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APPROXIMATE CONSTITUTIVE EQUATIONS FOR THE SHORT-TIME BEHAVIOR OF NONLINEAR VISCOELASTIC MATERIALS

by

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Summary

An approximation for nonlinear viscoelasticity over short time ranges, recently published by Huang and Lee, is extended to include large deformations and simplified by the dropping of insignificant terms. An alternative approximation for short times is developed systematically as the counterpart to Coleman and Noll's approximation for slow deformations. This method is applied to the study of acceleration waves.

1. Introduction

A few years ago Coleman and Noll [1,2,3] established some important theorems on the asymptotic behavior of nonlinear viscoelastic ("simple") materials. Using the axiom of fading memory, they showed that for slow deformations the behavior can be approximated by constitutive equations of the Rivlin-Ericksen type, tending to Newtonian viscosity in the limit, while for deformations (relative to the "present" configuration) which, in the recent past, are <u>infinitesimal</u>, the behavior is characterized by linear viscoelasticity. The latter approximation is also valid for large deformations if they take place slowly, but is superseded by the former.

A considerable amount of work in viscoelasticity is concerned with deformations that vary rapidly and by finite (though small) amounts. Constitutive equations for small finite deformations were presented by Pipkin [4], on the basis of the work of Green and Rivlin [5], in the form of multiple-integral expansions for functionals possessing sufficient continuity. In order to study the effects of rapid variations, Huang and Lee [6] presented an approximation to Pipkin's equations which is, in form, a special case of the "finite linear viscoelasticity" of Coleman and Noll [1]. This note will discuss the validity of Huang and Lee's approximation, present an alternative, and show some applications.

While all the references cited above [1-6] work with three-dimensional deformations, it is sufficient for the purposes of the present discussion to treat the situation characterized by one stress variable σ and one deformation variable ϵ, the former being a functional of the latter. The extension to the general case is made by a suitable mapping onto the space of second-rank tensors, providing that objectivity requirements are observed. Such a procedure may by now be regarded as elementary.

2. Discussion and Generalization of Huang and Lee's Approximation

If the functional mentioned above is continuous, then, to any desired accuracy, we have

$$\sigma = \sum_{n=1}^{N} S_n \tag{1}$$

with

$$S_n = \int_0^t \ldots \int_0^t K_n\,(t_1,\ldots t_n)\dot{\epsilon}(t-t_1)\,\ldots\dot{\epsilon}(t-t_n)\,dt_1\ldots dt_n \tag{2}$$

where t is the time, with the origin so chosen that $\epsilon=0$ for $t\leq 0$. The kernels $K_n(\ldots)$ are symmetric functions of their arguments.

The first term S_1 is the familiar relaxation integral of linear visco-elasticity. The next term,

$$S_2 = \int_0^t \int_0^t K_2(t_1,t_2)\dot{\epsilon}(t-t_1)\dot{\epsilon}(t-t_2)\,dt_1 dt_2,$$

may be integrated by parts to yield, after Huang and Lee (with ϵ written for $\epsilon(t)$),

$$S_2 = K_2(0,0)\,\epsilon^2 + 2\epsilon \int_0^t \frac{\partial K_2}{\partial t_1}\,(t_1,0)\,\epsilon\,(t-t_1)\,dt_1 + R_2, \tag{3}$$

where

$$R_2 = \int_0^t \int_0^t \frac{\partial^2 K_2}{\partial t_1 \partial t_2}\,(t_1,t_2)\epsilon(t-t_1)\epsilon(t-t_2)\,dt_1 dt_2.$$

Let

$$A = \max_{0\le t_1, t_2 \le t} \left| \frac{\partial^2 K_2}{\partial t_1 \partial t_2} \right| ,$$

$$M = \max_{0\le t_1 \le t} \left| \epsilon(t_1) \right| ;$$

then

$$\left| R_2 \right| \le AM^2 t^2 .$$

Similarly,

$$S_3 = K_3 \ (0,0,0) \ \epsilon^3$$

$$+ \ 3 \ \epsilon^2 \ {}_o\int^t \ \frac{\partial K_3}{\partial t_1} \ (t_1,0,0)\epsilon(t-t_1) \ dt_1 + R_3 \tag{4}$$

where

$$R_3 = 3 \ \epsilon \ {}_o\int^t {}_o\int^t \ \frac{\partial^2 K_3}{\partial t_1 \partial t_2} \ (t_1, t_2, 0)\epsilon(t-t_1)\epsilon(t-t_2) \ dt_1 dt_2$$

$$+ \ {}_o\int^t \ \frac{\partial^3 K_3}{\partial t_1 \partial t_2 \partial t_3} \ (t_1, t_2, t_3)\epsilon(t-t_1)\epsilon(t-t_2)\epsilon(t-t_3) dt_1 dt_2 dt_3 .$$

Let

$$B = \max_{0\le t_1, t_2 \le t} \left| \frac{\partial^2 K_3}{\partial t_1 \partial t_2} \right|_{t_3=0} ,$$

$$C = \max_{0\le t_1, t_2, t_3 \le t} \left| \frac{\partial^3 K_3}{\partial t_1 \partial t_2 \partial t_3} \right| ;$$

then

$$\left| R_3 \right| \le M^3 \ (3 \ Bt^2 + Ct^3).$$

The preceding analysis completes, in effect, Huang and Lee's approximation. But it can be readily seen that, for any n,

$$S_n = K_n\ (0,\ldots,0)\epsilon^n + n\ \epsilon^{n-1} \int_0^t \frac{\partial K_n}{\partial t_1}\ (t_1,0,\ldots,0)\epsilon(t-t_1)dt_1+R_n, \tag{5}$$

with

$$R_n = 0\ (M^n\ t^2)$$

as

$M,\ t \to 0$. Hence

$$\sigma = F(\epsilon) + \int_0^t G(\epsilon,t_1)\epsilon(t-t_1)\ dt_1 + R, \tag{6}$$

where

$$F(\epsilon) = \sum_{n=1}^{N} K_n\ (0,\ldots,0)\ \epsilon^n,$$

$$G(\epsilon,t_1) = \sum_{n=1}^{N} n\ \frac{\partial K_n}{\partial t_1}\ (t_1,0,\ldots,0)\ \epsilon^{n-1},$$

$$R = \sum_{n=2}^{N} R_n.$$

with

$$R = 0\ (t^2)$$

as $t \to 0$.

With R neglected, Eq. (6) is formally identical with the one-dimensional equation of finite linear viscoelasticity [2].

I will now show that, just as in the case of slow deformations, finite linear viscoelasticity is superseded by the Rivlin-Ericksen approximation, with the constants related to integrals of the relaxation function [3,7], so in the case of rapid deformations the generalized Huang-Lee approximation, Eq. (6), may in turn be simplified. Let us define

$$\epsilon_1(t) = \int_0^t \epsilon(t_1)dt_1$$

and by recursion,

$$\epsilon_{n+1}(t) = (n+1) \int_0^t \epsilon_n(t_1)dt_1 \ ;$$

hence by definition $\epsilon_o \equiv \epsilon$. Also,

$$\epsilon_n = \int_0^t t_1^{\,n} \, \dot{\epsilon}\,(t-t_1)\, dt_1 .$$

Clearly,

$$|\epsilon_n| \leq Mt^n .$$

Let us consider, now, the second term on the right-hand side of Eq. (5). Integration by parts leads to

$$n\epsilon^{n-1} \int_0^t \frac{\partial K_n}{\partial t_1} (t_1,0,\ldots,0)\epsilon(t-t_1)dt_1$$

$$= \ n\epsilon^{n-1} \ \epsilon_1(t) \ \frac{\partial K_n}{\partial t_1} \ (0,\ldots,0) + Q_n ,$$

with

$$Q_n = n\epsilon^{n-1} \int_0^t \frac{\partial^2 K_n}{\partial t_1^2} (t_1, 0, \ldots 0)\epsilon_1 (t-t_1) dt_1.$$

Letting

$$D_n = \max_{0 \le t_1 \le t} \left| \frac{\partial^2 K_n}{\partial t_1^{\,2}} (t_1, 0, \ldots, 0) \right| ,$$

we find

$$| Q_n | \le n D_n |\epsilon|^{n-1} M t^2;$$

hence

$$Q_n = 0 \,(M^n t^2)$$

exactly like R_n. There being no a priori reason to suppose that $\frac{\partial^2 K_n}{\partial t_1^{\,2}}$ differs substantially in order of magnitude from $\frac{\partial^2 K_n}{\partial t_1 \partial t_2}$, one should neglect Q_n if one neglects R_n. Equation (6), therefore, reduces to

$$\sigma = F(\epsilon) + G_1(\epsilon)\epsilon_1 + 0(t^2), \tag{7}$$

where $G_1(\epsilon) = G(\epsilon, 0)$. On defining the kernel "traces",

$$\overline{K}_n(t) = K_n(t, \ldots, t),$$

we also have

$$F(\epsilon) = \sum_{n=1}^{N} \overline{K}_n(0)\epsilon^n,$$

$$G_1(\epsilon) = \sum_{n=1}^{N} \bar{K}_n'(0)\epsilon^{n-1}.$$

The effect of the approximation (7) is that of the kernels $K_n(\ldots)$ the only significant parts are those which are linear in the arguments. In their calculations, Huang and Lee used quadratic and exponential kernels; we see now that the contributions of the nonlinear terms are of the same order of magnitude as the neglected terms.

3. A Systematic Short-Time Approximation

Let us expand all the kernels as power series

$$K_n(t_1,\ldots t_n) = \sum_{k_1=0}^{\infty} \ldots \sum_{k_n=0}^{\infty} K^{(n)}_{k_1\ldots k_n} t_1^{k_1} \ldots t_n^{k_n} . \tag{8}$$

The coefficients $K^{(n)}_{k_1 \ldots k_n}$ are symmetric with respect to their subscripts. Inserting into Eq. (2), we obtain

$$S_n = \sum_{k_1=0}^{\infty} \ldots \sum_{k_n=0}^{\infty} K^{(n)}_{k_1\ldots k_n} \epsilon_{k_1} \ldots \epsilon_{k_n} \tag{9}$$

The contribution of any one term is $O(M^n t^{k_1+\ldots . k_n})$.

It is now possible to gather all terms of any desired order in M and t, and to approximate by lopping off those of orders higher than a desired one (in M or t). For example, if terms up to t^2 are retained we have

$$\sigma = \sum_{n=1}^{N} K^{(n)}_{0\ldots0} \epsilon^n + \sum_{n=1}^{N} n\, K^{(n)}_{0\ldots01} \epsilon^{n-1} \epsilon_1$$

$$+ \sum_{n=1}^{N} nK^{(n)}_{0\ldots02} \epsilon^{n-1}\epsilon_2 + \sum_{n=2}^{N} \frac{n(n-1)}{2} K^{(n)}_{0\ldots011} \epsilon^{n-2} \epsilon_1^{\,2}$$

$$+ O(t^3). \tag{10}$$

The first two terms are equivalent to Eq. (7); the next two represent the next approximation.

If ϵ is represented by a polynomial,

$$\epsilon(t) = \sum_m a_m t^m ,$$

then

$$\epsilon_k = \sum_m \frac{k!m!}{(k+m)!} a_m t^{k+m} .$$

Consequently the stress may be represented by a power series,

$$\sigma(t) = \sum_m b_m t^m .$$

The first three coefficients may be obtained from Eq. (10), to wit:

$$b_o = \sum_{n=1}^{N} K^{(n)}_{o....o} a_o^n ,$$

$$b_1 = \sum_{n=1}^{N} n (K^{(n)}_{0...0} a_0^{n-1} a_n + K_{0....01} a_0^n),$$

$$b_2 = \sum_{n=1}^{N} n [K^{(n)}_{0...0} (a_0^{n-1} a_2 + \frac{n-1}{2} a_0^{n-2} a_1^2)$$

$$+ (n-\frac{1}{2}) K^{(n)}_{0...01} a_0^{n-1} a_1$$

$$+ (\frac{1}{2} K^{(n)}_{0...02} + \frac{n-1}{2} K^{(n)}_{0\ ...011}) a_0^n].$$

These expressions are particularly simple if the strain history is continuous, i.e., $a_0 = 0$:

$$b_0 = 0,$$

$$b_1 = K_0^{(1)} a_1, \tag{11}$$

$$b_2 = K_0^{(1)} a_2 + K^{(2)}_{00} a_1^2 + \frac{1}{2} K_1^{(1)} a_1 .$$

These expressions will be used in the following application.

4. Application to Wave Propagation

We consider the one-dimensional equations of wave propagation,

$$\frac{\partial \sigma}{\partial x} = \rho_o \frac{\partial v}{\partial t}, \quad \frac{\partial v}{\partial x} = \frac{\partial \epsilon}{\partial t}, \tag{12}$$

v being the velocity, in the material half-space $x > 0$ with uniform rest density ρ_o, and the boundary condition

$$v = \sum_{m=1}^{\infty} C_m t^m, \quad t > 0, \ x = 0.$$

Since the velocity input is continuous, we expect it to be propagated, at least for a certain distance, as an acceleration wave moving with material speed U. Hence we assume solutions in the form

$$\epsilon = \sum_{m=1}^{\infty} a_n(x)\tau^m ,$$

$$\sigma = \sum_{m=1}^{\infty} b_m(x)\tau^m ,$$

$$v = \sum_{m=1}^{\infty} c_m(x)\tau^m ,$$

with $\tau = t - x/U$, valid in some region $0 < \tau < T(x)$, $0 < x < X$. We are not concerned with the limits of validity $T(x), X$. Assuming term-by-term differentiability, we obtain, from Eqs. (12),

$$-\frac{1}{U}\, b_1 = \rho_o\, c_1,$$

$$-\frac{1}{U}\, c_1 = a_1,$$

$$-\frac{2}{U}\, b_2 + b_1' = 2\, \rho_o\, c_2,$$

$$-\frac{2}{U}\, c_2 + c_1' = 2\, a_2.$$

Since b_1 and b_2 are given by Eqs. (11), we obtain, after elimination,

$$(\rho_o\, U^2 - K_0^{(1)})\, c_1 = 0,$$

finding the material wave speed to be given, as expected, by

$$U = (K_0^{(1)}/\rho\,)^{1/2}\;;$$

and

$$K^{(1)}\, c_1' - \frac{1}{2}\, K_1^{(1)}\, c_1 + \frac{K_{00}^{(2)}}{U}\, c_1^{\,2} = 0. \tag{13}$$

Equation (13), which governs the variation of the acceleration discontinuity c_1 as the wave progresses, has been derived by other means by Varley [8] and by Coleman and Gurtin [9]. Further results have been obtained elsewhere by the use of the present method [10].

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