all $n \ge 1$

$$\begin{pmatrix} \phi^+(j+n+2) \\ \phi^+(j+n+1) \end{pmatrix} = A^E_n(j+2) \begin{pmatrix} \phi^+(j+2) \\ \phi^+(j+1) \end{pmatrix}.$$

So all such vectors form a space of dimensional at most 2. To show that ϕ^+ satisfies condition (3), i.e. ϕ^+ does not solve $J_{(j,+)}\phi^+ = E\phi^+$, notice that if it does, then $(\dots, 0, 0, \phi^+)$ becomes an eigenvector of $J_{a,b}$ which makes E an eigenvalue, which contradicts our assumption that $E \in \rho(J_{a,b})$. Thus, such ϕ^+ can only form a one-dimensional space since otherwise some linear combination of two linear independent such ϕ^+ can solve $J_{(j,+)}\phi^+ = E\phi^+$. Then, for all $m \geq j+1$, $g_m(n) = \phi_m^+(n)$ for all $n \geq m$, where $\phi_m^+(n)$ is some $\phi^+(n)$ described above.

Similarly, we can get solutions the one-dimensional space of ϕ^s which solves $J_{(-,j+1)}\phi^s = E\phi^s$ and the one-dimensional space of ϕ^- which satisfies

1. $[(J_{a,b} - E)\phi^-](k) = 0$ for all $k < j$,
2. $\phi^- \in \ell^2(\mathbb{Z}_-)$,
3. $a_{j-1}\phi^-(j-1) + (E - b_j)\phi^-(j) \neq 0$.

Let $k \leq j$ and $m > j$. Then

$$g_k(n) = \begin{cases} \phi_k^-(n), & n \leq k, \\ \phi_k^s(n), & k \leq n \leq j, \\ 0, & n > j. \end{cases}$$

$$g_m(n) = \begin{cases} \phi_m^+(n), & n \geq m, \\ \phi_m^u(n), & j < n \leq m, \\ 0, & n \leq j. \end{cases}$$

Now we can rewrite $(J_{a,b} - E)^{-1}$ as follows

$$
\begin{array}{c|cccc|}
 & j-1 & j & j+1 & j+2 \\
 & \ddots & & & \\
j-2 & \phi^{-}_{j-1}(j-2) & \phi^{-}_{j}(j-2) & 0 & 0 \\
j-1 & \phi^{s(-)}_{j-1}(j-1) & \phi^{-}_{j}(j-1) & 0 & 0 \\
j & \phi^{s}_{j-1}(j) & \phi^{s(-)}_{j}(j) & 0 & 0 \\
j+1 & 0 & 0 & \phi^{u(+)}_{j+1}(j+1) & \phi^{u}_{j+2}(j+1) \\
j+2 & 0 & 0 & \phi^{+}_{j+1}(j+2) & \phi^{u(+)}_{j+2}(j+2) \\
j+3 & 0 & 0 & \phi^{+}_{j+1}(j+3) & \phi^{+}_{j+2}(j+3) \\
 & & & & \ddots
\end{array}
$$

Note, all $\phi^{s(u)}$ are linear dependent, respectively. All $\phi^{\pm}$ are linear dependent, respectively.

To see the invariance of $E^s(n)$, we first have for all $n \neq j+1$:

$$
B^E(n)\vec{s}(n) = B^E(n)\begin{pmatrix} g_p(n) \\ g_p(n-1) \end{pmatrix} = \begin{cases} B^E(n)\binom{\phi_p^+(n)}{\phi_p^+(n-1)} = a_n\binom{\phi_p^+(n+1)}{\phi_p^+(n)}, & n > j+1 \\ B^E(n)\binom{\phi_p^s(n)}{\phi_p^s(n-1)} = \binom{0}{0}, & n = j \\ B^E(n)\binom{\phi_p^s(n)}{\phi_p^s(n-1)} = a_n\binom{\phi_p^s(n+1)}{\phi_p^s(n)}, & n < j, \end{cases}
$$

where $p = n-1$ or $n-2$. But in case $n = j+2$, by the discussion defining $\vec{s}(n)$ and by $a_j = a_{n-2} = 0$, we know we must have $p = n-1 = j+1$. Hence, everything in the first case above is still well-defined. In all the cases, we see $B^E(n)\vec{s}(n)$ must be linear dependent with $\vec{s}(n+1)$ which is

$$\vec{s}(n+1) = \begin{pmatrix} g_q(n+1) \\ g_q(n) \end{pmatrix} = \begin{cases} \binom{\phi_q^+(n+1)}{\phi_q^+(n)}, & n > j+1 \\ \binom{0}{\phi_q^s(j)}, & n = j \\ \binom{\phi_q^s(n+1)}{\phi_q^s(n)}, & n < j, \end{cases}$$

where $q = n$ or $n-1$. To see the invariance of $E^s(n)$ at the step $n = j+1$, notice that

$$\begin{aligned} B(j+1)\vec{s}(j+1) &= \begin{pmatrix} E - b_{j+1} & -a_j \\ a_{j+1} & 0 \end{pmatrix} \begin{pmatrix} g_p(j+1) \\ g_p(j) \end{pmatrix} \\ &= \begin{pmatrix} E - b_{j+1} & 0 \\ a_{j+1} & 0 \end{pmatrix} \begin{pmatrix} 0 \\ \phi_p^s(j) \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \end{aligned}$$

which is of course linearly dependent with $\vec{s}(j+2)$, where $p = j$ or $j-1$. Note that in any case, we either have $B^E(j)E^s(j) \subseteq E^s(j+1)$ or $B^E(j)E^s(j) = \{\vec{0}\}$.

To check the invariance of $E^u(n)$, we first have for all $n \neq j$:

$$B^E(n)\vec{u}(n) = B^E(n)\begin{pmatrix} g_p(n) \\ g_p(n-1) \end{pmatrix} = \begin{cases} B^E(n)\binom{\phi_p^u(n)}{\phi_p^u(n-1)} = a_n\binom{\phi_p^u(n+1)}{\phi_p^u(n)}, & n > j+1 \\ B^E(n)\binom{\phi_p^u(n)}{0} = a_n\binom{\phi_p^u(n+1)}{\phi_p^u(n)}, & n = j+1 \\ B^E(n)\binom{\phi_p^-(n)}{\phi_p^-(n-1)} = a_n\binom{\phi_p^-(n+1)}{\phi_p^-(n)}, & n < j, \end{cases}$$

where $p = n$ or $n+1$. In all these cases, $\vec{0} \neq B^E(n)\vec{u}(n)$ is linear dependent with $\vec{u}(n+1)$ since

$$\vec{u}(n+1) = \begin{pmatrix} g_q(n+1) \\ g_q(n) \end{pmatrix} = \begin{cases} \binom{\phi_q^u(n+1)}{\phi_q^u(n)}, & n \geq j+1 \\ \binom{\phi_q^-(n+1)}{\phi_q^-(n)}, & n < j, \end{cases}$$

where $q = n+1$ or $n+2$. To see the invariance of $E^u(n)$ at the step $n = j$, notice that

$$\begin{aligned} B(j)\vec{u}(j) &= \begin{pmatrix} E - b_j & -a_{j-1} \\ \\ a_j & 0 \end{pmatrix} \begin{pmatrix} g_j(j) \\ g_j(j-1) \end{pmatrix} \\ &= \begin{pmatrix} E - b_j & -a_{j-1} \\ \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \phi_j^-(j) \\ \phi_j^-(j-1) \end{pmatrix} \\ &= \begin{pmatrix} (E - b_j)\phi_j^-(j) - a_{j-1}\phi_j^-(j-1) \\ 0 \end{pmatrix}, \end{aligned}$$

where the first component cannot be zero by definition of ϕ^-. Then $B(j)\vec{u}(j)$ is linearly dependent with $\vec{u}(j+1) = \begin{pmatrix} g_q(j+1) \\ g_q(j) \end{pmatrix} = \begin{pmatrix} \phi_q^u(j+1) \\ 0 \end{pmatrix}$, where $q = j+1$ or $j+2$. Note, in all these cases,we always have $B^E(j)E^u(j) = E^u(j+1)$.

Hence, from the definition of $\vec{s}(j)$ and $\vec{u}(j)$, it follows that for all $j \in Z$, for all $\vec{s}(j) \in E^s(j)$ and all $\vec{u}(j) \in E^u(j)$

$$B^E(j) \cdot \vec{u}(j) = E^u(j+1) \text{ and } B^E(j) \cdot \vec{s}(j) \subseteq E^s(j+1).$$

Case 3. Assume $a_j = 0$ for $j = \{j_1, j_2\}$, $j_1 < j_2$ and $a_n \neq 0$ for all $n \neq j_1, j_2$.

In this case,

$$J_{a,b} = J_{a,b(-,j_1+1)} \oplus J_{a,b(j_1,j_2+1)} \oplus J_{a,b(j_2,+)}.$$

We deal with infinite restrictions $J_{a,b(-,j_1+1)}$ and $J_{a,b(j_2,+)}$ the same way as in case 2. In particular, we obtain a one-dimensional space of $\phi^{s(u)}$ solving $J_{(-,j_1+1)}\phi^s = E\phi^s$ and $J_{(j_2,+)}\phi^u = E\phi^u$ respectively. We also obtain one-dimensional spaces of $\phi^{\pm} \in \ell^2(\mathbb{Z}_{\pm})$ defined on $\mathbb{Z}_{\geq j_2+1}$ and $\mathbb{Z}_{\leq j_1}$, respectively, as we described in case 2. As for the finite restriction,

$$\dim\{\phi^f(n), j_1 < n \leq j_2 : (J_{a,b(j_1,j_2+1)} - E)\phi^f(n) = 0\} = 0$$

since if there exists a nonzero solution ϕ^f such that $(J_{a,b(j_1,j_2+1)}-E)\phi^f = 0$, then $(\dots, 0, \phi^f, 0, \dots)$ is a solution to $(J_{a,b} - E)\phi = 0$, which contradicts the fact that $E \in \rho(J_{a,b})$. However, when we consider $J_{a,b(j_1,j_2+1)}$, and start at the boundary at position $j_1 + 1$, we get the the one-dimensional space of solutions $\widetilde{\phi}^u : (j_1, j_2 + 1) \to \mathbb{C}$ of $J_{a,b(j_1,j_2+1)}\phi = E\phi$ that coincides with $g_j(n)$ for $n \geq j$ between $j_1 + 1$ and j_2 positions, but does not agree with the boundary conditions at the j_2 position. Similarly, if we start at the j_2 position, we can generate a one-dimensional space of solutions $\widetilde{\phi}^s : (j_1, j_2 + 1) \to \mathbb{C}$ of $J_{a,b(j_1,j_2+1)}\phi = E\phi$ that coincides with $g_j(n)$ for $n \leq j$ between $j_1 + 1$ and j_2 positions, but does not agree with the boundary conditions at the $j_1 + 1$ position. Hence we have the following.

Let $k \leq j_1$, $m > j_2$ and $j_1 < l \leq j_2$. Then

$$g_k(n) = \begin{cases} \phi_k^u(n), & n \leq k, \\ \phi_k^-(n), & k \leq n \leq j_1, \\ 0, & n > j_1. \end{cases}$$

$$g_m(n) = \begin{cases} \phi_m^s(n), & n \geq m, \\ \phi_m^+(n), & j_2 < n \leq m, \\ 0, & n \leq j_2. \end{cases}$$

$$g_l(n) = \begin{cases} \widetilde{\phi}_l^s(n), & l \leq n \leq j_2, \\ \widetilde{\phi}_l^u(n), & j_1 < n \leq l, \\ 0, & \text{otherwise}. \end{cases}$$

In this case, $(J_{a,b}-E)^{-1}$ becomes

$$\begin{array}{c} \\ \\ j_1-1 \\ j_1 \\ j_1+1 \\ \dots \\ j_2 \\ j_2+1 \\ j_2+2 \\ \\ \end{array}
\begin{array}{c} \begin{array}{ccccc} j_1 & j_1+1 & \dots & j_2 & j_2+1 \end{array} \\
\begin{pmatrix} \ddots & & & & & \\
& \phi^-_{j_1}(j_1-1) & 0 & \dots & 0 & 0 \\
& \phi^{-(s)}_{j_1}(j_1) & 0 & \dots & 0 & 0 \\
& 0 & \widetilde{\phi}^{u(s)}_{j_1+1}(j_1+1) & \dots & \widetilde{\phi}^{u}_{j_2}(j_1+1) & 0 \\
& 0 & \vdots & \dots & \vdots & 0 \\
& 0 & \widetilde{\phi}^{s}_{j_1+1}(j_2) & \dots & \widetilde{\phi}^{s(u)}_{j_2}(j_2) & 0 \\
& 0 & 0 & \dots & 0 & \phi^{+(u)}_{j_2+1}(j_2+1) \\
& 0 & 0 & \dots & 0 & \phi^{+}_{j_2+1}(j_2+2) \\
& & & & & & \ddots \end{pmatrix} \end{array}$$

The proof of invariance of $E^s(n)$ for $n \le j_1$ or $n > j_2+1$ is identical to case 2 as it only involves ϕ^+ or ϕ^s. For $j_1+1 < n \le j_2$, it's identical to the argument of case 1. We just need to replace ϕ^u by $\widetilde{\phi}^u$ and ϕ^s by $\widetilde{\phi}^s$. For the case $n = j_1+1$ or j_2+1, we have

$$\begin{aligned} B(j_1+1)\vec{s}(j_1+1) &= \begin{pmatrix} E-b_{j_1+1} & -a_{j_1} \\ a_{j_1+1} & 0 \end{pmatrix} \begin{pmatrix} g_p(j_1+1) \\ g_p(j_1) \end{pmatrix} \\ &= \begin{pmatrix} E-b_{j_1+1} & 0 \\ a_{j_1+1} & 0 \end{pmatrix} \begin{pmatrix} 0 \\ \phi^s_p(j_1) \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \end{aligned}$$

which is of course linearly dependent with $\vec{s}(j_1+2)$, where $p = j_1$ or j_1-1.

$$
\begin{aligned}
B(j_2+1)\vec{s}(j_2+1) &= \begin{pmatrix} E-b_{j_2+1} & -a_{j_2} \\ \\ a_{j_2+1} & 0 \end{pmatrix} \begin{pmatrix} g_q(j_2+1) \\ g_q(j_2) \end{pmatrix} \\
&= \begin{pmatrix} E-b_{j_2+1} & 0 \\ \\ a_{j_2+1} & 0 \end{pmatrix} \begin{pmatrix} 0 \\ \widetilde{\phi}_q^s(j_2) \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ 0 \end{pmatrix},
\end{aligned}
$$

where $q = j_2$ or $j_2 - 1$, which is linearly dependent with $\vec{s}(j_2+2)$.

Similarly, the proof of invariance of $E^u(n)$ for $n < j_1$ or $n \geq j_2 + 1$ is identical to case 2 as it only involves ϕ^- or ϕ^u. For $j_1 + 1 \leq n < j_2$, it's identical to the argument of case 1. For the case $n = j_1$ or j_2, we have

$$
\begin{aligned}
B(j_1)\vec{u}(j_1) &= \begin{pmatrix} E-b_{j_1} & -a_{j_1-1} \\ \\ a_{j_1} & 0 \end{pmatrix} \begin{pmatrix} g_{j_1}(j_1) \\ g_{j_1}(j_1-1) \end{pmatrix} \\
&= \begin{pmatrix} E-b_{j_1} & -a_{j_1-1} \\ \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \phi_{j_1}^-(j_1) \\ \phi_{j_1}^-(j_1-1) \end{pmatrix} \\
&= \begin{pmatrix} (E-b_{j_1})\phi_{j_1}^-(j_1) - a_{j_1-1}\phi_{j_1}^-(j_1-1) \\ 0 \end{pmatrix},
\end{aligned}
$$

where the first component cannot be zero by definition of ϕ^-. Then $B(j_1)\vec{u}(j_1)$ is linearly dependent with $\vec{u}(j_1+1) = \begin{pmatrix} g_p(j_1+1) \\ g_p(j_1) \end{pmatrix} = \begin{pmatrix} \widetilde{\phi}_p^u(j_1+1) \\ 0 \end{pmatrix}$, where $p = j_1 + 1$ or $j_1 + 2$.

$$B(j_2)\vec{u}(j_2) = \begin{pmatrix} E - b_{j_2} & -a_{j_2-1} \\ \\ a_{j_2} & 0 \end{pmatrix} \begin{pmatrix} g_{j_2}(j_2) \\ g_{j_2}(j_2-1) \end{pmatrix}$$
$$= \begin{pmatrix} E - b_{j_2} & -a_{j_2-1} \\ \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \widetilde{\phi}^u_{j_2}(j_2) \\ \widetilde{\phi}^u_{j_2}(j_2-1) \end{pmatrix}$$
$$= \begin{pmatrix} (E - b_{j_2})\widetilde{\phi}^u_{j_2}(j_2) - a_{j_2-1}\widetilde{\phi}^u_{j_2}(j_2-1) \\ 0 \end{pmatrix},$$

where the first component cannot be zero by definition of $\widetilde{\phi}^u$. Then $B(j_2)\vec{u}(j_2)$ is linearly dependent with $\vec{u}(j_2+1) = \begin{pmatrix} g_{j_2+1}(j_2+1) \\ g_{j_2+1}(j_2) \end{pmatrix} = \begin{pmatrix} \phi^u_q(j_2+1) \\ 0 \end{pmatrix}$, where $q = j_2+1$ or j_2+2.

Hence we see that when $a_j = 0$ for $j = \{j_1, j_2\}$, $j_1 < j_2$ and $a_n \neq 0$ for all $n \neq j_1, j_2$, it holds that for all $j \in Z$, for all $\vec{s}(j) \in E^s(j)$ and all $\vec{u}(j) \in E^u(j)$

$$B^E(j) \cdot \vec{u}(j) = E^u(j+1) \text{ and } B^E(j) \cdot \vec{s}(j) \subseteq E^s(j+1).$$

Case 4. Assume $a_j = 0$ for some $j \in K$ where $K \subseteq \mathbb{Z}$.

This case is an extension to case 3. We simply decompose $J_{a,b}$ into the direct sum of disjoint matrices. Let

$$j_{\max} := \sup\{j \in \mathbb{Z} : a_j = 0\} \text{ and } j_{\min} := \inf\{j \in \mathbb{Z} : a_j = 0\}.$$

Now for each $j \in \mathbb{Z}$, either $j \in (j_{\max}, \infty) \cup (-\infty, j_{\min}]$, or $j \in (j_k, j_{k+1}]$ where $a_{j_k} = a_{j_{k+1}} = 0$ and all $a_j \neq 0$ for all $j_k < j < j_{k+1}$ (if such j exists). No matter where j is, to show invariance of $E^{s(u)}$ at j, we only need to deal with one of the following type of vectors

$$\phi^u, \phi^s, \phi^+, \phi^-, \widetilde{\phi}^s, \text{ or } \widetilde{\phi}^u,$$

where $\phi^{s(u)}$ and $\phi^{\pm}$ are as defined in case 2 which are related to infinite half operators; $\widetilde{\phi}^{s(u)}$ are as defined in case 3 which are related to the finite restriction of $J_{a,b}$. In all of the above cases, we get the invariance of subspaces E^s and E^u with respect to B^E by the same logic as in case 3.

Combining the four cases above, we get that for all $j \in \mathbb{Z}$, it holds that

$$B^E(j) \cdot E^u(j) = E^u(j+1) \text{ and } B^E(j) \cdot E^s(j) \subseteq E^s(j+1).$$

■

Notice that from the proof of Lemma 14, we see that when $a_j = 0$ for some $j \in \mathbb{Z}$, then

$$B^E(j)E^s(j) = B^E(j+1)E^s(j+1) = \{\vec{0}\}. \tag{6.9}$$

Moreover, it holds that

$$E^s(j+1) = \operatorname{span}\left\{\begin{pmatrix}0\\1\end{pmatrix}\right\}. \tag{6.10}$$

and

$$E^u(j+1) = B^E(j)E^u(j) = \operatorname{span}\left\{\begin{pmatrix}1\\0\end{pmatrix}\right\}. \tag{6.11}$$

Define $\vec{s}^u(j) = \frac{\vec{s}(j)}{\|\vec{s}(j)\|}$ and $\vec{u}^u(j) = \frac{\vec{u}(j)}{\|\vec{u}(j)\|}$ to be the unit vectors. Then we have the following lemma.

Lemma 15 *There exists $N \in \mathbb{Z}_+$ such that*

$$\|B_N^E(j)\vec{u}^u(j)\| > 2\|B_N^E(j)\vec{s}^u(j)\|$$

for all $j \in \mathbb{Z}$, where $\vec{s}^u(j), \vec{u}^u(j)$ are the unit vectors of $E^s(j), E^u(j)$ respectively. In particular, $E^s(j) \neq E^u(j)$ for all $j \in \mathbb{Z}$.

Proof. Assume for some $j \in \mathbb{Z}$ and $n \in \mathbb{Z}_+$ that $a_i \neq 0$ for all $j-1 \leq i < j+n$. Then by the proof of Lemma 14, there is a ϕ such that

$$A_k^E(j)\begin{pmatrix}\phi(j)\\ \phi(j-1)\end{pmatrix} = \begin{pmatrix}\phi(j+k)\\ \phi(j+k-1)\end{pmatrix} \text{ for all } 1 \leq k \leq n$$

and $g_p(i) = \phi(i)$ for all $j-1 \leq i \leq j+n$ where $p = j-1$ or $j-2$. Hence, we have

$$\begin{aligned} \|B_n^E(j)\vec{s}(j)\| &= \left\| B_n^E(j) \begin{pmatrix} g_p(j) \\ g_p(j-1) \end{pmatrix} \right\| \\ &= \left(\prod_{i=j}^{j+n-1} a_i \right) \left\| \begin{pmatrix} g_p(j+n) \\ g_p(j+n-1) \end{pmatrix} \right\| \\ &= \|B_n^E(j)\vec{s}^u(j)\| \|\vec{s}(j)\| \end{aligned}$$

Similarly, for the same j and n, there is a ψ so that

$$A_{-k}^E(j+n) \begin{pmatrix} \psi(j+n) \\ \psi(j+n-1) \end{pmatrix} = \begin{pmatrix} \psi(j+n-k) \\ \psi(j+n-k-1) \end{pmatrix} \text{ for all } 1 \leq k \leq n$$

and $g_{q+n}(i) = \psi(i)$ for all $j-1 \leq i \leq j+n$ where $q = j-1$ or j. Hence, we have

$$\begin{aligned} \|B_n^E(j)\vec{u}^u(j)\| &= \frac{\|B_n^E(j)B_{-n}^E(j+n)\vec{u}(j+n)\|}{\|B_{-n}^E(j+n)\vec{u}(j+n)\|} \\ &= \frac{\|\vec{u}(j+n)\|}{\|B_{-n}^E(j+n)\vec{u}(j+n)\|} \\ &= \frac{\|\vec{u}(j+n)\|}{\left(\prod_{i=j+n-1}^{j} \frac{1}{a_i}\right) \left\| \binom{g_{q+n}(j)}{g_{q+n}(j-1)} \right\|} \\ &= \frac{\|\vec{u}(j+n)\|}{\left(\prod_{i=j+n-1}^{j} \frac{1}{a_i}\right) \left\| \binom{g_j(q+n)}{g_{j-1}(q+n)} \right\|}, \end{aligned}$$

Then, for all such j and n,

$$\begin{aligned} \frac{\|B_n^E(j)\vec{u}^u(j)\|}{\|B_n^E(j)\vec{s}^u(j)\|} &= \frac{\|\vec{u}(j+n)\|\|\vec{s}(j)\|}{\left(\prod_{i=j+n-1}^{j} \frac{1}{a_i}\right) \left\| \binom{g_j(q+n)}{g_{j-1}(q+n)} \right\| \left(\prod_{i=j}^{j+n-1} a_i\right) \left\| \binom{g_p(j+n)}{g_p(j+n-1)} \right\|} \\ &\geq \frac{\|\vec{u}(j+n)\|\|\vec{s}(j)\|}{K^2 e^{-2\gamma n}(1+e^{-2\gamma})} \\ &\geq \frac{e^{2\gamma n}}{C^2K^2(1+e^{-2\gamma})}, \end{aligned}$$

where first inequality follows from (6.6) and the last inequality follows from (6.8). If we let $n \geq \lceil \frac{\log(2CK)}{\gamma} \rceil$, then

$$\frac{e^{2\gamma n}}{C^2K^2(1+e^{-2\gamma})} > 2.$$

Notice that n depends only on $d(E, \sigma(J_{a,b}))$. We can then set $N = \lceil \frac{\log(2CK)}{\gamma} \rceil$. Then for all $j \in \mathbb{Z}$ such that $a_i \neq 0$ for all $j - 1 \leq i < j + N$, it holds

$$\|B_N^E(j)\vec{u}^u(j)\| > 2\|B_N^E(j)\vec{s}^u(j)\|.$$

Now consider all $j \in \mathbb{Z}$ such that $a_i = 0$ for some $j - 1 \leq i < j + N$. First, we note that $B_N^E(j)\vec{u}^u(j) \neq \vec{0}$ since Lemma 14 says $B^E(i)E^u(i) = E^u(i+1)$ for all $i \in \mathbb{Z}$. By (6.9), $a_i = 0$ implies that

$$B^E(i)\vec{s}(i) = B^E(i+1)\vec{s}(i+1) = \vec{0}.$$

In particular, if $i = j - 1$, then $B^E(j)\vec{s}(j) = 0$ which implies

$$B_N^E(j)\vec{s}(j) = B_{N-1}^E(j+1)B^E(j)\vec{s}(j) = \vec{0}.$$

If $i \geq j$, then $B_{i-j}^E(j)\vec{s}(j) \in E^s(i)$ which implies

$$B_N^E(j)\vec{s}(j) = B_{N-i+j-1}^E(i+1)B^E(i)B_{i-j}^E(j)\vec{s}(j) = \vec{0}.$$

In any case, we have $B_N^E(j)\vec{s}^u(j) = \vec{0}$ which implies for such j that

$$\|B_N^E(j)\vec{u}^u(j)\| > 0 = 2\|B_N^E(j)\vec{s}^u(j)\|,$$

concluding the proof. ■

The only thing that is left to show is that the invariant directions of $B^E(j)$ are separated uniformly for all $j \in \mathbb{Z}$.

Lemma 16 *Let $E \in \rho(J_{a,b})$. Let $E^u(j)$, $E^s(j)$ be two invariant directions of $B^E(j)$. Then, there exists $\tau > 0$ such that*

$$\inf_{j \in \mathbb{Z}} d(E^u(j), E^s(j)) > \tau.$$

Proof. Recall that $\vec{s}(j) = \binom{g_p(j)}{g_p(j-1)}$ and $\vec{u}(j) = \binom{g_q(j)}{g_q(j-1)}$ where $p = j-1$ or $j-2$ and $q = j$ or $j+1$. By (2.3), we have

$$d(E^u(j), E^s(j)) = \frac{|\det(\vec{s}(j), \vec{u}(j))|}{\|\vec{s}(j)\|\|\vec{u}(j)\|}.$$

Let $\delta = d(\sigma(J_{a,b}), E)$ and $M = \sup_{j\in Z}\{|a_j|, |b_j|, |E-b_j|, |E|\}$. By (6.8) and (6.6), there exists $C = C(\delta, M) > 0$, such that $C^{-1} < \|\vec{s}(j)\| < C$ and $C^{-1} < \|\vec{u}(j)\| < C$ for all $j \in \mathbb{Z}$. Thus, to show that $d(E^u(j), E^s(j)) > \tau$, we can instead show that

$$|\det(\vec{s}(j), \vec{u}(j))| > c,$$

for some $c > 0$. For the proof of this lemma, we will revisit the cases in Lemma 14.

Case 1. Assume $a_j \neq 0$ for all $j \in \mathbb{Z}$.

Let $G(E; n, m) = (J_{a,b} - E)^{-1}(n, m)$ be the Green's function of $J_{a,b}$ at E. Then, by (6.5),

$$g_j(n) = G(E; j, n) =< \delta_j, (J_{a,b} - E)^{-1}\delta_n > . \tag{6.12}$$

On the other hand, by case 1 of Lemma 14, for all $j \in \mathbb{Z}$

$$g_j(n) = \begin{cases} \phi_j^s(n), & j \le n, \\ \phi_j^u(n), & j \ge n. \end{cases}$$

Recall that ϕ_p^s and ϕ_j^s are linearly dependent; ϕ_q^u and ϕ_j^u are linearly dependent. Hence we can rewrite $g_j(n)$ as follows

$$g_j(n) = \frac{1}{W(\phi_p^s, \phi_q^u)} \begin{cases} \phi_p^s(n)\phi_q^u(j), & j \le n, \\ \phi_p^s(j)\phi_q^u(n), & j \ge n, \end{cases} \tag{6.13}$$

where

$$W(\phi_p^s, \phi_q^u)(n) = a(n) \det \begin{pmatrix} \phi_p^s(n+1) & \phi_q^u(n+1) \\ \phi_p^s(n) & \phi_q^u(n) \end{pmatrix}$$

is the modified Wronskian of ϕ_p^s and ϕ_q^u which is independent of n (see [16]). Indeed, for any m,

$$\begin{aligned}
W(\phi_p^s, \phi_q^u)(m+1) &= a_{m+1} \det \begin{pmatrix} \phi_p^s(m+2) & \phi_q^u(m+2) \\ \phi_p^s(m+1) & \phi_q^u(m+1) \end{pmatrix} \\
&= a_{m+1} \det \left[A^E(m+1) \begin{pmatrix} \phi_p^s(m+1) & \phi_q^u(m+1) \\ \phi_p^s(m) & \phi_q^u(m) \end{pmatrix} \right] \\
&= a_{m+1} \det(A^E(m+1)) \det \begin{pmatrix} \phi_p^s(m+1) & \phi_q^u(m+1) \\ \phi_p^s(m) & \phi_q^u(m) \end{pmatrix} \\
&= a_{m+1} \frac{a_m}{a_{m+1}} \det \begin{pmatrix} \phi_p^s(m+1) & \phi_q^u(m+1) \\ \phi_p^s(m) & \phi_q^u(m) \end{pmatrix} \\
&= a_m \det \begin{pmatrix} \phi_p^s(m+1) & \phi_q^u(m+1) \\ \phi_p^s(m) & \phi_q^u(m) \end{pmatrix} \\
&= W(\phi_p^s, \phi_q^u)(m).
\end{aligned}$$

Now, combining (6.12) and (6.13), and setting $m = j-1$ in $W(\phi_p^s, \phi_q^u)(m)$, we get

$$\begin{aligned}
|\phi_p^s(j)\phi_q^u(j)| &= |W(\phi_p^s, \phi_q^u)(j-1)| \cdot |g_j(j)| \\
&= \left| \det \begin{pmatrix} \phi_p^s(j) & \phi_q^u(j) \\ \phi_p^s(j-1) & \phi_q^u(j-1) \end{pmatrix} \right| \cdot |a_{j-1} g_j(j)| \\
&= |a_{j-1} \det(\vec{s}(j), \vec{u}(j))| \cdot | < \delta_j, (J_{a,b} - E)^{-1} \delta_j > | \qquad (6.14) \\
&\leq |a_{j-1} \det(\vec{s}(j), \vec{u}(j))| \|(J_{a,b} - E)^{-1}\| \\
&= |a_{j-1} \det(\vec{s}(j), \vec{u}(j))| \frac{1}{d(\sigma(J_{a,b}), E)} \\
&\leq \frac{M}{\delta} |\det(\vec{s}(j), \vec{u}(j))|,
\end{aligned}$$

where the last equality comes from (6.3). We also have

$$\begin{aligned}|\phi_p^s(j-1)\phi_q^u(j-1)| &= |W(\phi_p^s,\phi_q^u)(j-1)|\cdot|g_{j-1}(j-1)| \\ &= |a_{j-1}\det(\vec{s}(j),\vec{u}(j))|\cdot|<\delta_{j-1},(J_{a,b}-E)^{-1}\delta_{j-1}>| \\ &\leq \frac{M}{\delta}|\det(\vec{s}(j),\vec{u}(j))|.\end{aligned} \tag{6.15}$$

Now, we want to show that $|\det(\vec{s}(j),\vec{u}(j))| > c$, for some $c>0$. Assume $C=C(M,\delta)>0$ is sufficiently large.

If $|\phi_p^s(j)\phi_q^u(j)| \geq C^{-4}$, then (6.14) implies the desired estimate. So we assume that $|\phi_p^s(j)\phi_q^u(j)| < C^{-4}$.

If $|\phi_p^s(j)| < C^{-2}$ and $|\phi_q^u(j)| < C^{-2}$, then $\|\vec{s}(j)\| > C^{-1}$ and $\|\vec{u}(j)\| > C^{-1}$ imply that $|\phi_p^s(j-1)| > \frac{C^{-1}}{2}$ and $|\phi_q^u(j-1)| > \frac{C^{-1}}{2}$ which together with (6.15) implies the desired estimate.

If $|\phi_p^s(j)| \geq C^{-2}$ and $|\phi_q^u(j)| < C^{-2}$. Then $|\phi_q^u(j-1)| > \frac{C^{-1}}{2}$. Hence, if $|\phi_p^s(j-1)| \geq \frac{C^{-2}}{2}$, then (6.15) implies the desired estimate.

If $|\phi_p^s(j-1)| \leq \frac{C^{-2}}{2}$. Then we have

$$\begin{aligned}|\det(\vec{s}(j),\vec{u}(j))| &= |\phi_p^s(j)\phi_q^u(j-1)-\phi_q^u(j)\phi_p^s(j-1)| \\ &\geq |\phi_p^s(j)\phi_q^u(j-1)|-|\phi_q^u(j)\phi_p^s(j-1)| \\ &\geq C^{-2}\frac{C^{-1}}{2}-C^{-2}\frac{C^{-2}}{2} \\ &\geq \frac{C^{-3}}{4}.\end{aligned}$$

In all the possible cases, we have $|\det(\vec{s}(j),\vec{u}(j))| > c > 0$, hence, for all $j\in\mathbb{Z}$,

$$\inf_{j\in\mathbb{Z}} d(E^s(j),E^u(j)) > \tau.$$

Case 2. Assume $a_j = 0$ for some $j \in \mathbb{Z}$ and $a_n \neq 0$ for all $n \neq j$.

Recall from Lemma (14), that in this case we can decompose $J_{a,b}$ into the direct sum

$$J_{a,b} = J_{a,b(-,j+1)} \oplus J_{a,b(j,+)}$$

on $\ell^2(-\infty, j] \oplus \ell^2(j, +\infty)$, and rewrite $(J_{a,b} - E)^{-1}$ as follows

$$\begin{array}{c} \\ j-2 \\ j-1 \\ j \\ j+1 \\ j+2 \\ j+3 \\ \\ \end{array}
\begin{array}{c} \begin{array}{cccc} j-1 & j & j+1 & j+2 \end{array} \\
\begin{pmatrix}
\ddots & & & & \\
& \phi^-_{j-1}(j-2) & \phi^-_j(j-2) & 0 & 0 \\
& \phi^{s(-)}_{j-1}(j-1) & \phi^-_j(j-1) & 0 & 0 \\
& \phi^s_{j-1}(j) & \phi^{s(-)}_j(j) & 0 & 0 \\
& 0 & 0 & \phi^{u(+)}_{j+1}(j+1) & \phi^u_{j+2}(j+1) \\
& 0 & 0 & \phi^+_{j+1}(j+2) & \phi^{u(+)}_{j+2}(j+2) \\
& 0 & 0 & \phi^+_{j+1}(j+3) & \phi^+_{j+2}(j+3) \\
& & & & & \ddots
\end{pmatrix} \end{array}$$

We consider $J_{a,b(-,j+1)}$ first. In this case, when $k \leq j$, we have

$$g_k(n) = \begin{cases} \phi^-_k(n), & n \leq k, \\ \phi^s_k(n), & k \leq n \leq j, \\ 0, & n > j. \end{cases}$$

Note that ϕ^s_p and ϕ^s_k are linearly dependent and ϕ^-_q and ϕ^-_k are linearly dependent. To find Green's function, we first assume that

$$g_k(n) = G(E; k, n) = \begin{cases} c_k \phi^-_q(k)\phi^s_p(n), & k \leq n \leq j, \\ c_k \phi^-_q(n)\phi^s_p(k), & n \leq k \leq j, \end{cases}$$

We want to find c_k. For all $k < j$,

$$a_{k-1}g_k(k-1) + a_k g_k(k+1) + (b_k - E)g_k(k) = 1.$$

Hence,

$$c_k a_{k-1}\phi_q^-(k-1)\phi_p^s(k) + c_k a_k \phi_q^-(k)\phi_p^s(k+1) + c_k(b_k - E)\phi_q^-(k)\phi_p^s(k) = 1.$$

For $k < j$ we also have that

$$c_k a_{k-1}\phi_q^-(k-1)\phi_p^s(k) + c_k(b_k - E)\phi_q^-(k)\phi_p^s(k) = -c_k a_k \phi_p^s(k)\phi_q^-(k+1).$$

Hence,

$$c_k = \frac{1}{a_k(\phi_p^s(k+1)\phi_q^-(k) - \phi_p^s(k)\phi_q^-(k+1))} = \frac{1}{W(\phi_p^s, \phi_q^-)(k)}$$

where

$$W(\phi_p^s, \phi_q^-)(k) = a(k)\det\begin{pmatrix} \phi_p^s(k+1) & \phi_q^-(k+1) \\ \phi_p^s(k) & \phi_q^-(k) \end{pmatrix}$$

is the modified Wronskian of ϕ_p^s and ϕ_q^- which is again independent of $k < j$. Since such c_k works for all $k < j$, then by symmetry of Green's function,

$$g_{j-1}(j) = c_{j-1}\phi_q^-(j-1)\phi_p^s(j) = c_j\phi_p^s(j)\phi_q^-(j-1) = g_j(j-1)$$

Hence, $c_j = c_{j-1} = c_k$ for all $k < j$ and so the Green's function becomes

$$G(E;k,n) = \frac{1}{\widetilde{W}(\phi_p^s, \phi_q^-)(k)}\begin{cases} \phi_q^-(k)\phi_p^s(n), & k \le n \le j, \\ \phi_q^-(n)\phi_p^s(k), & n \le k \le j, \end{cases}$$

where

$$\widetilde{W}(\phi_p^s, \phi_q^-)(k) = \begin{cases} W(\phi_p^s, \phi_q^-)(k), & k < j, \\ W(\phi_p^s, \phi_q^-)(k-1), & k = j, \end{cases}$$

is independent of $k \leq j$.

Following the same steps as in case 1, we get that

$$|\phi_p^s(k)\phi_q^u(k)| \leq \frac{M}{\delta}|\det(\vec{s}(k), \vec{u}(k))|$$

and

$$|\phi_p^s(k-1)\phi_q^u(k-1)| \leq \frac{M}{\delta}|\det(\vec{s}(k), \vec{u}(k))|$$

This again implies that $|\det(\vec{s}(k), \vec{u}(k))| > c$ for $k \leq j$. Similarly, we get that $|\det(\vec{s}(m), \vec{u}(m))| > c$ for $m \geq j+2$. Now, at the step $j+1$, by (6.10) and (6.11) we have $E^s(j+1) = \operatorname{span}\left\{\binom{0}{1}\right\}$ and $E^u(j+1) = \operatorname{span}\left\{\binom{1}{0}\right\}$. Clearly, we then have

$$d(E^s(j+1), E^u(j+1)) = 2.$$

Hence, for all $j \in \mathbb{Z}$,

$$\inf_{j \in \mathbb{Z}} d(E^s(j), E^u(j)) > \tau.$$

Case 3. Assume $a_j = 0$ for $j = \{j_1, j_2\}$, $j_1 < j_2$ and $a_n \neq 0$ for all $n \neq j_1, j_2$.

In this case, $(J_{a,b}-E)^{-1}$ becomes

$$\begin{array}{c} \\ \\ j_1-1 \\ j_1 \\ j_1+1 \\ \dots \\ j_2 \\ j_2+1 \\ j_2+2 \\ \\ \end{array}
\begin{array}{c}
\begin{array}{ccccc} j_1 & j_1+1 & \dots & j_2 & j_2+1 \end{array} \\
\begin{pmatrix}
\ddots & & & & & \\
& \phi^-_{j_1}(j_1-1) & 0 & \dots & 0 & 0 \\
& \phi^{-(s)}_{j_1}(j_1) & 0 & \dots & 0 & 0 \\
& 0 & \widetilde{\phi}^{u(s)}_{j_1+1}(j_1+1) & \dots & \widetilde{\phi}^{u}_{j_2}(j_1+1) & 0 \\
& 0 & \vdots & \dots & \vdots & 0 \\
& 0 & \widetilde{\phi}^{s}_{j_1+1}(j_2) & \dots & \widetilde{\phi}^{s(u)}_{j_2}(j_2) & 0 \\
& 0 & 0 & \dots & 0 & \phi^{+(u)}_{j_2+1}(j_2+1) \\
& 0 & 0 & \dots & 0 & \phi^{+}_{j_2+1}(j_2+2) \\
& & & & & \ddots
\end{pmatrix}
\end{array}$$

By case 2, $|\det(\vec{s}(j),\vec{u}(j))|>c$ for all $j\le j_1$ and $j>j_2+1$. Moreover, for $j=j_1+1$ and $j=j_2+1$, $d(E^s(j),E^u(j))=2$. The case when $j_1+1<j\le j_2$ needs to be divided into the following two subcases:

Case 3a. Assume $j_2=j_1+2$.

Since $a_{j_1}=a_{j_1+2}=0$, then the eigenvalues of $J_{[j_1+1,j_1+2]}$ are eigenvalues of $J_{a,b}$. Let E_1 and E_2 be the two eigenvalues of $J_{[j_1+1,j_1+2]}$. Then we have $\delta\le|E-E_i|\le 2M$ for $i=1,2$. Note it holds that

$$(J_{[j_1+1,j_1+2]}-E)^{-1}=\begin{pmatrix} g_{j_1+1}(j_1+1) & g_{j_1+2}(j_1+1) \\ g_{j_1+1}(j_1+2) & g_{j_1+2}(j_1+2) \end{pmatrix}.$$

Then,

$$|\det(\vec{s}(j_1+2),\vec{u}(j_1+2))| = \left|\det\begin{pmatrix} g_{j_1+1}(j_1+1) & g_{j_1+2}(j_1+1) \\ g_{j_1+1}(j_1+2) & g_{j_1+2}(j_1+2)\end{pmatrix}\right|$$
$$= |\det(J_{[j_1+1,j_1+2]}-E)^{-1}|$$
$$= |\det(J_{[j_1+1,j_1+2]}-E)|^{-1}$$
$$= |(E-E_1)(E-E_2)|^{-1}$$
$$\geq \frac{1}{4M^2}.$$

Case 3b. Assume $j_2 > j_1+2$.

This case is similar to case 1. Indeed, by case 3 of Lemma 14, for all $j_1+1 \leq k \leq j_2$,

$$g_k(n) = \begin{cases} \widetilde{\phi}_k^s(n), & k \leq n \leq j_2, \\ \widetilde{\phi}_k^u(n), & j_1+1 \leq n \leq k, \\ 0, & \text{otherwise.}\end{cases}$$

Then, for all $j_1+1 \leq k \leq j_2$ we can define

$$G(E;k,n) = \frac{1}{\widetilde{W}(\widetilde{\phi}_p^s,\widetilde{\phi}_q^u)(k)}\begin{cases} \widetilde{\phi}_p^s(k)\widetilde{\phi}_q^u(n), & j_1+1 \leq n \leq k, \\ \widetilde{\phi}_p^s(n)\widetilde{\phi}_q^u(k), & k \leq n \leq j_2,\end{cases}$$

where

$$\widetilde{W}(\widetilde{\phi}_p^s,\widetilde{\phi}_q^u)(k) = \begin{cases} W(\widetilde{\phi}_p^s,\widetilde{\phi}_q^u)(k), & j_1+1 \leq k < j_2, \\ W(\widetilde{\phi}_p^s,\widetilde{\phi}_q^u)(k-1), & k = j_2,\end{cases}$$

is the modified Wronskian of $\widetilde{\phi}_p^s$ and $\widetilde{\phi}_q^u$ which is independent of $j_1+1 \leq k \leq j_2$. Following the same logic as in case 1, we can proceed to get

$$|\widetilde{\phi}_p^s(k)\widetilde{\phi}_q^u(k)| \leq \frac{M}{\delta}|\det(\vec{s}(k),\vec{u}(k))|$$

and

$$|\widetilde{\phi}_p^s(k-1)\widetilde{\phi}_q^u(k-1)| \leq \frac{M}{\delta}|\det(\vec{s}(k), \vec{u}(k))|$$

for $j_1 + 1 < k \leq j_2$. Then we get that $|\det(\vec{s}(k), \vec{u}(k))| > c$ for $j_1 + 1 < k \leq j_2$. Hence, for all $j \in \mathbb{Z}$,

$$\inf_{j \in \mathbb{Z}} d(E^s(j), E^u(j)) > \tau.$$

By the same logic we used in case 4 of Lemma 14, it is straightforward to extend case 3 to allow $a_j = 0$ for all $j \in K$, where $K \subseteq \mathbb{Z}$. Hence, we get the statement of the lemma. ∎

Lemma 16 concludes the proof that if $E \in \rho(J_{a,b})$, then $B^E(j)$ admits dominated splitting.

Chapter 7

Conclusions

In this paper we have shown that the spectrum of Jacobi operator $J_{a,b} : \ell^2(\mathbb{Z}, \mathbb{R}) \to \ell^2(\mathbb{Z}, \mathbb{R})$ can be identified by those energies $E \in \mathbb{R}$ whose cocycle map $B^E \in \mathrm{M}(2, \mathbb{R})$ does not admit dominated splitting, i.e.

$$\sigma(J_{a,b}) = \{E \in \mathbb{R} : B^E \notin \mathcal{DS}\}.$$

This result is the first known version of Johnson's theorem for possibly singular Jacobi operator defined by sequences. The result is stated for any Jacobi operator defined by sequences $a = (a_n)_{n\in\mathbb{Z}} \in \ell^\infty(\mathbb{Z}, \mathbb{R})$, and $b = (b_n)_{n\in\mathbb{Z}} \in \ell^\infty(\mathbb{Z}, \mathbb{R})$. However, the result can be extended to allow sequence $(a_n)_{n\in\mathbb{Z}}$ to assume complex values, i.e. $a = (a_n)_{n\in\mathbb{Z}} \in \ell^\infty(\mathbb{Z}, \mathbb{C})$. In this case, the correct way to define Jacobi operator $J_{a,b} : \ell^2(\mathbb{Z}, \mathbb{C}) \to \ell^2(\mathbb{Z}, \mathbb{C})$ is as follows:

$$(J_{a,b}\psi)(n) = \overline{a_{n-1}}\psi(n-1) + a_n\psi(n+1) + b_n\psi(n), \tag{7.1}$$

where $\psi = (\psi_n)_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z}, \mathbb{C})$, $a = (a_n)_{n\in\mathbb{Z}} \in \ell^\infty(\mathbb{Z}, \mathbb{C})$, and $b = (b_n)_{n\in\mathbb{Z}} \in \ell^\infty(\mathbb{Z}, \mathbb{R})$. In this case, $J_{a,b}$ is Hermitian, however, the associated tridiagonal matrix is not symmetric.

This posses some difficulties, since we used the symmetry in several places in this paper, for example when defining the invariant subspaces, E^u and E^s of $J_{a,b}$ in Section 6. Nevertheless, in [1] we work around this difficulty and prove our version of Johnson's theorem for Jacobi operators defined as in (7.1).

Another future development, as discussed earlier in this paper, can be to lift and extend our result for the dynamically defined Jacobi operators. A version of Johnson's theorem for dynamically defined Jacobi operators was proven in [10] with many restrictions to the base dynamics. However, in [1] we show that topological transitivity of the base dynamics should be the only restriction.

Indeed, one directions is clear, since if a dynamically defined cocycle (T, B) admits dominated splitting, then

$$B(T^{(\cdot)}\omega) : \mathbb{Z} \to \mathrm{M}(2, \mathbb{C})$$

is a sequence that satisfies all the conditions of Definition 6, meaning that it admits dominated splitting for each $\omega \in \Omega$.

To show the other direction, notice that in general, for a continuous cocycle map $A : \Omega \to \mathrm{M}(2, \mathbb{C})$, one may not be able to lift the dominated splitting from sequences (i.e. dominated splitting of $A(T^{(\cdot)}\omega) : \mathbb{Z} \to \mathrm{M}(2, \mathbb{C})$ for all $\omega \in \Omega$) to dominated splitting of cocycle (T, A). This is because one may lose the uniformity of certain constants. However, when the base dynamics is topologically transitive, it is possible to do so.

In particular, Theorem 6 in [18] indicates that for Jacobi operators we have the following:

If $\overline{\mathrm{Orb}(\omega_0)} = \Omega$, *then* $\sigma(J_\omega) \subset \sigma(J_{\omega_0})$ *for all* $\omega \in \Omega$.

In particular, if $E \in \rho(J_{\omega_0})$, then $d(E, \sigma(J_\omega)) > \delta > 0$ for all $\omega \in \Omega$. Even though the result is stated for Schrödinger operators, the proof works for Jacobi operators as well. Then, using the strategy of [18, Section 3.3.2], we can make all the estimates showing conditions (1)-(3) of Definition 6 depend only on $d(E, \sigma(J_{a,b}))$. In particular, those constants can be made uniform since they depend only on $\delta > 0$. This allows us to go from Theorem 5 to the following dynamical version:

Consider the dynamically defined Jacobi operators J_ω, $\omega \in \Omega$. Assume that T be a topologically transitive and let ω_0 be that $\overline{\mathrm{Orb}(\omega_0)} = \Omega$. Then

$$\rho(\omega_0) = \{E \in \mathbb{C} : (T, B^E) \in \mathcal{DS}\}.$$

In some sense, the statement above extends Theorem 3 to the most generality.

As we have seen, our version of Johnson's theorem is not only an innovative independent result on its own, it is also the stepping stone to perhaps the best version of Johnson's theorem for dynamically defined Jacobi operators one can hope to obtain.

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