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QUANTUM CURVES FOR HITCHIN FIBRATIONS AND THE EYNARD-ORANTIN THEORY

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Abstract. We generalize the topological recursion of Eynard-Orantin [20, 41] to the family of spectral curves of Hitchin fibrations. A *spectral curve* in the topological recursion, which is defined to be a complex plane curve, is replaced with a generic curve in the cotangent bundle T^*C of an arbitrary smooth base curve C. We then prove that these spectral curves are quantizable, using the new formalism. More precisely, we construct the canonical generators of the formal $\hbar$-deformation family of D-modules over an arbitrary projective algebraic curve C of genus greater than 1, from the geometry of a prescribed family of smooth Hitchin spectral curves associated with the $SL(2,\mathbb{C})$-character variety of the fundamental group $\pi_1(C)$. We show that the semi-classical limit through the WKB approximation of these $\hbar$-deformed D-modules recovers the initial family of Hitchin spectral curves.

Contents

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1. Introduction and the main results

A **quantum curve** [2, 11, 23, 24, 25, 39, 54, 61] is a magical object. It conjecturally captures information of quantum topological invariants in an effective and compact manner. Mathematically, it is a D-module defined on a formal family of complex holomorphic curves $C[[\hbar]]$ that *quantizes* a **spectral curve** Σ. It is automatically holonomic, and its **semi-classical limit** through the **WKB approximation** induces a holomorphic *Lagrangian immersion*

$$\begin{array}{ccc} \iota : \Sigma & \longrightarrow & T^*C \\ & & \Big\downarrow \pi \\ & & C \end{array} \tag{1.1}$$

with respect to the natural symplectic structure of the cotangent bundle T^*C. It is also closely related to an *oper* of [9, 45], a λ-*connection* of Deligne (see for example, [7]), a *quantum characteristic polynomial* in the theory of integrable models in statistical mechanics [21, 78], a *Cherednik algebra* [36], and the differential equation appearing in the context of determining the *Nekrasov partition function* [75] through the AGT *correspondence* [4, 12, 48].

We note that not every morphism of curves $\Sigma \longrightarrow C$ is quantizable. Clearly we need a Lagrangian immersion for the WKB method to work. Therefore, it is natural to ask what type of conditions we need for the existence of quantization.

The purpose of this paper is to show that the spectral curves associated with $SL(2,\mathbb{C})$-**Hitchin fibrations** [58, 59] are quantizable, by concretely constructing a canonical generator (which is related to the *conformal block* in the context of the AGT correspondence) of $\hbar$-deformed D-modules on an arbitrary smooth projective algebraic curve C of genus $g(C) \geq 2$. For this construction we first generalize the *topological recursion* mechanism proposed in [41], which is strictly restricted to the case of $C = \mathbb{C}$ or $C = \mathbb{C}^*$, to what we call the **Eynard-Orantin theory**, making it applicable to the spectral curves of (1.1) with an arbitrary base curve C. We show that this new formalism allows us to construct the desired quantization of Σ.

Since our work connects many different developments that took place in a vast array of mathematics in recent years, we present each component that forms our work in this introductory section.

1.1. Generalization of the topological recursion of Eynard and Orantin.

The Eynard-Orantin theory that we propose in this paper stems out of various physics literature, including [11, 16, 20, 26, 41, 67]. The key ingredient in both Hitchin fibrations and the Eynard-Orantin theory is the notion of spectral curves. By generalizing the original topological recursion of [41], we shall show that these spectral curves are exactly the same object. As a consequence of this identification, we obtain a purely geometric interpretation of *what the topological recursion does*. More precisely, we construct a quantum curve when the spectral curve (1.1) is non-singular and $\pi : \Sigma \longrightarrow C$ is a ramified double-sheeted covering. In this particular mechanism, the Eynard-Orantin theory that we propose solves

the all-order **Wentzel-Kramers-Brillouin (WKB) approximation** (see for example, [10]). The mechanism works as follows.

- The spectral curves of Hitchin fibrations are quantized by the WKB method.
- The Eynard-Orantin theory gives a solution to the *all-order, exact* WKB approximation from the geometry of spectral curves as the initial condition.
- Along the branched points of $\pi : \Sigma \longrightarrow C$, the WKB method does not work. Around these points, asymptotically, the Eynard-Orantin theory gives the expected **Airy function** solution [11], in the same way as it appears in [7]. This is because the local behavior of π around a branched point is the double-sheeted covering of a formal disc by another formal disc, ramified at the origin. The Airy function here is representing the Witten-Kontsevich theory of the cotangent ψ-class intersection numbers on $\overline{\mathcal{M}}_{g,n}$ [62, 80] (see also [32]).

We note that the relation between Hitchin systems and the WKB method is extensively studied in Gaiotto-Moore-Neitzke [49] and their subsequenct papers.

The first formulation of the topological recursion in [20, 41] assumes that the base curve C is always the complex line $\mathbb{C}$. A modification is proposed for the case of $C = \mathbb{C}^*$ in [16, 17]. Our current work provides a generalization of these theories to *compact* base curve C. The original case is just a restriction of our picture onto an affine piece of C. From this point of view, we develop a *global* topological recursion, utilizing the full global structure of the starting spectral curve. The main technical difficulty of the theory that we overcome in this paper is our global calculation of the residue integrals appearing in the topological recursion formula.

When we consider a spectral curve embedded in the principal $\mathbb{C}^*$-bundle associated with T^*C, such as those we find in [23, 54], even though a similar formalism works, the topological recursion acquires a different mathematical flavor. It is a relation to algebraic K-theory and the Bloch regulator appearing as a Bohr-Sommerfeld type quantization condition described in [2, 54]. We come back to this point later.

1.2. **Hitchin spectral curves.** In algebraic geometry, a spectral curve simply means the diagram (1.1) for an algebraic curve C. The curve Σ also appress as the **Seiberg-Witten curve** [48]. It is obvious that such a Σ *cannot* be the characteristic variety of a D-module defined over the base curve C, because $\dim C = 1$ and the characteristic varieties are necessarily $\mathbb{C}^*$ invariant with respect to the $\mathbb{C}^*$-action on T^*C. To capture the geometry of a spectral curve, we need to utilize Deligne's λ-connections. The idea of the λ-connections is parallel to that of the WKB method in analysis. This is explained in Section 5.

The notion of spectral curves was developed by Hitchin [58, 59] in the process of **Abelianization** of the moduli spaces of stable vector bundles on a projective algebraic curve C of genus greater than 1 (see also [8, 29, 55, 56, 65, 77]). Consider a **Higgs pair** (E, ϕ) consisting of a vector bundle E of rank r on C and a **Higgs field** $\phi \in H^0(C, \mathcal{E}nd(E) \otimes \Omega^1_C)$, where Ω^1_C denotes the sheaf of holomorphic 1-forms on C. The Higgs field here is defined on a curve through the dimensional reduction of the Higgs boson [57] on a 4-dimensional space-time. Let η be the tautological 1-form on T^*C such that $-d\eta$ gives the natural holomorphic symplectic form on T^*C. Then the characteristic equation

$\det(\eta - \phi) = 0$ defines a spectral curve Σ as the space of eigenvalues of ϕ. Under a good condition, Σ is nonsingular and the natural projection $\pi : \Sigma \longrightarrow C$ is a ramified covering of degree r with ramification divisor R. In symplectic geometry, a ramification point $p \in R$ is called a **Lagrangian singularity**, and the branch divisor $\pi(R) \subset C$ the **caustics** of π. Let $M \longrightarrow \Sigma$ be the eigenspace bundle of the Higgs field on Σ, and define $L = M \otimes \mathcal{O}_\Sigma(R)$. Then the original vector bundle E is recovered by $E = \pi_* L$. The Abelianization refers to the correspondence

$$(C, E, \phi) \longleftrightarrow (\pi : \Sigma \longrightarrow C, L, \iota^*\eta).$$

Let us denote by

$$s = (s_1, s_2, \dots, s_r) = \big(-\mathrm{tr}\phi, \mathrm{tr}(\wedge^2\phi), \dots, (-1)^r \mathrm{tr}(\wedge^r \phi)\big) \in V^*_{GL_r} := \bigoplus_{i=1}^{r} H^0\big(C, (\Omega^1_C)^{\otimes i}\big) \tag{1.2}$$

the characteristic coefficients of the Higgs field ϕ. The dual notation * on the vector space is due to the analogy with the dual Lie algebra we normally have as a target space of a moment map in real symplectic geometry. In algebraic geometry, a family of groups can act symplectomorphically, with the same Lie algebra. Here we have such a situation (see for example, [60]). The notation $\mathrm{tr}(\wedge^i\phi)$ of a matrix ϕ means the sum of all principal $i \times i$-minors of ϕ that is considered as an element of the symmetric power $H^0\big(C, (\Omega^1_C)^{\otimes i}\big)$. We are not talking about the exterior power $\phi \wedge \phi$ here, since all higher exterior powers of ϕ automatically vanish on C. The global section

$$\eta^{\otimes r} + \sum_{i=1}^{r} \eta^{\otimes(r-i)} \otimes \pi^* s_i \in H^0\big(T^*C \times V^*_{GL_r}, \pi^*(\Omega^1_C)^{\otimes r} \boxtimes \mathcal{O}_{V^*_{GL_r}}\big) \tag{1.3}$$

defines a family of spectral curves

$$\begin{array}{ccccc} \Sigma_s \subset \widetilde{\Sigma} & \xrightarrow{\ \iota\ } & T^*C \times V^*_{GL_r} & \longrightarrow & V^*_{GL_r} \\ & & \big\downarrow{\scriptstyle \pi\times\mathrm{id}} & & \\ C \times \{s\} & \longrightarrow & C \times V^*_{GL_r} & & \end{array} \tag{1.4}$$

on $V^*_{GL_r}$. The morphism $\pi : \Sigma_s \longrightarrow C$ has degree r. When there is no need to specify the rank r, we denote simply by $V^*_{GL_r} = V^*_{GL}$.

Our discovery of this paper is that when we restrict ourselves to the case of $r = 2$ and generic $s \in V^*_{GL_2}$ so that Σ_s is smooth and the covering is simply ramified, the generalized Eynard-Orantin theory precisely gives the quantization of a *family* of smooth spectral curves $\widetilde{\Sigma}\big|_V$ for a contractible open neighborhood $V \subset V^*_{GL_2}$ of s.

1.3. **The Generalized Eynard-Orantin theory.** In their seminal paper [41], Eynard and Orantin propose a geometric theory of computing quantum invariants using an integral recursion formula on a plane curve Σ which is realized as a simply ramified covering $\pi : \Sigma \longrightarrow \mathbb{C}$, i.e., when the base curve C of (1.1) is the complex line $\mathbb{C}$. In Section 3 we generalize the original topological recursion to a mathematical framework suitable for the

purpose of the current paper. The heart of this theory is an integral recursion formula, originally found in random matrix theory [5, 20, 37].

The topological nature of the formula itself is known to the mathematics community for a long time. It is the same degeneration on the Deligne-Mumford moduli stack $\overline{\mathcal{M}}_{g,n}$ of n-pointed stable curves of genus g as described in [6, Chapter 17, Section 5, Page 589]. It appears as the Dijkgraaf-Verlinde-Verlinde formula [27] for the Witten-Kontsevich intersection theory [62, 80], known as the *Virasoro constraint condition*, and also as a recursion formula for the Weil-Petersson volume of the moduli space of bordered hyperbolic surfaces in Mirzakhani's work [68, 69] (see also [66, 70]). The key difference between the topological recursion and the above mentioned formalisms is that the former is a **B-model theory** that exhibits a universal structure (cf. [16, 67]). Indeed, the B-model formalism is the *Laplace transform* [32, 40] of the geometric equations mentioned above.

In the context of the Hitchin spectral curves or the Seiberg-Witten curves (1.4), the generalized formalism goes as follows. The goal of the theory is to define symmetric differentials $W^s_{g,n}$ on Σ_s^n for $g \geq 0$ and $n \geq 1$. The starting point is the two *unstable* cases $2g-2+n \leq 0$. We first define $W^s_{0,1}(z_1) = \iota^*\eta(z_1)$, which is called the **Seiberg-Witten differential**. As $W^s_{0,2}(z_1,z_2)$ we take the Riemann fundamental form of the second kind [44, 74] with an appropriate normalization that we can choose on an open neighborhood of a generic point $s \in V^*_{GL}$. This is the unique differential form of degree 2 on $\Sigma_s \times \Sigma_s$ with double poles along the diagonal, and when considered as an integration kernel it operates as the exterior differentiation $f \longmapsto df$ for any meromorphic function on Σ_s. For the *stable* range $2g-2+n>0$, the differentials $W^s_{g,n}$ at a point $(z_1,\dots,z_n) \in \Sigma_s^n$ are recursively defined by the following integral recursion formula:

$$
\begin{aligned}
(1.5)\quad W^s_{g,n}(z_1,\dots,z_n) = \frac{1}{2}\,\frac{1}{2\pi i}\sum_{p\in R_s}\oint_{\gamma_p}\frac{\int_z^{\sigma_p(z)} W^s_{0,2}(\,\cdot\,,z_1)}{W^s_{0,1}\big(\sigma_p(z)\big)-W^s_{0,1}(z)}\\
\times\Bigg[\sum_{j=2}^{n}\Big(W^s_{0,2}(z,z_j)W^s_{g,n-1}\big(\sigma_p(z),z_{[\hat{1},\hat{j}]}\big)+W^s_{0,2}\big(\sigma_p(z),z_j\big)W^s_{g,n-1}\big(z,z_{[\hat{1},\hat{j}]}\big)\Big)\\
+W^s_{g-1,n+1}\big(z,\sigma_p(z),z_{[\hat{1}]}\big)+\sum_{\substack{g_1+g_2=g\\ I\sqcup J=[\hat{1}]}}^{\text{stable}}W^s_{g_1,|I|+1}(z,z_I)W^s_{g_2,|J|+1}\big(\sigma_p(z),z_J\big)\Bigg].
\end{aligned}
$$

Here R_s is the ramification divisor of the spectral curve $\pi:\Sigma_s \longrightarrow C$ which is assumed to be a simple ramified covering, γ_p is a small simple closed loop with the positive orientation around a Lagrangian singularity $p \in R_s \subset \Sigma_s$, and σ_p is the local Galois conjugation of the curve Σ_s near p. The residue integration is taken with respect to the z variable on γ_p. For the index set $[n]=\{1,\dots,n\}$, we indicate missing indices by the $\hat{}$ notation. For a subset $I \subset [n]$, we denote $z_I=(z_i)_{i\in I}$, and by $|I|$ the cardinality of I. The sum in the last line runs for all partitions of g and set partitions of $\{2,\dots,n\}$, subject to the condition that $2g_1-1+|I|>0$ and $2g_2-1+|J|>0$.

The **free energy** of type (g,n) is a (meromorphic) function $F^s_{g,n}$ on Σ_s^n satisfying that

$$d_1 \cdots d_n F^s_{g,n} = W^s_{g,n}. \tag{1.6}$$

Of course such $F^s_{g,n}$'s are never unique because of the constants of integration, and their existence is not even guaranteed because Σ_s has a nontrivial fundamental group. When $F^s_{g,n}$ exists, we impose the uniqueness condition by integration along the fiber:

$$(\pi_i)_* F^s_{g,n} := \sum_{z_i \in \pi^{-1}(x_i)} F^s_{g,n}(z_1, \dots, z_i, \dots, z_n) = 0, \qquad (g,n) \neq (0,2). \tag{1.7}$$

Here we choose an arbitrary point $x_i \in C$ that is not a branched point, and consider the integration of $F^s_{g,n}$ along the fiber of π at x_i for the i-th component of the product of Σ_s, while fixing all other z_j's, $j \neq i$.

1.4. **The main result.** We prove the following.

Theorem 1.1 (Main Theorem)**.** *Let C be an arbitrary smooth projective algebraic curve of genus $g \geq 2$ over $\mathbb{C}$. We consider the family* (1.4) *of degree 2 spectral curves on $C \times V^*_{SL_2}$ corresponding to the $SL(2,\mathbb{C})$ Hitchin fibration. If the spectral data $s \in V^*_{SL_2} := H^0\big(C, (\Omega^1_C)^{\otimes 2}\big)$ is generic so that Σ_s is non-singular and the covering $\pi : \Sigma_s \longrightarrow C$ is simply ramified, then there is an open neighborhood $s \in V \subset H^0\big(C, (\Omega^1_C)^{\otimes 2}\big)$ such that the family of spectral curves $\widetilde{\Sigma}\big|_V$ is quantizable by using the Eynard-Orantin theory.*

More precisely, we construct a quantum curve, or a **Schrödinger operator** $P_s(x,\hbar)$, as more commonly known, on a formal family $C[[\hbar]]$ of the curve C such that

$$E = \mathcal{D}^\hbar / \mathcal{D}^\hbar P_s \tag{1.8}$$

is a D-module of $\mathcal{O}_{C[[\hbar]]}$-rank 2 over $C[[\hbar]]$. Here we denote by $\mathcal{D}^\hbar = \mathcal{D}^\hbar_{C[[\hbar]]}$ the sheaf of differential operators on $C[[\hbar]]$ without $\hbar$-derivatives. We use local coordinates z on Σ_s, x on C, and a local section $z = z(x)$ of π. We prove that the canonical solution of the Schrödinger equation

$$P_s(x,\hbar)\big|_U \Psi_s\big(z(x),\hbar\big) = 0 \tag{1.9}$$

defined on an open subset $U \subset C$ that contains no caustics of $\pi : \Sigma_s \longrightarrow C$ is constructed by the formula of [41, 54]

$$\Psi_s(z,\hbar) = \exp\left(\sum_{g \geq 0} \sum_{n \geq 1} \frac{1}{n!} \hbar^{2g-2+n} F^s_{g,n}(z, \dots, z) \right). \tag{1.10}$$

In the context of the AGT correspondence, this seems to be related to the function known as a **conformal block**. We note that (1.10) is exactly a geometric refinement of the singular perturbation method known as the WKB approximation. Moreover, the **semi-classical limit** (i.e., the zeroth-order terms in the $\hbar$-expansion of the WKB approximation) of this Schrödinger equation recovers the spectral curve equation

$$\eta^{\otimes 2} + \pi^* s = 0$$

for $\Sigma_s \subset T^*C$.

The heart of the construction is Theorem 4.7, which is derived from the generalized integral recursion (1.5) by concretely evaluating the residue integration of the formula. We emphasize that the residue calculation of (1.5) is made possible only because we generalize the topological recursion formalism of [41] to the compact base curve C. We establish the unique existence of the free energy $F^s_{g,n}$ for every $(g,n) \neq (0,2)$, and construct the Schrödinger operator P_s from (4.7), after identifying $F^s_{0,2}$ through the first-order WKB approximation. Although in its expression, (1.10), depends on the choice of coordinates, Theorem 4.7 is coordinate independent, and establishes the quantization of the spectral curve in a coordinate-free manner.

We also remark that though our formalism is more general, the actual technical calculations are parallel to that of [11]. Indeed, we asked the following question: what would be the mathematical framework that would allow the analysis technique of [11, 32, 72] work? In the process of answering this question, we discover that the Hitchin spectral curves are the right framework.

The $SL(2,\mathbb{C})$ assumption we impose is due to a technical reason, but not by any conceptual reason. The formulation of [14], which assumes that the spectral curve is a compact plane algebraic curve, can easily be generalized to our situation of Hitchin spectral curves (1.4). However, the idea developed in [14] does not seem to directly provide the counterpart of our Theorem 4.7. We can also allow a base curve C with prescribed marked points, and consider the moduli space of *parabolic* Higgs bundles. In the context of the AGT correspondence and Seiberg-Witten curves [4, 48], such a setup naturally arises. In this paper, however, we stay with the simplest situation, avoiding too much technical complications. The case for parabolic Higgs bundles with singular Seiberg-Witten differentials will be treated in a forthcoming paper.

1.5. **The geometric significance of the topological recursion.** The significance of what the topological recursion does is first recognized in the string theory community [16, 26, 67, 76]. Mariño [67], and then Bouchard, Klemm, Mariño, and Pasquetti [16], have conjectured that when the spectral curve $\pi : \Sigma \longrightarrow \mathbb{C}^*$ is the mirror curve of a toric Calabi-Yau space X of dimension 3 (in this case it covers the punctured complex line $\mathbb{C}^*$), the topological recursion should calculate open Gromov-Witten invariants of X for all genera (*the remodeling conjecture*). Their conjecture is a concrete and universal mechanism to read off, from $W_{g,n}$ of (1.5), all open Gromov-Witten invariants of genus g with n boundary components of the source Riemann surface that are mapped to a Lagrangian in X.

Bouchard and Mariño then related the topological recursion with the counting problem of simple Hurwitz numbers [17]. They conjectured that certain generating functions of simple Hurwitz numbers should satisfy (1.5) for $C = \mathbb{C}^*$ with the spectral curve Σ defined by the *Lambert function* $x = ye^{-y}$.

The Hurwitz number conjecture of Bouchard and Mariño was solved in [40, 73]. The key discovery was that the topological recursion was equivalent to the *Laplace transform* of the combinatorial relation known as the *cut-and-join equation* [52, 53, 79] of Hurwitz numbers.

Here again, we emphasize that the proof of the conjecture is based on the global complex analysis of the Lambert curve, rather than the local behavior of the spectral curve.

Once the relation between a counting problem (A-model) and the integral recursion on a complex curve (B-model) is understood as the Laplace transform, the same idea is used to solve the remodeling conjecture of [16] for the case of topological vertex [19, 81]. Since the topological vertex method gives a combinatorial description of the Gromov-Witten invariants for an arbitrary smooth toric Calabi-Yau threefold [64], the smooth case of the remodeling conjecture was solved in [42] by identifying the combinatorial structure of the integral recursion with the localization method in open Gromov-Witten invariants. Most recently, the general orbifold case of the conjecture is solved in [43].

The mathematical structure of topological recursion has also been studied in [34, 38], when the spectral curve is considered as a collection of disjoint open discs. In particular, the discovery of the equivalence to the Givental formalism in this local case [34], and its application to obtaining a new proof of the ELSV formula [33], are significant. Compared to these structural analysis, the emphasis of our current work lies in noticing the importance of the *global structure* of the spectral curve that covers an *arbitrary* projective algebraic curve.

1.6. **Quantum curves, and the motivation of our current paper.** Although the topological recursion for simple Hurwitz numbers was conjectured from the consideration of open Gromov-Witten invariants of $\mathbb{C}^3$ at the infinity limit of the framing parameter, the Hurwitz case has a feature not shared with the geometry of toric Calabi-Yau spaces. This is *the existence of the quantum curve* [72]. The similar situation happens also for orbifold Hurwitz numbers [15, 71].

Gukov and Sułkowski [54] considered the **A-polynomial** of Cooper, Culler, Gillet, Long, and Shalen [22] associated with a knot K. The $SL(2,\mathbb{C})$-character variety of the fundamental group of the knot complement is mapped to the boundary torus

$$\mathrm{Hom}\big(\pi_1(S^3 \setminus K), SL(2,\mathbb{C})\big)/\!\!/SL(2,\mathbb{C}) \longrightarrow \mathrm{Hom}\big(\pi_1(T^2), SL(2,\mathbb{C})\big)/\!\!/SL(2,\mathbb{C}) \cong (\mathbb{C}^*)^2$$

and determines a (usually) singular plane algebraic curve in $(\mathbb{C}^*)^2$ defined over $\mathbb{Z}$. Its defining equation is the A-polynomial, which captures the *classical* knot invariant $\pi_1(S^3 \setminus K)$. The proposal of Gukov-Sułkowski is that by applying the topological recursion that is suitably modified for spectral curves in $(\mathbb{C}^*)^2$, one can quantize the A-polynomial into a Schrödinger equation, much like (1.9) above but of an infinite order due to the appearance of $\mathbb{C}^*$ in the fiber direction of π, whose semi-classical limit recovers precisely the A-polynomial. Moreover, they predict that the Schrödinger equation is equivalent to the **AJ-conjecture** of Garoufalidis [50, 51], which implies that the generator Ψ of the $\hbar$-deformed D-module is the colored Jones polynomial of the knot K!

We recall that the A-polynomial of a knot K is a polynomial in $\mathbb{Z}[x,y]$, where x and y are determined by the meridian and the longitude of the torus boundary of the knot complement in S^3. It is established in [22] that the Steinberg symbol $\{x,y\} \in K_2\big(\mathbb{C}(C_K)\big)$ is a torsion element of the second algebraic K-group of the function field of the projective curve

C_K determined by the A-polynomial of the knot K. Gukov and Sułkowski [54] attribute the quantizability of the A-polynomial to this algebraic K-theory condition, which plays a similar role of the Bohr-Sommerfeld quantization condition through the Bloch regulator.

We have constructed rigorous mathematical examples of the topological recursion in [32], for which we can test all physics predictions. A quantum curve construction is also carried out in [72], and for many other examples of counting problems of Hurwitz type [15, 71, 82]. For these cases the K_2 condition (the torsion property of the Steinberg symbol) holds. But it has to be remarked that all these rigorous examples have spectral curves of genus 0. So far no examples of quantum curves have been rigorously constructed for a spectral curve with a higher genus. This motivates our current paper. Although we do not address the question in this paper, the ultimate interest is to identify the quantum topological information that our Ψ must carry. In this context, establishing the relation to the Seiberg-Witten prepotential of Nekrasov [75] through the AGT correspondence [4] is the key [12, 48]. The Eynard-Orantin theory then provides an expansion formula for the conformal block Ψ from the geometric data of the Seiberg-Witten curve covering the Gaiotto curve.

We note that the relation between the topological recursion and knot invariants are growing at this moment [3, 13, 18, 23, 46, 47]. It is beyond our scope to make any comment in this direction.

1.7. **Organization of the paper.** The paper is organized as follows. We begin with gathering the classical geometric materials we use in this paper, recalling spectral curves, Riemann prime forms, and geometry of degree 2 spectral curves, in Section 2. Then in Section 3, we re-define the topological recursion with an arbitrary base curve. Section 4 is devoted to integrating the newly formulated recursion. We will establish a differential recursion formula for free energies. Here our generalization (1.5) of the topological recursion of [41] plays an essential role, due to the fact that our spectral curve and the base curve are both compact. The notion of quantum curves from physics requires us to utilize Deligne's λ-connections. We review the necessary materials in Section 5, following [7]. Finally in Section 6, we take the *principal specialization* of the formula established in Section 4. In this way we construct the quantum curve and the $\hbar$-deformed D-module, quantizing the spectral curve. This method is indeed the same as solving the exact WKB analysis.

2. Geometry of spectral curves

Let C be a non-singular complete algebraic curve over $\mathbb{C}$ of genus $g = g(C) \geq 2$. Although somewhat restrictive, since we need the smoothness and the simple ramification conditions, we adopt the following definition in this paper.

Definition 2.1. A **spectral curve** of degree r is a complete smooth algebraic curve Σ embedded in the cotangent bundle T^*C such that its projection

$$\begin{array}{ccc} \iota : \Sigma & \longrightarrow & T^*C \\ & & \downarrow \pi \\ & & C \end{array}$$

onto C is a simply ramified covering of degree r. We denote by $\eta \in H^0\big(T^*C, \pi^*\Omega^1_C\big)$ the tautological 1-form on T^*C such that $-d\eta$ is the canonical holomorphic symplectic form on T^*C. A **spectral data** is an element of a vector space

$$s = (s_1, s_2, \dots, s_r) \in V^*_{GL} := \bigoplus_{i=1}^{r} H^0\big(C, (\Omega^1_C)^{\otimes i}\big) \tag{2.1}$$

of dimension $r^2(g-1)+1$. We consider a spectral data **generic** if the characteristic equation

$$\eta^{\otimes r} + \sum_{i=1}^{r} s_i \eta^{\otimes (r-i)} = 0 \tag{2.2}$$

defines a spectral curve Σ in our sense. Here the characteristic polynomial is viewed as a global section

$$\eta^{\otimes r} + \sum_{i=1}^{r} \pi^* s_i \otimes \eta^{\otimes (r-i)} \in H^0\big(T^*C, \pi^*(\Omega^1_C)^{\otimes r}\big)$$

that defines Σ as its 0-locus. To indicate the $s \in V^*_{GL}$ dependence of the spectral curve, we use the notation $\Sigma = \Sigma_s$.

Remark 2.2. The smoothness assumption of Σ_s is crucial. The evaluation of the residue integrations of (1.5) that is necessary for defining the free energies would not go through if Σ_s has singularities. The assumption of simple ramification is imposed here only because of the simplicity of the formulation. We can generalize the framework to arbitrarily ramified coverings in a similar way as developed in [14], although it is restricted to the case when the spectral curve is a compact plane curve.

Remark 2.3. Note that for every 1-form $s_1 \in H^0(C, \Omega^1_C)$, $\eta + \pi^* s_1$ determines the same symplectic form, because

$$-d\eta = -d(\eta + \pi^* s_1). \tag{2.3}$$

The spectral curves are originally considered in the context of Abelianization of the moduli space of stable vector bundles on C in terms of Hitchin integrable systems [8, 29, 58, 59]. Recall that a Higgs pair (E, ϕ) of rank r and degree d consists of a vector bundle E on C of rank r and degree d and a Higgs field $\phi \in H^0\big(C, \mathcal{E}nd(E) \otimes \Omega^1_C\big)$. Stability conditions are appropriately defined so that for the case of $(r, d) = 1$ the moduli space $\mathcal{H}_C(r, d)$ of stable Higgs pairs form a smooth quasi-projective variety of dimension $2(r^2(g-1)+1)$. The space $\mathcal{H}_C(r, d)$ contains the cotangent bundle $T^*\mathcal{U}_C(r, d)$ of the moduli space $\mathcal{U}_C(r, d)$ of stable vector bundles of rank r and degree d on C as an open dense subset. We note that the character variety

$$\mathrm{Hom}\big(\pi_1(C), GL(r, \mathbb{C})\big) /\!\!/ GL(r, \mathbb{C})$$

has the same dimension $2(r^2(g-1)+1)$. We refer to [55, 56, 60] for more detail on the relation between the character variety and the Hitchin moduli spaces.

The **Hitchin fibration**

$$\mu_H : \mathcal{H}(r,d) \ni (E,\phi) \longmapsto \det(y-\phi) = y^r + \sum_{i=1}^{r} (-1)^i \mathrm{trace}(\wedge^i \phi) y^{r-1} \in V_{GL}^* \tag{2.4}$$

induces an algebraically completely integrable Hamiltonian system on $\mathcal{H}_C(r,d)$. A generic Higgs pair (E,ϕ) gives rise to a generic spectral data

$$s = (s_1, s_2, \dots, s_r) = \big((-1)^i \mathrm{trace}(\wedge^i \phi)\big)_{i=1}^r \in V_{GL}^*,$$

and the fiber of the Hitchin fibration μ_H is isomorphic to the Jacobian variety of the spectral curve:

$$\mu_H^{-1}(s) \cong \mathrm{Jac}(\Sigma_s).$$

In particular, the spectral curve has genus

$$\hat{g} = g(\Sigma_s) = r^2(g-1) + 1. \tag{2.5}$$

If we further assume that the projection $\pi : \Sigma_s \longrightarrow C$ is simply ramified, then the ramification divisor $R_s \subset \Sigma_s$ consists of $2r(r-1)(g-1)$ points. This shows that the spectral curves we are dealing with form a very special class of ramified coverings over C of a given degree r. If we were to consider the Givental formalism following [34] or the corresponding Frobenius manifold [30, 31], then for a fixed C, the cardinality of R_s should represent the degrees of freedom of the theory. However, we note that R_s is far from arbitrary as a divisor. Indeed the degrees of freedom of our case is less than the expected value from the Frobenius manifold theory, since

$$\dim V_{GL}^* - \dim \mathrm{Jac}(C) = (r^2-1)(g-1) < (2r^2-2r)(g-1) = \deg R_s$$

for $r \geq 2$. Here we subtract the dimension of $\mathrm{Jac}(C)$ because changing the vector bundle E to $E \otimes \mathcal{L}$ with $\mathcal{L} \in \mathrm{Jac}(C)$ does not change the spectral curve, because the Higgs field ϕ remains the same. As noted in [60], the family of spectral curves is effective only on the space

$$V_{SL}^* := \bigoplus_{i=2}^{r} H^0\big(C, (\Omega_C^1)^i\big), \tag{2.6}$$

which has the dimension $(r^2-1)(g-1)$.

This consideration also corresponds to the following. The application of a *symplectic transformation* $\eta \longmapsto \eta + \frac{1}{r}\pi^* s_1$ changes the characteristic equation

$$\eta^{\otimes r} + \sum_{i=1}^{r} \pi^* s_i \otimes \eta^{\otimes(r-i)} = \left(\eta + \frac{1}{r}\pi^* s_1\right)^{\otimes r} + \sum_{i=2}^{r} \pi^* s_i' \otimes \left(\eta + \frac{1}{r}\pi^* s_1\right)^{\otimes(r-i)}, \tag{2.7}$$

where $s_i' \in H^0\big(C, (\Omega_C^1)^{\otimes i}\big)$ is a polynomial in $s_1, \dots, s_i$ of the homogeneous degree i. Thus without loss of generality we can consider the **traceless** spectral data $s = (s_2, \dots, s_r) \in V_{SL}^*$ for the purpose of dealing with the spectral curve.

To introduce the Eynard-Orantin theory, we need a classical geometric ingredient, the **normalized fundamental differential of the second kind** $B_X(z_1, z_2)$ on a smooth complete algebraic curve X [44, Page 20], [74, Page 3.213]. This is a symmetric differential

2-form on $X \times X$ with second-order poles only along the diagonal. We identify the Jacobian variety of X as $\mathrm{Jac}(X) = Pic^0(X)$, which is isomorphic to $Pic^{g-1}(X)$. The *theta divisor* Θ of $Pic^{g-1}(X)$ is defined by

$$\Theta = \{L \in Pic^{g-1}(X) \mid \dim H^1(X, L) > 0\}.$$

We use the same notation for the translate divisor on $\mathrm{Jac}(X)$, also called the theta divisor. Consider the diagram

$$\begin{array}{ccccc} & & \mathrm{Jac}(X) & & \\ & & \uparrow{\scriptstyle \delta} & & \\ & & X \times X & & \\ & \swarrow^{pr_1} & & \searrow^{pr_2} & \\ X & & & & X, \end{array}$$

where pr_j denotes the projection to the j-th component, and

$$\delta : X \times X \ni (p, q) \longmapsto p - q \in \mathrm{Jac}(X).$$

The *prime form* $E_X(z_1, z_2)$ [44, Page 16] is defined as a holomorphic section

$$E_X(p, q) \in H^0\left(X \times X, pr_1^*(\Omega^1_X)^{-\frac{1}{2}} \otimes pr_2^*(\Omega^1_X)^{-\frac{1}{2}} \otimes \delta^*(\Theta)\right),$$

where we choose Riemann's spin structure (or the Szegö kernel) $(\Omega^1_X)^{\frac{1}{2}}$, which has a unique global section up to the constant multiplication (see [44, Theorem 1.1]). We have

(1) $E_X(p, q)$ vanishes only along the diagonal $\Delta \subset X \times X$, and has simple zeros along Δ.
(2) Let z be a local coordinate on X. Then $dz(p)$ gives the local trivialization of Ω^1_X around p. When q is near at p, $\delta^*(\Theta)$ is also trivialized around $(p, q) \in X \times X$, and we have a local expression

$$E_X\big(z(p), z(q)\big) = \frac{z(p) - z(q)}{\sqrt{dz(p)} \cdot \sqrt{dz(q)}} \left(1 + O\big((z(p) - z(q))^2\big)\right). \tag{2.8}$$

(3) $E_X\big(z(p), z(q)\big) = -E_X\big(z(q), z(p)\big)$.

The fundamental 2-form $B_X(p, q)$ is then defined by

$$B_X(p, q) = d_1 \otimes d_2 \log E_X(p, q) \tag{2.9}$$

(see [44, Page 20], [74, Page 3.213]). We note that $dz(p)$ appears in (2.8) just as the indicator of our choice of the local trivialization. With this local trivialization, we have

$$\begin{aligned} B_X\big(z(p), z(q)\big) &= d_1 \otimes d_2 \log E\big(z(p), z(q)\big) \\ &= \frac{dz(p) \cdot dz(q)}{\big(z(p) - z(q)\big)^2} + O(1)\, dz(p) \cdot dz(q) \\ &\in H^0\left(X \times X, pr_1^*\Omega^1_X \otimes pr_2^*\Omega^1_X \otimes \mathcal{O}(2\Delta)\right). \end{aligned} \tag{2.10}$$

As noted in the literature [44, 74], the local expression (2.10) alone does not uniquely determine the form. Riemann chose a symplectic basis $\langle A_1, \dots, A_g; B_1, \dots, B_g \rangle$ for $H_1(X, \mathbb{Z})$, and normalized the fundamental form by

$$\oint_{A_j} B_X(\,\cdot\,, q) = 0 \tag{2.11}$$

for every A-cycle A_j, $j = 1, \dots, g$. Because of the symmetry $B_X(p, q) = B_X(q, p)$, the A-cycle normalization uniquely determines the fundamental form.

In the theory of complex analysis in one variable, the most fundamental object is the Cauchy integration kernel. Ironically, we do not have a Cauchy kernel on a compact Riemann surface X. The best we can do is the meromorphic 1-form $\omega^{a-b}(z)$ uniquely defined by the following conditions. Let a and b be two distinct points of X.

(1) $\omega^{a-b}(z)$ is holomorphic except for $z = a$ and $z = b$.
(2) $\omega^{a-b}(z)$ has a simple pole of residue 1 at $z = a$.
(3) $\omega^{a-b}(z)$ has a simple pole of residue -1 at $z = b$.
(4) $\omega^{a-b}(z)$ is A-cycle normalized:

$$\oint_{A_j} \omega^{a-b}(z) = 0$$

for every $j = 1, \dots, g$.

The relation between $\omega^{a-b}(z)$ and Riemann's normalized second fundamental form is

$$d_1 \omega^{z_1 - b}(z_2) = B_X(z_1, z_2). \tag{2.12}$$

This equation does not depend on the point $b \in X$.

Now let us go back to our spectral curve

$$\begin{array}{ccc} \iota : \Sigma_s & \longrightarrow & T^*C \\ & & \big\downarrow \pi\,. \\ & & C \end{array} \tag{2.13}$$

In what follows, we concentrate our attention to the case of $r = 2$ traceless spectral data. Thus our spectral curve $\Sigma = \Sigma_s$ is a double sheeted ramified covering of C defined by a characteristic equation

$$\eta^{\otimes 2} + \pi^* s_2 = 0, \tag{2.14}$$

where the spectral data s consists of only one component $s = s_2 \in H^0\big(C, (\Omega^1_C)^{\otimes 2}\big)$, which is a generic quadratic differential on C so that the characteristic equation defines a smooth curve that is simply ramified over C. The genus of the spectral curve, calculated by (2.5), gives $\hat{g} = g(\Sigma_s) = 4g - 3$. The cotangent bundle T^*C has a natural involution

$$\sigma : T^*C \supset T^*_x C \ni (x, y) \longmapsto (x, -y) \in T^*_x C \subset T^*C. \tag{2.15}$$

The spectral curve Σ_s is invariant under σ, and it provides the deck-transformation of the ramified covering $\pi : \Sigma_s \longrightarrow C$.

Let $R_s \subset \Sigma_s$ denote the ramification divisor of this covering. Because of the simple covering assumption, R_s as a point set has $4g-4$ distinct points that are determined by $s_2 = 0$ on C. Since both C and Σ_s are divisors of T^*C, R_s is defined also as $C \cap \Sigma_s$. Note that η vanishes only along $C \subset T^*C$. As a holomorphic 1-form on Σ_s, $\iota^*\eta$ has $2\hat{g}-2 = 8g-8$ zeros on Σ_s. Thus it has a degree 2 zero at each point of R_s.

As mentioned above, the Eynard-Orantin theory requires a *normalized* second fundamental form of Riemann. To normalize differential forms, there are many different choices. Here we use the A-cycle normalization, following Riemann's original idea. The reason for this choice is its extendability to a *family of smooth* spectral curves

$$\widetilde{\Sigma}\big|_V = \{\Sigma_s\}_{s\in V}$$

on a contractible open subset $V \subset H^0\big(C, (\Omega^1_C)^{\otimes 2}\big)$.

To explain our choice of the symplectic basis of the first homology group of the family of spectral curves, let us start with choosing, once and for all, a symplectic basis

$$\langle A_1, \dots, A_g; B_1, \dots, B_g\rangle = H_1(C, \mathbb{Z}).$$

Let us label points of R_s and denote $R_s = \{p_1, p_2, \dots, p_{4g-4}\}$. We can connect p_{2i} and p_{2i+1}, $i = 1, \dots, 2g-3$, with a simple path on Σ_s that is mutually non-intersecting so that $\pi^*(\overline{p_{2i}p_{2i+1}})$, $i = 1, \dots, 2g-3$, form a part of the basis for $H_1(\Sigma_s, \mathbb{Z})$. We denote these cycles by $\alpha_1, \dots, \alpha_{2g-3}$. Since π is locally homeomorphic away from R_s, we have g cycles $a_1, \dots, a_g$ on Σ_s so that $\pi_*(a_j) = A_j$ for $j = 1, \dots, g$, where A_j's are previously chosen A-cycles of C. We define the A-cycles of Σ_s to be the set

$$\{a_1, \dots, a_g, \sigma_*(a_1), \dots, \sigma_*(a_g), \alpha_1, \dots, \alpha_{2g-3}\} \subset H_1(\Sigma_s, \mathbb{Z}). \tag{2.16}$$

Clearly, this set can be extended into a symplectic basis for $H_1(\Sigma_s, \mathbb{Z})$. This choice of the symplectic basis trivializes the homology bundle

$$\big\{H_1(\Sigma_s, \mathbb{Z})\big\}_{s\in V} \longrightarrow V \subset H^0\big(C, (\Omega^1_C)^{\otimes 2}\big)$$

globally on a contractible V.

The monodromy of the choice of the symplectic basis on the family of *all* smooth spectral curves leads us to considering the modular group action on the space of solutions to the Eynard-Orantin theory (1.5). In this paper we stay with the family on a contractible base.

3. The Eynard-Orantin integral recursion on an arbitrary base curve

The construction of the $\hbar$-deformed D-module over an arbitrary complete smooth curve C is carried out in three stages.

(1) Construction of the Eynard-Orantin differentials $W^s_{g,n}$ on Σ^n_s for all $g \geq 0$ and $n \geq 1$ using the geometry of the spectral curve Σ_s.
(2) Construction of the free energies $F^s_{g,n}$, which are meromorphic functions on Σ^n_s for $2g-2+n > 0$, and satisfies that $d_1 \cdots d_n F^s_{g,n} = W^s_{g,n}$.
(3) Construction of the exponential generating function of $F^s_{g,n}$ with $\hbar$ as the expansion parameter, in the way the WKB approximation dictates us to do, and take its

principal specialization. The principal specialization then gives the generator of the $\hbar$-deformed D-module.

Our point of departure is the spectral curve (1.4) defined by the characteristic equation (1.3) for generic values of a spectral data $s = V_{GL}^*$ so that Σ_s is smooth and the covering π is simply ramified along the divisor R_s. Since we do not consider the monodromy transformation and the modular property of the theory under the change of symplectic basis for $H_1(C,\mathbb{Z})$ in the current paper, for simplicity we assume that s belongs to a contractible open subset $V \subset V_{SL}^* = \bigoplus_{i=2}^r H^0\big(C,(\Omega_C^1)^{\otimes i}\big)$ (2.6). What we call the Eynard-Orantin theory in this paper is the following procedure of determining the Eynard-Orantin differentials.

Definition 3.1 (Eynard-Orantin differentials)**.** For every (g,n), $g \geq 0$ and $n \geq 1$, the quantity $W_{g,n}$ defined by one of the following formulas is what we call the **Eynad-Orantin differential** of type (g,n). To avoid extra cumbersome notation, we suppress the s-dependence of the Eynard-Orantin differentials. First, we define a holomorphic 1-form on the spectral curve Σ_s by

$$W_{0,1}(z_1) = \iota^*\eta \in H^0(\Sigma_s, \pi^*\Omega_C^1) \subset H^0(\Sigma_s, \Omega_{\Sigma_s}^1). \tag{3.1}$$

We define a symmetric 2-form $W_{0,2}$ on $\Sigma_s \times \Sigma_s$ using Riemann's normalized second fundamental form by

$$W_{0,2}(z_1,z_2) = B_{\Sigma_s}(z_1,z_2), \tag{3.2}$$

where $(z_1,z_2) \in \Sigma_s \times \Sigma_s$. For this definition we choose once and for all a symplectic basis for $H_1(\Sigma_s,\mathbb{Z})$ that is independent of $s \in V$ and use the A-cycle normalized second fundamental forms of Section 2.

For each $p \in R_s$ we choose a local neighborhood $p \in U_p \subset \Sigma_s$. Since the covering is simple, there is a local Galois conjugation

$$\sigma_p : U_p \longrightarrow U_p, \tag{3.3}$$

which is an involution. We define the **recursion kernel** for each $p \in R_s$ by

$$\begin{aligned} K_p(z,z_1) &= \frac{\int_z^{\sigma_p(z)} B_{\Sigma_s}(\,\cdot\,,z_1)}{\sigma_p^* W_{0,1}(z) - W_{0,1}(z)} \\ &\in H^0\left(U_p \times \Sigma_s, \left((\Omega_{\Sigma_s}^1)^{-1}(2R_s) \boxtimes \Omega_{\Sigma_s}^1\right) \otimes \mathcal{O}_{U_p\times\Sigma_s}(\Delta_s + (\sigma_p \times id)^*\Delta_s)\right), \end{aligned} \tag{3.4}$$

where $\Delta_s \subset \Sigma_s \times \Sigma_s$ is the diagonal. The reciprocal notation means

$$\frac{1}{W_{0,1}(z)} \in H^0\big(\Sigma_s, (\Omega_{\Sigma_s}^1)^{-1} \otimes \mathcal{O}_{\Sigma_s}(2R_s)\big).$$

Using the recursion kernel, we define the first two Eynard-Orantin differentials in the stable range $2g-2+n>0$.

$$W_{1,1}(z_1) = \frac{1}{2}\,\frac{1}{2\pi i} \sum_{p\in R_s} \oint_{\gamma_p} K_p(z,z_1) B_{\Sigma_s}(z,\sigma_p(z)), \tag{3.5}$$

$$W_{0,3}(z_1,z_2,z_3) = \frac{1}{2}\,\frac{1}{2\pi i}\sum_{p\in R_s}\oint_{\gamma_p} K_p(z,z_1) \times \big(B_{\Sigma_s}(z,z_2)B_{\Sigma_s}(\sigma_p(z),z_3) + B_{\Sigma_s}(z,z_3)B_{\Sigma_s}(\sigma_p(z),z_2)\big). \tag{3.6}$$

Here and in what follows, γ_q denotes a positively oriented simple closed loop around a point $q \in \Sigma_s$, and the integration is taken with respect to the variable z along the loop γ_p for each $p \in R_s$. For a general value of (g,n) subject to $2g-2+n \geq 2$, the Eynard-Orantin differential is recursively defined by

$$\begin{aligned} W_{g,n}(z_1,\dots,z_n) = \frac{1}{2}\,\frac{1}{2\pi i}\sum_{p\in R_s}\oint_{\gamma_p} K_p(z,z_1) \\ \times\left[\sum_{j=2}^{n}\Big(W_{0,2}(z,z_j)W_{g,n-1}(\sigma_p(z),z_{[\hat{1},\hat{j}]}) + W_{0,2}(\sigma_p(z),z_j)W_{g,n-1}(z,z_{[\hat{1},\hat{j}]})\Big)\right. \\ \left. + W_{g-1,n+1}\big(z,\sigma_p(z),z_{[\hat{1}]}\big) + \sum_{\substack{g_1+g_2=g \\ I\sqcup J=\{2,\dots,n\}}}^{\text{stable}} W_{g_1,|I|+1}(z,z_I)W_{g_2,|J|+1}(\sigma_p(z),z_J)\right]. \end{aligned} \tag{3.7}$$

Here we use the index convention that $[n] = \{1,\dots,n\}$, the hat notation $[\hat{j}]$ indicates deletion of the index, and for every subset $I \subset [n]$, $z_I = (z_i)_{i\in I}$, and $|I|$ is the cardinality of the subset. The sum in the third line is for indices in the stable range only.

Remark 3.2. $W_{0,1}$ is also known as the Seiberg-Witten differential, when we allow prescribed poles of s on C. In this paper we consider only holomorphic s. Spectral data with poles will be dealt with in a forthcoming paper.

In this definition, we need to clarify the ambiguity of the integration in (3.4). Since Σ_s has genus $r^2(g-1)+1$, the integration from z to $\sigma_p(z)$ of any 1-form is ambiguous. We use a systematic method to avoid this ambiguity. Let us recall the unique A-cycle normalized meromorphic 1-form $\omega_s^{z-b}(z_1)$ on Σ_s. Regardless the point $b \in \Sigma_s$, we have $d_z\omega_s^{z-b}(z_1) = B_{\Sigma_s}(z,z_1)$. Therefore, we *define* the integral to be

$$\int_z^{\sigma_p(z)} B_{\Sigma_s}(\,\cdot\,,z_1) = \omega_s^{\sigma_p(z)-b}(z_1) - \omega_s^{z-b}(z_1) = \omega_s^{\sigma_p(z)-z}(z_1). \tag{3.8}$$

The recursion kernel is now calculated to be

$$K_p(z,z_1) = \frac{\omega_s^{\sigma_p(z)-z}(z_1)}{\sigma_p^*\eta(z) - \eta(z)}. \tag{3.9}$$

From now on we omit the pull-back sign ι^* by the inclusion $\iota : \Sigma_s \longrightarrow T^*C$.

Remark 3.3. The existence of a canonical choice of the integral (3.8) for the family of spectral curves $\widetilde{\Sigma}\big|_V = \{\Sigma_s\}_{s\in V}$ is significant for the existence of the quantum curve, starting from the recursion formula (3.7). Our choice of the trivialization of the homology bundle $\{H_1(\Sigma_s,\mathbb{Z})\}_{s\in V}$ that we have made in the end of Section 2 assures this unique existence.

Remark 3.4. Recently many calculations have been performed to relate the Eynard-Orantin differentials with intersection numbers of certain tautological classes on $\overline{\mathcal{M}}_{g,n}$ [34, 38]. All these calculations assume that the spectral curve is a ramified covering over $\mathbb{C}$, and that the curve itself is just the disjoint union of small disks around each ramification point. The location of these ramification points are arbitrarily chosen to represent the degree of freedom for deformations.

Here we emphasize that the spectral curve Σ_s is a *global* object, and that the ramification divisor R_s on Σ_s is not an arbitrary set of points. We view that the heart of the Eynard-Orantin theory lies in the global structure of the spectral curve, and hence the calculation of the residues appearing in the definition above has to be carried out globally, not locally. In what follows, we perform this very calculation.

The relation between the local and global considerations mentioned above gives us a non-trivial formula of the result of our calculations in terms of tautological intersection numbers on $\overline{\mathcal{M}}_{g,n}$. The identification of this formula is one of the important questions that is not addressed in the current paper.

To actually compute integrals, it is convenient to consider the case when both z and z_1 are close to a ramification point $p \in R_s$, but not quite equal. Then we have local expressions

$$\omega_s^{\sigma_p(z)-z}(z_1) = \left(\frac{1}{z_1-\sigma_p(z)} - \frac{1}{z_1-z} + O(1)\right) dz_1, \tag{3.10}$$

$$B_{\Sigma_s}(z,\sigma_p(z)) = \left(\frac{1}{(z-\sigma_p(z))^2} + O(1)\right) dz d\sigma_p(z), \tag{3.11}$$

$$\eta(z) = h(z)dz. \tag{3.12}$$

We can also choose a small neighborhood of p such that

$$\sigma_p(z) = -z, \tag{3.13}$$

if necessary. In this case $z=0$ is the point $p \in R_s$. We also use formulas

$$\begin{aligned} K_p(z,z_1) &= K_p(\sigma_p(z),z_1) = -K_p(z,\sigma_p(z_1)), \\ B_{\Sigma_s}(z_1,\sigma_p(z_2)) &= B_{\Sigma_s}(\sigma_p(z_1),z_2), \\ h(\sigma_p(z)) &= h(z). \end{aligned} \tag{3.14}$$

Proposition 3.5. *For $2g-2+n>0$, the Eynard-Orantin differential $W_{g,n}(z_1,\dots,z_n)$ is a symmetric meromorphic n-form on Σ_s^n with poles only at $z_i \in R_s$, $i=1,\dots,n$. It satisfies the following balanced average property with respect to the deck transformation:*

$$\sum_{z_i \in \pi^{-1}(x_i)} W_{g,n}(z_1,\dots,z_i,\dots,z_n) = 0, \quad i=1,\dots,n, \quad (g,n) \neq (0,2). \tag{3.15}$$

Here we choose a non-branched point $x_i \in C$, and add $W_{g,n}(z_1,\dots,z_i,\dots,z_n)$ for all r-points $z_i \in \pi^{-1}(x_i)$ on the fiber of x_i. (This is commonly known as the integration along the fiber.)

Proof. Since the assertion of the Proposition is essentially a local statement, we can take an affine covering of the base curve C, and prove the statement on each affine piece. Although the proof is quite involved and requires many steps for an affine curve, the idea and the technique are exactly the same as those in [41]. □

4. The differential recursion for free energies

The global property of the spectral curve we are emphasizing in this paper is that we can actually *integrate and evaluate* the residue calculations appearing in the definition of the Eynard-Orantin differentials. The purpose of this section is to concretely perform this evaluation. We start with giving the definition of free energies. It is worth mentioning that all our calculations are actually performed on the family of spectral curves defined on a contractible base space V as explained in Section 2. Again to avoid cumbersome notations, we suppress the s-dependence in what follows.

Definition 4.1. The **free energy** of type (g, n) is a function $F_{g,n}(z_1, \dots, z_n)$ defined on Σ_s^n subject to the following two conditions:

$$d_1 \cdots d_n F_{g,n}(z_1, \dots, z_n) = W_{g,n}(z_1, \dots, z_n), \tag{4.1}$$

$$\sum_{z_i \in \pi^{-1}(x_i)} F_{g,n}(z_1, \dots, z_i, \dots, z_n) = 0, \quad i = 1, \dots, n, \quad (g, n) \neq (0, 2). \tag{4.2}$$

Here we choose a non-branched point $x_i \in C$, and consider the integration of $F_{g,n}$ along the fiber of x_i with respect to the projection $\pi : \Sigma \longrightarrow C$ applied to the i-th component.

Remark 4.2. The primitive condition (4.1) alone does not determine $F_{g,n}$ due to constants of integration. For example, one can add any function in less than n variables to $F_{g,n}$. It is obvious that the vanishing condition of the integration along the fiber (4.2), reflecting (3.15), uniquely determines the free energies. The authors are indebted to Paul Norbury and Brad Safnuk for the idea of imposing (4.2) to define the unique free energies. In the examples considered in [32], we know $F_{g,n}$ from the beginning because we start with an A-model counting problem that defines the free energies via the Laplace transform. In our current context, since we start with the Eynard-Orantin theory, i.e., from the B-model side, we have no knowledge of what the corresponding A-model is.

Remark 4.3. We exclude the case $(g, n) = (0, 2)$ from the balanced Galois average condition (4.2). How to define $F_{0,2}$ is an extremely subtle matter, and is also related to the heart of the quantizability of the spectral curve Σ_s. We discuss this issue in detail in Section 6. It is important to note that our choice of $F_{0,2}(z, z)$ differs from the definition given in [54].

From now on, we restrict ourselves to the case of degree 2 covering $\pi : \Sigma_s \longrightarrow C$. This restriction is necessary due to several technical reasons. Since the spectral curve Σ_s is a degree 2 covering, we have $R_s = \Sigma_s \cap C \subset T^*C$, and the Galois conjugation σ is global on Σ_s, which is the same as the (-1) involution

$$\sigma : T^*C \longrightarrow T^*C.$$

In particular,

$$\sigma^*\eta = -\eta. \tag{4.3}$$

We denote $\sigma_p = \sigma$, and drop the reference point p from the recursion kernel, because it does not depend on the ramification point any more. The following lemma indicates how we calculate the residues in the integration formulas.

Lemma 4.4. *We calculate*

$$W_{1,1}(z_1) = \frac{B_{\Sigma_s}(z_1, \sigma(z_1))}{2\eta(z_1)} \in H^0\big(\Sigma_s, \Omega^1_{\Sigma_s} \otimes \mathcal{O}_{\Sigma_s}(4R_s)\big). \tag{4.4}$$

Remark 4.5. Since our geometric setting is exactly the same, it is not surprising that the same formula appears in [63], though for a different purpose.

Proof. Taking the advantage of (3.9) and (4.3), let us first identify the poles of the differential form

$$-\frac{\omega_s^{\sigma(z)-z}(z_1)}{2\eta(z)} B_{\Sigma_s}(z, \sigma(z))$$

in z, where $z_1 \in \Sigma_s$ is a point arbitrarily chosen and fixed. We see that $z = p$ for every $p \in R_s$ is a pole, since η vanishes on R_s. The fundamental form $B_{\Sigma_s}(z, z_1)$ has poles only along the diagonal, thus $B_{\Sigma_s}(z, \sigma(z))$ also has poles at R_s. Besides R_s, the form has simple poles at $z = z_1$ and $z = \sigma(z_1)$. Since these are the only poles, and remembering that the integration variable is z, we use the Cauchy integration formula to calculate

$$\begin{aligned} W_{1,1}(z_1) &= \frac{1}{2}\,\frac{1}{2\pi i} \sum_{p \in R_s} \oint_{\gamma_p} K(z, z_1) B_{\Sigma_s}(z, \sigma(z)) \\ &= \frac{1}{2}\,\frac{1}{2\pi i} \oint_{\gamma_{z_1} \cup \gamma_{\sigma(z_1)}} \frac{\omega_s^{\sigma(z)-z}(z_1)}{2\eta(z)} B_{\Sigma_s}(z, \sigma(z)) \\ &= \frac{1}{2}\left(-\frac{B_{\Sigma_s}(z_1, \sigma(z_1))}{2\eta(\sigma(z_1))} + \frac{B_{\Sigma_s}(z_1, \sigma(z_1))}{2\eta(z_1)} \right) \\ &= \frac{B_{\Sigma_s}(z_1, \sigma(z_1))}{2\eta(z_1)}. \end{aligned}$$

It is important to note that $W_{1,1}(z_1)$ has poles only at the ramification divisor R_s. □

It is clear from the above example that integration against $\omega_s^{\sigma(z)-z}(z_1)$ is exactly the Cauchy integration formula. Similarly, integration against $B_{\Sigma_s}(z_1, z_2)$ is the differentiation. Let $f(z_1)$ be a meromorphic function on Σ_s. Then we have

$$\frac{1}{2\pi i} \oint_{\gamma_{z_2}} f(z_1) B_{\Sigma_s}(z_1, z_2) = d_2 f(z_2), \tag{4.5}$$

where the integration is taken with respect to the variable z_1. We note that the result is a meromorphic 1-form on Σ_s.

Lemma 4.6. *We have*

$$W_{0,3}(z_1,z_2,z_3) = \frac{1}{2\eta(z_1)}\Big(B_{\Sigma_s}(z_1,z_2)B_{\Sigma_s}(z_1,\sigma(z_3)) + B_{\Sigma_s}(z_1,z_3)B_{\Sigma_s}(z_1,\sigma(z_2))\Big) + d_2\left(\frac{\omega_s^{\sigma(z_2)-z_2}(z_1)B_{\Sigma_s}(z_2,\sigma(z_3))}{2\eta(z_2)}\right) + d_3\left(\frac{\omega_s^{\sigma(z_3)-z_3}(z_1)B_{\Sigma_s}(z_2,\sigma(z_3))}{2\eta(z_3)}\right). \tag{4.6}$$

Proof. This time the change of contour $\sqcup_{p\in R_s}\gamma_p$ to other poles picks up contributions from $z = z_i$ and $z = \sigma(z_i)$ for $i = 1, 2, 3$. As in the previous case, the contributions from $z = z_i$ and $z = \sigma(z_i)$ are always exactly the same, which are compensated by the overall factor $1/2$. Then the calculations are performed at each pole. For simple poles we use the Cauchy integration formula with respect to $\omega_s^{\sigma(z)-z}(z_1)$, which produces the first line of (4.6). The second line comes from the double poles of the Riemann fundamental form, as explained in (4.5). □

In terms of the local coordinate z of (3.10)-(3.13), we can approximate that $h(z) = z^2$. Then we have

$$W_{0,3}(z_1,z_2,z_3) = -\frac{dz_1dz_2dz_3}{z_1^2z_2^2z_3^2} + O(1)dz_1dz_2dz_3.$$

It is surprising that $W_{0,3}(z_1,z_2,z_3)$ has poles only at $z_i = p \in R_s$ for $i = 1, 2, 3$, and not along any diagonals.

Theorem 4.7. *For $2g - 2 + n \geq 2$, the free energies satisfy the following differential recursion formula:*

$$\begin{aligned} d_1F_{g,n}(z_1,\dots,z_n) &= -\sum_{j=2}^{n}\left[\frac{\omega_s^{z_j-\sigma(z_j)}(z_1)}{2\eta(z_1)}\cdot d_1F_{g,n-1}\big(z_{[\hat{j}]}\big) - \frac{\omega_s^{z_j-\sigma(z_j)}(z_1)}{2\eta(z_j)}\cdot d_jF_{g,n-1}\big(z_{[\hat{1}]}\big)\right] \\ &- \frac{1}{2\eta(z_1)}d_{u_1}d_{u_2}\left[F_{g-1,n+1}\big(u_1,u_2,z_{[\hat{1}]}\big) + \sum_{\substack{g_1+g_2=g\\ I\sqcup J=[\hat{1}]}}^{stable} F_{g_1,|I|+1}(u_1,z_I)F_{g_2,|J|+1}(u_2,z_J)\right]\Bigg|_{\substack{u_1=z_1\\ u_2=z_1}}. \end{aligned} \tag{4.7}$$

Remark 4.8. It has to be emphasized that (4.7) is given in terms of the exterior differentiation and contraction operations so that the equation is indeed coordinate independent. The labels $z_1, \dots, z_n$ are simply indicating which factor of the product Σ_s^n the operation is taking place. They are not a *coordinate* of the spectral curve.

Remark 4.9. Although we do not specify the $s \in V$ dependence of $F_{g,n}$ in the formula, (4.7) holds for the family of functions $\{F_{g,n}^s\}_{s\in V}$.

Proof. We wish to derive (3.7) from (4.7). We first recall the basic relations

$$d_z\omega_s^{z-b}(z_1) = B_{\Sigma_s}(z,z_1) \qquad \text{and} \qquad \omega_s^{z-b}(z_1) + \omega_s^{b-a}(z_1) = \omega_s^{z-a}(z_1).$$

Next let us apply the differentiation $d_2 \cdots d_n$ everywhere in (4.7). The result is

$$\begin{aligned}(4.8)\quad & W_{g,n}(z_1,\dots,z_n) \\ &= -\sum_{j=2}^{n}\left[\frac{1}{2\eta(z_1)}\Big(W_{0,2}(z_1,z_j)-W_{0,2}\big(z_1,\sigma(z_j)\big)\Big)W_{g,n-1}\big(z_{[\hat{j}]}\big)\right] \\ &\qquad -\sum_{j=2}^{n} d_j\left[\frac{1}{2\eta(z_j)}\omega_s^{\sigma(z_j)-z_j}W_{g,n-1}\big(z_{[\hat{1}]}\big)\right] \\ &\quad -\frac{1}{2\eta(z_1)}\left[W_{g-1,n+1}\big(u_1,u_2,z_{[\hat{1}]}\big)\Big|_{\substack{u_1=z_1\\u_2=z_1}} + \sum_{\substack{g_1+g_2=g\\ I\sqcup J=[\hat{1}]}}^{\text{stable}} W_{g_1,|I|+1}(z_1,z_I)W_{g_2,|J|+1}(z_1,z_J)\right].\end{aligned}$$

It is time to evaluate the residue integration in (3.7) for $2g-2+n>1$. First we change the integration contour from $\sum_{p\in R_s}\oint_{\gamma_p}$ to the diagonals $z=z_j$ and $z=\sigma(z_j)$ for $j=1,2,\dots,n$. We can do this, because of Proposition 3.5, we know that $W_{g,n}$ has poles only at R_s for $2g-2+n>0$. As noted in the example calculations Lemma 4.4 and Lemma 4.6 above, the residue contributions from $z=z_i$ and $z=\sigma(z_i)$ are always the same, and are compensated by the overall factor of $1/2$ in the formula. Thus we have

$$\begin{aligned}(4.9)\quad & W_{g,n}(z_1,\dots,z_n) = \frac{1}{2\pi\sqrt{-1}}\sum_{i=1}^{n}\oint_{\gamma_{z_i}}\frac{\omega_s^{\sigma(z)-z}(z_1)}{2\eta(z)} \\ &\quad\times\left[\sum_{j=2}^{n}\Big(W_{0,2}(z,z_j)W_{g,n-1}(\sigma(z),z_{[\hat{1},\hat{j}]})+W_{0,2}(\sigma(z),z_j)W_{g,n-1}(z,z_{[\hat{1},\hat{j}]})\Big)\right. \\ &\qquad \left. + W_{g-1,n+1}\big(z,\sigma(z),z_{[\hat{1}]}\big) + \sum_{\substack{g_1+g_2=g\\ I\sqcup J=\{2,\dots,n\}}}^{\text{stable}} W_{g_1,|I|+1}(z,z_I)W_{g_2,|J|+1}(\sigma(z),z_J)\right].\end{aligned}$$

The contribution from the integration around $z=z_1$ comes from the simple pole of the differential form $\omega_s^{\sigma(z)-z}(z_1)$. The integration is done by the Cauchy integration formula, and the result is

$$\begin{aligned}& -\frac{1}{2\eta(z_1)}\sum_{j=2}^{n}\Big(W_{0,2}(z_1,z_j)-W_{0,2}\big(z_1,\sigma(z_j)\big)\Big)W_{g,n-1}(z_{[\hat{j}]}) \\ &\quad -\frac{1}{2\eta(z_1)}\left[W_{g-1,n+1}\big(z_1,z_1,z_{[\hat{1}]}\big)+\sum_{\substack{g_1+g_2=g\\ I\sqcup J=\{2,\dots,n\}}}^{\text{stable}} W_{g_1,|I|+1}(z_1,z_I)W_{g_2,|J|+1}\big(\sigma(z_1),z_J\big)\right].\end{aligned}$$

Here we have used (3.15). We have thus recovered the first and the third lines of the right-hand side of (4.8).

The contribution in (4.9) from the integration around $z = z_j$, $j \geq 2$, comes from the diagonal double poles of $W_{0,2}(z, z_j)$. Since $W_{0,2} = B_{\Sigma_s}$ acts as the differentiation kernel (4.5), it is easy to see that the result is exactly the same as the second line of the right-hand side of (4.8). This completes the proof. □

5. The λ-connections and the WKB method

The precise notion we need to describe our quantum curve is Deligne's λ-connection, where λ is a formal parameter. In physics the notation $\lambda = \hbar$ is commonly used. Since the literature on quantum curves consistently use the Planck constant notation, we adopt it here as well. In this section we review the materials on λ-connections that we need in this paper, following the excellent article of Arinkin [7]. In what follows, when we say an $\hbar$-connection, we are indeed referring to a λ-connection with $\lambda = \hbar$. The most important feature of the $\hbar$-connections is that the WKB approximation method can be applied to this type of connections.

Definition 5.1 ($\hbar$-Connection)**.** Let (E, ϕ) be a Higgs pair defined on C. An **$\hbar$-connection** on E associated with the pair (E, ϕ) is a $\mathbb{C}$-linear homomorphism

$$\nabla^{\hbar} : E \longrightarrow E \otimes \Omega^1_C$$

subject to the following two conditions:

$$\nabla^{\hbar}(f \cdot v) = f \cdot \nabla^{\hbar}(v) + v \otimes (\hbar\, df) \tag{5.1}$$

for $f \in \mathcal{O}_C$ and $v \in E$, and

$$\phi = \nabla^{\hbar}\big|_{\hbar=0}. \tag{5.2}$$

For every tangent vector $X \in T_xC$ at $x \in C$, the $\mathbb{C}$-linear $\hbar$-covariant derivative

$$\nabla^{\hbar}_X : E \longrightarrow E$$

is defined by the derivation equation

$$\nabla^{\hbar}_X(f \cdot v) = f \cdot \nabla^{\hbar}_X(v) + \hbar X(f) \cdot v. \tag{5.3}$$

If $\hbar \neq 0$, then $\frac{1}{\hbar}\nabla^{\hbar}$ is a holomorphic connection in E. Hence E is flat, and it necessarily has $\deg(E) = 0$.

We consider the variable $\hbar$ as a deformation parameter. First we extend the base curve C to a formal family

$$C[[\hbar]] := \varinjlim_n C \times \mathrm{Spec}\left(\mathbb{C}[\hbar]/(\hbar^n)\right). \tag{5.4}$$

A $\mathbb{C}[[\hbar]]$-linear $\hbar$-connection on a vector bundle E over $C[[\hbar]]$ is defined in the same way as above. As a flat connection on a vector bundle makes the bundle a D-module, an $\hbar$-connection on $C[[\hbar]]$ gives E a D-module structure. Since we do not consider differentiations with respect to $\hbar$, we call a vector bundle with a $\mathbb{C}[[\hbar]]$-linear $\hbar$-connection a **$D^{\hbar}$-module**.

A D-module on a complex manifold M gives rise to a characteristic variety in T^*M. When the D-module is holonomic, the characteristic variety becomes a Lagrangian in T^*M. For our case, any D-module over a complete algebraic curve C is holonomic, and

defines a Lagrangian subvariety in T^*C. These Lagrangians are either the 0-section of the cotangent bundle T^*C, or a union of finite number of fibers. They satisfy the $\mathbb{C}^*$-invariance with respect to the $\mathbb{C}^*$-action on T^*C. The spectral curves we consider (1.4) are not those Lagrangians as the characteristic variety of a D-module. They do not satisfy the $\mathbb{C}^*$-invariance.

The sheaf of $\hbar$-differential operators $\mathcal{D}^\hbar$ on $C[[\hbar]]$ is constructed by gluing

$$\mathcal{D}^\hbar\big|_{U[[\hbar]]} = \mathcal{O}_{U[[\hbar]]}\left[\hbar\frac{d}{dx}\right], \tag{5.5}$$

where x is a coordinate of an affine open subscheme U of C. The **classical limit** of a $D^\hbar$-module is the mod $\hbar$-reduction, which simply is an $\mathcal{O}_C$-module. The passage between the spectral curves of Hitchin fibrations and D-modules is not the classical limit, or the characteristic variety. It is the semi-classical limit, and it requires the WKB method (see for example, [10]) to define.

Let $(E,\nabla^\hbar)$ be a $\mathbb{C}[[\hbar]]$-linear $\hbar$-connection on a vector bundle E over $C[[\hbar]]$. As a $D^\hbar$-module, it is easy to show that on an affine open $U\subset C$ we have a differential operator $P(x,\hbar)\in \mathcal{D}^\hbar\big|_{U[[\hbar]]}$ such that

$$E|_{U[[\hbar]]} \cong \left(\mathcal{D}^\hbar/\mathcal{D}^\hbar P\right)\Big|_{U[[\hbar]]}. \tag{5.6}$$

Usually we consider a solution of

$$P(x,\hbar)\Psi(x,\hbar)=0 \tag{5.7}$$

as an element

$$\Psi(x,\hbar)\in \mathrm{Hom}\big(E_{U[[\hbar]]},\mathcal{O}_{U[[\hbar]]}\big).$$

The WKB method is a mechanism to construct the solution of (5.7) that *does not* have a convergent limit as $\hbar\to 0$, by the *singular perturbation method*

$$\Psi(x,\hbar)=\exp\left(\sum_{m=0}^{\infty}\hbar^{m-1}S_m(x)\right). \tag{5.8}$$

Here $S_m(x)$ is a holomorphic function defined on an open subset $U\subset C$, but has poles at certain points of C. The parameter $\hbar$ is considered to be small, so the $m=0$ contribution is *singular*. The equation (5.7) is interpreted as

$$\left(e^{-\frac{1}{\hbar}S_0(x)}P(x,\hbar)e^{\frac{1}{\hbar}S_0(x)}\right)\exp\left(\sum_{m=1}^{\infty}\hbar^{m-1}S_m(x)\right)=0. \tag{5.9}$$

Since

$$P(x,\hbar)\in\mathcal{O}_{U[[\hbar]]}\left[\hbar\frac{d}{dx}\right],$$

both the operator and the solution of (5.9) are defined over $U[[\hbar]]$.

Definition 5.2. Consider an operator $P(x,\hbar)$ defined on an open subset $U \subset C$ that is in the normal ordering expression

$$P(x,\hbar) = \sum_{k=0}^{n} a_k(x,\hbar)\left(\hbar\frac{d}{dx}\right)^{n-k}, \tag{5.10}$$

where $a_k(x,\hbar) \in \mathcal{O}_{U[[\hbar]]}$. Then we have

$$e^{-\frac{1}{\hbar}S_0(x)}P(x,\hbar)e^{\frac{1}{\hbar}S_0(x)}\Big|_{\hbar=0} = \sum_{k=0}^{n} a_k(x,0)\big(S_0'(x)\big)^{n-k}, \tag{5.11}$$

where $'$ indicates the x-derivative. The **semi-classical limit** of the differential equation (5.7) at $\hbar = 0$ is the formula (5.11). If we use an indeterminate $y = S_0'(x)$, then the semi-classical limit is the mod $\hbar$-reduction

$$\sum_{k=0}^{n} a_k(x,0)y^{n-k} \tag{5.12}$$

of the **total symbol** of the normal ordered operator (5.10).

Note that the semi-classical limit (5.12) is neither the *symbol* nor the *characteristic variety* of the operator $P(x,\hbar)$. The passage from (5.12) to (5.10) is the **quantization** we are discussing in this paper. In an abstract setting, of course there is no way determining a differential operator from its total symbol (5.12) at $\hbar = 0$. In the next section we show that a $SL(2,\mathbb{C})$-Hitchin spectral curve has a *unique* quantization.

6. The WKB approximation and quantum curves

We are now ready to state and prove the main theorem of this paper.

Theorem 6.1. *Let $\mathcal{H}_C(2,0)_0$ denote the modui stack of rank 2 Higgs pairs of degree 0 vector bundles with a fixed determinant line bundle, and consider the $SL(2,\mathbb{C})$-Hitchin fibration*

$$\mu_H : \mathcal{H}_C(2,0)_0 \longrightarrow V_{SL}^* := H^0\big(C, (\Omega_C^1)^{\otimes 2}\big). \tag{6.1}$$

*For a generic spectral data $s \in V_{SL}^*$, there is a contractible open neighborhood $s \in V \subset V_{SL}^*$ such that the **family** of smooth spectral curves*

$$\widetilde{\Sigma}\big|_V = \{\Sigma_s\}_{s\in V}$$

is quantizable via the WKB method.

Remark 6.2. The most involved technical part of this paper is the reduction of the differential recursion (4.7) into an ordinary differential equation via the *principal specialization*

$$z_1 = z_2 = \cdots = z_n = z. \tag{6.2}$$

We note that for the case of simple and double Hurwitz numbers and related topics discussed in [15, 71, 72, 82], the principal specialization corresponds to the reduction of a summation over all Young diagrams (or partitions) into a sum over 1-row Young diagrams.

Thus the formulas dramatically simplify, and this is the key to constructing the quantum curves. For the case of Hitchin fibrations we do not have an interpretation as a sum over partitions, and the process of principal specialization becomes technically more difficult.

Remark 6.3. The $s \in V$ dependence does not pose any difficulty, because the only consideration we need is the consistent integration we have taken care of in Section 4 for the choice of the subset V with a consistent symplectic basis for $H_1(\Sigma_s, \mathbb{Z})$. The calculations in this section are thus all carried out over this family.

We first recall a trivial lemma from [72]:

Lemma 6.4. *Let $f(z_1, \dots, z_n)$ be a symmetric function in n variables. Then*

$$\begin{aligned} \frac{d}{dz} f(z, z, \dots, z) &= n \left[\frac{\partial}{\partial u} f(u, z, \dots, z) \right] \Big|_{u=z} ; \\ \frac{d^2}{dz^2} f(z, z, \dots, z) &= n \left[\frac{\partial^2}{\partial u^2} f(u, z, \dots, z) \right] \Big|_{u=z} \\ &\quad + n(n-1) \left[\frac{\partial^2}{\partial u_1 \partial u_2} f(u_1, u_2, z, \dots, z) \right] \Big|_{u_1=u_2=z} . \end{aligned} \tag{6.3}$$

For a function in one variable $f(z)$, we have

$$\lim_{z_2 \to z_1} \left[\omega^{z_2 - b}(z_1) \big(f(z_1) - f(z_2) \big) \right] = d_1 f(z_1), \tag{6.4}$$

where $\omega^{z_2-b}(z_1)$ is the 1-form of (2.12).

The rest of the section is devoted to proving Theorem 6.1.

Proof of Theorem 6.1. For the purpose of calculation, let us choose one of the ramification points $p \in R$ of the covering $\pi : \Sigma_s \longrightarrow C$ for a generic spectral data $s = s_2 \in H^0\big(C, (\Omega^1_C)^{\otimes 2}\big)$, and assume that all points $z_1, \dots, z_n$ are close to p, but not quite equal. As a consequence, their Galois conjugates $\sigma(z_j)$'s are also close to p. On a neighborhood we choose a local coordinate z around p such that $z = 0$ defines p and that $\sigma(z) = -z$. We use the local expressions (3.10), (3.11), (3.12), and the relations (3.14). Using the notation $\partial_z = \partial/\partial z$, we have a local formula equivalent to (4.7) that is valid for $2g - 2 + n \geq 2$:

$$\begin{aligned} &\partial_{z_1} F_{g,n}(z_1, \dots, z_n) \\ &\quad = -\sum_{j=2}^{n} \left[\frac{\omega_s^{z_j - \sigma(z_j)}(z_1)}{2h(z_1) dz_1} \cdot \partial_{z_1} F_{g,n-1}\big(z_{[\hat{j}]}\big) - \frac{\omega_s^{z_j - \sigma(z_j)}(z_1)}{dz_1 \cdot 2h(z_j)} \cdot \partial_{z_j} F_{g,n-1}\big(z_{[\hat{1}]}\big) \right] \\ &- \frac{1}{2h(z_1)} \frac{\partial^2}{\partial u_1 \partial u_2} \left[F_{g-1,n+1}\big(u_1, u_2, z_{[\hat{1}]}\big) + \sum_{\substack{g_1+g_2=g \\ I \sqcup J = [\hat{1}]}}^{\text{stable}} F_{g_1,|I|+1}(u_1, z_I) F_{g_2,|J|+1}(u_2, z_J) \right] \Bigg|_{\substack{u_1=z_1 \\ u_2=z_1}} . \end{aligned} \tag{6.5}$$

Let us apply (6.2). The left-hand side becomes $\frac{1}{n}\partial_z F_{g,n}(z,\dots,z)$. To calculate the contributions from the first line of the right-hand side of (6.5), we choose $j>1$ and set $z_i=z$ for all i except for $i=1,j$. Then take the limit $z_j\to z_1$. In this procedure, we note that the contributions from the simple pole of $\omega_s^{z_j-\sigma(z_j)}(z_1)$ at $z_1=\sigma(z_j)$ cancel at $z_1=z_j$. Thus we obtain

$$\begin{aligned}
-\sum_{j=2}^{n}\frac{1}{z_1-z_j}\left(\frac{1}{2h(z_1)}\partial_{z_1}F_{g,n-1}(z_1,z,\dots,z)-\frac{1}{2h(z_j)}\partial_{z_j}F_{g,n-1}(z_j,z,\dots,z)\right)\Bigg|_{z_1=z_j}&\\
=-\sum_{j=2}^{n}\partial_{z_1}\left(\frac{1}{2h(z_1)}\partial_{z_1}F_{g,n-1}(z_1,z,\dots,z)\right)&\\
=-(n-1)\partial_{z_1}\left(\frac{1}{2h(z_1)}\partial_{z_1}F_{g,n-1}(z_1,z,\dots,z)\right)&\\
=-(n-1)\partial_{z_1}\left(\frac{1}{2h(z_1)}\right)\partial_{z_1}F_{g,n-1}(z_1,z,\dots,z)-\frac{n-1}{2h(z_1)}\partial_{z_1}^2F_{g,n-1}(z_1,z,\dots,z).&
\end{aligned}$$

The limit $z_1\to z$ then produces

$$\begin{aligned}
(6.6)\quad -\partial_z\frac{1}{2h(z)}\cdot\partial_z F_{g,n-1}(z,\dots,z)-\frac{1}{2h(z)}\partial_z^2F_{g,n-1}(z\dots,z)&\\
+\frac{(n-1)(n-2)}{2h(z)}\frac{\partial^2}{\partial u_1\partial u_2}F_{g,n-1}(u_1,u_2,z\dots,z)\Big|_{u_1=u_2=z}&.
\end{aligned}$$

To calculate the principal specialization of the second line of the right-hand side of (6.5), we note that since all points z_i's for $i\geq 2$ are set to be equal, a set partition by index sets I and J becomes a partition of $n-1$ with a combinatorial factor that counts the redundancy. The result is

$$\begin{aligned}
(6.7)\quad -\frac{1}{2h(z)}\frac{\partial^2}{\partial u_1\partial u_2}F_{g-1,n+1}(u_1,u_2,z\dots,z)\Big|_{u_1=u_2=z}&\\
-\frac{1}{2h(z)}\sum_{\substack{g_1+g_2=g\\ n_1+n_2=n-1}}^{\text{stable}}\partial_z F_{g_1,n_1+1}(z,\dots,z)\cdot\partial_z F_{g_2,n_2+1}(z,\dots,z).&
\end{aligned}$$

Assembling (6.6) and (6.7) together, we obtain

$$\begin{aligned}
(6.8)\quad \frac{1}{2h(z)}\left[\partial_z^2F_{g,n-1}(z\dots,z)+\sum_{\substack{g_1+g_2=g\\ n_1+n_2=n-1}}^{\text{stable}}\partial_z F_{g_1,n_1+1}(z,\dots,z)\cdot\partial_z F_{g_2,n_2+1}(z,\dots,z)\right]&\\
+\frac{1}{n}\partial_z F_{g,n}(z,\dots,z)+\partial_z\frac{1}{2h(z)}\cdot\partial_z F_{g,n-1}(z,\dots,z)&
\end{aligned}$$

$$= \frac{(n-1)(n-2)}{2h(z)} \frac{\partial^2}{\partial u_1 \partial u_2} F_{g,n-1}(u_1, u_2, z \ldots, z)\Big|_{u_1=u_2=z} - \frac{1}{2h(z)} \frac{\partial^2}{\partial u_1 \partial u_2} F_{g-1,n+1}(u_1, u_2, z \ldots, z)\Big|_{u_1=u_2=z}.$$

Following the construction of the quantum curves of [54, 72], we now apply the operation $\sum_{2g-2+n=m} \frac{1}{(n-1)!}$ to (6.8) above, and write the result in terms of

$$S_m(z) := \sum_{2g-2+n=m-1} \frac{1}{n!} F_{g,n}(z, \ldots, z), \tag{6.9}$$

to fit into the WKB formalism. For $m \geq 2$, $S_m(z)$ is a meromorphic function on Σ_s with a pole at each ramification point (Lagrangian singularity) $p \in R_s$ of order $3m-3$. This can be easily seen by the fact that $F_{g,n}(z, \ldots, z)$ has a pole of order $6g-6+3n$ at each $p \in R_s$. And this fact follows by induction from the integral recursion (4.7) on $F_{g,n}$, and the initial conditions (3.5) and (3.6).

Our first remark is that summing over all possibilities of (g, n) with the fixed value of $2g-2+n$, the right-hand side of (6.8) becomes 0. Thus we have established

Theorem 6.5. *The functions $S_m(z)$ of* (6.9) *for $m \geq 2$ satisfy the recursion formula*

$$\frac{1}{2h(z)} \left(\frac{d^2 S_m}{dz^2} + \sum_{\substack{a+b=m+1 \\ a,b \geq 2}} \frac{dS_a}{dz} \frac{dS_b}{dz} \right) + \frac{dS_{m+1}}{dz} + \frac{d}{dz} \left(\frac{1}{2h(z)} \right) \frac{dS_m}{dz} = 0. \tag{6.10}$$

It can also be written as a coordinate-free manner as an equation for meromorphic 1-forms on Σ_s:

$$dS_{m+1} + \frac{1}{2\eta} \sum_{\substack{a+b=m+1 \\ a,b \geq 2}} dS_a \cdot dS_b + d\left(\frac{1}{2\eta} dS_m \right) = 0, \tag{6.11}$$

where $1/\eta$ is again the contraction operator with respect to the 1-form η.

Recall the local geometry of the spectral curve

$$p \in \Sigma_s \subset T^*C,$$

and that $p \in C$ is also on the 0-section of the cotangent bundle T^*C. We trivialize the cotangent bundle near $x = p$, where x is a local coordinate on C, and let y be the fiber coordinate of $T^*_x C$. The relation between $(x, y) \in T^*C$ and the local coordinate z of Σ_s around $p \in \Sigma_s$ is given by the formula

$$\eta = h(z)dz = ydx. \tag{6.12}$$

Let the local expression of the spectral data $s = s_2$ be $s_2 = s_2(x)(dx)^2$. Then the equation for the spectral curve Σ_s near $p \in \Sigma_s$ is given by

$$y^2 + s_2(x) = 0. \tag{6.13}$$

The local expression of the quantum curve, which is an $\hbar$-differential operator, becomes

$$P(x,\hbar) := \hbar^2 \left(\frac{d}{dx}\right)^2 + s_2(x). \tag{6.14}$$

Following the method of Bergère-Eynard [11] and the WKB formalism of Gukov-Sułkowski [54], we define

$$F(z,\hbar) = \sum_{m=0}^{\infty} \hbar^{m-1} S_m(z) = \sum_{g\geq 0}\sum_{n\geq 1} \hbar^{2g-2+n} \frac{1}{n!} F_{g,n}(z,\dots,z), \tag{6.15}$$

$$\Psi(z,\hbar) = \exp F(z,\hbar). \tag{6.16}$$

The truncated summation for $m \geq 2$ in (6.15), and the corresponding portion of (6.16), are functions on $C[[\hbar]]$ with essential singularities at each Lagrangian singularity of the spectral curve $\pi : \Sigma_s \longrightarrow C$. The factor $e^{\frac{1}{\hbar}S_0}$ in Ψ plays the role of determining the semi-classical limit, as explained in Section 5.

Using (6.12) we identify the derivation

$$\frac{d}{dx} = \frac{y}{h(z)}\,\frac{d}{dz}, \tag{6.17}$$

which comes from the push-forward $\pi_*(d/dz)$. The transformation (6.17) is singular at every ramification point. The Schrödinger equation is calculated as

$$P(x,\hbar)\Psi(z,\hbar) = 0 \tag{6.18}$$

$$\iff \quad \hbar^2 \left(\frac{d^2F}{dx^2} + \frac{dF}{dx}\cdot\frac{dF}{dx}\right) + s_2(x) = 0 \tag{6.19}$$

$$\iff \quad \sum_{m=0}^{\infty} \hbar^{m+1}\frac{d^2S_m}{dx^2} + \sum_{a,b\geq 0} \hbar^{a+b}\frac{dS_a}{dx}\cdot\frac{dS_b}{dx} + s_2(x) = 0. \tag{6.20}$$

Collecting the coefficient of the $\hbar^0$ terms in (6.20), we obtain the semi-classical limit

$$\left(\frac{dS_0}{dx}\right)^2 + s_2(x) = 0. \tag{6.21}$$

From (6.13) and (6.21) we conclude that

$$\frac{dS_0}{dx} = y = \sqrt{-s_2(x)}. \tag{6.22}$$

This is consistent with our choice of $W_{0,1}$ of the Eynard-Orantin theory (3.1):

$$dS_0 = dF_{0,1} = W_{0,1} = \eta = y dx.$$

Moreover, if we allow terms $a = 0$ or $b = 0$ in (6.10), then what we have in addition is

$$\frac{1}{2h(z)} 2\frac{dS_0}{dz}\frac{dS_{m+1}}{dz} = \frac{1}{h(z)}\frac{h(z)}{y}\frac{dS_0}{dx}\frac{dS_{m+1}}{dz} = \frac{dS_{m+1}}{dz}.$$

In other words, the $\frac{dS_{m+1}}{dz}$ term already there in (6.10) is absorbed in the split differentiation for $a = 0$ and $b = 0$.

Here we comment that $S_0 = \int \eta$ is not a function on Σ_s. Since η is a holomorphic 1-form on Σ_s, its integral is defined only on the universal covering of Σ_s. From (6.21), we calculate the conjugated operator (5.9)

$$e^{-\frac{1}{\hbar}S_0} P(x,\hbar) e^{\frac{1}{\hbar}S_0} = \hbar^2 \frac{d^2}{dx^2} + 2\hbar \frac{dS_0}{dx}\frac{d}{dx} + \hbar \frac{d^2 S_0}{dx^2}. \tag{6.23}$$

The $\hbar^1$ terms of (6.20) give what we call the **consistency condition**

$$\frac{d^2 S_0}{dx^2} + 2\frac{dS_0}{dx} \cdot \frac{dS_1}{dx} = 0, \tag{6.24}$$

which also follows from (6.23). We recall that until now we have never *defined* what we want to use as $F_{0,2}(z_1, z_2)$. The defining equation $d_1 d_2 F_{0,2} = W_{0,2}$ alone does not determine $F_{0,2}$ because we can add terms

$$F_{0,2}(z_1, z_2) + f(z_1) + f(z_2)$$

using an arbitrary function $f(z)$. The principal specialization then becomes $F_{0,2}(z,z) + 2f(z)$, which makes

$$S_1 = \frac{1}{2} F_{0,2}(z,z) + f(z).$$

This situation allows us to *define* the quantity S_1 by a solution of the consistency condition (6.24). Thus we define,

$$S_1 = \int^x \frac{dS_1}{dx} dx = -\frac{1}{2} \log \frac{dS_0}{dx}. \tag{6.25}$$

This makes

$$e^{S_1} = \frac{1}{\sqrt{y}}. \tag{6.26}$$

Remark 6.6. We note that the choice we need to make for S_1, the formula given in (6.25), is different from the choice of the *torsion* term of [54].

More importantly for our purpose, we read off from (6.24) that

$$\frac{dS_1}{dx} = -\frac{1}{2} \frac{\frac{d}{dx}\sqrt{-s_2(x)}}{\sqrt{-s_2(x)}}. \tag{6.27}$$

Note that $s_2(x)$ has a simple zero at each branch point $p \in C$. If x is chosen as a local coordinate centered at p, then (6.27) is a meromorphic function with a simple pole at p. The conjugation of (6.23) by e^{S_1} is calculated as

$$e^{-S_1} e^{-\frac{1}{\hbar}S_0} P(x,\hbar) e^{\frac{1}{\hbar}S_0} e^{S_1} = \hbar^2 \frac{d^2}{dx^2} + 2\left(\hbar \frac{dS_1}{dx} + \frac{dS_0}{dx}\right)\hbar \frac{d}{dx} \in \mathcal{D}^{\hbar}(U), \tag{6.28}$$

where $U \subset C$ is an open subset that does not contain any branch point of the covering π.

Finally we have

Lemma 6.7. *The consistency condition* (6.24) *makes* (6.10) *and* (6.20) *equivalent on any open subset $U \subset C$ that is away from the caustics.*

Proof. First we calculate the second differential operator, from (6.17) and (6.22):

$$\frac{d^2}{dx^2} = \frac{d}{dx}\left(\frac{S_0'}{h}\frac{d}{dz}\right) = \frac{(S_0')^2}{h^2}\frac{d^2}{dz^2} + \frac{S_0'}{h}\frac{d}{dz}\left(\frac{S_0'}{h}\right)\cdot\frac{d}{dz},$$

denoting by $S_0' = dS_0/dx$. The $\hbar^{m+1}$-terms of (6.20) then produce

$$\frac{(S_0')^2}{h^2}\left(\frac{d^2}{dz^2}S_m + \sum_{a+b=m+1}\frac{dS_a}{dz}\frac{dS_b}{dz}\right) + \frac{S_0'}{h}\frac{d}{dz}\left(\frac{S_0'}{h}\right)\cdot\frac{dS_m}{dz} = 0. \tag{6.29}$$

The coefficients of dS_m/dz in (6.29) are

$$\begin{aligned} 2\frac{(S_0')^2}{h^2}\frac{dS_1}{dz} + \frac{S_0'}{h}\frac{d}{dz}\left(\frac{S_0'}{h}\right) &= 2\frac{(S_0')^2}{h^2}\frac{h}{S_0'}S_1' + \frac{d}{dx}\left(\frac{S_0'}{h}\right) \\ &= \frac{1}{h}\big(2S_0'S_1' + S_0''\big) + S_0'\frac{d}{dx}\left(\frac{1}{h}\right) = S_0'\frac{d}{dx}\left(\frac{1}{h}\right) \\ &= \frac{(S_0')^2}{h^2}\cdot 2h\frac{d}{dz}\left(\frac{1}{2h}\right). \end{aligned}$$

This is exactly what the last term of (6.10) has, after adjusting the overall coefficient of $\frac{(S_0')^2}{h^2}\cdot 2h$. This complets the proof of Lemma. □

With the above lemma, we have competed the proof of the main theorem. □

Remark 6.8. The Schrödinger equation (6.18) has a holomorphic coefficient $s_2(x)$. Therefore, the solution is also holomorphic. The expression (6.16) is therefore valid only for points away from the caustics. In other words, the WKB method is not valid at the caustics. The local behavior of $\Psi(z,\hbar)$ at every Lagrangian singularity is universal, because $s_2(x)$ has a simple zero at each point $p \in R_s$ of the caustics. Here recall that $R_s = \Sigma_s \cap C$, so R_s is also the branch divisor in C. If we have chosen a local coordinate x of C at $p \in R_s$ so that $x = 0$ gives the point p, then on a small neighborhood of p we have an expression $s_2(x) = -x$. Since the differential equation becomes

$$\left(\hbar^2\frac{d^2}{dx^2} - x\right)\Psi(x,\hbar) = 0,$$

it is obvious that the local solution is given by the Airy function (see for example, [1]). This calculation has been carried out in [7, 11]. The spectral curve in this case is locally $x = y^2$, for which the Eynard-Orantin theory produces the cotangent ψ-class intersection numbers considered by Witten [80] and Kontsevich [62]. See for example, [32], on this connection.

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References

[1] M. Ablowitz and A. Fokas, *Complex variables: Introduction and applications*, Second Edition, Cambridge Texts in Applied Mathematics **35**, Cambridge University Press, 660 pages (2003)

[2] M. Aganagic, R. Dijkgraaf, A. Klemm, M. Mariño, and C. Vafa, *Topological Strings and Integrable Hierarchies*, [arXiv:hep-th/0312085], Commun. Math. Phys. **261**, 451–516 (2006).

[3] M. Aganagic and C. Vafa, *Large N duality, mirror symmetry, and a Q-deformed A-polynomial for knots*, arXiv:1204.4709v4 [physics.hep-th] (2012).

[4] L.F. Alday, D. Gaiotto, and Y. Tachikawa, *Liouville correlation functions from four-dimensional gauge theories*, Lett. Math. Phys. **91**, 167–197 (2010).

[5] A. Alexandrov, A. Mironov and A. Morozov, *Unified description of correlators in non-Gaussian phases of Hermitean matrix model*, arXiv:hep-th/0412099 (2004).

[6] E. Arbarello, M. Cornalba, and P.A. Griffiths, *Geometry of Algebraic Curves*, Volume II, Grundlehren der mathematischen Wissenschaften **268**, Springer-Verlag 2011.

[7] D. Arinkin, *On λ-connections on a curve where λ is a formal parameter*, Mathematical Research Letters **12**, 551–565 (2005).

[8] A. Beauville, M. S. Narasimhan, and S. Ramanan, *Spectral curves and the generalized theta divisor*, J. reine angew. Math. **398**, 169–179 (1989).

[9] A. Beilinson and V. Drinfeld, *Quantization of Hitchin integrable system and Hecke eigensheaves*, Preprint.

[10] C.M. Bender and S.A. Orszag, *Advanced mathematical methods for scientists and engineers: Asymptotic methods and perturbation theory*, vol. **I**, Springer, 593 pages, (1999).

[11] M. Bergère and B. Eynard, *Determinantal formulae and loop equations*, arXiv:0901.3273 [physics.math-ph] (2009).

[12] G. Bonelli, K. Maruyoshi, and A. Tanzini, *Quantum Hitchin systems via beta-deformed matrix models*, arXiv:1104.4016 [hep-th] (2011).

[13] G. Borot and B. Eynard, *All-order asymptotics of hyperbolic knot invariants from non-perturbative topological recursion of A-polynomials*, arXiv:1205.2261v1 [math-ph] (2012).

[14] V. Bouchard and B. Eynard, *Think globally, compute locally*, JHEP **02**, Article: 143 (34 pages), (2013).

[15] V. Bouchard, D. Hernández Serrano, X. Liu, and M. Mulase, *Mirror symmetry for orbifold Hurwitz numbers*, arXiv:1301.4871 [math.AG] (2013).

[16] V. Bouchard, A. Klemm, M. Mariño, and S. Pasquetti, *Remodeling the B-model*, Commun. Math. Phys. **287**, 117–178 (2008).

[17] V. Bouchard and M. Mariño, *Hurwitz numbers, matrix models and enumerative geometry*, Proc. Symposia Pure Math. **78**, 263–283 (2008).

[18] A. Brini, B. Eynard, and M. Mariño, *Torus knots and mirror symmetry* arXiv:1105.2012 (2011).

[19] L. Chen, *Bouchard-Klemm-Marino-Pasquetti Conjecture for* $\mathbb{C}^3$, arXiv:0910.3739 (2009).

[20] L. Chekhov, B. Eynard, and N. Orantin, *Free energy topological expansion for the 2-matrix model*, JHEP 0612:053 (2006).
[21] A. Chernov and D. Talalaev, *Quantum spectral curves, quantum integrable systems and the geometric Langlands correspondence*, arXiv:hep-th/0604128 [hep-th] (2006).
[22] D. Cooper, D.M. Culler, H. Gillet, D. Long, and P. Shalen, *Plane curves associated to character varieties of 3-manifolds*, Invent. Math. **118**, 47–84 (1994).
[23] R. Dijkgraaf, H. Fuji, and M. Manabe, *The volume conjecture, perturbative knot invariants, and recursion relations for topological strings*, arXiv:1010.4542 [hep-th] (2010).
[24] R. Dijkgraaf, L. Hollands, and P. Sułkowski, *Quantum curves and $\mathcal{D}$-modules*, Journal of High Energy Physics **0810.4157**, 1–58 (2009).
[25] R. Dijkgraaf, L. Hollands P. Sułkowski, and C. Vafa, *Supersymmetric gauge theories, intersecting branes and free Fermions*, Journal of High Energy Physics **0802.106**, (2008).
[26] R. Dijkgraaf and C. Vafa, *Two Dimensional Kodaira-Spencer Theory and Three Dimensional Chern-Simons Gravity*, arXiv:0711.1932 [hep-th] (2007).
[27] R. Dijkgraaf, E. Verlinde, and H. Verlinde, *Loop equations and Virasoro constraints in non-perturbative two-dimensional quantum gravity*, Nucl. Phys. **B348**, 435–456 (1991).
[28] N. Do, O. Leigh, and P. Norbury, *Orbifold Hurwitz numbers and Eynard-Orantin invariants*, arXiv:1212.6850 [math.AG (physics.math-ph)] (2012).
[29] R. Donagi and E. Markman, *Spectral covers, algebraically completely integrable Hamiltonian systems, and moduli of bundles*, Lecture Notes in Mathematics **1620**, Springer-Verlag, 1996.
[30] B. Dubrovin, *Geometry of 2d topological field theories*, arxiv:hep-th/9407018 [physics.hep-th] (1994).
[31] B. Dubrovin and Y. Zhang, *Normal forms of hierarchies of integrable PDEs, Frobenius manifolds and Gromov-Witten invariants*, arxiv:math/0108160 [math.DG] (2001).
[32] O. Dumitrescu, M. Mulase, A. Sorkin and B. Safnuk, *The spectral curve of the Eynard-Orantin recursion via the Laplace transform*, in "Algebraic and Geometric Aspects of Integrable Systems and Random Matrices," Dzhamay, Maruno and Pierce, Eds. Contemporary Mathematics **593**, 263–315 (2013).
[33] P. Dunin-Barkowski, M. Kazarian, N. Orantin, S. Shadrin, and L. Spitz, *Polynomiality of Hurwitz numbers, Bouchard-Mario conjecture, and a new proof of the ELSV formula*, arXiv:1307.4729 [math.AG (physics.hep-th physics.math-ph)] (2013).
[34] P. Dunin-Barkowski, N. Orantin, S. Shadrin, and L. Spitz, *Identification of the Givental formula with the spectral curve topological recursion procedure*, arXiv:1211.4021 [physics.math-ph (math.AG, physics.hep-th)] (2012).
[35] T. Ekedahl, S. Lando, M. Shapiro, A. Vainshtein, *Hurwitz numbers and intersections on moduli spaces of curves*, Invent. Math. **146**, 297–327 (2001).
[36] P. Etingof and X. Ma, *Lecture notes on Cherednik algebras*, Preprint.
[37] B. Eynard, *Topological expansion for the 1-hermitian matrix model correlation functions*, arXiv:0407261 [hep-th] (2004).
[38] B. Eynard, *Intersection numbers of spectral curves*, arXiv:1104.0176 (2011).
[39] B. Eynard and M. Mariño, *A holomorphic and background independent partition function for matrix models and topological strings*, J. Geom. Phys. **61**, 1181–1202 (2011).
[40] B. Eynard, M. Mulase and B. Safnuk, *The Laplace transform of the cut-and-join equation and the Bouchard-Mariño conjecture on Hurwitz numbers*, Publications of the Research Institute for Mathematical Sciences **47**, 629–670 (2011).
[41] B. Eynard and N. Orantin, *Invariants of algebraic curves and topological expansion*, Communications in Number Theory and Physics **1**, 347–452 (2007).
[42] B. Eynard and N. Orantin, *Computation of open Gromov-Witten invariants for toric Calabi-Yau 3-folds by topological recursion, a proof of the BKMP conjecture*, arXiv:1205.1103v2 [math-ph] (2013).
[43] B. Fang, C.-C. M. Liu, and Z. Zong, *All genus open-closed mirror symmetry for affine toric Calabi-Yau 3-orbifolds*, arXiv:1310.4818 [math.AG] (2013).
[44] J. Fay, *Theta functions on Riemann surfaces*, Lecture Notes in Mathematics **352**, Springer, 1973.
[45] E. Frenkel, *Lectures on the Langlands program and conformal field theory*, arXiv:hep-th/0512172 (2005).
[46] H. Fuji, S. Gukov, and P. Sułkowski, *Volume conjecture: refined and categorified*, arXiv:1203.2182v1 [hep-th] (2012).
[47] H. Fuji, S. Gukov, and P. Sułkowski, *Super-A-polynomial for knots and BPS states*, arXiv:1205.1515v2 [hep-th] (2012).
[48] D. Gaiotto, $N = 2$ *dualities*, arXiv:0904.2715 [hep-th] (2009).

[49] D. Gaiotto, G.W. Moore, and A. Neitzke, *Wall-crossing, Hitchin systems, and the WKB approximation*, arXiv:0907.3987 [hep-th] (2009).
[50] S. Garoufalidis, *On the characteristic and deformation varieties of a knot*, Geometry & Topology Monographs **7**, 291–309 (2004).
[51] S. Garoufalidis and T.T.Q. Lê, *The colored Jones function is qholonomic*, Geometry and Topology **9**, 1253–1293 (2005).
[52] I.P. Goulden, *A differential operato for symmetric functions and the combinatorics of multiplying transpositions*, Trans. A.M.S., **344**, 421–440 (1994).
[53] I.P. Goulden and D.M. Jackson, *Transitive factorisations into transpositions and holomorphic mappings on the sphere*, Proc. A.M.S., **125**, 51–60 (1997).
[54] S. Gukov and P. Sułkowski, *A-polynomial, B-model, and quantization*, arXiv:1108.0002v1 [hep-th] (2011).
[55] T. Hausel, *Global topology of the Hitchin system*, arXiv:1102.1717v12 [math.AG] (2011).
[56] T. Hausel and M. Thaddeus, *Mirror symmetry, Langlands duality, and the Hitchin system*, Inventiones Math. **153**, 197–229 (2003).
[57] P.W. Higgs, *Broken symmetries and the masses of gauge bosons*, Phys. Rev. Lett. **13**, 508–509 (1964).
[58] N.J. Hitchin, *The self-duality equations on a Riemann surface*, Proc. London Math. Soc. (Ser. 3) **55**, 59–126 (1987).
[59] N.J. Hitchin, *Stable bundles and integrable systems*, Duke Math. J. **54**, 91–114 (1987).
[60] A. Hodge and M. Mulase, *Hitchin integrable systems, deformations of spectral curves, and KP-type equations*, Advanced Studies in Pure Mathematics **59**, 31–77 (2010).
[61] L. Hollands, *Topological strings and quantum curves*, Ph.D. Thesis, University of Amsterdam arXiv:0911.3413 [hep-th] (2009).
[62] M. Kontsevich, *Intersection theory on the moduli space of curves and the matrix Airy function*, Communications in Mathematical Physics **147**, 1–23 (1992).
[63] D. Korotkin and P. Zograf, *Tau function and the Prym class*, arXiv:1302.0577 [math.AG (nlin.SI physics.math-ph)] (2013).
[64] J. Li, C.-C. M. Liu, K. Liu, and J. Zhou, *A mathematical theory of the topological vertex*, Geom. Topol. **13**, 527–621 (2009).
[65] Y. Li and M. Mulase, *Prym varieties and integrable systems*, Commun. Analysis and Geom. **5**, 279–332 (1997).
[66] K. Liu and H. Xu, *Recursion formulae of higher Weil-Petersson volumes*, Int. Math. Res. Notices **5**, 835–859 (2009).
[67] M. Mariño, *Open string amplitudes and large order behavior in topological string theory*, J. High Energy Physics **0803-060**, 1–33 (2008).
[68] M. Mirzakhani, *Simple geodesics and Weil-Petersson volumes of moduli spaces of bordered Riemann surfaces*, Invent. Math. **167**, 179–222 (2007).
[69] M. Mirzakhani, *Weil-Petersson volumes and intersection theory on the moduli space of curves*, J. Amer. Math. Soc. **20**, 1–23 (2007).
[70] M. Mulase and B. Safnuk, *Mirzakhani's Recursion Relations, Virasoro Constraints and the KdV Hierarchy*, Indian Journal of Mathematics **50**, 189–228 (2008).
[71] M. Mulase, S. Shadrin, and L. Spitz, *The spectral curve and the Schrödinger equation of double Hurwitz numbers and higher spin structures*, Communications in Number Theory and Physics **7**, 1–19 (2013).
[72] M. Mulase and P. Sułkowski, *Spectral curves and the Schrödinger equations for the Eynard-Orantin recursion*, arXiv:1210.3006 (2012).
[73] M. Mulase and N. Zhang, *Polynomial recursion formula for linear Hodge integrals*, Communications in Number Theory and Physics **4**, 267–294 (2010).
[74] D. Mumford, *Tata lectures on theta II, Jacobian theta functions and differential equations*, Progress in Mathematics **43**, Birkhäuser 1984.
[75] N.A. Nekrasov, *Seiberg-Witten prepotential from instanton counting*, Adv. Theor. Math. Phys. **7**, 831–864 (2004).
[76] H. Ooguri, P. Sulkowski, M. Yamazaki, *Wall Crossing As Seen By Matrix Models*, arXiv:1005.1293 (2010).
[77] C.T. Simpson, *Higgs bundles and local systems*, Publications Mathématiques de l'I.H.E.S. **75**, 5–95 (1992).
[78] D. Talalaev, *Quantization of the Gaudin System*, arXiv:hep-th/0404153 (2004).
[79] R. Vakil, Harvard Thesis 1997.
[80] E. Witten, *Two dimensional gravity and intersection theory on moduli space*, Surveys in Differential Geometry **1**, 243–310 (1991).

[81] J. Zhou, *Local Mirror Symmetry for One-Legged Topological Vertex*, arXiv:0910.4320 (2009).
[82] J. Zhou, *Quantum Mirror Curves for* $\mathbb{C}^3$ *and the Resolved Conifold*, arXiv:1207.0598v1 [math.AG] (2012).

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