NONEQUIVALENCE OF THE ONE-CHANNEL N/D EQUATIONS
WITH INELASTIC UNITARITY AND THE MULTICHANNEL ND<1 EQUATIONS

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Consider a partial-wave elastic-scattering amplitude¹ for two spinless particles of equal mass, \( M \), as a function of \( s = 4(k^2 + M^2) \):

\[
A = \frac{1}{2i\rho} (S-1) = \frac{1}{2i\rho} (\eta \rho^{2i\delta} - 1) = B + R_A,
\]

where \( \rho \) is a kinematical factor and the "generalized potential" \( B \) is regular in the physical region, whereas \( R_A \) has cuts only for \( s > 4M^2 \approx s_E \). The inelastic partial-wave cross section \( \sigma^I \) is determined by \( \eta \) alone:

\[
\sigma^I = \pi k^2 (2l + 1)(1 - \eta^2).
\]

Given \( B \) and \( \eta \), we can determine \( A = N/D \) using the Frye-Warnock equations²:

\[
\frac{2\eta(s)}{1 + \eta(s)} \text{Re} N(s) = \frac{B(s)}{1 + \eta(s)} \text{Re} N(s) = \frac{1}{1 + \eta(s)} \int_{s_E}^{\infty} \frac{\text{Im} N(s')}{s' - s} \text{ds}'
\]

\[
B(s) = B(s) + \frac{P}{\pi} \int_{s_I}^{\infty} \frac{1 - \eta(s')}{2\rho(s')} \text{ds}',
\]

\[
D(s) = 1 - \frac{P}{\pi} \int_{s_E}^{\infty} \frac{2\rho(s') \text{Re} N(s') \text{ds}'}{(s' - s)(1 + \eta(s'))}
\]

\[
\text{Im} N(s) = \frac{1 - \eta(s)}{2\rho(s)} \text{Re} D(s) \delta(s - s_E),
\]

(3)
\[ A_{ij} = \frac{(s - s_i)}{2i(p_i^2)} \]

may be determined by the multichannel ND\(^{-1}\) formalism from the \(B_{ij}\). Now take \(B_{11}\) and \(\eta\) determined from the \(|A_{11}|^2\) and calculate \(A\) from (3).

The purpose of this note is to demonstrate by a simple example that the solution \(A\) is not in general equal to \(A_{11}\). We find in our simple two-channel example described below that a sufficient condition for the two amplitudes \(A\) [calculated from (3)] and \(A_{11}\) (calculated by the multichannel ND\(^{-1}\) equations) to be equal is that the diagonal forces \(B_{22}\) are not strong enough to produce bound states in channel 2 in the absence of coupling between the channels. As one increases the strengths for the \(B_{22}\) beyond the value necessary to produce binding, complex-conjugate pairs of zeros in \(S_{11}\) move onto the physical sheet through the inelastic cut \((s > s_2)\). The two calculations then disagree. Thus the physical situation in which we have a \(B_{22}\) strong enough to produce a bound state in channel 2 and then weakly couple it to the open channel 1 to produce a narrow resonance in \(A_{11}\). cannot be reproduced in the one-channel calculation (3). In addition, we demonstrate that, in our simple example, there are no poles of the \(S\) matrix on the physical sheet for complex values of \(s\). (There did occur "ghost" poles on the negative real axis for some values of the \(B_{ij}\), in which case \(A_{11}\) and \(A\) disagree. However, we disregard these unphysical situations.)

In order to carry out a substantial amount of the calculations analytically, we consider a two-channel nonrelativistic \(s\)-wave \([\rho_i = (s - s_i)^{1/2}]\) system with the input (symmetric) \(B\) given by a single pole

\[ B_{ij} = \frac{g_{ij}}{(s + m)}. \]  

Then

\[ A_{ij} = \frac{g_{ik} (D^{-1})_{kj}}{s + m}, \]

\[ \frac{1}{4}(1 - \eta^2) = \rho_i \rho_j |A_{12}|^2 \delta(s - s_2), \]

\[ \delta_{ij} = \frac{\delta_{ij}}{\varphi_i (s + m)} \]

\[ \varphi_i = \frac{(s_i + m)^{1/2}}{s + m} + \frac{(s_i - s)^{1/2}}{s + m}. \]

The procedure is as follows: For given \(g_{ij}\) and \(m\), calculate \(A_{11}\) and \(\eta\) from (5). Then using \(B_{11}\) and \(\eta\) as input we calculate \(A\) from (3) and compare it with \(A_{11}\). [The integral equation (3) for \(ReN(s)\) is solved numerically by the matrix inversion technique.] The next step in the program is to locate the zeros and poles of \(S_{11}\) for a given input pole position \(m\). After solving for the zeros, we determine whether they correspond to poles or zeros of \(S_{11}\) on the physical sheet [where \(Im(s - s_2)^{1/2} \neq 0\) and \(Im(s - s_2)^{1/2} = 0\)].

For \(g_{11}\) and \(g_{12}\), take \(g_{22}\) small; then we find from our numerical calculations that \(A_{11}\) agrees with \(A\) as calculated from (3). Increase \(g_{22}\): For \(g_{22} > \text{some value} \bar{g}_{22}(\bar{g}_{11}, \bar{g}_{12}, m) = 2(s^2 + m)^{1/2}\) (the value for which channel 2 in the absence of coupling to channel 1 develops a bound state), the two amplitudes \(A_{11}\) and \(A\) disagree. Returning to the location of the zeros in \(S_{11}\), we find that \(\bar{g}_{22}\) corresponds to the value for which a (double) zero in \(S_{11}\) occurs along the real axis above the inelastic threshold, i.e., \(\eta\) for some \(s > s_2\) is equal to zero.

We see for this situation that the integral equation (3) for \(ReN\) is no longer Fredholm. For \(g_{22} > \bar{g}_{22}(\bar{g}_{11}, \bar{g}_{12}, m)\), a pair of zeros in \(S_{11}\) (at complex-conjugate points) move from the real axis onto the physical sheet.

We investigated in great detail the case \(g_{11} = 0\), i.e., no left-hand cut in channel 1. In this case the Ball-Frazer representation is applicable: We write a dispersion relation for the phase shift in channel 1 (which is valid when the \(S\) matrix has no zeros on the physical sheet):

\[ \delta = -\frac{1}{2} \int_{s_1}^{s_2} \frac{\ln |\eta(s')| ds'}{s - s' - s_1} - \frac{1}{2(s - s_2)}, \]

In addition, we note that the quartic equation for the zeros in \(S_{11}\) reduces to a cubic. We find that in all cases (\(g_{12} = 0\)) that both one-channel calculations (3) and (6) for \(A\) agree. They both break down and disagree with the two-channel \(A_{11}\) when zeros in \(S_{11}\) appear on the physical sheet, coming through the inelastic cut. It is clear that \(A\) as calculated from (6)
will disagree with \( A_{11} \) then, since zeros in \( S_{11} \) amount to cuts in \( \delta \) which are not taken into account by (6).

The appearance of zeros (at \( \alpha \) and \( \alpha^* \)) of \( S_{11} \) on the physical sheet through the inelastic cut will also cause the Froissart\(^8\) one-channel \( N/D \) formalism to disagree with \( A_{11} \). He introduces

\[
R = \exp \left[ -\frac{i(s-s_1)^{1/2}}{\pi} \int s_I^{\infty} ds' \ln \left( \frac{s'}{s-s_1} \right)^{1/2} \left( s'-s \right)^{1/2} \right],
\]

and notes that \( R^{-1} S \) satisfies elastic unitarity. However, \( R \) is not unique since we could multiply it by the factor

\[
G = \left[ \frac{\alpha-i(s-s_1)^{1/2}}{\alpha+i(s-s_1)^{1/2}} \right] \left[ \frac{\alpha^*+i(s-s_1)^{1/2}}{\alpha^*+i(s-s_1)^{1/2}} \right].
\]

This would presumably bring the one-channel calculation in agreement with the multichannel one. The \( G \) factor is clearly related to specifying the Castillejo-Dalitz-Dyson ambiguity.\(^9\)

In summary, we found in our simple example that a sufficient condition for the one-channel calculation (3) to agree with the multichannel amplitude \( A_{11} \) is that the diagonal forces in the channels not explicitly considered should not be strong enough to produce bound states in the absence of coupling to channel 1.\(^{10}\) We speculate that this condition will hold in general.

It would be of interest to investigate more complicated examples, in particular, more complicated input "potentials" \( B \) and systems with more than two channels.

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\(^1\)We use units \( \hbar = c = 1. \)
\(^4\)We consider the situation in which all the channels \( i \) have thresholds \( s_i \) greater than that of channel 1.
\(^5\)A similar problem has been considered from a different approach by G. Chew (private communication) and E. Squires (to be published).
\(^7\)Note that a numerical comparison between \( A_{11} \) and \( A \) as calculated by (6) can be made to more significant figures than the calculation (3) since (3) involves the solution of an integral equation in addition to numerical integration.
\(^{10}\)Thus, for example, it is felt that the conclusions of reference 3 are valid.