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NONEQUIVALENCE OF THE ONE-CHANNEL N/D EQUATIONS WITH INELASTIC UNITARITY AND THE MULTICHANNEL ND^{-1} EQUATIONS

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Consider a partial-wave elastic-scattering amplitude[1] for two spinless particles of equal mass, M, as a function of $s=4(k^2+M^2)$:

$$A=\frac{1}{2i\rho}(S-1)=\frac{1}{2i\rho}(\eta e^{2i\delta}-1)=B+{}^RA, \tag{1}$$

where ρ is a kinematical factor and the "generalized potential" B is regular in the physical region, whereas ${}^RA$ has cuts only for $s>4M^2\equiv s_E$. The inelastic partial-wave cross section $\sigma_r^{\,l}$ is determined by η alone:

$$\sigma_r^{\,l}=\pi k^2(2l+1)(1-\eta^2). \tag{2}$$

Given B and η, we can determine $A\equiv N/D$ using the Frye-Warnock equations[2,3]

$$\frac{2\eta(s)}{1+\eta(s)}\mathrm{Re}N(s)$$

$$=\bar{B}(s)+\frac{1}{\pi}\int_{s_E}^{\infty}\frac{[\bar{B}(s')-\bar{B}(s)]2\rho(s')\,\mathrm{Re}N(s')ds'}{(s'-s)[1+\eta(s')]},$$

$$\bar{B}(s)=B(s)+\frac{\mathrm{P}}{\pi}\int_{s_I}^{\infty}\frac{[1-\eta(s')]ds'}{2\rho(s)(s'-s)},$$

$$D(s)=1-\frac{\mathrm{P}}{\pi}\int_{s_E}^{\infty}\frac{2\rho(s')\,\mathrm{Re}N(s')ds'}{(s'-s)[1+\eta(s')]}$$

$$-i\frac{2\rho(s)}{1+\eta(s)}\mathrm{Re}N(s)\theta(s-s_E),$$

$$\mathrm{Im}N(s)=\frac{1-\eta(s)}{2\rho(s)}\mathrm{Re}D(s)\theta(s-s_I) \tag{3}$$

[in addition to the usual left-hand cut in $N(s)$], where s_I is the lowest inelastic threshold. On the other hand, consider a set of coupled two-body channels with potentials B_{ij}. The amplitudes

$$A_{ij}=(S_{ij}-\delta_{ij})\frac{1}{2i(\rho_i\rho_j)^{1/2}}$$

may be determined by the multichannel ND^{-1} formalism from the B_{ij}. Now take B_{11} and η determined from the $|A_{ij}|^2$ and calculate A from (3).[4]

The purpose of this note is to demonstrate by a simple example that the solution A is not in general equal to A_{11}.[5] We find in our simple two-channel example described below that a sufficient condition for the two amplitudes A [calculated from (3)] and A_{11} (calculated by the multichannel ND^{-1} equations) to be identical is that the diagonal forces B_{22} are not strong enough to produce bound states in channel 2 in the absence of coupling between the channels. As one increases the strengths for the B_{22} beyond the value necessary to produce binding, complex-conjugate pairs of zeros in S_{11} move onto the physical sheet through the inelastic cut ($s>s_I$). The two calculations then disagree. Thus the physical situation in which we have a B_{22} strong enough to produce a bound state in channel 2 and then weakly couple it to the open channel 1 to produce a narrow resonance in A_{11} cannot be reproduced in the one-channel calculation (3). In addition, we demonstrate that, in our simple example, there are no poles of the S matrix on the physical sheet for complex values of s. (There did occur "ghost" poles on the negative real axis for some values of the B_{ij}, in which case A_{11} and A disagree. However, we disregard these unphysical situations.)

In order to carry out a substantial amount of the calculations analytically, we consider a two-channel nonrelativistic s-wave [$\rho_i=(s-s_i)^{1/2}$] system with the input (symmetric) B given by a single pole

$$B_{ij}=g_{ij}/(s+m). \tag{4}$$

Then

$$A_{ij}=g_{ik}(D^{-1})_{kj}/s+m,$$

$$\tfrac{1}{4}(1-\eta^2)=\rho_1\rho_2|A_{12}|^2\theta(s-s_2),$$

$$D_{ij}=\delta_{ij}-g_{ij}\varphi_i,$$

$$\varphi_i=-\frac{1}{2(s_i+m)^{1/2}}+\frac{(s_i+m)^{1/2}}{s+m}-\frac{(s_i-s)^{1/2}}{s+m}. \tag{5}$$

The procedure is as follows: For given g_{ij} and m, calculate A_{11} and η from (5). Then using B_{11} and η as input we calculate A from (3) and compare it with A_{11}. [The integral equation (3) for Re$N(s)$ is solved numerically by the matrix inversion technique.] The next step in the program is to locate the zeros and poles of $S_{11}=2i\rho_1A_{11}+1$. This problem is easily reduced to solving a quartic equation in the variable $(s-s_2)^{1/2}$; the same equation gives both zeros and poles of S_{11} as a function of the three g_{ij}'s for a given input pole position m. After solving for the roots, we determine whether they correspond to poles or zeros of S_{11} on the physical sheet [where $\mathrm{Im}(s-s_2)^{1/2}\geqslant 0$ and $\mathrm{Im}(s-s_1)^{1/2}\geqslant 0$] by putting these values back into the expression for S_{11}. We find that there are no poles in S_{11} on the physical sheet for complex values of s.

Now for given g_{11} and g_{12}, take g_{22} small; then we find from our numerical calculations that A_{11} agrees with A as calculated from (3). Increase g_{22}: For all $g_{22}>$ some value $\bar{g}_{22}(g_{11},g_{12},m)>2(s_2+m)^{1/2}$ (the value for which channel 2 in the absence of coupling to channel 1 develops a bound state), the two amplitudes A_{11} and A disagree. Returning to the location of the zeros in S_{11}, we find that $\bar{g}_{22}$ corresponds to the value for which a (double) zero in S_{11} occurs along the real axis above the inelastic threshold, i.e., η for some $s>s_2$ is equal to zero. We see for this situation that the integral equation (3) for ReN is no longer Fredholm. For $g_{22}>\bar{g}_{22}(g_{11},g_{12},m)$, a pair of zeros in S_{11} (at complex-conjugate points) move from the real axis onto the physical sheet.

We investigated in great detail the case $g_{11}=0$, i.e., no left-hand cut in channel 1. In this case the Ball-Frazer[6] representation is applicable: We write a dispersion relation for the phase shift in channel 1 (which is valid when the S matrix has no zeros on the physical sheet):

$$\delta=-(s-s_1)^{1/2}\frac{\mathrm{P}}{2\pi}\int_{s_2}^{\infty}\frac{\ln\eta(s')ds'}{(s'-s_1)^{1/2}(s'-s)}. \tag{6}$$

In addition, we note that the quartic equation for the zeros in S_{11} reduces to a cubic. We find that in all cases ($g_{11}=0$) that both one-channel calculations (3) and (6) for A agree. They both break down and disagree[7] with the two-channel A_{11} when zeros in S_{11} appear on the physical sheet, coming through the inelastic cut. It is clear that A as calculated from (6) will disagree with A_{11} then, since zeros in S_{11} amount to cuts in δ which are not taken into account by (6).

The appearance of zeros (at α and α^*) of S_{11} on the physical sheet through the inelastic cut will also cause the Froissart[8] one-channel N/D formalism to disagree with A_{11}. He introduces

$$R=\exp\left[-\frac{i(s-s_1)^{1/2}}{\pi}\int_{s_I}^{\infty}\frac{ds'\ln\eta(s')}{(s'-s_1)^{1/2}(s'-s)}\right],$$

and notes that $R^{-1}S$ satisfies elastic unitarity. However, R is not unique since we could multiply it by the factor

$$G=\frac{[\alpha-i(s-s_1)^{1/2}][\alpha^*-i(s-s_1)^{1/2}]}{[\alpha+i(s-s_1)^{1/2}][\alpha^*+i(s-s_1)^{1/2}]}.$$

This would presumably bring the one-channel calculation in agreement with the multichannel one. The G factor is clearly related to specifying the Castillejo-Dalitz-Dyson ambiguity.[9]

In summary, we found in our simple example that a sufficient condition for the one-channel calculation (3) to agree with the multichannel amplitude A_{11} is that the diagonal forces in the channels not explicitly considered should not be strong enough to produce bound states in the absence of coupling to channel 1.[10] We speculate that this condition will hold in general. It would be of interest to investigate more complicated examples, in particular, more complicated input "potentials" B and systems with more than two channels.

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[1]We use units $\hbar=c=1$.

[2]G. Frye and R. Warnock, Phys. Rev. 130, 478 (1963).

[3]P. Coulter, A. Scotti, and G. Shaw, Phys. Rev. 136, B1399 (1964).

[4]We consider the situation in which all the channels i have thresholds s_i greater than that of channel 1.

[5]A similar problem has been considered from a different approach by G. Chew (private communication) and E. Squires (to be published).

[6]J. Ball and W. Frazer, Phys. Rev. Letters 7, 204 (1961).

[7]Note that a numerical comparison between A_{11} and A as calculated by (6) can be made to more significant figures than the calculation (3) since (3) involves the solution of an integral equation in addition to numerical integration.

[8]M. Froissart, Nuovo Cimento 22, 191 (1961).

[9]L. Castillejo, R. Dalitz, and F. Dyson, Phys. Rev. 101, 453 (1956).

[10]Thus, for example, it is felt that the conclusions of reference 3 are valid.