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Deformation and rigidity in von Neumann algebras: Cartan subalgebras and tensor product decompositions

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Deformation and rigidity in von Neumann algebras: Cartan subalgebras and tensor product decompositions

A dissertation submitted in partial satisfaction of the
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Doctor of Philosophy

in

Mathematics

by

Pieter Spaas

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2019

The dissertation of Pieter Spaas is approved, and it is acceptable in quality and form for publication on microfilm and electronically:

Chair

University of California San Diego

2019

DEDICATION

To my family.

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ACKNOWLEDGEMENTS

Looking back at 4 years as a PhD student at UC San Diego, I can only say that I am incredibly happy to have been given the opportunity to do so. During these years, I have been able to learn a lot. Part of that learning has resulted in the thesis in front of you. However, before engaging in any mathematical adventures, I would like to say “thank you” to a number of people who have helped me survive till this point.

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Pieter

May 9, 2019

Chapters 1, 2, and 3 contain material from: P. Spaas, "Non-classification of Cartan subalgebras for a class of von Neumann algebras", *Advances in Mathematics*, vol. 332, pp. 510-552, 2018. The dissertation author was the primary investigator and author of this paper.

Chapters 1, 2, and 4 contain material from: A. Ioana and P. Spaas, "A class of II_1 factors with a unique McDuff decomposition", preprint arXiv: 1808.02965, submitted. The dissertation author was one of the primary investigators and authors of this paper.

VITA

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ABSTRACT OF THE DISSERTATION

Deformation and rigidity in von Neumann algebras:
Cartan subalgebras and tensor product decompositions

by

Pieter Spaas

Doctor of Philosophy in Mathematics

University of California San Diego, 2019

Professor Adrian Ioana, Chair

We study structural properties and classification problems for von Neumann algebras. Using Popa's deformation/rigidity techniques, we investigate Cartan subalgebras and tensor product decompositions. Firstly, we prove a structural result for Cartan subalgebras inside tensor products using a direct integral decomposition. This result is used to provide the first examples of II_1 factors whose Cartan subalgebras are not classifiable by countable structures. Secondly, we study McDuff II_1 factors. We prove a structural

result for the stabilization of a II_1 factors whose central sequences are captured by a Cartan subalgebra, and use this result to provide many new examples of II_1 factors with a unique McDuff decomposition. Finally, along the way, we will discuss several related problems for group actions and equivalence relations. In particular, we settle a question by Jones and Schmidt from 1985 by providing examples of equivalence relations without the "Jones-Schmidt property".

Fortunately, a mathematician always has problems.

— Unknown

Chapter 1

Introduction

von Neumann algebras were first introduced in the 1930's by Murray and von Neumann [MvN36], who called them rings of operators. Even though the original motivation for introducing them came from quantum mechanics, it has become clear that the study of von Neumann algebras has ties to many other areas throughout mathematics, several of which we will encounter later. Already in their first paper, Murray and von Neumann introduced two extremely important constructions of von Neumann algebras. Associated to a group, they defined the *group von Neumann algebra*, and starting from a group action on a measure space, they introduced the *group measure space construction*. Starting from fairly mild and natural assumptions on the group, respectively the action, one gets a particular kind of von Neumann algebra out of these constructions, namely a so-called II_1 factor. It was clear that classifying II_1 factors was crucial for most applications, and Murray and von Neumann subsequently provided the first examples of two non-isomorphic such von Neumann algebras in [MvN43]. They did this through the introduction of their so-called *property Gamma*, studying *central sequences* in these von Neumann algebras.

Since then mathematicians all around the world have made many contributions towards a better understanding of these objects. Nevertheless, it turns out that telling von Neumann algebras apart is a very subtle issue. After the aforementioned work of Murray and von Neumann, it took more than 20 years, till the late 1960's, for the analysis of central sequences of [MvN43] to be refined to provide additional examples of non-isomorphic II_1 factors in [Chi69, DL69, Sak68, ZM69], culminating with McDuff's construction of a continuum of such factors in [McD69a, McD69b].

The most fundamental problems in the study of von Neumann algebras concern the question to what extent one can recover information about the group, respectively the group action, from the associated von Neumann algebra. In general the answer to this question is simply "nothing". Or at least not much. The problem seems to be that von Neumann algebras are such large objects, and that we almost literally "blow up" all information when passing from say a group to its von Neumann algebra.

Perhaps the most famous and striking such "lack of rigidity" result is the famous classification of amenable von Neumann algebras by A. Connes [Con76]. He shows that whenever you start from an amenable group, or an action of an amenable group, the associated von Neumann algebra is always the same one, namely the so-called hyperfinite II_1 factor R. This tells us that, from the associated von Neumann algebra, we cannot recover any additional information about the group nor the action, besides its amenability.

In a sense, von Neumann algebras treat all amenable groups as being "small".

On the other hand, the study of von Neumann algebras appears to tell us more about non-amenable groups. Of course we cannot expect a full one-to-one correspondence. For instance, associated to a free product of any two infinite amenable groups, we always find the same von Neumann algebra [Dyk93]. Nevertheless, as soon as we leave the amenable world behind, it seems that rigidity phenomena start showing up in lots of places. It took another three decades though, and the breakthrough work of S. Popa in [Pop06a, Pop06b], whose so-called *deformation/rigidity* techniques allowed for a new angle of attack on these problems. Over the last two decades, this assortment of techniques has resulted in many new rigidity results and solutions to problems that before seemed intractable.

One recurring theme in a lot of known classification results is the special role played by the so-called Cartan subalgebras. These are particularly nice maximal abelian subalgebras inside II_1 factors. The main reason for studying these algebras is that any II_1 factor associated to a free ergodic probability measure preserving (p.m.p.) action on a measure space, or more generally a countable ergodic p.m.p. equivalence relation, comes equipped with a canonical Cartan subalgebra. Moreover, I. Singer established in [Sin55] that two actions are *orbit equivalent*, i.e. give rise to isomorphic orbit equivalence relations, if and only if the inclusions of their Cartan subalgebras inside their respective group measure space II_1 factors are isomorphic. This result both gives an immediate link with orbit equivalence theory and highlights the importance of understanding the Cartan subalgebras of a given II_1 factor. For instance, if one can show that certain classes of group measure space II_1 factors have a unique Cartan subalgebra (up to conjugacy by an automorphism), their classification up to isomorphism reduces to the classification of the corresponding actions up to orbit equivalence.

It has been proven in [CFW81] that the hyperfinite II_1 factor has a unique Cartan subalgebra up to conjugation by an automorphism. Together with the aforementioned work of A. Connes, this results in a lack of rigidity for actions of amenable groups, namely any two such actions are orbit equivalent. On the other hand, for non-amenable groups Popa's deformation/rigidity theory has led to considerable progress in the classification of group measure space II_1 factors, see for instance the surveys [Pop07a, Vae10, Ioa12a]. In particular, several uniqueness results for Cartan subalgebras inside non-amenable II_1 factors have been established. The first such result was obtained by N. Ozawa and S. Popa in [OP10a]. They proved that the canonical Cartan subalgebra inside the group measure space construction is unique whenever we start from a non-abelian free group and a free ergodic *profinite* p.m.p. action. The class of groups whose profinite actions give II_1 factors with a unique Cartan subalgebra was subsequently extended in [OP10b, CS13]. Later the condition of profiniteness was removed by S. Popa and S. Vaes, who showed in [PV14a] that *any* free ergodic p.m.p. action of a non-abelian free group gives rise to a II_1 factor with a unique Cartan subalgebra up to unitary conjugacy. Together with earlier orbit equivalence rigidity results from D. Gaboriau [Gab00, Gab02], this resulted in new and surprising *von Neumann (super)rigidity* statements. Hereafter, additional uniqueness results were obtained in (among others) [PV14b, Ioa15, CIK15].

Now, of course not every II_1 factor has a unique Cartan subalgebra. The first II_1 factor containing at least two Cartan subalgebras that are not conjugate by an automorphism was constructed in [CJ82]. By now there are more examples known (see for instance [OP10b, PV10]). Nevertheless it is worth mentioning that at the moment, as soon as uniqueness fails, we do not have any examples of II_1 factors for which we can describe all Cartan subalgebras in a satisfactory way. Some progress in this direction was made by A. Krogager and S. Vaes in [KV17], where they construct II_1 factors for which they can describe all so-called *group measure space Cartan subalgebras*. In particular they construct a II_1 factor with exactly two group measure space Cartan subalgebras up to unitary conjugacy. Another result in the non-uniqueness direction was obtained by A. Speelman and S. Vaes in [SV12, Theorem 2]. They construct a class of II_1 factors for which the relations of unitary conjugacy and conjugacy by a (stable) automorphism on Cartan subalgebras are *not smooth* (or not concretely classifiable, see also Definition 2.3.3).

Other recurring themes in a lot of classification results are structural properties related to tensor product decompositions of II_1 factors and the aforementioned central sequences of a II_1 factor. One important instance where both of these notions come into play together are the so-called *McDuff* II_1 factors. By definition, a II_1 factor M is called McDuff if it absorbs the hyperfinite II_1 factor R tensorially, i.e. $M \cong M\bar{\otimes}R$. Since R tensorially absorbs itself, i.e. $R\bar{\otimes}R \cong R$, one sees that a II_1 factor M is McDuff if and only if we can write $M = N\bar{\otimes}R$ for some II_1 factor N. The main result of [McD70] provides a satisfactory characterisation of when such a decomposition exists: a II_1 factor M is McDuff if and only if it admits two non-commuting central sequences.

In order to obtain this result, McDuff introduced the *central sequence algebra* of a II_1 factor M as the relative commutant, $M' \cap M^\omega$, of M into its ultrapower M^ω ([Wri54, Sak62]). This has since allowed for a more structural approach to central sequences and led to significant progress in the study of II_1 factors. Indeed, the central sequence algebra was a crucial tool in A. Connes' aforementioned celebrated classification of amenable II_1 factors, where a crucial step was showing that any such factor has McDuff's property.

In the above framework of McDuff's property, a natural question to ask is the following: given two II_1 factors N and P, when are their "stabilisations" $N\bar{\otimes}R$ and $P\bar{\otimes}R$ isomorphic? Or in other words, can we retrieve, in any reasonable way, a II_1 factor N from its stabilisation $N\bar{\otimes}R$? If the answer to this question is affirmative, we will say that $M = N\bar{\otimes}R$ has a *unique McDuff decomposition* (see also Section 1.1.2). By a striking theorem of Popa [Pop07b, Theorem 5.1], if a II_1 factor N does not have Murray and von Neumann's aforementioned *property Gamma*, then $M = N\bar{\otimes}R$ admits a unique McDuff decomposition in a strong sense. The proof of this result relies on Popa's discovery of his influential spectral gap rigidity principle [Pop08, Pop07b], which, generalizing the central sequence algebra, relies on the study of the relative commutant $M' \cap \mathcal{M}^\omega$, of a subalgebra $M \subset \mathcal{M}$ inside the ultrapower $\mathcal{M}^\omega$ of $\mathcal{M}$.

Despite all the progress described above, several questions remained unanswered. How many Cartan subalgebras can a von Neumann algebra have? If a II_1 factor N has property Gamma, is it possible for $M = N\bar{\otimes}R$ to have a unique McDuff decomposition? Can we reasonably describe the structure of the central sequence algebra of a given II_1 factor? In this dissertation we address these questions, provide several new structural results for von Neumann algebras, and discuss numerous related notions and questions.

1.1 Discussion of the main results in this dissertation

1.1.1 Classifying Cartan subalgebras

In Chapter 3, motivated by the aforementioned results of [SV12], we continue exploring further the complexity, in the sense of descriptive set theory, of the classification problem for Cartan subalgebras. We will consider both the equivalence relations of unitary conjugacy and of conjugation by an automorphism on the space of Cartan subalgebras of a family of II_1 factors. Using different techniques, we will provide the first examples of II_1 factors whose Cartan subalgebras are not classifiable by countable structures for either notion of equivalence. Hereby we confirm the statement in [SV12, Remark 14] expressing the belief that such II_1 factors should exist. We refer to Section 2.3.2 for the definition of classifiability by countable structures. Intuitively this means we cannot construct complete invariants out of countable (or discrete) structures such as countable groups, graphs, etc. and that these equivalence relations are "beyond S_∞-actions".

We discuss here the setup, the motivation and formulation of the main results of Chapter 3, and discuss how they are related. Let Γ be a relatively strongly solid group (see Definition 2.1.4. This class includes all non-abelian free groups and more generally all non-elementary hyperbolic groups by [PV14a, PV14b]). Let further (X, μ) be a standard probability space, $\Gamma \curvearrowright X$ a free ergodic p.m.p. action and N an arbitrary II_1 factor. Consider

$$\mathcal{M} := (L^\infty(X) \rtimes \Gamma)\bar{\otimes} N.$$

It turns out that the structure of Cartan($\mathcal{M}$), the space of Cartan subalgebras of $\mathcal{M}$, is completely determined by the structure of Cartan(N) and the orbits of the action $\Gamma \curvearrowright X$. More precisely, we can write $\mathcal{M} = (L^\infty(X)\bar{\otimes} N) \rtimes \Gamma$ where Γ acts trivially on N. Taking the (in this case constant) integral decomposition of $L^\infty(X)\bar{\otimes} N$ over its center $L^\infty(X)$, we can write $L^\infty(X)\bar{\otimes} N = \int_X^\oplus N \, d\mu(x)$. We refer to section 2.1.4 where we collected the basic definitions and properties of direct integrals. The main structural result on the Cartan subalgebras of $\mathcal{M}$ is the following theorem.

Theorem 1.1.1. *Let $\Gamma \in \mathcal{C}_{rss}$ and N be a II_1 factor. Suppose (X, μ) is a standard probability space and $\Gamma \curvearrowright X$ is a free ergodic p.m.p. action. Consider $\mathcal{M} := (L^\infty(X) \rtimes \Gamma)\bar{\otimes} N$ and let $A \subseteq \mathcal{M}$ be a subalgebra. Then the following are equivalent:*

1. *A is a Cartan subalgebra of $\mathcal{M}$,*
2. *A is unitarily conjugate to a subalgebra B of the form $B = \int_X^\oplus B_x \, d\mu(x) \subseteq L^\infty(X)\bar{\otimes} N$ satisfying*
   - *B_x is a Cartan subalgebra of N for almost every x,*
   - *For every $g \in \Gamma$, B_x is unitarily conjugate to B_{gx} inside N for almost every x.*

Moreover it will follow from the proof of Theorem 1.1.1 that two Cartan subalgebras A, B of $\mathcal{M}$ contained in $L^\infty(X)\bar{\otimes}N$ are unitarily conjugate if and only if A_x is unitarily conjugate to B_x for almost every $x \in X$, where we wrote $A = \int_X^\oplus A_x\, dx$ and $B = \int_X^\oplus B_x\, dx$. It turns out that this is in general not the case anymore for conjugacy by an automorphism. However, in specific cases, a similar result will hold (see section 3.3). We also note that the assumption $\Gamma \in \mathcal{C}_{rss}$ is only needed for the proof of (1) $\Rightarrow$ (2). In particular, for any countable group Γ, every subalgebra B as in (2) of Theorem 1.1.1 will be a Cartan subalgebra of $(L^\infty(X) \rtimes \Gamma)\bar{\otimes}N$.

One can also derive the following corollary in case N has at most one Cartan subalgebra up to unitary conjugacy.

Corollary 1.1.2. *If N has either no Cartan subalgebras or a unique Cartan subalgebra up to unitary conjugacy, then the same holds for $\mathcal{M} = (L^\infty(X) \rtimes \Gamma)\bar{\otimes}N$ as in Theorem 1.1.1.*

For any Polish group G acting continuously on a Polish space Y we will write $\mathcal{R}(G \curvearrowright Y)$ for its corresponding orbit equivalence relation. The proof of theorem 1.1.1 will lead to a criterion guaranteeing that the Cartan subalgebras of $\mathcal{M}$ are not classifiable by countable structures. This will allow us to prove the following result. Note that we don't need the assumption $\Gamma \in \mathcal{C}_{rss}$ here, since we only use the part of Theorem 1.1.1 which doesn't need this assumption, namely the proof that (2) $\Rightarrow$ (1). For the definition of strong ergodicity, we refer to Section 2.2.2.

Theorem 1.1.3. *Let Γ be a countable group and N be a II$_1$ factor. Suppose (X, μ) is a standard probability space and $\Gamma \curvearrowright X$ is a free ergodic p.m.p. action that is not strongly* *ergodic. Assume furthermore that $\mathcal{R}(\mathcal{U}(N) \curvearrowright \mathrm{Cartan}(N))$ is not smooth. Then the Cartan subalgebras of $\mathcal{M} := (L^\infty(X) \rtimes \Gamma)\bar{\otimes} N$ up to unitary conjugacy are not classifiable by countable structures.*

Together with the results from [SV12] this gives the first concrete family of (non-hyperfinite) II_1 factors whose Cartan subalgebras up to unitary conjugacy are not classifiable by countable structures. Indeed, one can take any countable group Γ admitting an ergodic non-strongly ergodic action (i.e. without property (T), cf. [CW80]. In fact given any non-property (T) group Γ, the generic action of Γ is ergodic but not strongly ergodic, see [Kec10, Corollary 12.4].) One can then consider any free ergodic non-strongly ergodic p.m.p. action of Γ on a standard probability space (X, μ). Taking N to be the II_1 factor from [SV12, Theorem 2(1)], $(L^\infty(X) \rtimes \Gamma)\bar{\otimes} N$ will satisfy all assumptions in the above theorem.

In the course of the proof of Theorem 1.1.3 we will establish the following result, which appears to be of independent interest. We denote by $\mathrm{Hom}(\mathcal{R}(\Gamma \curvearrowright X), E_0)$ the space of homomorphisms (see Definition 2.3.14) from $\mathcal{R}(\Gamma \curvearrowright X)$ to E_0, where E_0 is the equivalence relation on $\{0,1\}^{\mathbb{N}}$ given by

$$\mathbf{x} E_0 \mathbf{y} \quad \Leftrightarrow \quad \exists N \in \mathbb{N}, \forall n \geq N : x_n = y_n.$$

Also, for $\varphi, \psi \in \mathrm{Hom}(\mathcal{R}(\Gamma \curvearrowright X), E_0)$ we let $\varphi \sim \psi$ if and only if $\varphi(x) E_0 \psi(x)$ almost everywhere.

Theorem 1.1.4. *Let Γ be a countable group, (X, μ) a standard probability space and $\Gamma \curvearrowright$*

X an ergodic p.m.p. action that is not strongly ergodic. Then $(\mathrm{Hom}(\mathcal{R}(\Gamma \curvearrowright X), E_0), \sim)$ *is not classifiable by countable structures.*

Since an action that is strongly ergodic (also called E_0-ergodic) does not admit any nontrivial homomorphisms to E_0 (see [JS87], or [HK05, Theorem A2.2] for a proof of exactly this statement), we get the following nice dichotomy for actions of countable groups.

Corollary 1.1.5. *Let Γ be a countable group, (X, μ) a standard probability space and $\Gamma \curvearrowright X$ an ergodic p.m.p. action. Then* $(\mathrm{Hom}(\mathcal{R}(\Gamma \curvearrowright X), E_0), \sim)$ *is either trivial or not classifiable by countable structures, depending on whether the action is strongly ergodic or not.*

Moreover, it follows easily from Theorem 1.1.1 and Theorem 1.1.3 that the II_1 factors involved there actually satisfy the following dichotomy property.

Theorem 1.1.6. *Let $\Gamma \in \mathcal{C}_{rss}$, (X, μ) be a standard probability space and N be a II_1 factor. Suppose $\Gamma \curvearrowright X$ is a free ergodic p.m.p. action that is not strongly ergodic and consider $\mathcal{M} := (L^\infty(X) \rtimes \Gamma)\bar{\otimes} N$. Then* $\mathcal{R}(\mathcal{U}(\mathcal{M}) \curvearrowright \mathrm{Cartan}(\mathcal{M}))$ *is either smooth or not classifiable by countable structures. Moreover, the former holds if and only if* $\mathcal{R}(\mathcal{U}(N) \curvearrowright \mathrm{Cartan}(N))$ *is smooth and this regardless of whether the action is strongly ergodic or not.*

In section 3.3 we will discuss the equivalence relation of conjugacy by an automorphism on Cartan subalgebras. Using slightly different methods it turns out that for specific choices of N, the same construction as in Theorems 1.1.3 and 1.1.6 also gives II_1 factors for which the Cartan subalgebras up to conjugacy by an automorphism are not classifiable by countable structures.

Theorem 1.1.7. *Suppose N is the II_1 factor constructed in [SV12, Theorem 2(1)]. Let $\Gamma \in \mathcal{C}_{rss}$, (X, μ) be a standard probability space and $\Gamma \curvearrowright X$ be a free ergodic p.m.p. action that is not strongly ergodic. Then the equivalence relation of Cartan subalgebras of $\mathcal{M} = (L^\infty(X) \rtimes \Gamma)\bar{\otimes}N$ up to conjugacy by an automorphism is not classifiable by countable structures.*

The proof of Theorem 1.1.7 relies more on the structure of N than the proof of Theorem 1.1.3. We will formulate the exact requirements on N in Theorem 3.3.1. In particular, the proof of this result will allow us to get the above specific examples of II_1 factors for which the Cartan subalgebras up to conjugacy by an automorphism are not classifiable by countable structures, but it will not allow us to get (in this way) a general dichotomy result as in Theorem 1.1.6.

Note that all our results we mentioned so far concern non-hyperfinite II_1 factors. As we already mentioned in the beginning of the introduction, the hyperfinite II_1 factor R has a unique Cartan subalgebra up to conjugacy by an automorphism. On the other hand, J. Packer constructed in [Pac85, Theorem 4.4] an uncountable family of Cartan subalgebras of R no two of which are unitarily conjugate. Combining her results with some turbulence results for cocycles from [Kec10], we will prove the following stronger statement in section 3.4.

Theorem 1.1.8. *Cartan subalgebras of the hyperfinite II_1 factor R up to unitary conjugacy are not classifiable by countable structures.*

It is quite straightforward to show that if M is a II_1 factor with a Cartan subalgebra A and N is any II_1 factor, then classifying the Cartan subalgebras of $M\bar{\otimes}N$ is at least as complicated as classifying the Cartan subalgebras of N (by considering Cartan subalgebras of the form $A\bar{\otimes}B$, with $B \subseteq N$ Cartan, see Proposition 3.4.5 for the precise statement). Together with Theorem 1.1.8 this then immediately implies the following result on McDuff II_1 factors.

Corollary 1.1.9. *Let M be a McDuff II_1 factor with at least one Cartan subalgebra. Then the Cartan subalgebras of M up to unitary conjugacy are not classifiable by countable structures.*

Ideas behind the proofs in Chapter 3

Let us sketch the proofs of the main structural result of Chapter 3 (Theorem 1.1.1) and of our non-classifiability result for homomorphisms to E_0 (Theorem 1.1.4). Afterwards, we indicate briefly how the non-classifiability results for Cartan subalgebras (Theorems 1.1.3 and 1.1.7) follow from these.

Theorem 1.1.1. Recall our setup for Theorem 1.1.1, where $\Gamma \in \mathcal{C}_{rss}$, N is a II_1 factor, (X, μ) a standard probability space and $\Gamma \curvearrowright X$ a free ergodic p.m.p. action. We consider $\mathcal{M} := (L^\infty(X) \rtimes \Gamma)\bar{\otimes}N$. The proof of Theorem 1.1.1 can be subdivided into the following two steps.

1. *Describe the Cartan subalgebras of* $\mathcal{M} \cong (L^\infty(X)\bar{\otimes}N) \rtimes \Gamma$ *contained in* $L^\infty(X)\bar{\otimes}N$.

Note that by step 2 it indeed suffices to characterize these Cartan subalgebras. Also note that we don't need the condition $\Gamma \in \mathcal{C}_{rss}$ until step 2, before which all results hold for any countable group Γ.

Writing $L^\infty(X)\bar{\otimes}N = \int_X^\oplus N\, d\mu(x)$, we will see in Lemmas 2.1.13 and 3.1.2 that it is quite straightforward to show from the basic properties of direct integrals that a Cartan subalgebra A contained in $L^\infty(X)\bar{\otimes}N$ can be written as a direct integral

$$A = \int_X^\oplus A_x\, d\mu(x) \tag{1.1}$$

where each A_x is a Cartan subalgebra of N. Moreover, one observes that the canonical unitaries u_g associated to the group elements of Γ act on $L^\infty(X)\bar{\otimes}N = \int_X^\oplus N\, d\mu(x)$ by shifting the integral components. Involving Popa's intertwining technique, this will imply the following result of independent interest, see also Lemma 3.1.3.

Lemma. *A subalgebra* A *of* $L^\infty(X)\bar{\otimes}N$ *of the form* (1.1) *is a Cartan subalgebra of* $(L^\infty(X) \rtimes \Gamma)\bar{\otimes}N$ *if and only if the integral components of* A *within the same* Γ*-orbit are unitarily conjugate.*

This in particular generalizes a result from Feldman and Moore ([FM77, Theorem II.10]) saying that two Cartan subalgebras A_1, A_2 of a II_1 factor N are unitarily conjugate if and only if $\mathrm{diag}(A_1, A_2)$ is a Cartan subalgebra of $M_2(N)$. Indeed, one can view $M_2(N)$ as $(L^\infty(\{0,1\}) \rtimes \mathbb{Z}/2\mathbb{Z})\bar{\otimes}N$.

2. *Show that every Cartan subalgebra of $\mathcal{M}$ is unitarily conjugate to one contained in $L^\infty(X)\bar{\otimes}N$.*

   This is where the assumption $\Gamma \in \mathcal{C}_{rss}$ comes into play. Using the dichotomy 2.1.4 for such groups, it follows easily that, given any Cartan subalgebra $A \subseteq \mathcal{M}$, we must have $A \prec_{\mathcal{M}} L^\infty(X)\bar{\otimes}N$ (see Theorem 2.1.1). Carefully exploiting the structure of $\mathcal{M}$, we will see in Lemma 3.1.4 that this implies the existence of a Cartan subalgebra B of $\mathcal{M}$ contained in $L^\infty(X)\bar{\otimes}N$ such that $A \prec_{\mathcal{M}} B$, which by [Pop06a, Theorem A.1] will allow us to finish the proof of Theorem 1.1.1.

Theorem 1.1.4. For the proof of Theorem 1.1.4 we start with any countable group Γ, a standard probability space (X, μ) and any free ergodic p.m.p. action $\Gamma \curvearrowright X$ that is not strongly ergodic. In order to construct homomorphisms from $\mathcal{R}(\Gamma \curvearrowright X)$ to E_0, the non-strong ergodicity of $\Gamma \curvearrowright X$ will come into play through the use of almost invariant sequences (see Section 2.2.2). Using the fundamental results of [Dye59, JS87] we will construct a lot of nontrivial almost invariant sequences for the action $\Gamma \curvearrowright X$, namely one for every element $\mathbf{t} \in (0,1)^{\mathbb{N}}$. Following [JS87] we can associate a map $X \to \{0,1\}^{\mathbb{N}}$ to every almost invariant sequence $(A_n)_n$ via

$$X \to \{0,1\}^{\mathbb{N}} : x \mapsto (\mathbb{1}_{A_n}(x))_n.$$

This construction will associate to every element of $(0,1)^{\mathbb{N}}$ a homomorphism from $\mathcal{R}(\Gamma \curvearrowright X)$ to E_0. Theorem 1.1.4 will now follow from Hjorth's theory of turbulence (see section 2.3.3 and [Hjo00]). More specifically we will see in Proposition 2.3.18 that the existence of a "nontrivial" homomorphism from $(0,1)^{\mathbb{N}}$ up to ℓ^1-equivalence to an equivalence relation E implies that E is not classifiable by countable structures. It turns out that this applies to our construction above, allowing us to finish the proof of Theorem 1.1.4.

Theorem 1.1.3. The proof of Theorem 1.1.3 will rely on the two aforementioned results in the following way.

- *Use Theorem 1.1.1 to reduce the problem to studying* $\mathrm{Hom}(\mathcal{R}(\Gamma \curvearrowright X), \mathcal{R}(\mathcal{U}(N) \curvearrowright \mathrm{Cartan}(N)))$.

  We will see in Lemma 3.2.1 that the proof of Theorem 1.1.1 implies that for getting the desired non-classifiability result on the equivalence relation of unitary conjugacy on $\mathrm{Cartan}(\mathcal{M})$ it is "enough" to study homomorphisms between $\mathcal{R}(\Gamma \curvearrowright X)$ and $\mathcal{R}(\mathcal{U}(N) \curvearrowright \mathrm{Cartan}(N))$ up to unitary conjugacy inside N almost everywhere.

- *Pick a II_1 factor N which already has "a lot" of Cartan subalgebras and use Theorem 1.1.4 to get enough such homomorphisms from them.*

  The standing assumption in the statement of Theorem 1.1.3 is that $\mathcal{R}(\mathcal{U}(N) \curvearrowright \mathrm{Cartan}(N))$ is not smooth, i.e. we cannot assign real numbers as complete invariants for this equivalence relation. It is known (see [HKL90, Theorem 1.1]) that this is equivalent to having a Borel reduction from E_0 to $\mathcal{R}(\mathcal{U}(N) \curvearrowright \mathrm{Cartan}(N))$. Intuitively this means we can find a copy of E_0 inside $\mathcal{R}(\mathcal{U}(N) \curvearrowright \mathrm{Cartan}(N))$. This will allow us to transfer the result in Theorem 1.1.4 to homomorphisms between $\mathcal{R}(\Gamma \curvearrowright X)$ and $\mathcal{R}(\mathcal{U}(N) \curvearrowright \mathrm{Cartan}(N))$, finishing the proof of Theorem 1.1.3.

Theorem 1.1.7. The proof of Theorem 1.1.7 is similar to the proof of Theorem 1.1.3, albeit slightly more technical. One of the differences is the first bullet point above. The reduction there works for Theorem 1.1.3 because of the fact that two Cartan subalgebras of $\mathcal{M}$ contained in $L^\infty(X)\bar{\otimes}N$ are unitarily conjugate in $\mathcal{M}$ if and only if the "slices" from their direct integral decompositions are unitarily conjugate in N. This does not hold for conjugation by an automorphism. Nevertheless it turns out that for specific choices of N, up to applying a partial automorphism of X, A being conjugate by an automorphism to B does imply that their slices are conjugate by a stable automorphism of N on a positive measure subset of X. Using this together with the II_1 factor N from [SV12] and an argument similar to that of the proof of Theorem 1.1.4 will give us the desired result.

1.1.2 Unique McDuff decompositions

In Chapter 4, we study in further detail McDuff's property for II_1 factors. As explained in the first part of the introduction, McDuff showed in [McD70] that M is McDuff, or in other words has a decomposition $M \cong N\bar{\otimes}R$ for some II_1 factor N, if and only if M admits two non-commuting central sequences. Our main goal is to investigate the complementary issue of when such a decomposition is unique.

To make this precise, following [HMV19], we say that a II_1 factor M admits a *McDuff decomposition* if it can be written as $M = N\bar{\otimes}R$, for some *non-McDuff* II_1 factor N. If two II_1 factors N and P are stably isomorphic, that is, if $N \cong P^t$ for some $t > 0$, then $N\bar{\otimes}R$ and $P\bar{\otimes}R$ are isomorphic. Thus, we say that a McDuff decomposition $M = N\bar{\otimes}R$

is unique if for any other McDuff decomposition $M = P\bar{\otimes}R$ we necessarily have that N and P are stably isomorphic. We restrict our attention to II_1 factors admitting a McDuff decomposition because if M is a McDuff II_1 factor not having a McDuff decomposition (see [Mar18, Corollary G] for examples), then any II_1 factor N satisfying $M \cong N\bar{\otimes}R$ is necessarily isomorphic to M and thus unique, up to isomorphism.

We discussed above the result of [Pop07b, Theorem 5.1], where S. Popa established the uniqueness of the McDuff decomposition of $M = N\bar{\otimes}R$ whenever N does not have property Gamma. In contrast to this result, the uniqueness problem for McDuff decompositions where the involved II_1 factor N has property Gamma is completely open. We make progress on this problem here, by establishing the first unique McDuff decomposition results in the spirit of [Pop07b] in the "property Gamma regime". Thus, we use methods from Popa's deformation/rigidity theory to provide the first classes of non-McDuff II_1 factors N with property Gamma such that $M = N\bar{\otimes}R$ has a unique McDuff decomposition, see Corollary 1.1.12 and Theorem 1.1.15. These classes will be obtained as a consequence of the main technical result of Chapter 4, which exploits in particular the by now familiar notion of a Cartan subalgebra.

Theorem 1.1.10. *Let N be a II_1 factor which admits a Cartan subalgebra A such that $N'\cap N^\omega \subset A^\omega$. Let P be a non-McDuff II_1 factor and $\theta : N\bar{\otimes}R \to P\bar{\otimes}R$ be an isomorphism. Then there exist a Cartan subalgebra $B \subset P$, a unitary $u \in P\bar{\otimes}R$, and some $t > 0$, such that $P' \cap P^\omega \subset B^\omega$ and $\theta(A^t) = uBu^*$, where we identify $N\bar{\otimes}R = N^t\bar{\otimes}R^{1/t}$.*

Moreover, $\mathcal{R}(B \subset P)$ is isomorphic to $\mathcal{R}(A \subset N)^t$.

Before giving examples of II_1 factors to which Theorem 1.1.10 applies, we briefly explain the notions used in its statement. We fix a free ultrafilter ω on $\mathbb{N}$, and denote by N^ω the *ultrapower* of a tracial von Neumann algebra N. Given a Cartan subalgebra $A \subset N$, we denote by $\mathcal{R}(A \subset N)$ the countable p.m.p. equivalence relation associated to the inclusion $A \subset N$ (see Sections 2.1 and 2.2.3 for details).

Example 1.1.11. Let Γ be a non-inner amenable group and $\Gamma \curvearrowright (X, \mu)$ be a free ergodic p.m.p. action. Consider the group measure space von Neumann algebra $N = L^\infty(X) \rtimes \Gamma$ associated to the action $\Gamma \curvearrowright (X, \mu)$ [MvN43]. Then N is a II_1 factor and $A = L^\infty(X) \subset N$ is a Cartan subalgebra. Moreover, [Cho82] implies that $N' \cap N^\omega \subset A^\omega$, and thus Theorem 1.1.10 applies to N. If the action $\Gamma \curvearrowright X$ is strongly ergodic, then N does not have property Gamma. In this case, Theorem 1.1.10 follows from [Pop07b, Theorem 5.1], which moreover shows that $\theta(N^t) = uPu^*$, for a unitary $u \in P\bar{\otimes}R$ and some $t > 0$. Theorem 1.1.10 is new whenever the action $\Gamma \curvearrowright X$ is not strongly ergodic.

To put Theorem 1.1.10 into a better perspective, let $A \subset N$ be an inclusion satisfying its hypothesis, and denote $\mathcal{R} = \mathcal{R}(A \subset N)$. A well-known result of Feldman and Moore [FM77] shows that N is isomorphic to the von Neumann algebra $L_w(\mathcal{R})$ associated to $\mathcal{R}$ and a 2-cocycle $w \in \mathrm{H}^2(\mathcal{R}, \mathbb{T})$. Theorem 1.1.10 thus leads to the following rigidity statement: any non-McDuff II_1 factor P such that $N\bar{\otimes}R \cong P\bar{\otimes}R$ is necessarily isomorphic to $L_v(\mathcal{R})^t$, for some $t > 0$ and a 2-cocycle $v \in \mathrm{H}^2(\mathcal{R}, \mathbb{T})$.

While we were unable to determine whether v must be cohomologous to w in general, this is automatically satisfied if the equivalence relation $\mathcal{R}$ is treeable, leading to the following:

Corollary 1.1.12. *Let $n \geq 2$ and $\mathbb{F}_n \curvearrowright (X, \mu)$ be a free ergodic p.m.p. action. Put $N = L^\infty(X) \rtimes \mathbb{F}_n$. Let P be any II_1 factor such that $N \bar{\otimes} R$ and $P \bar{\otimes} R$ are isomorphic. Then P is either isomorphic to N^t, for some $t > 0$, or to $N \bar{\otimes} R$.*

As a particular case of Corollary 1.1.12, we derive a new result for free group measure space factors. Consider a free ergodic p.m.p. action $\mathbb{F}_p \curvearrowright (Y, \nu)$, for some $p \geq 2$, and N as in the corollary. By the breakthrough theorem of S. Popa and S. Vaes [PV14a], mentioned above, $L^\infty(X)$ is the unique Cartan subalgebra of N, up to unitary conjugacy. Using this result and applying Corollary 1.1.12 to $P = L^\infty(Y) \rtimes \mathbb{F}_p$, we deduce that $N \bar{\otimes} R$ and $P \bar{\otimes} R$ are isomorphic if and only if the actions $\mathbb{F}_n \curvearrowright X$ and $\mathbb{F}_p \curvearrowright Y$ are stably orbit equivalent. A result of D. Gaboriau [Gab00] further implies that if exactly one of n or p is finite, then $N \bar{\otimes} R \ncong P \bar{\otimes} R$.

Remark. Let $N = L^\infty(X) \rtimes \mathbb{F}_n$, where $\mathbb{F}_n \curvearrowright (X, \mu)$ is a free ergodic but not strongly ergodic p.m.p. action. Corollary 1.1.12 shows that if we can decompose $N \bar{\otimes} R = P \bar{\otimes} R$, for some non-McDuff II_1 factor P, then there is an abstract isomorphism between N^t and P, for some $t > 0$. This result is optimal in the sense that it cannot be improved to deduce that N^t and P are unitarily conjugate, i.e., that the isomorphism between N^t and P is implemented by a unitary from $N \bar{\otimes} R$. Indeed, since N has property Gamma, [Hof16, Theorem B] implies that $N \bar{\otimes} R$ admits an (approximately inner) automorphism θ such that $\theta(N^t)$ is not unitarily conjugate to N, inside $N \bar{\otimes} R$, for any $t > 0$.

Before stating the next corollary to Theorem 1.1.10, we recall that the reason we could not deduce unique McDuff decomposition for every II_1 factor to which it applies, is the presence of a 2-cocycle that twists the von Neumann algebra. This difficulty does not appear at the level of the equivalence relations, which allows us to deduce the following:

Corollary 1.1.13. *Let $\mathcal{R}_1$ and $\mathcal{R}_2$ be countable ergodic p.m.p. equivalence relations on probability spaces (X_1, μ_1) and (X_2, μ_2), respectively. Assume that $L(\mathcal{R}_1)' \cap L(\mathcal{R}_1)^\omega \subset L^\infty(X_1)^\omega$ and that $L(\mathcal{R}_2)$ is not McDuff. Suppose that $\mathcal{R}_1 \times \mathcal{T}$ is isomorphic to $\mathcal{R}_2 \times \mathcal{T}$, where $\mathcal{T}$ is a hyperfinite ergodic p.m.p. equivalence relation on a probability space (Y, ν). Then $\mathcal{R}_2$ is isomorphic to $\mathcal{R}_1^t$, for some $t > 0$.*

Corollary 1.1.13 in particular implies that if $\mathcal{R}_1$ and $\mathcal{R}_2$ are the orbit equivalence relations of any free ergodic p.m.p. actions of any non-inner amenable groups, then $\mathcal{R}_1 \times \mathcal{T}$ is isomorphic to $\mathcal{R}_2 \times \mathcal{T}$ if and only if $\mathcal{R}_1$ is stably isomorphic to $\mathcal{R}_2$.

Next, we return to the uniqueness problem for McDuff decompositions of II_1 factors. Before providing another class of II_1 factors with a unique McDuff decomposition in Theorem 1.1.15 below, we first introduce a property for equivalence relations motivated by a problem posed by Jones and Schmidt. In [JS87] they proved that if $\mathcal{S}$ is an ergodic but not strongly ergodic (see Section 2.2.2) countable p.m.p. equivalence relation on a probability space (X, μ), then $\mathcal{S}$ admits a hyperfinite quotient. Specifically, there exist a hyperfinite ergodic p.m.p. equivalence relation $\mathcal{T}$ on a probability space (Y, ν) and a factor map $\pi : (X, \mu) \to (Y, \nu)$ such that $\pi([x]_{\mathcal{S}}) = [\pi(x)]_{\mathcal{T}}$, for almost every $x \in X$ (see [JS87, Theorem 2.1]). Jones and Schmidt asked whether one can always find such

$\mathcal{T}, (Y, \nu), \pi$ with the additional property that $\mathcal{S}_0 = \{(x_1, x_2) \in \mathcal{S} \mid \pi(x_1) = \pi(x_2)\}$ is strongly ergodic on almost every ergodic component of $\mathcal{S}_0$ (see [JS87, Problem 4.3]). This problem was also mentioned by Schmidt in [Sch85, Problem 2.1] as an important question left unanswered by [JS87, Theorem 2.1]. Motivated by this problem, we introduce the following:

Definition 1.1.14. We say that a countable ergodic p.m.p. equivalence relation $\mathcal{S}$ on a probability space (X, μ) has the *Jones-Schmidt property* if there exist a hyperfinite ergodic p.m.p. equivalence relation $\mathcal{T}$ on a probability space (Y, ν) and a factor map $\pi : (X, \mu) \to (Y, \nu)$ such that

1. $\pi([x]_{\mathcal{S}}) = [\pi(x)]_{\mathcal{T}}$, for almost every $x \in X$, and
2. the equivalence relation $\mathcal{S}_0 = \{(x_1, x_2) \in \mathcal{S} \mid \pi(x_1) = \pi(x_2)\}$ is strongly ergodic on almost every ergodic component of $\mathcal{S}_0$.

◀

Using this terminology, [JS87, Problem 4.3] can be reformulated as follows: does every countable ergodic but not strongly ergodic p.m.p. equivalence relation have the Jones-Schmidt property?

Our next main result shows that $N \bar{\otimes} R$ has a unique McDuff decomposition whenever the above question has an affirmative answer for the equivalence relation arising from the inclusion of a Cartan subalgebra $A \subset N$ to which Theorem 1.1.10 applies.

Theorem 1.1.15. *Let N be a II_1 factor which admits a Cartan subalgebra A such that $N' \cap N^\omega \subset A^\omega$ and $\mathcal{R}(A \subset N)$ has the Jones-Schmidt property. Let P be any II_1 factor such that $N\bar{\otimes}R$ and $P\bar{\otimes}R$ are isomorphic.*
Then P is either isomorphic to N^t, for some $t > 0$, or to $N\bar{\otimes}R$.

Moreover, assume that P is not McDuff. Then for any isomorphism $\theta : N\bar{\otimes}R \to P\bar{\otimes}R$, we can find isomorphisms $\theta_1 : N^s \to P$, $\theta_2 : R^{1/s} \to R$, for some $s > 0$, and an approximately inner automorphism $\Psi : N\bar{\otimes}R \to N\bar{\otimes}R$ such that $\theta = (\theta_1 \otimes \theta_2) \circ \Psi$, where we identify $N\bar{\otimes}R = N^s\bar{\otimes}R^{1/s}$.

Remark. The moreover part of Theorem 1.1.15 shows that if N is a II_1 factor as in its statement, then $N\bar{\otimes}R$ admits a unique McDuff decomposition, up to "stable approximately inner conjugacy": given any other McDuff decomposition $N\bar{\otimes}R = P\bar{\otimes}R$, there exists an isomorphism $N^s \cong P$, for some $s > 0$, which is implemented by an approximately inner automorphism of $N\bar{\otimes}R$. As in the Remark following Corollary 1.1.12, by [Hof16, Theorem B] this is optimal and cannot be improved to an implementation by an inner automorphism.

Example 1.1.16. Theorem 1.1.15 applies to a large class of group measure space II_1 factors. To see this, let Γ be an infinite group, Σ an infinite amenable group, and $\delta : \Gamma \to \Sigma$ an onto group homomorphism. Let $\Gamma \curvearrowright (Z, \eta)$, $\Sigma \curvearrowright (Y, \nu)$ be free ergodic p.m.p. actions such that the action $\ker\delta \curvearrowright (Z, \eta)$ is strongly ergodic. Let $(X, \mu) = (Y, \nu) \times (Z, \eta)$ and consider the action $\Gamma \curvearrowright (X, \mu)$ given by $g \cdot (y, z) = (\delta(g)y, gz)$, for all $g \in \Gamma, y \in Y, z \in Z$.

Then $\mathcal{S} = \mathcal{R}(\Gamma \curvearrowright X) := \{(x_1, x_2) \in X \times X \mid \Gamma \cdot x_1 = \Gamma \cdot x_2\}$ has the Jones-

Schmidt property. To see this, let $\mathcal{T} = \mathcal{R}(\Sigma \curvearrowright Y)$, and define the factor map $\pi : X \to Y$ by $\pi(y, z) = y$. Since Σ is infinite amenable and the action $\Sigma \curvearrowright Y$ is ergodic, $\mathcal{T}$ is a hyperfinite ergodic p.m.p. equivalence relation by [OW80]. Also, $\pi([x]_{\mathcal{S}}) = [\pi(x)]_{\mathcal{T}}$ for all $x \in X$. Moreover, if $\mathcal{S}_0 = \{(x_1, x_2) \in \mathcal{S} \mid \pi(x_1) = \pi(x_2)\}$, then $\mathcal{S}_0$ is equal to $\{(y, y) | y \in Y\} \times \mathcal{R}(\ker \delta \curvearrowright Z)$. Thus, every ergodic component of $\mathcal{S}_0$ is isomorphic to $\mathcal{R}(\ker \delta \curvearrowright Z)$, and hence is strongly ergodic.

Moreover, assume that Γ is not inner amenable, denote $N = L^\infty(X) \rtimes \Gamma$ and $A = L^\infty(X)$. Then $N' \cap N^\omega \subset A^\omega$ by [Cho82], and therefore Theorem 1.1.15 applies to N.

The above example gives a wide class of examples of countable ergodic p.m.p. equivalence relations with the Jones-Schmidt property. We next show that there exist countable ergodic but not strongly ergodic p.m.p. equivalence relations that fail to have the Jones-Schmidt property. This settles the question asked by Jones and Schmidt in [JS87, Problem 4.3]. More specifically, in Theorems 1.1.17 and 1.1.18 below, we give two large classes of free ergodic p.m.p. actions whose orbit equivalence relations do not have the Jones-Schmidt property. First, we prove that this holds for any infinite product of strongly ergodic actions:

Theorem 1.1.17. *For every $n \in \mathbb{N}$, let $\Gamma_n \curvearrowright (X_n, \mu_n)$ be a strongly ergodic free p.m.p. action of an infinite countable group Γ_n. Define $\Gamma = \bigoplus_{n\in\mathbb{N}} \Gamma_n$, $(X, \mu) = \prod_{n\in\mathbb{N}} (X_n, \mu_n)$, and consider the product action $\Gamma \curvearrowright (X, \mu)$ given by $g \cdot x = (g_n \cdot x_n)_n$, for all $g = (g_n)_n \in \Gamma$ and $x = (x_n)_n \in X$.*

Then the orbit equivalence relation $\mathcal{R}(\Gamma \curvearrowright X)$ does not have the Jones-Schmidt property.

A countable group Λ admits strongly ergodic free p.m.p. actions if and only if it is non-amenable. Moreover, if Λ is non-amenable, then the Bernoulli action $\Lambda \curvearrowright (Y_0, \nu_0)^{\Lambda}$ is strongly ergodic, for any non-trivial probability space (Y_0, ν_0). Theorem 1.1.17 therefore applies to any infinite direct sum $\Gamma = \bigoplus_{n\in\mathbb{N}} \Gamma_n$ of non-amenable groups and shows that any such group admits free ergodic p.m.p. actions whose orbit equivalence relations fail the Jones-Schmidt property. Next, we provide a second such result which, unlike Theorem 1.1.17, covers actions of non-abelian free groups (see Example 1.1.19).

Theorem 1.1.18. *Let Γ be a countable group. For every $n \in \mathbb{N}$, let $\Gamma \curvearrowright (X_n, \mu_n)$ be a free ergodic p.m.p. action such that the diagonal action $\Gamma \curvearrowright (X_n \times X_n, \mu_n \times \mu_n)$ has spectral gap. Assume that we can find $F_{n,k} \in L^{\infty}(X_n)$, for all $n, k \in \mathbb{N}$, such that $\sup_{n,k} \|F_{n,k}\|_{\infty} \leq 1$, $\inf_{n,k} \|F_{n,k}\|_2 > 0$,*

- *$F_{n,k} \to 0$ weakly in $L^2(X_n)$, as $k \to \infty$, for every $n \in \mathbb{N}$, and*
- *$\lim\limits_{n\to\infty} \left(\sup_{k\in\mathbb{N}} \|F_{n,k} \circ g - F_{n,k}\|_2\right) = 0$, for every $g \in \Gamma$.*

Consider the diagonal action $\Gamma \curvearrowright (X, \mu) = \prod_{n\in\mathbb{N}} (X_n, \mu_n)$ given by $g \cdot x = (g \cdot x_n)_n$, for all $g \in \Gamma$ and $x = (x_n)_n \in X$.

Then the orbit equivalence relation $\mathcal{R}(\Gamma \curvearrowright X)$ does not have the Jones-Schmidt property.

Example 1.1.19. Let $\Gamma = \mathbb{F}_m$ be the free group on $2 \leq m \leq \infty$ generators. Denote by $|g|$ the word length of an element $g \in \Gamma$ with respect to a free set of generators. Let

$t > 0$. By a well-known result of Haagerup [Haa79], the function $\varphi_t : \Gamma \to \mathbb{R}$ given by $\varphi_t(g) = e^{-t|g|}$ is positive definite. Denote by $\pi_t : \Gamma \to \mathcal{O}(\mathcal{H}_t)$ the GNS orthogonal representation associated to φ_t. Let $\tilde{\mathcal{H}}_t = \oplus_{i\in\mathbb{N}}\mathcal{H}_t$ and $\tilde{\pi}_t = \oplus_{i\in\mathbb{N}}\pi_t : \Gamma \to \mathcal{O}(\tilde{\mathcal{H}}_t)$ be the direct sum of infinitely many copies of π_t. Let $\Gamma \curvearrowright (X_t, \mu_t)$ be the Gaussian action associated to $\widetilde{\pi}_t$ (see, e.g., [Gla03, Chapter 3]). Let (t_n) be any sequence of positive numbers converging to 0. As we will prove in Remark 4.4.2, the diagonal action of Γ on $(X, \mu) = \prod_{n\in\mathbb{N}}(X_{t_n}, \mu_{t_n})$ satisfies the hypothesis of Theorem 1.1.18.

Let $\mathcal{T}$ denote a hyperfinite ergodic p.m.p. equivalence relation. Similar to the von Neumann algebra case considered above, one can ask the following question: given two countable ergodic p.m.p. equivalence relations $\mathcal{R}_1$ and $\mathcal{R}_2$, when are their "stabilisations" $\mathcal{R}_1 \times \mathcal{T}$ and $\mathcal{R}_2 \times \mathcal{T}$ isomorphic? We recall from [JS87] that a countable ergodic p.m.p. equivalence relation $\mathcal{S}$ is called *stable* if it can be decomposed as $\mathcal{S} = \mathcal{R} \times \mathcal{T}$, for some countable ergodic equivalence relation $\mathcal{R}$. In [JS87, Theorem 3.4] stability is shown to be equivalent to an asymptotic property: $\mathcal{S}$ is stable if and only if its full group contains a nontrivial asymptotically central sequence. In recent years, there has been a surge of interest in the study of stability for equivalence relations, see, e.g., the articles [Kid14, TD14, Kid17b, DV18, Mar18].

We contribute to this study here by investigating the problem of when a decomposition as above is unique. By analogy with the case of McDuff II_1 factors, we say that a countable ergodic p.m.p. equivalence relation $\mathcal{S}$ admits a *stable decomposition* if it can be written as $\mathcal{S} = \mathcal{R} \times \mathcal{T}$ for some non-stable countable ergodic p.m.p. equivalence relation $\mathcal{R}$. If two countable ergodic p.m.p. equivalence relations $\mathcal{R}_1$ and $\mathcal{R}_2$ are stably isomorphic, that is, if $\mathcal{R}_1 \cong \mathcal{R}_2^t$ for some $t > 0$, then $\mathcal{R}_1 \times \mathcal{T}$ and $\mathcal{R}_2 \times \mathcal{T}$ are isomorphic. Thus, we say that a stable decomposition $\mathcal{S} = \mathcal{R}_1 \times \mathcal{T}$ is unique if for any other stable decomposition $\mathcal{S} = \mathcal{R}_2 \times \mathcal{T}$ we necessarily have that $\mathcal{R}_1$ and $\mathcal{R}_2$ are stably isomorphic.

Note that Corollary 1.1.13 already gave the first result in this direction: let $\mathcal{R}_1$ be a countable ergodic p.m.p. equivalence relation on a probability space (X_1, μ_1) such that $L(\mathcal{R}_1)' \cap L(\mathcal{R}_1)^\omega \subset L^\infty(X_1)^\omega$. Then any countable ergodic p.m.p. equivalence relation $\mathcal{R}_2$ whose von Neumann algebra $L(\mathcal{R}_2)$ is not McDuff and satisfies $\mathcal{R}_1 \times \mathcal{T} \cong \mathcal{R}_2 \times \mathcal{T}$ must be stably isomorphic to $\mathcal{R}_1$. However, by [CJ82], there exist equivalence relations $\mathcal{R}_2$ which are strongly ergodic, and thus not stable, such that $L(\mathcal{R}_2)$ is McDuff. As such, Corollary 1.1.13 does not imply that $\mathcal{R}_1 \times \mathcal{T}$ has a unique stable decomposition.

Our next main result completely settles the uniqueness problem for stable decompositions of $\mathcal{R}_1 \times \mathcal{T}$ under the assumption that $\mathcal{R}_1$ is strongly ergodic. More precisely, we have the following:

Theorem 1.1.20. *Let $\mathcal{R}_1$ and $\mathcal{R}_2$ be countable ergodic p.m.p. equivalence relations on probability spaces (X_1, μ_1) and (X_2, μ_2), respectively. Assume that $\mathcal{R}_1$ is strongly ergodic. Suppose that $\mathcal{R}_1 \times \mathcal{T}$ is isomorphic to $\mathcal{R}_2 \times \mathcal{T}$, where $\mathcal{T}$ is a hyperfinite ergodic p.m.p. equivalence relation on a probability space (Y, ν).*

Then either

1. *$\mathcal{R}_2$ is also strongly ergodic and $\mathcal{R}_2 \cong \mathcal{R}_1^t$, for some $t > 0$, or*

2. *$\mathcal{R}_2$ is stable and $\mathcal{R}_2 \cong \mathcal{R}_1 \times \mathcal{T}$.*

At the end of Chapter 4, we provide new characterisations of property Gamma for II_1 factors and of strong ergodicity for countable ergodic p.m.p. equivalence relations. We include these results here, although they are not directly related to the results stated above, as they appear to be of independent interest. Moreover, their proofs rely on some similar techniques as the previous results of Chapter 4.

On [JS87, page 92], Jones and Schmidt pointed out that their characterisation of strong ergodicity for countable equivalence relations [JS87, Theorem 2.1] has no obvious analogue in the setting of von Neumann algebras. We address this problem here by giving one such analogue. In particular, we show that any II_1 factor M with property Gamma admits a regular diffuse abelian von Neumann subalgebra.

Theorem 1.1.21. *Let M be a separable II_1 factor. Then the following are equivalent:*

1. *M has property Gamma.*
2. *There exist a hyperfinite subfactor $R \subset M$ and a Cartan subalgebra $A \subset R$ such that M is generated by R and $A' \cap M$.*
3. *There exists a regular abelian von Neumann subalgebra $A \subset M$ such that $\mathcal{R}(A \subset M)$ is hyperfinite and ergodic.*

In order to state our last result, let $\mathcal{S}$ be a countable p.m.p. equivalence relation on a probability space (X, μ) and E be a Borel equivalence relation on a standard Borel space Y. We recall that $\mathcal{S}$ is *E-ergodic* if for any Borel homomorphism $\varphi : X \to Y$, there exists $y \in Y$ such that $(\varphi(x), y) \in E$, for almost every $x \in X$. As mentioned before, it is shown in [HK05, Theorem A.2.2], that a countable ergodic p.m.p. equivalence relation $\mathcal{S}$ is strongly ergodic if and only if it is E_0-ergodic. We prove that strong ergodicity can also be characterized as E-ergodicity, where E is the orbit equivalence relation of the left translation action $\mathrm{Inn}(R) \curvearrowright \mathrm{Aut}(R)$.

Theorem 1.1.22. *Let $\mathcal{S}$ be a countable ergodic p.m.p. equivalence relation on a probability space (X, μ). Let $\mathcal{T}$ be a countable hyperfinite ergodic p.m.p. equivalence relation on a probability space (Y, ν). Let R denote the hyperfinite II_1 factor. Then the following are equivalent:*

1. *$\mathcal{S}$ is strongly ergodic.*
2. *For any measurable map $\varphi : X \to Aut(\mathcal{T})$ satisfying $\varphi(x_1)^{-1}\varphi(x_2) \in [\mathcal{T}]$ for almost every $(x_1, x_2) \in \mathcal{S}$, there exists $\alpha \in Aut(\mathcal{T})$ such that $\varphi(x)\alpha \in [\mathcal{T}]$, for almost every $x \in X$.*
3. *For any measurable map $\varphi : X \to Aut(R)$ satisfying $\varphi(x_1)^{-1}\varphi(x_2) \in Inn(R)$ for almost every $(x_1, x_2) \in \mathcal{S}$, there exists $\alpha \in Aut(R)$ such that $\varphi(x)\alpha \in Inn(R)$, for almost every $x \in X$.*

This chapter contains material from: P. Spaas, "Non-classification of Cartan subalgebras for a class of von Neumann algebras", *Advances in Mathematics*, vol. 332, pp.

510-552, 2018. The dissertation author was the primary investigator and author of this paper.

This chapter also contains material from: A. Ioana and P. Spaas, "A class of II_1 factors with a unique McDuff decomposition", preprint arXiv: 1808.02965, submitted. The dissertation author was one of the primary investigators and authors of this paper.

Chapter 2

Preliminaries

This chapter is devoted to all preliminary material necessary for the proofs of the main results in this dissertation later on. We fix notation and recall several important definitions and notions related to von Neumann algebras. For future reference, we also include a few useful lemmas. Some of these were known before, others were established in the course of proving our main results. We similarly introduce the necessary preliminaries on equivalence relations, descriptive set theory, and their interactions with von Neumann algebras.

2.1 von Neumann algebras

Let H be a Hilbert space. We denote by $\mathbb{B}(H)$ the space of bounded operators on H, which forms a *-algebra under the usual operations of adding and composing operators, and taking the adjoint. $\mathbb{B}(H)$ also carries several natural topologies, including but not limited to:

- The *norm topology*, induced by the usual operator norm $\|T\| = \sup_{\|\xi\|\leq 1} \|T\xi\|$.
- The *strong operator topology*, for which T_n converges to T if and only if $T_n\xi$ converges in H to $T\xi$ for all $\xi \in H$.
- The *weak operator topology*, for which T_n converges to T if and only if $\langle T_n\xi, \eta\rangle$ converges to $\langle T\xi, \eta\rangle$ for all $\xi, \eta \in H$.

A unital *-subalgebra $M \subseteq H$, is called a *von Neumann algebra* if M is closed in the weak operator topology. von Neumann's famous *double commutant theorem* asserts that the following are equivalent:

- M is a von Neumann algebra,
- M is closed in the strong operator topology,
- $M = M''$.

Here the *commutant* S' of a subset $S \subseteq \mathbb{B}(H)$ is defined as $S' := \{T \in \mathbb{B}(H) \mid Tx = xT \text{ for all } x \in S\}$.

Notation

We will mostly be concerned with tracial von Neumann algebras, i.e. von Neumann algebras with a faithful normal tracial state $\tau : M \to \mathbb{C}$. Throughout, we will denote by $\mathcal{U}(M)$ the unitary group of M, by $(M)_1$ the operator norm unit ball of M, by $\mathcal{Z}(M) :=$

$M' \cap M$ the center of M, and by $L^2(M)$ the completion of M with respect to the Hilbert norm $\|x\|_2 = \tau(x^*x)^{\frac{1}{2}}$. We note the well-known fact that this 2-norm turns $\mathcal{U}(M)$ into a Polish group. Unless stated otherwise, we will always assume that M is a *separable* von Neumann algebra, i.e. $L^2(M)$ is a separable Hilbert space. For a projection $p \in M$, we denote by $z(p)$ its central support, i.e. the smallest projection $z \in \mathcal{Z}(M)$ such that $zp = p$.

Let $P, Q \subset M$ be von Neumann subalgebras, which we will always assume to be unital. We denote by $E_P : M \to P$ the unique τ-preserving *conditional expectation* from M onto P, by $P' \cap M := \{x \in M \mid xy = yx, \text{ for all } y \in P\}$ the *relative commutant* of P in M, and by $\mathcal{N}_M(P) := \{u \in \mathcal{U}(M) \mid uPu^* = P\}$ the *normalizer* of P in M. We say that P is *regular* in M if the von Neumann algebra generated by $\mathcal{N}_M(P)$ is equal to M. We write $Q \subset_\varepsilon P$, for some $\varepsilon > 0$, if we have $\|x - E_P(x)\|_2 \leq \varepsilon$, for every $x \in (Q)_1$. Further we write $P \vee Q$ for the von Neumann subalgebra of M generated by P and Q. Let $e_P : L^2(M) \to L^2(P)$ be the orthogonal projection from $L^2(M)$ onto $L^2(P)$. *Jones' basic construction* of the inclusion $P \subseteq M$ is the von Neumann subalgebra of $B(L^2(M))$ generated by M and e_P, and is denoted by $\langle M, e_P \rangle$.

We will write $\mathrm{Aut}(M)$ for the group of (trace preserving) automorphisms of (M, τ). We endow $\mathrm{Aut}(M)$ with the Polish topology given by pointwise $\|.\|_2$-convergence. We also denote by $\mathrm{Inn}(M)$ the group of *inner* automorphisms of M, i.e. those of the form $\mathrm{Ad}(u)(x) = uxu^*$, where $u \in \mathcal{U}(M)$. If (X, μ) is a standard measure space and G is a Polish group (e.g., $\mathcal{U}(M)$ or $\mathrm{Aut}(M)$), we say that a map $f : X \to G$ is measurable if $f^{-1}(B)$ is a measurable subset of X, for every Borel subset $B \subset G$. For a subgroup $H < \mathrm{Aut}(M)$, we denote $M^H = \{x \in M \mid \theta(x) = x, \text{ for all } \theta \in H\}$.

Of special interest will be II_1 factors, i.e. infinite dimensional tracial von Neumann algebras with trivial center $\mathcal{Z}(M) = \mathbb{C}1$. A *Cartan subalgebra* of a II_1 factor M is a maximal abelian von Neumann subalgebra that is *regular*, i.e. $\mathcal{N}_M(A)'' = M$. Given a II_1 factor M, we write $\mathrm{Cartan}(M)$ for the space of Cartan subalgebras of M. We say that two Cartan subalgebras A and B are *unitarily conjugate in* M if there exists $u \in \mathcal{U}(M)$ such that $uAu^* = B$, we say they are *conjugate by an automorphism of* M if there exists $\alpha \in \mathrm{Aut}(M)$ such that $\alpha(A) = B$ and we say they are *conjugate by a stable automorphism of* M if there exist nonzero projections $p \in A$, $q \in B$ and a $*$-isomorphism $\alpha : pMp \to qMq$ such that $\alpha(Ap) = Bq$. We will write $A \sim_u B$, $A \sim_a B$, and $A \sim_{sa} B$ respectively for these notions of equivalence.

Let ω be a free ultrafilter on $\mathbb{N}$. Consider the C*-algebra $\ell^\infty(\mathbb{N}, M) = \{(x_n) \in M^{\mathbb{N}} \mid \sup \|x_n\| < \infty\}$ together with its closed ideal

$$\mathcal{I}_\omega = \{(x_n) \in \ell^\infty(\mathbb{N}, M) \mid \lim_{n\to\omega} \|x_n\|_2 = 0\}.$$

Then the C*-algebra $M^\omega := \ell^\infty(\mathbb{N}, M)/\mathcal{I}_\omega$ is in fact a tracial von Neumann algebra, called the *ultrapower* of M, whose canonical trace is given by $\tau_\omega(x) = \lim_{n\to\omega} \tau(x_n)$, for all $x = (x_n) \in M^\omega$. Recall that a von Neumann algebra is called *diffuse* if it does not contain a minimal projection. We then note that if M is diffuse (or has a diffuse direct summand), M^ω is non-separable. For an automorphism θ of M we denote still by θ the automorphism of M^ω given by $\theta((x_n)) = (\theta(x_n))$. In this manner, we view any subgroup $G < \mathrm{Aut}(M)$

as a subgroup $G < \mathrm{Aut}(M^\omega)$. Finally, if $M_n, n \in \mathbb{N}$, is a sequence of von Neumann subalgebras of M, then their *ultraproduct*, denoted by $\prod_\omega M_n$, can be realized as the von Neumann subalgebra of M^ω consisting of $x = (x_n)$ such that $\lim\limits_{n\to\omega} \|x_n - E_{M_n}(x_n)\|_2 = 0$.

2.1.1 Intertwining-by-bimodules

In [Pop06b], Popa introduced a powerful theory for deducing unitary conjugacy of subalgebras of tracial von Neumann algebras. This theory, which we recall next, will be a key tool in the remainder of this dissertation.

Theorem 2.1.1 ([Pop06b, Theorem 2.1 and Corollary 2.3])**.** *Let (M, τ) be a tracial von Neumann algebra and $P \subseteq pMp$, $Q \subseteq qMq$ be von Neumann subalgebras. Let $\mathcal{U} \subseteq \mathcal{U}(P)$ be a subgroup such that $\mathcal{U}'' = P$. Then the following are equivalent.*

- *There exist projections $p_0 \in P$, $q_0 \in Q$, a *-homomorphism $\psi : p_0Pp_0 \to q_0Qq_0$ and a nonzero partial isometry $v \in q_0Mp_0$ such that $\psi(x)v = vx$ for all $x \in p_0Pp_0$.*
- *There is no sequence $(u_n)_n \in \mathcal{U}$ such that $\|E_Q(x^*u_ny)\|_2 \to 0$ for all $x, y \in pMq$.*

Terminology. If one of the equivalent conditions of the above theorem holds, we say that *a corner of P embeds into Q inside M*, and we write $P \prec_M Q$. If $Pp' \prec_M Q$ for any nonzero projection $p' \in P' \cap pMp$, then we say that P embeds strongly into Q inside M, and we write $P \prec^s_M Q$.

If $P \subset_\varepsilon Q$, for some $\varepsilon < 1$, then Theorem 2.1.1 implies that $P \prec_M Q$. Moreover, this can be made quantitative, as follows:

Lemma 2.1.2. *In the above setting, assume that $P \subset_\varepsilon Q$, for some $\varepsilon > 0$. Then there exists a projection $p' \in \mathcal{Z}(P' \cap M)$ such that $Pp' \prec^s_M Q$ and $\tau(p') \geq 1 - \varepsilon$.*

Proof. Let $p' \in \mathcal{Z}(P' \cap M)$ be the maximal projection such that $Pp' \prec^s_M Q$. If we let $p'' = 1 - p'$, then by [DHI16, Lemma 2.3(3)] we get that $Pp'' \nprec_M Q$. On the other hand, for every $u \in \mathcal{U}(P)$ we have that $\|u - E_Q(u)\|_2 \leq \varepsilon$, hence $\Re\tau(up''E_Q(u)^*) = \tau(p'') + \Re\tau(up''(E_Q(u) - u)^*) \geq \tau(p'') - \varepsilon$, and therefore $\|E_Q(up'')\|_2 \geq \tau(p'') - \varepsilon$, for all $u \in \mathcal{U}(P)$. If $\tau(p'') - \varepsilon > 0$, then by Theorem 2.1.1 we would get that $Pp'' \prec_M Q$, which is a contradiction. Hence, $\tau(p'') \leq \varepsilon$ and thus $\tau(p') \geq 1 - \varepsilon$. □

2.1.2 Relative amenability

Recall that a tracial von Neumann algebra (M, τ) is called *amenable* if there exists a positive linear functional $\varphi : B(L^2(M)) \to \mathbb{C}$ such that $\varphi|_M = \tau$ and φ is M-central, i.e. $\varphi(xT) = \varphi(Tx)$ for all $x \in M$, $T \in B(L^2(M))$. In [OP10a] Ozawa and Popa introduced the analogous notion of relative amenability:

Definition 2.1.3. Suppose $P \subseteq pMp$, $Q \subseteq M$ are von Neumann subalgebras. Then we say that P is *amenable relative to Q inside M* if there exists a positive linear functional $\varphi : p\langle M, e_Q\rangle p \to \mathbb{C}$ such that $\varphi|_{pMp} = \tau$ and φ is P-central. ◀

2.1.3 Relatively strongly solid groups

In [Oza04] Ozawa established that the group von Neumann algebra of a non-elementary hyperbolic group is *solid*: the relative commutant of any diffuse von Neumann subalgebra is amenable. Later Ozawa and Popa strengthened this result in [OP10b] for free groups by proving that $L\mathbb{F}_n$, $2 \leq n \leq \infty$, is *strongly solid*: the normalizer of any diffuse amenable von Neumann subalgebra is still amenable. Later Chifan and Sinclair showed in [CS13] that this actually holds for all non-elementary hyperbolic groups.

The next breakthrough was realized by Popa and Vaes who showed in [PV14a, PV14b] that non-abelian free groups and more generally non-elementary hyperbolic groups are *relatively strongly solid*. Here we use the terminology from [CIK15, Definition 2.7].

Definition 2.1.4. A countable non-amenable group Γ is called *relatively strongly solid* if for any tracial crossed product $M := B \rtimes \Gamma$ and all von Neumann subalgebras $A \subseteq M$ with A amenable relative to B inside M we have either

1. $A \prec_M B$, or
2. $\mathcal{N}_M(A)''$ is still amenable relative to B inside M.

We denote the class of relatively strongly solid groups by $\mathcal{C}_{rss}$. ◀

Note that it follows from Definition 2.1.4 that $L^\infty(X) \rtimes \Gamma$ has $L^\infty(X)$ as its unique Cartan subalgebra up to unitary conjugacy if $\Gamma \in \mathcal{C}_{rss}$ and the action is free and ergodic, see also [PV14a, PV14b].

2.1.4 Direct integrals

In this section we recall the basic definitions and properties we need from the theory of direct integral decompositions of von Neumann algebras. A lot of it is taken from chapter 14 of [KR97]. We start with the direct integral of Hilbert spaces. Recall that a standard probability space (X, μ) is a probability space whose (Borel) σ-algebra is generated by the open sets of some Polish topology on X.

Definition 2.1.5. Let (X, μ) be a standard probability space. Let $\{\mathcal{H}_x\}$ be a family of separable Hilbert spaces indexed by the points x of X. A separable Hilbert space $\mathcal{H}$ is the *direct integral* of $\{\mathcal{H}_x\}$ over (X, μ), written $\mathcal{H} = \int_X^{\oplus} \mathcal{H}_x \, d\mu(x)$, if for each $\xi \in \mathcal{H}$ there exists a function $x \mapsto \xi(x)$ on X such that $\xi(x) \in \mathcal{H}_x$ for every x and

(i) $x \mapsto \langle \xi(x), \eta(x) \rangle$ is integrable and $\langle \xi, \eta \rangle = \int_X \langle \xi(x), \eta(x) \rangle \, d\mu(x)$ for all $\xi, \eta \in \mathcal{H}$,

(ii) if $\xi_x \in \mathcal{H}_x$ for every $x \in X$ and $x \mapsto \langle \xi_x, \eta(x) \rangle$ is integrable for every $\eta \in \mathcal{H}$ then there exists $\xi \in \mathcal{H}$ such that $\xi(x) = \xi_x$ for almost every $x \in X$.

We call $\int_X^{\oplus} \mathcal{H}_x \, d\mu(x)$ and $x \mapsto \xi(x)$ the *direct integral decompositions* of $\mathcal{H}$ and ξ respectively. ◀

Two very easy examples are the following.

Example 2.1.6. 1. The (discrete) direct sum of countably many Hilbert spaces $\{\mathcal{H}_n\}$ can be viewed as the direct integral of $\{\mathcal{H}_n\}$ over the natural numbers with the counting measure.

2. Given (X, μ) as above it is easy to check that $L^2(X, \mu) = \int_X^{\oplus} \mathbb{C} \, d\mu(x)$.

Once we have a direct integral of Hilbert spaces, we can consider the appropriate notions of the direct integral of operators and after that of von Neumann algebras as well.

Definition 2.1.7. If $\mathcal{H} = \int_X^\oplus \mathcal{H}_x \, d\mu(x)$, an operator $T \in \mathcal{B}(\mathcal{H})$ is called *decomposable* if there is a function $x \mapsto T(x)$ on X such that $T(x) \in \mathcal{B}(\mathcal{H}_x)$ for every x and such that for every $\xi \in \mathcal{H}$, $T(x)\xi(x) = (T\xi)(x)$ for almost every x. If $T(x) = f(x)I_x$ for almost every x, we say that T is *diagonalizable.* ◀

Remark. It is easy to show that for $\xi, \eta \in \mathcal{H}$, respectively $S, T \in \mathcal{B}(\mathcal{H})$ decomposable, we have $\xi = \eta$ if and only if $\xi(x) = \eta(x)$ almost everywhere, respectively $S = T$ if and only if $S(x) = T(x)$ almost everywhere.

The following proposition tells us that direct integrals commute with all the basic operations, which allows us to manipulate them easily.

Proposition 2.1.8 ([KR97, Proposition 14.1.8])**.** *If T, T_1, T_2 are decomposable operators on $\mathcal{H} = \int_X^\oplus \mathcal{H}_x \, d\mu(x)$, then aT_1+T_2, T_1T_2 and T^* are decomposable. Moreover the following hold for almost every x:*

(i) $(aT_1 + T_2)(x) = aT_1(x) + T_2(x)$,

(ii) $(T_1T_2)(x) = T_1(x)T_2(x)$,

(iii) $T^*(x) = T(x)^*$.

Theorem 2.1.9 ([KR97, Theorem 14.1.10])**.** *If $\mathcal{H} = \int_X^\oplus \mathcal{H}_x \, d\mu(x)$, then the set $\mathcal{R}$ of decomposable operators is a von Neumann algebra. Moreover $\mathcal{R}$ has abelian commutant consisting of the algebra $\mathcal{C}$ of diagonalizable operators.*

Definition 2.1.10. A von Neumann algebra M on $\mathcal{H} = \int_X^\oplus \mathcal{H}_x \, d\mu(x)$ is called *decomposable* if it is a subalgebra of the (von Neumann) algebra of decomposable operators. ◀

Proposition 2.1.11 ([KR97, Proposition 14.1.18]). *If M is a decomposable von Neumann algebra on $\mathcal{H} = \int_X^\oplus \mathcal{H}_x \, d\mu(x)$ containing the algebra of diagonalizable operators, then there exist von Neumann algebras M_x on $\mathcal{B}(\mathcal{H}_x)$ such that $M = \int_X^\oplus M_x \, d\mu(x)$ in the following sense: If $T \in \mathcal{B}(\mathcal{H})$ is a decomposable operator, then $T \in M$ if and only if $T(x) \in M_x$ almost everywhere. Moreover, if N is a von Neumann algebra with decomposition $N = \int_X^\oplus N_x \, d\mu(x)$ and $M_x = N_x$ almost everywhere, then $M = N$.*

Remark. In [KR97] a decomposable von Neumann algebra M is defined through the existence of a norm-separable strong operator dense C*-subalgebra A such that the identity representation ι is decomposable and $\iota_x(A)$ is strong operator dense in M_x almost everywhere. It is then shown ([KR97, Theorem 14.1.16]) that this is equivalent to our definition above and that the decomposition $x \mapsto M_x$ is unique.

Proposition 2.1.12 ([KR97, Proposition 14.1.24]). *Let M be a decomposable von Neumann algebra on $\mathcal{H} = \int_X^\oplus \mathcal{H}_x \, d\mu(x)$ containing the algebra $\mathcal{C}$ of diagonalizable operators, with decomposition $x \mapsto M_x$. Then M' is also decomposable and $(M')_x = (M_x)'$ almost everywhere.*

The following easy lemma describes the maximal abelian subalgebras of a decomposable von Neumann algebra and will be used several times in section 3.1.

Lemma 2.1.13. *Let M be a decomposable von Neumann algebra on $\mathcal{H} = \int_X^\oplus \mathcal{H}_x \, d\mu(x)$ containing the algebra $\mathcal{C}$ of diagonalizable operators, with decomposition $x \mapsto M_x$. Let $A = \int_X^\oplus A_x \, d\mu(x)$ be a von Neumann subalgebra of M. Then A is maximal abelian inside*

M if and only if A_x is maximal abelian inside M_x for almost every $x \in X$.

Proof. By Proposition 2.1.12 it follows that

$$A' \cap M = \left(\int_X^{\oplus} A_x \, d\mu(x) \right)' \cap \int_X^{\oplus} M_x \, d\mu(x) = \int_X^{\oplus} A_x' \cap M_x \, d\mu(x).$$

If A_x is maximal abelian inside M_x for almost every $x \in X$, this implies that

$$A' \cap M = \int_X^{\oplus} A_x' \cap M_x \, d\mu(x) = \int_X^{\oplus} A_x \, d\mu(x) = A.$$

If on the other hand A is maximal abelian inside M, we get

$$\int_X^{\oplus} A_x \, d\mu(x) = A = A' \cap M = \int_X^{\oplus} A_x' \cap M_x \, d\mu(x),$$

and it follows from the uniqueness of the decomposition that $A_x = A_x' \cap M_x$ almost everywhere. □

We end this section with the following theorem which will be helpful for us in Section 3.3.

Theorem 2.1.14. *Suppose we have direct integrals $(M_1, \tau_1) = \int_X^{\oplus} (M_1(x), \tau_1(x)) d\mu_1(x)$ and $(M_2, \tau_2) = \int_Y^{\oplus} (M_2(x), \tau_2(x)) d\mu_2(x)$ of tracial von Neumann algebras and $\alpha : M_1 \to M_2$ is a trace preserving automorphism such that $\alpha(L^\infty(X)) = L^\infty(Y)$. Then there exist full measure sets $X' \subseteq X$, $Y' \subseteq Y$, and a Borel isomorphism $\Phi : Y' \to X'$ with $\Phi(\mu_2)$ equivalent to μ_1 such that α decomposes into tracial isomorphisms $\{\alpha_x : M_1(x) \to M_2(\Phi^{-1}(x))\}$.*

Proof. Since α is trace preserving, it gives rise to a unitary $U : L^2(M_1) \to L^2(M_2)$ which implements α. The result then immediately follows from [Tak01, Theorem IV.8.23]. □

2.1.5 Relative commutants in ultrapower II_1 factors

In this subsection, we establish several results concerning relative commutants in ultrapower II_1 factors. First, we record the following easy and well-known fact that will be used repeatedly in Chapter 4.

Lemma 2.1.15. *Let (M,τ) be a tracial von Neumann algebra, and $P \subset M$, $B_n \subset M$, for $n \in \mathbb{N}$, be von Neumann subalgebras. Assume that $\prod_\omega B_n \subset P^\omega$. Then there exists a sequence of positive real numbers (ε_n) such that $B_n \subset_{\varepsilon_n} P$, for every $n \in \mathbb{N}$, and $\lim\limits_{n\to\omega} \varepsilon_n = 0$.*

Proof. For every $n \in \mathbb{N}$, let $\varepsilon_n = \sup\{\|x - E_P(x)\|_2 | \ x \in (B_n)_1\}$, and $x_n \in (B_n)_1$ such that $\|x_n - E_P(x_n)\| \geq \varepsilon_n - 2^{-n}$. Let $x = (x_n) \in \prod_\omega B_n$. Then $\lim\limits_{n\to\omega} \|x_n - E_P(x_n)\|_2 = \|x - E_{P^\omega}(x)\|_2 = 0$, which implies that $\lim\limits_{n\to\omega} \varepsilon_n = 0$. $\square$

The next result is a particular case of [CD18, Lemma 2.2]. For completeness, we include a short argument based on Lemma 2.1.15.

Lemma 2.1.16 ([CD18]). *Let (M,τ) be a tracial von Neumann algebra, $R \subset M$ be the hyperfinite II_1 factor, and $P \subset M$ a von Neumann subalgebra. Assume that $R' \cap R^\omega \subset P^\omega$. Then $R \prec^s_M P$.*

Proof. Let $p \in R' \cap M$ be a non-zero projection. Write $R = \bar{\otimes}_{k\in\mathbb{N}} \mathbb{M}_2(\mathbb{C})$, and put $R_n = \bar{\otimes}_{k\geq n} \mathbb{M}_2(\mathbb{C})$, for $n \in \mathbb{N}$. Since $\prod_\omega R_n \subset R' \cap R^\omega$, Lemma 2.1.15 provides $n_0 \geq 1$ such that $\|u - E_P(u)\|_2 \leq \tau(p)/2$, for all $u \in \mathcal{U}(R_{n_0})$. This implies that $\|E_P(up)\|_2 \geq$

$\tau(upE_P(u)^*) \geq \tau(p)/2$, for all $u \in \mathcal{U}(R_{n_0})$. Thus, $R_{n_0}p \prec_M P$, and since $R_{n_0}p \subset Rp$ is a finite index subfactor, we get that $Rp \prec_M P$. $\square$

The following result is a particular case of Ocneanu's central freedom lemma, see [EK98, Lemma 15.25]. Nevertheless, we include a proof for the reader's convenience.

Theorem 2.1.17 ([EK98])**.** *Let (M, τ) be a tracial von Neumann algebra and $R \subset M$ be the hyperfinite II_1 factor. Then $(R' \cap R^\omega)' \cap M^\omega = (R' \cap M)^\omega \vee R$.*

Proof. Since clearly $(R' \cap M)^\omega \vee R \subset (R' \cap R^\omega)' \cap M^\omega$, we only have to show the reverse inclusion. To this end, we adapt the proof of [Pop14, Theorem 2.1.2] which deals with the case $M = R$. Write $R = \bar{\otimes}_{k\in\mathbb{N}}\mathbb{M}_2(\mathbb{C})$. Define $R_m = \bar{\otimes}_{k\geq m}\mathbb{M}_2(\mathbb{C})$ and $R^m = \bar{\otimes}_{k<m}\mathbb{M}_2(\mathbb{C})$, for every $m \in \mathbb{N}$.

Claim. For all $x \in (R' \cap R^\omega)' \cap M^\omega$ and $\varepsilon > 0$, there exists $k \in \mathbb{N}$ such that $\|x - E_{(R_k' \cap M)^\omega}(x)\|_2 \leq \varepsilon$.

Proof of the claim. We may clearly assume that $\lim\limits_{n\to\omega} \|x_n - E_{R_1'\cap M}(x_n)\|_2 = \|x - E_{(R_1'\cap M)^\omega}(x)\|_2 > \varepsilon$, otherwise the claim holds for $k = 1$. If we let $V = \{n \in \mathbb{N} | \; \|x_n - E_{R_1'\cap M}(x_n)\|_2 > \varepsilon\}$, then $V \in \omega$. Fix $n \in V$, and denote $F_n = \{k \in \mathbb{N} \mid \|x_n - E_{R_k'\cap M}(x_n)\|_2 > \varepsilon\}$. Then $1 \in F_n$, and we define

$$k(n) = \begin{cases} \max F_n & \text{if } F_n \text{ is finite} \\ n & \text{if } F_n \text{ is infinite.} \end{cases}$$

We claim that there is $u_n \in \mathcal{U}(R_{k(n)})$ such that $\|x_n - u_n x_n u_n^*\|_2 \geq \varepsilon/2$. Suppose by contradiction that $\|x_n - u x_n u^*\|_2 < \varepsilon/2$, for all $u \in \mathcal{U}(R_{k(n)})$. Let $C_n \subset L^2(M)$ be the $\|.\|_2$-closure of the convex hull of $\{ux_nu^*|u \in \mathcal{U}(R_{k(n)})\}$. Then $\|x_n - \xi\|_2 \leq \varepsilon/2$, $\|\xi\| \leq \|x_n\|$, and $E_{R'_{k(n)}\cap M}(\xi) = E_{R'_{k(n)}\cap M}(x_n)$, for all $\xi \in C_n$. Let η be the unique element of minimal $\|.\|_2$ in C_n. Since $u\xi u^* \in C_n$ and $\|u\xi u^*\|_2 = \|\xi\|_2$, for all $\xi \in C_n$ and $u \in \mathcal{U}(R_{k(n)})$, the uniqueness of η implies that $\eta = u\eta u^*$, for all $u \in \mathcal{U}(R_{k(n)})$. Thus, $\eta \in R'_{k(n)} \cap M$ and hence $\eta = E_{R'_{k(n)}\cap M}(\eta) = E_{R'_{k(n)}\cap M}(x_n)$. This however contradicts the fact that $\|x_n - E_{R'_{k(n)}\cap M}(x_n)\|_2 \geq \varepsilon$, by the construction of $k(n)$.

For every $n \in \mathbb{N} \setminus V$, we put $k(n) = 1$ and $u_n = 1$. We claim that $\lim\limits_{n\to\omega} k(n) < \infty$. Otherwise, if $\lim\limits_{n\to\omega} k(n) = \infty$, then since $u_n \in \mathcal{U}(R_{k(n)})$, for all $n \geq 1$, we would get that $u = (u_n) \in R' \cap R^\omega$. Since $\|x - uxu^*\|_2 = \lim\limits_{n\to\omega} \|x_n - u_nx_nu_n^*\|_2 \geq \varepsilon/2$, this would contradict that x commutes with $R' \cap R^\omega$. Now, if $k = \lim\limits_{n\to\omega} k(n) + 1 < \infty$, then $\|x - E_{(R'_k\cap M)^\omega}(x)\|_2 = \lim\limits_{n\to\omega} \|x_n - E_{R'_{k(n)+1}\cap M}(x_n)\|_2 \leq \varepsilon$. ◇

Finally, note that if $k \geq 1$, then $R'_k \cap M$ is generated by $R' \cap M$ and R^k. Since R^k is finite dimensional, it follows that $(R'_k \cap M)^\omega$ is generated by $(R' \cap M)^\omega$ and R^k. In particular, $(R'_k \cap M)^\omega \subset (R' \cap M)^\omega \vee R$, for every $k \geq 1$. In combination with the claim, this implies that $x \in (R' \cap M)^\omega \vee R$, as desired. □

Corollary 2.1.18. *Let $M = P\bar{\otimes}R$, where P is a tracial von Neumann algebra and R is the hyperfinite II_1 factor. Then $(R' \cap R^\omega)' \cap M^\omega = P^\omega\bar{\otimes}R$. Moreover, $\mathcal{Z}(M' \cap M^\omega) = \mathcal{Z}(P' \cap P^\omega)$.*

Proof. The first assertion is a consequence of Theorem 2.1.17. This implies that

$$\mathcal{Z}(M' \cap M^\omega) \subset (M' \cap M^\omega)' \cap M^\omega \subset (R' \cap R^\omega)' \cap M^\omega = P^\omega\bar{\otimes}R,$$

and thus $\mathcal{Z}(M'\cap M^\omega)\subset M'\cap(P^\omega\bar{\otimes}R)=P'\cap P^\omega$. Since clearly $P'\cap P^\omega\subset M'\cap M^\omega$, we deduce that $\mathcal{Z}(M'\cap M^\omega)\subset\mathcal{Z}(P'\cap P^\omega)$. Since $\mathcal{Z}(P'\cap P^\omega)\subset\mathcal{Z}(M'\cap M^\omega)$ by [Mar18, Lemma 5.3], the moreover assertion follows. $\square$

2.1.6 A result on normalizers

The next lemma identifies the normalizer of a Cartan subalgebra of a II_1 factor P in the tensor product of P with another II_1 factor. More generally, we have

Lemma 2.1.19. *Let $P\subset M$ be two II_1 factors, and $A\subset P$ be a Cartan subalgebra. Assume that $P\vee(A'\cap M)=M$. If $u\in\mathcal{N}_M(A)$, then we can find $u_1\in\mathcal{N}_P(A)$ and $u_2\in\mathcal{U}(A'\cap M)$ such that $u=u_1u_2$. In particular, $\mathcal{R}(A\subset M)=\mathcal{R}(A\subset P)$.*

Proof. Let $u\in\mathcal{N}_M(A)$ and θ be the automorphism of A given by $\theta(a)=uau^*$, for every $a\in A$. Let $p\in A$ be a non-zero projection.

Then $\theta(a)up=upa$, for every $a\in Ap$. Since $A\subset P$ is a Cartan subalgebra and $P\vee(A'\cap M)=M$, the linear span of $\mathcal{V}=\{wv\mid v\in\mathcal{N}_P(A), w\in\mathcal{U}(A'\cap M)\}$ is $\|.\|_2$-dense in M. Thus, we can find $v\in\mathcal{N}_P(A)$ and $w\in\mathcal{U}(A'\cap M)$ such that $\tau(upvw)\neq 0$, and hence we have $\xi:=E_A(upvw)\neq 0$. Since $\theta(a)upvw=upvw(v^*av)$, by applying E_A we conclude that $\theta(a)\xi=(v^*av)\xi$, for all $a\in Ap$. Thus, if $q\in A$ denotes the support projection of ξ, then $\theta(a)q=(v^*av)q$, for all $a\in Ap$. Since $\xi\in A(v^*pv)$, we deduce that $q\in A(v^*pv)$. Thus, $p'=vqv^*\in Ap$ and we have $\theta(p')q=(v^*p'v)q=q$. Thus, for all $a\in Ap'$ we have that $\theta(a)=\theta(ap')=\theta(a)q=(v^*av)q=v^*av$.

In conclusion, for every non-zero projection $p \in A$ there is a non-zero projection $p' \in Ap$ such that the restriction of θ to Ap' is equal to $\mathrm{Ad}(v)$, for some $v \in \mathcal{N}_P(A)$. Since $A \subset P$ is a Cartan subalgebra, this implies that there exists $u_1 \in \mathcal{N}_P(A)$ such that $\theta(a) = u_1 a u_1^*$, for all $a \in A$. Then $u_1^* u \in A' \cap M$, which proves the conclusion. $\square$

2.1.7 The standard Borel space of von Neumann algebras

In [Eff65] Effros showed that there is a standard Borel structure on the space of von Neumann algebras $\mathrm{vNa}(\mathcal{H})$ on a given separable Hilbert space $\mathcal{H}$. Moreover it follows from his results that the set of von Neumann subalgebras $\mathrm{vNa}(M)$ of a given separable II_1 factor (M, τ) is a standard Borel space. In this case one can check that its standard Borel structure is given by the smallest σ-algebra such that

$$A \mapsto \tau(E_A(x)y)$$

is measurable for all $x, y \in M$. Speelman and Vaes then showed the following for the space of Cartan subalgebras $\mathrm{Cartan}(M) := \{A \subseteq M \mid A \text{ is a Cartan subalgebra of } M\}$.

Proposition 2.1.20 ([SV12, Proposition 12]). *In the above setting the following hold.*

- $\mathrm{Cartan}(M) \subseteq \mathrm{vNa}(M)$ *is a Borel set and hence a standard Borel space.*
- *The equivalence relation of unitary conjugacy on* $\mathrm{Cartan}(M)$ *is Borel (i.e. as a subset of* $\mathrm{Cartan}(M) \times \mathrm{Cartan}(M)$*).*
- *The equivalence relation of conjugacy by an automorphism on* $\mathrm{Cartan}(M)$ *is analytic.*

Remark. In [SV12, Theorem 2] the authors construct a II_1 factor for which the equivalence relation of conjugacy by an automorphism on Cartan subalgebras is completely analytic and hence not Borel.

2.2 Group actions and equivalence relations

Let (X, μ) be a probability space, which we will always assume to be *standard*, i.e. its σ-algebra is generated by the open sets of some Polish topology. If $\Gamma \curvearrowright (X, \mu)$ is a p.m.p. action of a countable group Γ, then its *orbit equivalence relation* $\mathcal{R}(\Gamma \curvearrowright X) = \{(x, y) \in X^2 \mid \Gamma \cdot x = \Gamma \cdot y\}$ is countable p.m.p., where countable means that the equivalence classes are countable. Conversely, every countable p.m.p. equivalence relation arises in this way by [FM77]. We say an action $\Gamma \curvearrowright (X, \mu)$ is (essentially) *free* if the stabilizer $\{x \in X \mid gx = x\}$ has measure zero for every $g \in \Gamma \setminus \{e\}$, and is *ergodic* if every invariant subset $A \subset X$, i.e. a subset for which $\mu(gA \Delta A) = 0$ for every $g \in \Gamma$, satisfies $\mu(A) \in \{0, 1\}$.

Let $\mathcal{R}$ be a countable p.m.p. equivalence relation on (X, μ). For every $x \in X$, we denote by $[x]_{\mathcal{R}}$ its equivalence class. We endow $\mathcal{R}$ with an infinite measure $\bar{\mu}$ given by

$$\bar{\mu}(A) = \int_X \#\{y \in X \mid (x, y) \in A\} \, \mathrm{d}\mu(x), \text{ for every Borel set } A \subset \mathcal{R}.$$

The *automorphism group* of $\mathcal{R}$, denoted by $\mathrm{Aut}(\mathcal{R})$, consists of all measure space automorphisms α of (X, μ) such that $(\alpha(x), \alpha(y)) \in \mathcal{R}$, for $\bar{\mu}$-almost every $(x, y) \in \mathcal{R}$. The *full group* of $\mathcal{R}$, denoted by $[\mathcal{R}]$, is the subgroup of all $\alpha \in \mathrm{Aut}(\mathcal{R})$ such that $(\alpha(x), x) \in \mathcal{R}$, for

μ-almost every $x \in X$. The *full pseudogroup* of $\mathcal{R}$, denoted by $[[\mathcal{R}]]$, consists of all isomorphisms $\alpha : (A, \mu_{|A}) \to (B, \mu_{|B})$, where $\mu_{|A}, \mu_{|B}$ denote the restrictions of μ to measurable subsets $A, B \subset X$, such that $(\alpha(x), x) \in \mathcal{R}$, for almost every $x \in A$.

We say that $\mathcal{R}$ is *ergodic* if for every measurable set $A \subset X$ satisfying $\mu(\alpha(A) \triangle A) = 0$, for all $\alpha \in [\mathcal{R}]$, we have $\mu(A) \in \{0, 1\}$. Assume that $\mathcal{R}$ is ergodic, and let $t > 0$. Denote by $\#$ the counting measure of $\mathbb{N}$, and consider a measurable set $X^t \subset X \times \mathbb{N}$ with $(\mu \times \#)(X^t) = t$. The *t-amplification* of $\mathcal{R}$, denoted by $\mathcal{R}^t$, is defined as the equivalence relation on X^t given by $((x, m), (y, n)) \in \mathcal{R}^t \iff (x, y) \in \mathcal{R}$. Since $\mathcal{R}$ is ergodic, the isomorphism class of $\mathcal{R}^t$ only depends on $\mathcal{R}$ and t, but not on the choice of X^t.

2.2.1 Cocycles

We briefly recall the notion of (1-)cocycles, merely to fix notation. We refer to [Kec10, Chapter 20] for a more detailed exposition. Let Γ be a countable group with a (Borel) measure preserving action on a standard measure space (X, μ) and let G be a Polish group. A (Borel) *1-cocycle* for the action $\Gamma \curvearrowright X$ with values in G is a Borel map $c : \Gamma \times X \to G$ satisfying the *cocycle identity*

$$c(gh, x) = c(g, h \cdot x)c(h, x),$$

for all $g, h \in \Gamma$ and μ-almost every $x \in X$. Note that this in particular implies that $c(1, x) = 1$ and $c(g, x)^{-1} = c(g^{-1}, g \cdot x)$. We will identify cocycles that are equal almost everywhere and denote by $Z^1(\Gamma \curvearrowright X, G)$ the set of cocycles for the action $\Gamma \curvearrowright X$ with values in G. Observe that if $c \in Z^1(\Gamma \curvearrowright X, G)$ is independent of x, then it is given by a homomorphism $\Gamma \to G$. In particular when $G = S^1$ cocycles independent of x are given by characters $\gamma \in \hat{\Gamma}$. Denote by $L(X, \mu, G)$ the space of all Borel maps $f : X \to G$ (up to agreeing μ-almost everywhere). Then we have an action $L(X, \mu, G) \curvearrowright Z^1(\Gamma \curvearrowright X, G)$ given by

$$(f \cdot c)(g, x) = f(g \cdot x)c(g, x)f(x)^{-1}.$$

Two cocycles $c_1, c_2 \in Z^1(\Gamma \curvearrowright X, G)$ are called *cohomologous* if there exists $f \in L(X, \mu, G)$ such that $f \cdot c_1 = c_2$. If a cocycle c is cohomologous to the trivial cocycle we call c a *coboundary*. We denote the set of coboundaries by $B^1(\Gamma \curvearrowright X, G)$.

Analogously, we can define cocycles for a countable Borel equivalence relation E. For $C \subseteq E$ Borel we will write $C_x := \{y \in X \mid (x, y) \in C\}$ and $C^y := \{x \in X \mid (x, y) \in C\}$. Following [FM77] we can define two σ-finite measures on $E \subseteq X^2$ by

$$\nu_l(C) := \int_X |C_x| \, d\mu(x) \quad \text{and} \quad \nu_r(C) := \int_X |C^y| \, d\mu(y),$$

where $|S|$ denotes the cardinality of a set S. We say that E is measure preserving if $\nu_l = \nu_r$ and in that case we denote this uniquely defined measure by ν. Note that this definition is equivalent to the one discussed above, namely that the equivalence relation is induced by a p.m.p. action of a countable group. A *cocycle* of E with values in G is then defined to be a Borel map $c : E \to G$ satisfying the *cocycle identity*

$$c(x, z) = c(y, z)c(x, y)$$

for all $x, y, z \in Y$ belonging to a single E-class, and where $Y \subseteq X$ is an E-invariant Borel subset of X of measure 1. Like before we identify two cocycles that are equal ν-almost everywhere and we denote by $Z^1(E, G)$ the set of cocycles of E with values in G. Here we have an action $L(X, \mu, G) \curvearrowright Z^1(E, G)$ given by

$$(f \cdot c)(x, y) = f(y)c(x, y)f(x)^{-1}$$

and we call two cocycles $c_1, c_2 \in Z^1(E, G)$ cohomologous if there exists $f \in L(X, \mu, G)$ such that $f \cdot c_1 = c_2$. The cocycles cohomologous to the trivial cocycle are again called coboundaries, and the set of coboundaries is denoted by $B^1(E, G)$.

2.2.2 Almost invariant sequences

We next recall the notion of strong ergodicity and some of the results from [JS87] because we will use them crucially throughout this dissertation. Let $\Gamma \curvearrowright (X, \mu)$ be an ergodic p.m.p. action of a countable group Γ on a probability space (X, μ). A sequence $(A_n)_{n \geq 1}$ is called *almost invariant* (or asymptotically invariant) if

$$\lim_n \mu(gA_n \Delta A_n) = 0$$

for every $g \in \Gamma$. An almost invariant sequence is *trivial* if $\lim_n \mu(A_n)(1 - \mu(A_n)) = 0$. The action $\Gamma \curvearrowright X$ is called *strongly ergodic* if every almost invariant sequence is trivial. We say that a countable p.m.p. equivalence relation $\mathcal{R}$ is *strongly ergodic* if for every sequence of measurable sets $A_n \subset X$ satisfying $\lim_{n\to\infty} \mu(\alpha(A_n) \triangle A_n) = 0$, for all $\alpha \in [\mathcal{R}]$, we have $\lim_{n\to\infty} \mu(A_n)(1 - \mu(A_n)) = 0$. Note here that a p.m.p. action $\Gamma \curvearrowright (X, \mu)$ of a countable group Γ is ergodic (respectively, strongly ergodic) if and only if its orbit equivalence relation $\mathcal{R}(\Gamma \curvearrowright X)$ is.

Given two countable p.m.p. equivalence relations $\mathcal{R}$ on (X,μ) and $\mathcal{S}$ on (Y,ν), we say that a measurable function $\theta : X \to Y$ is a *factor map* from $\mathcal{R}$ onto $\mathcal{S}$ if it is measure preserving (so in particular essentially onto) and if $\theta([x]_{\mathcal{R}}) = [\theta(x)]_{\mathcal{S}}$ for almost every $x \in X$. The above notions are linked in the following way.

Theorem 2.2.1 ([JS87, Theorem 2.1])**.** *Let $\mathcal{R}$ be a countable ergodic p.m.p. equivalence relation on (X,μ). Then the following are equivalent.*

1. *$\mathcal{R}$ is not strongly ergodic,*

2. *There exists an ergodic hyperfinite p.m.p. equivalence relation $\mathcal{T}$ on a probability space (Y,ν) and a factor map $\theta : X \to Y$ from $\mathcal{R}$ onto $\mathcal{T}$.*

Lemma 2.2.2. *Let $\Gamma \curvearrowright (X,\mu)$ and $\Lambda \curvearrowright (Y,\nu)$ be ergodic p.m.p. actions of countable groups on probability spaces. Suppose $\theta : X \to Y$ is a factor map and $(B_n)_n$ is an almost invariant sequence for $\Lambda \curvearrowright (Y,\nu)$. Put $A_n = \theta^{-1}(B_n)$. Then $(A_n)_n$ is an almost invariant sequence for $\Gamma \curvearrowright (X,\mu)$. Moreover, if $\lim_n \nu(\cup_{k\geq n} sB_k \Delta B_k) = 0$ for every $s \in \Lambda$, then also $\lim_n \mu(\cup_{k\geq n} gA_k \Delta A_k) = 0$ for every $g \in \Gamma$.*

Proof. We will prove the moreover part, the fact that $(A_n)_n$ is an almost invariant sequence can be done in exactly the same way by dropping "$\cup_{k\geq n}$" everywhere. Fix $g \in \Gamma$ and let $\varepsilon > 0$. By the assumptions we can take $S \subseteq \Lambda$ finite such that

$$\mu(\{x \in X \mid \theta(g^{-1}x) \in S\theta(x)\}) \geq 1 - \varepsilon.$$

Writing $X_0 := \{x \in X \mid \theta(g^{-1}x) \in S\theta(x)\}$ we then get

$$\begin{aligned}
\mu(\cup_{k\geq n} gA_k \setminus A_k) &= \mu(\cup_{k\geq n} g\theta^{-1}(B_k) \setminus \theta^{-1}(B_k)) \\
&= \mu(\{x \in X \mid \exists k \geq n : \theta(g^{-1}x) \in B_k, \theta(x) \notin B_k\}) \\
&\leq \varepsilon + \mu(X_0 \cap \{x \in X \mid \exists k \geq n : \theta(g^{-1}x) \in B_k, \theta(x) \notin B_k\}) \\
&= \varepsilon + \mu(\cup_{s\in S}\{x \in X_0 \mid \theta(g^{-1}x) = s\theta(x) \text{ and} \\
&\qquad\qquad \exists k \geq n : s\theta(x) \in B_k, \theta(x) \notin B_k\}) \\
&\leq \varepsilon + \sum_{s\in S} \mu(\{x \in X_0 \mid \exists k \geq n : s\theta(x) \in B_k, \theta(x) \notin B_k\}) \\
&\leq \varepsilon + \sum_{s\in S} \mu(\cup_{k\geq n} \theta^{-1}(s^{-1}B_k) \setminus \theta^{-1}(B_k)) \\
&= \varepsilon + \sum_{s\in S} \nu(\cup_{k\geq n} s^{-1}B_k \setminus B_k),
\end{aligned}$$

which by assumption converges to ε as $n \to \infty$. By symmetry the same will hold for $\mu(\cup_{k\geq n} A_k \setminus gA_k)$. Since ε was arbitrary, this finishes the proof. $\square$

2.2.3 Equivalence relations and Cartan subalgebras

Let (M, τ) be a separable tracial von Neumann algebra and $A \subset M$ be an abelian von Neumann subalgebra. Identify $A = L^\infty(X)$, for some standard probability space (X, μ). Then for every $u \in \mathcal{N}_M(A)$, we can find an automorphism α_u of (X, μ) such that $a \circ \alpha_u = uau^*$, for every $a \in A$. The equivalence relation $\mathcal{R}(A \subset M)$ of the inclusion $A \subset M$ is the smallest countable p.m.p. equivalence relation on (X, μ) whose full group contains α_u, for every $u \in \mathcal{N}_M(A)$.

Now, assume that M is a II_1 factor and $A \subset M$ is a *Cartan subalgebra*, i.e. a maximal abelian regular von Neumann subalgebra. Then $\mathcal{R} := \mathcal{R}(A \subset M)$ is ergodic and there exists a 2-cocycle $w \in \mathrm{H}^2(\mathcal{R}, \mathbb{T})$ such that the inclusions $(A \subset M)$ and $(L^\infty(X) \subset L_w(\mathcal{R}))$ are isomorphic [FM77]. For $t > 0$, the *t-amplification* of M, denoted by M^t, is defined as the isomorphism class of $p(\mathbb{B}(\ell^2)\bar{\otimes}M)p$, where $p \in \mathbb{B}(\ell^2)\bar{\otimes}M$ is a projection with $(\mathrm{Tr} \otimes \tau)(p) = t$, Tr denoting the usual trace on $\mathbb{B}(\ell^2)$. Similarly, the inclusion $(A^t \subset M^t)$ is defined as the isomorphism class of the inclusion $(p(\ell^\infty \otimes A)p \subset p(\mathbb{B}(\ell^2)\bar{\otimes}M)p)$, where $p \in \mathbb{B}(\ell^2)\bar{\otimes}A$ is a projection with $(\mathrm{Tr} \otimes \tau)(p) = t$, and $\ell^\infty \subset \mathbb{B}(\ell^2)$ is the subalgebra of diagonal operators. With this terminology, we have that $A^t \subset M^t$ is a Cartan subalgebra, and $\mathcal{R}(A^t \subset M^t) \cong \mathcal{R}(A \subset M)^t$.

2.3 Complexity of classification

In the following we will review some of the set-theoretic notions allowing us to talk about the exact complexity of a classification problem. A good reference is [KTD13], where several of the definitions below are taken from.

2.3.1 First definitions and results

Given an equivalence relation E on a space X, a *(complete) classification* of X up to E consists of a set of invariants I and a map $f : X \to I$ such that $xEy \Leftrightarrow f(x) = f(y)$. Most often the base space X is a standard Borel space, i.e. a Polish space with the Borel

σ-algebra generated by the open sets, and the equivalence relation E is Borel or analytic (as a subspace of $X \times X$). The following important notion gives us a way of comparing equivalence relations.

Definition 2.3.1. Let (X, E) and (Y, F) be equivalence relations on standard Borel spaces. Then E is *(Borel) reducible* to F, written $E \leq_B F$, if there is a Borel map $f : X \to Y$ such that $xEy \Leftrightarrow f(x)Ff(y)$. ◀

This is exactly saying that the F-classes are complete invariants for the E-classes and intuitively means that the classification problem for E is at most as complicated as the one for F. If both $E \leq_B F$ and $F \leq_B E$ we say that E is *(Borel) bi-reducible* to F and write $E \sim_B F$. If $E \leq_B F$ but $F \not\leq_B E$, we write $E <_B F$.

For any Polish space Y we can consider the equality relation $=_Y$ on Y given by $\{(x, y) \in Y^2 \mid x = y\}$. Denoting by n, for $n \in \mathbb{N}$, any set of cardinality n we then have $E \sim_B (=_n)$ for any Borel equivalence relation E with n equivalence classes. We thus get (dropping $=$ for clarity) that

$$1 <_B 2 <_B 3 <_B \cdots <_B \mathbb{N}$$

is an initial segment of $\leq_B$, as $\mathbb{N} \leq_B E$ for any Borel equivalence relation E with infinitely many equivalence classes. The following dichotomy theorem of Silver extends this result and tells us that $\mathbb{R} \leq_B E$ if E has uncountably many equivalence classes.

Theorem 2.3.2 ([Sil80])**.** *Let E be a Borel equivalence relation. Then either $E \leq_B \mathbb{N}$ or $\mathbb{R} \leq_B E$.*

Definition 2.3.3. An equivalence relation E on X is *smooth* (or *concretely classifiable*) if $E \leq_B (=_Y)$ for some Polish space Y, i.e. there is a Borel map $f : X \to Y$ such that $xEy \Leftrightarrow f(x) = f(y)$. ◀

Note that this is equivalent to asking that $E \leq_B (=_{\mathbb{R}})$, since all Polish spaces are Borel isomorphic. In a way smooth equivalence relations can still be considered "easy". (Un)fortunately, not all equivalence relations are smooth.

Example 2.3.4. A very important non-smooth equivalence relation is the relation E_0 on $2^{\mathbb{N}}$ where

$$\mathbf{x} E_0 \mathbf{y} \quad \Leftrightarrow \quad \exists N \in \mathbb{N}, \forall n \geq N : x_n = y_n.$$

To see that E_0 is not smooth, assume that we have a Borel reduction $f : 2^{\mathbb{N}} \to [0,1]$ from E_0 to $(=_{[0,1]})$. Let μ be the usual product measure on $2^{\mathbb{N}}$. Then $f^{-1}([0, \frac{1}{2}])$ and $f^{-1}([\frac{1}{2}, 1])$ are both tail events, so applying Kolmogorov's zero-one law, we get that either $\mu(f^{-1}([0, \frac{1}{2}])) = 1$ or $\mu(f^{-1}([\frac{1}{2}, 1])) = 1$. Continuing cutting intervals in half we get in this way that f is μ-almost everywhere constant, which contradicts the fact that it is a Borel reduction.

The following result, usually referred to as the Glimm-Effros dichotomy, shows that E_0 is the "smallest" among non-smooth Borel equivalence relations.

Theorem 2.3.5 ([HKL90])**.** *If E is a Borel equivalence relation on a Polish space X and E is not smooth, then $E_0 \leq_B E$. Moreover one can find a continuous injective Borel reduction $f : 2^{\mathbb{N}} \to X$.*

It follows from the above discussion that

$$1 <_B 2 <_B 3 <_B \cdots <_B \mathbb{N} <_B \mathbb{R} <_B E_0$$

forms an initial segment for $\leq_B$ and moreover $E_0 \leq E$ for any non-smooth Borel equivalence relation E. Beyond E_0 the situation becomes way more complicated and one will for instance not find a unique minimal equivalence relation above E_0 (see [KL97, Theorem 2]). However, one natural thing to consider for general equivalence relations is *classification by countable structures*.

2.3.2 Classification by countable structures

Intuitively, an equivalence relation is classifiable by countable structures if we can assign complete invariants built out of countable (or "discrete") structures of some given type, e.g. groups, graphs, fields, etc. More concretely, we have the following.

Definition 2.3.6. A *countable signature* is a countable family $L = \{f_i\}_{i \in I} \cup \{R_j\}_{j \in J}$ of function symbols f_i and relation symbols R_j. We denote by $n_i \geq 0$, resp. $m_j \geq 1$, the arity of f_i, resp. R_j.

An *L-structure* is given by

$$\mathcal{A} := (A, \{f_i^{\mathcal{A}}\}_{i \in I}, \{R_j^{\mathcal{A}}\}_{j \in J}),$$

where A is a nonempty set, $f_i^{\mathcal{A}} : A^{n_i} \to A$ are functions, and $R_j^{\mathcal{A}} \subseteq A^{m_j}$ are relations. ◀

Example 2.3.7. Consider $L = \{\cdot, 1\}$, where $\cdot$ is a binary and 1 is a nullary function symbol. Then a group is any L-structure $(G, \cdot^G, 1^G)$ satisfying the group axioms. Similarly, using other signatures, one can describe rings, fields, graphs, etc.

When considering only countable structures, we can of course always take $A = \mathbb{N}$ up to isomorphism. This gives rise to the following.

Definition 2.3.8. Given a countable signature L, we consider the *space of countable L-structures*

$$X_L := \prod_{i \in I} \mathbb{N}^{(\mathbb{N}^{n_i})} \times \prod_{j \in J} 2^{(\mathbb{N}^{m_j})}.$$

Putting the discrete topologies on $\mathbb{N}$ and 2, X_L becomes a Polish space for the product topology. ◀

Now let S_∞ be the group of *all* permutations of $\mathbb{N}$. Then we have an obvious action $S_\infty \curvearrowright X_L$, called the *logic action*. One easily checks that this action induces the equivalence relation of isomorphism in X_L, i.e. for $\mathcal{A}, \mathcal{B} \in X_L$ we have $\mathcal{A} \cong \mathcal{B} \Leftrightarrow \exists g \in S_\infty : g \cdot \mathcal{A} = \mathcal{B}$. We will denote by $\cong_L$ the equivalence relation of isomorphism on X_L.

Definition 2.3.9. Let E be an equivalence relation on a standard Borel space X. Then we say that E is *classifiable by countable structures* if there exists a countable signature L such that $E \leq_B (\cong_L)$. ◀

Example 2.3.10.

- If E is smooth, then E admits classification by countable structures.

- ([Kec92]) If G is a Polish locally compact group and X is a Borel G-space, then the orbit equivalence relation of $G \curvearrowright X$ admits classification by countable structures.

2.3.3 Turbulence and generic ergodicity

The basic method for showing that some equivalence relation is not classifiable by countable structures was developed by Hjorth (see [Hjo00]) and is called *turbulence.* In the following, let G be a Polish group acting continuously on a Polish space X.

Definition 2.3.11. Let $U \subseteq X$ be a nonempty open set and $V \subseteq G$ be an open neighborhood of $1 \in G$. For $x \in U$ we define

$$\mathcal{O}(x, U, V) := \{y \in U \mid \exists k \in \mathbb{N}, x_0, x_1, \ldots, x_k \in U, g_0, \ldots, g_{k-1} \in V$$
$$\text{such that } x = x_0, y = x_k \text{ and } \forall i < k : x_{i+1} = g_i x_i\}.$$

We call $\mathcal{O}(x, U, V)$ the *local U, V-orbit of x.* ◀

Recall that a subspace of a Polish space is called *meager* if it is disjoint from a dense G_δ set and *comeager* if it includes one. We can think of these as being "small", respectively "large", sets.

Definition 2.3.12. The action $G \curvearrowright X$ is *turbulent* if

(i) every orbit is dense,

(ii) every orbit is meager,

(iii) for every $x \in X$, for every $U \subseteq X$ open and every $V \subseteq G$ open with $x \in U$, $1 \in V$: $\overline{\mathcal{O}(x, U, V)}$ has nonempty interior.

◀

Note that these conditions give rise to "turbulence" on different scales. Orbits being dense is a global phenomenon, telling us we can get to every region of our space by applying group elements to any chosen point in our space. Moreover, one can easily show (see Example 2.3.16 below) that an action satisfying (i) and (ii) from the above definition is already "turbulent enough", in the sense that it cannot be smooth. The main condition in the above definition is (iii) though, which says that even on a small scale, both in the group and in the space, the action exhibits turbulent behaviour.

Example 2.3.13. Consider $(\ell^1, +)$ with the usual 1-norm ($\|g\|_1 = \sum_i |g_i|$) acting by translation on $(\mathbb{R}^{\mathbb{N}}, +)$ with the product topology. It is not hard to see that this action is turbulent (see also [Hjo00, Proposition 3.25]). Every orbit is dense and meager because ℓ^1 is dense and meager. To get the third condition, fix $x \in \mathbb{R}^{\mathbb{N}}$, $U \subseteq \mathbb{R}^{\mathbb{N}}$ open containing x and $V \subseteq \ell^1$ an open neighborhood of the identity. By shrinking U and V if necessary, we can assume that for some $\varepsilon > 0, l \in \mathbb{N}$ we have

$$U = \{z \in \mathbb{R}^{\mathbb{N}} \mid \forall i < l : |z_i - x_i| < \varepsilon\},$$

$$V = \{z \in \ell^1 \mid \|z\|_1 < \varepsilon\}.$$

Now let $U_0 \subseteq U$ be any open set, then it suffices to find some $z \in \mathcal{O}(x, U, V) \cap U_0$. Using the density of the orbit of x, we can find $g \in \ell^1$ with $g \cdot x \in U_0$. Choose now $k \in \mathbb{N}$ large enough so that $\|g\|_1 / k < \varepsilon$ and put $h = g/k$, i.e. $h_i = g_i/k$ for every $i \in \mathbb{N}$. Then $h \in V$ and we can put $x_0 := x, x_{j+1} := h \cdot x_j$. By convexity of U, we have that $x_j \in U$ for every $0 \leq j \leq k$. Hence $x_k = g \cdot x \in \mathcal{O}(x, U, V) \cap U_0$ and the result follows.

Next we consider what is called *generic ergodicity*. For this we first need the obvious notion of a homomorphism between equivalence relations.

Definition 2.3.14. Let E and F be equivalence relations on Polish spaces X and Y. A *homomorphism* from E to F is a Borel function $f : X \to Y$ such that $xEy \Rightarrow f(x)Ff(y)$. ◀

Given E and F equivalence relations on Polish spaces X and Y, we will write $\mathrm{Hom}(E, F)$ for the space of homomorphisms from E to F.

Definition 2.3.15. For E, F as above, we say that E is *generically F-ergodic* if for every homomorphism f between E and F, there is a comeager set $A \subseteq X$ such that f maps A to a single F-class. ◀

Example 2.3.16. If E denotes the orbit equivalence relation for some continuous action of a Polish group G on a Polish space X with a dense orbit, then E is easily seen to be generically $(=_{\mathbb{R}})$-ergodic, see for instance [KTD13, Example 2.17]. In particular, if every orbit is also meager, then E is not smooth.

The following important result of Hjorth tells us that turbulence is an obstruction for classification by countable structures.

Theorem 2.3.17 ([Hjo00]). *If a Polish group G acts turbulently on a Polish space X, then $\mathcal{R}(G \curvearrowright X)$ is generically $\cong_L$-ergodic for any countable signature L. In particular, since turbulent actions have meager orbits by definition, $\mathcal{R}(G \curvearrowright X)$ does not admit classification by countable structures.*

2.3.4 A non-classifiability criterion

The following criterion follows easily from Example 2.3.13 and Theorem 2.3.17.

Proposition 2.3.18. *If E is an equivalence relation on a Polish space X such that there exists a Borel map $f : (0,1)^{\mathbb{N}} \to X$ satisfying*

1. *$f(\mathbf{x}) E f(\mathbf{y})$ whenever $\mathbf{x} - \mathbf{y} \in \ell^1$, and*

2. *there is no comeager set A in $(0,1)^{\mathbb{N}}$ which is mapped into a single E-class by f,*

then E is not classifiable by countable structures.

Proof. (This proof follows the same lines of reasoning as [Lup14, Criterion 3.3].)

Claim 1. Suppose F, G, R are equivalence relations on Polish spaces Y_1, Y_2, Z respectively such that G is generically R-ergodic. If there exists a homomorphism $g : Y_2 \to Y_1$ from G to F such that $g(C)$ is comeager in Y_1 for all comeager $C \subseteq Y_2$, then F is generically R-ergodic.

Proof of Claim 1. Suppose $h : Y_1 \to Z$ is a homomorphism from F to R. Then $h \circ g : Y_2 \to Z$ is a homomorphism from G to R. Since G is generically R-ergodic, there exists a comeager set $C \subseteq Y_2$ that is mapped onto one R-class. Hence $g(C)$ is a comeager subset of Y_1 that is mapped onto a single R-class by h. $\diamond$

Claim 2. Suppose E and F are equivalence relations on X and Y respectively, and F is generically $\cong_L$-ergodic for every countable signature L. If there is a homomorphism $f : Y \to X$ from F to E such that there is no comeager set in Y which is mapped onto a single E-class, then E is not classifiable by countable structures.

Proof of Claim 2. Let L be a countable signature and assume that we have a Borel reduction $g : X \to X_L$ from E to $\cong_L$. Then $g \circ f : Y \to X_L$ is a homomorphism from F to $\cong_L$ and since F is generically $\cong_L$-ergodic, it follows that there is a comeager set $C \subseteq Y$ which is mapped to a single $\cong_L$-class. Since g is a Borel reduction, this implies that $f(x)Ef(y)$ for all $x, y \in C$, contradicting the fact that no E-class has a comeager preimage. $\diamond$

Let now E be as in the proposition, G the relation of equivalence modulo ℓ^1 on $\mathbb{R}^{\mathbb{N}}$, and F the relation of equivalence modulo ℓ^1 on $(0,1)^{\mathbb{N}}$. Consider the functions

$$g' : \mathbb{R}^{\mathbb{N}} \to (-1,1)^{\mathbb{N}} : (t_n)_n \mapsto \left(\frac{t_n}{|t_n|+1}\right)_n,$$
$$\varphi : (-1,1) \to (0,1) : x \mapsto \frac{1}{2}x + \frac{1}{2},$$
$$g := \varphi^{\mathbb{N}} \circ g' : \mathbb{R}^{\mathbb{N}} \to (0,1)^{\mathbb{N}}.$$

It is easy to see that g satisfies the conditions of Claim 1. Since G is generically $\cong_L$-ergodic for any countable signature L by Example 2.3.13 and Theorem 2.3.17, the same holds for F. The result then follows from Claim 2. $\square$

This chapter contains material from: P. Spaas, "Non-classification of Cartan subalgebras for a class of von Neumann algebras", *Advances in Mathematics*, vol. 332, pp. 510-552, 2018. The dissertation author was the primary investigator and author of this paper.

This chapter also contains material from: A. Ioana and P. Spaas, "A class of II_1 factors with a unique McDuff decomposition", preprint arXiv: 1808.02965, submitted. The dissertation author was one of the primary investigators and authors of this paper.

Chapter 3

Non-classifiability of Cartan subalgebras

3.1 A structural result for Cartan subalgebras

For this section we fix the following.

- $\Gamma \curvearrowright X$ a free ergodic p.m.p. action of a countable group Γ on a standard probability space (X, μ),
- an arbitrary II_1 factor N,
- $\mathcal{M} := (L^\infty(X) \rtimes \Gamma)\bar{\otimes} N$.

Note that

$$\mathcal{M} \cong (L^\infty(X)\bar{\otimes} N) \rtimes \Gamma$$

where Γ acts trivially on N. Also, we can write $L^\infty(X)\bar{\otimes}N$ as the (constant) direct integral $L^\infty(X)\bar{\otimes}N = \int_X^\oplus N\,d\mu(x)$. For notational convenience we will drop the measure μ from the integrals from now on. In this section we will prove the main structural result, namely Theorem 1.1.1. For this we will need a few lemmas. Note that only Lemma 3.1.4 will assume that $\Gamma \in \mathcal{C}_{rss}$. The other results, in particular Lemma 3.1.3, hold for any countable group Γ.

Lemma 3.1.1. *Suppose $f_1, f_2 : X \to \mathrm{Cartan}(N)$ are measurable functions such that $f_1(x) \sim_u f_2(x)$ in N for almost every $x \in X$. Write $A_i = \int_X^\oplus f_i(x)\,dx$, $i = 1, 2$. Then there exists $u \in \mathcal{U}(L^\infty(X)\bar{\otimes}N)$ such that $uA_1u^* = A_2$.*

Proof. We will prove that we can intertwine arbitrarily small corners of A_1 into arbitrarily small corners of A_2 inside $L^\infty(X)\bar{\otimes}N$, after which we can conclude the proof with a maximality argument.

Claim. $A_1p_1 \prec_{L^\infty(X)\bar{\otimes}N} A_2p_2$ for all nonzero projections $p_1 \in A_1, p_2 \in A_2$ such that $z(p_1)z(p_2) \neq 0$.

Proof of the claim. For $i = 1, 2$, we can write $p_i = \int_X^\oplus p_{i,x}\,dx$, with $p_{i,x}$ a projection in $A_{i,x}$. We then have the decompositions $A_ip_i = \int_X^\oplus (A_ip_i)_x\,dx = \int_X^\oplus A_{i,x}p_{i,x}\,dx$. Assume the claim doesn't hold. Then by Theorem 2.1.1 we can find a sequence of unitaries $(u_n)_n \in \mathcal{U}(A_1p_1)$ such that $\|E_{A_2p_2}(vu_nw)\|_2 \to 0$ for all $v, w \in L^\infty(X)\bar{\otimes}N$. Writing

$v = \int_X^{\oplus} v_x \, dx$, $w = \int_X^{\oplus} w_x \, dx$ and $u_n = \int_X^{\oplus} u_{n,x} \, dx$ we get

$$\begin{aligned}
\|E_{A_2p_2}(vu_nw)\|_2^2 &= \left\| E_{A_2p_2}\left(\int_X^{\oplus} v_x u_{n,x} w_x \, dx \right) \right\|_2^2 \\
&= \left\| \int_X^{\oplus} E_{(A_2p_2)_x}(v_x u_{n,x} w_x) \, dx \right\|_2^2 \\
&= \int_X \left\| E_{(A_2p_2)_x}(v_x u_{n,x} w_x) \right\|_2^2 \, dx,
\end{aligned}$$

which by assumption converges to zero. Now let $\{t_i\}_{i\in\mathbb{N}}$ be a countable $\|.\|_2$-dense subset of N. Letting v and w range over $\{1 \otimes t_i \mid i \in \mathbb{N}\}$ we get a subsequence of $(u_n)_n$ such that for almost every $x \in X$ we have $\left\| E_{(A_2p_2)_x}(t_i u_{n,x} t_j) \right\|_2 \to 0$ for all $i, j \in \mathbb{N}$. Since $\{t_i\}_{i\in\mathbb{N}}$ is $\|.\|_2$-dense in N, this means that $(A_1p_1)_x \not\prec_N (A_2p_2)_x$ for almost every x, which is absurd since $(A_1)_x = f_1(x) \sim_u f_2(x) = (A_2)_x$ in N for almost every x, and p_1 and p_2 don't have disjoint central supports by assumption. $\diamond$

Now let $(p_i)_i$ and $(q_i)_i$ be maximal families of orthogonal projections such that $A_1p_i \sim_u A_2q_i$ inside $L^\infty(X)\bar{\otimes}N$ and put $p = \sum_i p_i$, $q = \sum_i q_i$. Then $A_1p \sim_u A_2q$ inside $L^\infty(X)\bar{\otimes}N$. In particular p and q are equivalent projections. Since $L^\infty(X)\bar{\otimes}N$ is a finite von Neumann algebra, also $1-p$ and $1-q$ are equivalent, and so in particular have equal central support. Assuming $p \neq 1$ (and hence also $q \neq 1$), it then follows from the claim that $A_1(1-p) \prec_{L^\infty(X)\bar{\otimes}N} A_2(1-q)$. Moreover, by Lemma 2.1.13, A_1 and A_2 are maximal abelian inside $L^\infty(X)\bar{\otimes}N$. Hence applying [Vae07, Theorem C.3] (see also [Pop06a, Theorem A.1]) yields the existence of a partial isometry $v \in L^\infty(X)\bar{\otimes}N$ such that $v^*v \in A_1(1-p)$, $vv^* \in A_2(1-q)$ and $vA_1(1-p)v^* = A_2(1-q)vv^*$, contradicting the maximality above. We conclude that $A_1 \sim_u A_2$ inside $L^\infty(X)\bar{\otimes}N$. $\square$

Lemma 3.1.2. *Suppose A is a Cartan subalgebra of $q(L^\infty(X)\bar{\otimes}N)q$ for some projection $q = \int_X^\oplus q_x\,dx \in L^\infty(X)\bar{\otimes}N$. Write $A = \int_X^\oplus A_x\,dx \subseteq \int_X^\oplus q_x N q_x\,dx$ and let $Y := \{x \in X \mid q_x \neq 0\}$. Then A_x is a Cartan subalgebra of $q_x N q_x$ for almost every $x \in Y$.*

Proof. First note that $A_x \subseteq q_x N q_x$ is maximal abelian for almost every $x \in Y$ by Lemma 2.1.13. To complete the proof we need to show that A_x is regular in $q_x N q_x$ for almost every $x \in Y$. Therefore take $u \in \mathcal{N}_{q(L^\infty(X)\bar{\otimes}N)q}(A)$ and write $u = \int_X^\oplus u_x\,dx$. Then

$$\int_X^\oplus A_x\,dx = A = uAu^* = u\left(\int_X^\oplus A_x\,dx\right)u^* = \int_X^\oplus u_x A_x u_x^*\,dx$$

and so $A_x = u_x A_x u_x^*$ almost everywhere. Let now $\{u^{(i)}\}$ be a countable $\|.\|_2$-dense subset of $\mathcal{N}_{q(L^\infty(X)\bar{\otimes}N)q}(A)$ and take null sets $E_i \subseteq X$ such that $A_x = u_x^{(i)} A_x u_x^{(i)*}$ for $x \notin E_i$. Put $E = \bigcup E_i$ so that $A_x = u_x^{(i)} A_x u_x^{(i)*}$ for all i and all $x \notin E$. Thus for $x \notin E$ we have that $\{u_x^{(i)} \mid i \in \mathbb{N}\} \subseteq \mathcal{N}_{q_x N q_x}(A_x)$ and so in order to prove that almost every A_x is regular it suffices to show that $\{u_x^{(i)} \mid i \in \mathbb{N}\} \subseteq \mathcal{N}_{q_x N q_x}(N_x)$ generates $q_x N q_x$ almost everywhere. For this note that if $T = \int_X^\oplus T_x\,dx \in q(L^\infty(X)\bar{\otimes}N)q = \{u^{(i)} \mid i \in \mathbb{N}\}''$ then $T_x \in \{u_x^{(i)} \mid i \in \mathbb{N}\}''$ almost everywhere by Propositions 2.1.8 and 2.1.11. Now take a countable $\|.\|_2$-dense subset $\{t_n\}_n \in N$ and consider $q(1 \otimes t_n)q = \int_X^\oplus q_x t_n q_x\,dx$. The above claim implies that $q_x t_n q_x \in \{u_x^{(i)} \mid i \in \mathbb{N}\}''$ almost everywhere. Let $F_n := \{x \in X \mid q_x t_n q_x \notin \{u_x^{(i)} \mid i \in \mathbb{N}\}''\}$ and $F := \bigcup_n F_n$. Then for all x not in the null set F the $u_x^{(i)}$ generate $q_x N q_x$. We conclude that A_x is regular in $q_x N q_x$ for almost every x, finishing the proof. $\square$

Lemma 3.1.3. *Suppose A is a von Neumann subalgebra of $L^\infty(X)\bar{\otimes}N$ and write $A = \int_X^\oplus A_x\,dx$. Then the following are equivalent:*

1. A is a Cartan subalgebra of $\mathcal{M}$,

2. A_x is a Cartan subalgebra of N for almost every $x \in X$ and for every $g \in \Gamma$, $A_{g^{-1}x} \sim_u A_x$ inside N for almost every $x \in X$.

Proof. $2 \Rightarrow 1$. First note that $A := \int_X^{\oplus} A_x \, dx$ is maximal abelian inside $\mathcal{M}$. Indeed, by Lemma 2.1.13 A is maximal abelian inside $L^\infty(X)\bar{\otimes}N$. In particular $L^\infty(X) \subseteq A$. Since $L^\infty(X)$ is maximal abelian inside $L^\infty(X) \rtimes \Gamma$ we then get

$$A = A' \cap (L^\infty(X)\bar{\otimes}N) \subseteq A' \cap \mathcal{M} \subseteq A' \cap L^\infty(X)' \cap \mathcal{M} = A' \cap (L^\infty(X)\bar{\otimes}N) = A,$$

and so A is maximal abelian inside $\mathcal{M}$. To get regularity, we first observe that

$$\mathcal{N}_{\mathcal{M}}(A) \supseteq \mathcal{N}_{L^\infty(X)\bar{\otimes}N}(A) = \int_X^{\oplus} \mathcal{N}_N(A_x) \, dx$$

where $\int_X^{\oplus} \mathcal{N}_N(A_x) \, dx$ is the subset of $L^\infty(X)\bar{\otimes}N$ consisting of all $u = \int_X^{\oplus} u_x \, dx$ such that $u_x \in \mathcal{N}_N(A_x)$ for almost every $x \in X$. We note here that the set $\mathrm{ClSub}(\mathcal{U}(N))$ of closed subsets of the Polish space $\mathcal{U}(N)$ is a standard Borel space when given the Effros Borel structure. Moreover, in the proof of [SV12, Proposition 12] it is shown that the map $\mathrm{Cartan}(N) \to \mathrm{ClSub}(\mathcal{U}(N)) : A \mapsto \mathcal{N}_N(A)$ is Borel. Together with the measurability of the field $x \mapsto A_x$, this gives the measurability of the field $x \mapsto \mathcal{N}_N(A_x)$.

Claim. $\mathcal{N}_{L^\infty(X)\bar{\otimes}N}(A)'' = L^\infty(X)\bar{\otimes}N$.

Proof of the claim. Assume by contradiction that the claim doesn't hold. Then we can write $\mathcal{N}_{L^\infty(X)\bar{\otimes}N}(A)'' = \int_X^{\oplus} N_x \, dx$ where $N_x \subsetneq N$ for all x in some subset of X of positive measure. Let $\varepsilon_x := \sup\{\|u - E_{N_x}(u)\|_2 \mid u \in \mathcal{N}_N(A_x)\}$ and define $Y := \{x \in$

$X \mid \varepsilon_x > 0\}$. By assumption $\mu(Y) > 0$. Since $X \to \mathrm{ClSub}(\mathcal{U}(N)) : x \mapsto \mathcal{N}_N(A_x)$ and $X \to \mathrm{vNa}(N) : x \mapsto N_x$ are Borel, it follows easily that the set $\mathbb{B} := \{(x, u) \in Y \times \mathcal{U}(N) \mid u \in \mathcal{N}_N(A_x), \|u - E_{N_x}(u)\|_2 \geq \varepsilon_x/2\}$ is a Borel subset of $Y \times \mathcal{U}(N)$. By construction $\pi_1(\mathbb{B}) = Y$, where π_1 is the projection onto the first coordinate. Hence by [Tak01, Theorem A.16] there exists a measurable function $\varphi : Y \to \mathcal{U}(N)$ such that $(x, \varphi(x)) \in \mathbb{B}$ for all $x \in Y$. Putting $\varphi(x) = 1$ for $x \notin Y$, we then get $T := \int_X^{\oplus} \varphi(x)\, dx \in \int_X^{\oplus} \mathcal{N}_N(A_x)\, dx$, but $T \notin \int_X^{\oplus} N_x\, dx$, which is impossible. $\diamond$

It follows from the claim that $L^\infty(X)\bar{\otimes}N \subseteq \mathcal{N}_{\mathcal{M}}(A)''$. Hence it suffices to argue that also $u_g \in \mathcal{N}_{\mathcal{M}}(A)''$ for every $g \in \Gamma$. Fix $g \in \Gamma$. For an element $a = \int_X^{\oplus} a_x\, dx \in L^\infty(X)\bar{\otimes}N$ we see that

$$u_g a u_g^* = (\sigma_g \otimes \mathrm{id})(a) = (\sigma_g \otimes \mathrm{id}) \left(\int_X^{\oplus} a_x\, dx \right) = \int_X^{\oplus} a_x\, d(gx) = \int_X^{\oplus} a_{g^{-1}x}\, dx,$$

so u_g acts by "shifting" the components of the direct integral, yielding $u_g A u_g^* = A_{g^{-1}}$ where $A_{g^{-1}} := \int_X^{\oplus} A_{g^{-1}x}\, dx$. On the other hand, from Lemma 3.1.1 and the given fact that $A_{g^{-1}x} \sim_u A_x$ for almost every $x \in X$, it follows that there exists $u(g) \in \mathcal{U}(L^\infty(X)\bar{\otimes}N)$ such that $u(g) A_{g^{-1}} u(g)^* = A$. Putting them together yields

$$(u(g)u_g)A(u(g)u_g)^* = u(g)A_{g^{-1}}u(g)^* = A. \tag{3.1}$$

Since $u(g) \in L^\infty(X)\bar{\otimes}N$ and we already know that $L^\infty(X)\bar{\otimes}N \subseteq \mathcal{N}_{\mathcal{M}}(A)''$ we get $u_g \in \mathcal{N}_{\mathcal{M}}(A)''$ which finishes the proof of one implication.

$1 \Rightarrow 2$. First of all it follows from [Dye63] that A, being Cartan in $\mathcal{M}$, is Cartan in $L^\infty(X)\bar{\otimes}N$ as well (see also [JP82]). So from Lemma 3.1.2 it follows that A_x is a Cartan subalgebra of N for almost every $x \in X$ and we are left with showing that for every $g \in \Gamma$, $A_{g^{-1}x} \sim_u A_x$ inside N for almost every $x \in X$. Fix $g \in \Gamma$. From [Pop06a, Theorem A.1] it follows that it suffices to show that $A_{g^{-1}x} \prec_N A_x$. Since A is a Cartan subalgebra of $\mathcal{M}$ we know that $u_g \in \mathcal{N}_{\mathcal{M}}(A)''$. Since $\mathcal{N}_{\mathcal{M}}(A)$ is closed under products, this means that for a given $n \in \mathbb{N}$, we can find $\lambda_1, \ldots, \lambda_m \in \mathbb{C}$ and $u_1, \ldots, u_m \in \mathcal{N}_{\mathcal{M}}(A)$ such that

$$\left\| \sum_{i=1}^{m} \lambda_i u_i - u_g \right\|_2^2 \leq \frac{1}{n}.$$

Writing $\sum_{i=1}^{m} \lambda_i u_i = \sum_{h \in \Gamma} c_h u_h$ with $c_h \in L^\infty(X) \bar{\otimes} N$ we get that

$$\frac{1}{n} \geq \left\| \sum_h c_h u_h - u_g \right\|_2^2 \geq \tau((c_g - 1)^*(c_g - 1)) = \int_X \tau_N((c_{g,x} - 1_N)^*(c_{g,x} - 1_N))\, dx,$$

where we have written $c_g = \int_X^{\oplus} c_{g,x}\, dx$. In particular $c_{g,x} \neq 0$ on a set of measure at least $1 - \frac{1}{n}$ and so for every such x at least one of the $u_i = \sum_h b_h^i u_h$ satisfies $b_{g,x}^i \neq 0$ (where again $b_h^i = \int_X^{\oplus} b_{h,x}^i\, dx \in L^\infty(X) \bar{\otimes} N$). Doing this for every $n \in \mathbb{N}$ implies that the set $E := \{x \in X \mid \exists u = \sum_h b_h u_h \in \mathcal{N}_{\mathcal{M}}(A) \text{ such that } b_{g,x} \neq 0\}$ has full measure inside X. Now take $x \in E$ and $u = \sum_h b_h u_h \in \mathcal{N}_{\mathcal{M}}(A)$ such that $b_{g,x} \neq 0$. Let $\theta \in \mathrm{Aut}(A)$ be such that

$$ua = \theta(a)u, \qquad \text{for all } a \in A. \tag{3.2}$$

Writing $a = \int_X^{\oplus} a_x\, dx$, $\theta(a) = \int_X^{\oplus} \theta(a)_x\, dx$ and substituting in equation (3.2), we find that for all $h \in \Gamma$ we have $b_{h,x} a_{h^{-1}x} = \theta(a)_x b_{h,x}$ almost everywhere. In particular this holds for g. Using the polar decomposition for $b_{g,x}$ it then follows that there exists a nonzero partial isometry $v \in N$ such that

$$v a_{g^{-1}x} = \theta(a)_x v, \qquad \text{for all } a \in A.$$

This relation implies that $vA_{g^{-1}x} \subseteq A_x v$ and hence gives the desired intertwining $A_{g^{-1}x} \prec_N A_x$, finishing the proof of the other implication. $\square$

Lemma 3.1.4. *Assume $\Gamma \in \mathcal{C}_{rss}$ and let $A \subseteq \mathcal{M}$ be a Cartan subalgebra. Then A is unitarily conjugate to a Cartan subalgebra of $\mathcal{M}$ contained in $L^\infty(X)\bar{\otimes}N$.*

Proof. Applying the dichotomy in Definition 2.1.4 for $\Gamma \in \mathcal{C}_{rss}$ we get that either $A \prec_{\mathcal{M}} L^\infty(X)\bar{\otimes}N$ or $\mathcal{N}_{\mathcal{M}}(A)'' = \mathcal{M}$ is amenable relative to $L^\infty(X)\bar{\otimes}N$. Since Γ is non-amenable the latter is not possible (see for instance [OP10a, Proposition 2.4]). We conclude that we can find projections $p \in A$, $q \in L^\infty(X)\bar{\otimes}N$, a nonzero partial isometry $v \in q\mathcal{M}p$ and a *-homomorphism $\psi : Ap \to q(L^\infty(X)\bar{\otimes}N)q$ such that $\psi(a)v = va$ for all $a \in Ap$, $v^*v = p$, and $vv^* \in \psi(Ap)' \cap q\mathcal{M}q$. Moreover, by [Ioa12b, Lemma 1.5], we can assume that $\psi(Ap)$ is maximal abelian in $q(L^\infty(X)\bar{\otimes}N)q$. In particular this means that $L^\infty(X)q \subseteq \psi(Ap)$ and taking relative commutants we get $\psi(Ap)' \cap q\mathcal{M}q \subseteq (L^\infty(X)q)' \cap q\mathcal{M}q = q(L^\infty(X)\bar{\otimes}N)q$. Since $\psi(Ap)$ is maximal abelian in $q(L^\infty(X)\bar{\otimes}N)q$ it then follows that

$$\psi(Ap)' \cap q\mathcal{M}q = \psi(Ap)' \cap q(L^\infty(X)\bar{\otimes}N)q = \psi(Ap).$$

As $vv^* \in \psi(Ap)' \cap q\mathcal{M}q$, we get $vv^* \in \psi(Ap)$ and so after possibly replacing q by a subprojection we can assume that $vv^* = q$. Hence $\mathrm{Ad}(v) : p\mathcal{M}p \to q\mathcal{M}q$ is an isomorphism and since Ap is a Cartan subalgebra of $p\mathcal{M}p$ we conclude that $\psi(Ap) = vApv^*$ is a Cartan subalgebra of $q\mathcal{M}q$.

Claim. There is a projection $p' \in Ap$ such that $\psi(Ap') \subseteq B$ where B is a Cartan subalgebra of $\mathcal{M}$ contained in $L^\infty(X)\bar{\otimes}N$.

Proof of the claim. Note that the central trace of $q = \int_X^{\oplus} q_x \, dx \in L^\infty(X)\bar{\otimes}N$ is given by $\mathrm{ctr}(q) = \int_X^{\oplus} \tau_N(q_x) \, dx \in L^\infty(X)$. Proposition 7.17 in [SZ79] tells us that for any $f \in L^\infty(X)$ with $f \leq \mathrm{ctr}(q)$, there exists a projection $q' \leq q$ in $\psi(Ap)$ such that $\mathrm{ctr}(q') = f$. Also, since q is nonzero, there exists $n \in \mathbb{N}$ such that $\mu(X_0) > 0$, where

$$X_0 := \{x \in X \mid \tau(q_x) \geq \frac{1}{n}\}.$$

Consider now $f = \frac{1}{n}\mathbb{1}_{X_0}$ and take $q' \in \psi(Ap)$ such that $q' \leq q$ and $\mathrm{ctr}(q') = f$. In particular we have $\tau(q'_x) = \frac{1}{n}$ for $x \in X_0$ and $\tau(q'_x) = 0$ otherwise.

Let now $p' = \psi^{-1}(q')$. Denoting the restriction of ψ still by ψ we get an injective homomorphism $\psi : Ap' \to q'(L^\infty(X)\bar{\otimes}N)q'$ such that $\psi(Ap') = \int_X^{\oplus} \tilde{A}_x \, dx$ is a Cartan subalgebra of $q'\mathcal{M}q'$. It then follows from Lemma 3.1.3 that $\tilde{A}_x$ is a Cartan subalgebra of $q'_x N q'_x$ for almost every $x \in X_0$. Put $p_1 := q'$ and consider $(1-p_1)(L^\infty(X_0)\bar{\otimes}N)(1-p_1)$. Applying again [SZ79, Proposition 7.17] we can find a projection $p_2 \leq 1 - p_1$ such that $\mathrm{ctr}(p_2) = \frac{1}{n}\mathbb{1}_{X_0}$. Continuing this we in the end find mutually orthogonal projections $p_1, \ldots, p_n \in L^\infty(X_0)\bar{\otimes}N$ such that $\mathrm{ctr}(p_i) = \frac{1}{n}\mathbb{1}_{X_0}$ for every i.

In particular $p_1, \ldots, p_n$ are equivalent and we can take partial isometries u_i such that $u_i^* u_i = p_1$ and $u_i u_i^* = p_i$. Define

$$\tilde{B} := \oplus_{i=1}^n u_i \psi(Ap') u_i^*.$$

Then $\tilde{B} = \int_X^{\oplus} \tilde{B}_x \, dx$ is a Cartan subalgebra of $L^\infty(X_0)\bar{\otimes}N$.

We want to upgrade this to a Cartan subalgebra of $\mathcal{M}$. For this we note that the proof of the implication $1 \Rightarrow 2$ in Lemma 3.1.3 goes through for a Cartan inclusion

$A \subseteq q\mathcal{M}q$ for some projection $q \in L^\infty(X)\bar{\otimes}N$. In the above situation this implies that we also had $\tilde{A}_{g^{-1}x} \prec_N \tilde{A}_x$ for almost every x when both sides are nonzero and hence the same holds for $\tilde{B}_x$ instead of $\tilde{A}_x$.

Summarizing, we now have a measurable map $X_0 \to \mathrm{Cartan}(N) : x \mapsto \tilde{B}_x$ such that $\tilde{B}_{g^{-1}x} \sim_u \tilde{B}_x$ inside N for almost every $x \in X_0 \cap gX_0$. Using the ergodicity of the action we can then extend this to a map

$$X \to \mathrm{Cartan}(N) : x \mapsto B_x$$

with the same properties. (A way to explicitly write this down is the following: Enumerate $\Gamma := \{e = g_1, g_2, \ldots\}$ and send $x \in X$ to the Cartan subalgebra associated to $g_i x$ where i is minimal among $\{n \in \mathbb{N} \mid g_n x \in X_0\}$.) Applying Lemma 3.1.3 again this implies that $B := \int_X^\oplus B_x \, dx$ is a Cartan subalgebra of $\mathcal{M}$ contained in $L^\infty(X)\bar{\otimes}N$ which by construction contains $\psi(Ap')$, thus finishing the proof of the claim. $\diamond$

It follows from the claim that $A \prec_{\mathcal{M}} B$, where A and B are both Cartan subalgebras of $\mathcal{M}$. Hence A and B are unitarily conjugate by [Pop06a, Theorem A.1], which finishes the proof of the lemma. $\square$

We are now ready to prove the main structural theorem of this Chapter from the introduction.

Theorem 1.1.1. *Let $\Gamma \in \mathcal{C}_{rss}$ and N be a II$_1$ factor. Suppose (X, μ) is a standard probability space and $\Gamma \curvearrowright X$ is a free ergodic p.m.p. action. Consider $\mathcal{M} := (L^\infty(X) \rtimes \Gamma)\bar{\otimes}N$ and let $A \subseteq \mathcal{M}$ be a subalgebra. Then the following are equivalent:*

1. *A is a Cartan subalgebra of* $\mathcal{M}$*,*

2. *A is unitarily conjugate to a subalgebra B of the form* $B = \int_X^{\oplus} B_x \, d\mu(x) \subseteq L^\infty(X)\bar{\otimes}N$ *satisfying*

   - B_x *is a Cartan subalgebra of N for almost every x,*

   - *For every* $g \in \Gamma$*,* B_x *is unitarily conjugate to* B_{gx} *inside N for almost every x.*

Proof. If A is a Cartan subalgebra of $\mathcal{M}$, then Lemma 3.1.4 implies that A is unitarily conjugate to a Cartan subalgebra B of $\mathcal{M}$ which is contained in $L^\infty(X)\bar{\otimes}N$. It follows from Lemma 3.1.3 that B has the desired form. The other implication is immediate from Lemma 3.1.3. □

3.2 A non-classifiability result for the equivalence relation of unitary conjugacy

Recall that for any Polish group G acting continuously on a Polish space Y we write $\mathcal{R}(G \curvearrowright Y)$ for its orbit equivalence relation. Consider a countable group Γ, a standard probability space (X, μ) and a free ergodic p.m.p. action $\Gamma \curvearrowright X$. Let N be a II_1 factor and consider $\mathcal{M} := (L^\infty(X) \rtimes \Gamma)\bar{\otimes}N$. Recall the equivalence relation E_0 on $\{0,1\}^{\mathbb{N}}$ where $\mathbf{x}E_0\mathbf{y}$ if there exists $N \in \mathbb{N}$ such that $x_n = y_n$ for all $n \geq N$. Theorem 1.1.3 says that if the relation of Cartan subalgebras of N up to unitary conjugacy is not smooth and the action $\Gamma \curvearrowright X$ is not strongly ergodic, then the equivalence relation of Cartan subalgebras of $\mathcal{M}$ up to unitary conjugacy cannot be classified by countable structures. The proof consists of two different parts. First of all we will use the structural results from section 3.1 to reduce the problem to studying the space of homomorphisms $\mathrm{Hom}(\mathcal{R}(\Gamma \curvearrowright X), \mathcal{R}(\mathcal{U}(N) \curvearrowright \mathrm{Cartan}(N)))$. Secondly we will prove that given any ergodic but not strongly ergodic p.m.p. action $\Gamma \curvearrowright X$, the space of homomorphisms $\mathrm{Hom}(\mathcal{R}(\Gamma \curvearrowright X), E_0)$ is not classifiable by countable structures. Using the Borel reduction $E_0 \leq_B \mathcal{R}(\mathcal{U}(N) \curvearrowright \mathrm{Cartan}(N))$ the desired result will then easily follow.

Convention. All standard Borel spaces will come equipped with a Borel measure, and we will be identifying Borel functions almost everywhere.

Given Polish spaces X, Y, and a measure μ on X, we write $B(X, Y)$ for the set of all Borel maps from X to Y, identified μ-almost everywhere. With the σ-algebra generated by the functions $f \mapsto \mu(A \cap f^{-1}(B))$ for $A \subseteq X$ and $B \subseteq Y$ Borel, this becomes a standard Borel space. We now first of all note that given two Borel equivalence relations E, F on X, Y respectively, also $\mathrm{Hom}(E, F)$ is a standard Borel space. One way to see this is the following. Since E is by assumption a Borel subset of $X \times X$, hence a standard Borel space, $B(E, Y \times Y)$ is a standard Borel space as well. Identifying $\mathrm{Hom}(E, F)$, $B(E, F)$ and $B(X, Y)$ with the image of their canonical embedding into $B(E, Y \times Y)$, it is then easy to see that $\mathrm{Hom}(E, F) = B(E, F) \cap B(X, Y)$. We conclude that $\mathrm{Hom}(E, F)$ is Borel, being the intersection of two Borel sets.

Lemma 3.2.1. *The equivalence relation* $(\mathrm{Hom}(\mathcal{R}(\Gamma \curvearrowright X), \mathcal{R}(\mathcal{U}(N) \curvearrowright \mathrm{Cartan}(N))), \sim_u$ $)$ *is Borel reducible to* $\mathcal{R}(\mathcal{U}(\mathcal{M}) \curvearrowright \mathrm{Cartan}(\mathcal{M}))$*, where, given* $\varphi, \psi \in \mathrm{Hom}(\mathcal{R}(\Gamma \curvearrowright$

$X), \mathcal{R}(\mathcal{U}(N) \curvearrowright \mathrm{Cartan}(N))$, we let $\varphi \sim_u \psi$ if and only if $\varphi(x) \sim_u \psi(x)$ for almost every $x \in X$. Moreover, if $\Gamma \in \mathcal{C}_{rss}$, the aforementioned equivalence relations are Borel bi-reducible.

Proof. We will write $\mathrm{Cartan}_{L^\infty(X)\bar{\otimes}N}(\mathcal{M}) := \{A \in \mathrm{Cartan}(\mathcal{M}) \mid A \subseteq L^\infty(X)\bar{\otimes}N\}$.

Step 1. $(\mathrm{Hom}(\mathcal{R}(\Gamma \curvearrowright X), \mathcal{R}(\mathcal{U}(N) \curvearrowright \mathrm{Cartan}(N))), \sim_u) \sim_B (\mathrm{Cartan}_{L^\infty(X)\bar{\otimes}N}(\mathcal{M}), \sim_u)$ *where the latter equivalence relation is unitary conjugacy in $\mathcal{M}$.*

First of all we note that $\mathrm{Cartan}_{L^\infty(X)\bar{\otimes}N}(\mathcal{M})$ is a standard Borel space. Indeed, we can write it as the intersection of $\mathrm{Cartan}(\mathcal{M})$ and $\mathrm{vNa}(L^\infty(X)\bar{\otimes}N)$ inside $\mathrm{vNa}(\mathcal{M})$. The former is Borel by Proposition 2.1.20 and the latter is Borel being the fixed points of the map $M \mapsto M \cap L^\infty(X)\bar{\otimes}N$ which is Borel by [Eff65, Corollary 2]. Using Lemma 3.1.3, given $A = \int_X^\oplus A_x\, dx \in \mathrm{Cartan}_{L^\infty(X)\bar{\otimes}N}(\mathcal{M})$ we get a function $[f_A : x \mapsto A_x] \in \mathrm{Hom}(\mathcal{R}(\Gamma \curvearrowright X), \mathcal{R}(\mathcal{U}(N) \curvearrowright \mathrm{Cartan}(N)))$. Conversely, given such a function we can build a Cartan subalgebra $A = \int_X^\oplus f(x)\, dx$. Moreover, by Lemma 3.1.1, we have that there exists $u \in \mathcal{U}(L^\infty(X)\bar{\otimes}N)$ such that $uAu^* = B$ if and only if $f_A \sim_u f_B$.

Suppose now that we have $A, B \in \mathrm{Cartan}_{L^\infty(X)\bar{\otimes}N}(\mathcal{M})$ and $u \in \mathcal{M}$ such that $uAu^* = B$. To complete the proof of step 1, we need to show that we can replace u by a unitary in $L^\infty(X)\bar{\otimes}N$. Write $u = \sum_g b_g u_g$ with $b_g \in L^\infty(X)\bar{\otimes}N$. Recall from (3.1) in the proof of $[2 \Rightarrow 1]$ of Lemma 3.1.3, that for every $g \in \Gamma$ we can find a unitary $u(g) \in \mathcal{U}(L^\infty(X)\bar{\otimes}N)$ such that

$$u_g^* u(g)^* A u(g) u_g = A.$$

From the fact that $uAu^* = B$, we then get $uu_g^* u(g)^* Au(g) = Buu_g^*$. Applying the conditional expectation onto $L^\infty(X)\bar{\otimes}N$ we get

$$E_{L^\infty(X)\bar{\otimes}N}(uu_g^*)[u(g)^* Au(g)] = BE_{L^\infty(X)\bar{\otimes}N}(uu_g^*).$$

Note that $E_{L^\infty(X)\bar{\otimes}N}(uu_g^*) = b_g$, so by taking the polar decomposition we get a partial isometry $v \in L^\infty(X)\bar{\otimes}N$ such that $vu(g)^* Au(g) = Bv$. Writing $v = \int_X^\oplus v_x \, dx$, $u(g)^* Au(g) = \int_X^\oplus [u(g)^* Au(g)]_x \, dx$ and $B = \int_X^\oplus B_x \, dx$ it follows that $v_x[u(g)^* Au(g)]_x = B_x v_x$. Since u is a unitary we can find for almost every $x \in X$ an element $g \in \Gamma$ such that $b_{g,x} \neq 0$, where we have written $b_g = \int_X^\oplus b_{g,x} \, dx$. Since $v_x \neq 0$ when $b_{g,x} \neq 0$, this means there exists for almost every $x \in X$ an element $g \in \Gamma$ such that

$$[u(g)^* Au(g)]_x \prec_N B_x.$$

It follows that $A_x \prec_N B_x$ for almost every $x \in X$ and since N is a factor we have $A_x \sim_u B_x$ inside N for almost every $x \in X$. Applying Lemma 3.1.1 we thus get a unitary $u' \in \mathcal{U}(L^\infty(X)\bar{\otimes}N)$ such that $u'Au'^* = B$ and we conclude that for $A, B \in \mathrm{Cartan}_{L^\infty(X)\bar{\otimes}N}(\mathcal{M})$, being unitarily conjugate inside $\mathcal{M}$ is equivalent to being unitarily conjugate inside $L^\infty(X)\bar{\otimes}N$.

Step 2. $(\mathrm{Cartan}_{L^\infty(X)\bar{\otimes}N}(\mathcal{M}), \sim_u) \leq_B \mathcal{R}(\mathcal{U}(\mathcal{M}) \curvearrowright \mathrm{Cartan}(\mathcal{M}))$.

We have the inclusion map $i : \mathrm{Cartan}_{L^\infty(X)\bar{\otimes}N}(\mathcal{M}) \hookrightarrow \mathrm{Cartan}(\mathcal{M})$ for which we obviously have $A \sim_u B \Leftrightarrow i(A) \sim_u i(B)$.

Step 1 and step 2 easily imply the first half of the lemma. For the moreover part, we conclude with the following.

Step 3. If $\Gamma \in \mathcal{C}_{rss}$, then $\mathcal{R}(\mathcal{U}(\mathcal{M}) \curvearrowright \mathrm{Cartan}(\mathcal{M})) \leq_B (\mathrm{Cartan}_{L^\infty(X)\bar{\otimes}N}(\mathcal{M}), \sim_u)$.

For this direction we consider

$$\begin{aligned}\mathbb{B} &:= \{(A,B) \in \mathrm{Cartan}(\mathcal{M}) \times \mathrm{Cartan}_{L^\infty(X)\bar{\otimes}N}(\mathcal{M}) \mid A \sim_u B\} \\ &= \{(A,B) \in \mathrm{Cartan}(\mathcal{M}) \times \mathrm{Cartan}(\mathcal{M}) \mid A \sim_u B\} \cap \mathrm{Cartan}(\mathcal{M}) \times \mathrm{Cartan}_{L^\infty(X)\bar{\otimes}N}(\mathcal{M})\end{aligned}$$

which is a Borel subset of $\mathrm{Cartan}(\mathcal{M}) \times \mathrm{Cartan}_{L^\infty(X)\bar{\otimes}N}(\mathcal{M})$ by Proposition 2.1.20(2) and the proof of step 1. Also, it follows from Lemma 3.1.4 that $\pi_1(\mathbb{B}) = \mathrm{Cartan}(\mathcal{M})$ where π_1 is the projection onto the first coordinate. Hence by [Tak01, Theorem A.16] there exists a measurable function $\Psi : \mathrm{Cartan}(\mathcal{M}) \to \mathrm{Cartan}_{L^\infty(X)\bar{\otimes}N}(\mathcal{M})$ such that $(A, \Psi(A)) \in \mathbb{B}$, i.e. $A \sim_u \Psi(A)$, for every $A \in \mathrm{Cartan}(\mathcal{M})$, which finishes step 3 and the proof of the moreover part of the lemma. $\square$

Consider now $\mathrm{Hom}(\mathcal{R}(\Gamma \curvearrowright X), E_0)$ with the equivalence relation given by $\varphi \sim \psi$ if and only if $\varphi(x) E_0 \psi(x)$ almost everywhere. We will prove the following more precise version of Theorem 1.1.4.

Theorem 3.2.2. *Let Γ be a countable group, (X, μ) a standard probability space and $\Gamma \curvearrowright X$ an ergodic p.m.p. action that is not strongly ergodic. Then there exists a map $f : (0,1)^{\mathbb{N}} \to \mathrm{Hom}(\mathcal{R}(\Gamma \curvearrowright X), E_0)$ satisfying all conditions from Proposition 2.3.18. In particular $(\mathrm{Hom}(\mathcal{R}(\Gamma \curvearrowright X), E_0), \sim)$ is not classifiable by countable structures.*

Proof. Consider the action $\oplus_{\mathbb{N}}\mathbb{Z} \curvearrowright (\Pi_{\mathbb{N}} S^1, \nu)$ where the i^{th} component of $\oplus_{\mathbb{N}}\mathbb{Z}$ acts on the i^{th} component of $\Pi_{\mathbb{N}} S^1$ by irrational rotation by $\alpha_i \in \mathbb{R}\backslash\mathbb{Q}$ and where $\nu := \otimes_{\mathbb{N}}\, m$ for

m the normalized Lebesgue (probability) measure on S^1. For $r \in (0,1)$ define the subset $C_r \subseteq S^1$

$$C_r := \left\{ e^{2\pi i(r+t)} \mid t \in [0,1/2] \right\},$$

i.e. C_r is half of the circle starting from the point $e^{2\pi i r}$. In the following, given any measure space Y we will write $\mathcal{B}(Y)$ for the space of all Borel subsets of Y, and we will equip it with the pseudo-metric given by the measure of the symmetric difference. Consider the function

$$\varphi : (0,1)^{\mathbb{N}} \to \{\text{sequences of Borel subsets of } \Pi_{\mathbb{N}} S^1\} = \prod_{\mathbb{N}} \mathcal{B}(\prod_{\mathbb{N}} S^1) : \mathbf{t} = (t_n)_n \mapsto (B_n^{\mathbf{t}})_n \tag{3.3}$$

where

$$B_n^{\mathbf{t}} := \underbrace{S^1 \times \cdots \times S^1}_{n-1} \times C_{t_n} \times S^1 \times \ldots .$$

The sequences $(B_n^{\mathbf{t}})_n$ are clearly almost invariant for the action of $\oplus_{\mathbb{N}} \mathbb{Z}$ and satisfy $\nu(B_n^{\mathbf{t}}) = \frac{1}{2}$ for every n. Also, for any element $a = (a_n)_n \in \oplus_{\mathbb{N}} \mathbb{Z}$ we have $\nu(aB_n^{\mathbf{t}} \Delta B_n^{\mathbf{t}}) = 0$ for n sufficiently large. Moreover, we note that

$$\nu(B_n^{\mathbf{s}} \Delta B_n^{\mathbf{t}}) = m(C_{s_n} \Delta C_{t_n}) = 2 \min\{|s_n - t_n|, 1 - |s_n - t_n|\}. \tag{3.4}$$

Using Theorem 2.2.1 together with the lack of strong ergodicity we get a factor map from the action $\Gamma \curvearrowright X$ to the action of $\mathbb{Z}$ on some standard probability space. By [Dye59] any two ergodic $\mathbb{Z}$-actions are orbit equivalent and so we also get a factor map $\theta : X \to \Pi_{\mathbb{N}} S^1$ for the above actions of Γ and $\oplus_{\mathbb{N}} \mathbb{Z}$ (and in fact to any ergodic action of any amenable group on some standard probability space by [OW80, Theorem 6]). Lifting the almost invariant sequence $(B_n^{\mathbf{t}})_n$ for $\oplus_{\mathbb{N}}\mathbb{Z} \curvearrowright (\Pi_{\mathbb{N}} S^1, \nu)$ we get by Lemma 2.2.2 an almost invariant sequence $(A_n^{\mathbf{t}})_n := \theta^{-1}(B_n^{\mathbf{t}})_n$ for the action $\Gamma \curvearrowright (X, \mu)$. Following [JS87] we then consider the map

$$A^{\mathbf{t}} : X \to \{0,1\}^{\mathbb{N}} : x \mapsto (\mathbb{1}_{A_n^{\mathbf{t}}}(x))_n. \tag{3.5}$$

We claim that for all $g \in \Gamma$ we have $A^{\mathbf{t}}(x) E_0 A^{\mathbf{t}}(gx)$ for almost every $x \in X$. Indeed the set of $x \in X$ satisfying the claim has measure

$$\mu(\{x \in X \mid \exists N \in \mathbb{N}, \forall n \geq N : x \notin gA_n^{\mathbf{t}} \Delta A_n^{\mathbf{t}}\}) = \mu\left(\bigcup_{N\in\mathbb{N}}\left(\bigcap_{n\geq N} X\backslash(gA_n^{\mathbf{t}}\Delta A_n^{\mathbf{t}})\right)\right) = 1. \tag{3.6}$$

For the last equality we used the moreover part of Lemma 2.2.2 and the fact that for every $a \in \oplus_{\mathbb{N}}\mathbb{Z}$, $\nu(aB_n^{\mathbf{t}} \Delta B_n^{\mathbf{t}}) = 0$ for n sufficiently large to deduce that $\lim_n \mu(\cup_{k\geq n} gA_k^{\mathbf{t}} \Delta A_k^{\mathbf{t}}) = 0$. In terms of the equivalence relations this means that $A^{\mathbf{t}}$ is a homomorphism (on a co-null subset of X) from $\mathcal{R}(\Gamma \curvearrowright X)$ to E_0. Recall that for two such elements we have $A^{\mathbf{s}} \sim A^{\mathbf{t}}$ if and only if $A^{\mathbf{s}}(x) E_0 A^{\mathbf{t}}(x)$ for almost every $x \in X$.

Altogether we now have a map $f : (0,1)^{\mathbb{N}} \to \mathrm{Hom}(\mathcal{R}(\Gamma \curvearrowright X), E_0) : \mathbf{t} \mapsto f(\mathbf{t}) := A^{\mathbf{t}}$. We claim that f satisfies all the conditions from Proposition 2.3.18. First of all we need that f is a Borel map. For this consider $(0,1)^{\mathbb{N}}$ with the metric given by

$$d((s_n)_n, (t_n)_n) = \sum_{n\in\mathbb{N}} 2^{-n} |s_n - t_n|,$$

and the space $\mathrm{Map}(X, \{0,1\}^{\mathbb{N}})$ of measurable functions from X to $\{0,1\}^{\mathbb{N}}$ with the metric given by

$$d(g,h) = \int_X \sum_{n\in\mathbb{N}} 2^{-n} |g(x)_n - h(x)_n| \, dx.$$

It is then easy to see that the map $(0,1)^{\mathbb{N}} \to \mathrm{Map}(X, \{0,1\}^{\mathbb{N}}) : \mathbf{t} \mapsto A^{\mathbf{t}}$ is actually continuous. The two conditions of Proposition 2.3.18 follow easily from the first part of the proof:

1. Suppose we have $\mathbf{s}$, $\mathbf{t} \in (0,1)^{\mathbb{N}}$ such that $\mathbf{s} - \mathbf{t} \in \ell^1$. Similar to (3.6) above, the points $x \in X$ such that $A^{\mathbf{s}}(x) E_0 A^{\mathbf{t}}(x)$ are given by

$$\{x \in X \mid \exists N \in \mathbb{N}, \forall n \geq N : x \notin A_n^{\mathbf{s}} \Delta A_n^{\mathbf{t}}\} = \bigcup_{N \in \mathbb{N}} \left(\bigcap_{n \geq N} X \backslash (A_n^{\mathbf{s}} \Delta A_n^{\mathbf{t}}) \right).$$

By (3.4) we have $\mu(X \backslash (A_n^{\mathbf{s}} \Delta A_n^{\mathbf{t}}) \geq 1 - 2\,|s_n - t_n|$. Since $\mathbf{s} - \mathbf{t} \in \ell^1$ it follows easily that $A^{\mathbf{s}}(x) E_0 A^{\mathbf{t}}(x)$ for almost every $x \in X$, i.e. $f(\mathbf{s}) \sim f(\mathbf{t})$.

2. Suppose we have $\mathbf{s}$, $\mathbf{t} \in (0,1)^{\mathbb{N}}$ such that $|s_n - t_n| \not\to 0$ and $|s_n - t_n| \not\to 1$. Then $\mu(A_n^{\mathbf{s}} \Delta A_n^{\mathbf{t}}) \not\to 0$ by (3.4), so we can find $\varepsilon > 0$ such that $\mu(A_n^{\mathbf{s}} \Delta A_n^{\mathbf{t}}) \geq \varepsilon$ for infinitely many n. Since X has measure 1, this means we can find a set Y of positive measure such that every element $y \in Y$ lies in infinitely many of the sets $A_n^{\mathbf{s}} \Delta A_n^{\mathbf{t}}$ and thus $A^{\mathbf{s}}(y) \not{E}_0 A^{\mathbf{t}}(y)$. Combining this with the observation that any comeager subset of $(0,1)^{\mathbb{N}}$ contains elements $\mathbf{s}$ and $\mathbf{t}$ such that $|s_n - t_n| \not\to 0, 1$ finishes the proof of the theorem. $\square$

This now enables us to prove the following two main results from the introduction.

Theorem 1.1.3. *Let Γ be a countable group and N be a II_1 factor. Suppose (X, μ) is a standard probability space and $\Gamma \curvearrowright X$ is a free ergodic p.m.p. action that is not strongly ergodic. Assume furthermore that $\mathcal{R}(\mathcal{U}(N) \curvearrowright \mathrm{Cartan}(N))$ is not smooth. Then the Cartan subalgebras of $\mathcal{M} := (L^\infty(X) \rtimes \Gamma) \bar{\otimes} N$ up to unitary conjugacy are not classifiable by countable structures.*

Proof. Recall that $\varphi, \psi \in \operatorname{Hom}(\mathcal{R}(\Gamma \curvearrowright X), \mathcal{R}(\mathcal{U}(N) \curvearrowright \operatorname{Cartan}(N)))$ are equivalent if and only if $\varphi(x) \sim_u \psi(x)$ almost everywhere. Given the map $f : (0,1)^{\mathbb{N}} \to \operatorname{Hom}(\mathcal{R}(\Gamma \curvearrowright X), E_0)$ from Theorem 3.2.2 we can use the given Borel reduction $\beta : \{0,1\}^{\mathbb{N}} \to \operatorname{Cartan}(N)$ from E_0 to $\mathcal{R}(\mathcal{U}(N) \curvearrowright \operatorname{Cartan}(N))$ to construct

$$g : (0,1)^{\mathbb{N}} \to \operatorname{Hom}(\mathcal{R}(\Gamma \curvearrowright X), \mathcal{R}(\mathcal{U}(N) \curvearrowright \operatorname{Cartan}(N))) : \mathbf{t} \mapsto g(\mathbf{t}) = \beta(f(\mathbf{t})) = \beta \circ A^{\mathbf{t}}.$$

Since β is a Borel reduction, it follows then immediately that g satisfies all conditions from Proposition 2.3.18 as well. Together with Lemma 3.2.1, this finishes the proof. $\square$

Remark. More generally, we see that with exactly the same proof, one can get the following statement. Suppose we have a countable group Γ, a standard probability space (X, μ) and an ergodic p.m.p. action $\Gamma \curvearrowright X$ that is not strongly ergodic. Then for any non-smooth equivalence relation E on a standard Borel space Y, the space of homomorphisms $\operatorname{Hom}(\mathcal{R}(\Gamma \curvearrowright X), E)$ up to E-equivalence almost everywhere is not classifiable by countable structures.

Theorem 1.1.6. *Let $\Gamma \in \mathcal{C}_{rss}$, (X, μ) be a standard probability space and N be a II_1 factor. Suppose $\Gamma \curvearrowright X$ is a free ergodic p.m.p. action that is not strongly ergodic and consider $\mathcal{M} := (L^\infty(X) \rtimes \Gamma) \bar{\otimes} N$. Then $\mathcal{R}(\mathcal{U}(\mathcal{M}) \curvearrowright \operatorname{Cartan}(\mathcal{M}))$ is either smooth or not classifiable by countable structures. Moreover, the former holds if and only if $\mathcal{R}(\mathcal{U}(N) \curvearrowright \operatorname{Cartan}(N))$ is smooth and this regardless of whether the action is strongly ergodic or not.*

Proof. It follows from Theorem 1.1.3 that $\mathcal{R}(\mathcal{U}(\mathcal{M}) \curvearrowright \operatorname{Cartan}(\mathcal{M}))$ is not classifiable by countable structures when $\mathcal{R}(\mathcal{U}(N) \curvearrowright \mathrm{Cartan}(N))$ is not smooth and $\Gamma \curvearrowright X$ is not strongly ergodic. Now assume $\mathcal{R}(\mathcal{U}(N) \curvearrowright \mathrm{Cartan}(N))$ is smooth, i.e. there is a Borel map $f : \mathrm{Cartan}(N) \to \mathbb{R}$ such that $C_1 \sim_u C_2 \Leftrightarrow f(C_1) = f(C_2)$. Then by steps 2 and 3 in Lemma 3.2.1 it is enough to show that $(\mathrm{Cartan}_{L^\infty(X)\bar{\otimes}N}(\mathcal{M}), \sim_u)$ is smooth. But given $A = \int_X^\oplus A_x\, dx \in \mathrm{Cartan}_{L^\infty(X)\bar{\otimes}N}(\mathcal{M})$ we can associate to it the function $[x \mapsto f(A_x)] \in L(X, \mu, \mathbb{R})$ which is a Polish space by [Kec10, ch. 19]. We already know that $A \sim_u B$ if and only if $A_x \sim_u B_x$ for almost every x. Hence $(\mathrm{Cartan}_{L^\infty(X)\bar{\otimes}N}(\mathcal{M}), \sim_u)$ is smooth, which finishes the proof. $\square$

3.3 Conjugacy by automorphisms

In this section we aim for a non-classification result for Cartan subalgebras up to conjugacy by an automorphism similar to Theorem 1.1.3 for unitary conjugacy. More specifically we will prove the following theorem which will easily imply Theorem 1.1.7.

Theorem 3.3.1. *Suppose N is a II_1 factor such that*

- *N has an irreducible regular amenable subfactor, i.e. an amenable subfactor $R \subseteq N$ such that $R' \cap N = \mathbb{C}1$ and $\mathcal{N}_N(R)'' = N$,*
- *there exists a Borel map $\beta : \{0,1\}^{\mathbb{N}} \to \mathrm{Cartan}(N)$ such that*

$$xE_0y \quad \Leftrightarrow \quad \beta(x) \sim_u \beta(y) \quad \Leftrightarrow \quad \beta(x) \sim_{sa} \beta(y).$$

Let $\Gamma \in \mathcal{C}_{rss}$, (X, μ) be a standard probability space and $\Gamma \curvearrowright X$ be a free ergodic p.m.p.

action that is not strongly ergodic. Then the equivalence relation of Cartan subalgebras of $\mathcal{M} = (L^\infty(X) \rtimes \Gamma)\bar{\otimes}N$ up to conjugacy by an automorphism is not classifiable by countable structures.

For the proof of Theorem 3.3.1 we start off with the following lemma.

Lemma 3.3.2. *In the setting of Theorem 3.3.1, suppose A and B are Cartan subalgebras of $\mathcal{M}$ contained in $L^\infty(X)\bar{\otimes}N$ such that $\alpha(A) = B$ for some $\alpha \in \mathrm{Aut}(\mathcal{M})$. Then $L^\infty(X) \prec^s_{\alpha(L^\infty(X)\bar{\otimes}N)} \alpha(L^\infty(X))$.*

Proof. Firstly note that the conclusion of the lemma is actually possible as $L^\infty(X) \subseteq \alpha(L^\infty(X)\bar{\otimes}N)$. Indeed $L^\infty(X) \subseteq B$ since B is a Cartan subalgebra of $L^\infty(X)\bar{\otimes}N$ whose center is $L^\infty(X)$, and $B = \alpha(A) \subseteq \alpha(L^\infty(X)\bar{\otimes}N)$. Now consider

$$\alpha(L^\infty(X)\bar{\otimes}R) \subseteq (L^\infty(X)\bar{\otimes}N) \rtimes \Gamma.$$

Since the left hand side is amenable, applying the dichotomy for $\Gamma \in \mathcal{C}_{rss}$ we get that either $\alpha(L^\infty(X)\bar{\otimes}R) \prec_{\mathcal{M}} L^\infty(X)\bar{\otimes}N$ or $\mathcal{N}_{\mathcal{M}}(\alpha(L^\infty(X)\bar{\otimes}R))''$ is amenable relative to $L^\infty(X)\bar{\otimes}N$. As this normalizer equals $\mathcal{M}$ by assumption and Γ is not amenable, the latter is not possible (see [OP10a, Proposition 2.4]). So we get that

$$\alpha(L^\infty(X)\bar{\otimes}R) \prec_{\mathcal{M}} L^\infty(X)\bar{\otimes}N.$$

Since $R' \cap N = \mathbb{C}1$, taking relative commutants and using [Vae08, Lemma 3.5] this implies that

$$L^\infty(X) \prec_{\mathcal{M}} \alpha(L^\infty(X)).$$

Since $\mathcal{N}_{\mathcal{M}}(L^\infty(X))'' = \mathcal{M}$ and $\mathcal{M}$ is a factor, [DHI16, Lemma 2.4(3)] implies that we actually have

$$L^\infty(X) \prec^s_{\mathcal{M}} \alpha(L^\infty(X)). \tag{3.7}$$

We will upgrade this to an embedding inside $\alpha(L^\infty(X)\bar{\otimes}N)$, i.e.

$$L^\infty(X) \prec^s_{\alpha(L^\infty(X)\bar{\otimes}N)} \alpha(L^\infty(X)), \tag{3.8}$$

which would finish the proof of the lemma. Assume (3.8) does not hold. Then there exists a projection $p \in L^\infty(X)' \cap \alpha(L^\infty(X)\bar{\otimes}N)$ such that $L^\infty(X)p \not\prec_{\alpha(L^\infty(X)\bar{\otimes}N)} \alpha(L^\infty(X))$. By Theorem 2.1.1 we can then find $u_n \in \mathcal{U}(L^\infty(X)p)$ such that

$$\left\| E_{\alpha(L^\infty(X))}(x^* u_n y) \right\|_2 \to 0$$

for all $x, y \in p\alpha(L^\infty(X)\bar{\otimes}N)$. We will show that in this case the same holds for all $x, y \in p\mathcal{M}$, contradicting (3.7). By density, we can assume that $x^* = \alpha(au_g)p$ and $y = p\alpha(bu_h)$ where $a, b \in L^\infty(X)\bar{\otimes}N$ and $g, h \in \Gamma$. Since $\alpha(L^\infty(X)) \subseteq \alpha(L^\infty(X)\bar{\otimes}N)$ we have $E_{\alpha(L^\infty(X))} = E_{\alpha(L^\infty(X))} \circ E_{\alpha(L^\infty(X)\bar{\otimes}N)}$. Using the fact that $\mathcal{U}(\alpha(N))$ normalizes $\alpha(L^\infty(X))$ we then get

$$\begin{aligned}
\left\| E_{\alpha(L^\infty(X))}(\alpha(au_g)u_n\alpha(bu_h)) \right\|_2 &= \left\| \alpha(u_g) E_{\alpha(L^\infty(X))}(\alpha(u_g^* a u_g) u_n \alpha(bu_{hg}))\alpha(u_g^*) \right\|_2 \\
&= \left\| E_{\alpha(L^\infty(X))}(E_{\alpha(L^\infty(X)\bar{\otimes}N)}(\alpha(u_g^* a u_g) u_n \alpha(b)\alpha(u_{hg}))) \right\|_2 \\
&= \left\| E_{\alpha(L^\infty(X))}(\alpha(u_g^* a u_g) u_n \alpha(b)\alpha(u_{hg})\delta_{hg,e}) \right\|_2 .
\end{aligned}$$

For the last equality we used the fact that $\alpha(u_g^* a u_g)u_n\alpha(b) \in \alpha(L^\infty(X)\bar{\otimes}N)$ to take it out of $E_{\alpha(L^\infty(X)\bar{\otimes}N)}$ together which the fact that the u_g's for $g \neq e$ are orthogonal to

$L^\infty(X)\bar{\otimes}N$. Now the last line converges to 0 by the discussion above, finishing the proof of the lemma. $\square$

Lemma 3.3.3. *In the setting of Lemma 3.3.2, there exists a nonzero projection $q \in B$ such that $\alpha(L^\infty(X))q = L^\infty(X)q$.*

Proof. Let $p \in L^\infty(X)\bar{\otimes}N \cap \alpha(L^\infty(X)\bar{\otimes}N)$ be any nonzero projection. From Lemma 3.3.2 it follows that there exist $p_0 \in L^\infty(X)p$, $p_1 \in \alpha(L^\infty(X))$, a *-homomorphism $\psi : L^\infty(X)p_0 \to \alpha(L^\infty(X))p_1$ and a nonzero partial isometry $v \in p_1\alpha(L^\infty(X)\bar{\otimes}N)p_0$ such that $\psi(x)v = vx$ for all $x \in L^\infty(X)p_0$. Note however that v in this case commutes with $\alpha(L^\infty(X))$, giving $v\psi(x) = vx$. Multiplying on the left by v^* and writing $q' := v^*v \in (L^\infty(X)p_0)' \cap p_0\alpha(L^\infty(X)\bar{\otimes}N)p_0 = p_0[(L^\infty(X)\bar{\otimes}N) \cap \alpha(L^\infty(X)\bar{\otimes}N)]p_0$, we get

$$L^\infty(X)q' \subseteq \alpha(L^\infty(X))q'. \tag{3.9}$$

Repeating the same arguments for α^{-1}, Lemma 3.3.2 tells us that also $\alpha(L^\infty(X)) \prec^s_{L^\infty(X)\bar{\otimes}N} L^\infty(X)$. In particular $\alpha(L^\infty(X))q' \prec_{L^\infty(X)\bar{\otimes}N} L^\infty(X)$ and with the same reasoning as above in the proof of this lemma, we get $0 \neq q \in (L^\infty(X)\bar{\otimes}N) \cap \alpha(L^\infty(X)\bar{\otimes}N)$ such that $q \leq q'$ and $\alpha(L^\infty(X))q \subseteq L^\infty(X)q$. Together with (3.9) this means

$$\alpha(L^\infty(X))q = L^\infty(X)q. \tag{3.10}$$

Applying $E := E_{L^\infty(X)\vee\alpha(L^\infty(X))}$ we get the same equality with $E(q)$ instead of q. Multiplying with $f_n(E(q))$ where $f_n(t) = t^{-1}\mathbb{1}_{[1/n,\infty)}(t)$ and taking the limit as $n \to \infty$, we also get the same for the support $s(E(q))$ of $E(q)$. Since $s(E(q)) \in L^\infty(X) \vee \alpha(L^\infty(X)) \subseteq B$

this means we can assume that q in equation (3.10) belongs to B, which finishes the proof of the lemma. $\square$

We now have all the ingredients to prove the main theorem of this section.

Proof of Theorem 3.3.1. We proceed in two steps. In the first step we show that, given two Cartan subalgebras of $\mathcal{M}$ contained in $L^\infty(X)\bar{\otimes}N$ that are conjugate by an automorphism of $\mathcal{M}$, there are positive measure subsets of X on which the Cartan subalgebras appearing in the respective integral decompositions are conjugate by a stable automorphism of N. In the second step we construct a map $f : (0,1)^{\mathbb{N}} \times (0,1)^{\mathbb{N}} \to \mathrm{Cartan}(\mathcal{M})$ similar to the one in the proof of Theorem 3.2.2. We then use the proof of Theorem 3.2.2 together with step 1 to verify that f satisfies the conditions of Proposition 2.3.18.

Step 1. Suppose that $A = \int_X^\oplus A_x \, dx$ and $B = \int_X^\oplus B_x \, dx$ are Cartan subalgebras of $\mathcal{M}$ contained in $L^\infty(X)\bar{\otimes}N$ (cf. Theorem 1.1.1) that are conjugate by an automorphism of $\mathcal{M}$. We will show that there exists a nonsingular Borel isomorphism $\Phi : Y_2 \to Y_1$ between positive measure subsets of X such that A_x and $B_{\Phi^{-1}x}$ are conjugate by a stable automorphism of N for all $x \in Y_1$. Take $\alpha \in \mathrm{Aut}(\mathcal{M})$ such that $\alpha(A) = B$. Using Lemma 3.3.3, we can take $q \in B$ such that $\alpha(L^\infty(X))q = L^\infty(X)q$. Taking relative commutants we get

$$q\alpha(L^\infty(X)\bar{\otimes}N)q = q(L^\infty(X)\bar{\otimes}N)q.$$

Writing $\tilde{q} = \alpha^{-1}(q) \in A$, $Y_1 := \{x \in X \mid \tilde{q}_x \neq 0\}$, and $Y_2 := \{x \in X \mid q_x \neq 0\}$ this gives the following diagram, where applying α gives an isomorphism from the first row to the second at every level.

$$\begin{tikzcd}[column sep=small]
L^\infty(Y_1)\tilde{q} = L^\infty(X)\tilde{q} & \subseteq & A\tilde{q} & \subseteq & \tilde{q}(L^\infty(X)\bar{\otimes}N)\tilde{q} = \int_{Y_1}^{\oplus} \tilde{q}_x N \tilde{q}_x \, dx \arrow[d, bend left, "\alpha", "\cong"'] \\
L^\infty(Y_2)q = L^\infty(X)q & \subseteq & Bq & \subseteq & q(L^\infty(X)\bar{\otimes}N)q = \int_{Y_2}^{\oplus} q_x N q_x \, dx
\end{tikzcd}$$

Possibly ignoring a zero measure subset, it then follows from Theorem 2.1.14 that we have a nonsingular Borel isomorphism $\Phi : Y_2 \to Y_1$ and isomorphisms

$$\alpha_x : \tilde{q}_x N \tilde{q}_x \xrightarrow{\sim} q_{\Phi^{-1}x} N q_{\Phi^{-1}x}$$

for every $x \in Y_1$. In particular

$$\alpha_x(A_x \tilde{q}_x) = B_{\Phi^{-1}x} q_{\Phi^{-1}x},$$

i.e. A_x is conjugate to $B_{\Phi^{-1}x}$ by a stable automorphism of N for every $x \in Y_1$.

Step 2. Recall from the proof of Theorem 3.2.2 that we defined the sets $C_r := \left\{ e^{2\pi i(r+t)} \mid t \in [0, 1/2] \right\} \subseteq S^1$ for $r \in (0, 1)$. Consider the map (cf. φ in (3.3))

$$\phi : (0,1)^{\mathbb{N}} \times (0,1)^{\mathbb{N}} \to \prod_{\mathbb{N}} \mathcal{B}(\prod_{\mathbb{N}} S^1) : (s_n, t_n)_n \mapsto (D_n^{\mathbf{s},\mathbf{t}})_n$$

where

$$D_{2n-1}^{\mathbf{s},\mathbf{t}} := (S^1)^{n-1} \times C_{s_n} \times S^1 \times S^1 \times \dots,$$
$$D_{2n}^{\mathbf{s},\mathbf{t}} := (S^1)^{n-1} \times C_{t_n} \times S^1 \times S^1 \times \dots.$$

Using the given Borel map $\beta : \{0,1\}^{\mathbb{N}} \to \mathrm{Cartan}(N)$, we can associate a Cartan subalgebra of $\mathcal{M}$ to every such sequence as follows (cf. Theorem 3.2.2). Using the non-strong ergodicity of $\Gamma \curvearrowright X$ we again get a factor map $\theta : X \to \Pi_{\mathbb{N}} S^1$ for the actions of Γ and $\oplus_{\mathbb{N}}\mathbb{Z}$. Let $A_n^{\mathbf{s},\mathbf{t}} := \theta^{-1}(D_n^{\mathbf{s},\mathbf{t}})$. Then we can consider the map

$$A^{\mathbf{s},\mathbf{t}} : X \to \{0,1\}^{\mathbb{N}} : x \mapsto (\mathbb{1}_{A_n^{\mathbf{s},\mathbf{t}}}(x))_n.$$

Arguing as in the proof of Theorem 3.2.2 we get in this way a map

$$(0,1)^{\mathbb{N}} \times (0,1)^{\mathbb{N}} \to \mathrm{Hom}(\mathcal{R}(\Gamma \curvearrowright X), \mathcal{R}(\mathcal{U}(N) \curvearrowright \mathrm{Cartan}(N))) : (\mathbf{s}, \mathbf{t}) \mapsto \beta \circ A^{\mathbf{s},\mathbf{t}}.$$

As in the proof of Step 1 of Lemma 3.2.1 we can associate a Cartan subalgebra of $\mathcal{M}$ to every homomorphism ψ from $\mathcal{R}(\Gamma \curvearrowright X)$ to $\mathcal{R}(\mathcal{U}(N) \curvearrowright \mathrm{Cartan}(N))$ via $A := \int_X^{\oplus} \psi(x)\,dx$. Altogether this gives us a measurable map

$$f : (0,1)^{\mathbb{N}} \times (0,1)^{\mathbb{N}} \to \mathrm{Cartan}(\mathcal{M}) : (\mathbf{s}, \mathbf{t}) \mapsto \int_X^{\oplus} \beta \circ A^{\mathbf{s},\mathbf{t}}(x)\,dx.$$

We claim that f satisfies the conditions in Proposition 2.3.18 for the relation of conjugacy by automorphisms on $\mathrm{Cartan}(\mathcal{M})$. The same proof as in Theorem 3.2.2 goes through to show that $f(\mathbf{s},\mathbf{t})$ and $f(\mathbf{v},\mathbf{w})$ are unitarily conjugate (and so certainly conjugate by an automorphism) whenever $\mathbf{s} - \mathbf{v} \in \ell^1$ and $\mathbf{t} - \mathbf{w} \in \ell^1$. Hence the first condition follows immediately.

For the second condition, take any comeager set $C \subseteq (0,1)^{\mathbb{N}} \times (0,1)^{\mathbb{N}}$. Then we can find sequences $(s_n)_n$, $(t_n)_n$, $(v_n)_n$ and $(w_n)_n$ in $(0,1)^{\mathbb{N}}$ such that $(s_n, t_n)_n \in C$, $(v_n, w_n)_n \in C$, $s_n, t_n, v_n \to 0$ and $w_n \to \frac{1}{2}$. Indeed, being comeager, C is dense and so in particular intersects arbitrary neighbourhoods of $(0,0)$ and $(0, \frac{1}{2})$. We claim that in this case $A := f(\mathbf{s},\mathbf{t})$ and $B := f(\mathbf{v},\mathbf{w})$ are not conjugate by an automorphism of $\mathcal{M}$, which would imply the second condition of Proposition 2.3.18. So assume $A \sim_a B$. Then from step 1 we get positive measure subsets $Y_{1,2} \subseteq X$ and a partial automorphism $\Phi : Y_2 \to Y_1$ such that

$$A_x \sim_{sa} B_{\Phi^{-1}(x)}$$

for all $x \in Y_1$. Looking at the construction of f above this means that we have

$$\mathbb{1}_{(A_n)_n}(x) E_0 \mathbb{1}_{(B_n)_n}(\Phi^{-1}(x))$$

for $x \in Y_1$. Here $(A_n)_n$ and $(B_n)_n$ are the almost invariant sequences used to construct A respectively B, i.e. $A_n = \theta^{-1}(D_n^{\mathbf{s},\mathbf{t}})$ and $B_n = \theta^{-1}(D_n^{\mathbf{v},\mathbf{w}})$ where $\theta : X \to \Pi_{\mathbb{N}} S^1$ is the factor map as before. Rephrasing the above, we have that for $x \in Y_1$,

$$x \in A_n \cap Y_1 \quad \Leftrightarrow \quad \Phi^{-1}(x) \in B_n \cap Y_2 \quad \Leftrightarrow \quad x \in \Phi(B_n \cap Y_2)$$

for n large enough. Hence by shrinking Y_1 to a positive measure subset if needed, we can assume that there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$

$$A_n \cap Y_1 = \Phi(B_n \cap Y_2). \tag{3.11}$$

Now recall that by construction we have

$$\begin{aligned} A_{2n-1} &:= \theta^{-1}((S^1)^{n-1} \times C_{s_n} \times S^1 \times \dots), \\ A_{2n} &:= \theta^{-1}((S^1)^{n-1} \times C_{t_n} \times S^1 \times \dots), \\ B_{2n-1} &:= \theta^{-1}((S^1)^{n-1} \times C_{v_n} \times S^1 \times \dots), \\ B_{2n} &:= \theta^{-1}((S^1)^{n-1} \times C_{w_n} \times S^1 \times \dots). \end{aligned}$$

Together with (3.11) we get

$$\{a \in Y_1 \mid \theta(a)_n \in C_{s_n}\} = A_{2n-1} \cap Y_1 = \Phi(B_{2n-1} \cap Y_2) = \Phi(\{b \in Y_2 \mid \theta(b)_n \in C_{v_n}\})$$

and

$$\{a \in Y_1 \mid \theta(a)_n \in C_{t_n}\} = A_{2n} \cap Y_1 = \Phi(B_{2n} \cap Y_2) = \Phi(\{b \in Y_2 \mid \theta(b)_n \in C_{w_n}\})$$

for $n \geq n_0$. Since $|s_n - t_n| \to 0$, we have $\mu(A_{2n-1} \Delta A_{2n}) \to 0$ (cf. (3.4)) and the above equalities then yield $\mu(\Phi(B_{2n-1} \cap Y_2) \Delta \Phi(B_{2n} \cap Y_2)) \to 0$. Since Y_2 has positive measure, this contradicts the fact that $|v_n - w_n| \to \frac{1}{2}$, finishing the proof. $\square$

Theorem 3.3.1 allows us to get the first family of II_1 factors whose Cartan subalgebras up to conjugacy by an automorphism are not classifiable by countable structures. More specifically we can use the II_1 factors of Speelman and Vaes constructed in [SV12] as follows. Let G be a countable group, K a compact abelian group and $\Lambda < K$ a countable dense subgroup. Consider the II_1 factor

$$N := L^\infty(K^G) \rtimes (G \times \Lambda) \tag{3.12}$$

where $G \curvearrowright K^G$ is the Bernoulli action and $\Lambda \curvearrowright K^G$ acts via $\lambda \cdot (k_g)_{g \in G} = (\lambda k_g)_{g \in G}$. If $K_1 < K$ is a closed subgroup such that $\Lambda_1 := \Lambda \cap K_1$ is dense in K_1, the subalgebra $\mathcal{C}(K_1) := L^\infty(K^G/K_1) \rtimes \Lambda_1$ is a Cartan subalgebra of N ([SV12, Lemma 6]). When G is a property (T) group such that $[G, G] = G$, [SV12, Theorem 1] tells us when two such Cartan subalgebras are unitarily conjugate or conjugate by a (stable) automorphism. This is then used to show in [SV12, Theorem 2] that for a specific choice of K and Λ one gets a Borel reduction β of E_0 into Cartan(N) for either the relation of being unitarily conjugate, conjugate by an automorphism or conjugate by a stable automorphism, i.e.

$$xE_0y \quad\Leftrightarrow\quad \beta(x)\sim_u\beta(y) \quad\Leftrightarrow\quad \beta(x)\sim_a\beta(y) \quad\Leftrightarrow\quad \beta(x)\sim_{sa}\beta(y). \tag{3.13}$$

Moreover one can easily check that $(L^\infty(K^G)\rtimes\Lambda)\subseteq N$ is an irreducible regular amenable subfactor. Hence the II_1 factor constructed in [SV12, Theorem 2] satisfies both conditions in Theorem 3.3.1. This now immediately implies Theorem 1.1.7.

Remark. Another approach would be to try to use the dichotomy for groups $\Gamma\in\mathcal{C}_{rss}$ as follows. Take an arbitrary II_1 factor N and suppose we have $A,B\in\mathrm{Cartan}((L^\infty(X)\rtimes\Gamma)\bar{\otimes}N)$ and $\alpha\in\mathrm{Aut}((L^\infty(X)\rtimes\Gamma)\bar{\otimes}N)$ such that

$$\alpha(A)=B.$$

Writing $(L^\infty(X)\bar{\otimes}N)\rtimes\Gamma=\alpha(L^\infty(X)\rtimes\Gamma)\bar{\otimes}\alpha(N)$ and using the fact that $\Gamma\in\mathcal{C}_{rss}$, it follows from [KV17, Lemma 5.2] that either $\alpha(L^\infty(X)\rtimes\Gamma)\prec_{\mathcal{M}}L^\infty(X)\bar{\otimes}N$ or $\alpha(N)$ is amenable relative to $L^\infty(X)\bar{\otimes}N$. Using once more that $\Gamma\in\mathcal{C}_{rss}$, the latter implies that $\alpha(N)\prec_{\mathcal{M}}L^\infty(X)\bar{\otimes}N$ (since $\mathcal{N}_{\mathcal{M}}(\alpha(N))''=\mathcal{M}$ is not amenable relative to $L^\infty(X)\bar{\otimes}N$). Hence either

$$\alpha(L^\infty(X)\rtimes\Gamma)\prec_{\mathcal{M}}L^\infty(X)\bar{\otimes}N, \tag{3.14}$$

or

$$\alpha(N)\prec_{\mathcal{M}}L^\infty(X)\bar{\otimes}N. \tag{3.15}$$

Given the latter, it is not difficult to get to the same conclusion as in Lemma 3.3.3. So if we can in some way exclude the first possibility, this could give an alternative way to our conclusion, avoiding to use the specific requirements on N in Theorem 3.3.1.

3.4 The hyperfinite II_1 factor

In [Pac85], J. Packer explicitly constructs an uncountable family of Cartan subalgebras of the hyperfinite II_1 factor R no two of which are unitarily conjugate. Combining the idea behind this construction with a turbulence result for cocycles from [Kec10] we will directly show that the Cartan subalgebras of R up to unitary conjugacy are in fact not classifiable by countable structures.

Recall that a p.m.p. action $\Gamma \curvearrowright (X, \mu)$ is called *compact* (or "has pure point spectrum" in the formulation of [Pac85]) if the image of Γ in $\mathrm{Aut}(X, \mu)$ is precompact in the weak topology (the smallest topology making the maps $T \mapsto \mu(T(A)\Delta B)$ continuous for all measurable sets $A, B \subseteq X$).

We first restate two results from [Pac85] used to construct a big family of Cartan subalgebras of R.

Theorem 3.4.1 ([Pac85, Corollary 2.6])**.** *Let Γ be a countable discrete abelian group, (Y, ν) a probability space and suppose $\Gamma \curvearrowright Y$ is a free ergodic p.m.p. action. Then $L\Gamma$ is a Cartan subalgebra of $L^\infty(Y) \rtimes \Gamma$ if and only if the action is compact.*

Given a free ergodic p.m.p. action $\Gamma \curvearrowright Y$ as in the theorem and a Borel 1-cocycle

$c : \Gamma \times Y \to S^1$ we can construct an automorphism A_c of $L^\infty(Y) \rtimes \Gamma$ given by

$$A_c(\sum_g a_g u_g) = \sum_g a_g^{(c)} u_g,$$

where $a_g^{(c)}(y) = c(g, y) a_g(y)$. If the action is also compact, we get in this way a family of Cartan subalgebras $(A_c(L\Gamma)) \subseteq L^\infty(Y) \rtimes \Gamma$ where c ranges over all 1-cocycles. The following result allows us to tell these Cartan subalgebras apart.

Theorem 3.4.2 ([Pac85, Theorem 3.8]). *Let Γ and Y be as above and suppose $\Gamma \curvearrowright Y$ is a free ergodic compact p.m.p. action. Let $c : \Gamma \times Y \to S^1$ be a Borel 1-cocycle. Then $A_c(L\Gamma)$ is unitarily conjugate to $L\Gamma$ inside $L^\infty(Y) \rtimes \Gamma$ if and only if c is cohomologous to a cocycle of the form $\gamma(g, y) = \gamma(g)$ for some $\gamma \in \hat{\Gamma}$.*

We will combine this theorem with the following results from [Kec10]. Recall that for a Polish group G and a standard measure space (Y, ν), we let $L(Y, \nu, G)$ be the space of all Borel maps $f : Y \to G$ up to agreeing ν-almost everywhere.

Theorem 3.4.3 ([Kec10, Corollary 27.4]). *Suppose E is an ergodic equivalence relation that is not strongly ergodic and let $G \neq \{1\}$ be a Polish group admitting an invariant metric. Then the action of $L(Y, \nu, G)$ on $\overline{B^1(E, G)}$ is turbulent. In particular, the cohomology relation on $\overline{B^1(E, G)}$ (and thus also on $Z^1(E, G)$) does not admit classification by countable structures.*

Proposition 3.4.4 ([Kec10, Corollary 22.2 and Proposition 23.5]). *Let G be a Polish group admitting an invariant metric and let G act continuously by isometries on a Polish* *metric space (M, ρ). Assume that the orbit $G \cdot x$ is not closed. Then the action of G on the invariant closed set $\overline{G \cdot x}$ is minimal and every orbit contained in $\overline{G \cdot x}$ is meager in $\overline{G \cdot x}$.*

We can now give a short proof of Theorem 1.1.8 from the introduction, establishing that the Cartan subalgebras of the hyperfinite II_1 factor R up to unitary conjugacy are not classifiable by countable structures.

Theorem 1.1.8. *Cartan subalgebras of the hyperfinite II_1 factor R up to unitary conjugacy are not classifiable by countable structures.*

Proof. Following [Pac85, section 4] we consider $\mathbb{Z}_2$ with the $(1/2, 1/2)$-measure. Let (Y, ν) be $\Pi_{i \in \mathbb{N}} \mathbb{Z}_2$ with product measure and put $\Gamma := \oplus_{i \in \mathbb{N}} \mathbb{Z}_2$. Then both Y and Γ are topological groups for addition modulo 2 and Γ embeds as a countable dense subgroup in Y. It is easy to check that the translation action $\Gamma \curvearrowright Y$ is free, ergodic, measure preserving and compact. Also we note that $L^\infty(Y) \rtimes \Gamma$ can be identified with $\bar{\otimes}_{\mathbb{N}}(L^\infty(\mathbb{Z}_2) \rtimes \mathbb{Z}_2) \cong \bar{\otimes}_{\mathbb{N}} M_2(\mathbb{C}) \cong R$ and that $\hat{\Gamma} \cong Y$. From Theorem 3.4.1 it now follows that $L\Gamma$ is a Cartan subalgebra of $L^\infty(Y) \rtimes \Gamma \cong R$, and so we can consider all Cartan subalgebras of the form $A_c(L\Gamma)$ for $c \in Z^1(\Gamma \curvearrowright Y, S^1)$. Note that Theorem 3.4.2 implies that $A_c(L\Gamma)$ is unitarily conjugate to $A_d(L\Gamma)$ if and only if $c^{-1}d$ is cohomologous to a cocycle $\gamma \in \hat{\Gamma}$. Consider now the action of $\hat{\Gamma} \times L(Y, \nu, S^1)$ on $\overline{B^1(\Gamma \curvearrowright Y, S^1)} = Z^1(\Gamma \curvearrowright Y, S^1)$ (see [Kec10, Theorem 26.4] for the equality) given by

$$(\gamma, f) \cdot c(g, y) = \gamma(g) f(gy) c(g, y) f(y)^{-1}.$$

From the above it follows that the orbit equivalence relation of this action is Borel reducible to the equivalence relation of unitary conjugacy on Cartan(R). Hence it suffices to show that the above action is turbulent. However, we know that the action of just the $L(Y, \nu, S^1)$-part is turbulent by Theorem 3.4.3. So if we can show that the orbits for the action of $\hat{\Gamma} \times L(Y, \nu, S^1)$ are still meager, the result will follow (since the other parts in the definition of turbulence are obviously satisfied). For this, note that the action is not transitive. Since the orbit of 1 is dense, it then immediately follows from Proposition 3.4.4 (and the remark below) that all orbits are meager, finishing the proof of the theorem. □

Remark. Concerning the conditions in the theorems we apply, note that

1. the orbit equivalence relation of $\Gamma \curvearrowright Y$ is hyperfinite and hence not strongly ergodic,
2. the Polish groups involved (namely $\hat{\Gamma}$, S^1 and $L(Y, \nu, S^1)$) admit invariant metrics,
3. $\hat{\Gamma} \times L(Y, \nu, S^1) \curvearrowright Z^1(\Gamma \curvearrowright Y, S^1)$ is an action by isometries. Indeed, on $Z^1(\Gamma \curvearrowright Y, S^1)$ we have the compatible metric

$$d(c, c') = \sum_{k=1}^{\infty} 2^{-k} \int_Y |c(g_k, y) - c'(g_k, y)| \, d\nu(y)$$

where $\Gamma := \{g_1, g_2, g_3, \dots\}$ (see also [Kec10, ch. 24]). It is then a straightforward calculation to check that $d((\gamma, f) \cdot c, (\gamma, f) \cdot c') = d(c, c')$.

We end with the following proposition, which together with Theorem 1.1.8 will easily imply Corollary 1.1.9 from the introduction.

Proposition 3.4.5. *Suppose M is a II_1 factor with at least one Cartan subalgebra and N is any II_1 factor. Then $\mathcal{R}(\mathcal{U}(N) \curvearrowright \mathrm{Cartan}(N)) \leq_B \mathcal{R}(\mathcal{U}(M\bar{\otimes}N) \curvearrowright \mathrm{Cartan}(M\bar{\otimes}N))$.*

Proof. Let $A \subseteq M$ be a Cartan subalgebra and consider the Borel map

$$f : \mathrm{Cartan}(N) \to \mathrm{Cartan}(M\bar{\otimes}N) : B \mapsto A\bar{\otimes}B.$$

We claim that this is a Borel reduction for the equivalence relations induced by unitary conjugacy, i.e. B and C are unitarily conjugate inside N if and only if $A\bar{\otimes}B$ and $A\bar{\otimes}C$ are unitarily conjugate inside $M\bar{\otimes}N$. One direction is trivial. For the other direction, suppose that $A\bar{\otimes}B$ is unitarily conjugate to $A\bar{\otimes}C$ inside $M\bar{\otimes}N$.

Claim. If $A \subseteq M$ and $B, C \subseteq N$ are any von Neumann algebras such that $A\bar{\otimes}B$ is unitarily conjugate to $A\bar{\otimes}C$ inside $M\bar{\otimes}N$, then $B \prec_N C$.

Proof of the claim. Suppose not, then by Theorem 2.1.1 there exist $u_n \in \mathcal{U}(B)$ such that $\|E_C(x^*u_ny)\|_2 \to 0$ for all $x, y \in N$. Take now $s_1, s_2 \in M$ and $t_1, t_2 \in N$. Then

$$\|E_{A\bar{\otimes}C}((s_1 \otimes t_1)^*(1 \otimes u_n)(s_2 \otimes t_2))\|_2 = \|E_A(s_1^*s_2) \otimes E_C(t_1^*u_nt_2)\|_2 \to 0.$$

Since elements of the form $s \otimes t$ are dense in $M\bar{\otimes}N$, it follows by normality of the conditional expectation that the same holds for any $x, y \in M\bar{\otimes}N$. Hence $A\bar{\otimes}B \not\prec_{M\bar{\otimes}N} A\bar{\otimes}C$, contradiction. $\diamond$

Since N is a II_1 factor and B, C are Cartan subalgebras, the claim implies that B and C are unitarily conjugate inside N, finishing the proof. $\square$

Corollary 1.1.9. *Let M be a McDuff II_1 factor with at least one Cartan subalgebra. Then the Cartan subalgebras of M up to unitary conjugacy are not classifiable by countable structures.*

Proof. Since a McDuff II_1 factor M satisfies $M \cong M \bar{\otimes} R$ by definition, this follows immediately from Proposition 3.4.5 and Theorem 1.1.8. □

This chapter contains material from: P. Spaas, "Non-classification of Cartan subalgebras for a class of von Neumann algebras", *Advances in Mathematics*, vol. 332, pp. 510-552, 2018. The dissertation author was the primary investigator and author of this paper.

Chapter 4

II_1 factors with a unique McDuff decomposition

4.1 The technical theorem

This section is devoted to the proof of our main technical result of this chapter, Theorem 1.1.10, which we reformulated here. We will end the section by deriving Corollaries 1.1.12 and 1.1.13 from it.

Theorem 4.1.1. *Let P_1 be a II_1 factor which admits a Cartan subalgebra A_1 such that $P_1' \cap P_1^\omega \subset A_1^\omega$. Let P_2 be a non-McDuff II_1 factor, and $\theta : P_1 \bar{\otimes} R_1 \to P_2 \bar{\otimes} R_2$ be an isomorphism, where R_1, R_2 are hyperfinite II_1 factors.*

Then P_2 admits a Cartan subalgebra A_2 satisfying $P_2' \cap P_2^\omega \subset A_2^\omega$ and we can find a unitary element $u \in P_2 \bar{\otimes} R_2$ and $t > 0$ such that $\theta(A_1^t) = uA_2u^*$, where we identify

$P_1\bar{\otimes}R_1 = P_1^t\bar{\otimes}R_1^{1/t}$.

Moreover, $\mathcal{R}(A_1 \subset P_1)^t$ *is isomorphic to* $\mathcal{R}(A_2 \subset P_2)$*.*

Proof. Note first that by Lemma 2.1.19, the moreover assertion is a consequence of the main assertion. To prove the main assertion of the theorem, put $M = P_1\bar{\otimes}R_1$ and identify $M = P_2\bar{\otimes}R_2$ using θ. Since $P_1' \cap P_1^\omega \subset A_1^\omega$, by applying[Mar18, Proposition 5.2] we deduce that

$$M' \cap M^\omega \subset (A_1\bar{\otimes}R_1)^\omega. \tag{4.1}$$

Thus, we have that $R_2' \cap R_2^\omega \subset (A_1\bar{\otimes}R_1)^\omega$. By applying Lemma 2.1.16, we get that $R_2 \prec_M A_1\bar{\otimes}R_1$. Using this fact, the first part of the proof of [Hof16, Proposition 6.3] provides some $s > 0$ such that if we identify $P_2\bar{\otimes}R_2 = P_2^s\bar{\otimes}R_2^{1/s}$, then we can find a non-zero projection $q_0 \in R_2^{1/s}$ and a unitary $v \in M$ such that $q := vq_0v^* \in A_1\bar{\otimes}R_1$ and we have

$$\mathrm{Ad}(v)(q_0R_2^{1/s}q_0) \subset q(A_1\bar{\otimes}R_1)q. \tag{4.2}$$

Denote $P_3 = \mathrm{Ad}(v)(P_2^s)$ and $R_3 = \mathrm{Ad}(v)(R_2^{1/s})$. Then $M = P_3\bar{\otimes}R_3$, $q \in R_3$, and (4.2) rewrites as $qR_3q \subset q(A_1\bar{\otimes}R_1)q$. By taking relative commutants inside M, this gives that $A_1q \subset P_3 \otimes q$. Therefore, we can write $A_1q = A_3 \otimes q$, where $A_3 \subset P_3$ is a unital von Neumann subalgebra.

Since P_2 is not McDuff, $P_3 \cong P_2^s$ is also not McDuff, i.e., $P_3' \cap P_3^\omega$ is abelian. Since $P_1' \cap P_1^\omega$ is abelian, Corollary 2.1.18 implies that $P_3' \cap P_3^\omega = \mathcal{Z}(M' \cap M^\omega) = P_1' \cap P_1^\omega \subset A_1^\omega$. From this we get that $(P_3' \cap P_3^\omega) \otimes q \subset A_1^\omega q = A_3^\omega \otimes q$, and since q is non-zero, it follows that

$$P_3' \cap P_3^\omega \subset A_3^\omega. \tag{4.3}$$

By combining (4.3) with [Mar18, Proposition 5.2] we derive that

$$M' \cap M^\omega \subset (A_3 \bar{\otimes} R_3)^\omega. \tag{4.4}$$

The rest of the proof is divided between three claims.

Claim 1. $A_3' \cap P_3 \prec_{A_3' \cap P_3} A_3$.

Proof of Claim 1. Since $A_3 \bar{\otimes} qR_3q \subset qMq$ is generated by $A_3 \otimes q = A_1q$ and $qR_3q \subset q(A_1 \bar{\otimes} R_1)q$, we get that $A_3 \bar{\otimes} qR_3q \subset q(A_1 \bar{\otimes} R_1)q$. Let us start with showing that $q(A_1 \bar{\otimes} R_1)q \prec_{q(A_1 \bar{\otimes} R_1)q} A_3 \bar{\otimes} qR_3q$. This is equivalent to proving that $(A_1 \bar{\otimes} R_1)\tilde{q} \prec_{(A_1 \bar{\otimes} R_1)\tilde{q}} A_3 \bar{\otimes} qR_3q$, where $\tilde{q} \in \mathcal{Z}(A_1 \bar{\otimes} R_1) = A_1$ denotes the central support of q. Let $v_1, .., v_m \in A_1 \bar{\otimes} R_1$ be partial isometries such that $v_i^* v_i \leq q$, for all $1 \leq i \leq m$, the projections $v_i v_i^*, 1 \leq i \leq m$, are mutually orthogonal, and $\|\tilde{q} - \sum_{i=1}^m v_i v_i^*\|_2 \leq \|\tilde{q}\|_2/4$.

Write $R_1 = \bar{\otimes}_{k \in \mathbb{N}} \mathbb{M}_2(\mathbb{C})$, and for $n \in \mathbb{N}$, define $R_{1,n} = \bar{\otimes}_{k \geq n} \mathbb{M}_2(\mathbb{C})$. Since $\prod_\omega R_{1,n} \subset M' \cap M^\omega$, by using (4.4) and Lemma 2.1.15, we can find $n \in \mathbb{N}$ such that $\|u - E_{A_3 \bar{\otimes} R_3}(u)\|_2 \leq \|\tilde{q}\|_2/(4m)$ and $\|v_i u - u v_i\|_2 \leq \|\tilde{q}\|_2/(4m)$, for all $u \in \mathcal{U}(R_{1,n})$ and every $1 \leq i \leq m$.

Let $a \in \mathcal{U}(A_1)$ and $u \in \mathcal{U}(R_{1,n})$. Then we have

$$\begin{aligned}\|u\tilde{q} - \sum_{i=1}^{m} v_i E_{A_3\bar{\otimes}R_3}(u)v_i^*\|_2 &\leq \|u\tilde{q} - \sum_{i=1}^{m} v_i u v_i^*\|_2 + \|\tilde{q}\|_2/4 \\ &\leq \|u\tilde{q} - u\sum_{i=1}^{m} v_i v_i^*\|_2 + \|\tilde{q}\|_2/2 \\ &\leq (3\|\tilde{q}\|_2)/4.\end{aligned}$$

Since $a \in A_1 = \mathcal{Z}(A_1\bar{\otimes}R_1)$, we get $\|au\tilde{q} - \sum_{i=1}^{m} v_i a E_{A_3\bar{\otimes}R_3}(u)v_i^*\|_2 \leq (3\|\tilde{q}\|_2)/4 < \|\tilde{q}\|_2$. Since $v_i \in (A_1\bar{\otimes}R_1)q$, for all $1 \leq i \leq m$ and $qa \in A_1 q = A_3 \otimes q$, we deduce that

$$\sum_{i=1}^{m} v_i a E_{A_3\bar{\otimes}R_3}(u)v_i^* = \sum_{i=1}^{m} v_i(aq)(qE_{A_3\bar{\otimes}R_3}(u)q)v_i^* \in \sum_{i=1}^{m} v_i(A_3\bar{\otimes}qR_3q)_1 v_i^*.$$

Since the subgroup $\{au\tilde{q} \mid a \in \mathcal{U}(A_1), u \in \mathcal{U}(R_{1,n})\}$ of $\mathcal{U}((A_1\bar{\otimes}R_{1,n})\tilde{q})$ generates $(A_1\bar{\otimes}R_{1,n})\tilde{q}$, by Theorem 2.1.1 we get that $(A_1\bar{\otimes}R_{1,n})\tilde{q} \prec_{(A_1\bar{\otimes}R_1)\tilde{q}} A_3\bar{\otimes}qR_3q$. Since $R_{1,n} \subset R_1$ is a finite index subfactor, we get that $(A_1\bar{\otimes}R_1)\tilde{q} \prec_{(A_1\bar{\otimes}R_1)\tilde{q}} A_3\bar{\otimes}qR_3q$, and thus $q(A_1\bar{\otimes}R_1)q \prec_{q(A_1\bar{\otimes}R_1)q} A_3\bar{\otimes}qR_3q$. Since $q(A_1\bar{\otimes}R_1)q = (A_1q)' \cap qMq = (A_3 \otimes q)' \cap qMq = (A_3' \cap P_3)\bar{\otimes}qR_3q$, the last embedding rewrites as $(A_3' \cap P_3)\bar{\otimes}qR_3q \prec_{(A_3' \cap P_3)\bar{\otimes}qR_3q} A_3\bar{\otimes}qR_3q$, which readily implies the claim. $\diamond$

Claim 2. There exist a Cartan subalgebra $A_4 \subset P_3$ such that $P_3' \cap P_3^\omega \subset A_4^\omega$, and a non-zero projection $p \in A_4$ such that $r = p \otimes q$ belongs to $q(A_1\bar{\otimes}R_1)q$ and $A_4 r = A_1 r$.

Proof of Claim 2. By Claim 1 we can find a non-zero projection $p \in A_3' \cap P_3$ such that $A_3 p$ is maximal abelian in pP_3p. Let $r = p \otimes q$. Then $r \in (A_3 \otimes q)' \cap qMq = (A_1q)' \cap qMq = q(A_1\bar{\otimes}R_1)q$ and $A_3 r = (A_3 \otimes q)r = (A_1q)r = A_1 r$. Since $A_1 \subset P_1$ is a Cartan subalgebra, A_1 is regular in $M = P_1\bar{\otimes}R_1$. By [Pop06b, Lemma 3.5.1] we get that

$A_3r = A_1r$ is quasi-regular in rMr. This implies that $A_3r = A_3p \otimes q$ is quasi-regular in $r(P_3 \otimes q)r = pP_3p \otimes q$, hence A_3p is quasi-regular in pP_3p. As $A_3p \subset pP_3p$ is maximal abelian, [PS03, Theorem 2.7] gives that $A_3p \subset pP_3p$ is a Cartan subalgebra.

Since P_3 is a II_1 factor, we can find a Cartan subalgebra $A_4 \subset P_3$ such that $p \in A_4$ and $A_4p = A_3p$. Thus, we have that $A_4r = A_3r = A_1r$. In order to finish the proof of the claim, it remains to argue that $P_3' \cap P_3^\omega \subset A_4^\omega$. First, by (4.3), we get that $(pP_3p)' \cap (pP_3p)^\omega \subset (A_3p)^\omega = (A_4p)^\omega$. Since P_3 is a II_1 factor and A_4 is a Cartan subalgebra, we can find partial isometries $v_1, ..., v_k \in P_3$ such that $v_i^*v_i \in A_4p$ and $v_iA_4v_i^* \subset A_4$, for every $1 \leq i \leq k$, and $\sum_{i=1}^k v_iv_i^* = 1$. If $x \in P_3' \cap P_3^\omega$, then $v_i^*xv_i \in (pP_3p)' \cap (pP_3p)^\omega \subset (A_4p)^\omega$ and thus $v_iv_i^*xv_iv_i^* = v_i(v_i^*xv_i)v_i^* \in A_4^\omega$, for every $1 \leq i \leq k$. Since $x = \sum_{i=1}^k v_iv_i^*xv_iv_i^*$, we get that $x \in A_4^\omega$, as claimed. $\diamond$

Since A_4p is a Cartan subalgebra of pP_3p, we deduce that $A_1r = A_4r = A_4p \otimes q$ is a regular subalgebra of rMr. Using this fact, we will next show the following:

Claim 3. There exist a projection $e \in A_1$ and $\alpha \geq 0$ such that $E_{A_1}(r) = \alpha e$.

Proof of Claim 3. Denote by $e \in A_1$ the support projection of $E_{A_1}(r)$. Let $u \in \mathcal{N}_{rMr}(A_1r)$. Since $A_1e \ni a \mapsto ar \in A_1r$ is an isomorphism, we can find an isomorphism $\theta : A_1e \to A_1e$ such that

$$uaru^* = \theta(a)r, \quad \text{for all } a \in A_1e. \tag{4.5}$$

Thus, $\theta(a) = E_{A_1}(uaru^*)E_{A_1}(r)^{-1}$, for all $a \in A_1e$. Combining this with the fact that $A_1e \subset eP_1e$ is a Cartan subalgebra, we can find $v \in \mathcal{N}_{eP_1e}(A_1e)$ such that $\theta(a) = vav^*$, for all $a \in A_1e$. Thus, $\tau(av^*rv) = \tau(vav^*r) = \tau(\theta(a)r) = \tau(uaru^*) = \tau(ar)$, for all $a \in A_1e$, hence $E_{A_1}(v^*rv) = E_{A_1}(r)$. Since v normalizes A_1e, we derive that $E_{A_1}(r)$ commutes with v. Since $E_{A_1}(r) \in A_1e$ we get that $\theta(E_{A_1}(r)) = E_{A_1}(r)$. In combination with (4.5), we get that $E_{A_1}(r)r = \theta(E_{A_1}(r))r = uE_{A_1}(r)ru^*$. This shows that $E_{A_1}(r)r \in rMr$ commutes with every $u \in \mathcal{N}_{rMr}(A_1r)$. Since A_1r is regular in rMr and rMr is a factor, we derive that $E_{A_1}(r)r = \alpha r$, for some $\alpha \geq 0$. Hence $E_{A_1}(r)^2 = \alpha E_{A_1}(r)$ and the claim follows. $\diamond$

Let $f \in R_1$ be a projection such that $\tau(f) = \alpha$. By Claim 3 we have $E_{A_1}(r) = E_{A_1}(e \otimes f)$, thus we can find a unitary $w \in A_1 \bar{\otimes} R_1$ such that $r = w(e \otimes f)w^*$. Altogether, we have found a Cartan subalgebra $A_4 \subset P_3$, projections $p \in A_4, q \in R_3, e \in A_1, f \in R_1$, and a unitary $w \in M$ such that

$$A_4p \otimes q = A_4r = A_1r = \mathrm{Ad}(w)(A_1e \otimes f). \tag{4.6}$$

If P and R are II_1 factors, $A \subset P$ is a Cartan subalgebra, and $a \in A, b \in R$ are projections, then we can identify $P\bar{\otimes}R = P^{1/\tau(b)}\bar{\otimes}R^{\tau(b)}$ such that $Aa \otimes b$ is identified with $A^{1/\tau(b)}c$, where $c \in A^{1/\tau(b)} \subset P^{1/\tau(b)}$ is a projection of trace $\tau(a)\tau(b)$. By combining this fact with (4.6) it follows that we can identify $M = P_1^t\bar{\otimes}R_1^{1/t}$, for $t = \tau(q)/\tau(f)$, such that A_4 and A_1^t are unitarily conjugate.

Finally, recall that $P_3 = \mathrm{Ad}(v)(P_2^s)$ and $R_3 = \mathrm{Ad}(v)(R_2^{1/s})$. Let $A_2 \subset P_2$ be a Cartan subalgebra such that $A_4 = \mathrm{Ad}(v)(A_2^s)$. Since $P_3' \cap P_3^\omega \subset A_4^\omega$ by Claim 2, we get that $(P_2^s)' \cap (P_2^s)^\omega \subset (A_2^s)^\omega$ and the argument from the proof of Claim 2 implies that $P_2' \cap P_2^\omega \subset A_2^\omega$. Moreover, identifying $M = P_1^t\bar{\otimes}R_1^{1/t} = P_2^s\bar{\otimes}R_2^{1/s}$ we have that A_1^t and A_2^s are unitarily conjugate. Thus, if we identify $M = P_1^{t/s}\bar{\otimes}R_1^{s/t}$, then $A_1^{t/s}$ is unitarily conjugate to A_2. This finishes the proof. $\square$

We can now derive the next two corollaries from the introduction.

Corollary 1.1.12. *Let $n \geq 2$ and $\mathbb{F}_n \curvearrowright (X, \mu)$ be a free ergodic p.m.p. action. Put $N = L^\infty(X) \rtimes \mathbb{F}_n$. Let P be any II_1 factor such that $N \bar{\otimes} R$ and $P \bar{\otimes} R$ are isomorphic. Then P is either isomorphic to N^t, for some $t > 0$, or to $N \bar{\otimes} R$.*

Proof. Let $M = L^\infty(X) \rtimes \mathbb{F}_n$, where $\mathbb{F}_n \curvearrowright (X, \mu)$ is a free ergodic p.m.p. action, for some $n \geq 2$. Let N be a II_1 factor such that $M \bar{\otimes} R \cong N \bar{\otimes} R$. If N is McDuff, then $N \cong N \bar{\otimes} R \cong M \bar{\otimes} R$. Thus, we may assume that N is not McDuff.

Since $\mathbb{F}_n$ is not inner amenable, [Cho82] implies that $M' \cap M^\omega \subset L^\infty(X)^\omega$. By Theorem 4.1.1 we can find a Cartan subalgebra $B \subset N$, some $t > 0$, and a unitary $u \in N \bar{\otimes} R$ such that $\theta(L^\infty(X)^t) = uBu^*$, where we identify $M \bar{\otimes} R = M^t \bar{\otimes} R^{1/t}$. Moreover, if we put $\mathcal{R} = \mathcal{R}(B \subset N)$, then $\mathcal{R} \cong \mathcal{R}(\mathbb{F}_n \curvearrowright X)^t$. Since $\mathcal{R}(\mathbb{F}_n \curvearrowright X)^t$ is a treeable equivalence relation, $\mathcal{R}$ is also treeable. Since $\mathcal{R}$ is treeable, we have that $\mathrm{H}^2(\mathcal{R}, \mathbb{T}) = 0$ (see e.g. [Kid17a, Corollary 2.4]). Therefore, [FM77] implies that $N \cong L(\mathcal{R})$ and thus $N \cong L(\mathcal{R}(\mathbb{F}_n \curvearrowright X)^t) \cong M^t$, as desired. $\square$

Corollary 1.1.13. *Let $\mathcal{R}_1$ and $\mathcal{R}_2$ be countable ergodic p.m.p. equivalence relations on probability spaces (X_1, μ_1) and (X_2, μ_2), respectively. Assume that $L(\mathcal{R}_1)' \cap L(\mathcal{R}_1)^\omega \subset L^\infty(X_1)^\omega$ and that $L(\mathcal{R}_2)$ is not McDuff. Suppose that $\mathcal{R}_1 \times \mathcal{T}$ is isomorphic to $\mathcal{R}_2 \times \mathcal{T}$, where $\mathcal{T}$ is a hyperfinite ergodic p.m.p. equivalence relation on a probability space (Y, ν). Then $\mathcal{R}_2$ is isomorphic to $\mathcal{R}_1^t$, for some $t > 0$.*

Proof. For $i \in \{1, 2\}$, put $A_i = L^\infty(X_i)$ and $M_i = L(\mathcal{R}_i)$. Denote $B = L^\infty(Y)$ and

$R = L(\mathcal{T})$. Since $\mathcal{T}$ is hyperfinite, ergodic and p.m.p., R is a hyperfinite II_1 factor. Since $\mathcal{R}_1 \times \mathcal{T} \cong \mathcal{R}_2 \times \mathcal{T}$, we have an isomorphism $\theta : M_1 \bar{\otimes} R \to M_2 \bar{\otimes} R$ such that $\theta(A_1 \bar{\otimes} B) = A_2 \bar{\otimes} B$. Since $M_1' \cap M_1^\omega \subset A_1^\omega$ and M_2 is not McDuff, by applying Theorem 1.1.10 we deduce the existence of a unitary $u \in M_2 \bar{\otimes} R$, a Cartan subagebra $\tilde{A}_2$ of M_2, and some $t > 0$, such that $\theta(A_1^t) = u\tilde{A}_2 u^*$, where we identify the inclusions $(A_1 \bar{\otimes} B \subset M_1 \bar{\otimes} R)$ and $(A_1^t \bar{\otimes} B^{1/t} \subset M_1^t \bar{\otimes} R^{1/t})$.

We claim that $\tilde{A}_2 \prec_{M_2} A_2$. Assume by contradiction that $\tilde{A}_2 \nprec_{M_2} A_2$. Then we can find a sequence $v_n \in \mathcal{U}(\tilde{A}_2)$ such that $\|E_{A_2}(xv_n y)\|_2 \to 0$, for all $x, y \in M_2$. Then $\|E_{A_2 \bar{\otimes} B}(zv_n t)\|_2 \to 0$, for all $z, t \in M_2 \bar{\otimes} R$. Indeed, it is enough to check this in the case when $z = z_1 \otimes z_2$ and $t = t_1 \otimes t_2$, for some $z_1, t_1 \in M_2$, $z_2, t_2 \in R$. In this case, we have that $E_{A_2 \bar{\otimes} B}(zv_n t) = E_{A_2}(z_1 v_n t_1) \otimes z_2 t_2$, and hence $\|E_{A_2 \bar{\otimes} B}(zv_n t)\|_2 = \|E_{A_2}(z_1 v_n t_1)\|_2 \|z_2 t_2\|_2 \to 0$. Since $\theta(A_1^t \bar{\otimes} B^{1/t}) = A_2 \bar{\otimes} B$, we get that $\|E_{A_1^t \bar{\otimes} B^{1/t}}(\theta^{-1}(uv_n u^*))\|_2 = \|E_{A_2 \bar{\otimes} B}(uv_n u^*)\|_2 \to 0$. This however contradicts the fact that $\theta^{-1}(uv_n u^*) \in A_1^t \subset A_1^t \bar{\otimes} B^{1/t}$, thus proving the claim.

Since $\tilde{A}_2$ and A_2 are Cartan subalgebras of M_2, by using the above claim and [Pop06a, Theorem A.1], we get that $\tilde{A}_2 = vA_2 v^*$, for some $v \in \mathcal{U}(M_2)$. Thus, $\theta(A_1^t) = \mathrm{Ad}(u)(\tilde{A}_2) = \mathrm{Ad}(uv)(A_2)$. Finally, Lemma 2.1.19 gives that

$$\mathcal{R}_1^t \cong \mathcal{R}(A_1^t \subset M_1^t \bar{\otimes} R^{1/t}) \cong \mathcal{R}(\theta(A_1^t) \subset M_2 \bar{\otimes} R) \cong \mathcal{R}(A_2 \subset M_2 \bar{\otimes} R) \cong \mathcal{R}_2,$$

which proves the conclusion. $\square$

4.2 A cocycle rigidity result

This section is devoted to the proof of the following cocycle rigidity result. This result, which seems to be of independent interest, will be used later on to derive Theorems 1.1.15 and 1.1.22.

Theorem 4.2.1. *Let $\mathcal{S}$ and $\mathcal{T}$ be countable ergodic p.m.p. equivalence relations on probability spaces (X, μ) and (Y, ν). Let $\pi : (X, \mu) \to (Y, \nu)$ be a factor map, consider the embedding $L^\infty(Y) \subset L^\infty(X)$ given by $f \mapsto f \circ \pi$, and denote $\mathcal{S}_0 = \{(x_1, x_2) \in \mathcal{S} \mid \pi(x_1) = \pi(x_2)\}$. Assume that*

1. *$[\pi(x)]_{\mathcal{T}} \subset \pi([x]_{\mathcal{S}})$, for almost every $x \in X$,*
2. *$L^\infty(X)^{[\mathcal{S}_0]} \subset L^\infty(Y)$, and*
3. *$(L^\infty(X)^\omega)^{[\mathcal{S}]} \subset L^\infty(Y)^\omega$.*

Let $\alpha : X \to Aut(R)$ be a measurable map satisfying $\alpha(x)^{-1}\alpha(y) \in Inn(R)$, for almost every $(x, y) \in \mathcal{S}$, where R denotes the hyperfinite II_1 factor.

Then we can find measurable maps $\tilde{\alpha} : Y \to Aut(R)$ and $U : X \to \mathcal{U}(R)$ such that we have $\alpha(x) = \tilde{\alpha}(\pi(x))Ad(U(x))$, for almost every $x \in X$.

Proof. In order to prove the conclusion, we may assume that $\alpha(x)^{-1}\alpha(y) \in \mathrm{Inn}(R)$, for all $(x, y) \in \mathcal{S}$. Consider the disintegration $\mu = \int_Y \mu_y \mathrm{d}\nu(y)$, where μ_y is a probability measure on X supported on $\pi^{-1}(\{y\})$, for all $y \in Y$. Put $\tilde{X} = \{(x_1, x_2) \in X \times X \mid \pi(x_1) = \pi(x_2)\}$

and endow $\tilde{X}$ with the probability measure $\tilde{\mu} = \int_Y (\mu_y \times \mu_y)\, \mathrm{d}\nu(y)$. Let $A = \{(x_1, x_2) \in \tilde{X} \mid \alpha(x_1)^{-1}\alpha(x_2) \in \mathrm{Inn}(R)\}$.

The first part of the proof consists of using condition (3) to establish the following:

Claim. $\tilde{\mu}(A) > 0$.

Proof of the claim. Denote by $L(\mathcal{S}) \subset \mathbb{B}(L^2(\mathcal{S}))$ the von Neumann algebra associated to $\mathcal{S}$ and by $\{u_\gamma\}_{\gamma \in [\mathcal{S}]} \subset L(\mathcal{S})$ the canonical unitaries [FM77]. Condition (3) gives that $L^\infty(X)^\omega \cap L(\mathcal{S})' \subset L^\infty(Y)^\omega$. By [Mar18, Proposition 5.2] we deduce that

$$(L^\infty(X)\bar{\otimes}R)^\omega \cap (L(\mathcal{S}) \otimes 1)' \subset (L^\infty(Y)\bar{\otimes}R)^\omega. \tag{4.7}$$

Next, we identify $L^\infty(X)\bar{\otimes}R = L^\infty(X, R)$, and define θ to be the automorphism of $L^\infty(X)\bar{\otimes}R$ given by $\theta(T)(x) = \alpha(x)(T(x))$, for every bounded measurable function $T : X \to R$ and all $x \in X$.

Fix $\gamma \in [\mathcal{S}]$, and define $w(\gamma, x) = \alpha(x)^{-1}\alpha(\gamma^{-1}x) \in \mathrm{Inn}(R)$, for $x \in X$. If $T \in R$, then we have $\mathrm{Ad}(u_\gamma \otimes 1)(\theta(1 \otimes T))(x) = \alpha(\gamma^{-1}x)(T) = \alpha(x)(w(\gamma, x)(T))$. Thus, if $(T_n)_n \in R' \cap R^\omega$, then as $w(\gamma, x) \in \mathrm{Inn}(R)$ we get that $\|\mathrm{Ad}(u_\gamma \otimes 1)(\theta(1 \otimes T_n))(x) - \theta(1 \otimes T_n)(x)\|_2 = \|w(\gamma, x)(T_n) - T_n\|_2 \to 0$, for almost every $x \in X$. Thus, $\|\mathrm{Ad}(u_\gamma \otimes 1)(\theta(1 \otimes T_n)) - \theta(1 \otimes T_n)\|_2 \to 0$, by the dominated convergence theorem. As this holds for all $\gamma \in [\mathcal{S}]$, we get that

$$\theta(1 \otimes R)' \cap \theta(1 \otimes R)^\omega \subset (L(\mathcal{S}) \otimes 1)' \cap (L^\infty(X)\bar{\otimes}R)^\omega.$$

By combining this with (4.7), we get that $\theta(1 \otimes R)' \cap \theta(1 \otimes R)^\omega \subset (L^\infty(Y)\bar{\otimes}R)^\omega$. Lemma 2.1.16 then gives that $\theta(1 \otimes R) \prec_{L^\infty(X)\bar{\otimes}R} L^\infty(Y)\bar{\otimes}R$. Thus, we can find a projection

$p \in R$, a $*$-homomorphism $\rho : pRp \to L^\infty(Y)\bar{\otimes}R$, and a non-zero partial isometry $v \in \rho(p)(L^\infty(X)\bar{\otimes}R)\theta(1 \otimes p)$ such that

$$\rho(T)v = v\theta(1 \otimes T), \quad \text{for all} \quad T \in pRp. \tag{4.8}$$

If we view $\rho(T)$ as a function $\rho(T) : Y \to R$, then (4.8) rewrites as

$$\rho(T)(\pi(x))v(x) = v(x)\alpha(x)(T), \quad \text{for all } T \in pRp \text{ and almost every } x \in X. \tag{4.9}$$

Since $v(x)v(x)^* \leq \rho(p)(\pi(x))$, for almost every $x \in X$, (4.9) gives that for all $T \in \mathcal{U}(pRp)$, we have

$$\alpha(x_1)(T)v(x_1)^*v(x_2) = v(x_1)^*v(x_2)\alpha(x_2)(T), \quad \text{for almost every } (x_1, x_2) \in \tilde{X}. \tag{4.10}$$

Now, note first that if $\alpha_1, \alpha_2 \in \mathrm{Aut}(R)$ satisfy $\alpha_1(T)w = w\alpha_2(T)$ for all $T \in pRp$, and for some non-zero partial isometry $w \in \alpha_1(p)R\alpha_2(p)$, then $\alpha_1^{-1}\alpha_2 \in \mathrm{Inn}(R)$. Second, note that

$$\begin{aligned}
\int_{\tilde{X}} \|v(x_1)^*v(x_2)\|_2^2 \, \mathrm{d}\tilde{\mu}(x_1, x_2) &= \int_Y \Big(\int_{X\times X} \tau(v(x_1)v(x_1)^*v(x_2)v(x_2)^*) \\
&\qquad\qquad \mathrm{d}(\mu_y \times \mu_y)(x_1, x_2) \Big) \, \mathrm{d}\nu(y) \\
&= \int_Y \tau(E_{L^\infty(Y)\bar{\otimes}R}(vv^*)(y)^2) \, \mathrm{d}\nu(y) \\
&= \|E_{L^\infty(Y)\bar{\otimes}R}(vv^*)\|_2^2 > 0.
\end{aligned}$$

By combining the last two facts and (4.10), the claim follows. $\diamond$

In the second part of the proof, we will use the ergodicity assumptions on $\mathcal{S}_0$ and $\mathcal{T}$ together with the claim to deduce the conclusion. First, we have that μ_y is $\mathcal{S}_0$-invariant, while condition (2) implies that μ_y is $\mathcal{S}_0$-ergodic, for almost every $y \in Y$. If $\gamma \in [\mathcal{S}_0]$ and $(x_1, x_2) \in A$, then since $\pi(\gamma x_1) = \pi(x_1)$ and $\alpha(\gamma x_1)^{-1}\alpha(x_1) \in \mathrm{Inn}(R)$, we derive that $(\gamma x_1, x_2) \in A$. Thus, for all $\gamma \in [\mathcal{S}_0]$, the set $A^{x_2} = \{x_1 \in \pi^{-1}(\{\pi(x_2)\}) \mid (x_1, x_2) \in A\}$ satisfies $\mu_{\pi(x_2)}(\gamma A^{x_2} \Delta A^{x_2}) = 0$, for almost every $x_2 \in X$. Since $\mu_{\pi(x_2)}$ is $\mathcal{S}_0$-ergodic, we get that $\mu_{\pi(x_2)}(A^{x_2}) \in \{0, 1\}$, for almost every $x_2 \in X$. Similarly, we get that $\mu_{\pi(x_1)}(A_{x_1}) \in \{0, 1\}$, for almost every $x_1 \in X$, where $A_{x_1} = \{x_2 \in \pi^{-1}(\{\pi(x_1)\}) \mid (x_1, x_2) \in A\}$. Since $\tilde{\mu}(A) > 0$, the last two facts imply the existence of a measurable set $B \subset Y$ such that $\nu(B) > 0$ and $A = \{(x_1, x_2) \in \tilde{X} \mid \pi(x_1) = \pi(x_2) \in B\}$.

We claim that B is $\mathcal{T}$-invariant. Let $\gamma \in [\mathcal{T}]$. Since $\gamma y \in [y]_{\mathcal{T}}$, by condition (1), for almost every $y \in Y$, for $(\mu_{\gamma y} \times \mu_{\gamma y})$-almost every $(x_1', x_2') \in \pi^{-1}(\{\gamma y\}) \times \pi^{-1}(\{\gamma y\})$, we can find $x_1 \in [x_1']_{\mathcal{S}}$ and $x_2 \in [x_2']_{\mathcal{S}}$ such that $\pi(x_1) = \pi(x_2) = y$. Since $(x_i, x_i') \in \mathcal{S}$, $\alpha(x_i)^{-1}\alpha(x_i') \in \mathrm{Inn}(R)$, for $i \in \{1, 2\}$. Additionally, for almost every $y \in B$, we have that $\alpha(x_1)^{-1}\alpha(x_2) \in \mathrm{Inn}(R)$, for $(\mu_y \times \mu_y)$-almost every $(x_1, x_2) \in \pi^{-1}(\{y\}) \times \pi^{-1}(\{y\})$. Combining these facts gives that for almost every $y \in B$, we have $\alpha(x_1')^{-1}\alpha(x_2') \in \mathrm{Inn}(R)$, for $(\mu_{\gamma y} \times \mu_{\gamma y})$-almost every $(x_1', x_2') \in \pi^{-1}(\{\gamma y\}) \times \pi^{-1}(\{\gamma y\})$. This shows that $\nu(\gamma B \Delta B) = 0$. Since $\gamma \in [\mathcal{T}]$ is arbitrary, we derive that B is indeed $\mathcal{T}$-invariant.

Since $\mathcal{T}$ is ergodic, we conclude that $B = Y$ and thus $A = \tilde{X}$, almost everywhere. In other words, $\alpha(x_1)^{-1}\alpha(x_2) \in \mathrm{Inn}(R)$, for almost every $(x_1, x_2) \in \tilde{X}$. Let $C = \{y \in Y \mid \mu_y$ is a non-atomic measure$\}$. Denote by Leb the Lebesgue measure on $[0, 1]$. Then we can find a measure space isomorphism $\sigma : (C \times [0, 1], \nu_{|C} \times \mathrm{Leb}) \to (\pi^{-1}(C), \mu_{|\pi^{-1}(C)})$ satisfying $\sigma(\{y\} \times [0, 1]) = \pi^{-1}(\{y\})$, for all $y \in C$. Since $\alpha(\sigma(y, t_1))^{-1}\alpha(\sigma(y, t_2)) \in \mathrm{Inn}(R)$, for

$(\nu \times \mathrm{Leb} \times \mathrm{Leb})$-almost every $(y, t_1, t_2) \in C \times [0,1] \times [0,1]$, Fubini's theorem implies the existence of $t_1 \in [0,1]$ such that $\alpha(\sigma(y,t_1))^{-1}\alpha(\sigma(y,t_2)) \in \mathrm{Inn}(R)$, for $(\mu \times \mathrm{Leb})$-almost every $(y, t_2) \in C \times [0,1]$. Since μ_y has atoms for every $y \in Y \setminus C$, we can find a measurable map $\eta : Y \setminus C \to X$ such that $\eta(y) \in \pi^{-1}(\{y\})$ and $\mu_y(\{\eta(y)\}) > 0$, for all $y \in Y \setminus C$. It is now easy to see that if we define $\tilde{\alpha} : Y \to \mathrm{Aut}(R)$ by

$$\tilde{\alpha}(y) = \begin{cases} \alpha(\sigma(y,t_1)) & \text{if } y \in C, \text{ and} \\ \\ \alpha(\eta(y)) & \text{if } y \in Y \setminus C \end{cases}$$

then $\beta(x) := \tilde{\alpha}(\pi(x))^{-1}\alpha(x) \in \mathrm{Inn}(R)$, for almost every $x \in X$.

Finally, let $\mathcal{U} \subset \mathcal{U}(R)$ be a countable $\|.\|_2$-dense set. Then an automorphism θ of R is inner if and only if $\inf_{u \in \mathcal{U}} \left(\sup_{x \in \mathcal{U}} \|\theta(x) - uxu^*\|_2 \right) = 0$. This implies that $\mathrm{Inn}(R)$ is a Borel subset of $\mathrm{Aut}(R)$. Endow $\mathrm{Inn}(R)$ with the Borel measure $\beta_*\mu$ and consider the Borel map $\zeta : \mathcal{U}(R) \to \mathrm{Inn}(R)$ given by $\zeta(u) = \mathrm{Ad}(u)$. Since ζ is onto, by applying [Tak01, Theorem A.16] we can find a $\beta_*\mu$-measurable map $\xi : \mathrm{Inn}(R) \to \mathcal{U}(R)$ such that $\zeta(\xi(\theta)) = \theta$, for all $\theta \in \mathrm{Inn}(R)$. After modifying ξ on a $\beta_*\mu$-null set, we can find a Borel map $\xi' : \mathrm{Inn}(R) \to \mathcal{U}(R)$ such that $\zeta(\xi'(\theta)) = \theta$, for $\beta_*\mu$-almost every $\theta \in \mathrm{Inn}(R)$. Then the map $U : X \to \mathcal{U}(R)$ given by $U(x) = \xi'(\beta(x))$ is measurable and satisfies $\tilde{\alpha}(\pi(x))\mathrm{Ad}(U(x)) = \tilde{\alpha}(\pi(x))\beta(x) = \alpha(x)$, for almost every $x \in X$, which proves the theorem. $\square$

4.3 The Jones-Schmidt property

The main goal of this section is to prove Theorem 1.1.15. We start by analyzing closer the structure of isomorphisms which satisfy the conclusion of Theorem 1.1.10.

Proposition 4.3.1. *Let M, N, P be II_1 factors. Assume that $A \subset M$ and $B \subset N$ are Cartan subalgebras. Denote $\mathcal{S} = \mathcal{R}(A \subset M)$, and identify $A = L^\infty(X)$ and $B = L^\infty(Z)$, for some probability spaces (X, μ) and (Z, η). For $g \in [\mathcal{S}]$, define $\sigma_g(a) = a \circ g^{-1}$, for every $a \in A$, and let $u_g \in \mathcal{N}_M(A)$ be such that $u_g a u_g^* = \sigma_g(a)$, for every $a \in A$.*

Let $\theta : M\bar{\otimes}P \to N\bar{\otimes}P$ be an isomorphism such that $\theta(A) = B$. Let $\alpha : (X, \mu) \to (Z, \eta)$ be the measure space isomorphism given by $\theta(a) = a \circ \alpha^{-1}$, for every $a \in A$.

Then $\theta(A\bar{\otimes}P) = B\bar{\otimes}P$, and we can find $w_g \in \mathcal{U}(A\bar{\otimes}P)$ and $v_g \in \mathcal{N}_N(B)$ for every $g \in [\mathcal{S}]$, and a measurable map $\varphi : X \to Aut(P)$ such that the following conditions hold:

1. *$\theta(w_g u_g) = v_g$ and $\theta(w_g(\sigma_g \otimes id_P)(w_h)w_{gh}^*) = v_g v_h v_{gh}^* \in B$, for all $g, h \in [\mathcal{S}]$.*
2. *$\theta^{-1}(T)(x) = \varphi_x(T(\alpha(x)))$, for every $T \in B\bar{\otimes}P$ and almost every $x \in X$, where we identify $A\bar{\otimes}P = L^\infty(X, P)$ and $B\bar{\otimes}P = L^\infty(Z, P)$.*
3. *$Ad(w_g(x)) = \varphi_x \varphi_{g^{-1}x}^{-1}$, for all $g \in [\mathcal{S}]$ and almost every $x \in X$.*

Proof. Since $\theta(A) = B$, we get that $\theta(A\bar{\otimes}P) = \theta(A)' \cap (N\bar{\otimes}P) = B' \cap (N\bar{\otimes}P) = B\bar{\otimes}P$. If $g \in [\mathcal{S}]$, then $\theta(u_g) \in \mathcal{N}_{N\bar{\otimes}P}(B)$, hence by Lemma 2.1.19 we can find $z_g \in \mathcal{U}(B\bar{\otimes}P)$ and $v_g \in \mathcal{N}_N(B)$ such that $\theta(u_g) = z_g v_g$. Then $w_g := \theta^{-1}(z_g^*) \in \mathcal{U}(A\bar{\otimes}P)$, and we have that $\theta(w_g u_g) = v_g$. Moreover, we get that $v_g v_h v_{gh}^* = \theta(w_g u_g w_h u_h u_h^* u_g^* w_{gh}^*) = \theta(w_g(\sigma_g \otimes \mathrm{id}_P)(w_h)w_{gh}^*) \in N \cap (B\bar{\otimes}P) = B$, proving (1).

Since $\theta^{-1}_{|(B\bar{\otimes}P)} : B\bar{\otimes}P \to A\bar{\otimes}P$ is a trace preserving isomorphism such that $\theta^{-1}(b) = b \circ \alpha$, for every $b \in B$, [Tak01, Theorem IV.8.23] provides the existence of a measurable map $\varphi : X \to \mathrm{Aut}(P)$ satisfying (2).

Let $T \in P \subset N\bar{\otimes}P$ and $g \in [\mathcal{S}]$. Since $v_g \in N$, we have that T commutes with v_g and therefore $\theta^{-1}(T) = \theta^{-1}(v_g T v_g^*) = w_g u_g \theta^{-1}(T) u_g^* w_g^*$. On the other hand, by (2) we get $\theta^{-1}(T)(x) = \varphi_x(T)$, for almost every $x \in X$. Combining these two facts implies that $\varphi_x(T) = w_g(x)\varphi_{g^{-1}x}(T)w_g(x)^*$, for almost every $x \in X$. Since this holds for all $T \in P$ and $g \in [\mathcal{S}]$, it follows that (3) holds. □

We continue with the following vanishing cohomology result.

Lemma 4.3.2. *Let $\mathcal{T}$ be a hyperfinite p.m.p. equivalence relation on a probability space (X, μ), and R be the hyperfinite II$_1$ factor. Let $c_n : \mathcal{T} \to \mathcal{U}(R)$ be a measurable cocycle, for every $n \geq 1$. Assume that $\|Ad(c_n(x,y))(v) - v\|_2 \to 0$, as $n \to \infty$, for every $v \in R$, and almost every $(x,y) \in \mathcal{T}$.*

Then we can find a subsequence $\{n_k\}_{k\geq 1}$ of $\mathbb{N}$ and measurable maps $d_k : X \to \mathcal{U}(R)$, for all $k \geq 1$, such that $\|Ad(d_k(x))(v) - v\|_2 \to 0$ and $\|c_{n_k}(x,y) - d_k(x)d_k(y)^{-1}\|_2 \to 0$, as $k \to \infty$, for every $v \in R$, and almost every $(x,y) \in \mathcal{T}$.

Proof. Let $\{\mathcal{T}_k\}_{k\geq 1} \subset \mathcal{T}$ be an increasing sequence of finite subequivalence relations such that we have $\mathcal{T} = \cup_{k\geq 1}\mathcal{T}_k$. Moreover, we may assume that $\sup_{x\in X} \#[x]_{\mathcal{T}_k} < \infty$, for all

$k \in \mathbb{N}$. For $k \geq 1$, let $\psi_k : X \to X$ be a measurable map such that $\psi_k(x) \in [x]_{\mathcal{T}_k}$ and $\psi_k(x) = \psi_k(y)$, for every $(x, y) \in \mathcal{T}_k$. Write $R = \bar{\otimes}_{n\geq 1}\mathbb{M}_2(\mathbb{C})$ and define $R_k = \bar{\otimes}_{n\geq k}\mathbb{M}_2(\mathbb{C})$, for all $k \geq 1$. For $n, k \geq 1$, define

$$X_{n,k} = \{x \in X \mid \|c_n(x, y) - E_{R_k}(c_n(x, y))\|_2 \leq 1/k, \text{ for all } y \in [x]_{\mathcal{T}_k}\}.$$

Then for every $k \geq 1$ we have $\mu(X_{n,k}) \to 1$, as $n \to \infty$. Thus, for all $k \geq 1$, we can find $n_k \geq 1$, such that $\mu(X_{n_k,k}) \geq 1 - 1/2^k$. For $k \geq 1$, we define $d_k : X \to \mathcal{U}(R)$ by letting $d_k(x) = c_{n_k}(x, \psi_k(x))$. Then for all $k \geq 1$ and $x \in X_{n_k,k}$ we have $\|d_k(x) - E_{R_k}(d_k(x))\|_2 \leq 1/k$. The Borel-Cantelli lemma implies that $\|d_k(x) - E_{R_k}(d_k(x))\|_2 \to 0$, for almost every $x \in X$. Moreover, for all $k \geq 1$ and $(x, y) \in \mathcal{T}_k$ we have $c_{n_k}(x, y) = d_k(x)d_k(y)^{-1}$. Since $\mathcal{T} = \cup_{k\geq 1}\mathcal{T}_k$, the conclusion follows. $\square$

We next show that the Jones-Schmidt property is equivalent with a formally stronger property.

Proposition 4.3.3. *Let $\mathcal{S}$ be a countable ergodic p.m.p. equivalence relation on a probability space (X, μ) with the Jones-Schmidt property. Then there exist a hyperfinite ergodic p.m.p. equivalence relation $\widetilde{\mathcal{T}}$ on a probability space $(\widetilde{Y}, \tilde{\nu})$ and a factor map $\widetilde{\pi} : (X, \mu) \to (\widetilde{Y}, \tilde{\nu})$ such that*

(a) $\widetilde{\pi}([x]_{\mathcal{S}}) = [\widetilde{\pi}(x)]_{\widetilde{\mathcal{T}}}$, for almost every $x \in X$, and

(b) $(L^\infty(X)^\omega)^{[\mathcal{S}_0]} = L^\infty(\widetilde{Y})^\omega$, where $\mathcal{S}_0 = \{(x_1, x_2) \in \mathcal{S} \mid \widetilde{\pi}(x_1) = \widetilde{\pi}(x_2)\}$ and we embed $L^\infty(\widetilde{Y}) \subset L^\infty(X)$ by $f \mapsto f \circ \widetilde{\pi}$.

Proof. Since $\mathcal{S}$ has the Jones-Schmidt property, we can find a hyperfinite ergodic p.m.p. equivalence relation $\mathcal{T}$ on a probability space (Y, ν) and a factor map $\pi : (X, \mu) \to (Y, \nu)$ such that we have $\pi([x]_{\mathcal{S}}) = [\pi(x)]_{\mathcal{T}}$, for almost every $x \in X$, and $\mathcal{S}_0 = \{(x_1, x_2) \in \mathcal{S} \mid \pi(x_1) = \pi(x_2)\}$ is strongly ergodic on almost every ergodic component of $\mathcal{S}_0$.

We may assume that $\pi([x]_{\mathcal{S}}) = [\pi(x)]_{\mathcal{T}}$, for all $x \in X$. We identify $L^\infty(X)^{[\mathcal{S}_0]} = L^\infty(\widetilde{Y})$, for a probability space $(\widetilde{Y}, \tilde{\nu})$. Let $\widetilde{\pi} : (X, \mu) \to (\widetilde{Y}, \tilde{\nu})$ and $\rho : (\widetilde{Y}, \tilde{\nu}) \to (Y, \nu)$ be the factor maps given by the embeddings $L^\infty(\widetilde{Y}) \subset L^\infty(X)$ and $L^\infty(Y) \subset L^\infty(\widetilde{Y})$. Then $\rho \circ \widetilde{\pi} = \pi$ and $\widetilde{\pi}(x_1) = \widetilde{\pi}(x_2)$, for every $(x_1, x_2) \in \mathcal{S}_0$. Therefore, $\mathcal{S}_0 = \{(x_1, x_2) \in \mathcal{S} |\ \widetilde{\pi}(x_1) = \widetilde{\pi}(x_2)\}$.

Consider the disintegration $\mu = \int_{\widetilde{Y}} \mu_y \, \mathrm{d}\tilde{\nu}(y)$ of μ with respect to $\widetilde{\pi}$, where μ_y is a probability measure on X supported on $\widetilde{\pi}^{-1}(\{y\})$, for every $y \in \widetilde{Y}$. We claim that (b) holds, i.e., we have

Claim 1. $(L^\infty(X)^\omega)^{[\mathcal{S}_0]} = L^\infty(\widetilde{Y})^\omega$.

Proof of Claim 1. In order to prove the claim, it suffices to show that if a sequence $f_n \in L^\infty(X)_1$ satisfies $\|\gamma f_n - f_n\|_2 \to 0$, for every $\gamma \in [\mathcal{S}_0]$, then $\|f_n - E_{L^\infty(\widetilde{Y})}(f_n)\|_2 \to 0$. Let $\{\gamma_m\}_{m \geq 1} \subset [\mathcal{S}_0]$ be a sequence which is dense with respect to the metric $d(\gamma, \gamma') = \mu(\{x \in X \mid \gamma(x) \neq \gamma'(x)\})$. If the assertion concerning (f_n) is false, then we can find a subsequence (g_k) of (f_n) such that $\|\gamma_m g_k - g_k\|_2 \leq 1/2^k$, for all $k \geq m$, and $\inf_k \|g_k - E_{L^\infty(\widetilde{Y})}(g_k)\|_2 > 0$. For $k \geq 1$ and $y \in \widetilde{Y}$, denote by $g_k^y \in L^\infty(\widetilde{\pi}^{-1}(\{y\}), \mu_y)_1$ the restriction of g_k to $\pi^{-1}(\{y\})$. Then for every $m \geq 1$, we have that

$$\int_{\widetilde{Y}} \sum_{k\geq 1} \|\gamma_m g_k^y - g_k^y\|_{2,\mu_y}^2 \, \mathrm{d}\tilde{\nu}(y) = \sum_{k\geq 1} \|\gamma_m g_k - g_k\|_2^2 < \infty.$$

Thus, for $\tilde{\nu}$-almost every $y \in \widetilde{Y}$, we have that $\|\gamma_m g_k^y - g_k^y\|_{2,\mu_y} \to 0$, for every $m \geq 1$. Therefore, for $\tilde{\nu}$-almost every $y \in \widetilde{Y}$, we have that $\|\gamma g_k^y - g_k^y\|_{2,\mu_y} \to 0$, for every $\gamma \in [\mathcal{S}_0]$. Since $\mathcal{S}_0$ is μ_y-strongly ergodic, we conclude that $\|g_k^y - \int g_k^y \, \mathrm{d}\mu_y\|_{2,\mu_y} \to 0$, for $\tilde{\nu}$-almost every $y \in \widetilde{Y}$. From this we deduce that $\|g_k - E_{L^\infty(\tilde{Y})}(g_k)\|_2^2 = \int_{\widetilde{Y}} \|g_k^y - \int g_k^y \, \mathrm{d}\mu_y\|_{2,\mu_y} \, \mathrm{d}\nu(y) \to 0$, which gives a contradiction. $\diamond$

We continue with the following claim:

Claim 2. Let $\alpha : A \to B$ be an element of $[[\mathcal{S}]]$ such that there exists $\beta, \gamma \in [[\mathcal{T}]]$ satisfying $\pi \circ \alpha = \beta \circ \pi$ and $\pi \circ \alpha^{-1} = \gamma \circ \pi$. Let $\widetilde{A} = \{y \in \widetilde{Y} | \mu_y(A) > 0\}$ and $\widetilde{B} = \{y \in \widetilde{Y} | \mu_y(B) > 0\}$. Then there exists a measure space isomorphism $\widetilde{\alpha} : (\widetilde{A}, \widetilde{\nu}_{|\widetilde{A}}) \to (\widetilde{B}, \widetilde{\nu}_{|\widetilde{B}})$ such that $\widetilde{\pi}(\alpha(x)) = \widetilde{\alpha}(\widetilde{\pi}(x))$, for almost every $x \in A$, and $\widetilde{\pi}(\alpha^{-1}(x)) = \widetilde{\alpha}^{-1}(\widetilde{\pi}(x))$, for almost every $x \in B$.

Proof of Claim 2. Note first that

$$\int_{\widetilde{Y}} \alpha_*(\mu_{y|A}) \, \mathrm{d}\tilde{\nu}(y) = \alpha_*(\mu_{|A}) = \mu_{|B} = \int_{\widetilde{Y}} \mu_{y|B} \, \mathrm{d}\tilde{\nu}(y). \tag{4.11}$$

Let $y \in \widetilde{A}$ and $x_1, x_2 \in \widetilde{\pi}^{-1}(\{y\}) \cap A$. Then $\widetilde{\pi}(x_1) = \widetilde{\pi}(x_2)$, hence $\pi(x_1) = \pi(x_2)$. Since $x_1, x_2 \in A$, we get that $\pi(\alpha(x_1)) = \pi(\alpha(x_2))$, thus $(\alpha(x_1), \alpha(x_2)) \in \mathcal{S}_0$, and hence $\widetilde{\pi}(\alpha(x_1)) = \widetilde{\pi}(\alpha(x_2))$. This proves that $\widetilde{\pi}$ is constant on $\alpha(\widetilde{\pi}^{-1}\{y\} \cap A)$. Thus, we can define a measurable map $\widetilde{\alpha} : \widetilde{A} \to \widetilde{Y}$ such that $\widetilde{\pi}(\alpha(x)) = \widetilde{\alpha}(y)$, for all $y \in \widetilde{A}$ and $x \in \widetilde{\pi}^{-1}\{y\} \cap A$. This implies that $\widetilde{\pi}(\alpha(x)) = \widetilde{\alpha}(\widetilde{\pi}(x))$, for almost every $x \in A$.

Since $\alpha_*(\mu_{y|A})$ is supported on $\alpha(\widetilde{\pi}^{-1}(\{y\}) \cap A) \subset \widetilde{\pi}^{-1}(\{\widetilde{\alpha}(y)\})$, by using (4.11) and the uniqueness of the disintegration of $\mu_{|B}$ we conclude that $\mu_{\widetilde{\alpha}(y)|B} = \alpha_*(\mu_{y|A})$ and $\widetilde{\alpha}(y) \in \widetilde{B}$, for almost every $y \in \widetilde{A}$, and that $\widetilde{\alpha} : \widetilde{A} \to \widetilde{B}$ is onto. Similarly, we can find an onto measurable map $\overline{\alpha} : \widetilde{B} \to \widetilde{A}$ such that $\widetilde{\pi}(\alpha^{-1}(x)) = \overline{\alpha}(\widetilde{\pi}(x))$, for almost every $x \in B$. It is easy to see that $\overline{\alpha} = \widetilde{\alpha}^{-1}$.

If $C \subset \widetilde{B}$ is a measurable set, then $\widetilde{\pi}^{-1}(\widetilde{\alpha}^{-1}(C)) = \{x \in A | \widetilde{\alpha}(\widetilde{\pi}(x)) \in C\} = \{x \in A | \widetilde{\pi}(\alpha(x)) \in C\}$. Since α is μ-preserving, we get that

$$\widetilde{\nu}(\widetilde{\alpha}^{-1}(C)) = \mu(\widetilde{\pi}^{-1}(\widetilde{\alpha}^{-1}(C))) = \mu(\{x \in A | \widetilde{\pi}(\alpha(x)) \in C\}) = \mu(\{x \in B | \widetilde{\pi}(x) \in C\}) = \widetilde{\nu}(C).$$

This shows that $\widetilde{\alpha}$ is $\widetilde{\nu}$-preserving, and finishes the proof of the claim. $\diamond$

Now, the first assumption on π implies we can find a sequence of elements $\alpha_n : A_n \to B_n, n \geq 1$, of $[[\mathcal{S}]]$ which satisfy the hypothesis of Claim 2 and such that $[x]_{\mathcal{S}} = \{\alpha_n(x)\}_{n \geq 1}$, for almost every $x \in X$. For $n \geq 1$, let $\widetilde{\alpha}_n : \widetilde{A}_n \to \widetilde{B}_n$ be the $\widetilde{\nu}$-preserving isomorphism obtained by applying Claim 2 to α_n.

Let $\widetilde{\mathcal{T}}$ be the countable p.m.p. equivalence relation of $(\widetilde{Y}, \widetilde{\nu})$ generated by $\{\widetilde{\alpha}_n\}_{n \geq 1}$. It is then immediate that condition (a) is satisfied. If $g \in L^\infty(\widetilde{Y})$ is $\widetilde{\mathcal{T}}$-invariant, then $g \circ \widetilde{\pi}$ is $\mathcal{S}$-invariant. Since $\mathcal{S}$ is ergodic, we get that $g \circ \widetilde{\pi} \in \mathbb{C}1$, hence $g \in \mathbb{C}1$, showing that $\widetilde{\mathcal{T}}$ is ergodic.

Finally, we show that $\widetilde{\mathcal{T}}$ is hyperfinite. Since $\mathcal{T}$ is hyperfinite, it suffices to show that the restriction of ρ to $[y]_{\widetilde{\mathcal{T}}}$ is injective and $\rho([y]_{\widetilde{\mathcal{T}}}) \subset [\rho(y)]_{\mathcal{T}}$, for all $y \in \widetilde{Y}$. To this end, let $y \in \widetilde{Y}$ and $y' \in [y]_{\widetilde{\mathcal{T}}}$. Then we can find $(x, x') \in \mathcal{S}$ such that $y = \widetilde{\pi}(x)$ and $y' = \widetilde{\pi}(y)$. Since $\rho(y) = \pi(x)$, $\rho(y') = \pi(x')$, and $\pi(x') \in [\pi(x)]_{\mathcal{T}}$, we get that $\rho(y') \in [\rho(y)]_{\mathcal{T}}$. If

$\rho(y') = \rho(y)$, then $\pi(x') = \pi(x)$, hence $(x, x') \in \mathcal{S}_0$ thus $y = \widetilde{\pi}(x) = \widetilde{\pi}(x') = y'$, showing that $\rho_{|[y]_{\widetilde{\mathcal{T}}}}$ is injective. This finishes the proof. $\square$

We are now ready to prove the next main theorem from the introduction.

Theorem 1.1.15. *Let N be a II_1 factor which admits a Cartan subalgebra A such that $N' \cap N^\omega \subset A^\omega$ and $\mathcal{R}(A \subset N)$ has the Jones-Schmidt property. Let P be any II_1 factor such that $N\bar{\otimes}R$ and $P\bar{\otimes}R$ are isomorphic.*

Then P is either isomorphic to N^t, for some $t > 0$, or to $N\bar{\otimes}R$.

Moreover, assume that P is not McDuff. Then for any isomorphism $\theta : N\bar{\otimes}R \to P\bar{\otimes}R$, we can find isomorphisms $\theta_1 : N^s \to P$, $\theta_2 : R^{1/s} \to R$, for some $s > 0$, and an approximately inner automorphism $\Psi : N\bar{\otimes}R \to N\bar{\otimes}R$ such that $\theta = (\theta_1 \otimes \theta_2) \circ \Psi$, where we identify $N\bar{\otimes}R = N^s\bar{\otimes}R^{1/s}$.

Proof. Since $N' \cap N^\omega \subset A^\omega$, Theorem 1.1.10 implies the existence of $u \in \mathcal{U}(P\bar{\otimes}R)$, $t > 0$, and a Cartan subalgebra $B \subset P^t$ such that $\theta(A) = uBu^*$, where we identify $P\bar{\otimes}R = P^t\bar{\otimes}R^{1/t}$. After replacing θ with $\mathrm{Ad}(u^*) \circ \theta$, we may assume that $\theta(A) = B$. We also identify $P^t\bar{\otimes}R^{1/t} = P^t\bar{\otimes}R$ using an isomorphism $R^{1/t} \mapsto R$. Thus, we see θ as an isomorphism $\theta : N\bar{\otimes}R \to P^t\bar{\otimes}R$ satisfying $\theta(A) = B$. Finally, we identify $A = L^\infty(X), B = L^\infty(Z)$, for some probability spaces $(X, \mu), (Z, \eta)$. Let $\alpha : (X, \mu) \to (Z, \eta)$ be the measure space isomorphism given by $\theta(a) = a \circ \alpha^{-1}$, for every $a \in A$.

Denote $\mathcal{S} = \mathcal{R}(A \subset N)$. For $g \in [\mathcal{S}]$, let $u_g \in \mathcal{N}_N(A)$ such that $u_g a u_g^* = \sigma_g(a) := a \circ g^{-1}$, for every $a \in A$. By Proposition 4.3.1, $\theta(A\bar{\otimes}R) = B\bar{\otimes}R$ and we can find

$w_g \in \mathcal{U}(A\bar{\otimes}R)$, for every $g \in [\mathcal{S}]$, and a measurable map $\varphi : X \to \mathrm{Aut}(R)$ such that the following conditions hold:

1. $\theta(w_g u_g) \in P^t$, for all $g \in [\mathcal{S}]$.

2. $\theta^{-1}(T)(x) = \varphi_x(T(\alpha(x))$, for every $T \in B\bar{\otimes}R$ and almost every $x \in X$.

3. $\mathrm{Ad}(w_g(x)) = \varphi_x\varphi_{g^{-1}x}^{-1}$, for all $g \in [\mathcal{S}]$ and almost every $x \in X$.

Since $\mathcal{S}$ has the Jones-Schmidt property, Proposition 4.3.3 provides a hyperfinite ergodic p.m.p. equivalence relation $\mathcal{T}$ on a probability space (Y, ν) and a factor map $\pi : (X, \mu) \to (Y, \nu)$ such that $\pi([x]_{\mathcal{S}}) = [\pi(x)]_{\mathcal{T}}$, for almost every $x \in X$, and if we let $\mathcal{S}_0 = \{(x_1, x_2) \in \mathcal{S} \mid \pi(x_1) = \pi(x_2)\}$, then $(L^\infty(X)^\omega)^{[\mathcal{S}_0]} = L^\infty(Y)^\omega$. In particular, $L^\infty(X)^{[\mathcal{S}_0]} \subset L^\infty(Y)$ and $(L^\infty(X)^\omega)^{[\mathcal{S}]} \subset L^\infty(Y)^\omega$.

Note that (3) implies that $\varphi_x\varphi_y^{-1} \in \mathrm{Inn}(R)$, for almost every $(x, y) \in \mathcal{S}$. By applying Theorem 4.2.1, we get measurable maps $\psi : Y \to \mathrm{Aut}(R)$ and $U : X \to \mathcal{U}(R)$ such that

$$\varphi_x = \mathrm{Ad}(U(x))\psi_{\pi(x)}, \quad \text{for almost every } x \in X. \tag{4.12}$$

View the function $U : X \to \mathcal{U}(R)$ as an element of $\mathcal{U}(A\bar{\otimes}R)$. Let $\Theta : N\bar{\otimes}R \to P^t\bar{\otimes}R$ be the isomorphism given by $\Theta = \theta \circ \mathrm{Ad}(U)$. For $g \in [\mathcal{S}]$, let $W_g = U^* w_g(\sigma_g \otimes \mathrm{id}_R)(U) \in \mathcal{U}(A\bar{\otimes}R)$. By using (4.12), the above conditions (1)-(3) rewrite as:

(a) $\Theta(W_g u_g) \in P^t$, for all $g \in [\mathcal{S}]$.

(b) $\Theta^{-1}(T)(x) = \psi_{\pi(x)}(T(\alpha(x))$, for every $T \in B\bar{\otimes}R$ and almost every $x \in X$.

(c) $\mathrm{Ad}(W_g(x)) = \psi_{\pi(x)}\psi^{-1}_{\pi(g^{-1}x)}$, for all $g \in [\mathcal{S}]$ and almost every $x \in X$.

By using condition (c) we get that $\psi_y\psi_z^{-1} \in \mathrm{Inn}(R)$, for almost every $(y,z) \in \mathcal{T}$. We claim that there is a cocycle $c : \mathcal{T} \to \mathcal{U}(R)$ such that

$$\psi_y\psi_z^{-1} = \mathrm{Ad}(c(y,z)), \quad \text{for almost every } (y,z) \in \mathcal{T}. \tag{4.13}$$

Since $\mathcal{T}$ is ergodic hyperfinite, we can find a free ergodic p.m.p. action $\mathbb{Z} \curvearrowright (Y,\nu)$ such that $\mathcal{T} = \mathcal{R}(\mathbb{Z} \curvearrowright Y)$. Let $T_0 : Y \to Y$ be the generator of the action $\mathbb{Z} \curvearrowright Y$. Since $T_0 \in [\mathcal{T}]$, by (c) we can find a measurable map $d : Y \to \mathcal{U}(R)$ such that $\psi_{T_0(y)}\psi_y^{-1} = \mathrm{Ad}(d(y))$, for almost every $y \in Y$. Then the claim holds for the unique cocycle $c : \mathcal{T} \to \mathcal{U}(R)$ satisfying $c(T_0(y), y) = d(y)$, for all $y \in Y$.

For $g \in [\mathcal{S}]$, we define $\widetilde{W}_g \in A\bar{\otimes}R$ by letting $\widetilde{W}_g(x) = c(\pi(x), \pi(g^{-1}x))$. By combining (c) and (4.13) we get that $\mathrm{Ad}(W_g(x)) = \mathrm{Ad}(\widetilde{W}_g(x))$, for almost every $x \in X$, and thus $\widetilde{W}_g W_g^* \in A$.

The rest of proof relies on the following claim:

Claim. There exists $\Psi \in \overline{\mathrm{Inn}(N\bar{\otimes}R)}$ such that $\Psi(A\bar{\otimes}R) = A\bar{\otimes}R$, $\Psi(T)(x) = \psi_{\pi(x)}(T(x))$, for all $T \in A\bar{\otimes}R$ and almost every $x \in X$, and $\Psi(u_g) = \widetilde{W}_g u_g$, for all $g \in [\mathcal{S}]$.

Proof of the claim. Write $R = \bar{\otimes}_{k\in\mathbb{N}}\mathbb{M}_2(\mathbb{C})$ and define $R^n = \bar{\otimes}_{k<n}\mathbb{M}_2(\mathbb{C})$, for $n \in \mathbb{N}$. Fix $n \in \mathbb{N}$ and consider the set $B_n := \{(y,u) \in Y \times \mathcal{U}(R) \mid \psi_{y|R^n} = \mathrm{Ad}(u)_{|R^n}\}$. As $\psi : Y \to \mathrm{Aut}(R)$ is measurable, we see that $B_n \subset Y \times \mathcal{U}(R)$ is a Borel subset. Moreover we have that $\pi_1(B_n) = Y$, where π_1 is the projection onto the first component. Applying [Tak01, Theorem A.16] we get a measurable map $u_n : Y \to \mathcal{U}(R)$ such that

$\psi_{y|R^n} = \mathrm{Ad}(u_n(y))_{|R^n}$, for all $y \in Y$. Thus, $\lim_{n\to\infty} \mathrm{Ad}(u_n(y))(v) = \psi_y(v)$, for all $y \in Y$ and $v \in R$.

Define a cocycle $c_n : \mathcal{T} \to \mathcal{U}(R)$ by $c_n(y,z) = u_n(y)^* c(y,z) u_n(z)$. Then we see that $\mathrm{Ad}(c_n(y,z))_{|R^n} = \mathrm{id}_{R^n}$, for all $(y,z) \in \mathcal{T}$. Since $\mathcal{T}$ is hyperfinite, by applying Lemma 4.3.2, we can find a subsequence $\{n_k\}_{k\geq 1}$ of $\mathbb{N}$ and measurable maps $d_k : Y \to \mathcal{U}(R)$, for $k \geq 1$, such that $\|c_{n_k}(y,z) - d_k(y)d_k(z)^{-1}\|_2 \to 0$ and $\|\mathrm{Ad}(d_k(y))(v) - v\|_2 \to 0$, as $k \to \infty$, for almost every $(y,z) \in \mathcal{T}$ and every $v \in R$.

For $k \geq 1$, define $U_k \in \mathcal{U}(A\bar{\otimes}R)$ by letting $U_k(x) = u_{n_k}(\pi(x))d_k(\pi(x))$. Then for every $g \in [\mathcal{S}]$ and almost every $x \in X$ we have that

$$(U_k(\sigma_g \otimes \mathrm{id}_R)(U_k^*))(x) = u_{n_k}(\pi(x))d_k(\pi(x))d_k(\pi(g^{-1}x))^* u_{n_k}(\pi(g^{-1}x))^* \to \widetilde{W}_g(x). \quad (4.14)$$

Thus, $\|U_k u_g U_k^* - \widetilde{W}_g u_g\|_2 \to 0$, for all $g \in [\mathcal{S}]$. If $T \in R$, then as $\|\mathrm{Ad}(d_k(\pi(x))(T) - T\|_2 \to 0$, for almost every $x \in X$, we get that $\lim_{k\to\infty} \mathrm{Ad}(U_k(x))(T) = \lim_{k\to\infty} \mathrm{Ad}(u_{n_k}(\pi(x)))(T) = \psi_{\pi(x)}(T)$. Also, $U_k T U_k^* = T$, for all $T \in A$. Since $\{u_g\}_{g\in[\mathcal{S}]} \cup R \cup A$ generates $N\bar{\otimes}R$ as a von Neumann algebra we conclude that the limit $\Psi = \lim_{k\to\infty} \mathrm{Ad}(U_k) \in \overline{\mathrm{Inn}(N\bar{\otimes}R)}$ exists and satisfies the claim. $\diamond$

Finally, let $g \in [\mathcal{S}]$. Since $\widetilde{W}_g W_g^* \in A$, we have $\Theta(\widetilde{W}_g W_g^*) = \theta(\widetilde{W}_g W_g^*) \in B$. Since $\Theta(W_g u_g) \in P^t$ by (a), we get that $(\Theta \circ \Psi)(u_g) = \Theta(\widetilde{W}_g u_g) = \Theta(\widetilde{W}_g W_g^*)\Theta(W_g u_g) \in P^t$. Since $(\Theta \circ \Psi)(A) = B$ and $\{u_g\}_{g\in[\mathcal{S}]} \cup A$ generates N as a von Neumann algebra, we deduce that $(\Theta \circ \Psi)(N) \subset P^t$. Let $T \in A\bar{\otimes}R$ and define $\widetilde{T} \in B\bar{\otimes}R$ by letting $\widetilde{T}(z) = T(\alpha^{-1}(z))$, for all $z \in Z$. Then by combining (b) and the claim we get that $\Theta^{-1}(\widetilde{T})(x) = \psi_{\pi(x)}(\widetilde{T}(\alpha(x)) = \psi_{\pi(x)}(T(x)) = \Psi(T)(x)$, for almost every $x \in X$. This shows that $(\Theta \circ \Psi)(T) = T \circ \alpha^{-1}$, for every $T \in A\bar{\otimes}R$, and in particular implies that $(\Theta\circ\Psi)(R) = R$. Since $\Theta\circ\Psi : N\bar{\otimes}R \to P^t\bar{\otimes}R$ is onto and $(\Theta\circ\Psi)(N) \subset P^t$, we derive that $(\Theta\circ\Psi)(N) = P^t$. Thus, we have that $\Theta\circ\Psi = \theta_1\otimes\theta_2$, where $\theta_1 : N \to P^t$ and $\theta_2 : R \to R$ are isomorphisms. This implies the conclusion of Theorem 1.1.15 for $s = 1/t$. $\square$

4.4 Equivalence relations without the Jones-Schmidt property

We begin with the following lemma, which will be key in the proofs of Theorems 1.1.17 and 1.1.18 below.

Lemma 4.4.1. *Let $\mathcal{S}$ be a countable ergodic p.m.p. equivalence relation on a probability space (X, μ) with the Jones-Schmidt property. Denote $M = L(\mathcal{S})$ and $A = L^\infty(X)$. Assume that $A_n \subset A$ are von Neumann subalgebras such that $A = \bar{\otimes}_{n\in\mathbb{N}}A_n$ and $M'\cap A^\omega \subset (\bar{\otimes}_{k\geq n}A_k)^\omega$, for every $n \in \mathbb{N}$.*

Then for every $\varepsilon > 0$, we can find finite dimensional subalgebras $\tilde{A}_n \subset A_n$, for $n \in \mathbb{N}$, such that

$$M' \cap A^\omega \subset_\varepsilon (\bar{\otimes}_{n\in\mathbb{N}}\tilde{A}_n)^\omega.$$

Proof. Let $(Y, \nu) = (Y_0, \nu_0)^{\mathbb{N}}$, where $Y_0 = \{0, 1\}$ and $\nu_0 = \frac{1}{2}(\delta_0 + \delta_1)$, and consider the equivalence relation $\mathcal{T}$ on (Y, ν) given by $(y_k)\mathcal{T}(z_k)$ if and only if $y_k = z_k$, for all but finitely many $k \in \mathbb{N}$. Since $\mathcal{S}$ has the Jones-Schmidt property, by applying Proposition 4.3.3 we can find a factor map $\pi : (X, \mu) \to (Y, \nu)$ such that (a) $\pi([x]_{\mathcal{S}}) = [\pi(x)]_{\mathcal{T}}$, for almost every $x \in X$, and (b) $(A^\omega)^{[\mathcal{S}_0]} = B^\omega$, where we embed $B = L^\infty(Y)$ into A via the map $f \mapsto f \circ \pi$ and denote $\mathcal{S}_0 = \{(x_1, x_2) \in \mathcal{S} \mid \pi(x_1) = \pi(x_2)\}$.

Note first that since $M' \cap A^\omega \subset (A^\omega)^{[\mathcal{S}]} \subset (A^\omega)^{[\mathcal{S}_0]}$, condition (b) implies that

$$M' \cap A^\omega \subset B^\omega. \tag{4.15}$$

Next, we identify $B = \bar{\otimes}_{k\in\mathbb{N}} L^\infty(Y_0, \nu_0)_k$ and put $B_n = \bar{\otimes}_{k\geq n} L^\infty(Y_0, \nu_0)_k$, for $n \in \mathbb{N}$. We claim that

$$\prod_\omega B_n \subset M' \cap A^\omega. \tag{4.16}$$

To see this, let $f \in \prod_\omega B_n$ and $\alpha \in [\mathcal{S}]$. If we view f as an element of A^ω, then we can represent it as $f = (f_n \circ \pi)$, where $f_n \in B_n$, for every $n \in \mathbb{N}$, and $\sup \|f_n\| < \infty$. By condition (a) above, we can find a measurable map $\beta : Y \to Y$ such that $\pi(\alpha^{-1}(x)) = \beta(\pi(x))$ and $\beta(y) \in [y]_\mathcal{T}$, for almost every $x \in X$ and $y \in Y$. Fix $n \in \mathbb{N}$, and denote $Y_n = \{(y_k) \in Y | \beta(y)_k = y_k, \text{ for all } k \geq n\}$. Then we have that $f_n(\beta(y)) = f_n(y)$, for all $y \in Y_n$. Since $u_\alpha(f_n \circ \pi)u_\alpha^* = f_n \circ \pi \circ \alpha^{-1} = f_n \circ \beta \circ \pi$, we get that

$$\|u_\alpha(f_n \circ \pi)u_\alpha^* - f_n \circ \pi\|_2 = \|f_n \circ \beta \circ \pi - f_n \circ \pi\|_2 = \|f_n \circ \beta - f_n\|_2 \leq 2\sqrt{\nu(Y \setminus Y_n)}\|f_n\|.$$

Since $\lim_{n\to\infty} \nu(Y_n) = 1$, we conclude that $\lim_{n\to\infty} \|u_\alpha(f_n \circ \pi)u_\alpha^* - f_n \circ \pi\|_2 = 0$, and hence $u_\alpha f u_\alpha^* = f$. Since this holds for every $f \in \prod_\omega B_n$ and $\alpha \in [\mathcal{S}]$, claim (4.16) follows.

For $n \in \mathbb{N}$, denote $C_n = \bar{\otimes}_{k\geq n} A_k$. We claim that

$$B \prec^s_A C_n, \text{ for every } n \in \mathbb{N}. \tag{4.17}$$

To see this, let $n \in \mathbb{N}$ and $p \in A$ be a non-zero projection. Put $\delta = \tau(p)/2 > 0$. Since $M' \cap A^\omega \subset C_n^\omega$, by combining (4.16) with Lemma 2.1.15, we deduce that $B_m \subset_\delta C_n$, for some $m \in \mathbb{N}$. Lemma 2.1.2 then implies the existence of a projection $q \in A$ such that $B_m q \prec_A^s C_n$ and $\tau(q) \geq 1 - \delta$. Since $B = (\bar{\otimes}_{k<m} L^\infty(Y_0, \nu_0)_k) \bar{\otimes} B_m$, it follows that $Bq \prec_A^s C_n$. Since $\tau(p) + \tau(q) > 1$, we have that $r = pq \in Ap$ is a non-zero projection such that $Br \prec_A C_n$. This proves the claim made in (4.17).

Now, for $n \in \mathbb{N}$, let $\{A_{n,k}\}_{k \in \mathbb{N}}$ be an increasing sequence of finite dimensional subalgebras of A_n such that $\cup_{k \in \mathbb{N}} A_{n,k}$ is weakly dense in A_n. Fix $\varepsilon > 0$ and $n \in \mathbb{N}$. Since A is abelian, (4.17) implies the existence of projections $\{p_{n,l}\}_{l \in \mathbb{N}}$ in A such that $\sum_{l \in \mathbb{N}} p_{n,l} = 1$ and $(B)_1 p_{n,l} \subset (C_n)_1 p_{n,l}$, for all $l \in \mathbb{N}$. We claim that we can find $k_{n,1}, k_{n,2}, ..., k_{n,n-1} \in \mathbb{N}$ such that

$$B \subset_{\varepsilon/2^n} \mathcal{A}_n := (\bar{\otimes}_{1 \leq m \leq n-1} A_{m,k_{n,m}}) \bar{\otimes} C_n, \text{ for every } n \in \mathbb{N}. \tag{4.18}$$

To justify this claim, let $N \in \mathbb{N}$ such that $\| \sum_{l > N} p_{n,l} \|_2 < \varepsilon/2^{n+1}$. Since by construction $A = (\bar{\otimes}_{1 \leq m \leq n-1} A_m) \bar{\otimes} C_n$, we have that $\cup_{k_1, ..., k_{n-1} \in \mathbb{N}} \big((\bar{\otimes}_{1 \leq m \leq n-1} A_{m,k_m}) \bar{\otimes} C_n \big)$ is weakly dense in A. Thus, we can find $k_{n,1}, k_{n,2}, ..., k_{n,n-1} \in \mathbb{N}$ such that $\|p_{n,l} - E_{\mathcal{A}_n}(p_{n,l})\|_2 \leq \varepsilon/(2^{n+1}N)$, for all $1 \leq l \leq N$, where $\mathcal{A}_n$ is defined as in (4.18). Let $x \in (B)_1$. Then for every $1 \leq l \leq N$, we can find $y_l \in (C_n)_1$ such that $x p_{n,l} = y_l p_{n,l}$. Since $\sum_{l=1}^N y_l E_{\mathcal{A}_n}(p_{n,l}) \in \mathcal{A}_n$, (4.18) is a consequence of the following estimate:

$$\|x - \sum_{l=1}^N y_l E_{\mathcal{A}_n}(p_{n,l})\|_2 \leq \|x - \sum_{l=1}^N y_l p_{n,l}\|_2 + \varepsilon/2^{n+1} = \|x - \sum_{l=1}^N x p_{n,l}\|_2 + \varepsilon/2^{n+1} \leq \varepsilon/2^n.$$

Put $\mathcal{A} = \cap_{n \in \mathbb{N}} \mathcal{A}_n$ and let $x \in (B)_1$. Then (4.18) gives that $\|x - E_{\mathcal{A}_k}(x)\|_2 \leq \varepsilon/2^k$, for all $k \in \mathbb{N}$. Since $E_{A_1} \circ E_{A_2} \circ ... \circ E_{A_n} = E_{\cap_{k=1}^n A_k}$, the last inequality implies that $\|x - E_{\cap_{k=1}^n \mathcal{A}_k}(x)\|_2 \leq \varepsilon$, for all $n \in \mathbb{N}$. Since $\lim\limits_{n \to \infty} \|E_{\cap_{k=1}^n \mathcal{A}_n}(x) - E_{\mathcal{A}}(x)\|_2 = 0$, we conclude that $\|x - E_{\mathcal{A}}(x)\|_2 \leq \varepsilon$. As this holds for all $x \in (B)_1$, we deduce that $B \subset_\varepsilon \mathcal{A}$. In combination with (4.15), it follows that $M' \cap A^\omega \subset_\varepsilon \mathcal{A}^\omega$. Finally, note that $\mathcal{A} = \bar{\otimes}_{m\in\mathbb{N}} A_{m,k_m}$, where $k_m = \min\{k_{n,m} | n \geq m+1\}$, for every $m \in \mathbb{N}$. This implies the conclusion of the lemma. □

We can now prove the absence of the Jones-Schmidt property for the following examples from the introduction.

Theorem 1.1.17. *For every $n \in \mathbb{N}$, let $\Gamma_n \curvearrowright (X_n, \mu_n)$ be a strongly ergodic free p.m.p. action of an infinite countable group Γ_n. Define $\Gamma = \bigoplus_{n\in\mathbb{N}} \Gamma_n$, $(X,\mu) = \prod_{n\in\mathbb{N}} (X_n, \mu_n)$, and consider the product action $\Gamma \curvearrowright (X,\mu)$ given by $g\cdot x = (g_n \cdot x_n)_n$, for all $g = (g_n)_n \in \Gamma$ and $x = (x_n)_n \in X$.*

Then the orbit equivalence relation $\mathcal{R}(\Gamma \curvearrowright X)$ does not have the Jones-Schmidt property.

Proof. Assume by contradiction that $\mathcal{S} := \mathcal{R}(\Gamma \curvearrowright X)$ has the Jones-Schmidt property. Denote $A = L^\infty(X)$ and $M = A \rtimes \Gamma$. For $n \in \mathbb{N}$, let $A_n = L^\infty(X_n)$ and $M_n = A_n \rtimes \Gamma_n$. Then $A = \bar{\otimes}_{n\in\mathbb{N}} A_n$. For $n \in \mathbb{N}$, let $C_n = \bar{\otimes}_{k\geq n} A_k$ and $D_n = \bar{\otimes}_{k\neq n} A_k$. We claim that

$$M' \cap A^\omega \subset C_n^\omega. \tag{4.19}$$

Indeed, if $k \in \mathbb{N}$, then since the action $\Gamma_k \curvearrowright (X_k, \mu_k)$ is strongly ergodic we have that $M_k' \cap A_k^\omega = \mathbb{C}1$. This easily implies that $M' \cap A^\omega \subset M_k' \cap A^\omega \subset D_k^\omega$ (by using, e.g., [Mar18, Proposition 5.2]). Thus, for every $n \in \mathbb{N}$, we have that $M' \cap A^\omega \subset \cap_{k=1}^{n-1} D_k^\omega = C_n^\omega$ and claim (4.19) follows.

We may thus apply Lemma 4.4.1 to deduce the existence of finite dimensional subalgebras $\tilde{A}_n \subset A_n$, for $n \in \mathbb{N}$, such that $M' \cap A^\omega \subset_{1/2} \mathcal{A}^\omega$, where $\mathcal{A} = \bar{\otimes}_{n\in\mathbb{N}}\tilde{A}_n$. On the other hand, since $\tilde{A}_n$ is finite dimensional and A_n is diffuse, we can find $a_n \in \mathcal{U}(A_n)$ such that $E_{\tilde{A}_n}(a_n) = 0$, for all $n \in \mathbb{N}$. Let $u_n \in \mathcal{U}(A)$ be given by $u_n = \otimes_{k\in\mathbb{N}}u_{n,k}$, where $u_{n,n} = a_n$ and $u_{n,k} = 1$, if $k \neq n$. Since $E_{\tilde{A}_n}(a_n) = 0$, we get that $E_{\mathcal{A}}(u_n) = 0$. Hence, if we put $u = (u_n)_n \in \mathcal{U}(A^\omega)$, then $E_{\mathcal{A}^\omega}(u) = 0$. Since we clearly have that $u \in M' \cap A^\omega$ this contradicts the fact that $M' \cap A^\omega \subset_{1/2} \mathcal{A}^\omega$. $\square$

Theorem 1.1.18. *Let Γ be a countable group. For every $n \in \mathbb{N}$, let $\Gamma \curvearrowright (X_n, \mu_n)$ be a free ergodic p.m.p. action such that the diagonal action $\Gamma \curvearrowright (X_n \times X_n, \mu_n \times \mu_n)$ has spectral gap. Assume that we can find $F_{n,k} \in L^\infty(X_n)$, for all $n, k \in \mathbb{N}$, such that $\sup_{n,k} \|F_{n,k}\|_\infty \leq 1$, $\inf_{n,k} \|F_{n,k}\|_2 > 0$,*

- *$F_{n,k} \to 0$ weakly in $L^2(X_n)$, as $k \to \infty$, for every $n \in \mathbb{N}$, and*
- *$\lim\limits_{n\to\infty} \big(\sup_{k\in\mathbb{N}} \|F_{n,k} \circ g - F_{n,k}\|_2\big) = 0$, for every $g \in \Gamma$.*

Consider the diagonal action $\Gamma \curvearrowright (X, \mu) = \prod_{n\in\mathbb{N}}(X_n, \mu_n)$ given by $g \cdot x = (g \cdot x_n)_n$, for all $g \in \Gamma$ and $x = (x_n)_n \in X$.

Then the orbit equivalence relation $\mathcal{R}(\Gamma \curvearrowright X)$ does not have the Jones-Schmidt property.

Proof. Assume by contradiction that $\mathcal{S} := \mathcal{R}(\Gamma \curvearrowright X)$ has the Jones-Schmidt property. Denote $A = L^\infty(X)$ and $M = A \rtimes \Gamma$. For $n \in \mathbb{N}$, let $A_n = L^\infty(X_n)$. Then $A = \bar{\otimes}_{n\in\mathbb{N}}A_n$. Let $F_{n,k} \in (A_n)_1$, $n, k \in \mathbb{N}$, be as in the hypothesis of Theorem 1.1.18 and put $c := \inf_{n,k} \|F_{n,k}\|_2 > 0$.

For $n \in \mathbb{N}$, let $C_n = \bar{\otimes}_{k \geq n} A_k$ and $D_n = \bar{\otimes}_{k \neq n} A_k$. We claim that

$$M' \cap A^\omega \subset C_n^\omega. \tag{4.20}$$

To see this, let $n \in \mathbb{N}$. Denote by π_n the Koopman representation of Γ on $L^2(X_n) \ominus \mathbb{C}1$. Then $\pi_n \otimes \bar{\pi}_n$ is a subrepresentation of the Koopman representation of Γ on $L^2(X_n \times X_n) \ominus \mathbb{C}1$. Since the latter representation is assumed to have spectral gap, we deduce that $\pi_n \otimes \bar{\pi}_n$ has spectral gap. By applying [Pop08, Lemmas 3.2 and 3.5], we get that $M' \cap A^\omega \subset D_n^\omega$. Thus, for every $n \in \mathbb{N}$, we have that $M' \cap A^\omega \subset \cap_{k=1}^{n-1} D_k^\omega = C_n^\omega$ and claim (4.20) follows.

We may thus apply Lemma 4.4.1 to deduce the existence of finite dimensional subalgebras $\tilde{A}_n \subset A_n$, for $n \in \mathbb{N}$, such that $M' \cap A^\omega \subset_{c/3} \mathcal{A}^\omega$, where $\mathcal{A} = \bar{\otimes}_{n \in \mathbb{N}} \tilde{A}_n$. We claim that there is $n \in \mathbb{N}$ such that

$$\|F_{n,k} - E_{\tilde{A}_n}(F_{n,k})\|_2 \leq c/2, \text{ for every } k \in \mathbb{N}. \tag{4.21}$$

If this were false, then for every $n \in \mathbb{N}$ we can find $k_n \in \mathbb{N}$ such that $\|F_{n,k_n} - E_{\tilde{A}_n}(F_{n,k_n})\|_2 > c/2$. But then $F = (F_{n,k_n})_n \in (A^\omega)_1$ satisfies $\|F - E_{\mathcal{A}^\omega}(F)\|_2 \geq c/2$. On the other hand, the hypothesis implies that $F \in M' \cap A^\omega$, which contradicts the fact that $M' \cap A^\omega \subset_{c/3} \mathcal{A}^\omega$ and thus proves (4.21).

Finally, since $F_{n,k} \to 0$ weakly in $L^2(X_n)$ and $\tilde{A}_n$ is finite dimensional, we get that $\|E_{\tilde{A}_n}(F_{n,k})\|_2 \to 0$, as $k \to \infty$. Since $c > 0$, this contradicts (4.21) and finishes the proof. $\square$

We end this section by justifying the claim made in Example 1.1.19 in the introduction.

Remark 4.4.2. Recall that $\widetilde{\pi}_t = \oplus_{i\in\mathbb{N}}\pi_t : \Gamma \to \mathcal{O}(\widetilde{\mathcal{H}}_t)$, where π_t is the orthogonal GNS representation of $\Gamma = \mathbb{F}_m$ associated to the positive definite function $\varphi_t(g) = e^{-t|g|}$, for $t > 0$. Let $\Gamma \curvearrowright (X_t, \mu_t)$ be the Gaussian representation associated to $\widetilde{\pi}_t$. Our goal is to show that the diagonal action $\Gamma \curvearrowright (X, \mu) := \prod_{n\in\mathbb{N}}(X_{t_n}, \mu_{t_n})$ satisfies the hypothesis of Theorem 1.1.18, for any sequence $t_n \to 0$.

First, we claim that the diagonal action $\Gamma \curvearrowright X_t \times X_t$ has spectral gap, for every $t > 0$. To this end, assume that $m < \infty$. Since the Koopman representation of Γ on $L^2(X_t) \ominus \mathbb{C}1$ is isomorphic to a subrepresentation of a multiple of $\bigoplus_{k\in\mathbb{N}} \pi_t^{\otimes k}$, the same is true for the Koopman representation of Γ on $L^2(X_t \times X_t) \ominus \mathbb{C}1$. Since $m < \infty$, we have that $\varphi_t^N \in \ell^1(\Gamma)$, for some $N \geq 1$, hence, $\pi_t^{\otimes N}$ is contained in the left regular representation of Γ. Since Γ is non-amenable, this implies that $\bigoplus_{k\in\mathbb{N}} \pi_t^{\otimes k}$ has spectral gap. In combination with the above it follows that the diagonal action $\Gamma \curvearrowright X_t \times X_t$ has spectral gap. If $m = +\infty$, then the same conclusion can be reached by replacing Γ with a finitely generated non-abelian subgroup in the above argument.

Secondly, let $t \in \mathbb{N}$. Note that we can find pairwise orthogonal vectors $\{\xi_{t,k}\}_{k\in\mathbb{N}}$ in $\tilde{\mathcal{H}}_t$ such that $\langle \widetilde{\pi}_t(g)\xi_{t,k}, \xi_{t,k}\rangle = \varphi_t(g)$, for all $g \in \Gamma$ and $k \in \mathbb{N}$. Recall that there is a map $u : \tilde{\mathcal{H}}_t \to \mathcal{U}(L^\infty(X_t))$ such that $u(\widetilde{\pi}_t(g)\xi) = u(\xi)\circ g^{-1}$, $u(\xi+\eta) = u(\xi)u(\eta)$, $\overline{u(\xi)} = u(-\xi)$, and $\int_{X_t} u(\xi) = \exp(-\|\xi\|_2^2/2)$, for all $g \in \Gamma$ and $\xi, \eta \in \widetilde{\mathcal{H}}_t$. For $k \in \mathbb{N}$, define

$$F_{t,k} := u(\xi_{t,k}) - \int_{X_t} u(\xi_{t,k}) \in L^\infty(X_t).$$

For all $k \in \mathbb{N}$ and $g \in \Gamma$, we then have $\|F_{t,k}\|_\infty \leq 2$, $\|F_{t,k}\|_2 = \sqrt{1 - \exp(-1)}$, and

$\|F_{t,k} \circ g - F_{t,k}\|_2 = \sqrt{2 - 2\exp(-1 + \varphi_t(g))}$. Moreover, $\int_{X_t} F_{t,k}\overline{F_{t,k'}} = 0$, for all $k \neq k'$, hence $F_{t,k} \to 0$ weakly in $L^2(X_t)$. It is now clear that $F_{n,k} := F_{t_n,k}/2 \in L^\infty(X_{t_n})$ satisfy the hypothesis of Theorem 1.1.18.

4.5 Equivalence relations with a unique stable decomposition

We start off with the following easy lemma, which we will use in the proof of Theorem 1.1.20 from the introduction below.

Lemma 4.5.1. *Suppose M, N are II_1 factors, and $A \subset M$, $B \subset N$ are Cartan subalgebras. Then*

$$\mathcal{R}(A \subset M) \times \mathcal{R}(B \subset N) \cong \mathcal{R}(A \bar{\otimes} B \subset M \bar{\otimes} N).$$

Proof. We identify A with $L^\infty(X)$ and B with $L^\infty(Y)$ for probability spaces (X, μ) and (Y, ν), and we write $\mathcal{R} := \mathcal{R}(A \subset M)$ and $\mathcal{S} := \mathcal{R}(B \subset N)$. Following [FM77] we can find 2-cocycles $v \in \mathrm{H}^2(\mathcal{R}, \mathbb{T})$ and $w \in \mathrm{H}^2(\mathcal{S}, \mathbb{T})$ such that

$$(A \subset M) \cong (L^\infty(X) \subset L_v(\mathcal{R})), \qquad \text{and} \qquad (B \subset N) \cong (L^\infty(Y) \subset L_w(\mathcal{S})).$$

It follows that

$$(A \bar{\otimes} B \subset M \bar{\otimes} N) \cong (L^\infty(X) \bar{\otimes} L^\infty(Y) \subset L_v(\mathcal{R}) \bar{\otimes} L_w(\mathcal{S})) \cong (L^\infty(X \times Y) \subset L_{v\times w}(\mathcal{R} \times \mathcal{S})),$$

where $v \times w \in \mathrm{H}^2(\mathcal{R} \times \mathcal{S}, \mathbb{T})$ is the cocycle defined by

$$(v \times w)((x_1, y_1), (x_2, y_2), (x_3, y_3)) = v(x_1, x_2, x_3)w(y_1, y_2, y_3).$$

Hence $\mathcal{R}(A\bar{\otimes}B \subset M\bar{\otimes}N) \cong \mathcal{R} \times \mathcal{S}$, as desired. $\square$

Theorem 1.1.20. *Let $\mathcal{R}_1$ and $\mathcal{R}_2$ be countable ergodic p.m.p. equivalence relations on probability spaces (X_1, μ_1) and (X_2, μ_2), respectively. Assume that $\mathcal{R}_1$ is strongly ergodic. Suppose that $\mathcal{R}_1 \times \mathcal{T}$ is isomorphic to $\mathcal{R}_2 \times \mathcal{T}$, where $\mathcal{T}$ is a hyperfinite ergodic p.m.p. equivalence relation on a probability space (Y, ν).*

Then either

1. *$\mathcal{R}_2$ is also strongly ergodic and $\mathcal{R}_2 \cong \mathcal{R}_1^t$, for some $t > 0$, or*

2. *$\mathcal{R}_2$ is stable and $\mathcal{R}_2 \cong \mathcal{R}_1 \times \mathcal{T}$.*

Proof. Let $\mathcal{R}_1, \mathcal{R}_2$ be countable ergodic p.m.p. equivalence relations on probability spaces $(X_1, \mu_1), (X_2, \mu_2)$. Let $\mathcal{T}_1, \mathcal{T}_2$ be hyperfinite ergodic p.m.p. equivalence relations on probability spaces $(Y_1, \nu_1), (Y_2, \nu_2)$. We assume that $\mathcal{R}_1$ is strongly ergodic and that $\mathcal{R} := \mathcal{R}_1 \times \mathcal{T}_1 \cong \mathcal{R}_2 \times \mathcal{T}_2$. We identify $X_1 \times Y_1 = X_2 \times Y_2$, and denote $M = L(\mathcal{R})$, $A = L^\infty(X_1 \times Y_1) = L^\infty(X_2 \times Y_2)$, $A_1 = L^\infty(X_1)$, $A_2 = L^\infty(X_2)$, $B_1 = L^\infty(Y_1)$, and $B_2 = L^\infty(Y_2)$. Lifting the isomorphism between $\mathcal{R}_1 \times \mathcal{T}_1$ and $\mathcal{R}_2 \times \mathcal{T}_2$ to the corresponding von Neumann algebras, we get an identification

$$[A_1\bar{\otimes}B_1 \subset L(\mathcal{R}_1)\bar{\otimes}L(\mathcal{T}_1)] = [A_2\bar{\otimes}B_2 \subset L(\mathcal{R}_2)\bar{\otimes}L(\mathcal{T}_2)].$$

Since $\mathcal{T}_2$ is a hyperfinite ergodic p.m.p. equivalence relation on (Y_2, ν_2), we can identify $(Y_2, \nu_2) = (Y_0, \nu_0)^{\mathbb{N}}$, where $Y_0 = \{0, 1\}$ and $\nu_0 = \frac{1}{2}(\delta_0 + \delta_1)$, in such a way that $(y_k)\mathcal{T}_2(z_k)$ if and only if $y_k = z_k$ for all but finitely many $k \in \mathbb{N}$. We can then identify $B_2 =$

$\bar{\otimes}_{k\in\mathbb{N}} L^\infty(Y_0,\nu_0)_k$, and we put $B_{2,n} = \bar{\otimes}_{k\geq n} L^\infty(Y_0,\nu_0)_k$, for $n \in \mathbb{N}$. As in the proof of Lemma 4.4.1, we observe that $\prod_\omega B_{2,n} \subset M' \cap A^\omega$. Moreover, by strong ergodicity of $\mathcal{R}_1$, we also know that $M' \cap A^\omega \subset B_1^\omega$. Putting these inclusions together, we deduce from Lemma 2.1.15 that $B_{2,n} \subset_{\varepsilon_n} B_1$ where $\varepsilon_n \to 0$. Since ε-containment does not change when passing to a common superalgebra, and $B_{2,n}, B_1 \subset A$, choosing k such that $\varepsilon_k < 1$ and applying Lemma 2.1.2 we get that

$$B_{2,k} \prec_A B_1.$$

Hence we can find nonzero projections $p_2 \in B_{2,k}$, $p_1 \in B_1$, a *-homomorphism $\psi : B_{2,k}p_2 \to B_1 p_1$, and a nonzero partial isometry $v \in p_1 A p_2$ such that $\psi(x)v = vx$ for all $x \in B_{2,k}p_2$. Observing that A is abelian and multiplying with v^* on both sides, we find a nonzero projection $p' \in A$ such that

$$B_{2,k}p' \subset B_1 p'.$$

Now, $B_{2,k} \subset B_2$ is the subalgebra of complex-valued functions on $Y_2 = Y_0^{\mathbb{N}}$ that do not depend on the first $k-1$ coordinates. Denoting for a fixed $b \in \prod_{i=1}^{k-1} Y_0$ by $e_b := \mathbb{1}_{\{b\}\times\prod_{i\geq k} Y_0}$ the indicator function of $\{b\} \times \prod_{i\geq k} Y_0$, we thus see that for any such b we have

$$B_2 e_b = B_{2,k} e_b.$$

As $\sum_{b\in\prod_{i=1}^{k-1} Y_0} e_b = 1_{Y_2}$, we can find b such that $p := e_b p' \neq 0$. From the above discussion it follows that

$$B_2 p \subset B_1 p.$$

Taking relative commutants, this implies that

$$p(L(\mathcal{R}_1)\bar{\otimes}B_1)p \subset p(L(\mathcal{R}_2)\bar{\otimes}B_2)p. \tag{4.22}$$

Since $p \in A$, we can write $p = \mathbb{1}_Z$ for some measurable subset $Z \subset X_1 \times Y_1$ of positive measure. Writing $Z_y = \{x \in X_1 \mid (x, y) \in Z\} \subset X_1$ and passing to a subprojection if necessary, we can assume that for every $y \in Y_1$ either $\mu_1(Z_y) = c > 0$ or $\mu_1(Z_y) = 0$. Choosing a subset $C_1 \subset X_1$ with $\mu_1(C_1) = c$, we consider

$$S := \{(y, \varphi) \in Y_1 \times [\mathcal{R}_1] \mid \varphi(Z_y) = C_1\}.$$

Denoting by π_1 the projection onto the first component, it follows from the ergodicity of $\mathcal{R}_1$ that $\pi_1(S) = \{y \in Y_1 \mid \mu_1(Z_y) \neq 0\}$. Hence by putting $\psi_y = \mathrm{id}_{X_1}$ when $\mu_1(Z_y) = 0$, it follows from [Tak01, Theorem A.16] that we can find a measurable map $\psi : Y_1 \to [\mathcal{R}_1]$ such that $\psi_y(Z_y) = C_1$ whenever $\mu_1(Z_y) \neq 0$. Identifying $L(\mathcal{R}_1)\bar{\otimes}B_1 = L^\infty(Y_1, L(\mathcal{R}_1))$ and putting

$$u = u_\psi : Y_1 \to L(\mathcal{R}_1) : y \mapsto u_{\psi_y},$$

we see that $u \in L(\mathcal{R}_1)\bar{\otimes}B_1$, $uAu^* = A$, and moreover there are projections $q \in A_1$, $\tilde{q} \in B_1$ such that $upu^* = q \otimes \tilde{q}$. Conjugating (4.22) by u, we thus get

$$qL(\mathcal{R}_1)q\,\bar{\otimes}B_1\tilde{q} \subset (q \otimes \tilde{q})u(L(\mathcal{R}_2)\bar{\otimes}B_2)u^*(q \otimes \tilde{q}). \tag{4.23}$$

Writing $P = \mathrm{Ad}(u)(L(\mathcal{R}_2))$, $Q = \mathrm{Ad}(u)(L(\mathcal{T}_2))$, $A_3 = \mathrm{Ad}(u)(A_2)$, and $B_3 = \mathrm{Ad}(u)(B_2)$, we get $M = P\bar{\otimes}Q$ and, since u normalizes A,

$$A_1\bar{\otimes}B_1 = A_2\bar{\otimes}B_2 = A_3\bar{\otimes}B_3. \tag{4.24}$$

Taking relative commutants in (4.23), we get $B_3(q \otimes \tilde{q}) \subset q \otimes B_1\tilde{q}$. In particular we can find a von Neumann subalgebra $B_4 \subset B_1$ such that $B_3(q \otimes \tilde{q}) = q \otimes B_4\tilde{q}$. Taking relative commutants again from both points of view, we get

$$qL(\mathcal{R}_1)q\,\bar{\otimes}[(B_4\tilde{q})' \cap \tilde{q}L(\mathcal{T}_1)\tilde{q}] = (q \otimes \tilde{q})(P\bar{\otimes}B_3)(q \otimes \tilde{q}).$$

In particular, since P is a factor, we see that the center of the above algebra equals $B_3(q \otimes \tilde{q}) = q \otimes B_4\tilde{q}$. Identifying both with $L^\infty(Y)$ for some probability space (Y, ν) and disintegrating in the above equality we get

$$\int_Y^\oplus qL(\mathcal{R}_1)q\,\bar{\otimes}N_y\,d\nu(y) = \int_Y^\oplus \bar{q}_y P\bar{q}_y\,d\nu(y),$$

where we decomposed $N := (B_4\tilde{q})' \cap \tilde{q}L(\mathcal{T}_1)\tilde{q} = \int_Y^\oplus N_y\,d\nu(y)$, and $q \otimes \tilde{q} = \int_Y^\oplus \bar{q}_y\,d\nu(y) \in P\bar{\otimes}B_3$. It follows from [Tak01, Theorem IV.8.23] that the above identification splits, i.e., for almost every $y \in Y$ we necessarily have

$$qL(\mathcal{R}_1)q\,\bar{\otimes}N_y = \bar{q}_y P\bar{q}_y.$$

Moreover, we see that $B_4\tilde{q} \subset B_1\tilde{q} \subset (B_4\tilde{q})' \cap \tilde{q}L(\mathcal{T}_1)\tilde{q}$, so we can decompose $B_1\tilde{q} = \int_Y^\oplus B_{1,y}\,d\nu(y) \subset \int_Y^\oplus N_y\,d\nu(y) = N$, where $B_{1,y} \subset N_y$ is a unital inclusion for all y. Now $B_1 \subset L(\mathcal{T}_1)$, and hence also $B_1\tilde{q} \subset \tilde{q}L(\mathcal{T}_1)\tilde{q}$, is a Cartan subalgebra. Since $B_1\tilde{q} \subset N \subset \tilde{q}L(\mathcal{T}_1)\tilde{q}$, it follows from [Dye63] that also $B_1\tilde{q} \subset N$ is a Cartan subalgebra. From [Spa18, Lemma 2.2] we then deduce that $B_{1,y} \subset N_y$ is a Cartan subalgebra for almost every y. Furthermore, it follows from (4.24) that

$$\int_Y^\oplus A_1 q\,\bar{\otimes}B_{1,y}\,d\nu(y) = \int_Y^\oplus A_3\bar{q}_y\,d\nu(y).$$

This identification again splits by [Tak01, Theorem IV.8.23], i.e. for almost every $y \in Y$ we have $A_1 q \,\bar{\otimes} B_{1,y} = A_3 \bar{q}_y$.

From the above discussion it now follows that we have the following identification of inclusions of Cartan subalgebras, for almost every $y \in Y$:

$$(A_1 q \,\bar{\otimes} B_{1,y} \subset qL(\mathcal{R}_1)q \,\bar{\otimes} N_y) = (A_3 \bar{q}_y \subset \bar{q}_y P \bar{q}_y).$$

Writing $\mathcal{T}_y = \mathcal{R}(B_{1,y} \subset N_y)$, $r = \tau(q)$, $s = \tau(\bar{q}_y)$, we thus get that for almost every y:

$$\mathcal{R}_2^s \cong \mathcal{R}(A_2^s \subset L(\mathcal{R}_2)^s) \cong \mathcal{R}(A_3^s \subset P^s) \cong \mathcal{R}(A_1^r \bar{\otimes} B_{1,y} \subset L(\mathcal{R}_1)^r \bar{\otimes} N_y) \cong \mathcal{R}_1^r \times \mathcal{T}_y,$$

where the last isomorphism follows from Lemma 4.5.1. Being a subalgebra of $L(\mathcal{T}_1) \cong R$, N is amenable, and so N_y is an amenable tracial factor for almost every y. Choosing y such that the above hold, [Con76] then implies that N_y is either isomorphic to $\mathbb{M}_n(\mathbb{C})$ for some $n \in \mathbb{N}$, or to R. In the first case, $\mathcal{T}_y$ is finite and we can find $t > 0$ such that $\mathcal{R}_2 \cong \mathcal{R}_1^t$. In the second case, $\mathcal{T}_y$ arises as the equivalence relation of a Cartan inclusion $B \subset R$, and as such is a hyperfinite ergodic p.m.p. equivalence relation by [CFW81]. Hence in this case we have $\mathcal{R}_2 \cong \mathcal{R}_1 \times \mathcal{T}$ where $\mathcal{T}$ is a hyperfinite ergodic p.m.p. equivalence relation. This finishes the proof of the theorem. $\square$

4.6 New characterisations of property Gamma and strong ergodicity

We now prove the last two main theorems mentioned in the introduction.

Theorem 1.1.21. *Let M be a separable II_1 factor. Then the following are equivalent:*

1. *M has property Gamma.*
2. *There exist a hyperfinite subfactor $R \subset M$ and a Cartan subalgebra $A \subset R$ such that M is generated by R and $A' \cap M$.*
3. *There exists a regular abelian von Neumann subalgebra $A \subset M$ such that $\mathcal{R}(A \subset M)$ is hyperfinite and ergodic.*

Proof. (1) $\Rightarrow$ (2) The proof of this implication builds on an argument due to Popa (see the proof of [Oza04, Proposition 7]) Assume that M has property Gamma and let $\{x_k\}_{k\geq 1} \subset M$ be a $\|.\|_2$-dense sequence. We construct inductively a sequence $\{B_n\}_{n\geq 1}$ of commuting $*$-subalgebras of M and a projection $p_n \in B_n$ such that $B_n \cong \mathbb{M}_2(\mathbb{C})$, $\tau(p_n) = \frac{1}{2}$, and $\|x_k p_n - p_n x_k\|_2 \leq \frac{1}{2^n}$, for every $n \geq 1$ and $1 \leq k \leq n$. Indeed, suppose that $B_1, ..., B_n$ and $p_1, ..., p_n$ are constructed. Define $C_n = B_1 \otimes ... \otimes B_n \cong \mathbb{M}_{2^n}(\mathbb{C})$. Since C_n is finite dimensional and M has property Gamma, we can find a projection $p_{n+1} \in C_n' \cap M$ such that $\tau(p_{n+1}) = \frac{1}{2}$, and $\|x_k p_{n+1} - p_{n+1} x_k\|_2 \leq \frac{1}{2^n}$, for every $1 \leq k \leq n+1$. Since $C_n' \cap M$ is a II_1 factor, we can also find a partial isometry $v_{n+1} \in C_n' \cap M$ such that $v_{n+1}^* v_{n+1} = p_{n+1}$ and $v_{n+1} v_{n+1}^* = 1 - p_{n+1}$. It is now clear that the algebra B_{n+1} generated by p_{n+1} and v_{n+1} has the desired property.

For $N \geq 1$, let $D_N = \mathbb{C}p_N \oplus \mathbb{C}(1 - p_N)$ be the algebra generated by p_N. Denote $R = \bar{\otimes}_{i\geq 1} B_i$, $A = \bar{\otimes}_{i\geq 1} D_i$, and $A_n = \bar{\otimes}_{i\geq n} D_i$, for $n \geq 1$. Then R is a hyperfinite II_1 factor and A is a Cartan subalgebra. Fix $n \geq k \geq 1$, and define $A_{n,N} = \bar{\otimes}_{N\geq i\geq n} D_i$, for

$N \geq n$. Then for every $N \geq n$ we have $\|E_{D'_N \cap M}(x_k) - x_k\|_2 \leq \frac{1}{2^{N-1}}$. Since $E_{A'_{n,N+1} \cap M} = E_{A'_{n,N} \cap M} \circ E_{D'_{N+1} \cap M}$, we get that $\|E_{A'_{n,N+1} \cap M}(x_k) - E_{A'_{n,N} \cap M}(x_k)\|_2 \leq \frac{1}{2^N}$, for every $N \geq n$. Since $\|E_{A'_{n,n} \cap M}(x_k) - x_k\|_2 \leq \frac{1}{2^{n-1}}$, by combining these inequalities we get $\|E_{A'_{n,N} \cap M}(x_k) - x_k\|_2 \leq \frac{1}{2^{n-1}} + ... + \frac{1}{2^{N-1}} < \frac{1}{2^{n-2}}$, for every $N \geq n$. Since $\lim_{N \to \infty} E_{A'_{n,N} \cap M}(x_k) = E_{A'_n \cap M}(x_k)$, we conclude that

$$\|E_{A'_n \cap M}(x_k) - x_k\|_2 \leq \frac{1}{2^{n-2}}, \quad \text{for every } n \geq k \geq 1. \tag{4.25}$$

Next, we claim that if N is a tracial von Neumann algebra and $C \subset N$ is a $*$-subalgebra isomorphic to $\mathbb{M}_\ell(\mathbb{C})$, for some $\ell \geq 1$, then N is generated by C and $C' \cap N$. To see this, let $p \in C$ be a projection of trace $\frac{1}{\ell}$ and $w_1, ..., w_\ell \in C$ be partial isometries such that $w_1 = p$, $w_i w_i^* = p$, for all $1 \leq i \leq \ell$, and $\sum_{i=1}^{\ell} w_i^* w_i = 1$. If $x \in pNp$, then $\sum_{i=1}^{\ell} w_i^* x w_i \in C' \cap N$, hence $x = p(\sum_{i=1}^{\ell} w_i^* x w_i)p \in C \vee (C' \cap N)$. If $x \in N$, then $x = \sum_{i,j=1}^{\ell} w_i^*(w_i x w_j^*) w_j$ and since $w_i x w_j^* \in pNp$, for all $1 \leq i, j \leq \ell$, we get that $x \in C \vee (C' \cap N)$, which proves our claim.

Finally, since $C_{n-1} = B_1 \otimes ... \otimes B_{n-1} \cong \mathbb{M}_{2^{n-1}}(\mathbb{C})$ is a $*$-subalgebra of $A'_n \cap M$, the claim implies that $A'_n \cap M = C_{n-1} \vee (C'_{n-1} \cap (A'_n \cap M))$, for every $n \geq 1$. Since $C_{n-1} \subset R$ and $C'_{n-1} \cap (A'_n \cap M) \subset A' \cap M$, we derive that $A'_n \cap M \subset R \vee (A' \cap M)$, for every $n \geq 1$. In combination with (4.25), it follows that $x_k \in R \vee (A' \cap M)$, for every $k \geq 1$, and hence that $M = R \vee (A' \cap M)$.

(2) $\Rightarrow$ (3) Assume that $R \subset M$ is a hyperfinite subfactor and $A \subset R$ is a Cartan subalgebra such that $M = R \vee (A' \cap M)$. By Lemma 2.1.19 we have that $\mathcal{R}(A \subset M) = \mathcal{R}(A \subset R)$ and thus $\mathcal{R}(A \subset M)$ is hyperfinite and ergodic.

(3) $\Rightarrow$ (1) Assume that $A \subset M$ is a regular von Neumann subalgebra such that $\mathcal{R} := \mathcal{R}(A \subset M)$ is ergodic and hyperfinite. Since $\mathcal{R}$ is hyperfinite, we can find a sequence $(a_n)_{n\geq 1} \subset \mathcal{U}(A)$ such that $\tau(a_n) = 0$, for every n, and $\|a_n \circ \alpha - a_n\|_2 \to 0$, for every $\alpha \in [\mathcal{R}]$. If $u \in \mathcal{N}_M(A)$, let $\alpha_u \in [\mathcal{R}]$ such that $uau^* = a \circ \alpha_u$, for every $a \in A$. Thus, $\|ua_nu^* - a_n\|_2 \to 0$, for every $u \in \mathcal{N}_M(A)$. Since $A \subset M$ is regular, we deduce that $(a_n)_n \in \mathcal{U}(M' \cap M^\omega)$. Since $\tau_\omega((a_n)_n) = 0$, it follows that M has property Gamma. $\square$

Theorem 1.1.22. *Let $\mathcal{S}$ be a countable ergodic p.m.p. equivalence relation on a probability space (X, μ). Let $\mathcal{T}$ be a countable hyperfinite ergodic p.m.p. equivalence relation on a probability space (Y, ν). Let R denote the hyperfinite II_1 factor. Then the following are equivalent:*

1. *$\mathcal{S}$ is strongly ergodic.*
2. *For any measurable map $\varphi : X \to Aut(\mathcal{T})$ satisfying $\varphi(x_1)^{-1}\varphi(x_2) \in [\mathcal{T}]$ for almost every $(x_1, x_2) \in \mathcal{S}$, there exists $\alpha \in Aut(\mathcal{T})$ such that $\varphi(x)\alpha \in [\mathcal{T}]$, for almost every $x \in X$.*
3. *For any measurable map $\varphi : X \to Aut(R)$ satisfying $\varphi(x_1)^{-1}\varphi(x_2) \in Inn(R)$ for almost every $(x_1, x_2) \in \mathcal{S}$, there exists $\alpha \in Aut(R)$ such that $\varphi(x)\alpha \in Inn(R)$, for almost every $x \in X$.*

Proof. (1) $\Rightarrow$ (2) Assume that $\mathcal{S}$ is strongly ergodic. Let $\varphi : X \to \mathrm{Aut}(\mathcal{T})$ be a measurable map satisfying $\varphi(x_1)^{-1}\varphi(x_2) \in [\mathcal{T}]$, for almost every $(x_1, x_2) \in \mathcal{S}$. For $y \in Y$, define the map $\varphi_y : X \to Y$ given by $\varphi_y(x) = \varphi(x)(y)$. Then $(\varphi_y(x_1), \varphi_y(x_2)) \in \mathcal{T}$, for almost every

$(x_1, x_2) \in \mathcal{S}$. Since $\mathcal{S}$ is strongly ergodic and $\mathcal{T}$ is hyperfinite, by [HK05, Theorem A.2.2], we get that $\varphi_y(x)$ is contained in a single $\mathcal{T}$-class, for almost every $x \in X$. In other words, for all $y \in Y$, we have that $(\varphi(x_1)(y), \varphi(x_2)(y)) \in \mathcal{T}$, for almost every $(x_1, x_2) \in X$. By Fubini's theorem, we can therefore find $x_2 \in X$ such that $(\varphi(x_1)(y), \varphi(x_2)(y)) \in \mathcal{T}$, for almost every $(x_1, y) \in X \times Y$. Hence, $\varphi(x_1)^{-1}\varphi(x_2) \in [\mathcal{T}]$, for almost every $x_1 \in X$, which proves (2).

(2) $\Rightarrow$ (1) Assume that $\mathcal{S}$ satisfies (2), and suppose by contradiction that $\mathcal{S}$ is not strongly ergodic. Let $Y_0 = \{0, 1\}$ together with the probability measure $\nu_0 = \frac{1}{2}(\delta_0 + \delta_1)$, and define $(Y, \nu) = (Y_0, \nu_0)^{\mathbb{N}}$. Consider the countable ergodic p.m.p. equivalence relation $\mathcal{T}$ on (Y, ν) given by $(y_n)\mathcal{T}(z_n)$ if and only if $y_n = z_n$, for all but finitely many $n \in \mathbb{N}$. Since $\mathcal{S}$ is not strongly ergodic, [JS87, Theorem 2.1] provides a factor map $\pi : (X, \mu) \to (Y, \nu)$ such that $\pi([x]_{\mathcal{S}}) = [\pi(x)]_{\mathcal{T}}$, for almost every $x \in X$.

Let $\rho_0 = \mathrm{id}_{Y_0}$ and $\rho_1 : Y_0 \to Y_0$ be given by $\rho_1(0) = 1$ and $\rho_1(1) = 0$. We define $\rho : Y \to \mathrm{Aut}(\mathcal{T})$ by letting $\rho(y)(t) = (\rho_{y_n}(t_n))$, for all $y = (y_n) \in Y$ and $t = (t_n) \in Y$. Then ρ satisfies

$$\rho(y)^{-1}\rho(z) \in [\mathcal{T}] \iff (y, z) \in \mathcal{T}, \ \text{ for all } y, z \in Y. \tag{4.26}$$

Define $\varphi : X \to \mathrm{Aut}(\mathcal{T})$ by letting $\varphi(x) = \rho(\pi(x))$. Then for almost every $(x_1, x_2) \in \mathcal{S}$, we have that $(\pi(x_1), \pi(x_2)) \in \mathcal{T}$, thus $\varphi(x_1)^{-1}\varphi(x_2) = \rho(\pi(x_1))^{-1}\rho(\pi(x_2)) \in [\mathcal{T}]$. On the other hand,

$$
\begin{aligned}
&(\mu \times \mu)(\{(x_1, x_2) \in X \times X \mid \varphi(x_1)^{-1}\varphi(x_2) \in [\mathcal{T}]\}) \\
&= (\mu \times \mu)(\{(x_1, x_2) \in X \times X \mid (\pi(x_1), \pi(x_2)) \in \mathcal{T}\}) \\
&= (\nu \times \nu)(\mathcal{T}) = 0.
\end{aligned}
$$

This implies that there does not exist $\alpha \in \mathrm{Aut}(\mathcal{T})$ such that $\varphi(x)\alpha \in [\mathcal{T}]$, for almost every $x \in X$, which contradicts the fact that $\mathcal{S}$ satisfies condition (2).

(1) $\Rightarrow$ (3) Assume that $\mathcal{S}$ is strongly ergodic. Then $(L^\infty(X)^\omega)^{[\mathcal{S}]} = \mathbb{C}1$, and (3) follows by applying Theorem 4.2.1 to the trivial factor map.

(3) $\Rightarrow$ (1) Assume that $\mathcal{S}$ satisfies (3), and suppose by contradiction that $\mathcal{S}$ is not strongly ergodic. Let $(Y, \nu) = (Y_0, \nu_0)^{\mathbb{N}}$ and $\mathcal{T}$ be defined as in the proof of (2) $\Rightarrow$ (1). Since $\mathcal{S}$ is not strongly ergodic, [JS87, Theorem 2.1] provides a factor map $\pi : (X, \mu) \to (Y, \nu)$ such that $\pi([x]_{\mathcal{S}}) = [\pi(x)]_{\mathcal{T}}$, for almost every $x \in X$.

Let $R = \bar{\otimes}_{n\in\mathbb{N}}\mathbb{M}_2(\mathbb{C})$ be the hyperfinite II_1 factor. Let $u \in \mathbb{M}_2(\mathbb{C})$ be a unitary with $\tau(u) = 0$. Let $\rho_0 = \mathrm{id}_{\mathbb{M}_2(\mathbb{C})}$ and $\rho_1 = \mathrm{Ad}(u) \in \mathrm{Aut}(\mathbb{M}_2(\mathbb{C}))$. We define $\rho : Y \to \mathrm{Aut}(R)$ by letting

$$\rho(y) = \otimes_{n\in\mathbb{N}}\, \rho_{y_n} \quad \text{for all } y = (y_n) \in Y.$$

If $y, z \in Y$ and $(y, z) \in \mathcal{T}$, then we clearly have that $\rho(y)^{-1}\rho(z) \in \mathrm{Inn}(R)$. Conversely, assume that $\rho(y)^{-1}\rho(z) \in \mathrm{Inn}(R)$, for some $y, z \in Y$. If $(y, z) \notin \mathcal{T}$, then we can find a subsequence (k_n) on $\mathbb{N}$ such that $y_{n_k} \neq z_{n_k}$, for all $k \geq 1$. Since $u \notin \mathbb{C}1$ we can find $v \in \mathbb{M}_2(\mathbb{C})$ such that $\rho_1(v) = uvu^* \neq v$. For $k \geq 1$, let $v_k = \otimes_{n\in\mathbb{N}} v_{n,k} \in R$, where

$v_{n,k} = v$, if $n = n_k$, and $v_{n,k} = 1$, if $n \neq n_k$. Then

$$\|\rho(y)^{-1}\rho(z)(v_k) - v_k\|_2 = \|\rho_{y_{n_k}}\rho_{z_{n_k}}^{-1}(v) - v\|_2 = \|uvu^* - v\|_2 > 0, \quad \text{for all } k \geq 1.$$

Since $(v_k)_k \in R' \cap R^\omega$ this contradicts the fact that $\rho(y)^{-1}\rho(z) \in \mathrm{Inn}(R)$. Altogether, we have shown that ρ satisfies $\rho(y)^{-1}\rho(z) \in \mathrm{Inn}(R) \iff (y, z) \in \mathcal{T}$, for all $y, z \in Y$.

Finally, define $\theta : X \to \mathrm{Aut}(R)$ by letting $\theta(x) = \rho(\pi(x))$. By repeating the argument from the end of the proof of (2) $\Rightarrow$ (1) one derives a contradiction with the fact that $\mathcal{S}$ satisfies condition (3). $\square$

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