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November 7, 1966

RECENT WORK ON REGGE POLES AND THE THREE-BODY PROBLEM*

(Lectures delivered at the Second Japanese Summer Institute of Theoretical Physics, Oiso, Japan)

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A. THREE-BODY N/D EQUATIONS

In my first talk I should like to construct three-body N/D equations, analogous to the familiar two-body N/D equations. We assume that all the singularities of a partial-wave amplitude, other than the right-hand unitarity cut are known, and we attempt to construct the amplitude therefrom.

Our object in constructing the N/D equations is not to perform numerical calculations with them as they stand. This alone would be a complicated numerical problem but, in addition, one does not know the left-hand cut explicitly. We shall not attempt here to obtain a self-consistent scheme for calculating the left- and right-hand cuts from crossing and unitarity, since no such a scheme, free from divergences and cut-off parameters, has been constructed even for the two-body problem. Our reasons for constructing N/D equations are mainly theoretical. They may possibly serve as a basis for simpler but cruder approximations which can be used to give an estimate of three-body effects. Further, it appears to be essential to treat resonances on a par with particles in two-body calculations if the results are to be at all accurate. Since a state consisting of a particle and a resonance is really a three-particle state, the three-body equations may be helpful in treating doubtful points in the equations for particle-resonance scattering.

Another application of three-body N/D equations is to the study of the complex J-plane in three-body systems. For this purpose it is unnecessary to know the left-hand and complex singularities, since the discontinuities across them can be proved to be holomorphic functions of J. At present we have not carried out the proof for complex J, as there are complications caused by the infinite number of helicity states which exist neither in the two-body problem for complex l or in the three-body problem for real J . In my next lecture I shall make some remarks on the three-body problem with complex J .

We may regard the three-body problem as analogous to a two-body problem with an infinite number of channels. However, there are complications in the three-body problem which do not exist in the two-body problem. Such complications are associated with disconnected diagrams in which two particles scatter which the third is unaffected. There will be unitarity discontinuities associated with the total energy and with all sub-energies, and we must choose our variables carefully if the unitarity equation is to have a form from which N/D equations can be constructed. We shall find that the kinematics associated with the correct variables are much simpler when the physical "Dalitz" regions retains its shape with increasing energy, as it does in the non-relativistic case, than when it does not. Since a system with a Dalitz region of fixed shape possesses all the essential features of the problem without the kinematical complications, we shall confine our discussion to such a system in this lecture.

Let us begin by writing down the many-channel two-body N/D equations, by analogy with which the three-body equations will be constructed. The equations are

$$N_{ij}(s) = F_{ij}(s) + \frac{1}{\pi} \int_R ds' \sum_{\ell} \frac{F_{i\ell}(s') - F_{i\ell}(s)}{s' - s} \, k_{\ell}(s') \, N_{\ell j}(s) \, , \qquad 1(a)$$

$$D_{ij}(s) = 1 - \frac{1}{\pi} \int_R ds' \, \frac{k_i(s') \, N_{ij}(s')}{s' - s} \qquad . \qquad 1(b)$$

The variables i, j and ℓ are the channel indices.

For the three-body problem, we shall begin by taking the Omnès variables. Besides the total angular momentum J there will be seven variables, which Omnès takes to be

(i) The total energy S

(ii) The partial energies s_1, s_2, s_1', s_2', where, for instance, $s_1 = -(p_2 + p_3)^2$.

(iii) The helicities M and M' of the initial and final states with respect to some body-centered axis.

The unprimed variables refer to the initial state, the primed variables to the final state. The third partial energy is not an independent variable, since it is given by the equation.

$$s_1 + s_2 + s_3 = s_1' + s_2' + s_3' = S + 3m^2 . \tag{2}$$

For simplicity we have taken all masses equal.

In constructing the N/D equation we shall take the variable S to be analogous to the variable s of the two-body problem, while s_1, s_2 and M will be analogous to the channel index i . Now, however, we have two continuous and one discrete channel indices.

The scattering amplitude will consist of a connected part A_c and three disconnected parts a_1, a_2 and a_3 as shown in Fig. 1:

$$A = A_c + \sum_{i=1}^{3} a_i . \tag{3}$$

The disconnected part a_i will contain a δ-function $\delta(s_i - s_i')$, so that

$$a_i = \tilde{a}_i \, \delta(s_i - s_i') \, . \qquad (4)$$

We may now write down the unitarity condition

$$\mathrm{Im}\,(A_c + \Sigma\ a_i) = k(S)\,(A_c^{*} A_c + \sum_i a_i^{*} A_c + \sum_i A_c^{*} a_i + \sum_{i,j} a_i^{*} a_j)\, . \qquad (5)$$

The function $k(S)$ is the kinematical factor. We have suppressed the integrations over the intermediate variables, so that the first term on the right of (5) is meant to be

$$\int ds_1'' \, ds_2'' \sum_{M''} A_c^{*}(S, J; s_1, s_2, M, s_1'', s_2'', M'') \times A_c(S, J; s_1'', s_2'', M'', s_1', s_2', M')\, . \qquad (6)$$

An integral such as (6) is always implied when products of scattering amplitudes are written down.

Equation (5), the unitarity equation, is straightforward. What we require for our N/D equations is the discontinuity equation, which involves more sophisticated concepts. We have to distinguish between the discontinuities in the total energy S and the six sub-energies s_i, s_i' . The usual rule relating a discontinuity to an imaginary part states that the change of the amplitude when the imaginary part of <u>all</u> <u>seven</u> variables changes sign is equal to the imaginary part. It might appear intuitively plausible that the discontinuity in S is equal to the first term on the right of (5), the discontinuity in s_i to the term $a_i^* A_c$ and the discontinuity in s_i' to the term $A_c^* a_i$. It has been shown, however, that this conjecture is not correct. One can indeed deform the contours of the intermediate-state integrations so that this correspondence is valid, but we would then have to integrate over complex values of the s''s, even when the initial- and final-state variables are in the physical region. As we wish to treat the s_i's as analogous to the channel indices of the two-body problem, we must restrict the integrations over them to the physical region.

We therefore cannot take the variable S as our dispersion variable and the s_i's as analogous to the channel indices of the two-body problem. In order to do this we would have to know the discontinuity of the amplitude in S, whereas we only know the sum of the discontinuities in all the energies. We can overcome this difficulty by replacing the s_i's and s_i''s by the variables

$$x_i = \frac{s_i - 4m^2}{S - 9m^2} , \quad x_i' = \frac{s_i' - 4m^2}{S - 9m^2} \tag{7}$$

If the x's are kept constant and positive and S is moved round the threshold in the complex plane, all the s_i's and s_i''s will also move round their thresholds. Thus the discontinuity in S with the x's kept constant is equal to the sum of the discontinuities in S and those in the s's and s''s, and it is given by the right-hand side of (6).

We therefore take our variables to be

$$S,\ J,\ x_1,\ x_2,\ M,\ x_1',\ x_2',\ M' .$$

Equation (2) rewritten in the terms of the $x's$ is

$$x_1 + x_2 + x_3 = x_1' + x_2' + x_3' = 1 \tag{8}$$

so that x_3 and x_3' are again not independent variables.

If we plot the three $x's$ in triangular co-ordinates, the physical Dalitz region has the form shown in Fig. 2. Within the triangle the values of the $x's$ range from 0 to 1. Near threshold the physical region is represented by the circle, as S is increased it expands eventually to fill the whole triangle at large energies. Unfortunately there is a difficulty associated with the change of shape of the Dalitz region, and the variables defined above are still not suited to our problem. The difficulty is associated with the disconnected terms in the unitarity equation. The range of integration in these terms is restricted by a δ-function in one of the s_i's, so that one has to integrate over a line such as AB in Fig. 2. If we now gradually decrease the value of S the physical region becomes smaller and the line AB shrinks, eventually becoming a point. If S is now decreased still farther, the analytic continuation of the integral would be along a path in the complex x_2-plane. Thus, even with our new variables, integrals over the intermediate variables of the unitarity equation have sometimes to be deformed into the complex plane. This is precisely the situation we are trying to avoid.

We therefore have to re-define variables in such a way that the shape of the Dalitz region does not depend on the energy. It is not difficult to find suitable variables, one can for instance take ratios of the magnitudes of the center-of-mass momenta. However, the kinematics associated with the new variables will be complicated and will involve the solution of quartic equations. In this talk we shall therefore confine ourselves to cases where the shape of the Dalitz region does not depend on S, such as the non-relativistic case. The problem has been treated with fully relativistic kinematics and N/D equations have been constructed, though the algebra is much more complicated.

Before writing down the N/D equations we shall mention one kinematic problem, to find the amplitudes associated with the disconnected diagrams a_i once the corresponding two-body amplitudes are known. Let us choose the amplitude a_1 and, for simplicity, we shall define the helicities M and M' to be taken with respect to a body-centered axis along the direction of motion of ℓ. Once we know the amplitude with one definition of the body-centered axis, we can easily obtain it with respect to another. In terms of the two-body variables, the amplitude a, will be

$$\Sigma_\ell \, (2\ell + 1) \; t_\ell(s_1) \; P_\ell(\cos \alpha_1) \; \delta(\underset{\sim}{p}_1 - \underset{\sim}{p}_1') \; , \qquad (9)$$

where t_ℓ is the two-body partial wave amplitude and α_1 is the angle of scattering of particles 2 and 3 in their own center-of-mass system. We rewrite (9) as

$$\Sigma_\ell (2\ell + 1) \; t_\ell(s_1) \sum_m \; P_\ell^m(\cos \theta_1 \;) \; P_\ell^m(\cos \theta_1') \; e^{im(\phi - \phi')} \; \delta^3(\underset{\sim}{p}_1 - \underset{\sim}{p}_1') \; , \qquad (10)$$

where ϕ_1 and ϕ_1' are the initial and final angles between the direction of motion of particles 1 and 2, in the center-of-mass system of

$$N(S) = F(S) + \frac{1}{\pi} \int_{9m^2}^{\infty} dS' \frac{F(S') - F(S)}{S' - S} k(S') N(S') , \tag{16a}$$

$$D(S) = 1 - \frac{1}{\pi} \int_{9m^2}^{\infty} dS' \frac{k(S') N(S')}{S' - S} \tag{16b}$$

Again, a product such as

$$\int dx_1'' dx_2'' \sum_{M''} F(S', J; x_1, x_2, M, x_1'', x_2'', M'')$$

$$\times N(S', J; x_1'', x_2'', M'', x_1, x_2, M')$$

is always implied when products NF are written down, so that our integral equation involves three continuous variables S, x_1, and x_2 and one discrete variable M.

The function F will contain disconnected parts and a connected part:

$$F = F_c + \sum_i f_i \tag{17}$$

where the subscript i indicates the presence of a delta function $\delta(y_i - y_i')$. It follows from (16) that the function N will also contain disconnected parts

$$N = N_c + \sum_i n_i . \tag{18}$$

We can now substitute (17) and (18) into (16). By equating coefficients of the three δ-functions, as well as the terms without δ-functions, we obtain the following equations for n_i and N_c :

$$n_i(S) = f(S) + \frac{1}{\pi} \int dS' \frac{f(S') - f(S)}{S' - S} k(S')\, n_i(S') , \tag{18}$$

$$N_c(S) = F_c(S) + \frac{1}{\pi} \sum_i \int dS' \frac{f_i(S') - f(S)}{S' - S} k(S')\, N_c(S')$$

$$+ \frac{1}{\pi} \sum_i \int dS' \frac{F_c(S') - F(S)}{S' - S} k(S')\, n_i(S) + \frac{1}{\pi} \sum_{i \neq j}$$

$$\times \int dS' \frac{f_i(S') - f(S)}{S' - S} k(S')\, n_j(S') + \frac{1}{\pi} \int dS'$$

$$\times \frac{F_c(S') - F_c(S)}{S' - S} k(S')\, N_c(S) . \tag{19}$$

It is not difficult to see that (18) is precisely the equation for the two-body n-function , with the correspondence between the functions in the spaces of two-body and three-body variables being given by (14). Thus n_i is known, and equation (19) can be solved. It is still not a Fredholm equation, as the terms f_i in the kernel contain delta functions. However, equations of this form can be reduced to Fredholm equations by the method used by Weinberg in the Schrödinger potential problem. We thus have a Fredholm equation for constructing N . The denominator function D can then be found from (16). The calculation of D^{-1} from D itself involves the solution of an integral equation in x_1 , x_2 and M . The disconnected parts can be separated off in the same way as before.

For simplicity we have confined ourselves to a pure three-body problem, but the generalization to the case of coupled two- and three-body channels is straightforward.

The S-integration in (16) is initially taken along the real axis, but we can now deform the contour into the complex plane. We thereby obtain a continuation of our amplitude onto the unphysical sheet. By examing the integral equation for D^{-1} in terms of D, we can show that D has a pole when any of the sub-energies is at a resonance value, and also that D has the expected cuts on the unphysical sheet at the threshold for particle-resonance scattering.

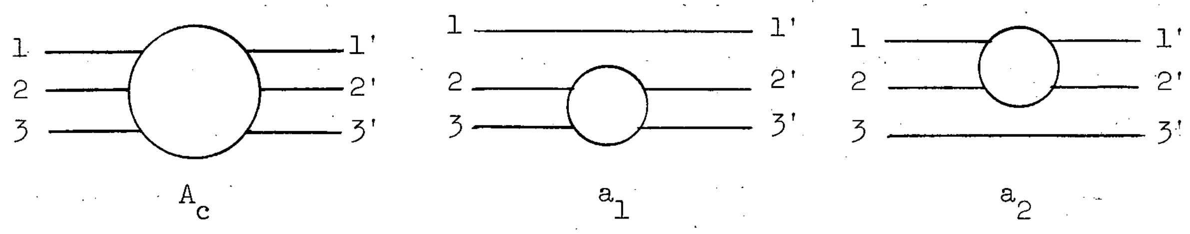

Fig. 1. Connected and disconnected diagrams for three-particle scattering.

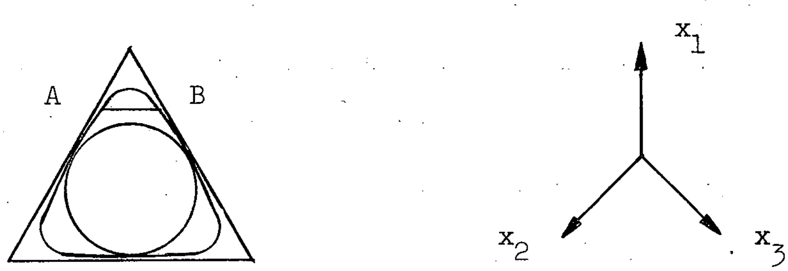

Fig. 2. The physical region for small, intermediate, and large values of s .

B. SOME FEATURES OF REGGE TRAJECTORIES WITH THREE-PARTICLE AND MULTI-PARTICLE INTERMEDIATE STATES

In this lecture I should like to make a few remarks on the behavior of Regge trajectories when there are three-particle as well as two-particle intermediate states present. Ideally we should like to prove from the N/D equations that the amplitude is meromorphic in the J-plane (except for cuts in the A.F.S. position). The principles of such an argument would be similar to those of the proof when only two-particle intermediate states are present. Unfortunately there are difficulties in the three-particle case which were not present in the two-particle case and, while I shall outline a procedure by which these difficulties may possibly be overcome, the problem is still far from solved. However, certain qualitative features of the Regge trajectories can be obtained by plausible reasoning without the complete solution of the problem. In particular, we can obtain information about the limit points of the trajectories at large values of the energy. We shall relate these properties to the hypothesis, suggested by experiment, that the trajectories rise indefinitely with increasing energy. We shall also make a few comments on the behavior of the phase shift at infinite energy in a system where the trajectories rise indefinitely, and we shall observe that this behavior is correlated with the decrease of form factors at high energy.

We begin by explaining why the three-body problem is more complicated than the corresponding two-body problem. Let me remind you of the procedure for establishing ℓ-plane meromorphy in the two-body problem. We can show that the elementary partial-wave projection formula

$$a(\ell, s) = \int_{-1}^{1} A(s, z)\, P_\ell(z)\, dz \quad , \tag{1}$$

though not valid for the complete amplitude when ℓ is not integral, is valid for the left-hand discontinuity. (If there is a third double-spectral function present the formula requires modification, but the argument still goes through). As the range of integration is finite, we can conclude that the left-hand discontinuity is a holomorphic function of ℓ. We then construct the complete amplitude from the left-hand cut by an N/D equation. If the amplitude approaches zero at high values of s the equation will be non-singular, so that N, D and N/D are meromorphic functions of ℓ.

The difficulty of the corresponding argument in the three-body problem is the infinite number of helicity states. Our amplitudes are functions of two helicities M and M' and the N/D equations will be integral equations in energy space and matrix equations in M-space. If J is integral the helicities will be restricted by the condition $M \leqslant J$, and the matrices in M-space will be finite. If J is non-integral the variable M can assume all integral values, and the matrices will become infinite.

It is unfortunately easy to show that the scattering amplitude diverges exponentially as M approaches infinity. The Froissart-Gribov formula for helicity states is

$$a(J, M, M') = \int A_{MM'}(z)\, e_{MM'}^{J}(z)\, dz \tag{2}$$

where the integral is to be taken around the cut in z, the cosine of the scattering angle, in the usual way. The function e is a hypergeometric function which bears the same relation the Wigner $d_{MM'}^{J}$'s as Q_ℓ does to P_ℓ . It tends to infinity as M and M' become infinite when z is fixed and complex. In the three-body problem there are always cuts in the complex z-plane , so that the amplitude approaches infinity exponentially with M and M' .

The above reasoning alone does not rule out the possibility of proving that the amplitude decreases with infinite M , since we may have overlooked a cancellation. However, we can quote an argument of Drummond to show that the singularities in the complex z-plane have an essential bearing on the problem, and the J-plane properties of the two-body amplitude cannot be taken into the three-body problem without modification. In the two-body problem we can construct positive- and negative-signature amplitudes, each of which is separately unitary. An equivalent statement is that we can divide the amplitude into contributions from the right- and left-hand cuts in the z-plane , so that two right-hand cuts or two left-hand cuts combine to give a right-hand cut, while a right-hand cut combines with a left-hand cut to give a left-hand cut. If we attempt such a division in the three-particle case we shall now demonstrate that we reach a contradiction.

First, if we take particles B and C to form a single system in our angular-momentum analysis, the diagram Fig. 1(a) clearly corresponds to a right-hand cut. It is not so clear whether Fig. 1(b) corresponds to a left- or a right-hand cut. We can settle the question by making a unitarity combination of Fig. 1(a) and 1(b), as in Fig. 1(c). This diagram corresponds to a right-hand cut, so that Fig. 1(b) must itself correspond to a right-hand cut. Similarly we observe that Fig. 1(d) corresponds to a right-hand cut, as we are treating particles

B and C symmetrically in our angular-momentum analysis. If we now combine Fig. 1(b) and 1(d) by unitarity we obtain Fig. 1(e), which corresponds to a left-hand cut. We have thus violated the rule that two right-hand cuts should combine to give a right-hand cut, and the simple rule that holds in the two-body case cannot be general when the angular momentum is complex.

We may ask whether it is possible to obtain a more complicated signature system in the three-body case, where more than two signatures are involved. Drummond has shown that this is indeed possible for the simple type of diagrams which we have just examined, but he has emphasized that this system is inadequate for diagrams such as Fig. 1(f). The signature formalism which applies in the two-body problem does not appear to possess a simple generalization to the three-body problem.

To circumvent the difficulties which we encounter when we attempt to treat the three-body problem in the J-plane, we propose to work in the space of the cosines of the angles rather than of the angular momenta. Let us first outline very briefly how the two-body problem may be treated in this fashion. The usual N/D equation is written down, and the N-function is expanded as ratio of two Fredholm series. We now change our variable from the angular momentum l to the cosine z of the scattering angle. Whenever a product $a(l,s)\ b(l,s)$ is encountered in the Fredholm series, we replace it by the integral $\int dz_1\, dz_2\ K(z, z_1, z_2)\, A(s, z_1)\, B(s, z_2)$, K being the kernel which occurs in the unitarity integral.

We now continue into the complex z-plane in the usual way. We can show that the Fredholm series converge, and that they have the same behavior at infinite z as the input. The next step is to invert the Fredholm denominator. (The Fredholm numerators and denominators must not be confused with the N and D of the N/D method, both N and D have a Fredholm numerator and a Fredholm denominator). We mean inversion in the sense of solving the integral equation defined above, if the Fredholm denominator has the form

$$\delta(z - 1) + D(s, z)$$

we have to solve the equation

$$F(s, z) + D(s, z) + \int dz' \, dz'' \, K(z, z', z'') \, F(s, z') \, D(s, z'') = 0 . \quad (3)$$

The reciprocal of the Fredholm denominator has been written as $\delta(z - 1) + F(s, z)$, the significance of the δ-function is that it is the Lengendre transform of unity. In solving the integral equation we can go to angular-momentum space, since it then becomes a numerical equation. As the Fredholm denominator in the three-body case does not involve the helicities, we should be able to carry out that part of the program there too. We then pass back to z-space again. If $D(s, z)$ behaves like z^{-1} at infinite z, its Legendre transform will be holomorphic in ℓ for $\mathrm{Re}\ell > 1$. The reciprocal $f(s, \ell)$ will therefore be meromorphic in ℓ, and its Legendre transform will behave like $\Sigma\, B(s)\, P_{\alpha(s)}(z)$ at infinite z. By this method we can show that the scattering amplitude has a Regge asymptotic behavior in z.

It is hoped that a similar program can be carried out for the three-body problem. The integration over angles will now be a three-dimensional integral $dz d\phi d\phi'$, where ϕ and ϕ' are the initial and final azimuthal angles. We shall certainly have to integrate over contours in the complex plane, but this in itself should not cause divergence difficulties as it did when we worked with helicity states.

We shall also have to deform contours of integration in such a way that the usual rule for combining left- and right-hand cuts in a unitarity integral does not hold, but this feature too does not appear to contradict the principles of the program. Thus, by examining asymptotic behavior in the z-plane rather than analytic properties in the J-plane, we might hope to establish Regge behavior in the three-body problem. We should emphasize that we are simply outlining a program here and are not reporting on completed work.

Even though we have not established that Regge's results can be extended to the three-body problem, we can use plausibility arguments to infer some properties of Regge trajectories. In this lecture we shall interest ourselves in the behavior of the trajectories at high energy. In the two-body problem without spin they approach the value $\ell = -1$ at high s. This is connected with the fact that the Born terms dominate the scattering amplitude at high energy. They have an asymptotic behavior t^{-1} and therefore have a pole in the angular-momentum plane at $\ell = -1$. In the three-body problem with spinless particles and only three-line vertices, a Born diagram is shown in Fig. 2. Each of its three internal lines gives one negative power of the momentum transfer, so that it has an asymptotic behavior t^{-3}. In the J-plane it has a pole at $J = -3$, so that the Regge trajectories in a problem with such a Born term would be expected to start and end at $J = -3$.

We can now begin to frame a hypothesis regarding the behavior of Regge trajectories in a "bootstrap" theory, where there are no elementary particles. The only way to distinguish between elementary and composite <u>external</u> particles in a scattering process is to couple the process with one involving a larger number of particles. Composite external particles would then be treated as bound states of the other external particles. Let us then treat a process $A + B \to A + B$, first in the two-particle approximation and then in a higher approximation where the AB channel is coupled with the three-body ACD channel. In the more accurate treatment B is treated as a bound state of C and D. The Regge trajectories in the two approximations are shown as curves (a) and (b) of Fig. 3. In the low-energy physical region they should be close to one another if the two-particle approximation is adequate. However, the more accurate trajectory should approach $\ell = -3$ rather than $\ell = -1$ at infinite s. We have also made the plausible assumption that the three-body trajectory does not begin to turn over until we are well above the three-particle threshold. The turn-over point is thus above the corresponding point of the two-body trajectory.

We may now envisage a still better treatment where the particles A, B or C are themselves regarded as composite, so that our particles are treated as bound sub-channels of an n-body system. If the process is continued indefinitely the trajectories of the fully bootstrapped system will go from $-\infty$ at $s = -\infty$ to $+\infty$ at $s = +\infty$. The indefinite rise of the Regge trajectories seems to be borne out by experiment. At any rate they show no signs of turning over below the highest energies where resonances have been looked for. It was partly this fact which led us to draw the curve (b) in Fig. (3) in the manner we did. The exponential decrease of scattering amplitudes at high energies and angles <u>may</u> be regarded as experimental evidence in favor of the limit $J = -\infty$ of the trajectories at $s = -\infty$, but here we have to be more careful, first because there are probably an infinite number of trajectories contributing in this limit, and second because the relation of the asymptotic behavior as both s and t become infinite to the Regge trajectories is far from clear. We feel, however, that it is plausible that Regge trajectories in a bootstrap system go from $-\infty$ to $+\infty$.

Another suggestion which we may make on an intuitive basis is that the point at which the phase shifts turn over is related to the point at which the Regge trajectories turn over. Thus, as we go to an approximation where systems containing more and more particles are treated, the phase shifts will take longer and longer to turn over. In fact, if all the lower Regge trajectories also rise indefinitely, each partial wave will contain an infinite number of resonances, and the phase shift will rise indefinitely. In the n-particle approximation it will eventually turn over and return to zero.

Under these circumstances we may ask whether the definition of a bootstrap system as one where the sum of the phase shifts returns to zero is reasonable. Properties which serve as a basis for fundamental definitions should be true, not only in the n-particle approximation, but exactly. We should therefore start with an alternative formulation of the bootstrap hypothesis such as the absence of polynomial terms as s or t approach infinity or, equivalently, the absence of non-Regge terms in the J-plane . In the two-particle approximation, and probably in the n-particle approximation, this is equivalent to the phase-shift criterion. If the number of particles now becomes infinite the phase-shift criterion may no longer apply, but analyticity in the J-plane should not be lost.

The hypothesis that the phase shifts of the exact problem never turn back is helpful in understanding the falling off of form factors at high energy. In fact, it appears to be essential in understanding this fact, which is borne out experimentally. The form factor is given by an equation of the Omnès type. Although Omnès' technique is not applicable to the multi-channel problem, much less the multi-body problem, all his results can be established by other methods for these problems. The multi-channel problem is treated in the book by Muskhelishvili, and Gross at Berkeley has recently shown that the results can be generalized to problems such as the n-body problem where the channel indices are continuous variables. Now it is a fundamental feature of these equations that they do not possess solutions with decreasing form factors in all channels when the sum of the phase shifts approaches zero at infinity. Unlike Fredholm equations, these equations have no exceptional cases. If the sum of the phase shifts approaches π, we have extra parameters at our disposal associated with coupling to the "elementary" particle, and we can obtain a solution which approaches zero at infinity. If the phase shift approaches 2π we can apply still stronger conditions at infinity, and so on.

We can illustrate the results just quoted by referring to the single-channel two-body problem. The form factor is then given by the function

$$F(s) = \mathrm{Exp}\left\{\frac{s - s_0}{\pi} \int_{m_0}^{\infty} \frac{\delta(s')\, ds'}{(s' - s_0)(s' - s)}\right\} \qquad (4)$$

If $\delta(\infty) = 0$, then

$$F(\infty) = \mathrm{Exp}\left\{-\frac{1}{\pi} \int \frac{\delta(s')\, ds'}{s' - s_0}\right\} \qquad (5)$$

which is a non-zero constant. If, on the other hand, $\delta(\infty) = \pi$, then

$$F(s) = \text{Exp}\left\{ \log \frac{s_0}{s} - \frac{1}{\pi} \int \frac{[\delta(s') - \pi]\, ds'}{s' - s_0} \right\}$$

$$= \frac{s_0}{s} \, \text{Exp}\left\{ - \frac{1}{\pi} \int \frac{[\delta(s') - \pi]\, ds'}{s' - s_0} \right\}, \quad s \to \infty \qquad (6)$$

and we have a falling form factor. Though this proof only applies to the single-channel problem, one can prove similar results for the multi-channel problem.

We must emphasize that, as long as we remain within the n-particle approximation, the hypothesis of no elementary particles makes it more difficult, not less, to explain falling form factors. In case you find this in conflict with your intuition (no elementary particles corresponding to no δ-function cores), let us take note of the channel we are examining. With the charge form factor of the proton, for instance, the intuitive arguments which suggest that non-elementary particles are soft refer to the process $\gamma + p \to p$. If we examine the channel $\gamma \to p + \bar{p}$, we do not obtain a decreasing form factor when the phase shift goes to zero at infinity, as it should in the elementary bootstrap picture.

If we now make the hypothesis that phase shifts do not turn back in the exact bootstrap solution, we can understand falling form factors even if we look at the channel $\gamma \to p + \bar{p}$. As we have just pointed out, equations of the Omnes type can be solved to give falling form factors when the phase shift approaches $n\pi$ $(n \geqslant 1)$ at infinity. We might even conjecture that we can obtain exponentially decreasing form factors if the phase shift increases indefinitely, but such a conjecture is far

K(z,z',z")

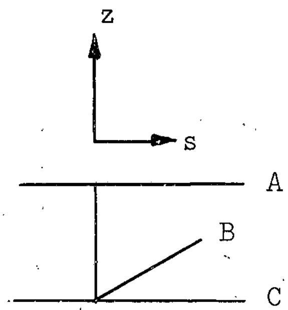

Fig. 1(a)

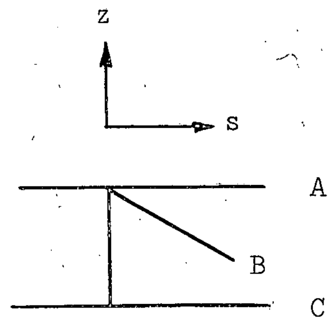

Fig. 1(b)

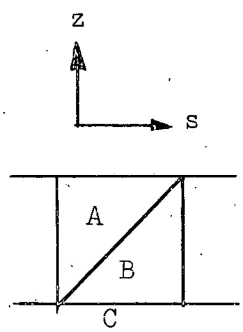

Fig. 1(c)

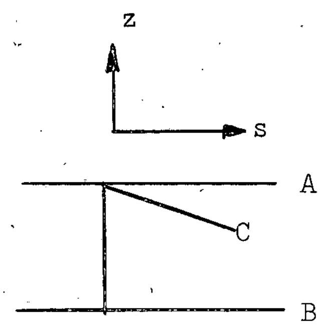

Fig. 1(d)

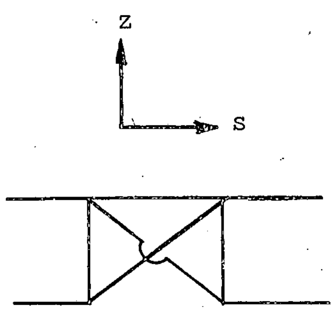

Fig. 1(e)

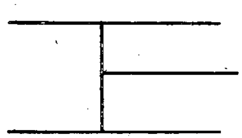

Fig. 1(f)

Fig. 1.(a,b,c,d,e,f). Feynman diagrams with three-particle intermediate states.

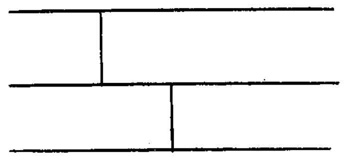

Fig. 2. Born diagram for a three-particle system.

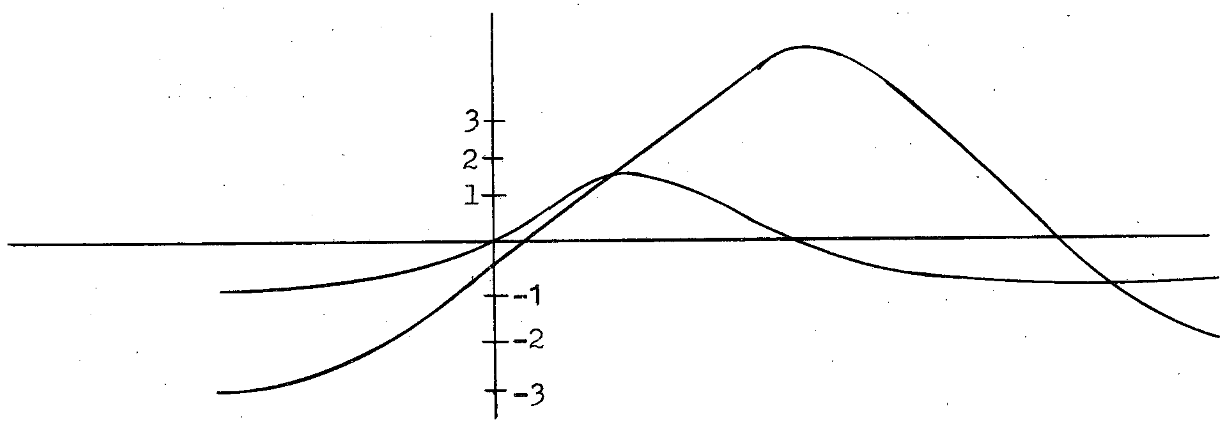

Fig. 3. Regge trajectories in the two- and three-particle approximations.

C. ASYMPTOTIC BEHAVIOR OF AMPLITUDES FOR BACKWARD SCATTERING

The subject of today's talk is the asymptotic behavior of scattering amplitudes for unequal mass particles in the backward direction as the energy becomes infinite. In this case the cosine of the scattering angle of the crossed channel remain finite (in fact zero) as s approaches infinity, so we may question whether the asymptotic formula $\frac{\beta(u)\ s^{\alpha(u)}}{\sin \pi\alpha(u)}$ is correct. We shall show that the asymptotic formula is correct in this case for, if it were correct at all values of u except $u = 0$, there would be violation of analyticity in s-t space. We shall also show that there must be further trajectories which pass through $u = 0$ at $\alpha = \alpha(0) - 1$, $\alpha(0) - 2$, etc., otherwise we would have unwanted singularities in s-t space. Finally, we shall report of some work by Freedman and Jiunn-Ming Wang, who showed that trajectories generated by a Bethe-Salpeter equation do possess this sequence of "daughters", with residues of just the right value to cancel the unwanted singularities in the s-t plane.

We begin by setting up the kinematics and explaining the problem. We treat a two-body scattering amplitude with pion-nucleon kinematics but without spin. The variables s and u are the energies in the two pion-nucleon channels, t is the energy of the channel $\pi + \pi \to N + \overline{N}$. The square of the center-of-mass momentum q_u in the u-channel is given by the formula

$$q_u^{\ 2} = \frac{1}{4u} \{u^2 - 2u\,(M^2 + \mu^2) + (M^2 - \mu^2)^2\} \tag{1}$$

while z_u, the cosine of the scattering angle in the u-channel, is related to s by the formula

$$z_u^{-1} = s/2q_u^2 . \tag{2}$$

If u is not zero at infinity, z_u goes to infinity with s , so that asymptotic behavior in z_u is equivalent to asymptotic behavior in s . If $u = 0$, however, q_u^2 is infinite and z_u is identically equal to unity. A naive reading of the Regge formula would therefore indicate that it could not be used to determine the asymptotic behavior in the limit $s \to \infty$, $u = 0$. Such a limit is of great practical interest, since it represents the infinite energy limit at backward scattering for the other pion-nucleon channel. This is a region easily accessible to experiment, and Frautschi has outlined some interesting results of experiments performed in this region. We shall now show that the Regge asymptotic behavior in the limit $s \to \infty$, $u \neq 0$, together with the usual assumptions of analyticity in the s-t plane, imply an asymptotic behavior of the form $\gamma(u)s^{\alpha(u)}$ at $u = 0$.

We start from the Regge asymptotic formula, valid for $u \neq 0$

$$A(s, u) = \frac{\beta(u)}{\sin \pi\alpha(u)} P_{\alpha(u)}(z_u) + \text{Background term.} \tag{3}$$

We now write

$$P_{\alpha(u)}(z_u) = C(z_u)^{\alpha(u)} + O(z_u)^{\alpha(u) - 1} \tag{4}$$

From the formula relating z_u and s , we find that

$$A(s, u) = \gamma(u)s^{\alpha(u)} + \text{Background term} \tag{5}$$

where

$$\gamma(u) = \frac{C\beta(u)}{(q_u)^{2\alpha(u)} \sin \pi\alpha(u)} \quad . \qquad (6)$$

The terms of order $(z_u)^{\alpha(u)-1}$ in the expansion of $P_{\alpha(u)}$ have been included in the background term.

Note that the formula (5), which has been derived from (3), is valid whether or not the asymptotic form $s^{\alpha(u)}$ is valid at $u = 0$. If, the function $\beta(u)$ were analytic at $u = 0$, the function $\gamma(u)$ would approach zero like $(q_u)^{2\alpha(u)}$. The background term of (5) would then dominate over the term $\gamma(u)s^{\alpha(u)}$ at $u = 0$, and the asymptotic form $s^{\alpha(u)}$ would not be valid. If, on the other hand, the function $\gamma(u)$ is analytic at $u = 0$, and if the asymptotic behavior of the background term at this point is the same as at other values of u, the first term of (5) will represent the asymptotic form. We wish to show from analyticity in the s-t plane that these conditions are in fact satisfied.

We shall assume that the Regge asymptotic form is valid when $u \neq 0$. In other words, we assume that the background term in (5) approaches zero as s approaches infinity like a negative power (say $s^{-\frac{1}{2}}$) or like $s^{\alpha(u)-1}$, whichever is larger. This is not implied unambiguously by meromorphy in the ℓ-plane, since we are in an unphysical region of the u-channel. The ℓ-integration of the background term

$$\int_{-\frac{1}{2} - i\infty}^{-\frac{1}{2} + i\infty} d\ell \; a(\ell, s) \, P_\ell(z)$$

may diverge, depending on the argument of z. Nevertheless, whenever one can prove meromorphy in the ℓ-plane one can also prove that the background term goes to zero (or to $s^{\alpha(u)-1}$) at infinite s, and that it does so uniformly in $u(u \neq 0)$. This is what one usually thinks of in connection with Regge behavior, and we shall adopt it as our assumption.

It is now not difficult to prove that the function $\gamma(u)$ must be analytic in u. If $\gamma(u)$ has a branch point at $u = 0$, the first term of (5) will not be an analytic function of u. It will have a cut which extends to values of u other than zero. Since we are assuming that the second term of (5) converges uniformly to zero at infinite s $(u \neq 0)$, we cannot cancel the cut with a cut in this second term. We thus conclude that $\gamma(u)$ cannot have a branch point at $u = 0$. A natural boundary can be similarly excluded, and one can also exclude a pole or essential singularity in γ at $u = 0$. Thus γ is an analytic function of u at $u = 0$.

We can now show that the first term of (5) dominates over the background term at $s = 0$. Since A is analytic in u at fixed s in a neighborhood of $u = 0$ and we have just seen that the first term of (5) is analytic in u, we can conclude that the second term is also analytic. Using the analyticity in u, Wang has shown from the maximum-modulus theorem that an asymptotic behavior of $s^{\alpha(u)-1}$ for $u \neq 0$ implies a similar asymptotic behavior for $u = 0$. Thus the first term of (5) dominates for s sufficiently large, and it represents correctly the asymptotic behavior. This is what we wished to prove.

Let us now observe what happens when we include further terms of the expansion of P_α in (3) in our main term. If we keep the first two terms of the expansion, we find that

$$A(s, u) = \gamma(u)\ \{s^{\alpha(u)} + 2q^2 s^{\alpha(u)-1} + \cdots\} + \text{Background term} \tag{7}$$

If $\alpha(u)-1 > \frac{1}{2}$, the first two terms of (7) will dominate over the background term. However, the second term of (7) has a pole at $u = 0$, since q^2 is infinite at this point. By going to sufficiently large s, we can show that the background term cannot cancel the pole. We thus appear to be contradicting the analytic properties in the s-t plane.

The phenomenon just discussed was first observed by Goldberger and Jones, who suggested that $\alpha(u)-1$ must be less than $\frac{1}{2}$. However, one can easily write the Regge formula with a background term which goes down as fast as we please as $u \to \infty$, so that the contradiction would still exist. Furthermore, experimental evidence strongly indicates that there are trajectories which rise above $\alpha = \frac{1}{2}$ at $u = 0$.

Another way out of the difficulty, which appears to be the correct one, is that the background term contains a second Regge trajectory which cancels the unwanted singularity. Equation (1) rewritten with contributions from two trajectories is

$$\begin{aligned} A(s, u) = {} & \gamma_1(u)\ \{s^{\alpha_1(u)} + 2q^2 s^{\alpha_1(u)-1} + \cdots\} \\ & + \gamma_2(u)\ \{s^{\alpha_2(u)} + \cdots\} + \text{Background term} \end{aligned} \tag{8}$$

The condition for cancellation of the pole in the second term of (8) is

$$\left.\begin{aligned} \alpha_2(u) &\approx \alpha_1(u)-1 \\ \gamma_2(u) &\approx \frac{(M^2 - \mu^2)^2}{2u}\ \gamma_1(u) \end{aligned}\right\} u = 0 \tag{9}$$

We are thus led to the conclusion that there is a second trajectory, which passes through the point $u = 0$ at one integer below the first. The residue $\gamma(u)$ associated with this second trajectory must itself have a pole in u at $u = 0$. One may ask whether it is possible for γ_2 to have a pole in u, since it has been proved that residues of Regge trajectories are analytic in u. The proof was based on analyticity in the s-t plane, however, and it did not examine the special conditions at $u = 0$. Our reasoning indicates that a pole in γ_2 at $u = 0$ is necessitated by analyticity in the s-t plane.

By similar reasoning one can show that further Regge trajectories must pass through the points $\alpha_1(0)-2$, $\alpha_1(0)-3$, etc., at $u = 0$. The corresponding γ's have poles of higher and higher order at $u = 0$. In our treatment we have not been very careful about signature but, by replacing the Legendre functions P_α in our derivations by the combinations $P_\alpha(z) \pm P_\alpha(-z)$, we can easily see that these trajectories alternate in signature. They have been called daughter trajectories by Freedman and Wang. As equations (8) and (9) show, their effect is to replace an asymptotic behavior $P_\alpha(z_u)$ by an asymptotic behavior $s^{\alpha(u)}$ at $u = 0$. Such a replacement is necessitated by analyticity in the s-t plane.

Our proof of the existence of daughter trajectories is based on analyticity in the s-t plane. As a two-body system satisfying the Bethe-Salpeter equation possesses these analyticity properties, it should be possible to show directly that the daughter trajectories are present in this system. The potential model is too simple, as the momentum q^2 never becomes infinite except at infinite energy. Freedman and Wang have shown directly that the Bethe-Salpeter system does possess daughter trajectories, and that the functions $\gamma(u)$ associated with the lower trajectories have poles at $u = 0$ with the correct residues.

The results of Freedman and Wang are based on the well-known four-dimensional symmetry of the Bethe-Salpeter equation at $u = 0$. The relevance of this symmetry to Regge trajectories was first pointed out by Domokos and Surányi. In the center-of-mass system, the Bethe-Salpeter kernel corresponding to Fig. 1 is

$$\frac{1}{(p - p')^2 + \mu^2} \{- (E - p_0)^2 + \underset{\sim}{p}^2 + \mu^2\}^{-1} \{- (E + p_0)^2 + \underset{\sim}{p}^2 + \mu^2\}^{-1} , \qquad (10)$$

where $E = u^{\frac{1}{2}}$. The usual method of treating this equation is to expand it into spherical harmonics involving the variables ℓ and m. The separated kernel depends on ℓ but not on m. However, the first factor of the kernel is invariant under four-dimensional rotations in p-space. We can therefore perform a Wick rotation and expand it in four-dimensional spherical harmonics or Gegenbauer polynomials, which depend on three integers $n(n > \ell)$, $\ell(\ell > 0)$ and $m(-\ell < m < \ell)$. Thus

$$\frac{1}{(p - p')^2 + \mu^2} = \frac{1}{|p|\ |p'|} \sum_{\ell=0}^{\infty} \sum_{n=\ell}^{\infty} F_{n-\ell} \{p^2 - p'^2 + \mu^2\}$$

$$\times\ D_{n-\ell}^{\ell+1}(w)\ D_{n-\ell}^{\ell+1}(w')\ P_\ell(z) , \qquad (11)$$

where w and w' are the initial and final angles in four-dimensional space with respect to some fixed axis. The functions D are Gegenbauer polynomials which we shall not define. Note that F depends only on

$\ell - n$ and not on ℓ and n separately. This is analogous to the fact that in three dimensions the kernel depends on ℓ but not on m. The propagator functions in (10) can be similarly expanded in a series

$$\sum_{\ell,n,n'} G_{\ell,n,n'} D^{\ell+1}_{n-\ell}(w) \; D^{\ell+1}_{n'-\ell}(w') . \qquad (12)$$

Since the propagator functions do not possess spherical symmetry, the summation in (12) is over two indices n and n'. We thus have an infinite series of coupled equations in n and n', depending on a parameter ℓ.

When u $(= E^2)$ is equal to zero, all functions in (10) possess four-dimensional spherical symmetry. Thus the equation (12) will have a form similar to (10), and terms with $n \neq n'$ are zero. We now have one equation for each value of n and ℓ, and the kernel depends only on $n - \ell$. For any solution with a given value ℓ_0 of ℓ there will be corresponding solutions with $\ell = \ell_0 - 1$, $\ell = \ell_0 - 2$, etc., since n can be decreased by integers and $\ell - n$ must remain constant. If there is a bound state at $\ell = \ell_0$, there will be corresponding bound states at $\ell_0 - 1$, $\ell_0 - 2$, etc. The same results is true when ℓ_0 is not integral, so that any trajectory passing through $\alpha(0)$ at $u = 0$ will possess daughters passing through $\ell = \alpha(0)-1$, $\ell = \alpha(0)-2$, etc., at $u = 0$. This is what we had inferred from our analyticity reasoning.

By considering the infinite set of coupled equations which we encounter when u is small but not zero, Freedman and Wang were able to show that the functions $\gamma(u)$ associated with the lower trajectories did possess poles with the correct residues at $u = 0$. These poles do not occur in the equal-mass case for, by equation (9), the residues would then vanish.

We may finally collect the threads of the different arguments of this talk to conclude that the problem of the asymptotic behavior at large energies in the backward direction is solved, and that the leading term in the asymptotic behavior is $s^{\alpha(0)}$.

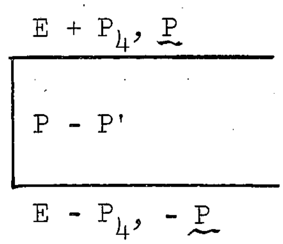

Fig. 1. Diagram for a simple Bethe-Salpeter kernel.