Title
PLASMA KINETIC EQUATIONS

Author
Price, John C.

Publication Date
1965-12-20

UNIVERSITY OF CALIFORNIA

Lawrence Radiation Laboratory
Berkeley, California

AEC Contract No. W-7405-eng-48

PLASMA KINETIC EQUATIONS

John C. Price

(Ph. D. Thesis)

December 20, 1965

UCRL-16631

UNIVERSITY OF CALIFORNIA

Lawrence Radiation Laboratory
Berkeley, California

AEC Contract No. W-7405-eng-48

PLASMA KINETIC EQUATIONS

John C. Price

(Ph. D. Thesis)

December 20, 1965

TABLE OF CONTENTS

Page

PLASMA KINETIC EQUATIONS

John C. Price

Lawrence Radiation Laboratory
University of California
Berkeley, California

December 20, 1965

ABSTRACT

In the major part of this work we derive a kinetic equation for a homogeneous field free plasma. This equation effectively joins the theory for a stable system (the Lenard-Balescu equation) and that for an unstable system (the quasilinear equation), for it contains terms from each equation, plus a term rather similar to that of quasilinear theory, but depending only on the one particle distribution function. The derivation follows the formal methods of Dupree, except that the collision term is evaluated for finite time, rather than in the limit $t \rightarrow \infty$. The behavior of a large number of terms which appear to oscillate in time is obtained by analytic continuation of various integrals in a manner similar to that used by Landau. We show that the kinetic equation satisfies the conservation laws, leads to an H theorem, and correctly reduces to the Lenard-Balescu equation in the asymptotic (long time) limit. We then generalize the equation to include the effect of a uniform magnetic field.

In a different calculation we obtain a collision term valid to order $\frac{1}{\ell n \Lambda}$ for small amplitude waves in a uniform plasma. This result generalizes the ordinary Fokker-Planck equation from the domain $0 \leqslant \omega \ll \omega_p$, $0 \leqslant k \ll k_d$ to the domain $0 \leqslant \omega \ll \Lambda\omega_p$, $0 \leqslant k \lesssim k_d$. We show the collisional correction to the behavior of small amplitude waves also appears in the description of fluctuations in a spatially uniform plasma. This correction applies to the kinetic equation described above, so that the collective effects arising from zeros of the dielectric function are affected by collisional damping as well as by Landau damping.

I. INTRODUCTION

Strictly speaking there can be no difference between the kinetic equation for a stable plasma and that for an unstable plasma. A marginally stable system is not abruptly different from an unstable system in which certain effects tend to grow exponentially (if slowly) in time, nor is it markedly different from a stable system in which the equivalent effects damp exponentially (and possibly slowly) in time. The transition of a given system from instability to stability is a smooth one, and this fact must be reflected by an adequate kinetic equation.

The major part of this work is devoted to the derivation of such an equation, valid for stable or unstable plasmas, for the case of an infinite homogeneous system. This equation includes as one special case the present kinetic equation for a stable plasma, the Lenard-Balescu[1,2] equation (see Chapter II-B), and as another special case an equation frequently used to describe unstable plasmas, the quasilinear equation (see Chapter II-D). In the general case the equation includes terms coming from each of the above cases, plus a term which may grow or damp (explicitly) in time, yet is qualitatively different from the term described by quasilinear theory. Thus the equation represents a true bridge between the equation for stable and unstable systems. Balescu[3] has derived a rather similar equation, but his result contains sufficient errors to make it insuitable to describe the long time behavior of a general system.

Chapter II contains a general discussion of the general techniques of plasma kinetic theory. In II-A the usual Fokker-Planck equation is derived from the Boltzmann collision integral, and the weaknesses of the derivation are pointed out. In II-B, the rigorous equations of the BBGKY hierarchy are discussed and applied to a new problem: the derivation of a collision term for small amplitude waves in a plasma. This (order $\frac{1}{\ell n\Lambda}$) collision term effectively extends the validity of the Fokker-Planck equation of II-A from the domain $0 \leqslant \omega \ll \omega_p$, $0 \leqslant k \ll k_d$, to the domain $0 \leqslant \omega \ll \Lambda\omega_p$, $0 \leqslant k \stackrel{>}{\sim} k_d$ where ω_p is the plasma frequency, k_d is the Debye wavenumber, and Λ is the plasma parameter, generally of order $10^4 - 10^8$. In II-C the equations of Klimontovich and Dupree, and their solution by Dupree[4] are discussed. Dupree's powerful methods provide the basis for the work carried out here, although Dupree made the usual Bogoliubov hypothesis regarding the behavior of correlation functions. This hypothesis is totally unsuitable for the case of an unstable plasma. Dupree's method is illustrated by the derivation of the Lenard-Balescu equation. Finally II-D presents a derivation of the quasilinear equation, plus a general criticism of this equation.

In Chapter III the general are of "kinetic theory" is separated from the far more general topic of the behavior of arbitrary systems for arbitrary time (for example very small times). For the latter case virtually anything less than the solution of the Liouville equation represents a restriction on the types of systems which may be studied.

In contrast "kinetic" systems may be described by a small number of relatively simple equations; furthermore the resultant kinetic equation is largely independent of the initial state of correlation functions in the system. Thus the general area of this thesis is established in Chapter III, as well as the restrictions on the work.

Chapter IV consists of a calculation of those effects, generally negligible, which must be retained in the derivation of the kinetic equation in order to assure the proper long-time behavior of the equation. In the general case of highly unstable plasmas these "small" terms are not small, and the approximations of this section limit the validity of the final result to systems which are not highly unstable. For the case considered the small terms obtained lead to the damping of fluctuations in the system identical to that predicted by the equation of II-B for small amplitude waves.

Chapter V contains the derivation of the general kinetic equation for a uniform system, as well as a discussion of the properties of this equation. Section A presents the analytic methods to be used in calculation of correlation functions in the absence of the Bogoliubov hypothesis (i.e., $t < \infty$), and the use of these methods for the calculation of the collision term of the kinetic equation. Section B discusses the elementary properties of the system, including conservation laws, H-theorem, and the approximations which are used in deriving a far simpler form of the equation. At this point the equation is not valid for times of the order of the time in which the one particle distribution function changes appreciably--this defect is remedied in section C to yield the principal result of this work. This kinetic equation is then generalized to include the effect of a uniform magnetic field. Section D contains an improved form of the quasilinear equation discussed earlier, plus a criticism of those aspects of quasilinear theory which are in contrast with the results of the rest of this chapter. Finally sections E, F, and G present a general criticism and discussion of the significance of this work.

In appendix A the equivalence is demonstrated between the equations of Klimontovich and Dupree, and the more commonly used equations of BBGKY . Appendix B contains several integrals which must be performed to obtain the collision term for small amplitude waves in a plasma.

II. THE FORMULATIONS OF PLASMA KINETIC THEORY

In this section we establish the basic subject matter of this thesis, and review some of the attempts to solve the basic problems of the kinetic theory of a plasma. The historical viewpoint is not logically necessary, but serves the purpose of introducing the reader to the lines of reasoning to be used in the body of this work. In addition the demonstration of deficiencies in previous work partially blunts the need for apologies regarding the results of this work. There are always questions left unanswered!

We shall define a plasma as a fully ionized gas, typically consisting of two species, ions and electrons. In general the requirement of complete ionization is not necessary, but the difficulties in treating the interactions between charged and neutral particles make the resulting subject beyond the scope of rigorous kinetic theory for the present. Typically an additional requirement is placed on the system of interest, that the Debye length $\lambda_d = \left(\frac{\kappa T}{4\pi n q^2}\right)^{1/2}$ be much less than L , where

κ is Boltzmann's constant

T is the temperature of the gas, in appropriate units

n is the macroscopic number density of ionized particles

q is the charge of a given charged particle (say electron)

L is any macroscopic length associated with the system.

We shall usually be concerned with an infinite uniform system, so this latter requirement is easily satisfied.

For purposes of simplifying notation we shall consider a coulomb plasma, so that the force of particle i on particle j is given by $\underset{\sim}{F}_{ij} = \frac{q_i q_j \underset{\sim}{r}_{ij}}{|\underset{\sim}{r}_{ij}|^3}$, where $\underset{\sim}{r}_{ij}$ is the directed length from particle i to particle j. We shall not consider the effects of relativistic particle motion. Formally these approximations are equivalent to letting the speed of light become infinite.

In brief, the goal of kinetic theory is to describe the behavior in time and space of a system of interest. Since a system will usually contain a very large number of particles, say 10^{23}, a detailed description of particle motion would be useless, even if we could obtain it. Thus we seek the behavior of an average system, where the average is taken over a large number of systems having those characteristics that we consider to be known, or given by experiment.

The subject of main interest is the average one-particle distribution function of the system, $f(\underset{\sim}{r},\underset{\sim}{v},t)$, where $nf(\underset{\sim}{r},\underset{\sim}{v},t)\, d\underset{\sim}{r}\, d\underset{\sim}{v}$ represents the probable number of particles in volume $d\underset{\sim}{r}\, d\underset{\sim}{v}$ at $\underset{\sim}{r},\underset{\sim}{v}$, at time t. (n is the average number density of the system, so that for a homogeneous system $\int f d\underset{\sim}{v} = 1$). Our main concern is to obtain an equation (kinetic) describing the evolution in time of this function. We now survey some past attempts to obtain such an equation, and point out associated difficulties.

A. The Fokker-Planck Equation

The kinetic equation which we discuss now may be derived in a number of ways; an outline of one derivation is sufficient because it illustrates the difficulties common to all. The sketch given here follows a more complete derivation in the excellent text by Montgomery and Tidman.[5] For more details the reader is referred to this source or to the original references.[6,7,8]

We choose as a starting point the Boltzmann equation for a single species plasma. The Boltzmann equation simply counts the particles flowing into and out of a given region of phase $(\underset{\sim}{r}, \underset{\sim}{v})$ space.

$$\frac{df}{dt} + \underset{\sim}{v} \cdot \nabla f + \frac{\underset{\approx}{F}}{m} \cdot \frac{df}{dv} = \left.\frac{df}{dt}\right|_{collisions} \qquad \text{(II-1)}$$

In words, the change of the distribution function with time $\frac{df}{dt}$ at point $\underset{\sim}{r}, \underset{\sim}{v}$, may be ascribed to three causes:

1. A net flux of particles moving $(\underset{\sim}{v})$ into or out of the volume element $d\underset{\sim}{r}$, in the absence of particle interactions.
2. A net flux of particles which are accelerated by macroscopic forces $(\frac{\underset{\approx}{F}}{m})$ into or out of the volume element $d\underset{\sim}{v}$, in the absence of particle interactions.
3. A net flux of particles into or out of the volume element $d\underset{\sim}{r}\, d\underset{\sim}{v}$, which is caused by interparticle collisions. For the present we consider the case in which the distribution function f is constant over a region in which a given pair of particles interact. In this case only the flux of particles in velocity space (i.e. through volume $d\underset{\sim}{v}$) need be considered.

Of course the equation is not useful until the form of $\left.\frac{df}{dt}\right|_{collisions}$ is specified. The Boltzmann collision term considers the effect of interactions between pairs of particles

$$\left.\frac{df}{dt}\right|_{collisions} = n\int d\underset{\sim}{v}_1 \int d\Omega \, \frac{d\sigma}{d\Omega} \, |\underset{\sim}{v} - \underset{\sim}{v}_1| \, (f'f'_1 - ff_1). \quad \text{(II-2)}$$

Here the term involving $f'f'_1$ gives the number of particles scattering into $d\underset{\sim}{v}$ per unit time. The notation signifies the sum of all two-particle collisions between a particle with velocity $\underset{\sim}{v}'$ and a particle with velocity $\underset{\sim}{v}'_1$, such that one acquires velocity $\underset{\sim}{v}$; the other acquires velocity $\underset{\sim}{v}_1$. $\frac{d\sigma}{d\Omega}$ is the cross section for such a two body collision, while the term $|\underset{\sim}{v} - \underset{\sim}{v}_1|$ is needed to give the flux of particles with velocity $\underset{\sim}{v}'$ incident on particles with velocity $\underset{\sim}{v}'$ (by conservation of energy $|\underset{\sim}{v} - \underset{\sim}{v}_1| = |\underset{\sim}{v}' - \underset{\sim}{v}'_1|$). Similarly the term involving ff_1 gives the number of particles scattering out of $d\underset{\sim}{v}$, per unit time, as a result of two-body collisions.

It is important to note a fundamental assumption made (explicitly or implicitly) in the derivation of the Boltzmann equation. Generally called the Stosszahlansatz or molecular chaos assumption, it states that there is no relation between the positions and relative velocities of a pair of particles before they collide. This assumption permits simple probability arguments to be used for estimating the number of particles which scatter through a given angle.

In principle the calculation of the collision term is now reduced to the evaluation of the integral over scattering angle in equation II-2. In fact we cannot perform the integral because we cannot calculate the orbits of particles, which we need to obtain the scattering cross section. The cross section for two-body scattering is not relevant because the range of the coulomb force is so long that many particles interact simultaneously.

In order to proceed we assume that two-body interactions occur independently. In effect this extends the molecular chaos assumption inside the region of two body interactions. The net force on a given particle is the sum of all two body forces, but it is possible that the forces are note independent of each other.

In order to simplify the mathematics, we consider the effect of small angle (large distance between particles) scattering as dominant. This permits a Taylor expansion of $f'(\underset{\sim}{v}') \; f'_1(\underset{\sim}{v}'_1)$ about $f(\underset{\sim}{v}) \; f(\underset{\sim}{v}_1)$ in powers of $\Delta \underset{\sim}{v} = \underset{\sim}{v} - \underset{\sim}{v}_1$. We keep terms through second order in $\Delta \underset{\sim}{v}$, and find:

$$\left.\frac{df}{dt}\right|_{\text{collisions}} = n\int d\underset{\sim}{v}_1 \int d\Omega \, \frac{d\sigma}{d\Omega} \, |\underset{\sim}{v}-\underset{\sim}{v}_1| \left\{ -f(\underset{\sim}{v})\Delta\underset{\sim}{v} \cdot \frac{df(\underset{\sim}{v}_1)}{d\underset{\sim}{v}_1} \mp \Delta\underset{\sim}{v} \frac{df}{d\underset{\sim}{v}} f(\underset{\sim}{v}_1) \right.$$
$$\left. + \frac{1}{2} \Delta\underset{\sim}{v}\Delta\underset{\sim}{v} : \frac{d^2 f(v_1)}{dv_1^{\,2}} \, f(v) - \Delta\underset{\sim}{v}\Delta\underset{\sim}{v} : \frac{df(\underset{\sim}{v}_1)}{dv_1} \, \frac{df(\underset{\sim}{v})}{dv} + \frac{1}{2} \Delta\underset{\sim}{v}\Delta\underset{\sim}{v} \cdot \frac{d^2 f(\underset{\sim}{v})}{dv^2} \, f(\underset{\sim}{v}_1) \right\} . \tag{II-3}$$

The cross section is simply the Rutherford cross section for the scattering of charged particles: $\frac{d\sigma}{d\Omega} = \frac{q^4}{m^2 |\underset{\sim}{v}-\underset{\sim}{v}_1|^4 \sin^4(\frac{\theta}{2})}$. The integration over solid angle leads to two integrals:

$$\int d\Omega \, \frac{d\sigma}{d\Omega} \, \Delta\underset{\sim}{v} \quad \text{and} \quad \int d\Omega \, \frac{d\sigma}{d\Omega} \, \Delta\underset{\sim}{v} \, \Delta\underset{\sim}{v} \quad . \tag{II-4}$$

When $\Delta\underset{\sim}{v}$ is expressed in a convenient reference frame these integrals may be performed. In both cases the integrals lead to terms which diverge as $\ell n\theta$ for $\theta \to 0$ (small scattering angle). We keep only these divergent terms, and cut off the integration at some angle θ_{min}. This cutoff is chosen on the basis of considerations which are not mentioned in this derivation. The concept of Debye shielding[4] suggests that we use for θ_{min} the value corresponding to an impact parameter of a Debye length, λ_d. After some manipulation, the collision term may be written in the form

$$\left.\frac{df}{dt}\right|_{\text{collisions}} = \frac{d}{d\underset{\sim}{v}} \cdot \frac{2\pi q^4 n(\sin[\frac{\theta_{min}}{2}])}{m^2} \int d\underset{\sim}{v}_1 \left[\frac{|\underset{\sim}{v}-\underset{\sim}{v}_1|^2 \underset{\approx}{I} - (\underset{\sim}{v}-\underset{\sim}{v}_1)(\underset{\sim}{v}-\underset{\sim}{v}_1)}{|\underset{\sim}{v}-\underset{\sim}{v}_1|^3} \right]$$

$$\cdot \left[\frac{df}{d\underset{\sim}{v}} f(\underset{\sim}{v}_1) - \frac{df}{d\underset{\sim}{v}_1} f(\underset{\sim}{v}) \right] . \qquad \text{(II-5)}$$

It is not difficult to criticize this derivation of the Fokker-Planck collision term. In essence this is a model calculation, with assumptions based on physical intuition rather than internal mathematical constraints. In view of better calculations (to be discussed) we may list the faults in this derivation.

1. The Stosszahlansatz, or molecular chaos assumption is not valid, in the sense that the range of the coulomb interaction (as expressed by the divergent integrals over θ) is infinite. Furthermore inside this (infinite) range the positions of particles are correlated--the ad hoc cutoff of the integral is replaced by a natural cutoff when this correlation is included in the theory.

2. The neglect of higher order terms in Taylor expansion in the Taylor expansion in $\Delta \underset{\sim}{v}$ is not justified when the pair of particles considered has a near collision. Large angle scattering, which causes difficulties in more exact theories, is simply ignored here.
3. We cannot estimate the domain of validity of the resulting equation. How fast may the time dependence be, and how rapid may spatial variation be? How do we distinguish $\underset{\sim}{F}$, the macroscopic force, for the microscopic forces which yield the collision term?
4. There is no way to obtain better accuracy. The equation is regarded as being accurate to order $\frac{1}{\ell n \Lambda}$ but it is not clear how this could be improved. (The quantity Λ is defined as the ratio of the Debye length λ_d to the distance of closest approach of two typical particles, $r_0 = \frac{q^2}{KT}$. In most physical applications $\ell n \Lambda$ has a value from 7 to 15). Other assumptions are possible (another derivation), but there is no way of knowing which set of assumptions is best.

It must be remarked that for all its faults the Fokker-Planck equation is quite satisfactory for many calculations. More sophisticated attempts at a kinetic equation have a common difficulty: the treatment of large angle scattering leads to formidable analytic results.[9,10,11] Frequently no attempt is made to handle large angle scattering, so that the kinetic equation contains an integral which tends to diverge for short range interactions. The integral is then cutoff at an appropriate distance corresponding to r_0 (defined above). In this work we shall follow this procedure, despite the fact that it limits the accuracy of our result to that of the Fokker-Planck equation: $\mathcal{O}(\frac{1}{\ell n\Lambda})$. The justification lies in the fact that the short range effects are purely quantitative, for they simply close off the divergent integral. The short range interaction between oppositely charged particles may be expected to cause difficulties for some time, as the treatment of bound states is very difficut. Again the effect on the kinetic equation is (presumably) strictly quantitative.

B. The BBGKY Kinetic Theory

1. The Equations of the Hierarchy

We consider now a rigorous set of equations for the behavior of a plasma. The BBGKY (Bogoliubov, Born, Green, Kirkwood and Yvon)[12,13,14,15,16] theory thus is a great step forward from the heuristic treatments that yield the standard Fokker-Planck equation. The difficulty now lies not in believing the equations, but in solving them.

The BBGKY kinetic theory consists of a set of $(N \rightarrow \infty)$ coupled equations for the probability distributions $\mathcal{F}_s(X_1,X_2 \cdots X_s,t)$ of $s = 1,2,3\ldots$ particles, where $X_1 = \left\{ \underset{\sim}{r}_1, \underset{\sim}{v}_1 \right\}$, etc. These equations are derived in rigorous fashion from Liouville's theorem for the probability distribution of N particles in $6N$ dimensional phase space. Since many derivations are available in the literature we simply quote the standard result. The equation for $\mathcal{F}_s$ is (we consider only one type of particle)

$$\frac{d\mathcal{F}_s}{dt} + \sum_{i=1}^{s} \underset{\sim}{v}_i \cdot \nabla_i \mathcal{F}_s - \sum_{i\neq j=1}^{s} \frac{1}{m} \nabla_i \phi_{ij} \cdot \frac{d\mathcal{F}_s}{d\underset{\sim}{v}_i} - \frac{n}{m} \sum_{i=1}^{s} \int d\underset{\sim}{X}_{s+1} \nabla_i \phi_{i,s+1} \cdot \frac{d\mathcal{F}_{s+1}}{d\underset{\sim}{v}_i} = 0 \quad . \tag{II-6}$$

Here $\nabla_i = \frac{d}{d\underset{\sim}{r}_i}$, for a plasma $\phi_{ij} = \frac{q^2}{|\underset{\sim}{r}_i - \underset{\sim}{r}_j|}$, and n = the average number density of particles in the system. Inclusion of more than one species leads to no difficulties. For purposes of solving these equations it is useful to introduce a cluster expansion of the functions $\mathcal{F}_s$. We write

$$\mathcal{F}_1(\underset{\sim}{X}_1) = f(\underset{\sim}{X}_1)$$

$$\mathcal{F}_2(\underset{\sim}{X}_1,\underset{\sim}{X}_2) = f(\underset{\sim}{X}_1)\, f(\underset{\sim}{X}_2) + g(\underset{\sim}{X}_1, \underset{\sim}{X}_2)$$

$$\mathcal{F}_3(\underset{\sim}{X}_1, \underset{\sim}{X}_2, \underset{\sim}{X}_3) = f(\underset{\sim}{X}_1)\, f(\underset{\sim}{X}_2)\, f(\underset{\sim}{X}_3) + f(\underset{\sim}{X}_1)\, g(\underset{\sim}{X}_2, \underset{\sim}{X}_3) + f(\underset{\sim}{X}_2) g(\underset{\sim}{X}_1,\underset{\sim}{X}_3)$$
$$f(\underset{\sim}{X}_3)\, g(\underset{\sim}{X}_1, \underset{\sim}{X}_2) + h(\underset{\sim}{X}_1, \underset{\sim}{X}_2, \underset{\sim}{X}_3)$$

$$\mathcal{F}_4(\underset{\sim}{X}_1, \underset{\sim}{X}_2, \underset{\sim}{X}_3, \underset{\sim}{X}_4) = \cdots \tag{II-7}$$

We may now convert the equations for the $\frac{d\mathcal{F}_s}{dt}$ into equations for $\frac{df}{dt}$, $\frac{dg}{dt}$, $\frac{dh}{dt}$, etc. When this is done the first two equations of the BBGKY hierarchy lead to the following equations for f and g.

$$\frac{df(\underset{\sim}{X}_1,t)}{dt} + \underset{\sim}{v}_1\cdot\nabla_1 f = \frac{n}{m}\int d\underset{\sim}{X}_2 \nabla_1 \phi_{12} \cdot \frac{df(\underset{\sim}{X}_1)}{d\underset{\sim}{v}_1} f(\underset{\sim}{X}_2)$$

$$+ \frac{n}{m}\int d\underset{\sim}{X}_2 \nabla_1 \phi_{12} \cdot \frac{d}{d\underset{\sim}{v}_1} g(\underset{\sim}{X}_1, \underset{\sim}{X}_2, t) \tag{II.8}$$

$$\left(\frac{d}{dt} + \underset{\sim}{v}_1\cdot\nabla_1 + \underset{\sim}{v}_2\cdot\nabla_2\right) g(\underset{\sim}{X}_1,\underset{\sim}{X}_2,t) = \frac{1}{m} \nabla_1 \phi_{12} \cdot \left[\frac{d}{d\underset{\sim}{v}_1} - \frac{d}{d\underset{\sim}{v}_2}\right]$$

$$\left[f(\underset{\sim}{X}_1) f(\underset{\sim}{X}_2) + g(\underset{\sim}{X}_1, \underset{\sim}{X}_2)\right]$$

$$+ \frac{n}{m}\left[\int d\underset{\sim}{X}_3 f(\underset{\sim}{X}_3) \nabla_1\phi_{13}\cdot \frac{d}{d\underset{\sim}{v}_1} + (1\leftrightarrow 2)\right] g(\underset{\sim}{X}_1,\underset{\sim}{X}_2) + \frac{n}{m}\int d\underset{\sim}{X}_3 \left[\nabla_1\phi_{13}\cdot \frac{d}{d\underset{\sim}{v}_1}\right.$$

$$\left. + (1\leftrightarrow 2)\right] h(\underset{\sim}{X}_1,\underset{\sim}{X}_2,\underset{\sim}{X}_3) + \frac{n}{m}\left[\frac{df}{d\underset{\sim}{v}}\cdot\int d\underset{\sim}{X}_3 \nabla_1\phi_{13}\, g(\underset{\sim}{X}_2,\underset{\sim}{X}_3) + (1\leftrightarrow 2)\right] . \tag{II.9}$$

Of course the f equation contains a term in g, the g equation is coupled to h, etc. We must terminate the infinite set of equations in some way. We do this by noting a property of the cluster expansion. We expect $g(\underset{\sim}{X}_1,\underset{\sim}{X}_2) \ll f(\underset{\sim}{X}_1)\, f(\underset{\sim}{X}_2)$ unless $|\underset{\sim}{r}_1 - \underset{\sim}{r}_2|$ is "small." Likewise we expect $h(\underset{\sim}{X}_1,\underset{\sim}{X}_2,\underset{\sim}{X}_3) \ll g(\underset{\sim}{X}_1,\underset{\sim}{X}_2)\, f(\underset{\sim}{X}_3)$ unless $|\underset{\sim}{r}_1 - \underset{\sim}{r}_2|$ and $|\underset{\sim}{r}_1 - \underset{\sim}{r}_3|$ are "small." Therefore as a first approximation we neglect g compared to f in equation II-8. In this approximation equation II-8 becomes the familiar Vlasov equation, and is not coupled to higher equations. The Vlasov equation has been worked over a good deal, and we will not go into its subtleties here.

As a second approximation we should like to keep g in equation I-8, but nelgect the higher order terms (g compared to $f\ f$

and h compared to fg) in equation II-9, so that the set of equations again terminates. We then solve II-9 to find g in terms of f, and insert the result in equation II-8, to obtain a kinetic equation valid for a sufficient time for the system to reach equilibrium. A rigorous treatment has proved possible for only two cases:

1. If f is independent of space, the time dependence of f comes only from the term involving g. Since g is small, f varies slowly in time. Therefore we solve the g equation while holding f fixed. This calculation has been done for the limit of infinite time $g(t=\infty)$, and leads to the Lenard Balescu equation, which will be discussed in a later section.

2. By linearizing about equilibrium $f = f_{eq} + f_1$, $g = g_{eq} + g_1$, with $g_1 \ll g_{eq}$ and $f_1 \ll f_{eq}$ Guernsey has been able to solve the g_1 equation for f_1, while permitting f_1 to have space and time dependence. Guernsey's solution runs to five lines in the published report[17] (involving many defined functions, at that), and he makes only general statements about the resulting equation for f_1.

Other results have been obtained. Rostoker[18] has shown that the solution of the g equation may be reduced to terms involving the solution of the Vlasov equation mentioned previously. This solution is formal, in the sense that some little labor is required just to recover the Lenard Balescu equation. Rostoker notes that his expression may be used to calculate g for the case of an unstable plasma, but he makes no attempt to do so.

Attempts[19,20] have been made to solve II-9 for the case of an unstable plasma. The results are unsatisfactory due to the omission of certain terms which are needed to keep certain integrals well defined. This will be discussed later in the section on quasilinear theory.

In general the equations of the hierarchy must be characterized as difficult. Case 1 above was proposed by Bogoliubov in 1946, and solved by Lenard in 1960. Case 2 was reported by Guernsey in 1962, but is too complicated to be of general interest. Furthermore one seldom sees the equation for h, much less an attempt to solve it . The Lenard Balescu equation breaks down for an unstable plasma--no plan for correcting this defect has been exploited. Thus progress in the hierarchy seems tediously slow, although the equations must be admitted to be exact. We shall leave this scheme after illustrating one of its virtues.

2. Extension of the Fokker-Planck Equation

The equation for f and g may easily be approximated to lead to a generalization of the Fokker-Planck equation (equation II-5) discussed in the preceding section. Before carrying out the calculation we indicate the known region of validity of equations of the general form of equation II-5.

a) Frieman[21] has shown that f may be spatially dependent, with a spatial dependence of the order of the mean free path of a particle. The time dependence is on the collisional time scale.

b) Berk[22] has derived an equation for the behavior of the electron distribution function. He treats ions as stationary, and neglects electron collisions. What is left is valid for a wide range of frequency and wavelength dependence, except that the result diverges at $\underset{\sim}{v} = 0$ unless $f(\underset{\sim}{v}) = f(|\underset{\sim}{v}|)$. Because of the approximations the result is not comparable with other Fokker-Planck equations.

c) Silin[23] has derived an equation for high frequency processes in the absence of spatial dependence. Since no external fields are considered, it is hard to see what could occur at a rate faster than the ordinary collision rate. We conclude that the resulting equation is mathematically correct, but is not applicable to any physical process (except at very low frequency).

Clearly an important problem has not been considered. Very frequently one is interested in the behavior of small amplitude waves in a nearly (spatially) uniform system. We may ask what effect the collisions might have on these waves. To be general we must permit the wavelength of the wave to vary from infinity to the Debye length. (Waves with wavelength shorter than a Debye length are Landau-damped very rapidly). Likewise the frequency of the wave may be high (much higher than the plasma frequency $\omega_p = \sqrt{\frac{4\pi nq^2}{m}}$), or low (essentially zero). We aks that the collision term be reasonably accurate (order $\frac{1}{\ell n\Lambda}$ for our calculation) and reasonably simple, i.e., of the general form of equation II-5.

A collision term valid in the broad domain indicated will provide an analytic result in a domain where many calculations are based on "model" collision terms. The collision term we construct here may be used for further analytic work, or to check and improve the validity of simplified models of collisions. Since we seek a result valid only to order $\frac{1}{\ell n\Lambda}$, it is clear that we have considerable leeway in making approximations. We shall therefore make further approximations on the g equation, which we rewrite here, with the higher order terms omitted.

$$\left(\frac{d}{dt} + \underset{\sim}{v}_1 \cdot \nabla_1 + \underset{\sim}{v}_2 \cdot \nabla_2\right) g - \frac{n}{m}\left[\frac{df}{d\underset{\sim}{v}_1} \cdot \int d\underset{\sim}{X}_3 \nabla_1 \phi_{13}\, g(\underset{\sim}{X}_1, \underset{\sim}{X}_3) + (1 \leftrightarrow 2)\right]$$

$$- \frac{n}{m}\left[\int d\underset{\sim}{X}_3\, f(\underset{\sim}{X}_3)\, \nabla_1 \phi_{13} \cdot \frac{d}{d\underset{\sim}{v}_1} + (1 \leftrightarrow 2)\right] g(\underset{\sim}{X}_1, \underset{\sim}{X}_2) = \frac{1}{m} \nabla_1 \phi_{12} \cdot \left[\frac{d}{d\underset{\sim}{v}_1} - \frac{d}{d\underset{\sim}{v}_2}\right]$$

$$f(\underset{\sim}{X}_1)\, f(\underset{\sim}{X}_2) \quad . \qquad \text{(II-10)}$$

We shall simplify the equation by estimating the relative magnitude of the various terms. The term $\frac{d}{dt}$ may be large or small--in general we must keep it. We estimate velocities (and velocity gradients) as comparable to a thermal velocity, V_{th}. Because $\int f dv = 1$, we take $f = \frac{1}{V_{th}^3}$ within a volume V_{th}^3. A more exact analysis, which we are trying to avoid here, shows that two particle interactions are cutoff (become unimportant) for an inter-particle spacing greater than a Debye length, while the neglect of the higher order term in g in equation II-9 means that short range interactions are not treated satisfactorily. We estimate distances for g by a length L much smaller than λ_d but longer than r_0.

It is true that g also varies on a scale given by the scale length of f. Thus we must have L much less than the scale length of f. We now estimate the magnitude of the terms in equation II-10

$$\left(\frac{d}{dt} : \frac{V_{th}}{L} : \frac{V_{th}}{L}\right)g : \frac{n}{m}\left[\frac{1}{V_{th}^{4}} V_{th}^{3} L^{3} \frac{q^{2}}{L^{2}} g + (1 \leftrightarrow 2)\right]$$

$$: \frac{n}{m}\left[\frac{V_{th}^{3} L^{3}}{V_{th}^{3}} \frac{q^{2}}{L^{2}} \frac{1}{V_{th}} + (1 \leftrightarrow 2)\right] g = \frac{1}{m}\frac{q^{2}}{L^{2}}\left[\frac{1}{V_{th}} : \frac{1}{V_{th}}\right] \frac{1}{V_{th}^{3}} \frac{1}{V_{th}^{3}} .$$

(II-11)

We wish to find g in terms of the right-hand side. We collect and cancel factors in the left-hand side, and use the fact that

$$\frac{nq^{2}}{m} \sim \omega_{p}^{2} \sim \frac{V_{th}^{2}}{\lambda_{d}^{2}}$$

to find

$$\left(\frac{d}{dt} : \frac{V_{th}}{L} : \frac{V_{th}}{L}\right)g : \left[\frac{V_{th}}{L}\left(\frac{L}{\lambda_{d}}\right)^{2} + \leftrightarrow \quad g\right] : \left[\frac{V_{th}}{L}\left(\frac{L}{\lambda_{d}}\right)^{2} + (1 \leftrightarrow 2)\right] g$$

$$= \frac{1}{m}\frac{q^{2}}{L^{2}}\left[\frac{1}{V_{th}} : \frac{1}{V_{th}}\right]\frac{1}{V_{th}^{6}} . \qquad \text{(II-12)}$$

As a good approximation we drop the second and third terms because $L < \lambda_{d}$. Therefore the simplified equation we consider for g is

$$\left(\frac{d}{dt} + \underset{\sim}{v}_{1}\cdot\nabla_{1} + \underset{\sim}{v}_{2}\cdot\nabla_{2}\right)g = \frac{1}{m}\nabla_{1}\,\phi_{12}\cdot\left[\frac{d}{d\underset{\sim}{v}_{1}} - \frac{d}{d\underset{\sim}{v}_{2}}\right] f(\underset{\sim}{X}_{1})\, f(\underset{\sim}{X}_{2}) . \qquad \text{(II-13)}$$

As a result of these approximations we must supply both short and long range cutoffs for the interaction between the particles. The resulting collision term will be valid to order $1/\ell n\Lambda$.

We now linearize f and g about a uniform stationary state.

$$f(\underset{\sim}{r}_1,\underset{\sim}{v}_1,t) = f_0(\underset{\sim}{v}_1) + f_1(\underset{\sim}{r}_1,\underset{\sim}{v}_1,t)$$

$$g(\underset{\sim}{r}_1,\underset{\sim}{r}_2,\underset{\sim}{v}_1,\underset{\sim}{v}_2,t) = g_0(\underset{\sim}{r}_1-\underset{\sim}{r}_2,\underset{\sim}{v}_1,\underset{\sim}{v}_2,) + g_1(\underset{\sim}{r}_1,\underset{\sim}{r}_2,\underset{\sim}{v}_1,\underset{\sim}{v}_2,t) \quad . \tag{II-14}$$

We shall treat the subscript 1 quantities as small, and keep only first order terms. The assumption that f_0 and g_0 have no time dependence simply requires that their actual time dependence be much slower than that of f_1 and g_1 . In particular when f_1 varies quite slowly f_0 must be the equilibrium (Maxwellian) distribution. We wish to consider equations II-8 and II-13. In keeping with the notation of the Vlasov equation we set

$$\frac{q}{m} \underset{\sim}{E}_1 = -\frac{n}{m}\int d\underset{\sim}{X}_2 \, \nabla_1\phi_{12} \, f_1(\underset{\sim}{X}_2,t) \, . \tag{II-15}$$

Throughout this work it will be convenient to use a Fourier transform on the spatial dependence of the equation we work with. We define the Fourier transform of a given function Φ .

$$\Phi(\underset{\sim}{K}_1\underset{\sim}{r}_2,\underset{\sim}{r}_3\cdots) = \int d\underset{\sim}{r}_1 \, e^{-i\underset{\sim}{K}_1\cdot\underset{\sim}{r}_1} \, \Phi(\underset{\sim}{r}_1,\underset{\sim}{r}_2,\underset{\sim}{r}_3\cdots) \tag{II-16}$$

which has the inverse transformation

$$\Phi(\underset{\sim}{r}_1,\underset{\sim}{r}_2,\underset{\sim}{r}_3\cdots) = \int \frac{d\underset{\sim}{K}_1}{(2\pi)^3} \; e^{i\underset{\sim}{K}_1\cdot\underset{\sim}{r}_1} \; \Phi(\underset{\sim}{K}_1,\underset{\sim}{r}_2,\underset{\sim}{r}_3\cdots) \; . \qquad \text{(II-17)}$$

We frequently wish to transform in more than one variable. For example

$$\Phi(\underset{\sim}{K}_1,\underset{\sim}{K}_2,\underset{\sim}{K}_3,\underset{\sim}{r}_4\cdots) = \int d\underset{\sim}{r}_1 \, d\underset{\sim}{r}_2 \, d\underset{\sim}{r}_3 \; e^{-i\underset{\sim}{K}_1\cdot\underset{\sim}{r}_1} \; e^{-i\underset{\sim}{K}_2\cdot\underset{\sim}{r}_2} \; e^{-i\underset{\sim}{K}_3\cdot\underset{\sim}{r}_3}$$

$$\Phi(\underset{\sim}{r}_1,\underset{\sim}{r}_2,\underset{\sim}{r}_3,\underset{\sim}{r}_4\cdots) \qquad \text{(II-18)}$$

$$\Phi(\underset{\sim}{r}_1,\underset{\sim}{r}_2,\underset{\sim}{r}_3,\underset{\sim}{r}_4\cdots) = \int \frac{d\underset{\sim}{K}_1}{(2\pi)^3} \frac{d\underset{\sim}{K}_2}{(2\pi)^3} \frac{d\underset{\sim}{K}_3}{(2\pi)^3} \; e^{i\underset{\sim}{K}_1\cdot\underset{\sim}{r}_1} \; e^{i\underset{\sim}{K}_2\cdot\underset{\sim}{r}_2} \; e^{i\underset{\sim}{K}_3\cdot\underset{\sim}{r}_3}$$

$$\Phi(\underset{\sim}{K}_1,\underset{\sim}{K}_2,\underset{\sim}{K}_3,\underset{\sim}{r}_4\cdots) \; . \qquad \text{(II-19)}$$

Of course the functional dependence of Φ on r, and on K, will be different, in general. We transform equation II-8 to find

$$\left(\frac{d}{dt} + i\underset{\sim}{k}\cdot\underset{\sim}{v}_1\right) f_1(\underset{\sim}{k},\underset{\sim}{v}_1,t) + \frac{q}{m} \underset{\sim}{E}_1(\underset{\sim}{k},t) \cdot \frac{df_0}{d\underset{\sim}{v}_1} = \frac{nq^2}{m} \int d\underset{\sim}{v}_2$$

$$\int \frac{d\underset{\sim}{K}}{2\pi^2} \; \frac{i\underset{\sim}{K}}{K^2} \; g_1\,(\underset{\sim}{k} - \underset{\sim}{K},\underset{\sim}{K},\underset{\sim}{v}_1,\underset{\sim}{v}_2,t) \; . \qquad \text{(II-20)}$$

We transform equation II-13 to produce the value of g required by equation II-20.

$$\left(\frac{d}{dt} + i(\underset{\sim}{k}-\underset{\sim}{K})\cdot\underset{\sim}{v}_1 + i\underset{\sim}{K}\cdot\underset{\sim}{v}_2\right) g_1(\underset{\sim}{k}-\underset{\sim}{K},\underset{\sim}{K},\underset{\sim}{v}_1,\underset{\sim}{v}_2,t) = -\frac{4\pi i q^2}{m}\left[\frac{d}{d\underset{\sim}{v}_1} - \frac{d}{d\underset{\sim}{v}_2}\right] \cdot \left[\frac{\underset{\sim}{K}}{K^2} f_0(\underset{\sim}{v}_2)\, f_1(\underset{\sim}{k},\underset{\sim}{v}_1,t) + \frac{\underset{\sim}{K}-\underset{\sim}{k}}{(K-k)^2} f_0(\underset{\sim}{v}_1)\, f_1(\underset{\sim}{k},\underset{\sim}{v}_2,t)\right] . \qquad \text{(II-21)}$$

The solution of equation II-21 for the time dependence of g_1 is elementary if we use a Laplace transform. We find that g_1 depends on its initial values, among other things. In fact we do not wish to treat these initial values, which generally die away rapidly (see section IIIA. Therefore we consider f_1 (and hence g_1) to have time dependence $e^{-i\omega t}$, with ω analytically continued from the upper half ω plane. The solution for g_1 is then arithmetic (we put $g_1(t) = g_1(\omega)\, e^{-i\omega t}$) .

$$g_1(\underset{\sim}{k}-\underset{\sim}{K},\underset{\sim}{K},\underset{\sim}{v}_1,\underset{\sim}{v}_2,\omega) = -\frac{4\pi i q^2}{m}\; \frac{\left[\frac{d}{d\underset{\sim}{v}_1} - \frac{d}{d\underset{\sim}{v}_2}\right]}{-i\omega + i(\underset{\sim}{k}-\underset{\sim}{K})\cdot\underset{\sim}{v}_1 + i\underset{\sim}{K}\cdot\underset{\sim}{v}_2} \cdot \left[\frac{\underset{\sim}{K}}{K^2} f_0(\underset{\sim}{v}_2) \times f_1(\underset{\sim}{k},\underset{\sim}{v}_1,\omega) + \frac{\underset{\sim}{K}-\underset{\sim}{k}}{(K-k)^2} f_0(\underset{\sim}{v}_1)\, f_1(\underset{\sim}{k},\underset{\sim}{v}_2,\omega)\right]. \qquad \text{(II-22)}$$

We insert this quantity into equation II-20 to find

$$(-\omega + i\underset{\sim}{k}\cdot\underset{\sim}{v}_1)\, f_1 + \frac{q}{m}\,\underset{\sim}{E}_1\cdot\frac{df_0}{d\underset{\sim}{v}_1} = \frac{2nq^4}{\pi m^2}\,\frac{d}{d\underset{\sim}{v}_1}\cdot\int d\underset{\sim}{v}_2 \int \frac{d\underset{\sim}{K}\,\underset{\sim}{K}}{K^2}\; \frac{\left[\frac{d}{d\underset{\sim}{v}_1} - \frac{d}{d\underset{\sim}{v}_2}\right]}{\left[-i\omega + i\underset{\sim}{k}\cdot\underset{\sim}{v}_1 + i\,\underset{\sim}{K}\cdot(\underset{\sim}{v}_2-\underset{\sim}{v}_1)\right]} \cdot \left[\frac{\underset{\sim}{K}}{K^2} f_0(\underset{\sim}{v}_2)\, f_1(\underset{\sim}{k},\underset{\sim}{v}_1,\omega) + \frac{\underset{\sim}{K}-\underset{\sim}{k}}{(K-k)^2} f_0(\underset{\sim}{v}_1) \times f_1(\underset{\sim}{k},\underset{\sim}{v}_2,\omega)\right] . \qquad \text{(II-23)}$$

We apply the Plemelj formula to the quantity $\frac{1}{-\omega + \underset{\sim}{k} \cdot \underset{\sim}{v}_1 + \underset{\sim}{K} \cdot (v_2 - \underset{\sim}{v}_1)}$ remembering that ω is analytically continued from the upper half plane

$$\frac{1}{-i(\omega - \underset{\sim}{k} \cdot \underset{\sim}{v}_1 + \underset{\sim}{K} \cdot (\underset{\sim}{v}_1 - \underset{\sim}{v}_2))} = \frac{P}{-i(\omega - \underset{\sim}{k} \cdot \underset{\sim}{v}_1 + \underset{\sim}{K} \cdot (\underset{\sim}{v}_1 - \underset{\sim}{v}_2))} + \pi\, \delta\left(\omega - \underset{\sim}{k} \cdot \underset{\sim}{v}_1 + \underset{\sim}{K} \cdot (\underset{\sim}{v}_1 - \underset{\sim}{v}_2)\right) .$$

(II-24)

Here P indicates the principal value is to be taken when the integral is performed and δ is the Dirac delta function. The limits of the k integration are

$$\int_{|K| = k_d}^{|k| = k_0} d\underset{\sim}{K}$$

where $k_d = \frac{2\pi}{\lambda_d}$ and $k_0 = \frac{2\pi}{r_0} = \frac{2\pi KT}{q^2}$.

The principal part integral is not divergent and may be neglected (it is odd for large $\underset{\sim}{K}$). The integration over the delta function is not difficult and is discussed in appendix B. In order to symmetrize the collision term in $\underset{\sim}{v}_1$ and $\underset{\sim}{v}_2$ we shift the origin of the integral over $\underset{\sim}{K}(\underset{\sim}{k}-\underset{\sim}{K})$ by the vector $\underset{\sim}{k}$ the effect is quantitatively negligible. The final result is the equation for perturbations on a uniform plasma, with the collision term valid to order $1/\ell n\Lambda$.

$$(-i\omega + i\mathbf{k}\cdot\mathbf{v}_1)\, f_1(\mathbf{k},\mathbf{v}_1,\omega) + \frac{q}{m}\, \mathbf{E}_1(\mathbf{k},\omega)\cdot\frac{df_0}{d\mathbf{v}_1} = \frac{2\pi n q^4}{m^2}\, \frac{d}{d\mathbf{v}_1}\cdot\int d\mathbf{v}_2$$

$$\Big[\mathbf{Q}(\omega-\mathbf{k}\cdot\mathbf{v}_1;\mathbf{v}_1-\mathbf{v}_2)\cdot\Big(\frac{d}{d\mathbf{v}_1}-\frac{d}{d\mathbf{v}_2}\Big)\, f_1(\mathbf{k},\mathbf{v}_1,\omega)\, f_0(\mathbf{v}_2)$$

$$+\,\mathbf{Q}(\omega-\mathbf{k}\cdot\mathbf{v}_2;\mathbf{v}_1-\mathbf{v}_2)\cdot\Big(\frac{d}{d\mathbf{v}_1}-\frac{d}{d\mathbf{v}_2}\Big)\, f_0(\mathbf{v}_1)\, f_1(\mathbf{k},\mathbf{v}_2,\omega)\Big] \qquad \text{(II-25)}$$

where $\mathbf{Q}$ is given by

$$\mathbf{Q}(\omega-\mathbf{k}\cdot\mathbf{v},\cdot\mathbf{v}') = \ell n\Lambda\left(\frac{v'^2\,\mathbf{I}-\mathbf{v}'\mathbf{v}'}{v'^3}\right) \text{ for } k_d > \frac{|\omega-\mathbf{k}\cdot\mathbf{v}|}{|\mathbf{v}'|}$$

$$= \ell n\,\frac{k_0|\mathbf{v}'|}{|\omega-\mathbf{k}\cdot\mathbf{v}|}\left(\frac{v'^2\,\mathbf{I}-\mathbf{v}'\mathbf{v}'}{v'^3}\right) \text{ for } k_0 > \frac{|\omega-\mathbf{k}\cdot\mathbf{v}|}{|\mathbf{v}'|}\; k_d$$

$$= 0 \qquad \text{for} \qquad \frac{|\omega-\mathbf{k}\cdot\mathbf{v}|}{|\mathbf{v}'|} \qquad k_0 \qquad \text{(II-26)}$$

We demonstrate that the collision term satisfies the conservation laws. If we integrate equation II-25 over $\mathbf{v}_1$, the contribution from the collision term is

$$\int d\mathbf{v}_1\, \frac{d}{d\mathbf{v}_1}\cdot\mathbf{J}(\mathbf{k},\mathbf{v}_1\omega) = -\,\mathbf{J}(\mathbf{k},\mathbf{v}_1,\omega)\Big|_{|\mathbf{v}_1|=\infty} = 0 \qquad \text{(II-27)}$$

since no particles have infinite velocity. Thus, particle number is conserved.

We multiply II-25 by $m\mathbf{v}_1$, and integrate over $\mathbf{v}_1$ to deomonstrate the conservation of momentum. The collision term contributes

$$\int d\mathbf{v}_1 \, m\mathbf{v}_1 \frac{d}{d\mathbf{v}_1} \cdot \frac{2\pi nq^4}{m^2} \int d\mathbf{v}_2 \Big[\mathbf{Q}(\omega-\mathbf{k}\cdot\mathbf{v}_1;\mathbf{v}_1-\mathbf{v}_2)\cdot\left(\frac{d}{d\mathbf{v}_1} - \frac{d}{d\mathbf{v}_2}\right)$$

$$\times f_0(\mathbf{v}_2)\ f_1(\mathbf{k},\mathbf{v}_1,\omega) + \mathbf{Q}(\omega-\mathbf{k}\cdot\mathbf{v}_2;\mathbf{v}_1-\mathbf{v}_2) \cdot \left(\frac{d}{d\mathbf{v}_1} - \frac{d}{d\mathbf{v}_2}\right) f_0(\mathbf{v}_1)\ f_1(\mathbf{k},\mathbf{v}_2,\omega)\Big]$$

$$= -\frac{2\pi nq^4}{m}\int d\mathbf{v}_1 \ d\mathbf{v}_2 \Big[\mathbf{Q}(\omega-\mathbf{k}\cdot\mathbf{v}_1;\mathbf{v}_1-\mathbf{v}_2) \cdot \left(\frac{d}{d\mathbf{v}_1} - \frac{d}{d\mathbf{v}_2}\right) f_1(\mathbf{k},\mathbf{v}_1,\omega)\ f_0(\mathbf{v}_2)$$

$$+ \mathbf{Q}(\omega-\mathbf{k}\cdot\mathbf{v}_2;\mathbf{v}_1-\mathbf{v}_2) \cdot \left(\frac{d}{d\mathbf{v}_1} - \frac{d}{d\mathbf{v}_2}\right) f_0(\mathbf{v}_1)\ f_1(\mathbf{k},\mathbf{v}_2,\omega)\Big]$$

$$= -\frac{2\pi nq^4}{m}\int d\mathbf{v}_1 \int d\mathbf{v}_2 \Big[\mathbf{Q}(\omega-\mathbf{k}\cdot\mathbf{v}_1;\mathbf{v}_1-\mathbf{v}_2) \cdot \left(\frac{d}{d\mathbf{v}_1} - \frac{d}{d\mathbf{v}_2}\right) f_0(\mathbf{v}_2)\ f_1(\mathbf{k},\mathbf{v}_1,\omega)\Big]$$

$$+ \mathbf{Q}(\omega-\mathbf{k};\mathbf{v}_1-\mathbf{v}_1-\mathbf{v}_2) \cdot \left(\frac{d}{d\mathbf{v}_1} - \frac{d}{d\mathbf{v}_2}\right) f_0(\mathbf{v}_2)\ f_1(\mathbf{k},\mathbf{v}_1,\omega)\Big]$$

$$= 0 \quad . \tag{II-28}$$

Therefore the collision term conserves momentum.

Finally we multiply by $\frac{1}{2}\, m\, v_1^{\,2}$, and integrate over $\mathbf{v}_1$ to demonstrate that the collision term conserves energy.

$$\int d\mathbf{v}_1 \ \frac{1}{2}\, m v_1^{\,2} \frac{d}{d\mathbf{v}_1} \cdot \mathbf{J}(\mathbf{k},\mathbf{v}_1,\omega) =$$

$$-\frac{2\pi nq^4}{m}\int d\mathbf{v}_1 \int d\mathbf{v}_2 \Big[\mathbf{v}_1 \cdot \mathbf{Q}(\omega-\mathbf{k}\cdot\mathbf{v}_1;\mathbf{v}_1-\mathbf{v}_2) \cdot \left(\frac{d}{d\mathbf{v}_1} - \frac{d}{d\mathbf{v}_2}\right) f_0(\mathbf{v}_2)\ f_1(\mathbf{k},\mathbf{v}_1,\omega)$$

$$+ \mathbf{v}_1 \cdot \mathbf{Q}(\omega-\mathbf{k}\cdot\mathbf{v}_2;\mathbf{v}_1-\mathbf{v}_2) \cdot \left(\frac{d}{d\mathbf{v}_1} - \frac{d}{d\mathbf{v}_2}\right) f_0(\mathbf{v}_1)\ f_1(\mathbf{k},\mathbf{v}_2,\omega)\Big]$$

$$= -\frac{2\pi nq^4}{m}\int d\mathbf{v}_1 \int d\mathbf{v}_2 \ (\mathbf{v}_1-\mathbf{v}_2) \cdot \mathbf{Q}(\omega-\mathbf{k}\cdot\mathbf{v}_1;\mathbf{v}_1-\mathbf{v}_2) \cdot \left(\frac{d}{d\mathbf{v}_1} - \frac{d}{d\mathbf{v}_2}\right) f_1(\mathbf{k},\mathbf{v}_1,\omega)$$

$$\times f_0(\mathbf{v}_2) = 0 \quad . \tag{II-29}$$

The result follows from the fact that $(\underset{\sim}{v}_1 - \underset{\sim}{v}_2) \cdot \underset{\approx}{Q}(\omega - \underset{\sim}{k} \cdot \underset{\sim}{v}_1 ; \underset{\sim}{v}_1 - \underset{\sim}{v}_2) = 0$. We have avoided the confusion of subscripts by considering only a single species. We can generalize equation II-25 immediately to more than one species, by using greek letter subscripts to designate the type of particle.

$$(-i\omega + i\underset{\sim}{k} \cdot \underset{\sim}{v}_1)\, f_\mu{}^1(\underset{\sim}{k}, \underset{\sim}{v}_1, \omega) + \frac{q_\mu}{m_\mu}\, \underset{\sim}{E}^1(\underset{\sim}{k}, \omega) \cdot \frac{df_\mu{}^0}{d\underset{\sim}{v}_1} = \frac{q_\mu{}^2}{m_\mu} \sum_\nu 2\pi n_\nu\, q_\nu{}^2$$

$$\times \frac{d}{d\underset{\sim}{v}_1} \cdot \int d\underset{\sim}{v}_2 \left[\underset{\approx}{Q}(\omega \cdot \underset{\sim}{k} \cdot \underset{\sim}{v}_1 ; \underset{\sim}{v}_1 - \underset{\sim}{v}_2) \cdot \left(\frac{1}{m_\mu} \frac{d}{d\underset{\sim}{v}_1} - \frac{1}{m_\nu} \frac{d}{d\underset{\sim}{v}_2} \right) f_\mu{}^1(\underset{\sim}{k}, \underset{\sim}{v}_1, \omega) f_\nu{}^0(\underset{\sim}{v}_2) \right.$$

$$\left. + \underset{\approx}{Q}(\omega \cdot \underset{\sim}{k} \cdot \underset{\sim}{v}_2 ; \underset{\sim}{v}_1 - \underset{\sim}{v}_2) \cdot \left(\frac{1}{m_\mu} \frac{1}{d\underset{\sim}{v}_1} - \frac{1}{m_\nu} \frac{d}{d\underset{\sim}{v}_2} \right) f_\mu{}^0(\underset{\sim}{v}_1) f_\nu{}^1(\underset{\sim}{k}, \underset{\sim}{v}_2, \omega) \right] . \qquad \text{(II-30)}$$

We may compare equations II-5, II-26, and II-30, and make a few general statements about the significance of the frequency and wavenumber dependence of the collision term of equation II-30. The essential difference between the collision term of equation II-30 and that of equation II-5 (beside the fact that the collision term of equation II-30 is linearized $f \rightarrow f^0 + f^1$, and it allows for more than one species of particle) lies in the logarithm which appears in $\underset{\approx}{Q}$. The logarithm of equation II-30 has frequency and wavenumber dependence. In the limit $\omega \rightarrow 0$, $k \rightarrow 0$, the logarithm reduces to $\ell n\Lambda$, in agreement with equation II-5. In the general case the logarithm is smaller than $\ell n\Lambda$ because of the frequency and wavenumber dependence. The actual dependence is quite complicated, as it involves relative velocity of the particles $|\underset{\sim}{v}_1 - \underset{\sim}{v}_2|$, and their velocity component along the phase velocity of the wave $|\omega - \underset{\sim}{k} \cdot \underset{\sim}{v}|$. We establish a rough criterion for the significance of the logarithmic dependence by requiring that most particles ($|v| < V_{th}$) be within the region where the logarithm is decreased from $\ell n\Lambda$. (We consider $k = 0$). We find $\omega \sim k_d V_{th}$, or $\omega \sim \omega_p$. The Fokker-Planck equation (equation II-5) is modified significantly for frequencies of the order of the plasma frequency (and above). More detailed statements must be based on further analytic work.

C. The Klimontovich Dupree Equations

The set of equations we consider now are formed by averaging (over an ensemble) the equation of motion for a particular system.[24,25,4] Of course the physical content of the equations is the same as that of the BBGKY set of equations. (See appendix A) An advantage of the Klimontovich Dupree set is that we can work with them more easily. Dupree[6] has indicated a formal method for solving the whole set, but we shall see that there are defects in his solution.

We begin by writing the distribution function in $\underset{\sim}{r}, \underset{\sim}{p}$ space for a particular system. The distribution function is normalized to volume $V \to \infty$. The superscript p indicates momentum--we shall drop it when we change to velocity as an independent variable.

$$F^p(\underset{\sim}{r}, \underset{\sim}{p}, t) = \frac{1}{n_\mu} \sum_{s=1}^{N_\mu} \delta\left(\underset{\sim}{r} - \underset{\sim}{r}_{\mu s}(t)\right) \delta\left(\underset{\sim}{p} - \underset{\sim}{p}_{\mu s}(t)\right) . \quad \text{(II-31)}$$

Here $\underset{\sim}{r}_{\mu s}(t)$, $\underset{\sim}{p}_{\mu s}(t)$ defines the position and momentum of particle $s = 1,2,3,\cdots N_\mu$ of species μ, and $n_\mu = \frac{N_\mu}{V}$. We shall be concerned with a two species plasma composed of ions (i) and electrons (e). We assume overall charge neutrality, so that $N_e = N_i = N$. The Hamiltonian of the system is

$$H = \sum_{\mu=e,i} \sum_{s=1}^{N} \frac{p_{\mu s}^2}{2m_\mu} + \frac{1}{2} \sum_{\mu,\nu} \sum_{\substack{\ell,s=1 \\ \mu\ell\neq\nu s}}^{N} \frac{q_\mu q_\nu}{|\underset{\sim}{r}_{\mu\ell} - \underset{\sim}{r}_{\nu s}|} \quad . \tag{II-32}$$

We take the time derivative of II-31 to find

$$\frac{dF_\mu^{\,p}}{dt} = \sum_s \left(\frac{dF_\mu}{d\underset{\sim}{r}_{\mu s}} \cdot \frac{d\underset{\sim}{r}_{\mu s}}{dt} + \frac{dF_\mu}{d\underset{\sim}{p}_{\mu s}} \cdot \frac{d\underset{\sim}{p}_{\mu s}}{dt} \right) \quad . \tag{II-33}$$

From Hamilton's equations we have

$$\frac{d\underset{\sim}{r}_{\mu s}}{dt} = \left[\underset{\sim}{r}_{\mu s}, H\right] = \frac{\underset{\sim}{p}_{\mu s}}{m_\mu} \tag{II-34}$$

$$\frac{d\underset{\sim}{p}_{\mu s}}{dt} = \left[\underset{\sim}{p}_{\mu s}, H\right] = -\sum_\nu \sum_{\substack{m=1 \\ \mu s\neq\nu m}}^{N} \frac{d}{d\underset{\sim}{r}} \frac{q_\mu q_\nu}{|\underset{\sim}{r}_{\mu s} - \underset{\sim}{r}_{\nu m}|} \quad . \tag{II-35}$$

Substituting these results in II-33 we have

$$\frac{dF_\mu^{\,p}}{dt} - \sum_s \frac{\underset{\sim}{p}_{\mu s}}{m_\mu} \cdot \frac{dF_\mu}{d\underset{\sim}{r}_{\mu s}} + \sum_{\substack{\nu \\ \mu s\neq\nu m}} \sum_m \frac{d}{d\underset{\sim}{r}} \frac{q_\mu q_\nu}{|\underset{\sim}{r}_{\mu s} - \underset{\sim}{r}_{\nu m}|} \cdot \frac{dF_\mu}{d\underset{\sim}{p}_{\mu s}} = 0 \; . \tag{II-36}$$

The final term may be simplified if we introduce the electric field.

$$\underset{\sim}{E} = -\sum_{\nu}\sum_{\ell} \frac{d}{d\underset{\sim}{r}} \frac{q_{\nu}}{|\underset{\sim}{r}-\underset{\sim}{r}_{\nu\ell}|} = -\sum_{\nu} n_{\nu}\, q_{\nu} \int d\underset{\sim}{p}'\, d\underset{\sim}{r}'\, \frac{d}{d\underset{\sim}{r}} \frac{1}{|\underset{\sim}{r}-\underset{\sim}{r}'|} F_{\nu}^{p}(\underset{\sim}{r}',\underset{\sim}{p}',t) \; . \qquad \text{(II-37)}$$

We now use the property of the delta function $f(x)\,\delta(x-a) = f(a)\,\delta(x-a)$ to rewrite the equation of motion (corresponding to the conservation of particles) in the familiar form

$$\frac{dF_{\mu}^{p}}{dt} + \frac{\underset{\sim}{p}}{m_{\mu}} \cdot \frac{dF_{\mu}^{p}}{d\underset{\sim}{r}} + q_{\mu} \frac{d}{d\underset{\sim}{p}} \cdot \left\{ \underset{\sim}{E}\, F_{\mu}^{p} \right\} = 0 \; . \qquad \text{(II-38)}$$

The curly brackets $\{\}$ indicate that the self force term is to be omitted.

Finally it is customary to work in the variables $\underset{\sim}{r},\underset{\sim}{v}$ rather than $\underset{\sim}{r},\underset{\sim}{p}$. Hence we change variables so that $\underset{\sim}{p} = m_{\nu}\,\underset{\sim}{v}$. The distribution function is now (we drop the superscript p).

$$F_{\mu}(\underset{\sim}{r},\underset{\sim}{v},t) = \frac{1}{n_{\mu}} \sum_{s=1}^{N} \delta\left(\underset{\sim}{r} - \underset{\sim}{r}_{\mu s}(t)\right) \delta\left(\underset{\sim}{v} - \underset{\sim}{v}_{\mu s}(t)\right) . \qquad \text{(II-39)}$$

The equation of motion becomes

$$\frac{dF_{\mu}}{dt} + \underset{\sim}{v} \cdot \nabla F_{\mu} + \frac{q_{\mu}}{m_{\mu}} \frac{d}{d\underset{\sim}{v}} \cdot \left\{ \underset{\sim}{E}\, F_{\nu} \right\} = 0 \quad . \qquad \text{(II-40)}$$

The electric field is given by

$$\underset{\sim}{E}(\underset{\sim}{r},t) = -\sum_{\nu} n_{\nu} q_{\nu} \int d\underset{\sim}{v}' \, d\underset{\sim}{r}' \, \frac{1}{|\underset{\sim}{r}-\underset{\sim}{r}'|} \, F_{\nu}(\underset{\sim}{r}',\underset{\sim}{v}',t) \quad . \qquad \text{(II-41)}$$

Equations II-40 and II-41 appear simple, in the sense that equation II-40 is an equation in $\underset{\sim}{r}, \underset{\sim}{v}, t$, while II-41 is an equation in $\underset{\sim}{r}, t$. Of course the complete solution would be the complete motion of our 2N particles, for the complexity is hidden in the definition of F. Since this complete solution cannot be found, we seek the behavior of an average system whose properties correspond to the small amount of information we might obtain about a particular system.

We let

$$F_{\mu}(\underset{\sim}{r},\underset{\sim}{v},t) = f_{\mu}(\underset{\sim}{r},\underset{\sim}{v},t) + \delta f_{\mu}(\underset{\sim}{r},\underset{\sim}{v},t) \qquad \text{(II-42)}$$

$$\underset{\sim}{E}(\underset{\sim}{r},t) = \underset{\sim}{E}^{0}(\underset{\sim}{r},t) + \delta \underset{\sim}{E}(\underset{\sim}{r},t) \qquad \text{(II-43)}$$

where f_{μ} is the average one particle distribution function of the system, and E^{0} the average electric field. Of course δ now represents a difference, not a Dirac delta function. If we denote the averaging process by $\langle\rangle$, we have

$$\langle F_{\mu} \rangle = f_{\mu} \qquad \text{(II-44)}$$

$$\langle \underset{\sim}{E} \rangle = \underset{\sim}{E}^{0} \qquad \text{(II-45)}$$

which implies that $\langle \delta f_\mu \rangle = 0$ and $\langle \delta \underset{\sim}{E} \rangle = 0$ but does not imply that an expression like $\langle \delta f \, \delta E \rangle$ is 0. We now substitute equations II-42 and II-43 into equations II-40 and II-41, and average. This yields the equations

$$\frac{df_\mu}{dt} + \underset{\sim}{v} \cdot \nabla f_\mu + \frac{q_\mu}{m_\mu} \frac{d}{d\underset{\sim}{v}} \cdot \underset{\sim}{E}^0 f_\mu + \frac{q_\mu}{m_\mu} \frac{d}{d\underset{\sim}{v}} \cdot \langle \{ \delta \underset{\sim}{E} \delta f_\mu \} \rangle = 0 \qquad \text{(II-46)}$$

$$\underset{\sim}{E}^0 = - \sum_\nu n_\nu q_\nu \int d\underset{\sim}{r}' \, d\underset{\sim}{v}' \, \nabla \frac{1}{|\underset{\sim}{r} - \underset{\sim}{r}'|} f_\nu(\underset{\sim}{r}', \underset{\sim}{v}', t) \, . \qquad \text{(II-47)}$$

It is often convenient to use the fact that $\nabla \cdot \nabla \frac{1}{|\underset{\sim}{r}|} = - 4\pi \, \delta(\underset{\sim}{r})$ to rewrite equation II-47 in the form of Poisson's equation.

$$\nabla \cdot \underset{\sim}{E}^0 = \sum_\nu 4\pi n_\nu q_\nu \int d\underset{\sim}{v} \, f_\nu(\underset{\sim}{r}, \underset{\sim}{v}, t) \, . \qquad \text{(II-48)}$$

The equations do not form a closed set, since equation II-46 depends on the quantity $\langle \delta f \, \delta \underset{\sim}{E} \rangle$. To proceed we must find an equation for this quantity. However this equation will depend on $\langle \delta f \delta f \delta \underset{\sim}{E} \rangle$, etc. In general we end up with a coupled set of equations similar to the BBGKY hierarchy. (The relation between the two sets of equations is described in appendix A. It is shown that the equations are equivalent). We indicate now the procedure for forming these equations.

We subtract equations II-46 and II-47 from equations II-40 and II-41 to produce the equations for the fluctuations δf_μ and $\delta \underset{\sim}{E}$.

$$\left(\frac{d}{dt} + \mathbf{v}\cdot\nabla\right) \delta f_\mu + \frac{q_\mu}{m_\mu} \frac{d}{d\mathbf{v}} \cdot \mathbf{E}^0 \delta f_\mu + \frac{q_\mu}{m_\mu} \frac{d}{d\mathbf{v}} \cdot \delta \mathbf{E} f_\mu$$

$$= \frac{q_\mu}{m_\mu} \frac{d}{d\mathbf{v}} \cdot \left(\{\delta \mathbf{E}\, \delta f_\mu\}\right) - \delta\mathbf{E}\, \delta f_\mu \qquad \text{(II-49)}$$

$$\delta\mathbf{E} = - \sum_\nu n_\nu q_\nu \int d\mathbf{r}'\, d\mathbf{v}'\, \nabla \frac{1}{|\mathbf{r}-\mathbf{r}'|} \delta f_\nu(\mathbf{r}', \mathbf{v}', t) \,. \qquad \text{(II-50)}$$

The equation for a quantity $\langle \delta f_e \delta f_e \cdots \delta f_i \rangle$ becomes unmanageably long unless we shorten notation. The procedure we use is very similar to that of Dupree. We abbreviate coordinate notation, $\mathbf{r}_1, \mathbf{v}_1 \to 1$, and consider δf_e, δf_i as a column vector by writing

$$\delta f(1) = \begin{bmatrix} \delta f_e(\mathbf{r}_1, \mathbf{v}_1, t) \\ \\ \delta f_i(\mathbf{r}_1, \mathbf{v}_1, t) \end{bmatrix} \,. \qquad \text{(II-51)}$$

We define a matrix operator T, which uses equation II-50 in expressing the electric field

$$T(1) = \begin{bmatrix} \left(\begin{array}{l} \mathbf{v}_1 \cdot \nabla_1 + \dfrac{q_e}{m_e} \dfrac{d}{d\mathbf{v}_1} \cdot \mathbf{E}^0 \\ - \dfrac{n_e q_e}{m_e} \dfrac{df_e}{d\mathbf{v}_1} \cdot \displaystyle\int d\mathbf{r}' d\mathbf{v}' \nabla_1 \frac{1}{|\mathbf{r}_1 - \mathbf{r}'|} \end{array} \right) & \left(- \dfrac{n_e q_e}{m_e} \dfrac{df_e}{d\mathbf{v}_1} \cdot \displaystyle\int d\mathbf{r}'\, d\mathbf{v}'\, \nabla_1 \frac{1}{|\mathbf{r}_1 - \mathbf{r}'|} \right) \\ \left(- \dfrac{n_i q_i}{m_i} \dfrac{df_i}{d\mathbf{v}_1} \cdot \displaystyle\int d\mathbf{r}' d\mathbf{v}'\, \nabla_1 \frac{1}{|\mathbf{r}_1 - \mathbf{r}'|} \right) & \left(\begin{array}{l} \mathbf{v}_1 \cdot \nabla_1 + \dfrac{q_i}{m_i} \dfrac{d}{d\mathbf{v}_1} \cdot \mathbf{E}^0 \\ - \dfrac{n_i q_i}{m_i} \dfrac{df_i}{d\mathbf{v}_1} \cdot \displaystyle\int d\mathbf{r}' d\mathbf{v}'\, \nabla_1 \frac{1}{|\mathbf{r}_1 - \mathbf{r}'|} \end{array} \right) \end{bmatrix}$$

(II.52)

The left-hand side of equation II-49 may now be written (for both species, with $\underset{\sim}{r}, \underset{\sim}{v} \rightarrow \underset{\sim}{r}_1, \underset{\sim}{v}_1$)

$$\left[\frac{d}{dt} + T(1)\right] \delta f(1) . \tag{II.53}$$

The integral term in $T\delta f$, involving $\int d\underset{\sim}{r}' \, d\underset{\sim}{v}' \, \nabla_1 \frac{1}{|\underset{\sim}{r}_1 - \underset{\sim}{r}'|} \delta f$ is taken to mean $\int d\underset{\sim}{r}' \, d\underset{\sim}{v}' \, \nabla_1 \frac{1}{|\underset{\sim}{r}_1 - \underset{\sim}{r}'|} \delta f(\underset{\sim}{r}', \underset{\sim}{v}', t)$. The remaining terms of equation II-49 may be shortened by defining an operator $\mathcal{O}$, which is a function of $\delta\underset{\sim}{E}$.

$$\underset{\sim}{\mathcal{O}}(1) = \begin{bmatrix} \frac{q_e}{m_e} \frac{d}{d\underset{\sim}{v}_1} \cdot \delta\underset{\sim}{E}(1) & 0 \\ & \\ 0 & \frac{q_i}{m_i} \frac{d}{d\underset{\sim}{v}_1} \cdot \delta\underset{\sim}{E}(1) \end{bmatrix} . \tag{II-54}$$

Equation II-49 now has the simple form

$$\left[\frac{d}{dt} + T(1)\right] \delta f(1) = \langle\{\underset{\sim}{\mathcal{O}}(1)\, \delta f(1)\}\rangle - \{\underset{\sim}{\mathcal{O}}(1)\, \delta f(1)\} . \tag{II-55}$$

We may now multiply equation II-55 by $\delta f(2) \cdots \delta f(n)$, and interchange arguments to produce equations for all $\delta f(1)\, \delta f(2) \cdots \frac{d}{dt} \delta f(i) \cdots \delta f(n)$. When we add these we have

$$\left[\frac{d}{dt} + \sum_{i=1}^{n} T(i)\right] \delta f(1)\, \delta f(2) \cdots \delta f(i) \cdots \delta f(n)$$

$$= \sum_{i=1}^{n} \langle \{ \underset{\sim}{\mathcal{A}}(i) \delta f(i) \} \rangle\, \delta f(1) \cdots \delta f(i-1) \delta f(i+1) \cdots \delta f(n)$$

$$- \sum_{i=1}^{n} \delta f(1) \cdots \{ \underset{\sim}{\mathcal{A}}(i)\, \delta f(i) \} \cdots \delta f(n) \quad . \qquad \text{(II-56)}$$

The average of this equation represents the equation for the general term $\langle \delta f(1) \cdots \delta f(n) \rangle$

$$\left[\frac{d}{dt} + \sum_{i=1}^{n} T(i)\right] \langle \delta f(1) \cdots \delta f(n) \rangle$$

$$= \sum_{i=1}^{n} \langle \{ \mathcal{A}(i) \delta f(i) \} \rangle \langle \delta f(1) \cdots \delta f(i-1) \cdot \delta f(i+1) \cdots \delta f(n) \rangle$$

$$- \sum_{i=1}^{n} \langle \delta f(1) \cdots \{ \mathcal{A}(i)\, \delta f(i) \} \cdots \delta f(n) \rangle \quad . \qquad \text{(II-57)}$$

The terms involving $\delta \underset{\sim}{E}$ may be found by appropriate use of equation II-50

$$\langle \delta f(1) \cdots \{ \delta f(i)\, \delta \underset{\sim}{E}(i) \} \cdots \delta f(n) \rangle = - \sum_{\nu} n_{\nu} q_{\nu} \int d\underset{\sim}{r}'\, d\underset{\sim}{v}'\, \nabla_i \frac{1}{|\underset{\sim}{r}_i - \underset{\sim}{r}'|}$$

$$\times \langle \delta f(1) \cdots \delta f(i)\, \delta f_{\nu}(\underset{\sim}{r}', \underset{\sim}{v}', t) \cdots \delta f(n) \rangle \quad . \qquad \text{(II-58)}$$

Note that we do not have to use curly brackets on the right hand side. The self force vanishes when the angular part of the $\underset{\sim}{r}'$

integration is performed. We now describe Dupree's method for solving these equations.

We may in principle solve the equation

$$\left[\frac{d}{dt} + T(1) \right] \psi(1) = 0 \tag{II-59}$$

to find $\psi(\underset{\sim}{r}_1, \underset{\sim}{v}_1, t)$ in terms of $\psi(\underset{\sim}{r}_1, \underset{\sim}{v}_1,\ t = 0)$. (A general analytic solution is out of the question. However the assumptions of spatial uniformity, etc., are not necessary until we wish to obtain this analytic solution). We define an operator P by the relation

$$\psi(\underset{\sim}{r}, \underset{\sim}{v}, t) = P(\underset{\sim}{r}, \underset{\sim}{v}, t)\, \psi(\underset{\sim}{r}, \underset{\sim}{v}, t = 0) \tag{II-60}$$

with the boundary condition $P(\underset{\sim}{r}, \underset{\sim}{v}, t = 0) = 1$.

It follows that

$$\left(\frac{d}{dt} + T \right) P = 0 \quad . \tag{II-61}$$

$P(t)$ gives the evolution of the system forward in time, while $P^{-1}(t) = P(-t)$ gives the evolution backward in time. We note also that when T is time independent

$$P(t_1)\, P(t_2) = P(t_1 + t_2) \quad . \tag{II-62}$$

If we multiply equation II-61 on the right and left by $P^{-1}(t)$, and use the fact that $\frac{d}{dt}(P^{-1}P) = \frac{d}{dt}(1) = 0$, we find

$$\frac{d}{dt} P^{-1}(t) - P^{-1}(t)\, T = 0 . \qquad \text{(II-63)}$$

We may generalize equations II-61 and II-63 immediately if the operators apply to different set of coordinates. Thus

$$\left[\frac{d}{dt} + \sum_{i=1}^{n} T(i)\right] P(1)\cdots P(i)\cdots P(n) = 0 \qquad \text{(II-64)}$$

$$\frac{d}{dt}\left(P^{-1}(1)\cdots P^{-1}(n)\right) - \left(P^{-1}(1)\cdots P^{-1}(n)\right)\sum_{i=1}^{n} T(i) = 0 . \qquad \text{(II-65)}$$

Using these operators we readily solve the general equation, II-57. We multiply on the left by $P^{-1}(1)\cdots P^{-1}(n)$ and use equation II-65 for the expression $P^{-1}(1)\cdots P^{-1}(n) \sum_{i=1}^{n} T(i)$.

$$P^{-1}(1)\cdots P^{-1}(n)\left[\frac{d}{dt} + \sum_{i=1}^{n} T(i)\right]\langle \delta f(1)\cdots \delta f(n)\rangle = - P^{-1}(1)\cdots P^{-1}(n)$$

$$\sum_{i=1}^{n} \langle \delta f(1)\cdots \{\underset{\sim}{\theta}(i)\delta f(i)\, \delta f(n)\rangle \qquad \text{(II-66)}$$

$$+ P^{-1}(1)\cdots P^{-1}(n) \sum_{i=1}^{n} \langle \{\underset{\sim}{\theta}(i)\delta f(i)\}\rangle\, \langle \delta f(1)\cdots \delta f(i-1)\delta f(i+1)\cdots \delta f(n)\rangle .$$

We shall abbreviate by setting $P(1)\cdots P(n) = P_n$. Then II-66 becomes

$$P_n^{-1}(t)\frac{d}{dt}\langle\delta f(1)\cdots\delta f(n)\rangle + \frac{dP_n^{-1}}{dt}\langle\delta f(1)\cdots\delta f(n)\rangle =$$

$$- P_n^{-1}\sum_{i=1}^{n}\langle\delta f(1)\cdots\{\underset{\sim}{\theta}(i)\delta f(i)\}\cdots\delta f(n)\rangle + P_n^{-1}\sum_{i=1}^{n}\langle\{\underset{\sim}{\theta}(i)\delta f(i)\}\rangle$$

$$\langle\delta f(1)\cdots\delta f(i-1)\delta f(i+1)\cdots\delta f(n)\rangle \quad . \qquad \text{(II-67)}$$

The left hand side is a perfect differential equal to $\frac{d}{dt}(P_n^{-1}$ $\langle\delta f(1)\cdots\delta f(n)\rangle)$, so that we may perform a time integral.

$$P_n^{-1}(t)\langle\delta f(1)\cdots\delta f(n)|t\rangle = \langle\delta f(1)\cdots\delta f(n)|t=0\rangle + \int_0^t d\tau\, P_n^{-1}(\tau)\,\cdot$$

$$\left[\sum_{i=1}^{n}\langle\{\underset{\sim}{\theta}(i)\delta f(i)\}\rangle\langle\delta f(1)\cdots\delta f(i-1)\delta f(i+1)\cdot\delta f(n)\rangle - \sum_{i=1}^{n}\right.$$

$$\langle\delta f(1)\cdots\{\delta f(i)\,\underset{\sim}{\theta}(i)\}\cdots\delta f(n)\rangle \quad . \qquad \text{(II-68)}$$

Multiplying by $P_n(t)$, we produce the general result

$$\langle\delta f(1)\cdots\delta f(n)|t\rangle = P_n(t)\,\langle\delta f(1)\cdots\delta f(n)|t=0\rangle + P_n(t)\int_0^t d\tau\, P_n^{-1}(\tau)\,\cdot$$

$$\left[\sum_{i=1}^{n}\langle\{\underset{\sim}{\theta}(i)\delta f(i)\}\rangle\langle\delta f(1)\cdots\delta f(i-1)\delta f(i+1)\cdots\delta f(n) - \sum_{i=1}^{n}\right.$$

$$\langle\delta f(1)\cdots\{\underset{\sim}{\theta}(i)\delta f(i)\}\cdots\delta f(n)\rangle \quad . \qquad \text{(II-69)}$$

Terms involving $\delta \underset{\sim}{E}$ through the $\underset{\sim}{\mathcal{D}}$ operator may be calculated using equation II-58. We note that $\langle\{P_2\delta f\delta f\}\rangle \neq P_2\langle\{\delta f\delta f\}\rangle$, etc. In calculating $\langle\{\delta f\delta \underset{\sim}{E}\}\rangle$ etc., using the P operators, we find that the self force term vanishes, as it should. We shall frequently omit the curly brackets for this reason.

We have discussed in some detail the methods for obtaining equation II-69 because we shall use them later in this work. We now sketch very briefly Dupree's conclusions regarding the solution to the original set of equations.

Of course equation II-69 does not represent a complete solution to the set of equations. The original set of coupled differential equations has been replaced by a set of coupled integral equations. Dupree suggests an iteration technique (i.e., perturbation theory) based on an ordering scheme similar to that of the BBGKY hierarchy. One may calculate in order $\langle\delta f\delta f\rangle$, $\langle\delta f\delta f\delta f\rangle$, etc., and by working up and down the set of equations find these quantities to any accuracy desired (in terms of the smallness parameter). It is clear that the validity of this scheme depends on the operator P always being of order 1, as it is at $t = 0$. Unfortunately this is not always true. In fact P becomes very large just at those points where the plasma exhibits collective behavior--a tendency toward coherent motion rather than individual particle motion. In addition the P operator is divergent for short range interactions. Dupree eliminates this divergence by the usual cutoff. However in view of this cutoff it seems hardly reasonable to try to carry the iteration technique to any great length, especially since the behavior of higher order terms is affected more and more strongly by short range effects.

A further difficulty crops up in the time integral in equation II-69. Only the lowest order approximation of $\langle \delta f \delta f \rangle$ does not require this time integral. All corrections and higher terms, which do involve this integral, cannot be calculated for arbitrary time because we cannot perform the integrals. With this in mind Dupree suggests that we calculate all quantities in the limit $t \to \infty$. This procedure, which is based on the Bogoliubov hypothesis (to be d iscussed shortly) breaks down completely for an unstable plasma. In general we may not accept the limit $t \to \infty$ unless the plasma is in equilibrium. However if the plasma is stable this limit is usually valid (with negligible error), and does permit the time integrals to be performed.

We conclude that Dupree's work is an elegant and largely successful attempt at a complete description of plasma behavior. In addition he is able to include electromagnetic effects without the slightest difficulty. This is indeed a step forward from the BBGKy hierarchy! Dupree's results are not perfect because he underestimatesthe difficulty of obtaining a valid description of plasma behavior. A brilliant formal method is no better than the foundation it rests on. In this work we shall attempt to patch up the foundations of plasma kinetic theory. In the process we must give up the claim to generality that Dupree makes.

Having criticized the general aspects of Dupree's theory, we turn now to those particulars for which the theory is admirally suited. We shall illustrate the direct and straightforward way in which Dupree's method yields the present plasma kinetic equation, the Lenard Balescu equation.

To construct the kinetic equation (equation II-46) we need the quantity $\langle\{\delta f \delta \underset{\sim}{E}\}\rangle$, which may be constructed from $\langle \delta f \delta f\rangle$ using equation II-58. From equation II-69 we have

$$\langle \delta f \delta f \rangle = P_2 \langle \delta f(1) \delta f(2) | t = 0 \rangle + P_2 \int_0^t d\tau \, P_2^{-1} \Big[\langle \{ \underset{\sim}{\mathcal{O}}(1) \delta f(1) \} \delta f(2) \rangle$$

$$+ \langle \delta f(1) \{ \underset{\sim}{\mathcal{O}}(2) \delta f(2) \} \rangle \quad . \qquad \text{(II-70)}$$

Since the $\underset{\sim}{\mathcal{O}}$ operator involves $\delta \underset{\sim}{E}$, the time integral involves $\langle \delta f \delta f \delta \underset{\sim}{E}\rangle$, which is higher order than $\langle \delta f \delta f\rangle$. (It is not generally true that $\langle \delta f(1) \cdots \delta f(n) \delta \underset{\sim}{E}\rangle$ is higher order than $\langle \delta f(1) \ldots \delta f(n)\rangle$. It <u>is</u> true for the single case $n = 2$. This is discussed in appendix A). Therefore in the first approximation we have

$$\langle \delta f \delta f \rangle = P(1,t) \, P(2,t) \, \langle \delta f(1, t = 0) \, \delta f(2, t = 0) \rangle \, . \qquad \text{(II-71)}$$

To evaluate this expression we must calculate the P operator defined by equation II-60. The P operator is obtained by solving equation II-61, which is simpler if we consider P operating on some quantity, say δf. Note that our purpose in solving this equation (II-59) is to obtain the P operator, not to obtain the behavior of δf. Dupree always uses P to obtain the behavior of averaged quantities. We now write out equation II-59 for particle species μ.

$$\left(\frac{d}{dt} + \underset{\sim}{v}\cdot\nabla\right) \delta f_\mu + \frac{q_\mu}{m_\mu} \frac{d}{d\underset{\sim}{v}} \cdot \underset{\sim}{E}^0 \, \delta f_\mu + \frac{q_\mu}{m_\mu} \frac{d}{d\underset{\sim}{v}} \cdot \delta\underset{\sim}{E} \, f_\mu = 0 \quad . \qquad \text{(II-72)}$$

The solution of this equation given an arbitrary $f_\mu(\underset{\sim}{r}, \underset{\sim}{v}, t)$ is beyond present analytic methods. For this reason we limit ourselves to distribution functions which are uniform in space, and assume no external electric field (the net field of the charges is 0, of course) so that $\underset{\sim}{E}^0 = 0$. These restrictions will remain through most of the rest of this work.

Despite the simplification the equation for δf_μ remains generally insoluble if we permit f_μ to have time dependence. We now drop this time dependence of f_μ, on the basis that δf_μ varies much more rapidly in time than f_μ. This "adiabatic hypothesis" permits us to proceed analytically, but is motivated by physical considerations.

We emphasize the distinction between dropping the space dependence and dropping the time dependence of f_μ. The omission of spatial dependence of f is simply a restriction on the class of distribution functions for which we develop a kinetic theory. It is made without any qualification whatever. In contrast the omission of time dependence for f is <u>never</u> valid unless f represents a stationary (equilibrium) state, with the trivial kinetic equation

$\frac{df}{dt} = 0$. It may be true as an approximation, but this must be checked a posteriori. We shall see later that this approximation is generally a very good one, with several rather unimportant exceptions.

The equation for δf_μ has now reduced to the simple form

$$\left(\frac{d}{dt} + \underset{\sim}{v}\cdot\nabla\right) \delta f_\mu(\underset{\sim}{r},\underset{\sim}{v},t) + \frac{q_\mu}{m_\mu} \frac{d}{d\underset{\sim}{v}} \cdot \delta\underset{\sim}{E}(\underset{\sim}{r},t)\, f_\mu(\underset{\sim}{v}) = 0 \qquad \text{(II-73)}$$

where $\delta\underset{\sim}{E}$ is determined by equation II-50.

Following Landau,[6] we solve the initial value problem with a Fourier transform in space and a Laplace transform in time.

$$\delta f_\mu(\underset{\sim}{k},\underset{\sim}{v},\omega) = \int d\underset{\sim}{r}\, e^{-i\underset{\sim}{k}\cdot\underset{\sim}{r}} \int_0^\infty dt\, e^{i\omega t}\, \delta f_\mu(\underset{\sim}{r},\underset{\sim}{v},t) \qquad \text{(II-74)}$$

$$\delta\underset{\sim}{E}(\underset{\sim}{k},\omega) = \int d\underset{\sim}{r}\, e^{-i\underset{\sim}{k}\cdot\underset{\sim}{r}} \int_0^\infty dt\, e^{i\omega t}\, \delta\underset{\sim}{E}(\underset{\sim}{r},t) \quad . \qquad \text{(II-75)}$$

ω is far enough in the upper half ω plane to insure convergence of the time integrals and may be analytically continued to the lower half plane. Later we shall have occasion to move k from the real axis. We note the general relation

$$A(\underset{\sim}{k},\omega) = A(-k^*, -\omega^*)^* \qquad \text{(II-76)}$$

where the star means the complex conjugate is to be taken. The inverse transforms are

$$\delta f_\mu(\underset{\sim}{r},\underset{\sim}{v},t) = \int \frac{d\underset{\sim}{k}}{(2\pi)^3} e^{i\underset{\sim}{k}\cdot\underset{\sim}{r}} \int_C \frac{d\omega}{2\pi} e^{-i\omega t} \delta f_\mu(\underset{\sim}{k},\underset{\sim}{v},\omega) . \tag{II-77}$$

$$\delta\underset{\sim}{E}(\underset{\sim}{r},t) = \int \frac{d\underset{\sim}{k}}{(2\pi)^3} \int_C \frac{d\omega}{2\pi} e^{-i\omega t} \delta\underset{\sim}{E}(\underset{\sim}{k},\omega) \tag{II-78}$$

where C is a contour parallel to the real ω axis and above all singularities of δf_μ or $\delta\underset{\sim}{E}$. The electric field is now expressed as

$$\delta\underset{\sim}{E}(\underset{\sim}{k},\omega) = \sum_\nu \left(\frac{-i\underset{\sim}{k}}{k^2}\right) 4\pi\, n_\nu q_\nu \int d\underset{\sim}{v}\, \delta f_\nu(\underset{\sim}{k},\underset{\sim}{v},\omega) \tag{II-79}$$

while the equation for δf_μ becomes

$$(-i\omega + i\underset{\sim}{k}\cdot\underset{\sim}{v})\delta f_\mu + \frac{q_\mu}{m_\mu} \frac{d}{d\underset{\sim}{v}} \cdot \delta\underset{\sim}{E}\, f_\mu = \delta f_\mu(\underset{\sim}{k},\underset{\sim}{v},t=0) . \tag{II-80}$$

The solution for δf and $\delta\underset{\sim}{E}$ in terms of $\delta f(t=0)$ is simple algebra,

$$\delta f_\mu(\underset{\sim}{r},\underset{\sim}{v},t) = \int \frac{d\underset{\sim}{k}}{(2\pi)^3} e^{i\underset{\sim}{k}\cdot\underset{\sim}{r}} \int \frac{d\omega}{2\pi} e^{-i\omega t} \Bigg[\frac{1}{-i\omega + i\underset{\sim}{k}\cdot\underset{\sim}{v}} + \frac{\sum_\nu \frac{q_\mu}{m_\mu} \frac{i\underset{\sim}{k}}{k^2} \cdot \frac{df_\mu}{d\underset{\sim}{v}} 4\pi\, n_\nu q_\nu}{(-i\omega + i\underset{\sim}{k}\cdot\underset{\sim}{v})\,\epsilon(\underset{\sim}{k},\omega)} \int \frac{d\underset{\sim}{v}'}{(-i\omega + i\underset{\sim}{k}\cdot\underset{\sim}{v}')} \Bigg] \delta f_\mu(t=0) \tag{II.81}$$

$$\delta\underset{\sim}{E}(\underset{\sim}{r},t) = \int \frac{d\underset{\sim}{k}}{(2\pi)^3} e^{i\underset{\sim}{k}\cdot\underset{\sim}{r}} \int \frac{d\omega}{2\pi} e^{-i\omega t} \left[\sum_\nu \left(\frac{-i\underset{\sim}{k}}{k^2}\right) \frac{4\pi n_\nu q_\nu}{\epsilon(\underset{\sim}{k},\omega)} \int \frac{d\underset{\sim}{v}}{-i\omega + i\underset{\sim}{k}\cdot\underset{\sim}{v}} \right] \times \delta f_\nu(t=0) \tag{II-82}$$

where

$$\epsilon(\underset{\sim}{k},\omega) = 1 + \sum_{\nu} \frac{\omega_{\nu}^{2}}{k^{2}} \int \frac{\underset{\sim}{k} \cdot \frac{df_{\nu}}{d\underset{\sim}{v}} \; d\underset{\sim}{v}}{\omega - \underset{\sim}{k} \cdot \underset{\sim}{v}} \; . \tag{II.83}$$

From equation II-81 we may readily identify the Fourier-Laplace transform of the P operator.

$$P(k,v,\omega) = \left[\frac{\sum_{\nu} \frac{q_{\mu}}{m_{\mu}} \frac{i\underset{\sim}{k}}{k^{2}} \cdot \frac{df_{\mu}}{d\underset{\sim}{v}} \; 4\pi \, n_{\nu} q_{\nu}}{(-i\omega+i\underset{\sim}{k}\cdot v) \;\; \epsilon(\underset{\sim}{k},\omega)} \int \frac{d\underset{\sim}{v}'}{-i\omega+i\underset{\sim}{k}\cdot\underset{\sim}{v}'} + \frac{1}{-i\omega+i\underset{\sim}{k}\cdot\underset{\sim}{v}} \right] . \tag{II-84}$$

The explicit form of the P operator is

$$P(\underset{\sim}{r},\underset{\sim}{v},t) = \int \frac{d\underset{\sim}{k}}{(2\pi)^{3}} \, e^{i\underset{\sim}{k}\cdot\underset{\sim}{r}} \int_{C} \frac{d\omega}{2\pi} \, e^{-i\omega t} \, P(\underset{\sim}{k},\underset{\sim}{v},\omega) \; . \tag{II.85}$$

In calculations we apply the P operator to the Fourier transform of the initial value of δf. Frequently we wish to calculate $\delta\underset{\sim}{E}$ from δf, after using the P operator. This is given by equation II-82.

We may now write a more explicit expression for the collision term for a uniform plasma.

$$\frac{q_\mu}{m_\mu}\frac{d}{d\underset{\sim}{v}_1}\cdot\langle\{\delta f_\mu \delta \underset{\sim}{E}\}\rangle = \frac{q_\mu}{m_\mu}\frac{d}{d\underset{\sim}{v}_1}\cdot\int\frac{d\underset{\sim}{k}_1}{(2\pi)^3}\int\frac{d\underset{\sim}{k}_2}{(2\pi)^3}\, e^{i\underset{\sim}{k}_1\cdot\underset{\sim}{r}}\, e^{i\underset{\sim}{k}_2\cdot\underset{\sim}{r}}$$

$$\times\int_C\frac{d\omega_1}{2\pi}\,e^{-i\omega_1 t}\int_C\frac{d\omega_2}{2\pi}\,e^{-i\omega_2 t}\left[\frac{\sum_\nu \frac{q_\mu}{m_\mu}\frac{i\underset{\sim}{k}_1}{k_1^2}\cdot\frac{df_\mu}{d\underset{\sim}{v}_1}4\pi\, n_\nu q_\nu}{(-i\omega_1 + i\underset{\sim}{k}_1\cdot\underset{\sim}{v}_1)\epsilon(\underset{\sim}{k}_1,\omega_1)}\int\frac{d\underset{\sim}{v}_1{}'}{-i\omega_1+i\underset{\sim}{k}_1\cdot\underset{\sim}{v}_1{}'}\right.$$

$$\left.+\frac{1}{-i\omega_1+i\underset{\sim}{k}_1\cdot\underset{\sim}{v}_1}\right]\left[\sum_\alpha\frac{-i\underset{\sim}{k}_2}{k_2^2}\frac{4\pi\, n_\alpha q_\alpha}{\epsilon(\underset{\sim}{k}_2,\omega_2)}\int\frac{d\underset{\sim}{v}_2}{(-i\omega_2+i\underset{\sim}{k}_2\cdot\underset{\sim}{v}_2)}\right]$$

$$\times\langle\delta f_\mu(k,\underset{\sim}{v}_1,t=0)\,\delta f_\alpha(\underset{\sim}{k}_2,\underset{\sim}{v}_2,t=0)\rangle \quad . \qquad \text{(II.86)}$$

As it stands $\langle\delta f\delta\underset{\sim}{E}\rangle$ will have explicit time dependence, and will, of course, depend on the initial value $\langle\delta f\delta f|t=0\rangle$. We come now to a second hypothesis (due to Bogoliubov);[12] we may calculate $\langle\delta f\delta\underset{\sim}{E}\rangle$ in the limit $t\to\infty$. This assumption, like the preceding one, is based on physical considerations, and may be tested a posteriori. It amounts to a statement that $\langle\delta f\delta\underset{\sim}{E}\rangle$ becomes a function of f much faster than the characteristic time in which f changes. Note the distinction: the first (adiabatic) hypothesis states that δf varies much more rapidly than f, while the second (time asymptotic) hypothesis says that $\langle\delta f\delta\underset{\sim}{E}\rangle$ reaches a stationary value much more rapidly than f varies in time. The latter hypothesis, which is obviously much stronger than the former, is not verifiable, and will result in considerable labor later in this work. For purposes of demonstrating the use of the P operator we accept the hypothesis at present.

Granted this hypothesis, we may invert the Laplace transform in equation II-86, ignoring all zeros of $\epsilon(k,\omega)$ since their contribution will vanish in the limit $t \to \infty$. Note that we are forced to assume a stable plasma, such that the zeros of ϵ are in the lower half ω plane. We have now

$$\frac{q_\mu}{m_\mu} \frac{d}{d\underset{\sim}{v}_1} \cdot \langle \delta f_\mu \delta \underset{\sim}{E} \rangle = \lim_{t \to \infty} \frac{q_\mu}{m_\mu} \frac{d}{d\underset{\sim}{v}_1} \cdot \int \frac{d\underset{\sim}{k}_1}{(2\pi)^3} e^{i\underset{\sim}{k}_1 \cdot \underset{\sim}{r}} \int \frac{d\underset{\sim}{k}_2}{(2\pi)^3} e^{i\underset{\sim}{k}_2 \cdot \underset{\sim}{r}}$$

$$+ \left[e^{-i\underset{\sim}{k}_1 \cdot \underset{\sim}{v}_1 t} + \sum_\nu \frac{q_\mu}{m_\mu} \frac{i\underset{\sim}{k}_1}{k_1^2} \cdot \frac{df_\mu}{d\underset{\sim}{v}_1} 4\pi n_\nu q_\nu \int \frac{d\underset{\sim}{v}'_1}{i\underset{\sim}{k}_1 \cdot \underset{\sim}{v}_1 - i\underset{\sim}{k}_1 \cdot \underset{\sim}{v}'_1} \right.$$

$$\left. \left(\frac{e^{-i\underset{\sim}{k}_1 \cdot \underset{\sim}{v}'_1 t}}{\epsilon(\underset{\sim}{k}_1, \underset{\sim}{k}_1 \cdot \underset{\sim}{v}'_1)} - \frac{e^{-i\underset{\sim}{k}_1 \cdot \underset{\sim}{v}_1 t}}{\epsilon(\underset{\sim}{k}_1, \underset{\sim}{k}_1 \cdot v_1)} \right) \right] \left[\sum_\alpha \frac{-i\underset{\sim}{k}_2}{k_2^2} 4\pi n_\alpha q_\alpha \right.$$

$$\left. \times \int \frac{d\underset{\sim}{v}_2 \, e^{-i\underset{\sim}{k}_2 \cdot \underset{\sim}{v}_2 t}}{\epsilon(\underset{\sim}{k}_2, \underset{\sim}{k}_2 \cdot \underset{\sim}{v}_2)} \right] \langle \delta f_\mu(\underset{\sim}{k}_1, \underset{\sim}{v}_1, t=0) \, \delta f_\nu(\underset{\sim}{k}_2, \underset{\sim}{v}_2, t=0) \rangle . \qquad \text{(II-87)}$$

It appears that the expression goes to 0 by the Riemann-Lebesque[26] lemma, which states

$$\lim_{t \to \infty} \int_a^b f(\alpha) \, e^{i\alpha t} \, d\alpha = 0 \qquad \text{(II-88)}$$

where f is Riemann integrable. We must use the explicit expression for $\langle \delta f \delta f \rangle$

$$\langle \delta f_\mu(\underset{\sim}{r}_1,\underset{\sim}{v}_1,t)\delta f_\alpha(\underset{\sim}{r}_2,\underset{\sim}{v}_2,t)\rangle = g_{\mu\alpha}(\underset{\sim}{r}_1,\underset{\sim}{r}_2,\underset{\sim}{v}_1,\underset{\sim}{v}_2,t) + f_\mu(\underset{\sim}{r}_1,\underset{\sim}{v}_1,t)\,\frac{\delta_{\mu\alpha}}{n_\mu}$$

$$\times\ \delta(\underset{\sim}{r}_1 - \underset{\sim}{r}_2)\ \delta(\underset{\sim}{v}_1 - \underset{\sim}{v}_2) \qquad \text{(II-89)}$$

where g is the two particle correlation function of the BBGKY hierarchy. This result is obtained by direct calculation in appendix A.

It is clear that the terms involving g do go to 0 by the Riemann-Lebesque lemma for any reasonable choice of g. We make use of the delta function in velocity space, and the Fourier transform of the delta function in space $\delta(\underset{\sim}{r}_1 - \underset{\sim}{r}_2) \rightarrow (2\pi)^3\ \delta(\underset{\sim}{k}_1 + \underset{\sim}{k}_2)$ to find:

$$\frac{q_\mu}{m_\mu}\,\frac{d}{d\underset{\sim}{v}_1}\cdot\langle\delta f_\mu \delta\underset{\sim}{E}\rangle = \lim_{t\to\infty}\frac{q_\mu}{m_\mu}\,\frac{d}{d\underset{\sim}{v}_1}\cdot\int\frac{d\underset{\sim}{k}_1}{(2\pi)^3}\left[\frac{i\underset{\sim}{k}_1}{k_1^{\,2}}\,\frac{4\pi\, q_\mu f_\mu(\underset{\sim}{v}_1)}{\epsilon(-\underset{\sim}{k}_1,-\underset{\sim}{k}_1\cdot\underset{\sim}{v}_1)}\right.$$

$$+\sum_\nu \frac{q_\mu}{m_\mu}\,\frac{i\underset{\sim}{k}_1}{k_1^{\,2}}\cdot\frac{df_\mu}{d\underset{\sim}{v}_1}\,(4\pi\, q_\nu)^2\, n_\nu\,\frac{i\underset{\sim}{k}_1}{k_1^{\,2}}\int\frac{d\underset{\sim}{v}_2\, f_\nu(\underset{\sim}{v}_2)}{(-i\underset{\sim}{k}_1\cdot\underset{\sim}{v}_1 - i\underset{\sim}{k}_2\cdot\underset{\sim}{v}_2)}$$

$$\left.\times\left(\frac{1}{\epsilon(\underset{\sim}{k}_1,\underset{\sim}{k}_1\cdot\underset{\sim}{v}_2)\ \epsilon(-\underset{\sim}{k}_1,-\underset{\sim}{k}_1\cdot\underset{\sim}{v}_2)} - \frac{e^{-i\underset{\sim}{k}_1\cdot\underset{\sim}{v}_1 t}\ e^{-i\underset{\sim}{k}_1\cdot\underset{\sim}{v}_2 t}}{\epsilon(\underset{\sim}{k}_1,\underset{\sim}{k}_1\cdot\underset{\sim}{v}_1)\ \epsilon(-\underset{\sim}{k}_1,-\underset{\sim}{k}_1\cdot\underset{\sim}{v}_2)}\right)\right].$$

(II-90)

Note that there is no singularity at $\underset{\sim}{k}\cdot\underset{\sim}{v}_1 = \underset{\sim}{k}\cdot\underset{\sim}{v}_2$, for the two expressions involving this quantity cancel each other at this point. Thus the $\underset{\sim}{v}_2$ integral goes directly through this point.

We consider again the kinetic equation we seek (no space dependence)

$$\frac{df_\mu}{dt} = - \frac{q_\mu}{m_\mu} \frac{d}{d\underset{\sim}{v}_1} \cdot \langle \delta f_\mu \delta \underset{\sim}{E} \rangle \tag{II-91}$$

and observe that it is an equation for a real quantity. Therefore in equation II-90 only the real part of the expression need be kept. This is consistent, of course, for the imaginary parts vanish, being odd in $\underset{\sim}{k}_1$. The infinite self force term also vanishes for this reason. One term now vanishes completely, and we may take the real part of the first term.

$$\text{Real}\left[\frac{i\underset{\sim}{k}_1}{k_1^2} \frac{4\pi q_\mu f_\mu(\underset{\sim}{v}_1)}{\epsilon(-\underset{\sim}{k}_1, -\underset{\sim}{k}_1\cdot\underset{\sim}{v}_1)}\right] = \frac{i\underset{\sim}{k}_1}{k_1^2} \sum_\nu \frac{\omega_\nu^2}{k^2} \int \frac{d\underset{\sim}{v}_2 \; i\underset{\sim}{k}_1 \cdot \frac{df_\nu}{d\underset{\sim}{v}_2} f_\mu(\underset{\sim}{v}_1)\delta(\underset{\sim}{k}_1\cdot\underset{\sim}{v}_1 - \underset{\sim}{k}_1\cdot\underset{\sim}{v}_2)}{|\epsilon(\underset{\sim}{k}_1, \underset{\sim}{k}_1\cdot\underset{\sim}{v}_1)|^2} . \tag{II-92}$$

We have used the fact that

$$\epsilon^*(\underset{\sim}{k}_1, \underset{\sim}{k}_1\cdot\underset{\sim}{v}_1) = \epsilon(-\underset{\sim}{k}_1, -\underset{\sim}{k}_1\cdot\underset{\sim}{v}_1) \tag{II-93}$$

where the star means that the complex conjugate is to be taken. We still must consider the term

$$\int \frac{d\underset{\sim}{v}_2 \; e^{i\underset{\sim}{k}_1 \cdot (\underset{\sim}{v}_1 - \underset{\sim}{v}_2)\, t}}{(i\underset{\sim}{k}_1 \cdot \underset{\sim}{v}_1 - i\underset{\sim}{k}_1 \cdot \underset{\sim}{v}_2)\, \epsilon(\underset{\sim}{k}_1, \underset{\sim}{k}_1 \cdot \underset{\sim}{v}_1)\, \epsilon(-\underset{\sim}{k}_1, -\underset{\sim}{k}_1 \cdot \underset{\sim}{v}_2)} \; . \qquad \text{(II-94)}$$

We split the exponential into its real and imaginary parts.

$$e^{ixt} = \cos(xt) + i \sin(xt) \; . \qquad \text{(II-95)}$$

The cos term contributes

$$\int \frac{d\underset{\sim}{v}_2 \; \cos\left[\underset{\sim}{k}_1 \cdot (\underset{\sim}{v}_1 - \underset{\sim}{v}_2)\, t\right]}{(i\underset{\sim}{k}_1 \cdot \underset{\sim}{v}_1 - i\underset{\sim}{k}_1 \cdot \underset{\sim}{v}_2)\, |\epsilon(\underset{\sim}{k}_1, \underset{\sim}{k}_1 \cdot \underset{\sim}{v}_1)|^2 |\, \epsilon(\underset{\sim}{k}_1, \underset{\sim}{k}_1 \cdot \underset{\sim}{v}_2)|^2}$$

$$\times \Big\{ \mathrm{Real}\left[\epsilon(\underset{\sim}{k}_1, \underset{\sim}{k}_1 \cdot \underset{\sim}{v}_2)\right] \mathrm{Imag}\left[\epsilon(\underset{\sim}{k}_1, \underset{\sim}{k}_1 \cdot \underset{\sim}{v}_1)\right] - \mathrm{Real}\left[\epsilon(\underset{\sim}{k}_1, \underset{\sim}{k}_1 \cdot \underset{\sim}{v}_1)\right]$$

$$\mathrm{Imag}\left[\epsilon(-\underset{\sim}{k}_1, -\underset{\sim}{k}_1 \cdot \underset{\sim}{v}_2)\right] \Big\} \qquad \text{(II-96)}$$

where we have rationalized the denominator and used equation II-93. This term vanishes by the Riemann Lebesque lemma. (The integrand is a smooth function $\underset{\sim}{k}_1 \cdot (\underset{\sim}{v}_1 - \underset{\sim}{v}_2)$).

Finally we have the sin term--again we rationalize the denominator.

$$\int \frac{d\underset{\sim}{v}_2 \; \sin\left[\underset{\sim}{k}_1 \cdot (\underset{\sim}{v}_1 - \underset{\sim}{v}_2)\, t\right]}{(i\underset{\sim}{k}_1 \cdot \underset{\sim}{v}_1 - i\underset{\sim}{k}_1 \cdot \underset{\sim}{v}_2)\, |\epsilon(\underset{\sim}{k}_1, \underset{\sim}{k}_1 \cdot \underset{\sim}{v}_1)|^2 \, |\epsilon(\underset{\sim}{k}_1, \underset{\sim}{k}_1 \cdot \underset{\sim}{v}_2)|^2} \qquad \text{(II-97)}$$

$$\times \Big\{ \mathrm{Real}\left[\epsilon(-\underset{\sim}{k}_1, -\underset{\sim}{k}_1 \cdot \underset{\sim}{v}_2)\right] \mathrm{Real}\left[\epsilon(\underset{\sim}{k}_1, \underset{\sim}{k}_1 \cdot \underset{\sim}{v}_1)\right] + \mathrm{Imag}\left[\epsilon(-\underset{\sim}{k}_1, -\underset{\sim}{k}_1 \cdot \underset{\sim}{v}_1)\right] \mathrm{Imag}\left[\epsilon(\underset{\sim}{k}_1, \underset{\sim}{k}_1 \cdot \underset{\sim}{v}_2)\right] \Big\} ;$$

This term may be calculated using Jordan's theorem:[27]

If g is of bounded bariation on $[0,\delta]$, then

$$\lim_{t\to\infty} \int_0^{\delta} g(x) \frac{\sin xt}{x} dx = \frac{\pi}{2} g(0) . \qquad \text{(II-98)}$$

The final result is the Lenard-Belescu equation, generalized to two species.

$$\frac{df_\mu}{dt} = \sum_\nu \frac{q_\mu^2}{m_\mu} 2n_\nu q_\nu^2 \frac{d}{d\underset{\sim}{v}_1} \cdot \int d\underset{\sim}{v}_2 \int \frac{d\underset{\sim}{k}_1 \, \underset{\sim}{k}_1 \underset{\sim}{k}_1 \delta(\underset{\sim}{k}_1 \cdot \underset{\sim}{v}_1 - \underset{\sim}{k}_1 \cdot \underset{\sim}{v}_2)}{k_1^4 \left|\epsilon(\underset{\sim}{k}_1, \underset{\sim}{k}_1 \cdot \underset{\sim}{v}_1)\right|^2}$$

$$\left[\frac{f_\nu(\underset{\sim}{v}_2)}{m_\mu} \frac{df_\mu}{d\underset{\sim}{v}_1} - \frac{f_\mu(\underset{\sim}{v}_1)}{m_\nu} \frac{df_\nu}{d\underset{\sim}{v}_2} \right] . \qquad \text{(II-99)}$$

Having completed the right side with aid of the adiabatic hypothesis, we now reinsert the time dependence $f(\underset{\sim}{v}) \to f(\underset{\sim}{v}, t)$. The $\underset{\sim}{k}_1$ integration is divergent at large $\underset{\sim}{k}_1$ and should be cutoff at the closest approach distance $k_0 = \frac{2\pi Kt}{q^2}$.

Equation II-99 is generally a quantitatively satisfactory equation for describing the behavior of a stable plasma, with the exception of the large $\underset{\sim}{k}_1$ divergence. We raise now the question of its faults, with an eye toward correcting them.

We see first that for certain choices of f we may cause ϵ to become very small, so that $\frac{df}{dt}$ becomes arbitrarily large. This fact discourages us from using the perturbation technique suggested by Dupree, for the first correction could be larger than the lowest order term.

Secondly, the theory cannot treat an unstable plasma, because the collision term blows up in the limit $t \to \infty$. It is clear from this derivation that one may make calculations of $\langle \delta f \delta \underset{\sim}{E} \rangle$ for t less than infinity. (The derivation from the BBGKY is tedious). The breakdown of the Bogoliubov hypothesis must be regarded as the cause of the difficulty for a barely stable plasma, as well as for an unstable plasma. A third difficulty lies in the transition to equilibrium. It is well known that long wavelength plasma oscillations damp very slowly in an equilibrium plasma. If we do not use the Bogoliubov hypothesis these modes will remain almost indefinitely, even though the plasma is very near equilibrium. It seems unreasonable for the approach to equilibrium to be so slow. We may expect to remedy this defect by including higher order terms in calculating $\langle \delta f \delta \underset{\sim}{E} \rangle$.

Before attempting a general discussion of our line of attack we discuss briefly the present kinetic theory for an unstable plasma.

D. The Quasilinear Theory

We now discuss the present kinetic theory for an unstable plasma in order to bring out some of the difficulties we shall have later. We make no attempt to "patch up" these difficulties here, for the work would have to be repeated later when a more general kinetic equation is derived. Apparently the problems are now well known, for present research is directed toward the calculation of higher order corrections, without regard to the validity of the whole scheme. We will not discuss the work of Balescu on the unstable plasma, for the author is not familiar with his methods. We shall ultimately derive an equation which is similar to that derived by Balescu, and shall compare the results there.

The quasilinear theory takes as a starting point the Vlasov equation.

$$\frac{df_\mu}{dt} + \underset{\sim}{v} \cdot \nabla f_\mu + \frac{q_\mu}{m_\mu} \frac{d}{d\underset{\sim}{v}} \cdot \underset{\sim}{E} f_\mu = 0 \quad . \qquad \text{(II-100)}$$

This is an approximation to begin with, and has the effect of limiting the time for which the theory may be considered valid. It might seem that the time for which the theory is valid could be estimated simply by estimating the size of the neglected term. If this is done, for a plasma near equilibrium, one obtains the estimate $t \sim \frac{\omega_p \ell n\Lambda}{\Lambda}$. This time may be made very long, simply by making Λ large. However, as we shall see later, the neglected term contains factors which grow exponentially in time. This has the effect of cutting down the time of validity of equation II-100. How much it is cut down depends in general on the initial value of the two body correlation function (g of the BBGKY hierarchy). In view of the methods (discussed later)used by the theory this problem must be regarded as serious.

We arrive at once at a second difficulty. Equation II-100 is an equation for an ensemble average quantity $f_\mu(\underset{\sim}{r}, \underset{\sim}{v}, t)$. Presumably f is given by experimental measurement or else chosen because it is of theoretical interest. In either case we would (it seems) desire to solve equation II-100 to find $f_\mu(\underset{\sim}{r}, \underset{\sim}{v}, t)$ in terms of $f_\mu(\underset{\sim}{r}, \underset{\sim}{v}, t = 0)$. The fact that this is not done is indeed perplexing. In general the theory is directed toward an equation for $f^0(\underset{\sim}{v}, t)$, representing the spatially homogeneous part of $f(\underset{\sim}{r}, \underset{\sim}{v}, t)$, while no effort is expended toward making statements about $f(\underset{\sim}{r}, \underset{\sim}{v}, t)$ itself. Confusion is increased because some authors[28,29] speak of deriving a kinetic equation for a spatially homogeneous system, though they expand f in a Fourier series and keep terms $e^{i\underset{\sim}{k}\cdot\underset{\sim}{r}}$ with $k \neq 0$.

This difficulty is more pronounced when we consider possible experimental verification of the quasilinear theory. We would have to determine $f(\underset{\sim}{r}, \underset{\sim}{v}, t = 0)$ by actual measurement, in order to predict the subsequent behavior of $f^0(\underset{\sim}{v}, t)$. This is so because the quasilinear theory drops the two-particle correlation function (g) and uses the Vlasov equation to describe the system. In order to eliminate the effects of g we must determine the one particle distribution function (throughout the system) with sufficient accuracy so that we may say that the effects of statistical fluctuations of g are negligible. If we cannot find the spatial dependence of f at $t = 0$, then the Vlasov equation predicts $\frac{df_0}{dt} = 0$ (alternatively is undetermined). In this case the evolution of the system would be described by the statistical nonuniformities, i.e., "g".

We note for emphasis that it is not possible to make a general statement about the size of the collision term (g) in an unstable plasma. Though it is possible to assign an effective collision frequency to the effects of short range ($r < \lambda_d$) encounters in a plasma, the effects of long range ($r \geq \lambda_d$) encounters may be large or small in an unstable plasma. The question is simply whether we are able to measure nonuniformities in the system, thereby determining f . If we cannot perform the measurement the nonuniformities still drive the system, but we must consider their effects in a statistical sense. Note that the spatial averaging of quasilinear theory does, in a sense, give a statistical result for $\frac{df^0}{dt}$. We return to the standard path and seek an equation for $\frac{df^0}{dt}$.

We decompose $f(\underset{\sim}{r}, \underset{\sim}{v}, t)$ by means of a Fourier sum. Frequently the sum is converted to an integral after the analysis is completed.

$$f_\mu(\underset{\sim}{r}, \underset{\sim}{v}, t) = f_\mu^{\,0}(\underset{\sim}{v}, t) + \sum_{k \neq 0} f_\mu^{\,k}(\underset{\sim}{v}, t)\, e^{i\underset{\sim}{k}\cdot\underset{\sim}{r}} \tag{II-101}$$

where

$$f_\mu^{\,k}(\underset{\sim}{v}, t) = \frac{1}{V}\int d\underset{\sim}{r}\; f(\underset{\sim}{r}, \underset{\sim}{v}, t)\, e^{-i\underset{\sim}{k}\cdot\underset{\sim}{r}} \quad . \tag{II-102}$$

Equation II-100 may be broken into spatially dependent and spatially independent parts

$$\frac{df_\mu^{\,0}}{d\underset{\sim}{v}} = -\frac{q_\mu}{m_\mu}\frac{d}{d\underset{\sim}{v}} \cdot \sum_k f_\mu^{\,k}\, \underset{\sim}{E}^{-k} \tag{II-103}$$

$$\frac{df_\mu^{\;k}}{d\underset{\sim}{v}} + i\underset{\sim}{k}\cdot\underset{\sim}{v}\, f_\mu^{\;k} + \frac{q_\mu}{m_\mu}\,\frac{d}{d\underset{\sim}{v}}\cdot\underset{\sim}{E}^k\, f_\mu^{\;0} + \frac{q_\mu}{m_\mu}\sum_\ell \underset{\sim}{E}^{k-\ell}\cdot\frac{df_\mu^{\;\ell}}{d\underset{\sim}{v}} = 0 \,. \qquad \text{(II-104)}$$

It is assumed that no external field is applied, so that $\underset{\sim}{E}^{k=0} = 0$. In deriving the basic quasilinear equation one neglects the term in equation II-104 which involves a sum on ℓ. We will follow this procedure, making a few remarks about the present means for including them after obtaining the equation for f^0. We make the usual adiabatic hypothesis: $f^{k\neq 0}$ varies much more rapidly than f^0, so that we may solve equation II-104 for $f^{k\neq 0}$ while holding f^0 fixed. Equation II-104 now has the form

$$\frac{df_\mu^{\;k}}{dt} + i\underset{\sim}{k}\cdot\underset{\sim}{v}\, f_\mu^{\;k} + \frac{q_\mu}{m_\mu}\,\frac{d}{d\underset{\sim}{v}}\cdot\underset{\sim}{E}^k\, f_\mu^{\;0} = 0 \quad . \qquad \text{(II-105)}$$

This is formally identical to equation II-73, and we may write down the solution immediately by comparison with equation II-81. The Fourier integral is replaced by a sum, and $\delta f(t = 0)$ is replaced by $f_\mu^{\;k}(\underset{\sim}{v}, t=0)$.

$$f_\mu(\underset{\sim}{r},\underset{\sim}{v},t) = \sum_k e^{i\underset{\sim}{k}\cdot\underset{\sim}{r}} \int \frac{d\omega}{(2\pi)}\, e^{-i\omega t}\left[\frac{1}{-i\omega + i\underset{\sim}{k}\cdot\underset{\sim}{v}}\right.$$

$$\left. + \sum_\nu \frac{\dfrac{q_\mu}{m_\mu}\,\dfrac{i\underset{\sim}{k}}{k^2}\cdot\dfrac{df_\mu^{\;0}}{d\underset{\sim}{v}}\; 4\pi\, n_\nu q_\nu}{(-i\omega + i\underset{\sim}{k}\cdot\underset{\sim}{v})\,\epsilon(\underset{\sim}{k},\omega)} \int \frac{d\underset{\sim}{v}'}{(-i\omega + i\underset{\sim}{k}\cdot\underset{\sim}{v}')}\right] f_\mu^{\;k}(t = 0) \,. \qquad \text{(II-106)}$$

The solution for $\underset{\sim}{E}(\underset{\sim}{r},t)$ is formally the same as that for $\delta\underset{\sim}{E}(\underset{\sim}{r},t)$ given by equation II-82.

$$\underset{\sim}{E}(\underset{\sim}{r},t) = \sum_k e^{i\underset{\sim}{k}\cdot\underset{\sim}{r}} \int \frac{d\omega}{2\pi} e^{-i\omega t} \left[\sum_\nu \left(\frac{-i\underset{\sim}{k}}{k^2} \right) \frac{4\pi n_\nu q_\nu}{\epsilon(\underset{\sim}{k},\omega)} \int \frac{d\underset{\sim}{v}}{(-i\omega + i\underset{\sim}{k}\cdot\underset{\sim}{v})} \right] f_\nu^{\,k}(t=0) \quad . \tag{II-107}$$

Substitution of equations II-106 and II-107 into equation II-103 leads immediately to a form of the kinetic equation for $f_\mu^{\,0}$.

$$\frac{df_\mu^{\,0}}{dt} = \frac{q_\mu}{m_\mu} \frac{d}{d\underset{\sim}{v}} \cdot \sum_k \int \frac{d\omega_1}{2\pi} \int \frac{d\omega_2}{2\pi} e^{-i\omega_1 t} e^{-i\omega_2 t} \left[\frac{1}{-i\omega_1 + i\underset{\sim}{k}\cdot\underset{\sim}{v}} \right.$$

$$\left. + \frac{\sum_\nu \frac{q_\mu}{m_\mu} \frac{i\underset{\sim}{k}}{k^2} \cdot \frac{df_\mu^{\,0}}{d\underset{\sim}{v}} 4\pi\, n_\nu q_\nu}{(-i\omega_1 + i\underset{\sim}{k}\cdot\underset{\sim}{v})\epsilon(\underset{\sim}{k},\omega_1)} \int \frac{d\underset{\sim}{v}_1}{(-i\omega_1 + i\underset{\sim}{k}\cdot\underset{\sim}{v}_1)} \right] \left[\sum_\alpha \frac{i\underset{\sim}{k}}{k^2} \frac{4\pi n_\alpha q_\alpha}{\epsilon(-\underset{\sim}{k},\omega_2)} \right.$$

$$\left. \times \int \frac{d\underset{\sim}{v}_2}{(-i\omega_2 - i\underset{\sim}{k}\cdot\underset{\sim}{v}_2)} \right] f_\mu^{\,k}(\underset{\sim}{v},t=0)\, f_\alpha^{\,-k}(\underset{\sim}{v}_2,t=0) \ . \tag{II-108}$$

The reader has not failed to notice the similarity between equation II-108 and equation II-86. We now consider the evaluation of the inverse Laplace transforms. The quasilinear theory states that they should be evaluated in the limit $t \to \infty$. This is meaningless for an unstable plasma, of course, for then these terms blow up for some values of ω_k. A better description is provided by the words large time, or time comparable to the time in which f^0 itself changes. One may now invert the transforms, picking up zeros of $\epsilon(\underset{\sim}{k},\omega)$ as well as those at $\omega = \underset{\sim}{k}\cdot\underset{\sim}{v}$.

The quasilinear theory keeps only contributions from $\epsilon(\underset{\sim}{k},\omega) = 0 = \epsilon(\underset{\sim}{k},\omega_k)$ which defines the complex frequency $\omega_k = \Omega_k + i\gamma_k$. The result is obtained directly:

$$\frac{df_\mu^0}{dt} = -\frac{q_\mu^2}{m_\mu}\frac{d}{d\underset{\sim}{v}}\cdot\sum_k \frac{\underset{\sim}{k}\,\underset{\sim}{k}\cdot\frac{df_\mu^0}{d\underset{\sim}{v}}\,\gamma_k\, e^{2\gamma_k t}\sum_{\alpha,\nu}(4\pi)^2 n_\alpha n_\nu q_\alpha q_\nu}{\left.\frac{d\epsilon}{d\omega}\right|_{\omega_k}\left.\frac{d\epsilon}{d\omega}\right|_{\omega_{-k}}\left[(\Omega_k-\underset{\sim}{k}\cdot\underset{\sim}{v})^2+\gamma_k^2\right]}$$

$$\times\int\frac{d\underset{\sim}{v}_1\, f_\nu^k(v_1,t=0)}{(-i\omega_k+i\underset{\sim}{k}\cdot\underset{\sim}{v}_1)}\int\frac{dv_2\, f_\alpha^{-k}(\underset{\sim}{v}_2,t=0)}{(-i\omega_{-k}-i\underset{\sim}{k}\cdot\underset{\sim}{v}_2)}\quad. \qquad \text{(II-109)}$$

It is convenient to express the derivatives of the dielectric function in terms of a real quantity. The following is adequate, though not elegant. From the definitions of the transforms we have

$$\epsilon(\underset{\sim}{k},\omega) = \epsilon(-\underset{\sim}{k};-\omega^*)^* \qquad \text{(II-110)}$$

It follows that

$$\epsilon(k,\omega_k+\delta) = \epsilon(-k,-\omega_k^*-\delta^*)^*\quad. \qquad \text{(II-111)}$$

We now consider δ a small real quantity, and use the fact that $\omega_k = -\omega_{-k}^*$.

$$\epsilon(\underset{\sim}{k},\omega_k+\delta) = \epsilon(-\underset{\sim}{k},\omega_{-k}-\delta)^*\quad. \qquad \text{(II-112)}$$

We now Taylor expand about $\delta = 0$.

$$\epsilon(\underset{0}{\underset{\shortparallel}{k,\omega_k}}) + \frac{d\epsilon}{d\omega}\Big|_{\omega_k} \delta + \cdots = \epsilon(\underset{0}{\underset{\shortparallel}{-k,\omega_{-k}}})^* - \delta \frac{d\epsilon}{d\omega}\Big|_{\omega_{-k}}^{*} + \cdots \qquad \text{(II-113)}$$

It follows that

$$\frac{d\epsilon}{d\omega}\Big|_{\omega_k} = - \frac{d\epsilon}{d\omega}\Big|_{\omega_{-k}}^{*} \qquad \text{(II-114)}$$

which is the desired relation.

Thus equation II-109 may be written in the form

$$\frac{df_\mu^{\,0}}{dt} = \frac{q_\mu^{\,2}}{m_\mu^{\,2}} \frac{d}{d\underset{\sim}{v}} \cdot \sum_{\underset{\sim}{k}} \frac{\underset{\sim}{k}\,\underset{\sim}{k} \cdot \frac{df_\mu^{\,0}}{d\underset{\sim}{v}} \sum_{\alpha,\nu} (4\pi)^2\, n_\alpha n_\nu q_\alpha q_\nu\, \gamma_k\, e^{2\gamma_k t}}{\left|\frac{d\epsilon}{d\omega}(\omega_k)\right|^2 \left[(\Omega_k - k\cdot v)^2 + \gamma_k^{\,2}\right]}$$

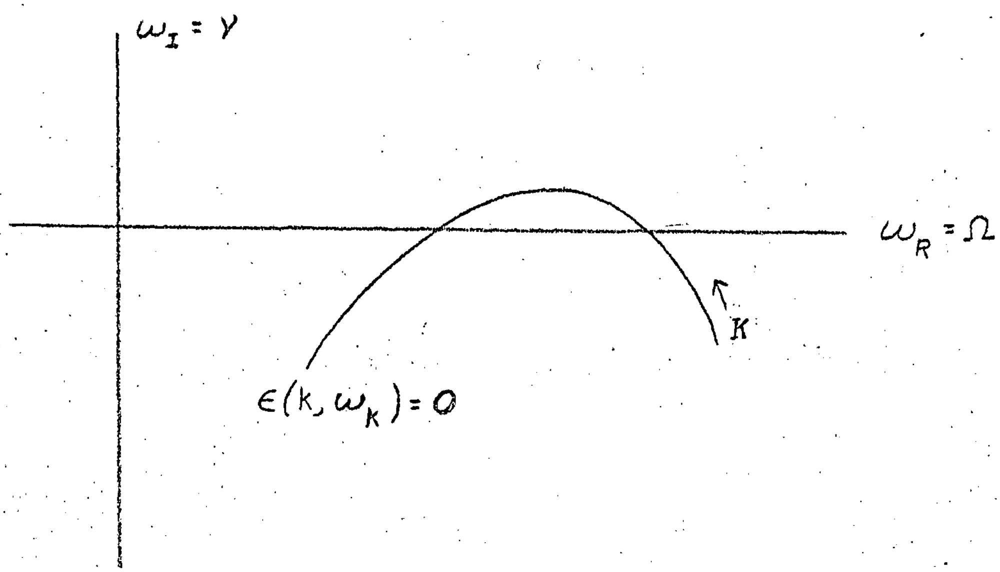
$$\times \int \frac{d\underset{\sim}{v}_1\, f_\nu^{\,k}(\underset{\sim}{v}_1, t=0)}{-i\underset{\sim}{k}\cdot\underset{\sim}{v}_1 - i\Omega_k + \gamma_k} \int \frac{d\underset{\sim}{v}_2\, f_\alpha^{\,-k}(\underset{\sim}{v}_2, t=0)}{-i\underset{\sim}{k}\cdot\underset{\sim}{v}_2 + i\Omega_k + \gamma_k} \; . \qquad \text{(II-115)}$$

This equation is usually written in terms of the electric field:

$$\frac{df_\mu^{\,0}}{dt} = \frac{q_\mu^{\,2}}{m_\mu^{\,2}} \frac{d}{d\underset{\sim}{v}} \cdot \sum_{k} \frac{\underset{\sim}{E}^k(t=0)\, \underset{\sim}{E}^{-k}(t=0) \cdot \frac{df_\mu^{\,0}}{d\underset{\sim}{v}}\, \gamma_k\, e^{2\gamma_k t}}{\left[(\Omega_k - \underset{\sim}{k}\cdot\underset{\sim}{v})^2 + \gamma_k^{\,2}\right]} \; . \qquad \text{(II-116)}$$

The quasilinear theory ignores all terms coming from the zeroes at $\omega = \underset{\sim}{k}\cdot\underset{\sim}{v}$. The statement is made that these terms $e^{-i\underset{\sim}{k}\cdot\underset{\sim}{v}t}$ disappear in the asymptotic limit. This statement is false. This misconception leads to a difficulty which we now point out (it is well known).

The quantity $\epsilon(\underset{\sim}{k},\omega)$ is an analytic function (by definition). The zeroes of this function will trace out lines in the ω plane, in general. We plot a 0 of ϵ for a typical case of an unstable plasma.



The plasma is unstable (has exponentially growing modes) for $\gamma > 0$. We consider the point of neutral stability $\gamma = 0$. At this point the velocity integrals are undefined at the point $\Omega = \underset{\sim}{k}\cdot\underset{\sim}{v}_1$, $\Omega = \underset{\sim}{k}\cdot\underset{\sim}{v}_2$. Likewise the factor $\frac{\gamma_k}{(\Omega_k-\underset{\sim}{k}\cdot\underset{\sim}{v})^2+\gamma_k^2}$ is not defined at the point $\underset{\sim}{v} = \frac{\Omega}{k^2}\underset{\sim}{k}$. The difficulty is unavoidable since an unstable plasma always has a point of neutral stability. Frequently the $\underset{\sim}{k}$ integration over the factor $\frac{\gamma_k}{(\Omega_k-\underset{\sim}{k}\cdot\underset{\sim}{v})^2+\gamma_k^2}$, is approximated by $\pi\,\delta(\Omega_k-\underset{\sim}{k}\cdot\underset{\sim}{v})$ despite the fact that the resulting equation no longer conserves momentum. Note also that for $\gamma_k < 0$, the approximation would lead to $-\pi\,\delta(\Omega_k - \underset{\sim}{k}\cdot\underset{\sim}{v})$, representing a negative diffusion coefficient.

The theory has other peculiar features. It is customary to prove that the energy of the system $\int \frac{1}{2} m v^2 f \, d\underset{\sim}{v} + \frac{1}{8\pi} \sum_k E^{k^2}$ is conserved, but a satisfactory proof for momentum conservation has not appeared. Kadomtsev[30] has proved that momentum is conserved by making approximations suitable to resonance and non-resonance particles. He does not discuss the transition region. At present stable modes $(\gamma < 0)$ are ignored, so that the theory disappears when the plasma becomes stable. We shall see later that all these difficulties are corrected by the inclusion of the terms proportional to $e^{i\underset{\sim}{k}\cdot\underset{\sim}{v}t}$.

We conclude by discussing the inclusion of the higher order terms (usually called mode coupling) in equation II-104. It is conventional practice to use perturbation theory to calculate these terms. Thus we substitute solutions II-107 and II-108 for f and E into the terms in question. One includes only those terms coming from zeroes of $\epsilon(\underset{\sim}{\ell},\omega)$ in the perturbation series, and neglects terms coming from zeroes at $\omega = \underset{\sim}{\ell}\cdot\underset{\sim}{v}$. The reason for this is made clear by a quote.[31]

> "It is also to be noted that if terms in f_1 like $e^{-i\underset{\sim}{\ell}\cdot\underset{\sim}{v}t}$ had been kept and substituted into the last term on the left in equation (14) (our equation II-104), then the velocity derivative would give rise to terms growing like

t in equation (14). This is a fundamental difficulty in pursuing a perturbation expansion to higher order in these "continium modes," and was first noted by Backus."[32]

It is true that these terms lead to secular behavior in the perturbation expansion. On the other hand it is not clear that one should simply drop the terms for that reason. It seems to this author more reasonable to conclude that the direct perturbation expansion breaks down, and other mathematical techniques are called for. This breakdown presumably represents some physical phenomenon, or else the lowest order theory is incorrect.

Later in this work we shall demonstrate a method for handling terms $e^{i\underset{\sim}{k}\cdot\underset{\sim}{v}t}$, which is satisfactory for testing spatially uniform systems. We shall avoid the perturbation technique, except when we want to demonstrate the appearance of certain terms in the dielectric function. In carrying out the analysis for the uniform system we shall correct many of the difficulties with the quasilinear theory. An improved form of equation II-116 will be presented on page 210.

III. THE GOALS OF PLASMA KINETIC THEORY

The present discussion should have come at the beginning of this work, logically speaking. We have postponed it because many of the arguments we make here are taken for granted in the theories discussed previously. On the other hand we shall find that when we weaken the basic assumptions we find difficulties which have not appeared before. It is well to make our assumptions explicit.

The discussion divides naturally into three topics: What is a kinetic equation; how do we go about constructing such an equation; and what mathematical pitfalls must we avoid?

A. What Is a Kinetic Equation?

This seemingly innocuous equestion was brushed aside casually at the beginning of this work (page 6). We wish "to describe the behavior in time and space of a system of interest." The answer means nothing until we decide what is an appropriate description. If we do not want the precise orbit of each of 10^{23} particles, what are we willing to settle for?

It is generally assumed (implicitly) that an appropriate description consists of an equation for the average one particle distribution $f(\underset{\sim}{v},t)$, (we consider spatially uniform systems), where this equation contains only independent variables, constants, possible external constraints (e.g., forces), and f itself. The Boltzmann, Fokker-Planck, and Lenard-Balescu equations meet these criteria. We now question not the desirability of obtaining this sort of equation, but the mathematical and physical assumptions used in deriving these equations.

It is clear, of course, that the macroscopic as well as microscopic properties of a system depend on all the initial conditions we impose on the system. The statement is equally true for a particular physical and an idealized (ensemble averaged) system. For this reason Dupree's proof[4] that all higher correlation functions become functionals of $f(\underset{\sim}{v}, t = 0)$ is intrinsically false. It is quite possible (though statistically unlikely) that a system may have a set of initial values for correlation functions which cause $f(\underset{\sim}{v}, t)$ itself to change very rapidly for some period of time. f might "jump" at $t = 0$ as a result of the higher correlations.

In general we wish to avoid such behavior. Note that there is no need to do so. The equations we have available, either BBGKY or Klimontovich-Dupree, are perfectly adequate to describe the evolution of an arbitrarily small volume of $\Gamma = \{X_1, X_2, \cdots, X_N\}$ space. We may keep all initial values if we desire. In fact our chief objective is to avoid this complete description. The reasons are practical, of course. It is this desire which leads us to attempt description of only certain types of systems--kinetic systems.

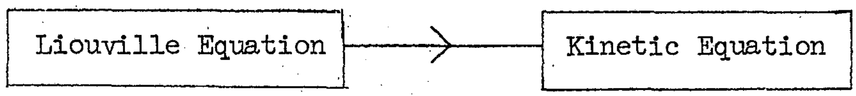
The problem has been discussed in some detail by Sandri.[33] We summarize by quoting:[34]

"We say that a gas is in a kinetic regime if the one-body distribution satisfies an equation of the form

$$\frac{\partial F^1}{\partial t} = A[F^1] \tag{III-1}$$

where $A[F^1]$ is a functional of F^1 only, $\cdots$ "

"The requirement that a gas should approach thermodynamical equilibrium via a kinetic regime defines sharply the class of correlation functions whose presence at $t = 0$ will be 'forgotten' so as to permit the contraction shown in Figure 9. The class of these correlations will be called kinetic."



Sandri - Figure 9

The prospects for obtaining a kinetic equation for a plasma (in Sandri's sense of the word) are quite bleak.

For some types of distribution functions (which are not pathological) the plasma will be unstable. In such a case the initial correlations do not go away rapidly. In fact their effect may persist until the system approaches equilibrium. In addition long wavelength plasma oscillations damp extremely slowly even for a plasma near equilibrium. In fact the damping rate goes to zero as the wavelength of the oscillation goes to infinity. Here we shall improve matters somewhat by including higher order terms. Nevertheless we shall not be able to eliminate all effects of initial conditions.

How can we obtain a consistent kinetic theory when some components of the initial correlation functions persist for long times? We simply keep these components.

B. The Construction of a Kinetic Equation

We consider now the means for obtaining the desired kinetic equation, where we have (perforce) weakened the condition that $\frac{df}{dt}$ should depend only on f. The discussion is directed toward the previous derivation of the Lenard Balescu equation (pages 38-43). We state first of all that the evaluation of $\langle \delta f \delta \underset{\sim}{E} \rangle$ in the limit $t \to \infty$ is not acceptable. The reason is quite clear: the adiabatic hypothesis, which holds $f(\underset{\sim}{v}, t)$ fixed in time for the calculation of $\langle \delta f \delta \underset{\sim}{E} \rangle$, does not remain valid indefinitely. Standard arguments[35] show that the change of $f_\mu(\underset{\sim}{v}, t)$ with time may be described by a collision frequency, or relaxation (to equilibrium) rate of the order of $\omega_\mu \ell n\, \Lambda/\Lambda$. The figure certainly must be revised upward for unstable distribution functions. Of course as a system approaches equilibrium the rate of change of f goes to zero. We shall use the figure given for a crude estimate of the time for which the adiabatic hypothesis is valid

$$t_{ad} \lesssim \frac{\Lambda}{\omega_p \ell n\, \Lambda} \quad .$$

Thus we must calculate $\langle \delta f \delta E \rangle$ for some time less than t_{ad}. The results will be largely independent of this time, of course.

The mathematically correct procedure is to evaluate $\langle \delta f \delta \underset{\sim}{E} \rangle$ without regard to the value of t, i.e., from $t = 0$. This leads to the following difficulty. $\langle \delta f \delta \underset{\sim}{E} \rangle$ depends on the initial values of the correlation functions. We wish to avoid these initial values, in so far as is possible. However if we simply put in an arbitrary value for $g(t = 0)$, we may generally expect the system to "jump" until $g(t)$ is in quasi-equilibrium with $f(t)$. In the meantime the adiabatic hypothesis has been broken and we have lost track of f, unless we can solve for it for short times. Note that arbitrarily setting $g(t = 0) = 0$ does not avoid this difficulty, for the following reason. When we set $g = 0$ we are stating that the particles are uncorrelated. (The statement applies to two body correlations, but we are ignoring higher correlations for the present). The particles are now distributed randomly through space. But in this case we may expect the potential energy of the system to be somewhat higher than it would be if the particles were given a brief period of time to rearrange their positions. The particles will minimize their potential energy very rapidly for $t \gtrsim 0$. Where does this energy go? It must go into kinetic energy. The plasma "heats" itself very rapidly while establishing quasi-equilibrium with its correlation functions. As expected, the adiabatic hypothesis fails. The fact that the heating of the plasma is very slight indicates that $g(t = 0) = 0$ is far from an extreme choice.

The cure for this difficulty is fairly obvious. We evaluate $\langle \delta f \delta \underset{\sim}{E} \rangle$ for time t greater than the time it takes the initial correlations to go away, but smaller than the time for which the adiabatic hypothesis is valid. We then treat t as a continuous variable starting at $t = 0$. Alternatively we may evaluate $\langle \delta f \delta E \rangle$

for all t (less than t_{ad}, of course) but neglect all terms which die out rapidly.

In either case we expect terms which do not go away even for $t \sim t_{ad}$. Thus we will generally end up keeping some of the initial values, and the question which to keep and which to throw away must be answered somewhat arbitrarily. In fact t_{ad} is a very long time, and almost any requirement will set a reasonable dividing line. For example it seems reasonable to throw away terms proportional to $e^{-|\gamma|t}$ if they decay by a factor $\frac{1}{\Lambda}$ in time t_{ad}. This leaves only a very small part of the initial correlation function in the kinetic equation.

We turn now to a second problem concerning the construction of a kinetic equation. How do we terminate the endless set of equations that describe the behavior of the system? To quote Montgomery and Tidman[36]

> "Expectation values of most measurable quantities are calculable in terms of f_1 and f_2. (In our notation $\mathfrak{F}_1$ and $\mathfrak{F}_2$ of the BBGKY hierarchy). If there were some scheme by which f_1 and f_2 could be calculated without knowning $f_3, f_4 \ldots$ (i.e., if we could break the chain of equations represented by (4.7)), it is apparent that a vast and practical simplification would have been achieved. It should be stated unequivocally that there is not yet even one non-equilibrium situation

where a clear proof of the correctness of such a procedure has been given."

In general f_3, f_4, etc., may be chosen arbitrarily at $t = 0$. Thus we can, in principle, choose these arbitrary functions in such a way that they have a considerable effect on the system, at least for small time $(t \lesssim \frac{1}{\omega p})$. However we shall see that the two-body correlation function relaxes to become a functional of f in this short time (with the exception of some long range effects). Presumably higher order correlation functions relax at approximately the same rate. Thus we shall exclude the possibility that higher order correlation functions have an important effect by considering only systems in which most of this relaxation has taken place. In terms of an initial value problem, we wish to find the behavior of $f(\underset{\sim}{v}, t)$ for t greater than this relaxation time, which may be taken as order $\frac{1}{\omega_p}$. The behavior of f for $t < \frac{1}{\omega_p}$ is a legitimate question, but we will not consider it. It is difficult to imagine preparing a plasma in the lab in which correlation functions would cause f to change on the time scale $\frac{1}{\omega_p}$.

Granted that the higher correlation functions become functionals of f, we must still justify a method for terminating our set of equations.

We are now on solid ground. We eliminate the effects of initial values from our equations because we are not interested in these effects. f changes slowly in time because we construct our theory only for this case. We may now make crude estimates of the size of terms in the equations for f, g, h, or f, $\langle \delta f \delta E \rangle$, $\langle \delta f \delta f \delta \underset{\sim}{E} \rangle$, etc. We find that successive terms are in the ratio $1 : \frac{1}{\Lambda}$. Thus the ultimate justification for truncating the set of equations is (and must be) quantitative. If successive terms did not become less and less important, there would be no point in working with present theories.

In this work we shall find an equation for $\frac{df}{dt}$ valid to order $\frac{1}{\Lambda}$ for the long range effects of particle interactions. The short range difficulty remains, and limits the quantitative accuracy of the equation to order $\frac{1}{\ell n \Lambda}$. In order to obtain realistic behavior from the terms with explicit time dependence $(e^{\gamma_k t})$ we shall have to include some of the effects of the three particle correlation function h (in terms of the K.D. equations $\langle \delta f \delta f \delta f \rangle$).

C. The Breakdown of Perturbation Theory

Our problem is now sufficiently limited to appear simple. We wish to derive a collision term for a uniform plasma. We truncate the infinite set of equations describing the evolution of the system by considering the usual case in which successively higher order terms become less and less important. We hold $f(\underset{\sim}{v}, t)$ fixed while calculating the collision term for t large enough so that most of the effects of initial values of $\langle \delta f \delta f \rangle$, $\langle \delta f \delta f \delta f \rangle$, etc., are negligible. All that remains is to carry out the mathematics. In fact this is not quite so simple, for a standard technique of mathematical physics (perturbation theory) often breaks down in the case of plasma kinetic theory. We now illustrate this statement:

1. The breakdown of the mode-coupling analysis of quasilinear theory has already been mentioned (p.53). The secular terms which appear are generally ignored, but this may be expected to change as better mathematical techniques are developed.

2. Montgomery and Tidman[37] demonstrate a method (due to Guernsey)[38] for calculating the pair correlation function to order $\frac{1}{\Lambda}$ for a plasma in equilibrium. The treatment is statistical, and leads to the conclusion that the two-body function is given by

$$\mathcal{F}_2(\underset{\sim}{r}_1,\underset{\sim}{r}_2,\underset{\sim}{v}_1,\underset{\sim}{v}_2) = \left(\frac{m}{2\pi\kappa T}\right)^3 e^{-\frac{mv_1^2}{2\kappa T}} e^{-\frac{mv_2^2}{2\kappa T}} \left\langle e^{-\frac{q^2}{\kappa T|\underset{\sim}{r}_1-\underset{\sim}{r}_2|}} \right\rangle \qquad \text{(III-2)}$$

where the bracketed (ensemble averaged) quantity is to be calculated. The calculation via perturbation theory is sufficiently long that we will not repeat it. The result is

$$\left\langle e^{-\frac{q^2}{\kappa T|\underset{\sim}{r}_1-\underset{\sim}{r}_2|}} \right\rangle = 1 - \frac{q^2}{\kappa T|\underset{\sim}{r}_1-\underset{\sim}{r}_2|} e^{-\frac{2\pi|\underset{\sim}{r}_1-\underset{\sim}{r}_2|}{\lambda_d}} + \mathcal{O}\left(\frac{1}{\Lambda^2}\right) . \qquad \text{(III-3)}$$

Again quoting:[39]

"The difficulties arise when one tries to go beyond first order in (in our notation $\frac{q^2}{\kappa T}$)... Much analytical work remains to be done before the structure of any possible perturbation series is accurately known, and the subject is very much an open one."

We note that for $|\underset{\sim}{r}_1-\underset{\sim}{r}_2|$ small, collective effects are unimportant, and we must have

$$\left\langle e^{-\frac{q^2}{\kappa T|\underset{\sim}{r}_1-\underset{\sim}{r}_2|}} \right\rangle \underset{\substack{|\underset{\sim}{r}_1-\underset{\sim}{r}_2| \\ \text{small}}}{\cong} e^{-\frac{q^2}{\kappa T|\underset{\sim}{r}_1-\underset{\sim}{r}_2|}} \quad . \qquad \text{(III-4)}$$

The right hand quantity simply is not expandable by perturbation theory. This is not to say that perturbation theory is completely useless, for higher terms give us more significant decimal places in the region where the series converges. However it tells us nothing about the behavior in the interesting region $|\underset{\sim}{r}_1-\underset{\sim}{r}_2| = (\frac{q^2}{\kappa T})$.

3. Misawa[40] has calculated the effects of three body correlations on the mean free path of particles in a plasma, using perturbation theory. He finds that the correction is represented by a factor of $\sim 10^4$ (exponent positive). He does state that he has not investigated the validity of perturbation theory. Presumably the calculation of a few more terms would enable him to make a decision. Rostoker[41] has repeated the calculation without using perturbation theory, and finds only a minor correction to the lowest order theory.

4. Sandri[33] has investigated the possibility of including three body effects in plasma kinetic theory, while keeping the Bogoliubov hypothesis (g,h etc. very rapidly functionals of f). He concludes that this is not possible, as their inclusion leads to divergent integrals. We shall show that this is false by calculating some of these effects: the effects of three body correlations on long wavelength fluctuation. Sandri's error lies in the method of analysis used, which corresponds to pertrubation theory. Sandri blames the Bogoliubov[12] expansion technique for his difficulties, although he does not use it. Sandri does demonstrate that expansions may be made in either the domain of a function (the independent variables) as well as in the range (e.g., perturbation theory). If Sandri had used the Bogoliubov expansion instead of a similar one due to Frieman[42] he would have avoided repeated arguments which are necessary to derive the desired equations. It is quite possible that the Bogoliubov expansion would not break down so quickly as the one used by Sandri.

We have not exhausted the subject, but it is clear that some underlying feature is being ignored. We state one difficulty: The exact equations of either hierarchy (BBGKY or KD) are never singular integral equations. (A singular integral equation has the form $(x - a)\, f(x) + u(x) \int f(x')\, dx' = 0$ where $u(x)$ is given. The singularity is at $x = a$). The equations of either hierarchy resemble singular integral equations after transforms are taken so that $\frac{d}{dt} + \underset{\sim}{v}\cdot\nabla \to -i\omega + i\underset{\sim}{k}\cdot\underset{\sim}{v}$. The statement is true a fortiori for the first equation. The lowest order equation is not a singular integral equation even after the omission of the higher order term (q or $\langle \delta f \delta E \rangle$). It has been and still is fashionable to discuss analytic properties of the linearized Vlasov equation (transformed so that $f_1 = f_1(\underset{\sim}{k},\underset{\sim}{v},\omega)$). Van Kampen[43] has demonstrated an elegant mathematical technique for solving these singular equations, and has applied the technique to the linearized Vlasov equation. The work is referred to frequently and the techniques are often used. However virtually all work along these lines ignores the fact that the expansion(s) used in obtaining the singular integral equation break down precisely at the singular point. The solution of Van Kampen is mathematically elegant but physically monstrous. The straightforward (the word is meant as a compliment) work of Landau[6] is physically adequate, and it makes no attempt to exploit a property that a real system does not have.

We trust it is clear why perturbation theory fails so often in plasma kinetic theory. If the lowest order equation is singular, its solution will contain a singularity. The perturbation expansion in this singular solution will then lead to worse and worse behavior in the neighborhood of the singularity.

There is another feature which is not commonly recognized. Small (higher order) terms may not be neglected over a long period of time. In physical terms when we neglect a small term we make a small error in $\frac{d}{dt}$. The effect may be, in a sense, cumulative. Statements are frequently made about the asymptotic behavior (in time) of solutions to plasma equations. Virtually no one examines the conditions under which the approximate equation is valid into the asymptotic region. Backus[32] and Stix[44] have considered this problem.

We have concluded that perturbation theory is generally to be avoided in plasma kinetic theory. In the next section we shall derive an equation which would (if we could solve it) permit the kinetic equation to be made accurate to order $(\frac{1}{\Lambda^2})$, neglecting the short range $\mathcal{O}(\frac{1}{\ell n\Lambda})$ difficulty, of course. The only means available (that the author is familiar with) for solving the equation is perturbation theory. We will proceed with care.

IV. CALCULATION OF HIGHER ORDER EFFECTS

A. Procedure for Deriving a Better Kinetic Equation

In section II-C the Lenard-Balescu equation was derived in a rather straightforward way. To explain the procedure for deriving a better kinetic equation we describe the steps we shall perform differently.

1. We shall not assume a stable plasma. As a result we cannot calculate $\langle \delta f \delta \underset{\sim}{E} \rangle$ in the limit $t \to \infty$. We shall calculate $\langle \delta f \delta \underset{\sim}{E} \rangle$ for t large, but., for a time somewhat less than the time for which the adiabatic hypothesis is valid. This implies that we shall find terms with explicit time dependence. In fact we shall find terms proportional to $e^{2\gamma_k t}$, where γ_k is the imaginary part of the frequency $\omega_k = \Omega_k + i\gamma_k$ satisfying $\epsilon(\underset{\sim}{k}, \omega_k) = 0$.
2. We shall attempt to estimate the effect of higher order terms in determining the evolution of the system. In particular we wish to establish the fact that long wavelength plasma oscillations do die away as the system approaches equilibrium. To do this we need an accurate estimate of γ.

The method for estimating higher order effects is suggested by Dupree's method for solving the set of equations for $\langle \delta f(1) \cdot \delta f(n) \rangle$.

Recall that we wish to evaluate $\langle \delta f \delta \underset{\sim}{E} \rangle$, which we obtain from the solution of the equation for $\langle \delta f \delta f \rangle$. We write down the exact equation for $\langle \delta f(1)\, \delta f(2) \rangle$, which is given by equation II-57, with $n = 2$.

$$\left[\frac{d}{dt} + T(1) + T(2)\right] \langle \delta f(1)\, \delta f(2) \rangle =$$

$$- \langle \delta f(1) \{ \underset{\sim}{\varnothing}(2) \delta f(2) \} \rangle - \langle \delta f(2) \{ \underset{\sim}{\varnothing}(1) \delta f(1) \} \rangle \quad . \tag{IV-1}$$

Previously the solution for $\langle \delta f \delta f \rangle$ was obtained by neglecting the right side of equation IV-1. This lead to the Lenard-Balescu equation. We now wish to investigate the significance of the right hand terms. To do so we use the solution for $\langle \delta f \delta f \delta f \rangle$ obtained by Dupree. (Equation II-69, with $n = 3$).

$$\langle \delta f(1) \delta f(2) \delta f(3) | t \rangle = P_3(t) \langle \delta f(1) \delta f(2) \delta f(3) | t = 0 \rangle$$

$$+ \int_0^t d\tau \; P_3(t-\tau) \; \Big[\langle \{ \underset{\sim}{\varnothing}(1) \delta f(1) \} \rangle \langle \delta f(2) \delta f(3) \rangle$$

$$+ \langle \{ \underset{\sim}{\varnothing}(2) \delta f(2) \} \rangle \langle \delta f(1) \delta f(3) \rangle + \langle \{ \underset{\sim}{\varnothing}(3) \delta f(3) \} \rangle \langle \delta f(1) \delta f(2) \rangle$$

$$- \langle \delta f(1) \delta f(2) \{ \underset{\sim}{\varnothing}(3) \delta f(3) \} \rangle - \langle \delta f(1) \delta f(3) \{ \underset{\sim}{\varnothing}(2) \delta f(2) \} \rangle$$

$$- \langle \delta f(2) \delta f(3) \{ \underset{\sim}{\varnothing}(1) \delta f(1) \} \rangle \Big] \quad . \tag{IV-2}$$

The terms involving $\langle \delta f \delta f \{ \underset{\sim}{\mathcal{D}} \delta f \} \rangle$ may be expanded according to the cluster expansion described in appendix I. From equation I-12 we have

$$\langle \delta f(1)\, \delta f(2) \cdots \delta f(n) \rangle = \sum_{\substack{\text{cluster} \\ a,b,c \cdots > 1 \\ a+b+c \cdots = n}} H_a H_b H_c \cdots \quad . \tag{A-12}$$

Thus

$$\langle \delta f(1)\, \delta f(2)\, \delta f(3)\, \delta f(4) \rangle = H_4 + \sum_{\text{cluster}} H_2\, H_2 \quad . \tag{IV-3}$$

Picking one of the terms of equation IV-2, we have

$$\langle \delta f(1) \delta f(2) \{ \underset{\sim}{\mathcal{D}}(3) \delta f(3) \} \rangle = \langle \{ \delta f(1) \delta f(2) \delta f(3) \delta f(4) \} \rangle$$

$$+ \langle \delta f(1) \delta f(2) \rangle \langle \{ \underset{\sim}{\mathcal{D}}(3) \delta f(3) \} \rangle + \langle \delta f(1) \delta f(3) \rangle \langle \delta f(2) \underset{\sim}{\mathcal{D}}(3) \rangle$$

$$\langle \delta f(1) \underset{\sim}{\mathcal{D}}(3) \rangle \langle \delta f(2) \delta f(3) \rangle \quad . \tag{IV-4}$$

Using this result in equation IV-2 we have

$$\langle \delta f(1)\, \delta f(2)\, \delta f(3) | t \rangle = P_3(t) \langle \delta f(1)\, \delta f(2)\, \delta f(3) | t = 0 \rangle -$$

(continued Eq. IV-5)

$$\int_0^t d\tau \; P_3(t-\tau) \Big[\langle \delta f(1) \delta f(3) \rangle \langle \delta f(2) \underset{\sim}{\mathcal{O}}(1) \rangle + \langle \delta f(1) \delta f(2) \rangle \langle \delta f(3) \underset{\sim}{\mathcal{O}}(1) \rangle$$

$$+ \langle \delta f(1) \delta f(2) \rangle \langle \delta f(3) \underset{\sim}{\mathcal{O}}(2) \rangle + \langle \delta f(1) \underset{\sim}{\mathcal{O}}(2) \rangle \langle \delta f(2) \delta f(3) \rangle$$

$$+ \langle \delta f(1) \delta f(3) \rangle \langle \delta f(2) \underset{\sim}{\mathcal{O}}(3) \rangle + \langle \delta f(1) \underset{\sim}{\mathcal{O}}(3) \rangle \langle \delta f(2) \delta f(3) \rangle$$

$$+ \langle \{ \delta f(1) \delta f(2) \underset{\sim}{\mathcal{O}}(3) \delta f(3) \} \rangle + \langle \{ \delta f(1) \delta f(3) \underset{\sim}{\mathcal{O}}(2) \delta f(2) \} \rangle$$

$$+ \langle \{ \delta f(2) \delta f(3) \underset{\sim}{\mathcal{O}}(1) \delta f(1) \} \rangle \Big] . \qquad \text{(IV-5)}$$

We now may, in principle, use equation IV-5 to obtain the right side of equation IV-1; then solve the resulting equation (which is still linked to higher equations) to find the behavior of $\langle \delta f \; \delta \underset{\sim}{E} \rangle$, the collision term for a plasma. In fact this is a task of enormous magnitude, which must be simplified by various approximations. The approximation we make now is well defined, though restricted in validity. We shall keep the effects corresponding to short range interactions between particles, while discarding the long range effects corresponding to collective effects in the plasma. This is essentially the same as the approximation leading to the Fokker-Planck collision term for a plasma, instead of the Lenard-Balescu collision term.

The validity of this approximation may be described briefly. In a stable plasma we may estimate the effect of higher order terms, and find that they are successively less important by a factor $1/\Lambda$. The dominant contribution to each term comes from the effects which we shall keep: the short range interactions $(r_0 < r < \lambda_d)$.

However the situation is quite different in the case of an unstable plasma, for two reasons. In the first place the terms coming from initial values may be assigned any size we choose, for they do not die away rapidly--in fact they grow in time. Secondly the correlation functions all contain source terms which grow exponentially in time, with the result that we cannot neglect higher order terms for long time, even if we set the initial values of each correlation function equal to zero. Both of these effects result from the behavior of the long range (collective) interactions, which we are excluding. We may describe the validity of our approximation in the following terms.

We may neglect the long range of interactions when:

1. The usual plasma ordering is maintained for the size of the initial value terms. The expansion parameter does not have to be $1/\Lambda$, but we cannot permit the initial value term from h to be so large as to dominate terms arising from g.
2. Any unstable plasma must return to stability in a sufficiently short time so that higher order terms do not have to be considered. If the system remains in an unstable or marginally stable state for a sufficiently long time, we may not neglect the effect of higher order terms.
3. The correlations which we shall calculate must dominate those corrections coming from collective interactions in the plasma. In the general case of an unstable plasma the effects of long range interactions increase in time, then decrease after the system stabilizes itself. We may be sure that the correction

we compute is significant only after the system has been stable for some time. Thus we are calculating a correction which is significant only for a stable plasma.

We proceed with the calculation. The left side of equation IV-1 involves $\langle \delta f\ \delta f \rangle$, and hence is of order η , where η is our smallness parameter. The right side of equation IV-1 is to be obtained using equation IV-5. All terms of equation IV-5 are of order η^2 , with the exception of these three terms of the form $\langle \{\delta f\ f \delta \mathcal{O}\ \delta f\} \rangle$ which are of order η^3 . We drop these latter terms.

Among the terms which remain are those which depend on initial values $\langle \delta f\ \delta f\ \delta f \mid t = 0 \rangle$. We now omit these terms because we shall consider only the case in which they are smaller (order η^2) than the initial value terms which we shall keep $\langle \delta f\ \delta f \mid t = 0 \rangle$. For the same reason we shall later omit terms of the form $\frac{\ell n\ \Lambda}{\Lambda} \langle \delta f\ \delta f \mid t = 0 \rangle$. They represent a higher order correction to the initial value terms which we shall keep. Again we state that the long range effects contained in $\langle \delta f\ \delta f\ \delta f \mid t = 0 \rangle$ tend to grow in time, as do those of $\langle \delta f\ \delta f \mid t = 0 \rangle$. Thus the correction which we are computing will be significant only after the system has been stable long enough so that the effects of the initial value terms on the evolution of the system are not important. (In a stable system the initial value terms die out exponentially with time.) See Chapter V). The terms which remain from equation IV-5 are now given by

$$\langle \delta f(1)\delta f(2)\delta f(3)|t\rangle = -\int_0^t d\tau \; P_3(t-\tau) \Big[\langle \delta f(1)\delta f(3)\rangle\langle \delta f(2)\underset{\sim}{\mathcal{O}}(1)\rangle$$

$$+ \langle \delta f(1)\delta f(2)\rangle\langle \delta f(3)\underset{\sim}{\mathcal{O}}(1)\rangle + \langle \delta f(1)\delta f(2)\rangle\langle \delta f(3)\underset{\sim}{\mathcal{O}}(2)\rangle$$

$$+ \langle \delta f(1)\underset{\sim}{\mathcal{O}}(2)\rangle\langle \delta f(2)\delta f(3)\rangle + \langle \delta f(1)\delta f(3)\rangle\langle \delta f(2)\underset{\sim}{\mathcal{O}}(3)\rangle$$

$$+ \langle \delta f(1)\underset{\sim}{\mathcal{O}}(3)\rangle\langle \delta f(2)\delta f(3)\rangle\Big]. \qquad \text{(IV-6)}$$

We may now verify that all terms on the right side of equation IV-6 are of the form $\langle \delta f \; \delta f\rangle\langle \delta f \; \delta \underset{\sim}{E}\rangle$, and hence are of order $\eta \times \eta = \eta^2$. Thus the right side of equation IV-1 represents an η^2 correction to the left side, which is of order η. We now inquire when the correction terms may be significant.

If we neglect the right side of equation IV-1, we may split the equation for $\langle \delta f \; \delta f\rangle$ into two equations for $\delta f(\underset{\sim}{r}_1, \underset{\sim}{v}_1, t)$ and $\delta f(\underset{\sim}{r}_2, \underset{\sim}{v}_2, t)$. In this case the P operator is given (formally) by the solution of the linearized Vlasov equation. We now regard the right side of equation IV-1)as coming from the "collisional correction" to the equation for δf. In this way we may observe when this "collision correction" should be considered in deriving the kinetic equation. It is easier to estimate effects in the space $\{\underset{\sim}{r}_1, \underset{\sim}{v}_1\}$, than it is to consider $\{\underset{\sim}{r}_1, \underset{\sim}{v}_1\}$ and $\{\underset{\sim}{r}_2, \underset{\sim}{v}_2\}$ simultaneously. We now give the transformed equation for $\delta f(\underset{\sim}{k}, \underset{\sim}{v}_1, \omega)$, where we have set $\langle \underset{\sim}{E}^0\rangle = 0$, and chosen $k \neq 0$. The $k = 0$ mode contributes nothing to the kinetic equation, due to the shielding effects present in the dielectric function.

$$(-i\omega+i\underset{\sim}{k}\cdot\underset{\sim}{v}_1)\ \delta f + \frac{q}{m}\,\delta\underset{\sim}{E}\cdot\frac{df}{d\underset{\sim}{v}} = \delta f(t=0) - \frac{q}{m}\,\frac{d}{d\underset{\sim}{v}}\cdot\{\delta f\delta E|k,\omega\}\ . \tag{IV-7}$$

The right hand side should not be neglected for

a. Low frequency $\omega \approx 0$. The left hand size = 0 (formally identical to the linearized Vlasov equation) is not satisfactory for very slow processes.

b. Long wavelength $k \sim 0$. In this case the term $\underset{\sim}{k}\cdot\underset{\sim}{v}\ \delta f$ becomes smaller than the collision term.

c. Field free motion $\delta\underset{\sim}{E} \sim 0$. In this case neglecting the right hand side yields free streaming as a solution.

d. For the value of $\underset{\sim}{v}$ such that $\delta\underset{\sim}{E}\cdot\frac{df}{d\underset{\sim}{v}} = 0$.

e. For any mode in the region of $\underset{\sim}{v}$ space $\omega = \underset{\sim}{k}\cdot\underset{\sim}{v}$.

f. For cases where $\gamma_k = \text{Imag}\ \omega_k$ is nearly 0 . The collision term will yield a small correction to the imaginary part of the frequency.

It is possible to go into more detail but we shall not do so.

What, then, would be the effect of including the collision term in the equation for δf ? In general very little. The inclusion of an order η term will lead to corrections of order η to the solution for δf (the correction will show up mostly at the regions mentioned above). This will lead to corrections of order η to the kinetic equation. Is there any case where the corrections must be kept? Yes! In general we must keep the higher order correction to the dielectric function if we wish to have an accurate estimate of the growth (damping) rate γ_k . In particular we must keep the correction in order to make long wavelength plasma oscillations die out on the collisional time scale.

We may now trace this required correction back to the equation for δf . In general short wavelength waves $(k > k_d)$ damp out rapidly in a plasma. The which damp slowly are those of long wavelength $(k \lesssim k_d)$. Thus we wish to calculate the corrections to the long wavelength behavior of the plasma. We shall calculate these corrections by considering only the dominant (large K) part of these corrections. This approximation limits the accuracy of our results to order $\frac{1}{\ell n\Lambda}$. In terms of equation IV-7 we wish to calculate $\{\delta f\delta \underset{\sim}{E}|k \lesssim k_d\}$ where the wavenumber integration used in obtaining the collision term will be over large wavenumbers (K). We indicate the integration explicitly.

$$\{\delta f\delta \underset{\sim}{E}|k\}_{\text{collision}} = \int_{k_d}^{k_0} \frac{d\underset{\sim}{K}}{(2\pi)^3} \{\delta f(\underset{\sim}{k} - \underset{\sim}{K})\delta \underset{\sim}{E}(\underset{\sim}{K})\} \quad . \qquad \text{(IV-8)}$$

This term, when multiplied by $\delta f(-\underset{\sim}{k})$ and averaged, will yield the small correction to the equation for $\langle\delta f(\underset{\sim}{k})\delta f(-\underset{\sim}{k})\rangle$, where $\underset{\sim}{k} \lesssim k_d$. In terms of equation IV-6

$$\langle \delta f(-\underset{\sim}{k}, \underset{\sim}{v}_2, t) \{\delta f \delta \underset{\sim}{E} | \underset{\sim}{k}, \underset{\sim}{v}_1, t\} \rangle = \sum_{\nu} 4\pi n_\nu q_\nu \int d\underset{\sim}{v}_3$$

$$\times \int_{K_d}^{K_0} \frac{d\underset{\sim}{K}}{(2\pi)^3} \left(\frac{i\underset{\sim}{K}}{K^2}\right) \langle \delta f(-\underset{\sim}{k}, \underset{\sim}{v}_2, t)\ \delta f(\underset{\sim}{k}-\underset{\sim}{K}, \underset{\sim}{v}_1, t)\ \delta f_\nu(\underset{\sim}{K}, \underset{\sim}{v}_3, t) \rangle =$$

$$\sum_{\nu} 4\pi n_\nu q_\nu \int d\underset{\sim}{v}_3 \int \frac{d\underset{\sim}{K}}{(2\pi)^3} \frac{i\underset{\sim}{K}}{K^2} \int_0^t d\tau\ P(-\underset{\sim}{k}, \underset{\sim}{v}_2, t-\tau)$$

$$\times P(\underset{\sim}{k} - \underset{\sim}{K}, \underset{\sim}{v}_1, t-\tau)\ P(\underset{\sim}{K}, \underset{\sim}{v}_3, t-\tau) \left[\frac{q}{m} \langle \delta f(-k, v_2, \tau)\ \delta \underset{\sim}{E}(\underset{\sim}{k}, \tau) \rangle \cdot \right.$$

$$\left\{ \frac{d}{d\underset{\sim}{v}_1} \langle \delta f(-\underset{\sim}{K}, \underset{\sim}{v}_1, \tau)\ \delta f(K, \underset{\sim}{v}_3, \tau) \rangle + \frac{d}{d\underset{\sim}{v}_3} \langle \delta f(\underset{\sim}{k} - \underset{\sim}{K}, \underset{\sim}{v}_1, \tau)\ \delta f(\underset{\sim}{K} - \underset{\sim}{k}, \underset{\sim}{v}_3, \tau) \rangle \right\}$$

$$+ \frac{q}{m} \langle \delta f(\underset{\sim}{K}, \underset{\sim}{v}_3, \tau)\ \delta \underset{\sim}{E}(-\underset{\sim}{K}, \tau) \rangle \cdot \left\{ \frac{d}{d\underset{\sim}{v}_1} \langle \delta f(\underset{\sim}{k}, \underset{\sim}{v}_1, \tau)\ \delta f(-\underset{\sim}{k}, \underset{\sim}{v}_2, \tau) \rangle + \right.$$

$$\left. \frac{d}{d\underset{\sim}{v}_2} \langle \delta f(\underset{\sim}{k}-\underset{\sim}{K}, \underset{\sim}{v}_1, \tau)\ \delta f(\underset{\sim}{K} - \underset{\sim}{k}, \underset{\sim}{v}_2, \tau) \rangle \right\} +$$

$$\frac{q}{m} \langle \delta f(\underset{\sim}{k}-\underset{\sim}{K}, \underset{\sim}{v}_1, \tau) \delta \underset{\sim}{E}(\underset{\sim}{K}-\underset{\sim}{k}, \tau) \rangle \cdot \left\{ \frac{d}{d\underset{\sim}{v}_2} \langle \delta f(-\underset{\sim}{K}, \underset{\sim}{v}_2, \tau) \delta f(\underset{\sim}{K}, \underset{\sim}{v}_3, \tau) \rangle \right.$$

$$\left.\left. \frac{d}{d\underset{\sim}{v}_3} \langle \delta f(-\underset{\sim}{k}, \underset{\sim}{v}_2, \tau)\ \delta f(\underset{\sim}{k}, \underset{\sim}{v}_3, \tau) \rangle \right\} \right] . \qquad \text{(IV-9)}$$

We now proceed by a perturbation technique similar to that suggested by Dupree. We shall insert for the expressions $\langle \delta f(\underset{\sim}{K})\ \delta f(-\underset{\sim}{K}) | \tau \rangle$ and $\langle \delta f(\underset{\sim}{K})\ \delta \underset{\sim}{E}(-\underset{\sim}{K}) | \tau \rangle$ the long time values $\langle \delta f(\underset{\sim}{K})\ \delta f(-\underset{\sim}{K}) | \tau = \infty \rangle$, $\langle \delta f(K)\ \delta E(-K) | \tau = \infty \rangle$. The physical motivation is simple: we expect the large K terms to evolve rapidly to become functionals of f. Thus we may insert the asymptotic forms for these expressions, then calculate $\langle \delta f(k)\,\delta f(-k)\rangle$ in terms of these long time values.

In fact even the large K terms do not approach a limiting value as $t \to \infty$, for they contain expressions which oscillate indefinitely. This comes from the fact that the P opeators are obtained from the linearized Vlasov equation, which is a singular equation. We shall avoid the difficulty by substituting the condition $\frac{d}{dt} = 0$ for the condition $t \to \infty$. Of course either condition represents an ad hoc approximation that is used on occasion in order to permit a calculation. The completely correct procedure is to solve for $\langle \delta f\,\delta f\rangle$ and $\langle \delta f\,\delta f\,\delta f\rangle$ simultaneously. This appears extremely difficult, though we expect that the result would justify the requirement used here. We note for emphasis that the requirement $\frac{d}{dt} = 0$ is a legitimate one, which may be satisfied by the appropriate choice of $g(t = 0)$. Thus we will substitute in equation IV-9 the stationary values of $\langle \delta f(K)\delta f(-K)\rangle$ and $\langle \delta f(K)\delta E(-K)\rangle$. The latter expression was calculated in detail on pages 45-50. We do not require the whole expression, only that part which leads to the dominant $(\ell n\Lambda)$ contribution to $\langle \delta f\{\delta f\delta E\}\rangle$. This part is given by

$$\langle \delta f_\nu(\underset{\sim}{K},\underset{\sim}{v})\ \delta \underset{\sim}{E}(-\underset{\sim}{K})\rangle = \frac{4\pi\ i\ \underset{\sim}{K}\ q_\nu\ f_\nu(\underset{\sim}{v})}{K^2\ \epsilon(-\underset{\sim}{K}, -\underset{\sim}{K}\cdot\underset{\sim}{v})} \quad . \qquad \text{(IV-10)}$$

If this term itself is integrated over $\underset{\sim}{K}$ (as in the calculation of the Lenard-Balescu collision term) the result diverges as K^2 for large $|K|$, but vanishes when the angular K integration is performed. We cannot ignore this term in the present context because other factors of K will appear before the $\underset{\sim}{K}$ integration is performed. We may set $\epsilon = 1$, since we cut off the K integral for small K .

We also need the expression $\frac{d}{d\underset{\sim}{v}} \langle \delta f(\underset{\sim}{K}, \underset{\sim}{v}) \, \delta f(-\underset{\sim}{K}, \underset{\sim}{v}')\rangle$, where we obtain $\langle \delta f \, \delta f \rangle$ by using equation II-71, which involves the P operators. Again there is a subtle point, for the velocity derivative does not commute with the P operators. (This point was overlooked by Dupree). We note the fact that the velocity derivative applies to particles, not operators, and write

$$\frac{d}{d\underset{\sim}{v}} \langle \delta f_\nu(\underset{\sim}{K}, \underset{\sim}{v}, \tau = \infty) \, \delta f_\alpha(-\underset{\sim}{K}, \underset{\sim}{v}', \tau = \infty)\rangle =$$

$$\langle \frac{1}{n^2} \sum_{i,j} \frac{d}{d\underset{\sim}{v}} \delta\left[\underset{\sim}{v} - v_{\nu i}(\infty)\right] \delta\left[\underset{\sim}{v}' - v_{\alpha j}(\infty)\right]$$

$$\exp\left[i \, \underset{\sim}{K} \cdot \left\{\underset{\sim}{r}_{\nu i}(\infty) - \underset{\sim}{r}_{\alpha j}(\infty)\right\}\right]\rangle$$

$$= \langle \frac{1}{n^2} \sum_{i,j} \frac{d}{d\underset{\sim}{v}_{\nu i}} \delta\left[\underset{\sim}{v} - \underset{\sim}{v}_{\nu i}(\infty)\right] \delta\left[\underset{\sim}{v}' - \underset{\sim}{v}_{\alpha j}(\infty)\right]$$

$$\times \exp\left[i \, \underset{\sim}{K} \cdot \left\{\underset{\sim}{r}_{\nu i}(\infty) - \underset{\sim}{r}_{\alpha j}(\infty)\right\}\right]\rangle \quad . \qquad \text{IV-II}$$

We may now use the P operators of equation II-71, and the fact that the result must be stationary in time (to avoid factors of the form $e^{i\mathbf{K}\cdot\mathbf{v}t}$) to compute

$$\frac{d}{d\mathbf{v}} \langle \delta f_\nu(\mathbf{K}, \mathbf{v}, \tau = \infty)\, \delta f_\alpha(-\mathbf{K}, \mathbf{v}', \tau = \infty) \rangle =$$

$$\frac{\delta(\mathbf{v}-\mathbf{v}_1)}{n_\nu}\, \delta_{\alpha\nu} \frac{df_\nu}{d\mathbf{v}} + \frac{\frac{4\pi\, q_\alpha q_\nu}{m_\nu} \frac{\mathbf{K}}{K^2} \cdot \frac{df_\alpha}{d\mathbf{v}'} \frac{df_\nu}{d\mathbf{v}}}{\mathbf{K} \cdot (\mathbf{v}' - \mathbf{v})} \,. \qquad \text{(IV-12)}$$

Among the terms of IV-9 are two which depend only on large K values. These terms do not contribute to the equation of motion for $\langle \delta f(k)\, \delta f(-k) \rangle$, being in effect a correction to the initial value term. We discard these two terms. We then substitute the results of IV-10 and IV-12 into equation IV-9 to find

$$\langle \delta f(-\mathbf{k}, \mathbf{v}_2, t) \;\{ \delta f \delta \mathbf{E} | \mathbf{k}, \mathbf{v}_1, t \} \rangle = \sum_\nu 4\pi\, n_\nu q_\nu$$

$$\times \int d\mathbf{v}_3 \int \frac{d\mathbf{K}}{(2\pi)^3} \frac{i\mathbf{K}}{K^2} \int_0^t d\tau\; P(-\mathbf{k}, \mathbf{v}_2, t-\tau)\; P(\mathbf{k}-\mathbf{K}, \mathbf{v}_1, t-\tau)$$

$$\times P(\mathbf{K}, \mathbf{v}_3, t-\tau) \left[\frac{q}{m} \langle \delta f(-\mathbf{k}, \mathbf{v}_2, \tau)\; \delta \mathbf{E}(\mathbf{k}, \tau) \rangle \right.$$

$$\times \left\{ \frac{\delta(\mathbf{v}_1 - \mathbf{v}_3)}{n} \left[\frac{df}{d\mathbf{v}_1} + \frac{df}{d\mathbf{v}_3} \right] + \frac{4\pi q^2}{m} \left[\frac{\frac{\mathbf{K}}{K^2} \cdot \frac{df}{d\mathbf{v}_3} \frac{df}{d\mathbf{v}_1}}{\mathbf{K} \cdot (\mathbf{v}_3 - \mathbf{v}_1)} \right. \right.$$

$$\left. \left. + \frac{\frac{(\mathbf{k}-\mathbf{K})}{(k-K)^2} \cdot \frac{df}{d\mathbf{v}_1} \frac{df}{d\mathbf{v}_3}}{(\mathbf{K}-\mathbf{k}) \cdot (\mathbf{v}_3 - \mathbf{v}_1)} \right] \right\} + \frac{4\pi\, i q^2}{m} \frac{\mathbf{K}}{K^2} \left\{ f(\mathbf{v}_3) \right.$$

(continued Eq. IV-13)

$$\frac{d}{d\underset{\sim}{v}_1} \langle \delta f(-\underset{\sim}{k}, \underset{\sim}{v}_2, \tau)\ \delta f(\underset{\sim}{k}, \underset{\sim}{v}_1, \tau)\rangle + \frac{(\underset{\sim}{k}-\underset{\sim}{K})}{(k-K)^2}\ f(\underset{\sim}{v}_1)\ .$$

$$\left.\left.\frac{d}{d\underset{\sim}{v}_3} \langle \delta f(-\underset{\sim}{k}, \underset{\sim}{v}_2, \tau)\ \delta f(\underset{\sim}{k}, \underset{\sim}{v}_3, \tau)\rangle\right\}\right]. \qquad \text{(IV-13)}$$

The result is largely a formal one, since the quantities $\langle \delta f(\underset{\sim}{k})\ \delta \underset{\sim}{E}(-\underset{\sim}{k})|\tau\rangle$ and $\langle \delta f(\underset{\sim}{k})\ \delta f(-\underset{\sim}{k})|\tau\rangle$ are in fact the quantities we wish to find. We now state the philosophy which enables us to carry through the calculation. We wish to redefine the P operator to include the effect of higher order terms on the small wavenumber (k) behavior. To this end we generalize the P operator to include these effects, and call the result P'.

$$\langle \delta f(\underset{\sim}{k}, \underset{\sim}{v}_1, t)\ \delta f(-\underset{\sim}{k}, \underset{\sim}{v}_2, t)\rangle = P'(\underset{\sim}{k}, \underset{\sim}{v}_1, t)\ P'(-k, \underset{\sim}{v}_2, t)$$

$$\times \langle \delta f\ \delta f|t = 0\rangle \quad . \qquad \text{(IV-14)}$$

For large wavenumbers (K) the P operator will not be affected. Now using the result of equation II-62 for the new operators P', we may rewrite IV-13 in the form

$$\langle \delta f(-\underset{\sim}{k}, \underset{\sim}{v}_2, t) \left\{\delta f\ \delta \underset{\sim}{E}|\underset{\sim}{k}, \underset{\sim}{v}_1, t\right\}\rangle =$$

$$\langle \delta f(-\underset{\sim}{k}, \underset{\sim}{v}_2, t)\left[\sum_{\nu} 4\pi\ n_\nu q_\nu \int d\underset{\sim}{v}_3 \int \frac{d\underset{\sim}{K}}{(2\pi)^3} \quad \frac{i\underset{\sim}{K}}{K^2}\right.$$

(continued Equation IV-15)

$$\times \int_0^t d\tau \; P(\underset{\sim}{k}-\underset{\sim}{K}, \underset{\sim}{v}_1, t-\tau) \; P(\underset{\sim}{K}, \underset{\sim}{v}_3, t-\tau) \left\{ \frac{q}{m} \; \delta\underset{\sim}{E}(\underset{\sim}{k}, \tau) \cdot \right.$$

$$\left[\frac{\delta(\underset{\sim}{v}_1 - \underset{\sim}{v}_3)}{n} \left(\frac{df}{d\underset{\sim}{v}_1} + \frac{df}{d\underset{\sim}{v}_3} \right) \right] + \frac{4\pi q^2}{m} \left[\frac{\frac{\underset{\sim}{K}}{K^2} \cdot \frac{df}{d\underset{\sim}{v}_3} \; \frac{df}{d\underset{\sim}{v}_1}}{\underset{\sim}{K} \cdot (\underset{\sim}{v}_3 - \underset{\sim}{v}_1)} + \right.$$

$$\left. \frac{\frac{(\underset{\sim}{k}-\underset{\sim}{K})}{(k-K)^2} \cdot \frac{df}{d\underset{\sim}{v}_1} \; \frac{df}{d\underset{\sim}{v}_3}}{(\underset{\sim}{k}-\underset{\sim}{K}) \cdot (\underset{\sim}{v}_1 - \underset{\sim}{v}_3)} \right] + \frac{4\pi i q^2}{m} \left[\frac{\underset{\sim}{K}}{K^2} \cdot \frac{d}{d\underset{\sim}{v}_1} \; \delta f(\underset{\sim}{k}, \underset{\sim}{v}_1, \tau) \; f(\underset{\sim}{v}_3) \right.$$

$$\left. \left. + \frac{(\underset{\sim}{k}-\underset{\sim}{K})}{(k-K)^2} \cdot \frac{d}{d\underset{\sim}{v}_3} \; \delta f(\underset{\sim}{k}, \underset{\sim}{v}_3, \tau) \; f(\underset{\sim}{v}_1) \right] \right\} \rangle \; . \qquad \text{(IV-15)}$$

The result follows because

$$\langle \delta f \delta f \langle \delta f \delta E \rangle \rangle = \langle \delta f \delta f \rangle \langle \delta f \delta E \rangle \; . \qquad \text{(IV-16)}$$

It is now clearly easier and shorter to remove the factor $\langle \delta f(-\underset{\sim}{k}, \underset{\sim}{v}_2, t)$ from equation IV-15, and regard the remainder as the collision term for the quantity δf . Thus we write

$$\left(\frac{d}{dt} + \underset{\sim}{v}_1 \cdot \nabla_1 \right) \delta f + \frac{q}{m} \; \delta\underset{\sim}{E} \cdot \frac{df}{d\underset{\sim}{v}_1} = - \frac{q}{m} \left\{ \delta\underset{\sim}{E} \cdot \frac{d\delta f}{d\underset{\sim}{v}_1} \right\}_{\text{collision}} \qquad \text{(IV-17)}$$

where

$$\left\{\delta f \delta \underset{\sim}{E}\right\}_{\text{collision}} = \sum_{\nu} 4\pi\, n_{\nu} q_{\nu} \int d\underset{\sim}{v}_3 \int \frac{d\underset{\sim}{K}}{(2\pi)^3} \; \frac{i\underset{\sim}{K}}{K^2}$$

$$\times \int_0^t d\tau\; P(\underset{\sim}{k}-\underset{\sim}{K}, \underset{\sim}{v}_1, t-\tau)\; P(\underset{\sim}{K}, \underset{\sim}{v}_3, t-\tau) \left[\frac{q}{m} \;\delta\underset{\sim}{E}(\underset{\sim}{k},\tau) \;\cdot\right.$$

$$\left\{ \frac{\delta(\underset{\sim}{v}_1 - \underset{\sim}{v}_3)}{n} \left(\frac{df}{d\underset{\sim}{v}_1} + \frac{df}{d\underset{\sim}{v}_3} \right) + \frac{4\pi q^2}{m} \left[\frac{\frac{\underset{\sim}{K}}{K^2} \cdot \frac{df}{d\underset{\sim}{v}_3} \; \frac{df}{d\underset{\sim}{v}_1}}{\underset{\sim}{K} \cdot (\underset{\sim}{v}_3 - \underset{\sim}{v}_1)} + \right.\right.$$

$$\left.\left. \frac{\frac{(\underset{\sim}{k}-\underset{\sim}{K})}{(k-K)^2} \cdot \frac{df}{d\underset{\sim}{v}_1} \; \frac{df}{d\underset{\sim}{v}_3}}{(k-\underset{\sim}{K}) \cdot (\underset{\sim}{v}_1 - \underset{\sim}{v}_3)} \right]\right\} + \frac{4\pi i q^2}{m} \left\{ \frac{\underset{\sim}{K}}{K^2} \cdot \frac{d}{d\underset{\sim}{v}_1} \;\; \delta f(\underset{\sim}{k}, \underset{\sim}{v}_1, \tau)\; f(\underset{\sim}{v}_3) \right.$$

$$\left.\left. + \frac{(\underset{\sim}{k}-\underset{\sim}{K})}{(k-K)^2} \cdot \frac{d}{d\underset{\sim}{v}_3} \;\; \delta f(\underset{\sim}{k}, \underset{\sim}{v}_3, \tau)\; f(\underset{\sim}{v}_1) \right\}\right]. \qquad \text{(IV-18)}$$

The procedure is justified because equation IV-17 does lead to the correct equation for $\langle \delta f \delta f \rangle$. Likewise we may solve equation IV-17 to find the new operator P'. Note that the operator P' for equation IV-17 will be a function of $\{r_1, v_1\}$, so that it will commute with the P' operator acting on the coordinates $\{\underset{\sim}{r}_2, \underset{\sim}{v}_2\}$. The entire procedure is thus self consistent. We are now faced with solving equations IV-17 and IV-18. Such a solution is all but impossible. We must determine the behavior of two quantities, δf_e and δf_i in terms of the rather arbitrary functions f_e and f_i. One encouraging feature is the fact that the equations are linear in δf. A second encouraging feature lies in the fact that the collision term is "small", so that the linearized Vlasov equation provides a decent description for the behavior of δf. With this in mind we shall solve IV-17 and IV-18 by means of a Laplace transform in time, followed by a weak form of perturbation theory. In general equation IV-17 is a "little kinetic equation," involving diffusion in velocity space. For lack of analytic methods we shall not attempt the general solution.

The transform of IV-17 yields

$$(i\omega + i\underset{\sim}{k}\cdot\underset{\sim}{v}_1)\,\delta f + \frac{q}{m}\,\delta\underset{\sim}{E}\cdot\frac{df}{d\underset{\sim}{v}} = \delta f(t=0) - \frac{q}{m}\,\frac{d}{d\underset{\sim}{v}_1}\cdot\{\delta f\delta E\,|\,\underset{\sim}{k},\omega\}_{\text{collision}} \quad . \tag{IV-19}$$

The bracketed quantity in equation IV-18 has time dependence only through $\delta f(\tau)$. We shall write this time dependence in terms of the inverse Laplace transform, and indicate the rest of the expression by the bracket $[\omega_3]$. Thus the transform of IV-18 yields:

$$\int_0^\infty dt\; e^{i\omega t}\,\{\delta f\delta E|t\}_{\text{collision}} = \{\delta f\delta E|\omega\}_{\text{collision}}$$

$$= \sum_\nu 4\pi\, n_\nu q_\nu \int d\underset{\sim}{v}_3 \int \frac{d\underset{\sim}{K}}{(2\pi)^3}\,\frac{i\underset{\sim}{K}}{K^2} \int_0^\infty dt\; e^{i\omega t}\int_0^\infty d\tau\; P_2(t-\tau)\int\frac{d\omega_3}{2\pi}\; e^{-i\omega_3\tau}\,[\omega_3] \quad . \tag{IV-20}$$

After the substitution $\tau' = t-\tau$ and an integration by parts on τ', we find

$$\{\delta f \delta \underset{\sim}{E} | \omega\}_{\text{collision}} = \int_0^{\infty} dt\, e^{i\omega t} \sum_{\nu} 4\pi\, n_\nu q_\nu$$

$$\int d\underset{\sim}{v}_3 \int \frac{d\underset{\sim}{K}}{(2\pi)^3} \left(\frac{i\underset{\sim}{K}}{K^2}\right) P_2(t)\, [\omega] \;. \qquad \text{(IV-21)}$$

We have used the fact that $\delta f(\omega_3)$, $\delta E(\omega_3) \to 0$ for $|\omega_3| \to 0$. The remaining time integral may be performed when we express $P_2(t)$ in terms of $P(\omega_1)\, P(\omega_2)$.

$$\{\delta f \delta \underset{\sim}{E} | \omega\}_{\text{collision}} = \sum_{\nu} 4\pi\, n_\nu q_\nu \int \frac{d\underset{\sim}{K}}{(2\pi)^3} \; \frac{i\underset{\sim}{K}}{K^2} \int d\underset{\sim}{v}_3$$

$$\times \int_0^{\infty} dt\, e^{i\omega t} \int \frac{d\omega_1}{2\pi}\, e^{-i\omega_1 t} P(\omega_1) \int \frac{d\omega_2}{2\pi}\, e^{-i\omega_2 t} P(\omega_2)\;\; [\omega] \;. \qquad \text{(II-22)}$$

We now perform the velocity integration over the delta function in equation IV-21, and omit terms which lead to order one expressions, while keeping the dominant $(\ell n\Lambda)$ terms. We also indicate particle species for definiteness.

$$\{\delta f_\mu \delta \underset{\sim}{E} | \underset{\sim}{k}, \underset{\sim}{v}_1, \omega\}_{\text{collision}} = \int \frac{d\underset{\sim}{K}}{(2\pi)^3} \int \frac{d\omega_1}{2\pi}$$

$$\left[\frac{ -\dfrac{i\underset{\sim}{K}}{K^2} \; \dfrac{4\pi\, q_\mu^{\;2}}{m_\mu} \; 2\delta \underset{\sim}{E}(\underset{\sim}{k},\omega) \cdot \dfrac{df_\mu}{d\underset{\sim}{v}_1} }{ (-\omega_1 + i[k-K]\cdot v_1)\; \epsilon(K, \omega - \omega)\; (-i[\omega - \omega_1] + i\, K \cdot v_1) } \right]$$

$$+ \int \frac{d\underset{\sim}{K}}{(2\pi)^3} \int \frac{d\omega_1}{2\pi} \left[\frac{ \dfrac{q_\mu}{m_\mu} \; \dfrac{i(\underset{\sim}{k}-\underset{\sim}{K})}{(k-K)^2} \cdot \dfrac{df_\mu}{dv_1} \left(\dfrac{-i\underset{\sim}{K}}{K^2}\right) \sum_{\nu} (4\pi\, q_\nu)^2\, n_\nu }{ (-\, i\, \omega_1 + i\, [\underset{\sim}{k}-\underset{\sim}{K}]\cdot \underset{\sim}{v}_1)\; \epsilon(\underset{\sim}{K}, \omega - \omega_1)\, \epsilon(\underset{\sim}{k}-\underset{\sim}{K}, \omega_1) } \right.$$

(continuted Eq. IV-23)

$$\times \int \frac{d\underset{\sim}{v}_3 \; 4 \frac{q_\nu}{m_\nu} \; \delta\underset{\sim}{E}(\underset{\sim}{k},\omega) \cdot \frac{df_\nu}{d\underset{\sim}{v}_3}}{(-\, i\, \omega_1 + i[\underset{\sim}{k}-\underset{\sim}{K}]\cdot \underset{\sim}{v}_3)\, (-\, i[\omega-\omega_1] + i\, \underset{\sim}{K}\cdot\underset{\sim}{v}_3)} \Bigg]$$

$$+ \int \frac{d\underset{\sim}{K}}{(2\pi)^3} \int \frac{d\omega_1}{2\pi} \sum_\nu \left[\frac{\frac{-iK}{K^2} \; 4\pi\, n_\nu q_\nu}{\epsilon(\underset{\sim}{K},\omega-\omega_1)(-i\omega_1 + i\left[\underset{\sim}{k}-\underset{\sim}{K}\right]\cdot \underset{\sim}{v}_1)} \right.$$

$$\left. \times \int \frac{d\underset{\sim}{v}_3}{(-\, i[\omega-\omega_1] + i\, \underset{\sim}{K}\cdot\underset{\sim}{v}_3)} \right] \left[4\pi\, i\, q_\mu q_\nu \left\{ \frac{\underset{\sim}{K}}{m_\mu K^2} \cdot \frac{d}{d\underset{\sim}{v}_1} \; \delta f_\mu(\underset{\sim}{k},\underset{\sim}{v}_1,\omega)\; f_\nu(\underset{\sim}{v}_3) \right. \right.$$

$$\left. + \frac{(\underset{\sim}{k}-\underset{\sim}{K})}{m_\nu (k-K)^2} \cdot \frac{d}{d\underset{\sim}{v}_3} \cdot \delta f_\nu(\underset{\sim}{k},\underset{\sim}{v}_3,\omega)\; f_\mu(v_1) \right\}$$

$$+ \frac{4\pi\, q_\mu q_\nu}{m_\mu\, m_\nu} \left\{ \frac{\frac{\underset{\sim}{K}}{K^2} \cdot \frac{df_\nu}{d\underset{\sim}{v}_3} \; q_\mu\, \delta\underset{\sim}{E}(\underset{\sim}{k},\omega) \cdot \frac{df_\mu}{d\underset{\sim}{v}_1}}{\underset{\sim}{K} \cdot (\underset{\sim}{v}_3 - \underset{\sim}{v}_1)} \right.$$

$$\left.\left. + \frac{\frac{(\underset{\sim}{k}-\underset{\sim}{K})}{(k-K)^2} \cdot \frac{df_\mu}{d\underset{\sim}{v}_1} \; q_\nu\, \delta\underset{\sim}{E}\, (\underset{\sim}{k},\omega) \cdot \frac{df_\nu}{d\underset{\sim}{v}_3}}{(\underset{\sim}{k}-\underset{\sim}{K}) \cdot (\underset{\sim}{v}_1 - \underset{\sim}{v}_3)} \right\} \right] . \qquad \text{(IV-23)}$$

In the first and third terms we close the ω_1 contour in the lower half plane since $\epsilon(K,\omega-\omega_1)$ has zeros in the upper half plane. In the second term the effects of the dielectric function may be neglected because of the small K cutoff. Here we set $\epsilon = 1$ and close the ω_1 contour in the upper half ω_1 plane. For the same reason we set $\epsilon = 1$ in the third term. In the first term we write

$$\frac{1}{\epsilon(\underset{\sim}{K}, \omega-[\underset{\sim}{k}-\underset{\sim}{K}]\cdot\underset{\sim}{v}_1)} = \frac{\text{Real } \epsilon(-\underset{\sim}{K}, [\underset{\sim}{k}-\underset{\sim}{K}]\cdot\underset{\sim}{v}_1-\omega) + i \text{ Imag } \epsilon(-\underset{\sim}{K}, [\underset{\sim}{k}-\underset{\sim}{K}]\cdot\underset{\sim}{v}_1-\omega)}{|\epsilon(\underset{\sim}{K}, \omega-[\underset{\sim}{k}-\underset{\sim}{K}]\cdot\underset{\sim}{v}_1)|^2} . \tag{IV-24}$$

The term involving the real part of ϵ vanishes, being odd in $\underset{\sim}{K}$, while the term involving the imaginary part leads to $\ell n\Lambda$ dependence, and must be kept. We then set $\epsilon = 1$ in the denominator of term one. Finally we write

$$\frac{1}{(-\omega + \underset{\sim}{k}\cdot\underset{\sim}{v}_1 + \underset{\sim}{K}\cdot[\underset{\sim}{v}_3 - \underset{\sim}{v}_1])} \times \frac{1}{(K\cdot[v_3 - v_1])} = \frac{1}{(\omega-\underset{\sim}{k}\cdot\underset{\sim}{v}_1)}$$

$$\times\left[\frac{1}{(-\omega + \underset{\sim}{k}\cdot\underset{\sim}{v}_1 + \underset{\sim}{K}\cdot[\underset{\sim}{v}_3 - \underset{\sim}{v}_1])} - \frac{1}{\underset{\sim}{K}\cdot[\underset{\sim}{v}_3 - \underset{\sim}{v}_1]}\right] \tag{IV-25}$$

and drop those terms which vanish, being odd in $\underset{\sim}{K}$. As a result of these operations we have

$$\{\delta f_\mu \delta \underset{\sim}{E} | \underset{\sim}{k}, \underset{\sim}{v}_1, \omega\}_{\text{collision}} = \int \frac{d\underset{\sim}{K}}{(2\pi)^3} \left[\frac{\underset{\sim}{K}}{K^2} \frac{4\pi\, q_\mu^{\ 2}}{m_\mu} \right.$$

$$\times \frac{2\delta\underset{\sim}{E}(\underset{\sim}{k},\omega)\cdot \dfrac{df_\mu}{d\underset{\sim}{v}_1}}{(-i\omega_1 + i\underset{\sim}{k}\cdot\underset{\sim}{v}_1)} \text{ Imag } \epsilon(-\underset{\sim}{K}, [\underset{\sim}{k}-\underset{\sim}{K}]\cdot\underset{\sim}{v}_1 - \omega)$$

(continued Eq. IV-26)

$$+\int \frac{d\underset{\sim}{K}}{(2\pi)^3} \int d\underset{\sim}{v}_3 \left[\frac{\frac{q_\mu}{m_\mu} \frac{(\underset{\sim}{k}-\underset{\sim}{K})}{(k-K)^2} \cdot \frac{df_\mu}{dv_1} \frac{\underset{\sim}{K}}{K^2} \sum_\nu (4\pi q_\nu)^2 n_\nu}{(-i\omega + i\underset{\sim}{k}\cdot\underset{\sim}{v}_3)(-i\omega + i\underset{\sim}{k}\cdot\underset{\sim}{v}_1 + i\underset{\sim}{K}\cdot[\underset{\sim}{v}_3 - \underset{\sim}{v}_1])} \right.$$

$$\left. \times\ 2\frac{q_\nu}{m_\nu} \delta\underset{\sim}{E}(k,\omega) \cdot \frac{df_\nu}{d\underset{\sim}{v}_3} \right] + \int \frac{d\underset{\sim}{K}}{(2\pi)^3} \left(\frac{-i\underset{\sim}{K}}{K^2} \right)$$

$$\times \sum_\nu (4\pi q_\nu)^2 n_\nu q_\mu \int \frac{d\underset{\sim}{v}_3}{(-i\omega + i\underset{\sim}{k}\cdot\underset{\sim}{v}_1 + i\underset{\sim}{K}\cdot[\underset{\sim}{v}_3 - \underset{\sim}{v}_1])}$$

$$\times \left[\frac{i\underset{\sim}{K}}{m_\mu K^2} \cdot \frac{d}{d\underset{\sim}{v}_1} \delta f_\mu(\underset{\sim}{k},\underset{\sim}{v}_1,\omega) f_\nu(\underset{\sim}{v}_3) + \frac{i(\underset{\sim}{k}-\underset{\sim}{K})}{m_\nu (k-K)^2} \cdot \frac{d}{d\underset{\sim}{v}_3} \delta f_\nu(\underset{\sim}{k},\underset{\sim}{v}_3,\omega) f_\mu(\underset{\sim}{v}_1) \right.$$

$$+ \frac{\frac{q_\mu}{m_\mu m_\nu} \delta\underset{\sim}{E}(\underset{\sim}{k},\omega) \cdot \frac{df_\mu}{d\underset{\sim}{v}_1} \frac{\underset{\sim}{K}}{K^2} \cdot \frac{df_\nu}{d\underset{\sim}{v}_3}}{\omega - \underset{\sim}{k}\cdot\underset{\sim}{v}_1}$$

$$\left. + \frac{\frac{q_\nu}{m_\nu m_\mu} \delta\underset{\sim}{E}(\underset{\sim}{k},\omega) \cdot \frac{df_\nu}{d\underset{\sim}{v}_3} \frac{(\underset{\sim}{k}-\underset{\sim}{K})}{(k-K)^2} \cdot \frac{df_\mu}{d\underset{\sim}{v}_1}}{\omega - \underset{\sim}{k}\cdot\underset{\sim}{v}_3} \right] . \qquad \text{(IV-26)}$$

We use the Plemelj formula (equation II-24), and keep only the delta function, which leads to log dependence when we perform the $\underset{\sim}{K}$ integration. The integration is discussed in appendix B . It also simplifies the result to use the fact that

$$\delta f_\mu(\underset{\sim}{k},\underset{\sim}{v},\omega) = \frac{-\frac{q_\mu}{m_\mu} \delta\underset{\sim}{E}(\underset{\sim}{k},\omega) \cdot \frac{df_\mu}{d\underset{\sim}{v}}}{-i\omega + i\underset{\sim}{k}\cdot\underset{\sim}{v}} . \qquad \text{(IV-27)}$$

Here we have used the solution to the linearized Vlasov equation as a first approximation for δf, and have dropped the term involving the initial value of δf. This initial value term would lead to a correction of order $\frac{\ell n \Lambda}{\Lambda} \langle \delta f \delta f \rangle$ to the initial value term which we keep, $\langle \delta f \delta f | t = 0 \rangle$. This has been discussed on pages 79, 80. The perturbation solution for δf will be discussed shortly. We now collect terms and find

$$\{\delta f_\mu \delta \underset{\sim}{E} | \underset{\sim}{k}, \underset{\sim}{v}_1, \omega\}_{\text{collision}} = - \sum_\nu 2\pi \, n_\nu q_\nu^2 q_\mu \int d\underset{\sim}{v}_3$$

$$\times \Big[\underset{\approx}{Q}(\omega - \underset{\sim}{k} \cdot \underset{\sim}{v}_1 ; \underset{\sim}{v}_1 - \underset{\sim}{v}_3) \cdot \left(\frac{1}{m_\mu} \frac{1}{d\underset{\sim}{v}_1} - \frac{1}{m_\nu} \frac{d}{d v_3} \right) \delta f_\mu(\underset{\sim}{k}, \underset{\sim}{v}_1, \omega) f_\nu(\underset{\sim}{v}_3) \tag{IV-28}$$

$$+ \underset{\approx}{Q}(\omega - \underset{\sim}{k} \cdot \underset{\sim}{v}_3 ; \underset{\sim}{v}_1 - \underset{\sim}{v}_3) \cdot \left(\frac{1}{m_\mu} \frac{d}{d\underset{\sim}{v}_1} - \frac{1}{m_\nu} \frac{d}{d\underset{\sim}{v}_3} \right) f_\mu(\underset{\sim}{v}_1) \, \delta f_\nu(\underset{\sim}{k}, \underset{\sim}{v}_3, \omega) \Big]$$

where

$$\begin{aligned} \underset{\approx}{Q}(x; \underset{\sim}{y}) &= \ell n \Lambda \left(\frac{y^2 \, \underset{\approx}{I} - \underset{\sim}{y}\underset{\sim}{y}}{y^3} \right) && \text{for} \quad k_d > \frac{x}{y} \\ &= \ell n \left(\frac{k_0 y}{x} \right) \left(\frac{y^2 \, \underset{\approx}{I} - \underset{\sim}{y}\underset{\sim}{y}}{y^3} \right) && \text{for} \quad k_0 > \frac{x}{y} > k_d \\ &= 0 && \text{for} \quad \frac{x}{y} > k_0 \; . \end{aligned} \tag{IV-29}$$

B. The Behavior of Fluctuations in the Plasma

We have already committed ourselves to a perturbation solution for δf. We may, if we choose, keep all small corrections to the P operator, with the result that this operator would contain order one terms plus order $\frac{\ell n \Lambda}{\Lambda}$ terms. This seems hardly worthwhile, in view of other difficulties which will appear in the kinetic equation, e.g., the large k cutoff. Accordingly we restrict our attention to the point which was the original cause for the calculation of $\frac{1}{\Lambda}$ terms. We need the correction to the dielectric function.

The lowest order solution to equation IV-19 is given by

$$\delta f_\mu^{\;0} = \frac{\frac{q_\mu}{m_\mu}\,\delta \underset{\sim}{E}(\underset{\sim}{k},\omega)\cdot\frac{df_\mu}{d\underset{\sim}{v}_1}}{i\,\omega - i\underset{\sim}{k}\cdot\underset{\sim}{v}_1} + \frac{\delta f_\mu(t=0)}{-i\omega + i\underset{\sim}{k}\cdot\underset{\sim}{v}_1} \quad . \qquad \text{(IV-30)}$$

We substitute this result into the collision term for fluctuations in the plasma (equation IV-28), and neglect the terms coming from the initial value of δf. (See pages 79-80 and 96). This leads to a correction to δf given by

$$\delta f_\mu^{\;1} = \frac{1}{-i\omega + i\underset{\sim}{k}\cdot\underset{\sim}{v}_1}\sum_\nu \frac{2\pi\, n_\nu q_\nu^{\,2} q_\mu^{\,2}}{m_\mu}\,\frac{d}{d\underset{\sim}{v}_1}\cdot\int d\underset{\sim}{v}_3$$

$$\times\Bigg[\underset{\approx}{Q}(\omega-\underset{\sim}{k}\cdot\underset{\sim}{v}_1;\ \underset{\sim}{v}_1-\underset{\sim}{v}_3)\cdot\left(\frac{1}{m_\mu}\frac{d}{d\underset{\sim}{v}_1}-\frac{1}{m_\nu}\frac{d}{d\underset{\sim}{v}_3}\right)\frac{\frac{q_\mu}{m_\mu}\,\delta\underset{\sim}{E}(\underset{\sim}{k},\omega)\cdot\frac{df_\mu}{d\underset{\sim}{v}_1}\,f_\nu(\underset{\sim}{v}_3)}{(i\,\omega - i\underset{\sim}{k}\cdot\underset{\sim}{v}_1)}$$

$$+\underset{\approx}{Q}(\omega-\underset{\sim}{k}\cdot\underset{\sim}{v}_3;\ \underset{\sim}{v}_1-\underset{\sim}{v}_3)\cdot\left(\frac{1}{m_\mu}\frac{d}{d\underset{\sim}{v}_1}-\frac{1}{m_\nu}\frac{d}{d\underset{\sim}{v}_3}\right)\frac{\frac{q_\nu}{m_\nu}\,\delta\underset{\sim}{E}(k,\omega)\cdot\frac{df_\nu}{d\underset{\sim}{v}_3}}{(i\omega - i\underset{\sim}{k}\cdot\underset{\sim}{v}_3)}\Bigg]. \qquad \text{(IV-31)}$$

Putting IV-30 and IV-31 into Poisson's equation, we find the dielectric function including the collisional damping of waves.

$$\epsilon(\underset{\sim}{k},\omega) = 1 + \sum_{\mu} \frac{\omega_{\mu}^{2}}{k^{2}} \int \frac{\underset{\sim}{k} \cdot \frac{df_{\mu}}{d\underset{\sim}{v}} \, d\underset{\sim}{v}}{\omega - \underset{\sim}{k}\cdot\underset{\sim}{v}}$$

$$+ i \sum_{\mu,\nu} \frac{\omega_{\mu}^{2}}{2k^{2}} \int d\underset{\sim}{v} \int d\underset{\sim}{v}' \left[\frac{\omega_{\nu}^{2}\, q_{\mu} q_{\nu}}{(\omega - \underset{\sim}{k}\cdot\underset{\sim}{v}')^{2}} - \frac{\omega_{\mu}^{2}\, q_{\nu}^{2}}{(\omega - k\cdot v)^{2}} \right]$$

$$\times \underset{\sim}{k} \cdot \underset{\approx}{Q}(\omega - \underset{\sim}{k}\cdot\underset{\sim}{v};\ \underset{\sim}{v}-\underset{\sim}{v}') \cdot \left(\frac{1}{m_{\mu}} \frac{d}{d\underset{\sim}{v}} - \frac{1}{m_{\nu}} \frac{d}{dv'} \right)$$

$$\times \frac{f_{\nu}(\underset{\sim}{v}')\, \underset{\sim}{k} \cdot \frac{df_{\mu}}{d\underset{\sim}{v}}}{(\omega - \underset{\sim}{k}\cdot\underset{\sim}{v})} \quad . \qquad \text{(IV-32)}$$

Since we keep only the correction to the dielectric function in the new operators P', we shall simply call the operators P, and the reader may decide which dielectric function suits his problem. Recall that the $\frac{\ell n\Lambda}{\Lambda}$ corrections to equation IV-32 were obtained by cutting off the K integration at the Debye wavenumber. We may be sure that this procedure is valid only after a sufficient time for the large K contribution to the integral to dominate the small K contribution. In general the small K contributions tend to die out slowly. Thus the $\frac{\ell n\Lambda}{\Lambda}$ corrections are generally valid only for a system near equilibrium.

V. THE KINETIC EQUATION FOR A UNIFORM SYSTEM

A. Construction of the Kinetic Equation for Short Time ($t < t_{ad}$)

Having modified the P operator to include higher order effects in the dielectric function, we now turn to the calculation of the kinetic equation. The formal procedure is essentially that discussed in the derivation of the Lenard-Balescu equation in chapter II, except that we do not take the limit $t \to \infty$. We wish to calculate

$$\frac{df_\mu}{dt} = -\frac{q_\mu}{m_\mu} \frac{d}{d\underset{\sim}{v}} \cdot \langle \delta f_\mu \delta \underset{\sim}{E} \rangle \quad . \tag{V-1}$$

where

$$\langle \delta f_\mu \delta \underset{\sim}{E} \rangle = \int \frac{d\underset{\sim}{k}}{(2\pi)^3} \langle \delta f_\mu(\underset{\sim}{k}, \underset{\sim}{v}, t)\, \delta \underset{\sim}{E}(-\underset{\sim}{k}, t) \rangle \quad . \tag{V-2}$$

We write out the explicit form of $\langle \delta f_\mu \delta \underset{\sim}{E} \rangle$, in terms of the P operator.

$$\langle \delta f_\mu \delta \underset{\sim}{E} \rangle = \int \frac{d\omega_1}{2\pi} e^{-i\omega_1 t} \left[\frac{\sum_\nu \frac{q_\mu}{m_\mu} \frac{i\underset{\sim}{k}}{k^2} \cdot \frac{df_\mu}{d\underset{\sim}{v}} 4\pi n_\nu q_\nu}{(-i\omega_1 + i\underset{\sim}{k}\cdot\underset{\sim}{v})\, \epsilon(\underset{\sim}{k}, \omega_1)} \int \frac{dv_1}{-i\omega_1 + i\underset{\sim}{k}\cdot\underset{\sim}{v}_1} + \right.$$

$$\left. \frac{1}{-i\omega_1 + i\underset{\sim}{k}\cdot\underset{\sim}{v}} \right] \int \frac{d\omega_2}{2\pi} e^{-i\omega_2 t} \left[\sum_\alpha \frac{i\underset{\sim}{k}}{k^2} \frac{4\pi n_\alpha q_\alpha}{\epsilon(-\underset{\sim}{k}, \omega_2)} \int \frac{d\underset{\sim}{v}_2}{-i\omega_2 - i\underset{\sim}{k}\cdot\underset{\sim}{v}_2} \right] X$$

$$\left[\frac{1}{n_\alpha}\,\delta_{\mu\alpha}f_\alpha(\underset{\sim}{v}_2)\;\delta(\underset{\sim}{v}-\underset{\sim}{v}_2) \;+\; g_{\mu\alpha}(\underset{\sim}{k},\,\underset{\sim}{v},\,\underset{\sim}{v}_2,\,t\;=\;0)\right]. \tag{V-3}$$

The last factor contains a singular term (involving $\delta(\underset{\sim}{v} - \underset{\sim}{v}_2)$) and a term coming from the initial value of the two particle correlation function. We shall call the latter term the "initial value term", and defer its calculation until later. We may perform a velocity integration over the singular term, with the result:

$$\langle\delta f_\mu \delta E\rangle_s = \int \frac{d\omega_1}{2\pi}\, e^{-i\omega_1 t} \int \frac{d\omega_2}{2\pi}\, e^{-i\omega_2 t} \qquad X$$

$$\left[\frac{i\underset{\sim}{k}}{k^2}\;\frac{4\pi q_\mu f_\mu(\underset{\sim}{v})}{\epsilon(-\underset{\sim}{k},\,\omega_2)(-i\omega_1 + i\underset{\sim}{k}\cdot\underset{\sim}{v})(-i\omega_2 - i\underset{\sim}{k}\cdot\underset{\sim}{v})} \qquad + \right.$$

$$\left.\sum_\nu \frac{\frac{q_\mu}{m_\mu}\;\frac{i\underset{\sim}{k}}{k^2}\cdot\frac{df_\mu}{d\underset{\sim}{v}}\,(4\pi q_\nu)^2\, n_\nu\,\frac{i\underset{\sim}{k}}{k^2}}{\epsilon(\underset{\sim}{k},\,\omega_1)\;\epsilon(-\underset{\sim}{k},\,\omega_2)(-i\omega_1 + i\underset{\sim}{k}\cdot\underset{\sim}{v})} \int \frac{d\underset{\sim}{v}_1 f_\nu(\underset{\sim}{v}_1)}{(-i\omega_1 + i\underset{\sim}{k}\cdot\underset{\sim}{v}_1)(-i\omega_2 - i\underset{\sim}{k}\cdot\underset{\sim}{v}_1)}\right]. \tag{V-4}$$

Our attention will be focused for some time on the results of the ω integrations. Therefore we shorten notation by defining operators A_1 and A_2 .

$$A_1 = \frac{i\mathbf{k}}{k^2} 4\pi q_\mu f_\mu(\mathbf{v}) \; .$$

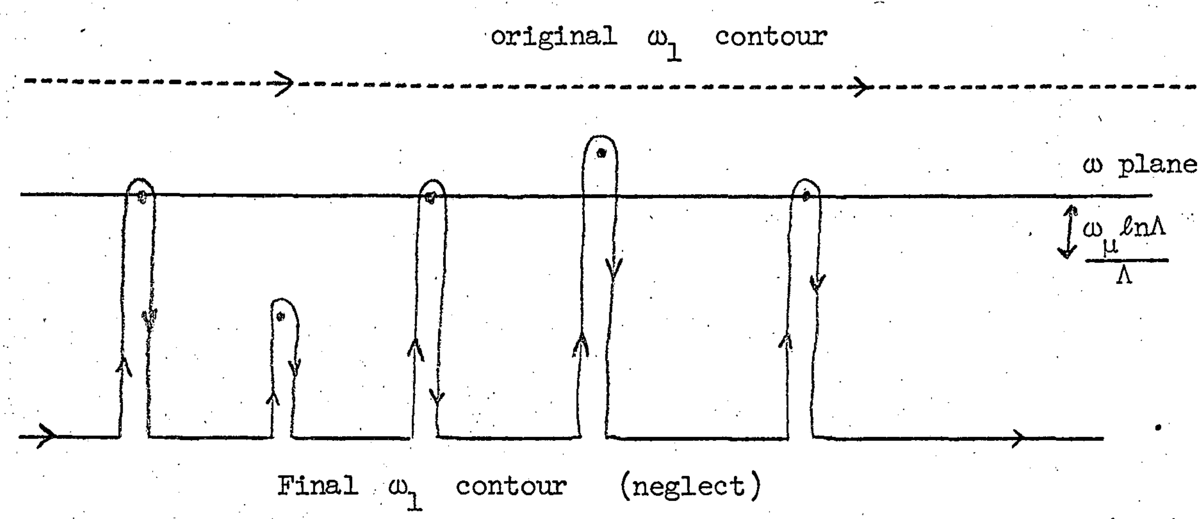
$$A_2 = \sum_\nu \frac{q_\mu}{m_\mu} \; \frac{i\mathbf{k}}{k^2} \cdot \frac{df_\mu}{d\mathbf{v}}(4\pi q_\nu)^2 n_\nu \; \frac{i\mathbf{k}}{k^2} \int d\mathbf{v} f_\nu(\mathbf{v}_1) \; \cdot \tag{V-5}$$

In terms of these definitions we have

$$\langle \delta f_\mu \delta \mathbf{E} \rangle_s + \int \frac{d\omega_1}{2\pi} e^{-i\omega_1 t} \int \frac{d\omega_2}{2\pi} e^{-i\omega_2 t} \left[A_1 \frac{1}{\epsilon(-\mathbf{k}, \omega_2)(i\omega_1 + i\mathbf{k}\cdot\mathbf{v})} \times \frac{1}{(-i\omega_2 - i\mathbf{k}\cdot\mathbf{v})} \right.$$

$$\left. + A_2 \frac{1}{\epsilon(\mathbf{k}, \omega_1)\, \epsilon(-\mathbf{k}, \omega_2)(-i\omega_1 + i\mathbf{k}\cdot\mathbf{v})(-i\omega_1 + i\mathbf{k}\cdot\mathbf{v}_1)(-i\omega_2 - i\mathbf{k}\cdot\mathbf{v}_1)} \right] . \tag{V-6}$$

We now evaluate the ω integrations by pushing the ω contours down in their respective ω planes. We keep the residues from the poles which each contour sweeps over, but we neglect the contour itself as soon as it is rapidly damped in time. (Imag $\omega < \frac{-\omega_\mu \ell n\Lambda}{\Lambda}$). We picture a possible result of the ω_1 integration. Poles in the upper half plane yield residues which grow in time, while poles in the lower half plane yield damped residues. There are also poles on the real axis.



(V-7)

As a result of the ω_1 and ω_2 integrations we obtain the following terms. Since we discard the final ω_1 and ω_2 contours, the terms $e^{-i\omega_k t}$ and $e^{-i\omega_{-k} t}$ do not include very rapidly damped terms. Thus the k integration over these terms is cut off (the integrand disappears) when ω_k and ω_{-k} have large negative imaginary parts. Imag$\begin{bmatrix}\omega_k \\ \omega_{-k}\end{bmatrix} < \frac{-\omega_\mu \ell n\Lambda}{\Lambda}$.

$$\langle \delta f_\mu \delta \underset{\sim}{E} \rangle_s = A_1 \left[\frac{1}{\epsilon(-\underset{\sim}{k}, -\underset{\sim}{k}\cdot\underset{\sim}{v})} + \frac{e^{-ik\cdot vt} \, e^{-i\omega_{-k}t}}{\left.\frac{d\epsilon}{d\omega}\right|_{\omega_{-k}} (\omega_{-k} + \underset{\sim}{k}\cdot\underset{\sim}{v})} \right]$$

$$+ \frac{A_2}{(-i)} \left[\frac{e^{-i\underset{\sim}{k}\cdot\underset{\sim}{v}t}}{\epsilon(\underset{\sim}{k}, \underset{\sim}{k}\cdot\underset{\sim}{v})(\underset{\sim}{k}\cdot\underset{\sim}{v} - \underset{\sim}{k}\cdot\underset{\sim}{v}_1)} + \frac{e^{-i\underset{\sim}{k}\cdot\underset{\sim}{v}_1 t}}{\epsilon(\underset{\sim}{k}, \underset{\sim}{k}\cdot\underset{\sim}{v}_1)(\underset{\sim}{k}\cdot\underset{\sim}{v}_1 - \underset{\sim}{k}\cdot\underset{\sim}{v})} \right.$$

$$\left. + \frac{e^{-i\omega_k t}}{\left.\frac{d\epsilon}{d\omega}\right|_{\omega_k} (\omega_k - \underset{\sim}{k}\cdot\underset{\sim}{v})(\omega_k - \underset{\sim}{k}\cdot\underset{\sim}{v}_1)} \right] \times \left[\frac{e^{ik\cdot v_1 t}}{\epsilon(-\underset{\sim}{k}, -\underset{\sim}{k}\cdot\underset{\sim}{v}_1)} + \right.$$

$$\left. \frac{e^{-i\omega_{-k} t}}{\left.\frac{d\epsilon}{d\omega}\right|_{\omega_{-k}} (\omega_{-k} + \underset{\sim}{k}\cdot\underset{\sim}{v}_1)} \right] . \tag{V-8}$$

In general the dielectric function may have several poles in the region swept over by the ω contours. Thus the terms which we have written as $e^{-i\omega_k t}$ and $e^{-i\omega_{-k} t}$ would be given more accurately by $\sum_i e^{-i\omega_k^i t}$ and $\sum_j e^{-i\omega_{-k}^j t}$. We shall omit the summation, which has no effect on the forthcoming analysis. We shall also consider only the term $i = j$, so that $\sum_i e^{-i\omega_k^i t} \sum_j e^{i\omega_{-k}^j t} \Rightarrow \sum_i e^{-i(\omega_k^i + \omega_{-k}^i)t}$ where the summation will again be implicit. The behavior of terms involving $e^{-i(\omega_k^i + \omega_{-k}^j)t}$ will be considered after the rest of the analysis is completed.

We now note the important fact that $\langle \delta f_{\mu} \delta \underset{\sim}{E} \rangle$ (the sum of all terms) is well defined for all values of $\underset{\sim}{v}$, $\underset{\sim}{v}_1$, and $\underset{\sim}{k}$. (The author wishes to thank Dr. David Sachs for emphasizing this point). Although individual terms are singular for some values of these arguments, the singularities are always cancelled out by other singularities, leaving the sum perfectly regular. In essence the ω contours have picked up all the poles of the original quantity $\langle \delta f_{\mu}(\underset{\sim}{k}, \underset{\sim}{v}, \omega_1)$ $\delta \underset{\sim}{E}(-\underset{\sim}{k}, \omega_2) \rangle$.

In practice it is inconvenient to work with individual terms which are undefined (singular) for some values of their arguments. We shall therefore replace $\underset{\sim}{k} \cdot \underset{\sim}{v}$ by $\underset{\sim}{k} \cdot \underset{\sim}{v} - i\rho$ whenever there is a question as to the meaning of a given expression (ρ is to be a positive real infinitesimal, although the choice ρ negative would lead to the same result). This changes the value of the quantity $\langle \delta f_{\mu} \delta \underset{\sim}{E} \rangle$ by an infinitesimal amount, but it makes our calculation much simpler. There is no way of eliminating singularities coming from zeros of the dielectric function (e.g., $\underset{\sim}{k} \cdot \underset{\sim}{v} = \omega_k$). This will not cause difficulty, for the behavior at such a point may be defined as the limit as the variables (e.g., $\underset{\sim}{k}$ or $\underset{\sim}{v}$) approach the value yielding a singularity. We shall see that the sum of all terms remains well defined.

When we carry out the multiplication of factors in equation V-8, we find eight terms. Two of these terms are independent of time, and one has time dependence $e^{2\gamma_k t}$ (from the fact that $\omega_k + \omega_{-k} = 2i\gamma_k$). These terms may be evaluated as they stand without difficulty.

We require the real part of these terms for the kinetic equation. Note that the A_1 operator is imaginary, while the A_2 operator is real. We use the relations

$$\text{Imag} \frac{1}{\epsilon(-\underset{\sim}{k}, -\underset{\sim}{k}\cdot\underset{\sim}{v})} = \frac{\text{Imag}\ \epsilon(\underset{\sim}{k}, \underset{\sim}{k}\cdot\underset{\sim}{v})}{\left|\epsilon(\underset{\sim}{k}, \underset{\sim}{k}\cdot\underset{\sim}{v})\right|^2} \quad . \tag{V-9}$$

$$\text{Real} \int d\underset{\sim}{v} \frac{1}{(-i)(\underset{\sim}{k}\cdot\underset{\sim}{v}_1 - \underset{\sim}{k}\cdot\underset{\sim}{v} + i\rho)} = \pi \int d\underset{\sim}{v}\ \delta(\underset{\sim}{k}\cdot\underset{\sim}{v}_1 - \underset{\sim}{k}\cdot\underset{\sim}{v}) \tag{V-10}$$

$$\text{Real} \frac{1}{(-i)(\omega_k - \underset{\sim}{k}\cdot\underset{\sim}{v})} \quad \frac{\gamma_k}{(\Omega_k - \underset{\sim}{k}\cdot\underset{\sim}{v})^2 + \gamma_k^2} \tag{V-11}$$

We also use the fact that $\left.\frac{d\epsilon}{d\omega}\right|_{\omega_k} = - \left.\frac{d\epsilon}{d\omega}\right|_{\omega_k}^{*}$ from equation II-114. The contribution to $\frac{df_\mu}{dt}$ coming from these three terms is given by

$$- \frac{q_\mu}{m_\mu} \frac{d}{d\underset{\sim}{v}} \cdot \langle \delta f_\mu \delta E \rangle_{III} = \frac{q_\mu^2}{m_\mu} \frac{d}{d\underset{\sim}{v}} \cdot \int d\underset{\sim}{k} \int \frac{d\underset{\sim}{v}\ \underset{\sim}{k}\,\underset{\sim}{k}\cdot \frac{df_\mu}{d\underset{\sim}{v}}\ \delta(\underset{\sim}{k}\cdot\underset{\sim}{v}-\underset{\sim}{k}\cdot\underset{\sim}{v}_1)}{k^4 \left|\epsilon(\underset{\sim}{k}, \underset{\sim}{k}\cdot\underset{\sim}{v}_1)\right|^2} \times$$

$$\sum_\nu \frac{2n_\nu q_\nu^2}{m_\mu} f_\nu(\underset{\sim}{v}_1)$$

$$- \frac{4\pi q_\mu^2}{m_\mu} \frac{d}{d\underset{\sim}{v}} \cdot \int \frac{d\underset{\sim}{k} \; \underset{\sim}{k} \; \underset{\sim}{f}_\mu(\underset{\sim}{v}) \; \text{Imag} \; \epsilon(\underset{\sim}{k}, \underset{\sim}{k} \cdot \underset{\sim}{v})}{k^2 \; \left| \epsilon(\underset{\sim}{k}, \underset{\sim}{k} \cdot \underset{\sim}{v}) \right|^2} + \frac{q_\mu^2}{m_\mu} \frac{d}{d\underset{\sim}{v}} \cdot$$

$$\int \frac{d\underset{\sim}{k} \; \underset{\sim}{k} \, \underset{\sim}{k} \cdot \frac{df_\mu}{d\underset{\sim}{v}} \gamma_k e^{2\gamma_k t}}{k^4 \left| \frac{d}{d\omega} \right|_{\omega_k}^2 \left[(\Omega_k - \underset{\sim}{k} \cdot \underset{\sim}{v})^2 + \gamma_k^2 \right]} \sum_\nu \frac{2n_\nu q_\nu^2}{\pi m_\mu} \int \frac{d\underset{\sim}{v}_1 \; f_\nu(\underset{\sim}{v}_1)}{(\Omega_{\underset{\sim}{k}} - \underset{\sim}{k} \cdot \underset{\sim}{v}_1)^2 + \gamma_k^2} . \tag{V-12}$$

The first and second terms are identical to those of the Lenard-Balescu equation, if we omit the higher order effects which are present in the dielectric function. The third (with the neglect of higher order effects) has been calculated by Rutherford and Frieman[45]. Rutherford and Frieman state that all terms involving $e^{i\underset{\sim}{k} \cdot \underset{\sim}{v} t}$ go to zero by a "phase mixing process". The statement is incorrect. These latter terms are precisely the ones which must be included to render the analysis consistant.

We turn now to the calculation of these other terms. We write them out and number them, as we shall consider them separately.

$$\langle \delta f_\mu \delta \underset{\sim}{E} \rangle_s \text{ "others" } = A_1 \frac{e^{-i\underset{\sim}{k}\cdot\underset{\sim}{v}t}\, e^{-i\omega_{-k}t}}{\left.\frac{d\epsilon}{d\omega}\right|_{\omega_{-k}} (\omega_{-k} + \underset{\sim}{k}\cdot\underset{\sim}{v})} \quad (1) \quad +$$

$$\frac{A_2}{(-i)} \Bigg[\frac{e^{-i\underset{\sim}{k}\cdot\underset{\sim}{v}t}\, e^{i\underset{\sim}{k}\cdot\underset{\sim}{v}_1 t}}{\epsilon(\underset{\sim}{k},\ \underset{\sim}{k}\cdot\underset{\sim}{v})\ \epsilon(-\underset{\sim}{k},\ -\underset{\sim}{k}\cdot\underset{\sim}{v}_1)(\underset{\sim}{k}\cdot\underset{\sim}{v} - \underset{\sim}{k}\cdot\underset{\sim}{v}_1 - i\rho)} \quad (2) \quad +$$

$$\frac{e^{-i\underset{\sim}{k}\cdot\underset{\sim}{v}t}\, e^{-i\omega_{-k}t}}{\epsilon(\underset{\sim}{k},\ \underset{\sim}{k}\cdot\underset{\sim}{v}) \left.\frac{d\epsilon}{d\omega}\right|_{\omega_{-k}} (\omega_{-k} + \underset{\sim}{k}\cdot\underset{\sim}{v}_1)(\underset{\sim}{k}\cdot\underset{\sim}{v} - \underset{\sim}{k}\cdot\underset{\sim}{v}_1 - i\rho)} \quad (3) \quad +$$

$$\frac{e^{-i\underset{\sim}{k}\cdot\underset{\sim}{v}_1 t}\, e^{-i\omega_{-k}t}}{\epsilon(\underset{\sim}{k},\ \underset{\sim}{k}\cdot\underset{\sim}{v}_1) \left.\frac{d\epsilon}{d\omega}\right|_{\omega_{-k}} (\omega_{-k} + \underset{\sim}{k}\cdot\underset{\sim}{v}_1)(\underset{\sim}{k}\cdot\underset{\sim}{v}_1 - \underset{\sim}{k}\cdot\underset{\sim}{v} + i\rho)} \quad (4) \quad +$$

$$\frac{e^{i\underset{\sim}{k}\cdot\underset{\sim}{v}_1 t}\, e^{-i\omega_{-k}t}}{\epsilon(-\underset{\sim}{k},\ -\underset{\sim}{k}\cdot\underset{\sim}{v}_1) \left.\frac{d\epsilon}{d\omega}\right|_{\omega_{k}} (\omega_{k} - \underset{\sim}{k}\cdot\underset{\sim}{v})(\omega_{k} - \underset{\sim}{k}\cdot\underset{\sim}{v}_1)} \quad (5) \Bigg] .$$

(V-13)

Our principal task consists of the calculation of these terms. We shall evaluate them by deforming contours of integration. The A_1 operator does not involve any integrations, while A_2 contains an integration over $\underset{\sim}{v}_1$. The entire expression (equation V-13) is then integrated over $\underset{\sim}{k}$ to provide a contribution to the collision term. We shall obtain simpler expressions for the terms in equation V-13 by deforming integrations over $\underset{\sim}{k}$ and $\underset{\sim}{v}_1$ into complex vector integrals.

It is pertinent to ask for the physical significance of integrals over functions of complex vectors. The author assigns to them neither more nor less significance than the terms which have already been calculated (equation V-12). The quantities 1 - 5 of equation V-13 are well defined as they stand, and may be left in this form, if we choose. Instead we shift contours of integration because this allows a great simplification in the explicit form of the collision term, and because it makes the resultant expression relatively tractable. The result is quantitatively unchanged.

Essential to the procedure is the fact that we may calculate the terms of equation V-8 for t very large (almost t_{ad}). When we let $\underset{\sim}{k}$ or $\underset{\sim}{v}_1$ (real) become $\underset{\sim}{k}$ or $\underset{\sim}{v}_1$ (complex) in the proper fashion we may cause the exponential time factor to cause rapid damping. The resultant contour of integration may then be thrown away, just as the ω_1 and ω_2 contours were thrown away once they became rapidly damped.

This does not mean that the terms given in equation V-8 vanish. As $\underset{\sim}{k}$ and $\underset{\sim}{v}_1$ move from real to complex values they pass over poles which lead to residues which are not sufficiently rapidly damped to be thrown away. In fact some of the resulting terms will not be damped at all (except that they will disappear on the implicit ($t>t_{ad}$) time scale. Our result for equation V-13 will consist of the contributions from poles which are swept over as we displace the variables of integration.

Thus far we have spoken of shifting $\underset{\sim}{k}$ and $\underset{\sim}{v}_1$ into complex values as though they are scalars. In fact both variables are vectors, and we must consider vector (volume) integrals rather than scalar (line) integrals. As the transformation from a real volume integration to a complex volume integration may tend to cause mental indigestion, we do what we can to make the procedure more palatable.

Note the fact that all terms of equation V-13 contain the factor $e^{-i\underset{\sim}{k}\cdot\underset{\sim}{v}t}$, or $e^{i\underset{\sim}{k}\cdot\underset{\sim}{v}_1 t}$, or both. If we wish to cause damping in the exponential we need displace only one component of $\underset{\sim}{k}$, or of $\underset{\sim}{v}_1$. Displacement of a component of $\underset{\sim}{v}_1$ perpendicular to $\underset{\sim}{k}$ leads to no change in the behavior of the exponential. Displacement of a component of $\underset{\sim}{k}$ perpendicular to $\underset{\sim}{v}_1$ does affect the time behavior, as it affects the value of ω_k and ω_{-k} . In this work we shall follow the simplest course, by expressing the results of all contour integrations in terms of a shift of $\underset{\sim}{v}_1 \| \underset{\sim}{k}$, or $\underset{\sim}{k} \| \underset{\sim}{v}_1$. This permits us to use the usual nomenclature of scalar contour integration and to draw pictures of the movement of contours in the respective complex planes. We simply ignore the components of $\underset{\sim}{v}_1 \perp \underset{\sim}{k}$ and $\underset{\sim}{k} \perp \underset{\sim}{v}_1$.

We will have to displace a component of $\underset{\sim}{k} \perp \underset{\sim}{v}$ on occasion, in order to make certain contours disappear sufficiently rapidly. However our final result will consist only of the residues of poles swept over by the displaced contours, so that this result will not contain terms involving two complex components of $\underset{\sim}{k}$. This will become clear presently.

A further question arises. When the only variable of integration is $\underset{\sim}{k}$, we must perforce shift the $\underset{\sim}{k}$ "contour" in order to produce damping in the exponential. However in many cases there are two variables of integration, $\underset{\sim}{k}$ and $\underset{\sim}{v}_1$. When we consider the initial value terms we shall find three variables of integration $\underset{\sim}{k}$, $\underset{\sim}{v}_1$, and $\underset{\sim}{v}_2$. Which contour(s) shall we move, and how should we express the result?

The question is academic. We move any of the contours in any way we see fit, as long as we perform legitimate mathematical operations. In this work we shall move either one or two contours (for each expression) in order to produce a result in the most direct fashion. Generally we shall seek a result containing the smallest number of terms. In some cases a different choice of contours, with a resultant larger number of terms, leads to more simple pictures of contours of integration. No clear advantage results from such choices, and we choose the contours which lead to the greatest economy of expression.

We now consider how far we may displace k and v_1 (we shall henceforth use scalar notation for the relevant components) from the real axis. For the $\underset{\sim}{k}$ integration we observe equation V-13, and the definitions of the operators A_1 and A_2 . There are no difficulties and we may displace $\underset{\sim}{k}$ as far as we choose. (Of course we pick up poles in the process). On the other hand the $\underset{\sim}{v}_1$ dependence is not given, for $f(\underset{\sim}{v}_1)$ is an unknown of the equation. How far may we push $\underset{\sim}{v}_1$ without running into difficulties? If we consider the equilibrium distribution $f_{eq} \sim \exp\left[-\frac{mv_1^2}{2\mathcal{K}T}\right]$, we see that f becomes large in magnitude as the imaginary part of v_1 becomes large. $f_{eq} \sim \exp\left[\frac{mv_I^2}{2\mathcal{K}T}\right]$. We do not want to throw away "large" terms, even if they are rapidly damped. For this reason we place a limit on the imaginary part of v_1 : Imag $v_1 \lesssim v_{th}$. In this work we shall not consider the possibility that $f(v_1)$ has poles within this range of v_1 = real. In essence this places a limit on the types of distribution functions for which the theory holds. The author doubts that the restriction is physically meaningful.

Because we evaluate all expressions for large times ($t \tilde{<} t_{ad}$), we seldom have to displace k or v_1 very far from real values. In general we want Imag $(kv_1) \gg \frac{\omega_\mu \ell n \Lambda}{\Lambda}$. If $v_1 \sim v_{th}$ then we displace k by approximately $k_d \frac{\ell n\Lambda}{\Lambda}$. If $k \sim k_d$, we displace v_1 by approximately $v_{th} \frac{\ell n\Lambda}{\Lambda}$. When k or v_1 are very small we must displace the corresponding variable a large distance from real values. This is not surprising, for our description of the fluctuation phenomena is not valid in this region - see page 82. Since there is no apparent inconsistency, we simply omit the contribution of contours in this region. Alternately, we permit k and v_1 to be displaced arbitrarily far.

We turn now to the explicit form of the terms we shall calculate. We have already performed contour integrals over ω_1 and ω_2, picking up poles of the form $\frac{1}{\omega - \underset{\sim}{k}\cdot\underset{\sim}{v}}$ and $\frac{1}{\epsilon(\underset{\sim}{k},\omega)}$. In either case the result was elementary, since ω_1 and ω_2 were bona fide scalars. We simply took the residue at the pole, replacing all ω's by the value at the pole.

This situation is modified when we consider moving the k and v_1 contours. Even though we replace the component in question by its value at the pole, we still must perform integrations over the remaining vector components. Instead we shall leave the residue theorem in implicit form. Thus

$$\text{Residue} \int \frac{d\underset{\sim}{k}\; A(\underset{\sim}{k})}{(\underset{\sim}{k}\cdot\underset{\sim}{v} - a)} = \int d\underset{\sim}{k}\; A(\underset{\sim}{k})\; \delta(\underset{\sim}{k}\cdot\underset{\sim}{v} - a) \quad . \tag{V-14}$$

or

$$\text{Residue} \int \frac{d\underset{\sim}{v}_1\; B(\underset{\sim}{v}_1)}{(\underset{\sim}{k}\cdot\underset{\sim}{v}_1 - a)} = \int d\underset{\sim}{v}_1\; B(\underset{\sim}{v}_1)\; \delta(\underset{\sim}{k}\cdot\underset{\sim}{v}_1 - a) \; . \tag{V-15}$$

The explicit form of several of the terms of equation V-13 is now determined. One point remains.

We have already found the zeros of the function $\epsilon(\underset{\sim}{k},\omega)$ in the ω plane. We call these values of ω "ω_k", where $\underset{\sim}{k}$ was real.

$$\epsilon(\underset{\sim}{k}, \omega_k) = 0 \quad . \tag{V-16}$$

Of course we shall use the same notation when $\underset{\sim}{k}$ is complex. The difficulty is the following. The zeros of $\epsilon(\underset{\sim}{k}, \omega)$ in the ω plane lead to residues containing $\left.\frac{d\epsilon}{d\omega}\right|_{\omega_k}$. But $\epsilon(\underset{\sim}{k}, \underset{\sim}{k}\cdot\underset{\sim}{v}_1)$, which appears in V-13, has zeros in the $\underset{\sim}{k}$ plane and in the $\underset{\sim}{v}_1$ plane. When we pick up the poles of $\frac{1}{\epsilon}$, we are lead to residues involving $\left.\frac{d\epsilon}{dv}\right|_{v_1(k)}$, $\left.\frac{d\epsilon}{dk}\right|_{k(v_1)}$, as well as the previously calculated $\left.\frac{d\epsilon}{d\omega}\right|_{\omega(k)}$. The terminology is inconvenient, and we seek a uniform means of expression for the poles of $\frac{1}{\epsilon}$. We procede by expanding ϵ about a zero in the respective planes. (ϵ is an analytic function by definition).

$$\text{Residue} \int \frac{d\omega\, A(\omega)}{\epsilon(\underset{\sim}{k}, \omega)} = \text{Res} \int \frac{d\omega\, A(\omega)}{\left.\frac{d\epsilon}{d\omega}\right|_{\omega_k} (\omega - \omega_k) + \cdots} = \frac{A(\omega_k)}{\left.\frac{d\epsilon}{d\omega}\right|_{\omega_k}} . \tag{V-17}$$

$$\text{Residue} \int \frac{d\underset{\sim}{v}_1 \; B(\underset{\sim}{v}_1)}{\epsilon(\underset{\sim}{k}, \; \underset{\sim}{k}\cdot\underset{\sim}{v}_1)} = \text{Residue} \int \frac{d\underset{\sim}{v}_1 \; B(\underset{\sim}{v}_1)}{\frac{d\epsilon}{d\omega}\Big|_{\omega_k} (\underset{\sim}{k}\cdot\underset{\sim}{v}_1 - \omega_k) + \cdot \cdot} =$$

$$\int \frac{d\underset{\sim}{v}_1 \; B(\underset{\sim}{v}_1)}{\frac{d\epsilon}{d\omega}\Big|_{\omega_k}} \, \delta(k\cdot v_1 - \omega_k) \; . \tag{V-18}$$

$$\text{Residue} \int \frac{d\underset{\sim}{k} \; C(\underset{\sim}{k})}{\epsilon(\underset{\sim}{k}, \; \underset{\sim}{k}\cdot\underset{\sim}{v}_1)} = \text{Res} \int \frac{d\underset{\sim}{k} \quad C(\underset{\sim}{k})}{\frac{d\epsilon}{dk}\Big|_k (k - k(v_1)) + \cdot \cdot} =$$

$$\int \frac{d\underset{\sim}{k} \; C(\underset{\sim}{k})}{\frac{d\epsilon}{dk}\Big|_{k(\underset{\sim}{v}_1)}} \, \delta(k - k(\underset{\sim}{v}_1)) \; . \tag{V-19}$$

We show that this expression may be written in the form of equation V-18. We have

$$\left.\frac{d\epsilon}{dk}\right|_{k(\underset{\sim}{v}_1)} = \left.\frac{\partial\epsilon}{\partial k}\right|_{k(\underset{\sim}{v}_1)} + \frac{d\epsilon}{d\omega} \left.\frac{\partial(\underset{\sim}{k}(\underset{\sim}{v}_1)\cdot\underset{\sim}{v}_1)}{\partial k}\right|_{k(\underset{\sim}{v}_1)} =$$

$$\left.\frac{\partial\epsilon}{\partial k}\right|_{k(v_1)} + v_1 \left.\frac{d\epsilon}{d\omega}\right|_{\omega_k} = \left.\frac{d\epsilon}{d\omega}\right|_{\omega_k} \left(v_1 + \left.\frac{\frac{\partial\epsilon}{\partial k}}{\frac{d\epsilon}{d\omega}}\right|_{\omega_k} \right) . \tag{V-20}$$

From equation V-16 we have

$$\left(\frac{\partial\epsilon}{\partial k} + \frac{d\epsilon}{d\omega}\frac{d\omega_k}{dk} \right)_{\omega_k} = 0 .$$

$$\frac{d\omega_k}{dk} = \left.\frac{\frac{\partial\epsilon}{\partial k}}{\frac{d\epsilon}{d\omega}}\right|_{\omega_k} . \tag{V-21}$$

and combining V-19, V-20 , and V-21 , we have

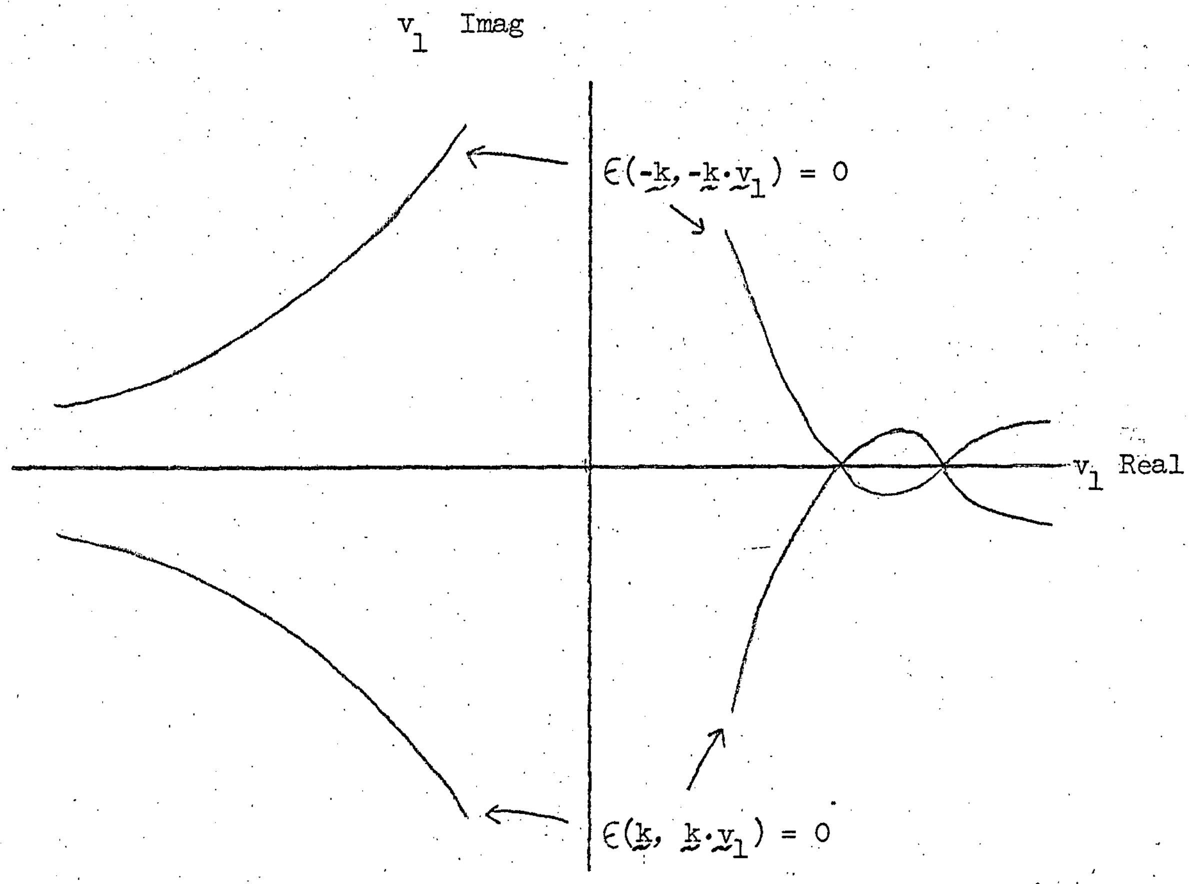
$$\text{Residue} \int \frac{d\underset{\sim}{k}\ C(\underset{\sim}{k})}{\epsilon(\underset{\sim}{k},\ \underset{\sim}{k}\cdot\underset{\sim}{v})} = \int \frac{d\underset{\sim}{k}\ C(\underset{\sim}{k})\ \delta(\underset{\sim}{k} - \underset{\sim}{k}(\underset{\sim}{v}_1))}{\frac{d\epsilon}{d\omega}\Big|_{\omega_k}\left(v_1 - \frac{d\omega_k}{dk}\right)} \; . \tag{V-22}$$

We use the property of the delta function $\int dxA(x)\delta\left[f(x)\right] = \int \frac{dxA(x)\delta(x - x_o)}{f'(x_o)}$ to write V-23 in the desired form

$$\text{Residue} \int \frac{d\underset{\sim}{k}\ C(\underset{\sim}{k})}{\epsilon(\underset{\sim}{k},\ \underset{\sim}{k}\cdot\underset{\sim}{v}_1)} = \int \frac{dk\ C(\underset{\sim}{k})\delta(\underset{\sim}{k}\cdot\underset{\sim}{v}_1 - \omega_k)}{\frac{d\epsilon}{d\omega}\Big|_{\omega_k}\left(v_1 - \frac{d\omega_k}{dk}\right)} \; . \tag{V-22}$$

We shall express all residues of $\frac{1}{\epsilon}$ in the form of equation V-18 or V-23.

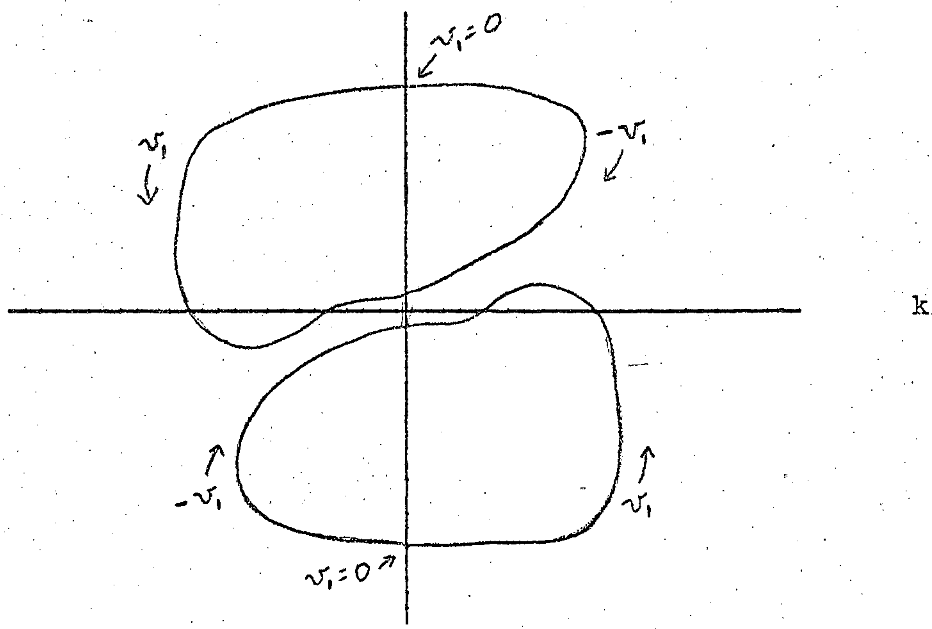
Before proceding with the calculation we illustrate where the zeros of $\epsilon(\underset{\sim}{k},\ \underset{\sim}{k}\cdot\underset{\sim}{v}_1)$ and $\epsilon(-\underset{\sim}{k},\ -\underset{\sim}{k}\cdot\underset{\sim}{v}_1)$ might be found in the complex $\underset{\sim}{k}$ and $\underset{\sim}{v}_1$ planes. No detailed picture is possible, since we do not assume a particular choice for the distribution functions $f_e(\underset{\sim}{v}_1)$ and $f_i(\underset{\sim}{v}_1)$. In general the plasma may become unstable for plasma oscillations $(\omega \sim \omega_e)$, ion waves $\frac{\omega}{k} \sim \sqrt{\frac{\mathcal{K}T_e}{m_i}}$, and the two stream instability (low frequency). We will draw pictures appropriate to the first case. We consider first the v_1 plane, with k real.



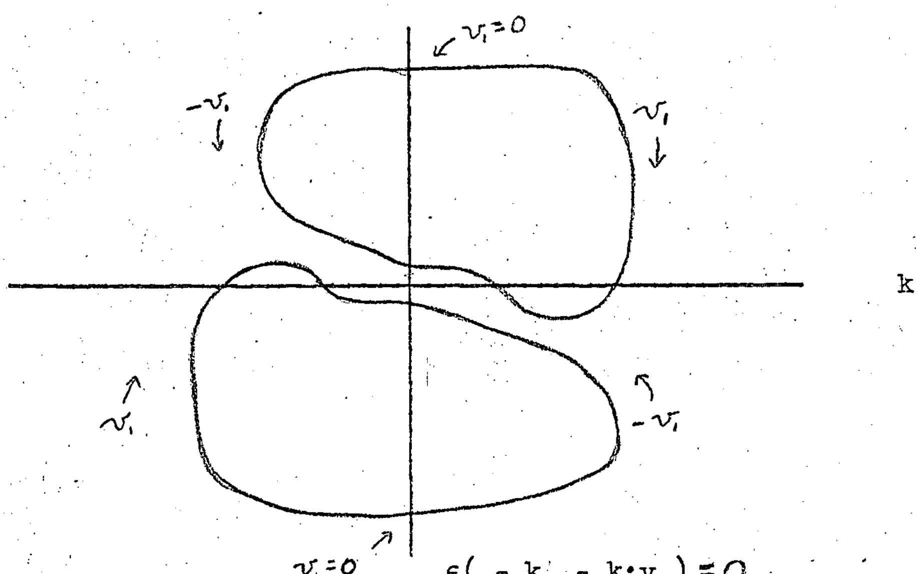
(V-24)

The symmetry follows from the fact that $\epsilon(\underset{\sim}{k}, \underset{\sim}{k}\cdot\underset{\sim}{v}_1) = \epsilon(-\underset{\sim}{k}, -\underset{\sim}{k}\cdot\underset{\sim}{v}_1^{*})^{*}$. As shown here the plasma is unstable for some waves with positive phase velocity, and stable for all waves with negative phase velocity. Note that in the unstable case the zeros of ϵ cross the real axis. The region enclosed by this crossing may be identified with an unstable volume of k, v_1 space. Waves (solutions to $\epsilon(\underset{\sim}{k}, \omega_k) = 0$). with this phase velocity are unstable, i.e., the factor $e^{-i\omega_k t}$ grows in time. In what follows we shall consider only the case of an unstable plasma, as it evidently includes the stable plasma as a special case.

The behavior in the k plane is somewhat different. There are two functions which we must consider $\epsilon(\underset{\sim}{k}, \underset{\sim}{k}\cdot\underset{\sim}{v}_1)$ and $\epsilon(-\underset{\sim}{k}, -\underset{\sim}{k}\cdot\underset{\sim}{v}_1)$. We consider the case of an unstable plasma.



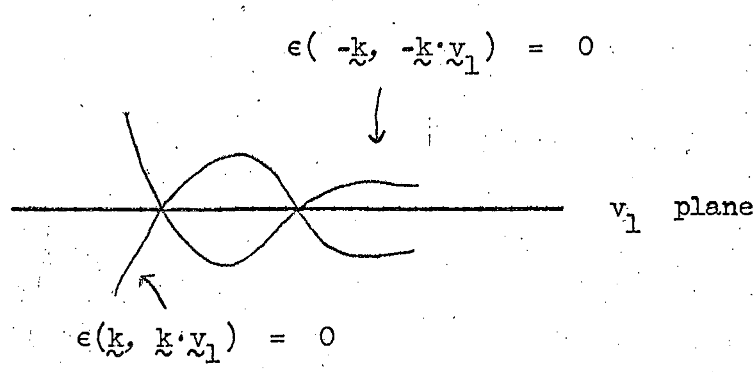
$$\epsilon(\underset{\sim}{k}, \underset{\sim}{k}\cdot\underset{\sim}{v}_1) = 0$$

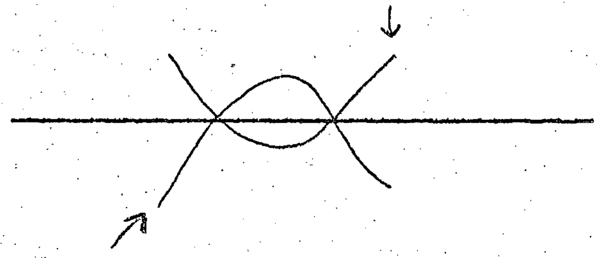


$$\epsilon(-\underset{\sim}{k}, -\underset{\sim}{k}\cdot\underset{\sim}{v}_1) = 0 \qquad \text{(V-25)}$$

The curves represent possible solutions to the equations indicated. As shown $\underset{\sim}{v}_1$ is essentially a parameter taking on values $-\infty$ to $+\infty$. There are two curves for each equation because for a given v_1 there are two solutions to the equation $\epsilon(\underset{\sim}{k}, \underset{\sim}{k}\cdot\underset{\sim}{v}_1) = 0$ or to the equation $\epsilon(-\underset{\sim}{k}, -\underset{\sim}{k}\cdot\underset{\sim}{v}_1) = 0$. The symmetry between the two pictures, and between the two curves in a given picture follows from the fact that $\epsilon(\underset{\sim}{k}, \underset{\sim}{k}\cdot\underset{\sim}{v}_1) = \epsilon(-\underset{\sim}{k}^*, -\underset{\sim}{k}^*\cdot\underset{\sim}{v}_1)^*$. Again the region(s) in which the zeros of ϵ cross the real axis may be identified with an unstable volume in $\underset{\sim}{k}$, $\underset{\sim}{v}_1$ space.

In the analysis to follow we must perform integrations over $\underset{\sim}{k}$ and $\underset{\sim}{v}_1$, where both vectors take on both positive and negative values. Since it is inconvenient to consider all possible combinations of signs, <u>we shall henceforth choose k and v_1 positive</u> with respect to some arbitrary axis. This is in no way a restriction, as the other cases follow trivially from the basic properties of the dielectric function. The other cases follow directly from the pictures given previously. Henceforth we need consider only the pictures:





(V-26)

We now consider the procedure for moving contours in the k plane so as to produce rapid damping. The terms which we will consider have the time dependence of the form $e^{i\underset{\sim}{k}\cdot\underset{\sim}{v}_1 t}$, and $e^{-i\underset{\sim}{k}\cdot\underset{\sim}{v}_1 t}\, e^{-i\omega_{-k} t}$. It might seem reasonable to move k up in the first case, and down in the second. This is not always correct, for ω_k and ω_{-k} also vary with k. We use the Cauchy-Riemann equations, with $\omega_k = \Omega_k + i\gamma_k$.

$$\frac{d\Omega_k}{dk_R} = \frac{d\gamma_k}{dk_I} \cdot \qquad \frac{d\gamma_k}{dk_R} = -\frac{d\Omega_k}{dk_I} \cdot \tag{V-27}$$

For plasma oscillations $\Omega_k \sim \sqrt{\omega_e^2 + \frac{3KTk^2}{m}}$ so

$$\frac{d\Omega_k}{dk_R} \sim \frac{\frac{3KT}{m}\, k}{\sqrt{\omega_e^2 + \frac{3KTk^2}{m}}} \sim \mathcal{O}\left(\frac{k}{k_d}\, v_{th}^e\right) \tag{V-28}$$

For the imaginary part of ω_k we have

$$\frac{d\gamma_k}{dk_R} \sim \mathcal{O}\left(\frac{\gamma_k}{kd}\right) . \tag{V-29}$$

Since we do not push k far off the real k axis in order to pick up poles, we will neglect the change of Ω_k with k_I, that is we neglect $\frac{d\Omega_k}{dk_I} \Delta k_I$ compared to γ_k. A further point along this line must be mentioned. In general the relation $\omega_k = -\omega_{-k}^{*}$ holds only for k a real vector. When k is complex we have $\omega_k = -\omega_{k^*}^{*}$. Thus we ask what is the value of $\omega_k + \omega_k = 2i\gamma_k$ for k off the real axis? From equation 211, for both ω_k and ω_{-k} we have

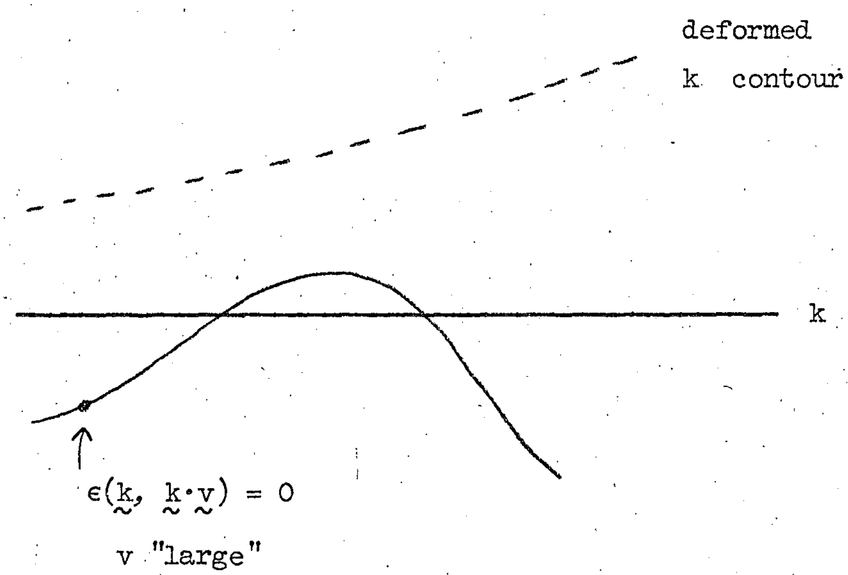
$$\begin{aligned} \frac{d\gamma_k}{dk_I} + \frac{d\gamma_{-k}}{dk_I} &= \frac{d\Omega_k}{dk_I} + \frac{d\Omega_{-k}}{dk_I} \\ &= \frac{d\Omega_k}{dk_I} + \frac{d(-\Omega_k)}{dk_I} \\ &= 0 \end{aligned} \tag{V-30}$$

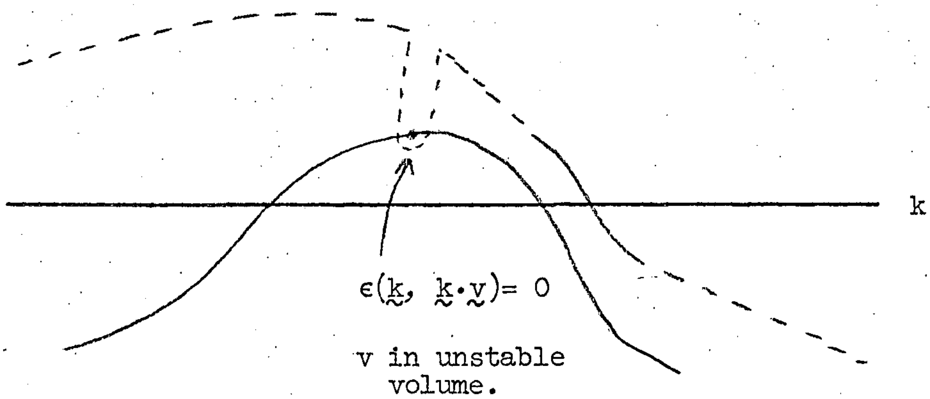
for k on the real axis. Thus when we push k off the real axis the value of $\omega_k + \omega_{-k}$ is effectively unchanged. In all expressions involving $e^{2\gamma_k t}$, the value of γ_k should be determined from $\epsilon(k = \text{real}, \omega_k) = 0$, rather than from a zero of the dielectric function at a point where k is complex.

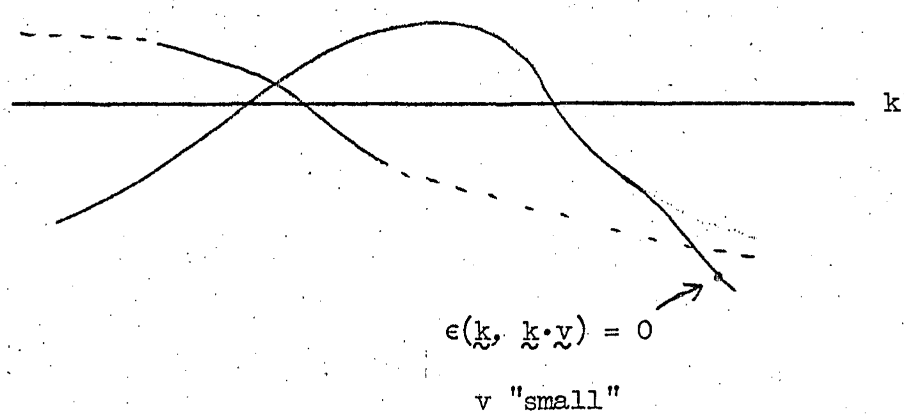
If we pick a representative term, say $e^{i\underset{\sim}{k}\cdot\underset{\sim}{v}t}e^{-i\omega_k t}$, it is clear from the preceding that a change Δk_I in the imaginary part of k causes a change in the real part of the exponent given by

$$-v\Delta k_I + \frac{d\Omega_k}{dk_R}\Delta k_I \quad .$$

Thus we want to move the k contour up (into positive imaginary values) for v larger than $\frac{d\Omega_k}{dk_R}$, and down for v less than $\frac{d\Omega_k}{dk_R}$. We illustrate possible motions of the k contour for varying values of v.







(V-31)

In some regions (specifically those regions where the deformed contour crosses the real axis) the contour integral is not sufficiently damped, and may not be neglected. These regions are unimportant, for we may still displace a component of k perpendicular to v. This does not affect the behavior of the factor $e^{i\underset{\sim}{k}\cdot\underset{\sim}{v}t}$, but it does affect the factor $e^{-i\omega_k t}$. Since we may always displace a component of k perpendicular to v so as to cause rapid damping, we shall simply neglect vestiges of contours of the type shown by solid lines in picture V-31.

We now consider the zeros which the k contour may pick up while moving in the k plane. Evidently there are two possibilities: $\epsilon(\underset{\sim}{k}, \underset{\sim}{k}\cdot\underset{\sim}{v}) = 0$ and $\epsilon(-\underset{\sim}{k}, -\underset{\sim}{k}\cdot\underset{\sim}{v}) = 0$. (Of course the zeros of $\underset{\sim}{k}\cdot\underset{\sim}{v} - \omega_k$ and $\underset{\sim}{k}\cdot\underset{\sim}{v} + \omega_{-k}$ are at the same respective points). In view of the picture just drawn it may not always be clear what terms the k contour will pick up while being displaced toward rapid damping. Representative pictures are fine, but we need a more systematic approach, for all pictures depend on the particular distribution function $f(\underset{\sim}{v})$ which we happen to choose. The point of view which we take now is essentially pragmatic, and applies equally well to the initial value terms which we shall calculate later.

The process is the following. We test the behavior of each integral by pushing the k contour both up and down. In one case we pass over the point where $\epsilon(\underset{\sim}{k}, \underset{\sim}{k}\cdot\underset{\sim}{v}) = 0$, in the other case we pass over the point where $\epsilon(-\underset{\sim}{k}, -\underset{\sim}{k}\cdot\underset{\sim}{v}) = 0$. (We cannot say which is which - it depends on whether $\underset{\sim}{v}$ is inside or outside the unstable volume in $\underset{\sim}{v}$ space) . The time behavior of the expressions $e^{i\underset{\sim}{k}\cdot\underset{\sim}{v}t}$ $e^{-i\omega_k t}$ or $e^{-i\underset{\sim}{k}\cdot\underset{\sim}{v}t}e^{-i\omega_k t}$ is now very simple. In the case where the k integral passes over $\epsilon(\underset{\sim}{k}, \underset{\sim}{k}\cdot\underset{\sim}{v}) = 0$, we find $\underset{\sim}{k}\cdot\underset{\sim}{v} = \omega_k$, so that $e^{i\underset{\sim}{k}\cdot\underset{\sim}{v}t}e^{-i\omega_k t} =$ constant, and $e^{-i\underset{\sim}{k}\cdot\underset{\sim}{v}t}e^{-i\omega_{-k}t} = e^{-i(\omega_k + \omega_{-k})t} = e^{2\gamma_k t}$. In the case where the k integral passes over $\epsilon(-\underset{\sim}{k}, -\underset{\sim}{k}\cdot\underset{\sim}{v}) = 0$, we find $-\underset{\sim}{k}\cdot\underset{\sim}{v} = \omega_{-k}$ so that $e^{-\underset{\sim}{k}\cdot\underset{\sim}{v}t}e^{-i\omega_k t} = e^{-i(\omega_k + \omega_{-k})t} = e^{2\gamma_k t}$, while $e^{-i\underset{\sim}{k}\cdot\underset{\sim}{v}t}e^{-i\omega_{-k}t} = e^{o}$ = constant. Thus in all cases the time behavior is either e^{o} = constant or $e^{2\gamma_k t}$. We now choose the k contour which has the more rapidly damped behavior, for the given value of $\underset{\sim}{v}$. Thus if $\gamma_k > 0$ for this value of $\underset{\sim}{v}$, we choose the $\underset{\sim}{k}$ contour leading to time behavior e^{o} = constant , while if $\gamma_k < 0$, we choose the contour leading to time behavior $e^{2\gamma_k t}$. The argument works equally well for the terms $e^{i\underset{\sim}{k}\cdot\underset{\sim}{v}t}e^{-i\omega_k t}$ and $e^{-i\underset{\sim}{k}\cdot\underset{\sim}{v}t}e^{-i\omega_{-k}t}$. Note that we do not have to consider taking residues from poles - we simply want to know which direction to push the k contour to produce damping.

The argument for throwing away contours resulting from the motion of $k\perp v$ is essentially similar. If pushing $k\perp v$ leads to a damped residue coming from a zero of ϵ , then this zero could (and should) have been picked up when we deformed $k\parallel v$. If we did not pick up the zero of ϵ with the original $k\parallel v$ contour, then we should have pushed the component of k parallel to v farther into complex values.

This may be observed from the time behavior of the exponentials. The reader may wonder if we could not pick up poles by a more or less arbitrary displacement of the vector k. We could, but we have chosen to leave the result with only one component of k complex.

In general a displaced k contour may pick up contours in more than one velocity space. Terms like $e^{i\underset{\sim}{k}\cdot\underset{\sim}{v}_1 t}\ e^{-i\omega_k t}\ \delta(\underset{\sim}{k}\cdot\underset{\sim}{v} - \omega_k)$ may be made to damp rapidly by moving the v_1 contour of integration. Again we may pick up poles in the process.

A final statement is necessary. In order to avoid confusion we adopt a convention expressing which variable(s) is(are) complex. In the delta functions we place a bar over the complex variable. Thus $\delta(\bar{\underset{\sim}{k}}\cdot\underset{\sim}{v}_1 - \omega_k)$ means that the component of k parallel to v_1 is analytically continued into complex values. Also we need a means for separating terms which appear for variables in the unstable volume (ω_k and ω_{-k} having positive imaginary parts) from terms which appear when the variables are in the stable volume (ω_k and ω_{-k} having negative imaginary parts). We shall indicate the difference by placing a plus or minus superscript on ω when it appears in a delta function. Thus $\delta(\underset{\sim}{k}\cdot\bar{\underset{\sim}{v}}_2 + \omega_{-k}^{\ +})$ means that v_2 is a complex vector, and ω_{-k} has a positive imaginary part.

We procede now to the calculation of the terms of equation 196). We consider first terms 4 and 5 .

$$4 + 5 = \frac{e^{-i\underset{\sim}{k}\cdot\underset{\sim}{v}_1 t} e^{-i\omega_{-k} t}}{\epsilon(\underset{\sim}{k}, \underset{\sim}{k}\cdot\underset{\sim}{v}) \frac{d\epsilon}{d\omega}\Big|_{\omega_{-k}} (\omega_{-k} + \underset{\sim}{k}\cdot\underset{\sim}{v})(\underset{\sim}{k}\cdot\underset{\sim}{v}_1 - \underset{\sim}{k}\cdot\underset{\sim}{v} + i\rho)} + \frac{e^{-i\underset{\sim}{k}\cdot\underset{\sim}{v}_1 t} e^{-i\omega_k t}}{\epsilon(-\underset{\sim}{k}, -\underset{\sim}{k}\cdot\underset{\sim}{v}_1)\frac{d\epsilon}{d\omega}\Big|_{\omega_k} (\omega_k - \underset{\sim}{k}\cdot\underset{\sim}{v})(\omega_k - \underset{\sim}{k}\cdot\underset{\sim}{v}_1)} . \tag{V-32}$$

Both terms are to be integrated over $\underset{\sim}{k}$ and $\underset{\sim}{v}_1$, so that we may move either contour. We choose to move the k contour as it leads to fewer terms. There are two possibilities, depending on whether $\underset{\sim}{v}_1$ is inside or outside the unstable volume of velocity space. We first consider the case of $\underset{\sim}{v}_1$ inside the unstable volume. If we push the respective k contours toward the zeros of $\epsilon(\underset{\sim}{k}, \underset{\sim}{k}\cdot\underset{\sim}{v}_1)$ and $\epsilon(-\underset{\sim}{k}, -\underset{\sim}{k}\cdot\underset{\sim}{v}_1)$ we are lead to time behavior $e^{2\gamma_k t}$ (γ_k positive since v_1 is inside the unstable volume of velocity space). If we push k in the opposite direction we come to the zeros of $\omega_{-k} + \underset{\sim}{k}\cdot\underset{\sim}{v}_1$ and $\omega_k - \underset{\sim}{k}\cdot\underset{\sim}{v}_1$ which produce time behavior $e^0 =$ constant. The latter case is more rapidly damped, and we push the k contours on until they are so rapidly damped they may be neglected. The result is given by the residues of the poles. (As noted previously we may neglect the zero at $\underset{\sim}{k}\cdot\underset{\sim}{v} = \omega_k$ because we may also push the v_1 contour to produce rapid damping).

Thus we have

$$4 + 5 = \frac{-2\pi i\,\delta(\bar{\underset{\sim}{k}}\cdot\underset{\sim}{v}_1 + \omega_{-k}^{+})}{\left.\frac{d\epsilon}{d\omega}\right|_{\omega_{-k}} \epsilon(\underset{\sim}{k},\ \underset{\sim}{k}\cdot\underset{\sim}{v}_1)(\underset{\sim}{k}\cdot\underset{\sim}{v}_1 - \underset{\sim}{k}\cdot\underset{\sim}{v} + i\rho)} -$$

$$\frac{2\pi i\,\delta(\bar{\underset{\sim}{k}}\cdot\underset{\sim}{v}_1 - \omega_k^{+})}{\left.\frac{d\epsilon}{d\omega}\right|_{\omega_k} \epsilon(-\underset{\sim}{k},\ -\underset{\sim}{k}\cdot\underset{\sim}{v}_1)(\underset{\sim}{k}\cdot\underset{\sim}{v}_1 - \underset{\sim}{k}\cdot\underset{\sim}{v} + i\rho)} \quad \text{(V-33)}$$

In the second term we have used the property of the delta function $\delta(x-a)f(x) = \delta(x-a)f(a)$, and inserted the $i\rho$ according to the prescription given earlier. We may rewrite equation V-33 using the decomposition $\frac{1}{x + i\rho} = P\frac{1}{x} - i\pi\delta(x)$ to find

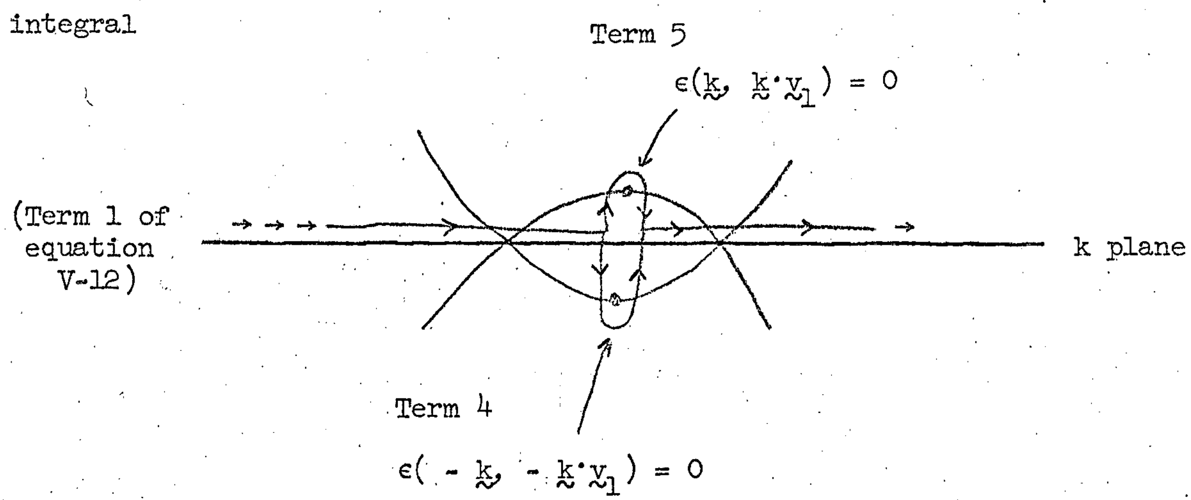
$$4 + 5 = -2\pi i \left\{ \frac{P}{\underset{\sim}{k}\cdot\underset{\sim}{v}_1 - \underset{\sim}{k}\cdot\underset{\sim}{v}} - \pi i\,\delta(\underset{\sim}{k}\cdot\underset{\sim}{v}_1 - \underset{\sim}{k}\cdot\underset{\sim}{v}) \right\} \left[\frac{\delta(\bar{\underset{\sim}{k}}\cdot\underset{\sim}{v}_1 - \omega_{-k}^{+})}{\left.\frac{d\epsilon}{d\omega}\right|_{\omega_{-k}} \epsilon(\underset{\sim}{k},\ \underset{\sim}{k}\cdot\underset{\sim}{v}_1)} + \frac{\delta(\underset{\sim}{k}\cdot\underset{\sim}{v}_1 - \omega_k^{+})}{\left.\frac{d\epsilon}{d\omega}\right|_{\omega_k} \epsilon(-\underset{\sim}{k},\ -\underset{\sim}{k}\cdot\underset{\sim}{v}_1)} \right]. \quad \text{(V-34)}$$

The contribution to the kinetic equation is given by $\int \frac{d\underset{\sim}{k}}{2\pi^3} \quad \frac{A_2}{(-i)} (4 + 5)$.

However if we write out the principal value terms we find that they lead to a contribution of the form $P (\alpha - \alpha^*)$, which is imaginary, and hence may be neglected (it must be 0). The term involving $\delta(\underset{\sim}{k}\cdot\underset{\sim}{v} - \underset{\sim}{k}\cdot\underset{\sim}{v}_1)$ leads to a real contribution to the kinetic equation given by

$$4 + 5 = - 2\pi^2\delta(\underset{\sim}{k}\cdot\underset{\sim}{v} - \underset{\sim}{k}\cdot\underset{\sim}{v}_1) \left[\frac{\delta(\underset{\sim}{k}\cdot\underset{\sim}{v}_1 + \omega_{-k}^{+})}{\left.\frac{d\epsilon}{d\omega}\right|_{\omega_{-k}} \epsilon(\underset{\sim}{k}, \underset{\sim}{k}\cdot\underset{\sim}{v}_1)} + \frac{\delta(\underset{\sim}{k}\cdot\underset{\sim}{v}_1 - \omega_k^{+})}{\left.\frac{d\epsilon}{d\omega}\right|_{\omega_k} \epsilon(-\underset{\sim}{k}, - \underset{\sim}{k}\cdot\underset{\sim}{v}_1)} \right] \tag{V-35}$$

If we compare these terms with the first term of equation V-12 we see that the three together may be represented by a single contour integral



(V-36)

where the contour is split only when v_1 is in the unstable region of velocity space.

There remains the case when v_1 is outside the unstable volume. In this case we push the k contour toward the zeros of $\epsilon(\underset{\sim}{k}, \underset{\sim}{k}\cdot\underset{\sim}{v}_1)$ and $\epsilon(-\underset{\sim}{k}, -\underset{\sim}{k}\cdot\underset{\sim}{v}_1)$, in order to produce damped residues having time behavior $e^{2\gamma_k t}$ (γ_k negative). The result is now given by

$$4 + 5 = \frac{2\pi i\,\delta(\bar{\underset{\sim}{k}}\cdot\underset{\sim}{v}_1 - \omega_k^-)\, e^{2\gamma_k t}}{\left|\frac{d\epsilon}{d\omega}(\omega_k)\right|^2 (\omega_k - \underset{\sim}{k}\cdot\underset{\sim}{v})(\omega_{-k} + \underset{\sim}{k}\cdot\underset{\sim}{v}_1)} +$$

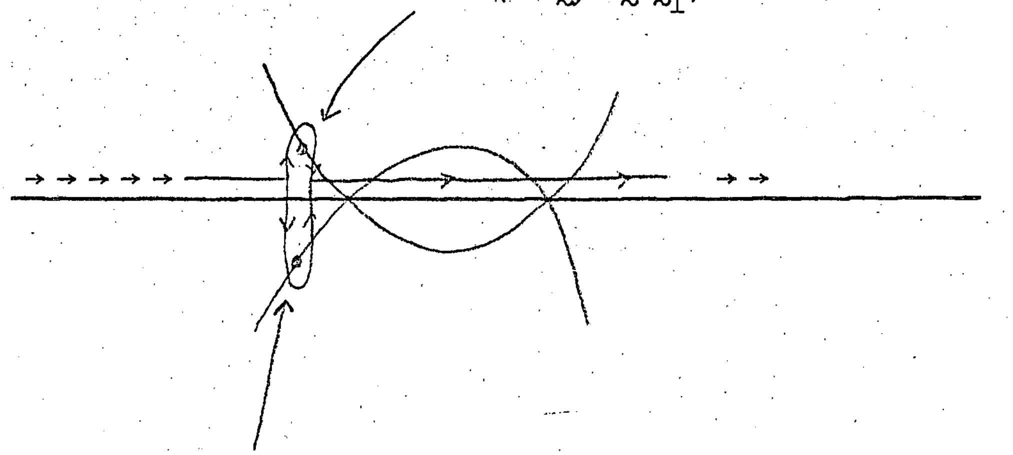
$$\frac{2\pi i\,\delta(\bar{\underset{\sim}{k}}\cdot\underset{\sim}{v}_1 + \omega_{-k}^-)\, e^{2\gamma_k t}}{\left|\frac{d\epsilon}{d\omega}(\omega_k)\right|^2 (\omega_k - \underset{\sim}{k}\cdot\underset{\sim}{v}_1)(\omega_k - \underset{\sim}{k}\cdot\underset{\sim}{v} + i\rho)} \; . \qquad \text{(V-37)}$$

Comparing these terms with the third term of equation V-12, we see that the sum may be represented by a single contour integral:

Term 5

$\epsilon(-\underset{\sim}{k}, -\underset{\sim}{k}\cdot\underset{\sim}{v}_1) = 0$

(Term 3 of Equation V-12)



Term 4

$$\epsilon(\underset{\sim}{k}, \underset{\sim}{k}\cdot\underset{\sim}{v}_1) = 0 \qquad \text{(V-38)}$$

where the contour is split only when v_1 is outside the unstable volume of v_1 space. (We have arbitrarily chosen $\epsilon(\underset{\sim}{k}, \underset{\sim}{k}\cdot\underset{\sim}{v}_1) = 0$ to the left of the unstable volume). Using the fact that $\omega_k + \omega_{-k} = 2i\gamma_k$ we may rewrite equation V-37)

$$4 + 5 = \frac{\pi e^{2\gamma_k t}}{\gamma_k \left|\frac{d\epsilon}{d\omega}(\omega_k)\right|^2} \left\{ \frac{(\bar{k}\cdot v_1 - \omega_k^{-})}{(\omega_k - \underset{\sim}{k}\cdot\underset{\sim}{v})} + \frac{(\bar{\underset{\sim}{k}}\cdot\underset{\sim}{v}_1 + \omega_{-k}^{-})}{(\omega_k - \underset{\sim}{k}\cdot\underset{\sim}{v} + i\rho)} \right\} .$$

(V-39)

We now consider terms 2 and 3 .

$$2 + 3 = \frac{e^{-i\underset{\sim}{k}\cdot\underset{\sim}{v}t} e^{i\underset{\sim}{k}\cdot\underset{\sim}{v}_1 t}}{\epsilon(\underset{\sim}{k}, \underset{\sim}{k}\cdot\underset{\sim}{v})\epsilon(-\underset{\sim}{k}, -\underset{\sim}{k}\cdot\underset{\sim}{v}_1)(\underset{\sim}{k}\cdot\underset{\sim}{v} - \underset{\sim}{k}\cdot\underset{\sim}{v}_1 - i\rho)} +$$

$$\frac{e^{-i\underset{\sim}{k}\cdot\underset{\sim}{v}t} e^{-i\omega_{-k}t}}{\left.\frac{d\epsilon}{d\omega}\right|_{\omega_{-k}} \epsilon(\underset{\sim}{k}, \underset{\sim}{k}\cdot\underset{\sim}{v})(\underset{\sim}{k}\cdot\underset{\sim}{v}_1 + \omega_{-k})(\underset{\sim}{k}\cdot\underset{\sim}{v} - \underset{\sim}{k}\cdot\underset{\sim}{v}_1 - i\rho)} . \tag{V-40}$$

We first push the $\underset{\sim}{v}_1$ contour up (into positive imaginary values) in term 2 . We obtain a non zero result only when the contour picks up a pole of $\epsilon(-\underset{\sim}{k}, -\underset{\sim}{k}\cdot\underset{\sim}{v}_1)$. (We discard the rapidly damped contour). We will pick up the pole only when ω_{-k} has a negative imaginary part, corresponding to damping. Thus

$$2 + 3 = \frac{-2\pi i\delta(\underset{\sim}{k}\cdot\bar{\underset{\sim}{v}}_1 + \omega_{-k}^{-})\, e^{-i\underset{\sim}{k}\cdot\underset{\sim}{v}t} e^{-i\omega_{-k}t}}{\left.\frac{d\epsilon}{d\omega}\right|_{\omega_{-k}} \epsilon(\underset{\sim}{k}, \underset{\sim}{k}\cdot\underset{\sim}{v})(\underset{\sim}{k}\cdot\underset{\sim}{v} + \omega_{-k})} +$$

$$\frac{e^{-i\underset{\sim}{k}\cdot\underset{\sim}{v}t} e^{-i\omega_{-k}t}}{\left.\frac{d\epsilon}{d\omega}\right|_{\omega_{-k}} \epsilon(\underset{\sim}{k}, \underset{\sim}{k}\cdot\underset{\sim}{v})(\omega_{-k} + \underset{\sim}{k}\cdot\underset{\sim}{v}_1)(\underset{\sim}{k}\cdot\underset{\sim}{v} - \underset{\sim}{k}\cdot\underset{\sim}{v}_1 - i\rho)} . \tag{V-41}$$

We now displace the k contour in each expression, obtaining a non zero result only when we pick up a pole from $\epsilon(\underset{\sim}{k}, \underset{\sim}{k}\cdot\underset{\sim}{v}) = 0$. (The zero of $\underset{\sim}{k}\cdot\underset{\sim}{v} + \omega_{-k}$ would lead to time dependence e^{0}, telling us we had pushed the contour the wrong way.) Thus

$$2 + 3 = \frac{-2\pi^2 i e^{2\gamma_k t}\,\delta(\bar{\underset{\sim}{k}}\cdot\bar{\underset{\sim}{v}}_1 + \omega_{-k}^{-})\,\delta(\bar{\underset{\sim}{k}}\cdot\underset{\sim}{v} - \omega_k^{-})}{\left|\frac{d\epsilon}{d\omega}(\omega_k)\right|^2 \gamma_k} +$$

$$\frac{2\pi i\delta(\bar{\underset{\sim}{k}}\cdot\underset{\sim}{v} - \omega_k^{-})e^{2\gamma_k t}}{\left|\frac{d\epsilon}{d\omega}(\omega_k)\right|^2 (\omega_{-k} + \underset{\sim}{k}\cdot\underset{\sim}{v}_1)(\underset{\sim}{k}\cdot\underset{\sim}{v} - \underset{\sim}{k}\cdot\underset{\sim}{v}_1 - i\rho)} . \qquad \text{(V-42)}$$

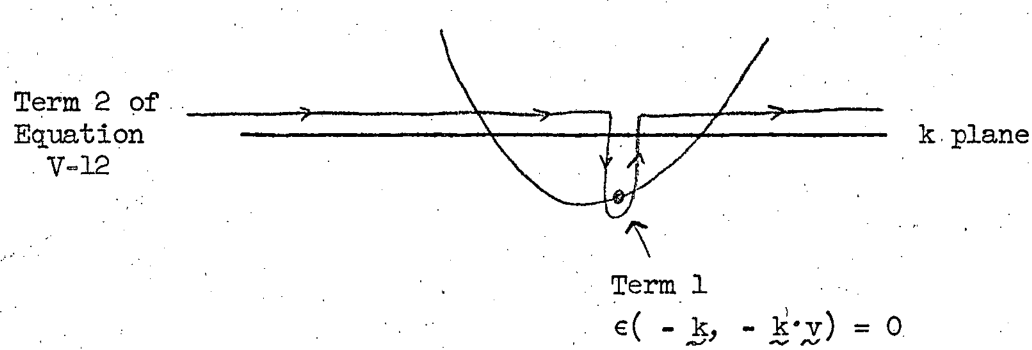
Terms 2 and 3 are related to the terms of picture V-38 by further deformations of contours, but we shall not draw a picture. Term 2 involves analytic continuation of two contours of integration, $\underset{\sim}{k}$ and $\underset{\sim}{v}_1$. There remains term 1.

$$\text{Term 1} = \frac{e^{-i\underset{\sim}{k}\cdot\underset{\sim}{v}t}\, e^{-i\omega_{-k}t}}{\left.\frac{d\epsilon}{d\omega}\right|_{\omega_{-k}} (\omega_{-k} + \underset{\sim}{k}\cdot\underset{\sim}{v})} . \qquad \text{(V-43)}$$

If we push the k contour to the point where $\epsilon(\underset{\sim}{k}, \underset{\sim}{k}\cdot\underset{\sim}{v}) = 0$, we find time dependence $e^{2\gamma_k t}$, while if we push k to the point $\epsilon(- \underset{\sim}{k}, -\underset{\sim}{k}\cdot\underset{\sim}{v}) = 0$ we find time dependence e^0 = constant. Since $\epsilon(\underset{\sim}{k}, \underset{\sim}{k}\cdot\underset{\sim}{v})$ does not appear in the expression, the result is given by

$$\text{Term 1} = -\frac{2\pi i\delta(\bar{\underset{\sim}{k}}\cdot\underset{\sim}{v} + \omega_{-k}^{+})}{\left.\frac{d\epsilon}{d\omega}\right|_{\omega_{-k}}} \qquad \text{(V-44)}$$

This term plus the second term of equation V-12 may be represented by the single contour integral:



(V-45)

This completes the calculation of the terms in equation V-13.

We evaluate now the terms of equation V-3 which we have called "initial value terms". The calculation is simply an application of the methods we have been using to a different set of terms. A few statements are necessary before we procede.

The initial value terms are different from those previously considered in that their k dependence is not explicit. This appears to be of little consequence, and we shall assume that $g(k, t = 0)$ does not have poles in the complex k plane for k approximately real $(|k_I| < k_d)$. We emphasize now that this need not be the case. A single (rather extreme) example will be sufficient. Suppose that $g(\underset{\sim}{r}_1 - \underset{\sim}{r}_2)$ contains periodic terms. Thus

$$g(\underset{\sim}{r}_1 - \underset{\sim}{r}_2) \sim \sum_i e^{i\underset{\sim}{\ell}_i \cdot (\underset{\sim}{r}_1 - \underset{\sim}{r}_2)} + g' \text{ (non periodic)}. \tag{V-46}$$

then

$$g(\underset{\sim}{k}) \sim \sum_i \delta(\underset{\sim}{k} - \underset{\sim}{\ell}_i) + g'(\underset{\sim}{k}) . \tag{V-47}$$

When we move the k contour the terms coming from the delta functions remain. In particular we find terms in the kinetic equation which are proportional to $e^{i\ell_i \cdot \underset{\sim}{v}t}\, e^{i\omega_{\ell i}t}$. Velocity derivatives of these terms lead to secular terms in the kinetic equation.

In this work we have been forced to make repeated assumptions about the initial value terms. It may not be clear why the treatment of initial values is intrinsically much more difficult than the treatment of the "f dependent" terms, so we give a brief explanation.

The adiabatic hypothesis ($\frac{df}{dt}$ is small) is in fact a very powerful one. This is not apparent in the derivation of the " f dependent" terms which usually dominate the kinetic equation. We assume $\frac{df}{dt}$ is small; the calculations bear this out; furthermore, higher order effects are usually negligible. The self consistency appears built in.

This is <u>not</u> the case when we consider initial value terms. Although we have imposed the condition that the adiabatic hypothesis should hold for $t \sim 0$, there is no general way of insuring this result. Initial values are initial values - they can be what they like. Although we wish to limit initial value terms to those which maintain the adiabatic hypothesis, we cannot do so unless we calculate (directly) their effects. The self consistency of the dependent terms is here replaced by explicit mathematical restrictions on the type of initial value terms we permit. Can these restrictions be broken, while the adiabatic hypothesis remains valid? The author doubts it. In the example chosen three-body effects (h) may be expected to have an important effect for $t \gtrsim 0$.

We now write out the "initial value terms" from equation V-3 .

$$\langle \delta f_\mu \delta E \rangle_I = \int \frac{d\omega_1}{2\pi} \int \frac{d\omega_2}{2\pi} e^{-i\omega_1 t} e^{-i\omega_2 t} \times \left[\frac{1}{-i\omega_1 + i\underset{\sim}{k}\cdot\underset{\sim}{v}} \right] \left[\frac{i\underset{\sim}{k}}{k^2} \times \right.$$

$$\left. \sum_\alpha \frac{4\pi n_\alpha q_\alpha}{\epsilon(-\underset{\sim}{k}, \omega_2)} \int \frac{d\underset{\sim}{v}_2}{(-i\omega_2 - i\underset{\sim}{k}\cdot\underset{\sim}{v}_2)} \right] g(\underset{\sim}{k}, \underset{\sim}{v}, \underset{\sim}{v}_2, t = 0) +$$

$$\int \frac{d\omega_1}{2\pi} \int \frac{d\omega_2}{2\pi} e^{-i\omega_1 t} e^{-i\omega_2 t} \left[\sum_\nu \frac{\frac{q_\mu}{m_\mu} \frac{i\underset{\sim}{k}}{k^2} \cdot \frac{df_\mu}{d\underset{\sim}{v}} 4\pi n_\nu q_\nu}{(-i\omega_1 + i\underset{\sim}{k}\cdot\underset{\sim}{v})\epsilon(\underset{\sim}{k}, \omega_1)} \int \frac{d\underset{\sim}{v}_1}{(-i\omega_1 + i\underset{\sim}{k}\cdot\underset{\sim}{v}_1)} \right] X$$

$$\left[\sum_\alpha \frac{i\underset{\sim}{k}}{k^2} \frac{4\pi n_\alpha q_\alpha}{\epsilon(-\underset{\sim}{k}, \omega_2)} \int \frac{d\underset{\sim}{v}_2}{(-i\omega_2 - i\underset{\sim}{k}\cdot\underset{\sim}{v}_2)} \right] g(\underset{\sim}{k}, \underset{\sim}{v}_1, \underset{\sim}{v}_2, t = 0) \cdot \qquad \text{(V-48)}$$

It is convenient to define operators which act on the factors containing poles in the ω planes.

$$B_1 = \sum_\alpha \frac{i\underset{\sim}{k}}{k^2} 4\pi n_\alpha q_\alpha \int d\underset{\sim}{v}_2 \qquad \text{(V-49)}$$

$$B_2 = \sum_{\alpha,\ \nu} \frac{q_\mu}{m_\mu} \frac{i\underset{\sim}{k}}{k^2} \cdot \frac{df_\mu}{d\underset{\sim}{v}} 4\pi n_\nu q_\nu \frac{i\underset{\sim}{k}}{k^2} 4\pi n_\alpha q_\alpha \int d\underset{\sim}{v}_1 \int d\underset{\sim}{v}_2 \qquad \text{(V-50)}$$

As before we perform the ω integrations by pushing the ω_1 and ω_2 contours down until they may be neglected. The result comes from the poles.

$$\langle \delta f_\mu \delta E\rangle_I = B_1 \left[\frac{e^{-i\underset{\sim}{k}\cdot\underset{\sim}{v}t}\, e^{i\underset{\sim}{k}\cdot\underset{\sim}{v}_2 t}}{\epsilon(-\underset{\sim}{k}, -\underset{\sim}{k}\cdot\underset{\sim}{v}_2)} + \frac{e^{-i\underset{\sim}{k}\cdot\underset{\sim}{v}t}\, e^{i\omega_{-k}t}}{\left.\frac{d\epsilon}{d\omega}\right|_{\omega_{-k}} (\omega_{-k} + \underset{\sim}{k}\cdot\underset{\sim}{v}_2)} \right] g(\underset{\sim}{k}, \underset{\sim}{v}, \underset{\sim}{v}_2, t=0)$$

$$+ \frac{B_2}{(-i)} \left[\frac{e^{-i\underset{\sim}{k}\cdot\underset{\sim}{v}t}\, e^{i\underset{\sim}{k}\cdot\underset{\sim}{v}_2 t}}{\epsilon(\underset{\sim}{k}, \underset{\sim}{k}\cdot\underset{\sim}{v})\epsilon(-\underset{\sim}{k}, -\underset{\sim}{k}\cdot\underset{\sim}{v}_2)(\underset{\sim}{k}\cdot\underset{\sim}{v} - \underset{\sim}{k}\cdot\underset{\sim}{v}_1)} + \right.$$

$$\frac{e^{-i\underset{\sim}{k}\cdot\underset{\sim}{v}_1 t}\, e^{i\underset{\sim}{k}\cdot\underset{\sim}{v}_2 t}}{\epsilon(\underset{\sim}{k}, \underset{\sim}{k}\cdot\underset{\sim}{v}_1)\epsilon(-\underset{\sim}{k}, -\underset{\sim}{k}\cdot\underset{\sim}{v}_2)(\underset{\sim}{k}\cdot\underset{\sim}{v}_1 - \underset{\sim}{k}\cdot\underset{\sim}{v})} +$$

$$\frac{e^{-i\underset{\sim}{k}\cdot\underset{\sim}{v}t}\, e^{-i\omega_{-k}t}}{\epsilon(\underset{\sim}{k}, \underset{\sim}{k}\cdot\underset{\sim}{v}) \left.\frac{d\epsilon}{d\omega}\right|_{\omega_{-k}} (\underset{\sim}{k}\cdot\underset{\sim}{v} - \underset{\sim}{k}\cdot\underset{\sim}{v}_1)(\omega_{-k} + \underset{\sim}{k}\cdot\underset{\sim}{v}_2)} +$$

$$\frac{e^{-i\underset{\sim}{k}\cdot\underset{\sim}{v}_1 t}\, e^{-i\omega_{-k}t}}{\epsilon(\underset{\sim}{k},\ \underset{\sim}{k}\cdot\underset{\sim}{v}_1)\frac{d\epsilon}{d\omega}\Big|_{\omega_{-k}}(\underset{\sim}{k}\cdot\underset{\sim}{v}_1 - \underset{\sim}{k}\cdot\underset{\sim}{v})(\omega_{-k} + \underset{\sim}{k}\cdot\underset{\sim}{v}_2)} +$$

$$\frac{e^{-i\omega_k t}\, e^{i\underset{\sim}{k}\cdot\underset{\sim}{v}_2 t}}{\epsilon(-\underset{\sim}{k},\ -\underset{\sim}{k}\cdot\underset{\sim}{v}_2)\frac{d\epsilon}{d\omega}\Big|_{\omega_k}(\omega_k - \underset{\sim}{k}\cdot\underset{\sim}{v})(\omega_k - \underset{\sim}{k}\cdot\underset{\sim}{v}_1)} +$$

$$\left.\frac{e^{-i\omega_k t}\, e^{-i\omega_{-k}t}}{\frac{d\epsilon}{d\omega}\Big|_{\omega_{-k}}\frac{d\epsilon}{d\omega}\Big|_{\omega_k}(\omega_k - \underset{\sim}{k}\cdot\underset{\sim}{v})(\omega_k - \underset{\sim}{k}\cdot\underset{\sim}{v}_1)(\omega_{-k} + \underset{\sim}{k}\cdot\underset{\sim}{v}_2)}\right] \times$$

$$g(\underset{\sim}{k},\ \underset{\sim}{v}_1,\ \underset{\sim}{v}_2,\ t = 0) \quad . \tag{V-51}$$

Again the result is well defined for all values of all arguments, although individual terms may be undefined at certain points. It simplifies matters to replace $\underset{\sim}{k}\cdot\underset{\sim}{v}$ by $\underset{\sim}{k}\cdot\underset{\sim}{v} + i\rho$ in all cases where an individual term is undefined. We consider ρ positive, though ρ negative leads to the same final result.

We now move contours of integration to produce damping in the terms containing $e^{i\underset{\sim}{k}\cdot\underset{\sim}{v}t}$, etc. Note that the B_1 operator contains an integration over $\underset{\sim}{v}_2$, while the B_2 operator contains integrations over $\underset{\sim}{v}_1$ and $\underset{\sim}{v}_2$. The final result is obtained by integrating all terms over $\underset{\sim}{k}$.

We consider first the terms involving the B_1 operator.

$$\langle \delta f \delta E \rangle_{B_1} \sim \frac{e^{-i\underset{\sim}{k}\cdot\underset{\sim}{v}t}\, e^{i\underset{\sim}{k}\cdot\underset{\sim}{v}_2 t}}{\epsilon(-\underset{\sim}{k}, -\underset{\sim}{k}\cdot\underset{\sim}{v}_2)} + \frac{e^{-i\underset{\sim}{k}\cdot\underset{\sim}{v}t}\, e^{-i\omega_{-k}t}}{\left.\frac{d\epsilon}{d\omega}\right|_{\omega_{-k}} (\omega_{-k} + \underset{\sim}{k}\cdot\underset{\sim}{v}_2)} \quad . \tag{V-52}$$

We push the $\underset{\sim}{v}_2$ contour up in the first term, obtaining a non zero result only when $\underset{\sim}{v}_2$ is in the stable volume

$$\langle \delta f \delta E \rangle_{B_1} \sim - \frac{2\pi i \delta(\underset{\sim}{k}\cdot\bar{\underset{\sim}{v}}_2 + \omega_{-k}^{-})\, e^{-i\underset{\sim}{k}\cdot\underset{\sim}{v}t}\, e^{-i\omega_{-k}t}}{\left.\frac{d\epsilon}{d\omega}\right|_{\omega_{-k}}} + \frac{e^{-i\underset{\sim}{k}\cdot\underset{\sim}{v}t}\, e^{-i\omega_{-k}t}}{\left.\frac{d\epsilon}{d\omega}\right|_{\omega_{-k}} (\omega_{-k} + \underset{\sim}{k}\cdot\underset{\sim}{v}_2)} \quad . \tag{V-53}$$

We then displace the k integration in the first term in order to make $\underset{\sim}{v}_2$ real ($\underset{\sim}{k}$ and $\underset{\sim}{v}_2$ are connected through the delta function), and displace $\underset{\sim}{k}$ to produce rapid damping in the second term, obtaining a non zero result only from the pole at $\underset{\sim}{k}\cdot\underset{\sim}{v}_2 + \omega_{-k} = 0$.

$$\langle \delta f \delta \underset{\sim}{E} \rangle_{B_1} \sim - \frac{2\pi i \delta(\bar{\underset{\sim}{k}}\cdot\underset{\sim}{v}_2 + \omega_{-k}^{-})\, e^{-i\underset{\sim}{k}\cdot\underset{\sim}{v}t}\, e^{-i\omega_{-k}t}}{\left.\frac{d\epsilon}{d\omega}\right|_{\omega_{-k}}} +$$

$$\frac{2\pi i \delta(\bar{\underset{\sim}{k}}\cdot\underset{\sim}{v}_2 + \omega_{-k}^{-})\, e^{-i\underset{\sim}{k}\cdot\underset{\sim}{v}t}\, e^{-i\omega_{-k}t}}{\left.\frac{d\epsilon}{d\omega}\right|_{\omega_{-k}}} . \qquad \text{(V-54)}$$

The sum of the two terms is zero, and the terms involving the B_1 operator do not contribute to the result.

We must still calculate the terms coming from the B_2 operator. Since the methods are now quite familiar we will not calculate them separately. In terms containing the factor $e^{-i\underset{\sim}{k}\cdot\underset{\sim}{v}_2 t}$ we push the $\underset{\sim}{v}_2$ contour up, obtaining a result from the poles the contour passes over. In terms containing $e^{-i\underset{\sim}{k}\cdot\underset{\sim}{v}_1 t}$ we push $\underset{\sim}{v}_1$ down. The last term is well defined as written.

$$\langle \delta f_\mu \delta \underset{\sim}{E} \rangle_I = \frac{B_2}{(-i)} \left[- \frac{2\pi i \delta(\underset{\sim}{k}\cdot\bar{\underset{\sim}{v}}_2 + \omega_{-k}^{-})\, e^{-i\underset{\sim}{k}\cdot\underset{\sim}{v}t}\, e^{-i\omega_{-k}t}}{\left.\frac{d\epsilon}{d\omega}\right|_{\omega_{-k}} \epsilon(\underset{\sim}{k}, \underset{\sim}{k}\cdot\underset{\sim}{v})(\underset{\sim}{k}\cdot\underset{\sim}{v} - \underset{\sim}{k}\cdot\underset{\sim}{v}_1 + i\rho)} \right.$$

$$- \frac{(2\pi i)^2\, e^{2\gamma_k t}\, \delta(\underset{\sim}{k}\cdot\bar{\underset{\sim}{v}}_1 - \omega_k^{-})\delta(\underset{\sim}{k}\cdot\bar{\underset{\sim}{v}}_2 + \omega_{-k}^{-})}{\left|\frac{d\epsilon}{d\omega}(\omega_k)\right|^2 (\omega_k - \underset{\sim}{k}\cdot\underset{\sim}{v})}$$

$$+ \frac{e^{-i\underset{\sim}{k}\cdot\underset{\sim}{v}t}\, e^{-i\omega_{-k}t}}{\epsilon(\underset{\sim}{k}, \underset{\sim}{k}\cdot\underset{\sim}{v}) \left.\frac{d\epsilon}{d\omega}\right|_{\omega_{-k}} (\underset{\sim}{k}\cdot\underset{\sim}{v} - \underset{\sim}{k}\cdot\underset{\sim}{v}_1 + i\rho)(\omega_{-k} + \underset{\sim}{k}\cdot\underset{\sim}{v}_2)}$$

$$+ \frac{2\pi i \delta(\underset{\sim}{k}\cdot\bar{\underset{\sim}{v}}_1 - \omega_k^{-})\; e^{2\gamma_k t}}{\left|\frac{d\epsilon}{d\omega}(\omega_k)\right|^2 (\omega_k - \underset{\sim}{k}\cdot\underset{\sim}{v})(\omega_{-k} + \underset{\sim}{k}\cdot\underset{\sim}{v}_2)}$$

$$+ \frac{2\pi i\, \delta(\underset{\sim}{k}\cdot\bar{\underset{\sim}{v}}_2 + \omega_{-k}^{-})\; e^{2\gamma_k t}}{\left|\frac{d\epsilon}{d\omega}(\omega_k)\right|^2 (\omega_k - \underset{\sim}{k}\cdot\underset{\sim}{v})(\omega_k - \underset{\sim}{k}\cdot\underset{\sim}{v}_1)}$$

$$\left. + \frac{e^{2\gamma_k t}}{\left|\frac{d\epsilon}{d\omega}(\omega_k)\right|^2 (\omega_k - \underset{\sim}{k}\cdot\underset{\sim}{v})(\omega_{-k} + \underset{\sim}{k}\cdot\underset{\sim}{v}_2)(\underset{\sim}{k}\cdot\underset{\sim}{v}_1 - \omega_k)}\right] .$$

(V-55)

In the first and third terms we displace the k contour in order to produce rapid damping. The term which persists comes from the pole at $\epsilon(\underset{\sim}{k}, \underset{\sim}{k}\cdot\underset{\sim}{v}) = 0$. Thus the final result is given by

$$\langle \delta f_\mu \delta E\rangle_I = \frac{B_2}{(-i)} \frac{e^{2\gamma_k t}}{\left|\frac{d\epsilon}{d\omega}(\omega_k)\right|^2} \left[\frac{1}{(\omega_k - \underset{\sim}{k}\cdot\underset{\sim}{v})(\underset{\sim}{k}\cdot\underset{\sim}{v}_1 - \omega_k)(\underset{\sim}{k}\cdot\underset{\sim}{v}_2 + \omega_{-k})} + \right.$$

$$\frac{2\pi i\delta(\underset{\sim}{k}\cdot\bar{\underset{\sim}{v}}_1 - \omega_k^{-})}{(\underset{\sim}{k}\cdot\underset{\sim}{v}_2 + \omega_{-k})(\omega_k - \underset{\sim}{k}\cdot\underset{\sim}{v})} + \frac{2\pi i\delta(\underset{\sim}{k}\cdot\tilde{\underset{\sim}{v}}_2 + \omega_{-k}^{-})}{(\omega_k - \underset{\sim}{k}\cdot\underset{\sim}{v})(\omega_k - \underset{\sim}{k}\cdot\underset{\sim}{v}_1)} + \frac{2\pi i\delta(\bar{\underset{\sim}{k}}\cdot\underset{\sim}{v} - \omega_k^{-})}{(\omega_{-k} + \underset{\sim}{k}\cdot\underset{\sim}{v}_2)(\omega_k - \underset{\sim}{k}\cdot\underset{\sim}{v}_1)}$$

$$- \frac{(2\pi i)^2 \delta(\underset{\sim}{k}\cdot\bar{\underset{\sim}{v}}_1 - \omega_k^-)\delta(\underset{\sim}{k}\cdot\bar{\underset{\sim}{v}}_2 + \omega_{-k}^-)}{(\omega_k - \underset{\sim}{k}\cdot\underset{\sim}{v})} - \frac{(2\pi i)^2 \delta(\bar{\underset{\sim}{k}}\cdot\underset{\sim}{v} - \omega_k^-)\delta(\bar{\underset{\sim}{k}}\cdot\bar{\underset{\sim}{v}}_2 + \omega_{-k}^-)}{(\omega_k - \underset{\sim}{k}\cdot\underset{\sim}{v}_1)}\Bigg] .$$

(V-56)

We shall not attempt to describe the result by pictures of the various complex planes. Note that all continuations into the complex plane appear in the stable volume of the respective spaces.

The analysis of equation V-3 is essentially complete except for one argument. We have not allowed for the fact that the original ω_1 and ω_2 contours can pick up a number of zeros of $\epsilon(k, \omega_1)$ and $\epsilon(-\underset{\sim}{k}, \omega_2)$. Thus where an ω contour integration produced a term $e^{-i\omega_k t}\, \mathcal{A}(\omega_k)$, the actual result is given by $\sum_i e^{-i\omega_k{}^i t}\, \mathcal{A}(\omega_k{}^i)$, where the superscript i labels the particular pole of $\epsilon(k, \omega)$. Likewise in displacing the $\underset{\sim}{k}$, $\underset{\sim}{v}_1$ and $\underset{\sim}{v}_2$ contours we may pick up various poles of ϵ . This affects our result in two ways.

1. For all terms we have calculated which involve a zero of the dielectric function, we have a sum on all roots of the dielectric function which are not too rapidly damped. We shall continue to leave this sum implicit.

2. In carrying our the preceding analysis, we also produce terms of the form $e^{-i\omega_k{}^i t}\, e^{-i\omega_{-k}{}^j t}$, with $i \neq j$. (These results generally come from the poles passed over by displaced contours; the contours are neglected as before). These terms may also be

made to damp rapidly because it is generally true that $\frac{d\gamma_k^{\,i}}{dk_I} \neq \frac{d\gamma_{-k}^{\,j}}{dk_I}$. We displace the k contour in the appropriate direction to cause $\gamma_k^{\,i} + \gamma_{-k}^{\,j}$ to become large and negative, and neglect these terms involving different roots of the dielectric function. It is possible that in certain regions of k space we may have $\frac{d\gamma_k^{\,i}}{dk_I} + \frac{d\gamma_{-k}^{\,j}}{dk_I} = 0$. (The requirement of equality leads to

$$\frac{d\Omega_k^{\,i}}{dk_R} \doteq \frac{d\Omega_{-k}^{\,j}}{dk_R} .$$

Thus plasma oscillations of wavenumber $\sim k_d\sqrt{\frac{m_i}{m_e}}$ have the same group velocity as ion waves). We neglect this possibility because it occurs only for specific values of k. Thus we simply exclude a small volume of integration from our result. The resultant error is comparable to that produced by the inadequate treatment of higher order effects.

We now state the result of all the preceding analysis. The collision term is extremely complicated in form, but it is a real function of the real variable $\underset{\sim}{v}$. The kinetic equation is given by:

$$\frac{df_\mu}{dt} = \frac{q_\mu^{\,2}}{m_\mu^{\,2}} \frac{d}{d\underset{\sim}{v}} \cdot \int \frac{d\underset{\sim}{k}\, \underset{\sim}{k}\, \underset{\sim}{k} \cdot \frac{df_\mu}{d\underset{\sim}{v}}}{k^4} \sum_\nu 2n_\nu q_\nu^{\,2} \int d\underset{\sim}{v}_1 f_\nu(\underset{\sim}{v}_1)\, \delta(\underset{\sim}{k}\cdot\underset{\sim}{v} - \underset{\sim}{k}\cdot\underset{\sim}{v}_1) \qquad \text{X}$$

$$\left[\frac{1}{|\epsilon(\underset{\sim}{k}, \underset{\sim}{k}\cdot\underset{\sim}{v}_1)|^2} - \frac{2\pi i\delta(\bar{\underset{\sim}{k}}\cdot\underset{\sim}{v}_1 - \omega_k^+)}{\left.\frac{d\epsilon}{d\omega}\right|_{\omega_k}\epsilon(-\underset{\sim}{k}, -\underset{\sim}{k}\cdot\underset{\sim}{v}_1)} - \frac{2\pi i\delta(\bar{\underset{\sim}{k}}\cdot\underset{\sim}{v}_1 + \omega_{-k}^+)}{\left.\frac{d\epsilon}{d\omega}\right|_{\omega_{-k}}\epsilon(\underset{\sim}{k}, \underset{\sim}{k}\cdot\underset{\sim}{v}_1)}\right] -$$

$$\frac{q_\mu^2}{m_\mu}\frac{d}{d\underset{\sim}{v}}\int\frac{d\underset{\sim}{k}}{2\pi^2}\frac{i\underset{\sim}{k}}{k^2}f_\mu(\underset{\sim}{v})\left[\frac{1}{\epsilon(-\underset{\sim}{k}, -\underset{\sim}{k}\cdot\underset{\sim}{v})} - \frac{2\pi i\delta(\bar{\underset{\sim}{k}}\cdot\underset{\sim}{v} + \omega_{-k}^+)}{\left.\frac{d\epsilon}{d\omega}\right|_{\omega_{-k}}}\right] +$$

$$\frac{q_\mu^2}{m_\mu^2}\frac{d}{d\underset{\sim}{v}}\cdot\int\frac{d\underset{\sim}{k}\,\underset{\sim}{k}\,\underset{\sim}{k}\cdot\frac{df_\mu}{d\underset{\sim}{v}}\,e^{2\gamma_k t}}{k^4\left|\frac{d\epsilon}{d\omega}(\omega_k)\right|^2}\sum_\nu\frac{2n_\nu q_\nu^2}{\pi}\int d\underset{\sim}{v}_1 f_\nu(\underset{\sim}{v}_1) \quad X$$

$$\left[\frac{1}{i(\underset{\sim}{k}\cdot\underset{\sim}{v} - \omega_k)\,|\underset{\sim}{k}\cdot\underset{\sim}{v}_1 - \omega_k|^2} - \frac{\pi i\delta(\bar{\underset{\sim}{k}}\cdot\underset{\sim}{v}_1 + \omega_{-k}^-)}{\gamma_k(\underset{\sim}{k}\cdot\underset{\sim}{v} - \omega_k)} - \frac{\pi i\delta(\bar{\underset{\sim}{k}}\cdot\underset{\sim}{v}_1 - \omega_k^-)}{\gamma_k(\underset{\sim}{k}\cdot\underset{\sim}{v} - \underset{\sim}{k}\cdot\underset{\sim}{v}_1 - i\rho)} -\right.$$

$$\left.\frac{2\pi\delta(\bar{\underset{\sim}{k}}\cdot\underset{\sim}{v} - \omega_k^-)}{(\underset{\sim}{k}\cdot\underset{\sim}{v}_1 + \omega_{-k})(\underset{\sim}{k}\cdot\underset{\sim}{v} - \underset{\sim}{k}\cdot\underset{\sim}{v}_1 - i\rho)} + \frac{2\pi^2\delta(\bar{\underset{\sim}{k}}\cdot\bar{\underset{\sim}{v}}_1 + \omega_{-k}^-)\delta(\bar{\underset{\sim}{k}}\cdot\underset{\sim}{v} - \omega_k^-)}{\gamma_k}\right] +$$

$$\frac{q_\mu^2}{m_\mu^2}\frac{d}{d\underset{\sim}{v}}\cdot\int\frac{d\underset{\sim}{k}\,\underset{\sim}{k}\,\underset{\sim}{k}\;\frac{df_\mu}{d\underset{\sim}{v}}\,e^{2\gamma_k t}}{k^4\left|\frac{d\epsilon}{d\omega}(\omega_k)\right|^2}\sum_{\alpha,\,\nu}\frac{2q_\nu q_\alpha n_\nu n_\alpha}{\pi}\int d\underset{\sim}{v}_1\int d\underset{\sim}{v}_2\;g(\underset{\sim}{k},\underset{\sim}{v}_1,\underset{\sim}{v}_2,t=0)$$

$$\times\left[\frac{1}{i(\omega_k-\underset{\sim}{k}\cdot\underset{\sim}{v})(\omega_k-\underset{\sim}{k}\cdot\underset{\sim}{v}_1)(\omega_{-k}+\underset{\sim}{k}\cdot\underset{\sim}{v}_2)}-\frac{2\pi\delta(\underset{\sim}{k}\cdot\bar{\underset{\sim}{v}}_1-\bar{\omega}_k)}{(\underset{\sim}{k}\cdot\underset{\sim}{v}_2+\omega_{-k})(\omega_k-\underset{\sim}{k}\cdot\underset{\sim}{v})}\right.$$

$$-\frac{2\pi\delta(\underset{\sim}{k}\cdot\underset{\sim}{v}_2+\bar{\omega}_{-k})}{(\omega_k-\underset{\sim}{k}\cdot\underset{\sim}{v})(\omega_k-\underset{\sim}{k}\cdot\underset{\sim}{v}_1)}-\frac{2\pi\delta(\bar{\underset{\sim}{k}}\cdot\underset{\sim}{v}-\bar{\omega}_k)}{(\omega_k-\underset{\sim}{k}\cdot\underset{\sim}{v}_1)(\omega_{-k}+\underset{\sim}{k}\cdot\underset{\sim}{v}_2)}+$$

$$\left.\frac{4\pi^2 i\,\delta(\underset{\sim}{k}\cdot\bar{\underset{\sim}{v}}_1-\bar{\omega}_k)\,\delta(\underset{\sim}{k}\cdot\bar{\underset{\sim}{v}}_2+\bar{\omega}_{-k})}{(\omega_k-\underset{\sim}{k}\cdot\underset{\sim}{v}-i\rho)}+\frac{4\pi^2 i\,\delta(\bar{\underset{\sim}{k}}\cdot\underset{\sim}{v}-\bar{\omega}_k)\,\delta(\bar{\underset{\sim}{k}}\cdot\bar{\underset{\sim}{v}}_2+\bar{\omega}_{-k})}{(\omega_k-\underset{\sim}{k}\cdot\underset{\sim}{v}_1+i\rho)}\right]. \qquad \text{(V-57)}$$

This equation constitutes one of the principal new results of this work. We have seen that the derivation of the proper contours of integration for the respective variables of integration is a complicated task. The author has not seen equivalent results derived from the KD (or equivalent BBGKY) set of equations. Thus we cannot compare the result to other work along this line.

By using very different techniques Balescu[46] has derived a kinetic equation for a homogeneous unstable plasma. The author is not familiar with the techniques used by Balescu, except that they are based on a formal solution of the Liouville equation, followed by an approximation scheme for picking out terms which lead to a kinetic (long time) equation. Since we cannot discuss the methods used by Balescu, we concentrate on the equation itself. We cite differences between our equation V-57 and Balescu's equations A9.18 - A9.21, in ascending order of importance.

a) In the first term of equation V-57 we find the terms

$$\frac{2\pi i\,\delta(\bar{\underset{\sim}{k}}\cdot\underset{\sim}{v}_1 - \omega_k^+)}{\frac{d\epsilon}{d\omega}\Big|_{\omega_k}\epsilon(-\underset{\sim}{k}, -\underset{\sim}{k}\cdot\underset{\sim}{v}_1)} \quad \text{and} \quad \frac{2\pi i\,\delta(\underset{\sim}{k}\cdot\underset{\sim}{v}_1 + \omega_{-k}^+)}{\frac{d\epsilon}{d\omega}\Big|_{\omega_{-k}}\epsilon(\underset{\sim}{k}, \underset{\sim}{k}\cdot\underset{\sim}{v}_1)} .$$

The equivalent term in Balescu's equation are given by

$$\frac{\pi\delta(\underset{\sim}{k}\cdot\bar{\underset{\sim}{v}}_1 - \omega_k^+)}{\gamma\left|\frac{d\epsilon}{d\omega}(\omega_k)\right|^2} \quad \text{and} \quad \frac{\pi\delta(\underset{\sim}{k}\cdot\bar{\underset{\sim}{v}}_1 + \omega_{-k}^+)}{\gamma_k\left|\frac{d\epsilon}{d\omega}(\omega_k)\right|^2} .$$

The quantitative difference is small, for by making use of the delta function we see that Balescu has replaced $\epsilon(k, \omega_k^*)$ and $\epsilon(-k, \omega_{-k}^*)$ by $\frac{d\epsilon}{d\omega}\Big|_{\omega_k}(-2i\gamma_k)$. We shall see later that this difference shows up clearly in the conservation laws, and our result is correct.

b) The second difference lies in the treatment of all terms which involve the factor $\delta(\bar{\underset{\sim}{k}}\cdot\underset{\sim}{v} - \omega_k)$ in equation V-57 . Balescu does not recognize that the $\underset{\sim}{k}$ (in Balescu's notation $\underset{\sim}{\ell}$) may be analytically continued. He is thus forced to use a contrived (and incorrect)argument. In essence he performs a "pseudo integration" over $\underset{\sim}{v}$, then analytically continues this $\underset{\sim}{v}$ integration, picking up poles but throwing away the rapidly damped contour. He then REMOVES the $\underset{\sim}{v}$ integration. This procedure is obviously incorrect, and the result shows in the fact that his equation is not an equation at all, for the independent variable $\underset{\sim}{v}$ is sometimes real, and sometimes complex.

c) Since Balescu does not analytically continue $\underset{\sim}{k}$, he cannot treat terms of the form $e^{i\omega_k{}^i t} e^{-i\omega_{-k}{}^j t}$. He therefore restricts himself to the case where only one zero of ϵ is not rapidly damped. Equation V-57 does not have this restriction, though we choose to leave the sum on the roots of ϵ implicit.

We shall see later that equation V-57 has a theoretical defect. In addition it is much too complicated to be of use for a practical calculation. Before going into these difficulties we investigate the basic properties of the equation.

B. Elementary Properties of the Equation

The equation we have derived is long and complicated in structure. It may be expected to fulfill certain basic conditions; likewise certain aspects of the equation should be investigated and compared to other work in plasma kinetic theory. We explore in some detail because the demonstrations are not always easy.

We note first the relation between our equation and that developed in Chapter II (equation II-99, the Lenard-Balescu equation). If we make the following approximations:

a) Neglect higher order effects. This eliminates the effect of "collisional" damping on the collective " wave" behavior in the plasma, and hence tends to make electron plasma oscillations less rapidly damped.

b) Restrict our attention to stable plasmas ($\gamma_k<0$ for all k).

c) Evaluate the collision term in the limit $t \to \infty$,

we recover equation II-99 . The significance of approximation c) will be considered later.

1. The Conservation Laws

We turn now to the basic laws which a valid collision term must obey. The collision term must conserve number density, momentum, and energy. Strictly speaking the result follows from a rigorous analysis, and the explicit calculation is simply a check. Our equation is sufficiently complicated to justify this check. In addition we will later obtain an approximate, but much more simple form of the kinetic equation. The individual who does not like the approximations should have confidence in the primary result.

We note at once that the collision term (the right side of equation V-57) may be split into three functionally independent parts.

1) Terms which do not have explicit time dependence.

2) Terms with time dependence $e^{2\gamma_k t}$, which depend only on f.

3) Terms with time dependence $e^{2\gamma_k t}$, which depend on the initial value of g .

Since the three parts are functionally independent, the conservation laws must hold for each part separately, as well as for the sum.

This more stringent requirement makes the analysis easier, for we may consider the three parts separately. We prove conservation of number density first, and shall write the collision term as $\frac{d}{d\underset{\sim}{v}} \cdot \underset{\sim}{J}$ where convenient.

$$\frac{d\eta_{\mu}}{dt} = n_{\mu} \int \frac{df_{\mu}}{dt} d\underset{\sim}{v} = n_{\mu} \int \frac{d}{d\underset{\sim}{v}} \cdot \underset{\sim}{J}_{\mu} \, d\underset{\sim}{v} = \int_{S} \underset{\sim}{J}_{\mu} \cdot N \, d\underset{\sim}{v} = 0 \quad . \tag{V-58}$$

The result follows because all functions of $\underset{\sim}{v}$ vanish at $|v| = \infty$. $\underset{\sim}{N}$ is a unit vector normal to the surface $\underset{\sim}{v} = \infty$.

We next demonstrate conservation of momentum. After an integration by parts we have

$$\frac{d}{dt} \sum_{\mu} n_{\mu} m_{\mu} \langle v \rangle = - \sum_{\mu} n_{\mu} m_{\mu} \int \underset{\sim}{J}_{\mu} \, d\underset{\sim}{v} \tag{V-59}$$

We consider first the part of $\underset{\sim}{J}$ which has implicit time dependence. For this part we have

$$\frac{d\underset{\sim}{P}_I}{dt} = - \sum_\mu n_\mu q_\mu^2 \int \frac{d\underset{\sim}{k}}{2\pi^2} \int \frac{d\underset{\sim}{v}\, i\underset{\sim}{k}}{k^2} f_\mu(\underset{\sim}{v}) \left[\frac{1}{\epsilon(-\underset{\sim}{k}, -\underset{\sim}{k}\cdot\underset{\sim}{v})} - \frac{2\pi i\, \delta(\bar{\underset{\sim}{k}}\cdot\underset{\sim}{v} + \omega_{-k}^{+})}{\left.\frac{d\epsilon}{d\omega}\right|_{\omega_{-k}}} \right] + \sum_\mu \frac{n_\mu q_\mu^2}{m_\mu} \int d\underset{\sim}{k} \int \frac{d\underset{\sim}{v}\, \underset{\sim}{k}\, \underset{\sim}{k}\cdot \frac{df_\mu}{d\underset{\sim}{v}}}{k^4} \sum_\nu 2n_\nu q_\nu^2 \int d\underset{\sim}{v}_1 f(\underset{\sim}{v}_1) \times$$

$$\delta(\underset{\sim}{k}\cdot\underset{\sim}{v} - \underset{\sim}{k}\cdot\underset{\sim}{v}_1) \times$$

$$\left[\frac{1}{|\epsilon(\underset{\sim}{k}, \underset{\sim}{k}\cdot\underset{\sim}{v}_1)|^2} - \frac{2\pi i\, \delta(\bar{\underset{\sim}{k}}\cdot\underset{\sim}{v}_1 - \omega_k^{+})}{\left.\frac{d\epsilon}{d\omega}\right|_{\omega_k} \epsilon(-\underset{\sim}{k}, -\underset{\sim}{k}\cdot\underset{\sim}{v}_1)} - \frac{2\pi i\, \delta(\bar{\underset{\sim}{k}}\cdot\underset{\sim}{v}_1 + \omega_{-k}^{+})}{\left.\frac{d\epsilon}{d\omega}\right|_{\omega_{-k}} \epsilon(\underset{\sim}{k}, \underset{\sim}{k}\cdot\underset{\sim}{v}_1)} \right] . \qquad \text{(V-60)}$$

Lenard considered the terms which do not involve derivatives of ϵ, and showed that they conserve both energy and momentum, i.e., the energy and momentum integrals are zero. In two of the remaining three terms we use the relation

$$\epsilon(\underset{\sim}{k}, \underset{\sim}{k}\cdot\underset{\sim}{v}_1) = 1 + \sum_\mu \frac{\omega_\mu^2}{k^2} P \int \frac{\underset{\sim}{k} \cdot \frac{df_\mu}{d\underset{\sim}{v}}\, d\underset{\sim}{v}}{\underset{\sim}{k}\cdot\underset{\sim}{v}_1 - \underset{\sim}{k}\cdot\underset{\sim}{v}} - \pi i \sum_\mu \frac{\omega_\mu^2}{k^2} \int \underset{\sim}{k}\cdot\frac{df_\mu}{d\underset{\sim}{v}} \times$$

$$\delta(\underset{\sim}{k}\cdot\underset{\sim}{v} - \underset{\sim}{k}\cdot\underset{\sim}{v}_1)\, d\underset{\sim}{v} \quad . \tag{V-61}$$

We now have

$$\frac{d\underset{\sim}{P}_I}{dt} = -\sum_{\mu} n_\mu q_\mu^{\,2} \int d\underset{\sim}{k} \int \frac{d\underset{\sim}{v}\ \underset{\sim}{k}}{\pi k^2} f_\mu(\underset{\sim}{v}) \ \frac{\delta(\bar{\underset{\sim}{k}}\cdot\underset{\sim}{v} + \omega_{-k}^{\;+})}{\left.\frac{d\epsilon}{d\omega}\right|_{\omega_{-k}}} \ +$$

$$\sum_{\nu} n_\nu q_\nu^{\,2} \int d\underset{\sim}{k} \int \frac{d\underset{\sim}{v}_1\ \underset{\sim}{k}}{\pi k^2} f_\nu(\underset{\sim}{v}_1) \left[\frac{\delta(\bar{\underset{\sim}{k}}\cdot\underset{\sim}{v}_1 - \omega_k^{\;+})}{\left.\frac{d\epsilon}{d\omega}\right|_{\omega_k} \epsilon(-\underset{\sim}{k}, -\underset{\sim}{k}\cdot\underset{\sim}{v}_1)} + \frac{\delta(\bar{\underset{\sim}{k}}\cdot\underset{\sim}{v}_1 + \omega_{-k}^{\;+})}{\left.\frac{d\epsilon}{d\omega}\right|_{\omega_{|k}} \epsilon(\underset{\sim}{k}, \underset{\sim}{k}\cdot\underset{\sim}{v}_1)} \right] X$$

$$\left[\epsilon(\underset{\sim}{k}, \underset{\sim}{k}\cdot\underset{\sim}{v}_1) - 1 - \sum_{\mu} \frac{\omega_\mu^{\,2}}{k^2} P \int \frac{\underset{\sim}{k}\cdot \frac{df_\mu}{d\underset{\sim}{v}}\, d\underset{\sim}{v}}{\underset{\sim}{k}\cdot\underset{\sim}{v}_1 - \underset{\sim}{k}\cdot\underset{\sim}{v}} \right] \quad . \tag{V-62}$$

In the first term we let $\mu \rightarrow \nu$ and $\underset{\sim}{v} \rightarrow \bar{\underset{\sim}{v}}_1$. In the second we note that $\epsilon(\underset{\sim}{k}, \underset{\sim}{k}\cdot\underset{\sim}{v}_1)\delta(\underset{\sim}{k}\cdot\underset{\sim}{v}_1 - \omega_k^{+}) = 0$. We now have

$$\frac{d\underset{\sim}{P}_I}{dt} = \sum_{\nu} n_\nu q_\nu^{\,2} \int d\underset{\sim}{k} \int \frac{d\underset{\sim}{v}_1\ \ \underset{\sim}{k}}{\pi k^2} f_\nu(\underset{\sim}{v}_1) - \left[\left\{ \frac{\delta(\bar{\underset{\sim}{k}}\cdot\underset{\sim}{v}_1 + \omega_{-k}^{\;+})}{\left.\frac{d\epsilon}{d\omega}\right|_{\omega_{-k}}} + \frac{\delta(\bar{\underset{\sim}{k}}\cdot\underset{\sim}{v}_1 + \omega_{-k}^{\;+})}{\left.\frac{d\epsilon}{d\omega}\right|_{\omega_{-k}}} \right\} - \right.$$

$$\left\{ 1 + \sum_\mu \frac{\omega_\mu^2}{k^2} P \int \frac{\underset{\sim}{k} \cdot \frac{df_\mu}{d\underset{\sim}{v}}}{\underset{\sim}{k}\cdot\underset{\sim}{v}_1 - \underset{\sim}{k}\cdot\underset{\sim}{v}} d\underset{\sim}{v} \right\} \left\{ \frac{\delta(\bar{\underset{\sim}{k}}\cdot\underset{\sim}{v}_1 - \omega_k^+)}{\left.\frac{d\epsilon}{d\omega}\right|_{\omega_k} \epsilon(-\underset{\sim}{k}, -\underset{\sim}{k}\cdot\underset{\sim}{v}_1)} + \frac{\delta(\bar{\underset{\sim}{k}}\cdot\underset{\sim}{v}_1 + \omega_{-k}^+)}{\left.\frac{d\epsilon}{d\omega}\right|_{\omega_{-k}} \epsilon(\underset{\sim}{k}, \underset{\sim}{k}\cdot\underset{\sim}{v}_1)} \right\} \Bigg] . \qquad \text{(V-63)}$$

The first two terms cancel identically. The remaining terms are of the form $A - A^*$ (from the fact that $\left.\frac{d\epsilon}{d\omega}\right|_{\omega_k} = - \left.\frac{d\epsilon}{d\omega}\right|^*_{\omega_{-k}}$), and hence are zero also. Note that the zero result (and hence momentum conservation) may not be proved if we replace $\frac{1}{\epsilon(\underset{\sim}{k}, \underset{\sim}{k}\cdot\underset{\sim}{v}_1)}$ by $\frac{1}{\left.\frac{d\epsilon}{d\omega}\right|_{\omega_k}(\underset{\sim}{k}\cdot\underset{\sim}{v}_1 - \omega_k)}$, as in Balescu's equation.

We now consider the terms of the kinetic equation which have explicit time dependence. For the terms which depend only on f we have

$$\frac{d\underset{\sim}{P}_{II}}{dt} = - \sum_\mu \frac{n_\mu q_\mu^2}{m_\mu} \int d\underset{\sim}{k} \int \frac{d\underset{\sim}{v}\, \underset{\sim}{k}\,\underset{\sim}{k} \cdot \frac{df_\mu}{d\underset{\sim}{v}}}{k^4 \left| \frac{d\epsilon}{d\omega}(\omega_k) \right|^2} e^{2\gamma_k t} \sum_\nu \frac{2 n_\nu q_\nu^2}{\pi} \int d\underset{\sim}{v}_1 f_\nu(\underset{\sim}{v}_1) \times$$

$$\left[\frac{1}{i(\underset{\sim}{k}\cdot\underset{\sim}{v} - \omega_k)|\underset{\sim}{k}\cdot\underset{\sim}{v}_1 - \omega_k|^2} - \frac{\pi i \delta(\bar{\underset{\sim}{k}}\cdot\underset{\sim}{v}_1 + \omega_{-k}^-)}{\gamma_k(\underset{\sim}{k}\cdot\underset{\sim}{v} - \omega_k)} - \frac{\pi i \delta(\bar{\underset{\sim}{k}}\cdot\underset{\sim}{v}_1 - \omega_k^-)}{\gamma_k(\underset{\sim}{k}\cdot\underset{\sim}{v} - \underset{\sim}{k}\cdot\underset{\sim}{v}_1 - i\rho)} + \right.$$

$$\frac{2\pi^2\delta(\bar{\mathbf{k}}\cdot\mathbf{v}_1 + \omega_{-k}^{-})\delta(\bar{\mathbf{k}}\cdot\mathbf{v} - \omega_k^{-})}{\gamma_k} - \frac{2\pi\delta(\bar{\mathbf{k}}\cdot\mathbf{v} - \omega_k^{-})}{(\mathbf{k}\cdot\mathbf{v}_1 + \omega_{-k})(\mathbf{k}\cdot\mathbf{v} - \mathbf{k}\cdot\mathbf{v}_1 - i\rho)}\Bigg] . \qquad \text{I}(V\text{-}64)$$

We may break the $\mathbf{k}$ integral into two parts:

1) The unstable volume of $\mathbf{k}$ space. This region is present only in the first term of equation V-64 . In all other terms the delta functions restrict $\mathbf{k}$ to the stable volume.

2) The stable volume of $\mathbf{k}$ space. This region is represented by all terms of equation V-64 .

We consider region 1) first. For ω in the upper half ω plane (corresponding to unstable values of $\mathbf{k}$), we have

$$\sum_\mu \frac{\omega_\mu^2}{k^2}\int \frac{\mathbf{k}\cdot\frac{df_\mu}{d\mathbf{v}}\,d\mathbf{v}}{\omega_k - \mathbf{k}\cdot\mathbf{v}} = \epsilon(\mathbf{k}, \omega_k) - 1 = -1 \qquad (V\text{-}65)$$

because $\epsilon(\mathbf{k}, \omega_k) \equiv 0$. When we substitute V-65 into the first term of V-64 , we find an imaginary result, which must be zero (it is odd in $\mathbf{k}$). This completes the proof for region 1 .

For region 2) we shall obtain a zero result by moving the various contours of integration. We first make substitutions similar to V-65. For ω_k in the lower half ω plane (corresponding to stable values of $\mathbf{k}$)

we have

$$\sum_\mu \frac{\omega_\mu^2}{k^2} \int \frac{\underset{\sim}{k} \cdot \frac{df_\mu}{d\underset{\sim}{v}} \, d\underset{\sim}{v}}{\omega_k - \underset{\sim}{k} \cdot \underset{\sim}{v}} = \epsilon(\underset{\sim}{k}, \omega_k) - 1 - 2\pi i \sum_\mu \frac{\omega_\mu^2}{k^2} \int \underset{\sim}{k} \, \frac{df_\mu}{d\underset{\sim}{v}} \, \delta(\underset{\sim}{k} \cdot \bar{\underset{\sim}{v}} - \omega_k^-) \, d\underset{\sim}{v}$$

$$= -1 - 2\pi i \sum_\mu \frac{\omega_\mu^2}{k^2} \int \underset{\sim}{k} \cdot \frac{df_\mu}{d\underset{\sim}{v}} \, \delta(\underset{\sim}{k} \cdot \bar{\underset{\sim}{v}} - \omega_k^-) \, d\underset{\sim}{v} \ . \qquad \text{(V-66)}$$

We note also that

$$\delta(\bar{k} \cdot v_1 - \omega_k^-) \sum_\mu \frac{\omega_\mu^2}{k^2} \int \frac{\underset{\sim}{k} \cdot \frac{df_\mu}{d\underset{\sim}{v}} \, d\underset{\sim}{v}}{\underset{\sim}{k} \cdot \underset{\sim}{v} - \underset{\sim}{k} \cdot \underset{\sim}{v}_1 - i\rho} = \delta(\bar{\underset{\sim}{k}} \cdot \underset{\sim}{v}_1 - \omega_k^-) \left[1 - \epsilon(\underset{\sim}{k}, \underset{\sim}{k} \cdot \underset{\sim}{v}_1) \right]$$

$$= \delta(\bar{\underset{\sim}{k}} \cdot \underset{\sim}{v}_1 - \omega_k^-) \left[1 - \epsilon(\underset{\sim}{k}, \omega_k) \right] = \delta(\bar{\underset{\sim}{k}} \cdot \underset{\sim}{v}_1 - \omega_k^-) \ . \qquad \text{(V-67)}$$

We substitute these results into V-65 to find

$$\frac{dP_{II}}{dt} = -\int \frac{d\underset{\sim}{k}}{2\pi^2} \; \frac{\underset{\sim}{k} \; e^{2\gamma_k t}}{k^2 \left| \frac{d\epsilon}{d\omega}(\omega_k) \right|^2} \sum_\nu n_\nu q_\nu^2 \int d\underset{\sim}{v}_1 f_\nu(\underset{\sim}{v}_1) \left[\frac{1}{i \left| \underset{\sim}{k} \cdot \underset{\sim}{v}_1 - \omega_k \right|^2} - \right.$$

$$\left. \frac{\pi i\delta(\bar{\underset{\sim}{k}}\cdot \underset{\sim}{v}_1 + \omega_{-k}^-)}{\gamma_k} - \frac{\pi i\delta(\bar{\underset{\sim}{k}}\cdot \underset{\sim}{v}_1 - \omega_k^-)}{\gamma_k} \right] + \sum_\mu \frac{n_\mu q_\mu^2}{m_\mu} \int d\underset{\sim}{k}$$

$$\int \frac{d\underset{\sim}{v} \;\; \underset{\sim}{k}\,\underset{\sim}{k}\cdot \frac{df_\mu}{d\underset{\sim}{v}}}{k^4 \left|\frac{d\epsilon}{d\omega}(\omega_k)\right|^2} e^{2\gamma_k t} \sum_\nu \frac{2n_\nu q_\nu^2}{\pi} \int dv_1 \; f_\nu(v_1) \left[\delta(\bar{\underset{\sim}{k}}\cdot \underset{\sim}{v} - \omega_k^-) \right.$$

$$\left\{ \frac{2\pi}{(\underset{\sim}{k}\cdot \underset{\sim}{v}_1 + \omega_{-k})(\omega_k - \underset{\sim}{k}\cdot \underset{\sim}{v}_1 - i\rho)} - \frac{2\pi^2\delta(\underset{\sim}{k}\cdot \underset{\sim}{v}_1 + \omega_{-k}^-)}{\gamma_k} \right\} +$$

$$\left. 2\pi i\delta(\underset{\sim}{k}\cdot \bar{\underset{\sim}{v}} - \omega_k^-) \left\{ \frac{1}{i|\underset{\sim}{k}\cdot \underset{\sim}{v}_1 - \omega_k|^2} - \frac{\pi i\delta(\bar{\underset{\sim}{k}}\cdot \underset{\sim}{v}_1 + \omega_{-k}^-)}{\gamma_k} \right\} \right] \quad \text{(V-68)}$$

The first term vanishes because it is imaginary, and hence is odd in k . (The bracketed part has the form $\left[\frac{1}{-i|A|^2} - i\mathcal{B} - i\mathcal{B}^*\right]$.

In the second term we deform the v contour to make v real in those terms involving $\delta(\underset{\sim}{k}\cdot \bar{\underset{\sim}{v}} - \omega_k^-)$. Due to the $i\rho$ prescription we do not pick up a pole. Thus the remaining terms vanish identically.

The initial value terms are essentially identical to those just considered, in terms of the methods for demonstrating the conservation laws. The momentum integral is given by

$$\frac{d\underset{\sim}{P}_{III}}{dt} = - \sum_{\mu} \frac{n_\mu q_\mu^2}{m_\mu} \int d\underset{\sim}{k} \int \frac{d\underset{\sim}{v}\ \underset{\sim}{k}\ \underset{\sim}{k}\cdot \frac{df_\mu}{d\underset{\sim}{v}}}{k^4 \left|\frac{d\epsilon}{d\omega}(\omega_k)\right|^2} e^{2\gamma_k t} \sum_{\alpha,\nu} \frac{2q_\alpha q_\nu n_\alpha n_\nu}{\pi}$$

$$\int d\underset{\sim}{v}_1 \int d\underset{\sim}{v}_2 \; g_{\alpha\nu}(\underset{\sim}{k}, \underset{\sim}{v}_1, \underset{\sim}{v}_2, t = 0) \; X \left[\frac{1}{i(\omega_k - \underset{\sim}{k}\cdot\underset{\sim}{v})(\omega_k - \underset{\sim}{k}\cdot\underset{\sim}{v}_1)(\omega_{-k} + \underset{\sim}{k}\cdot\underset{\sim}{v}_2)} \right.$$

$$- \frac{2\pi\delta(\underset{\sim}{k}\cdot\bar{\underset{\sim}{v}}_1 - \omega_k^-)}{(\omega_k - \underset{\sim}{k}\cdot\underset{\sim}{v})(\omega_{-k} + \underset{\sim}{k}\cdot\underset{\sim}{v}_2)} - \frac{2\pi\delta(\underset{\sim}{k}\cdot\bar{\underset{\sim}{v}}_2 + \omega_{-k}^-)}{(\omega_k - \underset{\sim}{k}\cdot\underset{\sim}{v})(\omega_k - \underset{\sim}{k}\cdot\underset{\sim}{v}_1)} -$$

$$\frac{2\pi\delta(\bar{\underset{\sim}{k}}\cdot\underset{\sim}{v} - \omega_k^-)}{(\omega_k - \underset{\sim}{k}\cdot\underset{\sim}{v}_1 - i\rho)(\omega_{-k} + \underset{\sim}{k}\cdot\underset{\sim}{v}_2)} + \frac{4\pi^2 i\delta(\underset{\sim}{k}\cdot\bar{\underset{\sim}{v}}_1 - \omega_k)\delta(\underset{\sim}{k}\cdot\bar{\underset{\sim}{v}}_2 + \omega_{-k}^-)}{(\omega_k - \underset{\sim}{k}\cdot\underset{\sim}{v})} +$$

$$\left. \frac{4\pi^2 i\delta(\bar{\underset{\sim}{k}}\cdot\underset{\sim}{v}_1 - \omega_k^-)\delta(\underset{\sim}{k}\cdot\bar{\underset{\sim}{v}}_2 + \omega_{-k}^-)}{(\omega_k - \underset{\sim}{k}\cdot\underset{\sim}{v}_1 - i\rho)} \right] . \qquad \text{(V-69)}$$

For $\underset{\sim}{k}$ in the unstable volume of $\underset{\sim}{k}$ space only the first term is non zero. Making use of equation V-65, we find an imaginary result, which must be zero. For $\underset{\sim}{k}$ in the stable volume of $\underset{\sim}{k}$ space we use equations V-66 and V-67 to find

$$\frac{d\underset{\sim}{P}_{III}}{dt} = - \sum_{\mu} \int \frac{d\underset{\sim}{k}\, \underset{\sim}{k}\; e^{2\gamma_k t}}{k^2 \left|\frac{d\epsilon}{d\omega}(\omega_k)\right|^2} \sum_{\alpha,\nu} \frac{q_\alpha q_\nu n_\alpha n_\nu}{\pi} \int d\underset{\sim}{v}_1 \int d\underset{\sim}{v}_2\; g_{\alpha\nu}(\underset{\sim}{k}, \underset{\sim}{v}_1, \underset{\sim}{v}_2, t = 0)$$

$$X \left[\frac{1}{i(\omega_k - \underset{\sim}{k}\cdot\underset{\sim}{v}_1)(\omega_{-k} + \underset{\sim}{k}\cdot\underset{\sim}{v}_2)} - \frac{2\pi\delta(\underset{\sim}{k}\cdot\bar{\underset{\sim}{v}}_1 - \omega_k^-)}{(\underset{\sim}{k}\cdot\underset{\sim}{v}_2 + \omega_{-k})} - \frac{2\pi\delta(\underset{\sim}{k}\cdot\bar{\underset{\sim}{v}}_2 + \omega_{-k}^-)}{(\omega_k - \underset{\sim}{k}\cdot\underset{\sim}{v}_1)} - \right.$$

$$\left. 4\pi^2 i\delta(\underset{\sim}{k}\cdot\bar{\underset{\sim}{v}}_1 - \omega_k^-)\delta(\underset{\sim}{k}\cdot\bar{\underset{\sim}{v}}_2 + \omega_{-k}^-) \right] + \sum_{\mu} \frac{n_\mu q_\mu^2}{m_\mu} \int d\underset{\sim}{k} \int d\underset{\sim}{v} \quad X$$

$$\frac{\underset{\sim}{k}\, \underset{\sim}{k}\cdot \frac{df_\mu}{d\underset{\sim}{v}}\; e^{2\gamma_k t}}{k^4 \left|\frac{d\epsilon}{d\omega}(\omega_k)\right|^2} \sum_{\alpha,\nu} \frac{2\, q_\alpha q_\nu n_\alpha n_\nu}{\pi} \int d\underset{\sim}{v}_1 \int d\underset{\sim}{v}_2\; g_{\alpha\nu}(\underset{\sim}{k}, \underset{\sim}{v}_1, \underset{\sim}{v}_2, t = 0) \quad X$$

$$\left[\frac{2\pi\delta(\bar{\underset{\sim}{k}}\cdot\underset{\sim}{v} - \omega_k^-)}{(\omega_k - \underset{\sim}{k}\cdot\underset{\sim}{v}_1 - i\rho)(\omega_{-k} + \underset{\sim}{k}\cdot\underset{\sim}{v}_2)} - \frac{2\pi\delta(\underset{\sim}{k}\cdot\bar{\underset{\sim}{v}} - \omega_k^-)}{(\omega_k - \underset{\sim}{k}\cdot\underset{\sim}{v}_1)(\omega_{-k} + \underset{\sim}{k}\cdot\underset{\sim}{v}_2)} + \right.$$

$$\frac{4\pi^2 i\delta(\underset{\sim}{k}\cdot\bar{\underset{\sim}{v}} - \omega_k^-)\delta(\underset{\sim}{k}\cdot\bar{\underset{\sim}{v}}_1 - \omega_k^-)}{(\underset{\sim}{k}\cdot\underset{\sim}{v}_2 + \omega_{-k})} - \frac{4\pi^2 i\delta(\bar{\underset{\sim}{k}}\cdot\underset{\sim}{v} - \omega_k^-)\delta(\bar{\underset{\sim}{k}}\cdot\bar{\underset{\sim}{v}}_2 + \omega_{-k}^-)}{(\omega_k - \underset{\sim}{k}\cdot\underset{\sim}{v}_1 - i\rho)} +$$

$$\frac{4\pi^2 i\delta(\underset{\sim}{k}\cdot\bar{\underset{\sim}{v}} - \omega_k^-)\delta(\underset{\sim}{k}\cdot\bar{\underset{\sim}{v}}_2 + \omega_{-k}^-)}{(\omega_k - \underset{\sim}{k}\cdot\underset{\sim}{v}_1)} + 8\pi^3\delta(\underset{\sim}{k}\cdot\bar{\underset{\sim}{v}}_1 - \omega_k^-)\delta(\underset{\sim}{k}\cdot\bar{\underset{\sim}{v}}_2 + \omega_{-k}^-) \times \delta(\underset{\sim}{k}\cdot\bar{\underset{\sim}{v}} - \omega_{-k}^-)\Bigg] . \tag{V-70}$$

We make use of the fact that

$$\sum_{\alpha,\nu} g_{\alpha\nu}(\underset{\sim}{k}, \underset{\sim}{v}_1, \underset{\sim}{v}_2) = \text{Real} . \tag{V-71}$$

to observe that the first group of terms in equation V-70 is odd in $\underset{\sim}{k}$, and therefore gives a zero result. In the remaining terms we move the $\underset{\sim}{v}$ contour in those terms containing $\delta(\underset{\sim}{k}\cdot\bar{\underset{\sim}{v}} - \omega_k^-)$, so as to make $\underset{\sim}{v}$ a real vector. The remaining terms then cancel. This completes the proof that the collision term conserves momentum.

We now turn to the demonstration that equation V-57 conserves energy. Of course the plasma contains electrostatic energy as well as kinetic energy. The appropriate conservation law is given by

$$\frac{d}{dt}\left(\sum_{\mu} \tfrac{1}{2} m_\mu n_\mu \langle v^2\rangle + \frac{\langle \delta E^2\rangle}{8\pi}\right) = 0 . \tag{V-72}$$

For the kinetic energy (T) of the plasma, we have, after an integration by parts

$$\frac{dT}{dt} = \frac{d}{dt}\left(\sum_\mu \tfrac{1}{2} m_\mu n_\mu \langle v^2 \rangle\right) = - \sum_\mu n_\mu m_\mu \int \underset{\sim}{v} \cdot \underset{\sim}{J}_\mu \, d\underset{\sim}{v} \quad . \tag{V-73}$$

The integrals we obtain from the explicit form of $\underset{\sim}{v} \cdot \underset{\sim}{J}_\mu \, d\underset{\sim}{v}$ may be reduced to those considered in the demonstration of momentum conservation. The relations are given by

$$\int \frac{\underset{\sim}{k}\cdot\underset{\sim}{v}\, \mathcal{A}(\underset{\sim}{v})\, d\underset{\sim}{v}}{\omega_k - \underset{\sim}{k}\cdot\underset{\sim}{v}} = \omega_k \int \frac{\mathcal{A}(\underset{\sim}{v})\, d\underset{\sim}{v}}{\omega_k - \underset{\sim}{k}\cdot\underset{\sim}{v}} - \int \mathcal{A}(\underset{\sim}{v})\, d\underset{\sim}{v} \quad . \tag{V-74}$$

$$\int \underset{\sim}{k}\cdot\underset{\sim}{v}\, \mathcal{B}(\underset{\sim}{v})\, \delta(\underset{\sim}{k}\cdot\bar{\underset{\sim}{v}} - x) d\underset{\sim}{v} = x \int \mathcal{B}(\underset{\sim}{v})\, \delta(\underset{\sim}{k}\cdot\bar{\underset{\sim}{v}} - x) d\underset{\sim}{v} \quad . \tag{V-75}$$

Integrals equivalent to the right side of V-74 and V-75 were evaluated previously. For this reason we will not calculate energy conservation explicitly. However we must still calculate the electrostatic energy of the plasma. We have

$$\langle \delta \underset{\sim}{E}_s^2 \rangle = \int \frac{d\underset{\sim}{k}}{(2\pi)^3} \langle \delta \underset{\sim}{E}(\underset{\sim}{k}) \delta \underset{\sim}{E}(-\underset{\sim}{k}) \rangle = \int \frac{d\underset{\sim}{k}}{(2\pi)^3} \int \frac{d\omega_1}{2\pi} \int \frac{d\omega_2}{2\pi} e^{i\omega_1 t} e^{-i\omega_2 t} \times$$

$$\left[\sum_\alpha \frac{i\underset{\sim}{k}}{k^2} \frac{4\pi n_\alpha q_\alpha}{\epsilon(\underset{\sim}{k},\omega_1)} \int \frac{d\underset{\sim}{v}_1}{(-i\omega_1 + i\underset{\sim}{k}\cdot\underset{\sim}{v}_1)}\right] \times \left[\sum_\nu \frac{i\underset{\sim}{k}}{k^2} \frac{4\pi n_\nu q_\nu}{\epsilon(-\underset{\sim}{k}, \omega_2)} \int \frac{d\underset{\sim}{v}_2}{(-i\omega_2 - i\underset{\sim}{k}\cdot\underset{\sim}{v}_2)}\right]$$

$$\left[\frac{\delta(\underset{\sim}{v}_1 - \underset{\sim}{v}_2)}{n_\alpha}\delta_{\alpha\nu}f_\alpha(\underset{\sim}{v}_1) + g_{\alpha\nu}(\underset{\sim}{k}, \underset{\sim}{v}_1, \underset{\sim}{v}_2, t = 0)\right]. \tag{V-76}$$

Since the method of calculation has been discussed fully we will not explain the steps in detail. We consider the singular term first, and invert the transforms.

$$\langle \delta \underset{\sim}{E}_s{}^2 \rangle = \int \frac{d\underset{\sim}{k}}{(2\pi)^3} \sum_\nu \frac{(4\pi q_\nu)^2 \underset{\sim}{k}\,\underset{\sim}{k}}{k^4} \int d\underset{\sim}{v}_1 f_\nu(\underset{\sim}{v}) \times \left[\frac{e^{-i\underset{\sim}{k}\cdot\underset{\sim}{v}_1 t}}{\epsilon(\underset{\sim}{k}, \underset{\sim}{k}\cdot\underset{\sim}{v}_1)} + \frac{e^{-i\omega_k t}}{\left.\frac{d\epsilon}{d\omega}\right|_{\omega_k}(\underset{\sim}{k}\cdot\underset{\sim}{v}_1 - \omega_k)} \right] \left[\frac{e^{i\underset{\sim}{k}\cdot\underset{\sim}{v}_1 t}}{\epsilon(-\underset{\sim}{k}, -\underset{\sim}{k}\cdot\underset{\sim}{v}_1)} - \frac{e^{-i\omega_{-k} t}}{\left.\frac{d\epsilon}{d\omega}\right|_{\omega_{-k}}(\underset{\sim}{k}\cdot\underset{\sim}{v}_1 + \omega_{-k})} \right]. \tag{V-77}$$

The product consists of four terms, two of which are well defined as written. We displace the k contour until it is rapidly damped, obtaining the result:

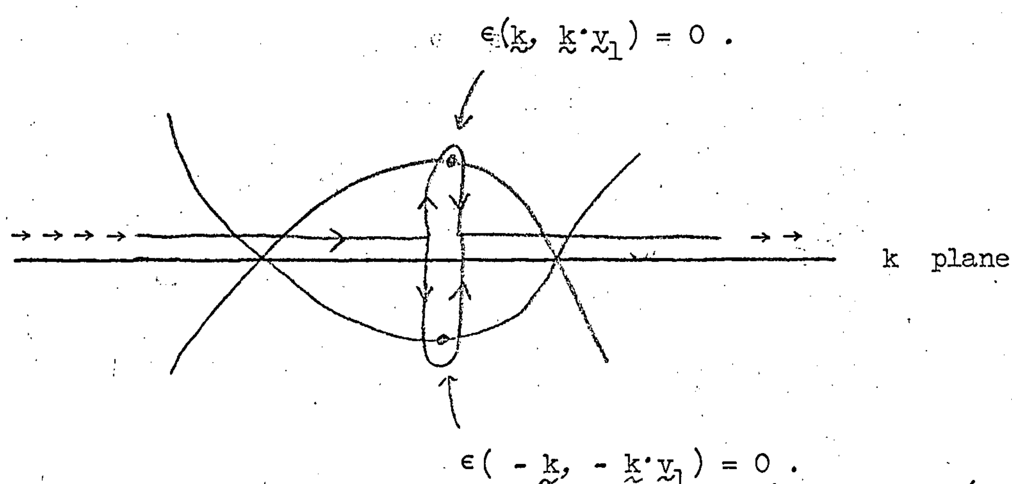
$$\langle \delta \underset{\sim}{E}_s{}^2 \rangle = \sum_\nu \frac{2n_\nu q_\nu^2}{\pi} \int \frac{d\underset{\sim}{k}\,\underset{\sim}{k}\,\underset{\sim}{k}}{k^4}\, d\underset{\sim}{v}_1 f_\nu(\underset{\sim}{v}_1) \left[\frac{1}{|\epsilon(\underset{\sim}{k}, \underset{\sim}{k}\cdot\underset{\sim}{v}_1)|^2} - \right.$$

$$\frac{2\pi i\delta(\bar{\underset{\sim}{k}}\cdot\underset{\sim}{v}_1 - \omega_k{}^+)}{\left.\frac{d\epsilon}{d\omega}\right|_{\omega_k} \epsilon(-\underset{\sim}{k}, -\underset{\sim}{k}\cdot\underset{\sim}{v}_1)} - \left.\frac{2\pi i\delta(\bar{\underset{\sim}{k}}\cdot\underset{\sim}{v}_1 + \omega_{-k}{}^+)}{\left.\frac{d\epsilon}{d\omega}\right|_{\omega_{-k}} \epsilon(\underset{\sim}{k}, \underset{\sim}{k}\cdot\underset{\sim}{v}_1)}\right] + \sum_\nu \frac{2n_\nu q_\nu{}^2}{\pi} \int d\underset{\sim}{k} \quad X$$

$$\frac{\underset{\sim}{k}\,\underset{\sim}{k}\,e^{2\gamma_k t}}{k^4\left|\frac{d\epsilon}{d\omega}(\omega_k)\right|^2} \int d\underset{\sim}{v}_1 f_\nu(\underset{\sim}{v}_1) \left[\frac{1}{|\underset{\sim}{k}\cdot\underset{\sim}{v}_1 - \omega_k|^2} + \frac{\pi}{\gamma_k}\left\{\delta(\bar{\underset{\sim}{k}}\cdot\underset{\sim}{v}_1 + \omega_{-k}{}^-) + \right.\right.$$

$$\left.\left.\delta(\bar{\underset{\sim}{k}}\cdot\underset{\sim}{v}_1 - \omega_k{}^-)\right\}\right] . \tag{V-78}$$

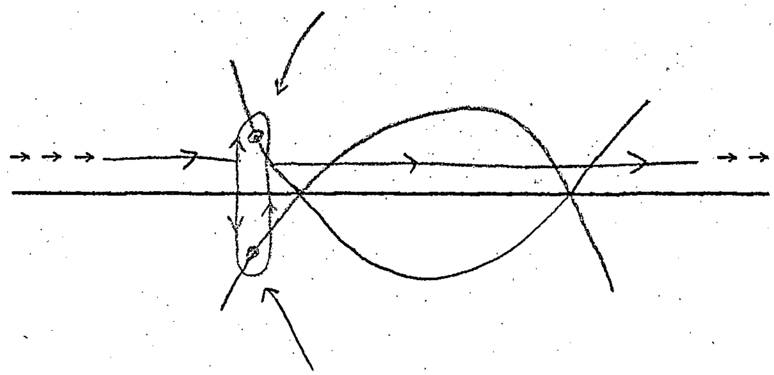
The first term may be represented by the contour integral



(V-79)

while the second may be expressed by the contour integral

$$\epsilon(-\mathbf{k}, -\mathbf{k}\cdot\mathbf{v}_1) = 0 .$$



$$\epsilon(\mathbf{k}, \mathbf{k}\cdot\mathbf{v}_1) = 0 . \tag{V-80}$$

We have arbitrarily chosen $\mathbf{v}_1$ in the unstable volume of velocity space in V-79 , and in the stable volume in V-80 .

We now calculate the $\langle \delta \mathbf{E}^2 \rangle$ coming from the initial value term. After inverting the Laplace transform we have

$$\langle \delta E_I^{\,2} \rangle = \sum_{\alpha,\nu} \frac{2n_\alpha n_\nu q_\alpha q_\nu}{\pi} \int \frac{d\mathbf{k}\,\mathbf{k}\,\mathbf{k}}{k^4} \int d\mathbf{v}_1 \int d\mathbf{v}_2\, g_{\alpha\nu}(\mathbf{k}, \mathbf{v}, \mathbf{v}, t = 0^-) \;\; X$$

$$\left[\frac{e^{-i\mathbf{k}\cdot\mathbf{v}_1 t}}{\epsilon(\mathbf{k}, \mathbf{k}\cdot\mathbf{v}_1)} + \frac{e^{-i\omega_k t}}{\left.\frac{d\epsilon}{d\omega}\right|_{\omega_k} (\mathbf{k}\cdot\mathbf{v}_1 - \omega_k)} \right] \left[\frac{e^{+i\mathbf{k}\cdot\mathbf{v}_2 t}}{\epsilon(-\mathbf{k}, -\mathbf{k}\cdot\mathbf{v}_2)} - \frac{e^{-i\omega_{-k} t}}{\left.\frac{d\epsilon}{d\omega}\right|_{\omega_{-k}} (\mathbf{k}\cdot\mathbf{v}_2 + \omega_{-k})} \right] . \tag{V-81}$$

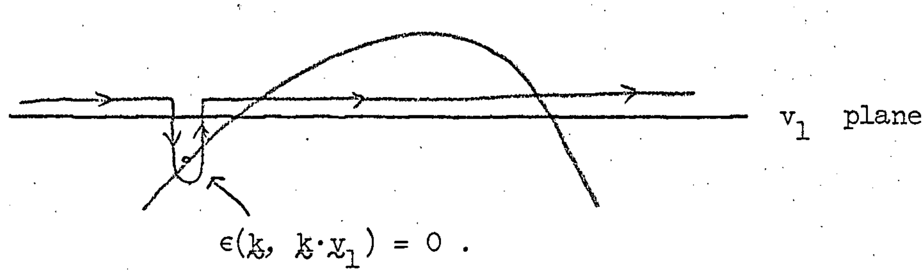
Where appropriate we deform the v_1 contour down, and the v_2 contour up, to produce the result:

$$\langle \delta E_I^{\,2} \rangle = \sum_{\alpha,\nu} \frac{2 n_\alpha n_\nu q_\alpha q_\nu}{\pi} \int \frac{d\underset{\sim}{k}\ \underset{\sim}{k}\ \underset{\sim}{k}\ e^{2\gamma_k t}}{k^4 \left| \frac{d\epsilon}{d\omega}(\omega_k) \right|^2} \int d\underset{\sim}{v}_1 \int d\underset{\sim}{v}_2\ g_{\alpha\nu}(\underset{\sim}{k}, \underset{\sim}{v}_1, \underset{\sim}{v}_2, t = 0)$$

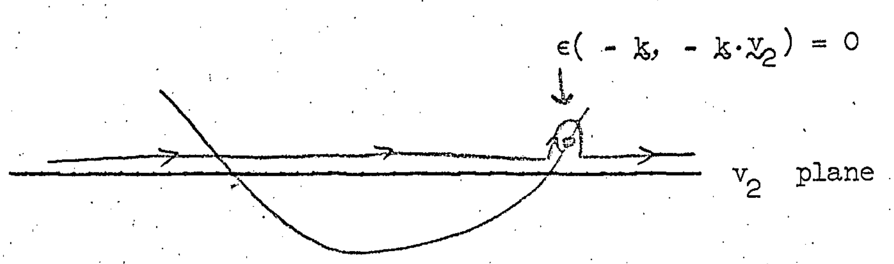
$$\times \left[\frac{1}{(\underset{\sim}{k}\cdot\underset{\sim}{v}_1 - \omega_k)(\underset{\sim}{k}\cdot\underset{\sim}{v}_2 + \omega_{-k})} + \frac{2\pi i \delta(\underset{\sim}{k}\cdot\bar{\underset{\sim}{v}}_1 - \omega_k^{\,-})}{\underset{\sim}{k}\cdot\underset{\sim}{v}_2 + \omega_{-k}} + \frac{2\pi i \delta(\underset{\sim}{k}\cdot\bar{\underset{\sim}{v}}_2 + \omega_{-k}^{\,-})}{\omega_k - \underset{\sim}{k}\cdot\underset{\sim}{v}_1} \right.$$

$$\left. + 4\pi^2 \delta(\underset{\sim}{k}\cdot\bar{\underset{\sim}{v}}_2 + \omega_{-k}^{\,-})\delta(\underset{\sim}{k}\cdot\bar{\underset{\sim}{v}}_1 - \omega_k^{\,-}) \right] . \tag{V-82}$$

These four terms may be represented by the following pictures in the v_1 and v_2 planes.



(V-83)



(V-84)

It is a straightforward but tedious calculation to demonstrate the conservation of energy for the system. Since the analytic work has been carried out we leave the verification to the interested reader.

The conservation laws are basic to the validity of a kinetic equation. We turn now to some rather subtle questions concerning the validity of equation V-57. We consider first the effect of initial conditions on the behavior of the system.

2. The Treatment of Initial Value Terms

We now treat the collision term (the sum of all terms on the right side of the kinetic equation) as an integral over $\underset{\sim}{k}$ of a sum of terms. We may arbitrarily break the $\underset{\sim}{k}$ integration into two regions:

1) A region, generally characterized by large $\underset{\sim}{k}$, and implicit time dependence, in which the initial value terms do not appear. In this region $g(\underset{\sim}{k}, \underset{\sim}{v}, \underset{\sim}{v}_1, t)$ rapidly becomes a functional of $f(\underset{\sim}{v}, t)$, and we have let this relaxation occur instantaneously. This is not an error, but a physical restriction valid for certain interesting systems (kinetic systems).

2) A region, always characterized by small $k (k \lesssim k_d)$, and by explicit time dependence $e^{2\gamma_k t}$, in which the initial value terms may have a significant effect on the system. For this region we must know the initial value term $g(t = 0)$ to insert into the kinetic equation.

The reason for the discussion is the following: region 2 appears to be counted twice. The initial value terms have time dependence arising from the zeros of the dielectric function; they represent collective effects.

But so do the time dependent terms which depend only on f. If we have obtained a correct expression for the effects of the initial value of g, then only these initial value terms should be significant for small times. We now demonstrate that this is the case.

We shall consider γ_k a small quantity, and shall not attempt to estimate the error caused by finite γ_k. (Balescu has considered the point of marginal stability $\gamma_k = 0$ in order to demonstrate continuity of the collision term for $\underset{\sim}{v}$ inside and outside an unstable volume. Of course this is essential for a valid result. Since Balescu's equation does not contain initial value terms the demonstration is simply a check and has no physical significance.) We will carry out the calculation for $\underset{\sim}{v}$ inside the unstable volume of $\underset{\sim}{v}$ space, because fewer terms appear. We state our approximations and objective precisely:

Given $t = 0$ (early time), and γ_k a positive infinitesmal and $\underset{\sim}{v}$ inside the unstable volume in velocity space (to simplify the calculation), we wish to show that the effects coming from the zero's of ϵ are given by the initial value term, while the remaining terms involving zeros of ϵ are canceled by still other terms in the kinetic equation.

We consider first the initial value term. With v inside the unstable volume all terms involving $\delta(\bar{\underset{\sim}{k}}\cdot\underset{\sim}{v} - \omega_k^-)$ drop out and we have

$$\langle \delta f_\mu \delta \underset{\sim}{E} \rangle_I = \frac{q_\mu}{m_\mu} \int \frac{d\underset{\sim}{k}\ \underset{\sim}{k}\ \underset{\sim}{k}\cdot \frac{df_\mu}{d\underset{\sim}{v}}}{k^4 \left| \frac{d\epsilon}{d\omega}(\omega_k) \right|^2 i(\omega_k - \underset{\sim}{k}\cdot\underset{\sim}{v})} \left[\sum_{\alpha,\nu} \frac{2 q_\alpha q_\nu n_\alpha n_\nu}{\pi} \int d\underset{\sim}{v}_1 \int d\underset{\sim}{v}_2 \quad X \right.$$

$$g_{\alpha\nu}(\underset{\sim}{k}, \underset{\sim}{v}_1, \underset{\sim}{v}_2, t = 0) \left\{ \frac{1}{(\omega_k - \underset{\sim}{k}\cdot\underset{\sim}{v}_1)(\omega_{-k} + \underset{\sim}{k}\cdot\underset{\sim}{v}_2)} - \frac{2\pi i\delta(\underset{\sim}{k}\cdot\bar{\underset{\sim}{v}}_1 - \omega_k^-)}{(\underset{\sim}{k}\cdot\underset{\sim}{v}_2 + \omega_{-k})} - \right.$$

$$\left. \frac{2\pi i\delta(\underset{\sim}{k}\cdot\bar{\underset{\sim}{v}}_2 + \omega_{-k}^-)}{(\omega_k - \underset{\sim}{k}\cdot\underset{\sim}{v}_1)} - 4\pi^2\delta(\underset{\sim}{k}\cdot\bar{\underset{\sim}{v}}_1 - \omega_k^-)\delta(\underset{\sim}{k}\cdot\bar{\underset{\sim}{v}}_2 + \omega_{-k}^-) \right\} \Bigg] . \tag{V-85}$$

The expression in square brackets is real, for the first and fourth terms have the form AA^*, while the second plus third may be written $B + B^*$. We indicate the bracketed term by $[\;]$. Since the collision term is real, we take the real part of V-85 .

$$\langle \delta f_\mu \delta \underset{\sim}{E} \rangle_I = - \frac{q_\mu}{m_\mu} \int \frac{d\underset{\sim}{k}\, \underset{\sim}{k}\, \underset{\sim}{k}\cdot \frac{df_\mu}{d\underset{\sim}{v}} \quad \gamma_k}{k^4 \left|\frac{d\epsilon}{d\omega}(\omega_k)\right|^2 \left|\omega_k - \underset{\sim}{k}\cdot\underset{\sim}{v}\right|^2} \Big[\quad \Big] . \tag{V-86}$$

For γ_k a positive infinitesmal we may approximate the k integration by a resonance integration.

$$\langle \delta f_\mu \delta \underset{\sim}{E} \rangle_I = - \frac{q_\mu}{m_\mu} \int \frac{d\underset{\sim}{k}\, \underset{\sim}{k}\, \underset{\sim}{k}\cdot \frac{df_\mu}{d\underset{\sim}{v}} \pi\delta(\Omega_k - \underset{\sim}{k}\cdot\underset{\sim}{v})}{k^4 \left|\frac{d\epsilon}{d\omega}(\omega_k)\right|^2 \left|\omega_k - \underset{\sim}{k}\cdot\underset{\sim}{v}\right|^2} \Big[\quad \Big] . \tag{V-87}$$

We see that the initial value term is not zero in general, even though $\gamma_k \to 0$.

We now pick out of the kinetic equation the remaining terms which have explicit time dependence, and set $t = 0$ in these terms.

$$\langle \delta f_\mu \delta \underset{\sim}{E} \rangle_T = - \frac{q_\mu}{m_\mu} \int \frac{d\underset{\sim}{k}\, \underset{\sim}{k}\, \underset{\sim}{k} \cdot \frac{df_\mu}{d\underset{\sim}{v}}}{k^4 \left| \frac{d}{d\omega}(\omega_k) \right|^2 i(\underset{\sim}{k}\cdot\underset{\sim}{v} - \omega_k)} \sum_\nu \frac{2n_\nu q_\nu^{\,2}}{\pi} \int d\underset{\sim}{v}_1 f_\nu(\underset{\sim}{v}_1) \;\; X$$

$$\left[\frac{1}{|\underset{\sim}{k}\cdot\underset{\sim}{v}_1 - \omega_k|^2} + \frac{\pi\delta(\bar{\underset{\sim}{k}}\cdot\underset{\sim}{v}_1 + \omega_{-k}^{\;-})}{\gamma_k} + \frac{\pi\delta(\bar{\underset{\sim}{k}}\cdot\underset{\sim}{v}_1 - \omega_k^{\;-})}{\gamma_k} \right] \qquad \text{(V-88)}$$

When $\underset{\sim}{v}$ is in the unstable volume of $\underset{\sim}{v}$ space the quantity $\frac{\gamma_k}{|\underset{\sim}{k}\cdot\underset{\sim}{v}_1 - \omega_k|^2}$ is resonant only for $\underset{\sim}{k}$ in the unstable volume of $\underset{\sim}{k}$ space. The terms involving $\delta(\bar{\underset{\sim}{k}}\cdot\underset{\sim}{v}_1 - \omega_k^{\;-})$ and $\delta(\bar{\underset{\sim}{k}}\cdot\underset{\sim}{v}_1 + \omega_{-k}^{\;-})$ drop out since they are non zero for k in the stable volume of k space. We perform the k integral in the resonance approximation to find

$$\langle \delta f_\mu \delta \underset{\sim}{E} \rangle_T = - \frac{q_\mu}{m_\mu} \int \frac{d\underset{\sim}{k}\, \underset{\sim}{k}\, \underset{\sim}{k} \cdot \frac{df_\mu}{d\underset{\sim}{v}}}{k^4 \left| \frac{d}{d\omega}(\omega_k) \right|^2} \sum_\nu \frac{2\pi n_\nu q_\nu^{\,2}\, \delta(\underset{\sim}{k}\cdot\underset{\sim}{v} - \Omega_k)}{\gamma_k} \;\; X$$

$$\int d\underset{\sim}{v}_1 f_\nu(\underset{\sim}{v}_1) \delta(\underset{\sim}{k}\cdot\underset{\sim}{v}_1 - \Omega_k) \; . \qquad \text{(V-89)}$$

The expression blows up as $\gamma_k \to 0$. We now show that this (divergent) term is canceled by other similar terms in the kinetic equation. These terms are given by

$$\langle \delta f_\mu \delta \underset{\sim}{E} \rangle_P = -\frac{q_\mu}{m_\mu} \int \frac{d\underset{\sim}{k}\, \underset{\sim}{k}\, \underset{\sim}{k} \cdot \frac{df_\mu}{d\underset{\sim}{v}}}{k^4} \sum_\nu 2n_\nu q_\nu^2 \int d\underset{\sim}{v}_1 f_\nu(\underset{\sim}{v}_1) \delta(\underset{\sim}{k}\cdot\underset{\sim}{v} - \underset{\sim}{k}\cdot\underset{\sim}{v}_1) \times$$

$$\left[\frac{1}{|\epsilon(\underset{\sim}{k}, \underset{\sim}{k}\cdot\underset{\sim}{v}_1)|^2} - \frac{2\pi i \delta(\bar{\underset{\sim}{k}}\cdot\underset{\sim}{v}_1 - \omega_k^{+})}{\left.\frac{d\epsilon}{d\omega}\right|_{\omega_k} \epsilon(-\underset{\sim}{k}, -\underset{\sim}{k}\cdot\underset{\sim}{v}_1)} - \frac{2\pi i \delta(\bar{\underset{\sim}{k}}\cdot\underset{\sim}{v}_1 + \omega_{-k}^{+})}{\left.\frac{d\epsilon}{d\omega}\right|_{\omega_{-k}} \epsilon(\underset{\sim}{k}, \underset{\sim}{k}\cdot\underset{\sim}{v}_1)} \right] \tag{V-90}$$

We are concerned with the resonant part of the expression.

In the resonance approximation we expand the dielectric function about a zero. Similarly for $\gamma \to 0$ we have $\delta(\bar{\underset{\sim}{k}}\cdot\underset{\sim}{v}_1 - \omega_k^{+}) \to \delta(\underset{\sim}{k}\cdot\underset{\sim}{v}_1 - \Omega_k)$, etc. We now have

$$\langle \delta f_\mu \delta \underset{\sim}{E} \rangle_P = -\frac{q_\mu}{m_\mu} \int \frac{d\underset{\sim}{k}\, \underset{\sim}{k}\, \underset{\sim}{k} \cdot \frac{df_\mu}{d\underset{\sim}{v}}}{k^4 \left|\frac{d\epsilon}{d\omega}(\omega_k)\right|^2} \sum_\nu 2n_\nu q_\nu^2 \int d\underset{\sim}{v}_1 f_\nu(\underset{\sim}{v}_1) \delta(\underset{\sim}{k}\cdot\underset{\sim}{v} - \underset{\sim}{k}\cdot\underset{\sim}{v}_1) \times$$

$$\left[\frac{1}{|\underset{\sim}{k}\cdot\underset{\sim}{v}_1 - \omega_k|^2} - \frac{\pi\delta(\underset{\sim}{k}\cdot\underset{\sim}{v}_1 - \Omega_k)}{\gamma_k} - \frac{\pi\delta(\underset{\sim}{k}\cdot\underset{\sim}{v}_1 + \Omega_{-k})}{\gamma_k} \right] . \tag{V-91}$$

We add equations V-89 and V-91 to find

$$\langle \delta f_\mu \delta \underset{\sim}{E} \rangle_T + \langle \delta f_\mu \delta \underset{\sim}{E} \rangle_P = - \frac{q_\mu}{m_\mu} \int \frac{d\underset{\sim}{k}\, \underset{\sim}{k}\, \underset{\sim}{k} \cdot \frac{df_\mu}{d\underset{\sim}{v}}}{k^4 \left| \frac{d\epsilon}{d\omega}(\omega_k) \right|^2} \sum_\nu 2n_\nu q_\nu^2 \int d\underset{\sim}{v}_1 f_\nu(\underset{\sim}{v}_1) \times$$

$$\left[\delta(\underset{\sim}{k} \cdot \underset{\sim}{v} - \underset{\sim}{k} \cdot \underset{\sim}{v}_1) \left\{ \frac{1}{|\underset{\sim}{k} \cdot \underset{\sim}{v}_1 - \omega_k|^2} - \frac{\pi \delta(\underset{\sim}{k} \cdot \underset{\sim}{v}_1 - \Omega_k)}{\gamma_k} - \frac{\pi \delta(\underset{\sim}{k} \cdot \underset{\sim}{v}_1 + \Omega_{-k})}{\gamma_k} \right\} + \right.$$

$$\left. \frac{\pi \delta(\underset{\sim}{k} \cdot \underset{\sim}{v} - \Omega_k) \delta(\underset{\sim}{k} \cdot \underset{\sim}{v}_1 - \Omega_k)}{\gamma_k} \right] . \qquad \text{(V-92)}$$

Two of the terms cancel when we write $\delta(\underset{\sim}{k} \cdot \underset{\sim}{v} - \Omega_k)\delta(\underset{\sim}{k} \cdot \underset{\sim}{v}_1 - \Omega_k) = \delta(\underset{\sim}{k} \cdot \underset{\sim}{v} - \underset{\sim}{k} \cdot \underset{\sim}{v}_1)\delta(\underset{\sim}{k} \cdot \underset{\sim}{v}_1 - \Omega_k)$. The remaining terms cancel when we perform the k resonance integral on the term $\frac{1}{(\underset{\sim}{k} \cdot \underset{\sim}{v}_1 - \Omega_k)^2 + \gamma_k^2}$. This completes the proof that for small time the explicit time dependence of the collision term is given by the initial value terms. For small times the other terms proportional to $e^{2\gamma_k t}$ contribute nothing to the evolution of the system. We may define a "small time" as a time for which $e^{2\gamma_k t} - 1 \ll 1$.

We note a second conclusion. The infinite resonance of the Lenard Balescu equation (see page) is in fact a zero for $t = 0$. For small times $e^{2\gamma_k t} - 1 = 2\gamma_k t$, and the contribution due to the terms of equation V-92 tends to grow linearly with time. Note in addition that this term grows in time even at the point $\gamma_k = 0$, while the initial value term, which is proportional to $e^{2\gamma_k t}$ does not. Thus as time goes on the initial value terms becomes less and less significant in a marginally stable system. We shall return to this point when we discuss quasi-linear theory.

3. Invariance under Translation of the Origin of Time

We turn now to the basic form of the equation. The equation seems peculiar (or at least unusual) in two respects; it depends on the initial value of the pair correlation function, and it contains explicit time dependence. Thus it is not obvious that the equation predicts the same behavior for observers who begin observing a given system at different times. The equation is not explicitly invariant under a translation of the origin of time (i.e., for different choices of the initial value of time). We carry out the demonstration that the equation is invariant under time translation, for we shall use this result in Section C.1.

Since our concern is with the time dependence of the kinetic equation, we drop the dependence on all other quantities ($\underset{\sim}{v}$, variables of integration, etc.). For generality we choose t_o as the origin of time, rather than $t = 0$. We may write the kinetic equation in the form

$$\left.\frac{df}{dt}\right|_{t_1} = \frac{d}{d\underset{\sim}{v}} \cdot \mathcal{J}(t_1) = \frac{d}{d\underset{\sim}{v}} \cdot \underset{\sim}{J}(t_1 \; ; \; t_o) \quad . \qquad \text{(V-93)}$$

Here $\mathcal{J}(t_1)$ is the collision term at the time in question, while $J(t_1\ ;\ t_o)$ indicates that the actual collision term depends on the origin of time through the initial value $g(t_o)$ and the explicit time dependence $(t_1 - t_o)$ in the term containing $e^{2\gamma_k(t_1 - t_o)}$. According to time translation invariance we should now be able to demonstrate that for any different choice of the origin of time (t_2) we obtain the same collision term.

$$\left.\frac{df}{dt}\right|_{t_1} = \frac{d}{d\underset{\sim}{v}} \cdot \underset{\sim}{\mathcal{J}}(t_1) = \frac{d}{d\underset{\sim}{v}} \cdot \underset{\sim}{J}(t_1\ ;\ t_2) \qquad \text{(V-94)}$$

A preliminary discussion is in order. The kinetic equation (V-57) was obtained by the inversion of two Laplace transforms in the limit of large positive time (see equation V-3). On the other hand the equation may be considered a valid <u>kinetic</u> (long time) equation for any time $(t_1 \lessgtr t_o)$. If we run time backward the system simply evolves away from equilibrium. This is not a surprise, for kinetic equations generally do have a preferred direction in time. We are now concerned with the mathematical structure. We write $L_2^{-1}(t_1 - t_o)$ as the (double) inverse Laplace transform (plus associated integrations, etc. which are not relevant here) which formed the basis for the derivation of the kinetic equation, while $\mathcal{L}_2^{-1}(t_1 - t_o)$ shall be the same operation evaluated with $t_1 - t_o$ considered large and positive. Of course the two are not the same, in general. According to the process used in the derivation of the kinetic equation we have

$$\mathcal{J}(t_1) = \underset{\sim}{J}(t_1 ; t_o) = \mathcal{L}_2^{-1}(t_1 - t_o)\left[f(t_o ; t_1) \frac{\delta(\underset{\sim}{v} - \underset{\sim}{v}_1)}{n} + g(t_o) \right] .$$

any t . (V-95)

f depends on t_o through its own initial value and $g(t_o)$, and on t_1 through the kinetic equation itself. We now point out that the collision term at the initial time t_o :

$$\mathcal{J}(t_o) = \underset{\sim}{J}(t_o ; t_o) = \lim_{t_1 \to t_o} \mathcal{L}_2^{-1}(t_1 - t_o) \left[f(t_o ; t_1) \frac{\delta(\underset{\sim}{v} - \underset{\sim}{v}_1)}{n} + g(t_o) \right] . \tag{V-96}$$

which we may calculate directly from V-95 by setting $t_1 = t_o$, is not equal to the expression

$$L_2^{-1}(o) \left[f(t_o ; t_o) \frac{\delta(\underset{\sim}{v} - \underset{\sim}{v}_1)}{n} + g(t_o) \right] . \tag{V-97}$$

because $L_2^{-1}(o) = 1$.

Likewise for t_1 less than t_o, $\mathcal{L}_2^{-1}(t_1 - t_o)$ is well defined and non zero, while $L_2^{-1}(t_1 - t_o)$ _is_ zero, by a basic property of the Laplace transform. We note an essential feature which will be sufficient for the demonstration of time translation invariance. By a basic property of the $\mathcal{L}_2^{-1}$ operation (equivalent to equation II-62 for the P operator) we have, for any times t_i , t_j :

$$\mathcal{L}_2^{-1}(t_i)\,\mathcal{L}_2^{-1}(t_j) = \mathcal{L}_2^{-1}(t_i + t_j) \quad . \tag{V-98}$$

We now carry out the demonstration. By virtue of equations V-96 and V-98 we have

$$\underset{\sim}{\mathcal{J}}(t_1) = \underset{\sim}{J}(t_1 ; t_o) = \mathcal{L}_2^{-1}(t_1 - t_o)\,\mathcal{L}_2^{-1}(o)\left[f(t_o ; t_1)\frac{\delta(\underset{\sim}{v} - \underset{\sim}{v}_1)}{n} + g(t_o)\right] = \mathcal{L}_2^{-1}(t_1 - t_o)\,\underset{\sim}{\mathcal{J}}(t_o) \quad . \tag{V-99}$$

Since t_o was an arbitrarily chosen origin for time, we have for any other choice of origin t_2

$$\underset{\sim}{\mathcal{J}}(t_1) = \mathcal{L}_2^{-1}(t_1 - t_2)\,\underset{\sim}{\mathcal{J}}(t_2) \tag{V-100}$$

We must find $\underset{\sim}{\mathcal{J}}(t_2)$ in order to carry out the required proof. We simply use equation V-99 to find $\underset{\sim}{\mathcal{J}}(t_2)$.

$$\underset{\sim}{\mathcal{J}}(t_2) = \mathcal{L}_2^{-1}(t_2 - t_o)\,\underset{\sim}{\mathcal{J}}(t_o) \quad . \tag{V-101}$$

From V-100 and V-101 we have

$$\underset{\sim}{\mathcal{J}}(t_1) = \underset{\sim}{J}(t_1 ; t_2) = \mathcal{L}_2^{-1}(t_1 - t_2)\,\underset{\sim}{\mathcal{J}}(t_2) = \mathcal{L}_2^{-1}(t_1 - t_2)\,\mathcal{L}_2^{-1}(t_2 - t_o)\,\underset{\sim}{\mathcal{J}}(t_o) \quad . \tag{V-102}$$

Using V-98 we find

$$\mathcal{J}(t_1) = \underset{\sim}{J}(t_1\ ;\ t_2) = \mathcal{L}_2^{-1}(t_1 - t_o)\mathcal{J}(t_o) = \underset{\sim}{J}(t_1\ ;\ t_o) \quad . \qquad \text{(V-103)}$$

which is the required relation. The collision term at a given time $\mathcal{J}(t_1)$ does not depend on our choice for the initial value of time. The kinetic equation is invariant under translation of the origin of time.

We may now consider Balescu's result and state an important conclusion. Balescu's equation is not invariant under time translation. If we arbitrarily set the initial value term equal to zero, as does Balescu, we find the system immediately starts generating correlations, and observers who start observing the system at later times will not be able to use Balescu's equation. This error has been pointed out recently.[47] We emphasize that equation V-57 is invariant because it will simplify a calculation still to be performed.

4. The H Theorem and Techniques for Approximating the Kinetic Equation

There is a final significant question which should be answered: Does the kinetic equation drive the system toward equilibrium? Abraham[48] has investigated the equation derived by Balescu. He considers the following problem:

1) Two translating Lorentz distributions such that the system is weakly unstable to the two stream instability.

2) No initial value terms, so that Balescu's equation is applicable.

He then shows that the immediate (for times much less than the time to reach equilibrium) tendency of the system is to stabilize the unstable modes.

The result is not a very useful one. We obtain a more general conclusion by proving an H theorem. This theorem is sufficient to show that the system approaches equilibrium; it therefore includes the stabilizing tendency of the system.

We consider the time derivative of H , where

$$H = \sum_{\mu} n_{\mu} \int d\underset{\sim}{v}\, f_{\mu}(\underset{\sim}{v}) \;\ln\left[n_{\mu} f_{\mu} \right] . \tag{V-104}$$

We have

$$\begin{aligned} \frac{dH}{dt} &= \sum_{\mu} n_{\mu} \int d\underset{\sim}{v} \;\frac{df_{\mu}}{dt}\left[1 + \ell n\,(n_{\mu} f_{\mu}) \right] \\ &= \sum_{\mu} n_{\mu} \int d\underset{\sim}{v} \left[1 + \ell n(n_{\mu} f_{\mu}) \right] \frac{d}{d\underset{\sim}{v}} \cdot \underset{\sim}{J}_{\mu} \\ &= \sum_{\mu} n_{\mu} \int d\underset{\sim}{v}\; \ell n(n_{\mu} f_{\mu}) \frac{d}{d\underset{\sim}{v}} \cdot \underset{\sim}{J}_{\mu} \\ &= -\sum_{\mu} n_{\mu} \int d\underset{\sim}{v} \frac{\frac{df_{\mu}}{d\underset{\sim}{v}} \cdot \underset{\sim}{J}_{\mu}(v)}{f_{\mu}(\underset{\sim}{v})} . \end{aligned} \tag{V-105}$$

We wish to show that $\frac{dH}{dt} \leq 0$, with equality only when the system is in the equilibrium state. The demonstration is not trivial, as a significant number of possibilities must be considered. We follow the path that seems most direct.

We will consider the initial value terms first. For these terms it is convenient to define an operator R_1 .

$$R_1 = \sum_\mu \frac{n_\mu q_\mu^2}{m_\mu^2} \int d\underset{\sim}{v} \int \frac{d\underset{\sim}{k} \left| \underset{\sim}{k} \cdot \frac{df_\mu}{d\underset{\sim}{v}} \right|^2 e^{2\gamma_k t}}{f_\mu(\underset{\sim}{v})\, k^4 \left| \frac{d\epsilon}{d\omega}(\omega_k) \right|^2} \sum_{\alpha,\nu} \frac{2q_\alpha q_\nu n_\alpha n_\nu}{\pi} \int d\underset{\sim}{v}_1 \quad X$$

$$\int d\underset{\sim}{v}_2\, g_{\alpha\nu}(\underset{\sim}{k}, \underset{\sim}{v}_1, \underset{\sim}{v}_2, t = 0) \, . \tag{V-106}$$

The contribution to $\frac{dH}{dt}$ coming from initial value terms is given by

$$\frac{dH_I}{dt} = - R_1 \Bigg[\frac{1}{i(\omega_k - \underset{\sim}{k}\cdot\underset{\sim}{v})(\omega_k - \underset{\sim}{k}\cdot\underset{\sim}{v}_1)(\omega_{-k} + \underset{\sim}{k}\cdot\underset{\sim}{v}_2)} -$$

$$\frac{2\pi\delta(\underset{\sim}{k}\cdot\bar{\underset{\sim}{v}}_1 - \omega_k^-)}{(\underset{\sim}{k}\cdot\underset{\sim}{v}_2 + \omega_{-k})(\omega_k - \underset{\sim}{k}\cdot\underset{\sim}{v})} - \frac{2\pi\delta(\underset{\sim}{k}\cdot\bar{\underset{\sim}{v}}_2 + \omega_{-k}^-)}{(\omega_k - \underset{\sim}{k}\cdot\underset{\sim}{v})(\omega_k - \underset{\sim}{k}\cdot\underset{\sim}{v}_1)} + \frac{2\pi\delta(\bar{\underset{\sim}{k}}\cdot\underset{\sim}{v} - \omega_k^-)}{(\omega_k - \underset{\sim}{k}\cdot\underset{\sim}{v}_1 - i\rho)(\omega_{-k} + \underset{\sim}{k}\cdot\underset{\sim}{v}_2)}$$

$$+ \frac{4\pi^2 i\delta(\underset{\sim}{k}\cdot\bar{\underset{\sim}{v}}_1 - \omega_k^-)\delta(\underset{\sim}{k}\cdot\bar{\underset{\sim}{v}}_2 + \omega_{-k})}{(\omega_k - \underset{\sim}{k}\cdot\underset{\sim}{v})} + \frac{4\pi^2 i\delta(\bar{\underset{\sim}{k}}\cdot\underset{\sim}{v} - \omega_k^-)\delta(\underset{\sim}{k}\cdot\bar{\underset{\sim}{v}}_2 + \omega_{-k}^-)}{\omega_k - \underset{\sim}{k}\cdot\underset{\sim}{v}_1 - i\rho} \Bigg] \, . \tag{V-107}$$

When k is in the unstable volume of k space only the first term in brackets appears. We take the real part to find

$$\frac{dH_I}{dt} = - R_1 \frac{\gamma_k}{|\omega_k - \underset{\sim}{k}\cdot\underset{\sim}{v}|^2 (\underset{\sim}{k}\cdot\underset{\sim}{v}_1 - \omega_k)(\underset{\sim}{k}\cdot\underset{\sim}{v}_2 + \omega_{-k})} \qquad \text{(V-108)}$$

In the unstable volume of k space γ_k is positive. The resultant expression involves a product of complex conjugate integrals, with the result $\frac{dH_I}{dt} < 0$.

When k is in the stable volume of k space all six terms of equation V-107 appear. It is now convenient to make the approximation that the poles of ϵ in the v_1 and v_2 planes are near the real variable axis. Thus we write

$$\frac{1}{i(\underset{\sim}{k}\cdot\underset{\sim}{v}_1 - \omega_k)} + 2\pi\delta(\underset{\sim}{k}\cdot\bar{\underset{\sim}{v}}_1 - \omega_k^-) \simeq \frac{1}{i(\underset{\sim}{k}\cdot\underset{\sim}{v}_1 + \omega_{-k})} = \frac{1}{i(\underset{\sim}{k}\cdot\underset{\sim}{v}_1 - \Omega_k - i|\gamma_k|)} . \qquad \text{(V-109)}$$

By using this approximation repeatedly in V-107 we find

$$\frac{dH_I}{dt} = - R_1 \left[\frac{1}{i(\underset{\sim}{k}\cdot\underset{\sim}{v} - \omega_k)(\underset{\sim}{k}\cdot\underset{\sim}{v}_1 - \Omega_k - i|\gamma_k|)(\underset{\sim}{k}\cdot\underset{\sim}{v}_2 - \Omega_k + |\gamma_k|)} - \frac{2\pi\,\delta(\bar{\underset{\sim}{k}}\cdot\underset{\sim}{v} - \omega_k^-)}{(\omega_k - \underset{\sim}{k}\cdot\underset{\sim}{v}_1 - i\rho)(\underset{\sim}{k}\cdot\underset{\sim}{v}_2 - \Omega_k + i|\gamma_k|)} \right] . \qquad \text{(V-110)}$$

We now make an equivalent approximation on the k integration of equation V-110. Here the approximation is not so good, for γ_k itself is a function of k. In computing $\frac{dH}{dt}$ we are interested in the sign of quantities. In the kinetic equation this approximation would cause some error for γ_k 0, though it would lead to simpler results. We write

$$\frac{1}{i(\underset{\sim}{k}\cdot\underset{\sim}{v} - \omega_k)} + 2\pi\delta(\bar{\underset{\sim}{k}}\cdot\underset{\sim}{v} - \omega_k^{-}) = \frac{1}{i(\underset{\sim}{k}\cdot\underset{\sim}{v} - \Omega_k - i|\gamma_k|} \quad . \tag{V-111}$$

We use this approximation on equation V-110, and take the real part of the result.

$$\frac{dH_I}{dt} = R_1 \frac{\gamma_k}{|\underset{\sim}{k}\cdot\underset{\sim}{v} - \omega_k|^2 (\underset{\sim}{k}\cdot\underset{\sim}{v}_1 - \Omega_k - i|\gamma_k|)(\underset{\sim}{k}\cdot\underset{\sim}{v}_2 - \Omega_k + i|\gamma_k|)} \quad . \tag{V-112}$$

Since γ_k is negative in the stable volume of k space, we again find $\frac{dH_I}{dt} < 0$. We note in passing that the approximations on the velocity integrations, when made on the initial value terms in the kinetic equation, lead to the following simple expression.

$$J_\mu^{\ I} \simeq \frac{q_\mu^{\ 2}}{m_\mu^{\ 2}} \int \frac{d\underset{\sim}{k}\, \underset{\sim}{k}\, \underset{\sim}{k} \cdot \frac{df_\mu}{d\underset{\sim}{v}}\, e^{2\gamma_k t}}{k^4 \left|\frac{d}{d\omega}(\omega_k)\right|^2} \left[\frac{\gamma_k}{|\omega_k - \underset{\sim}{k}\cdot\underset{\sim}{v}|^2} + 2\pi\delta(\underset{\sim}{k}\cdot\underset{\sim}{v} - \Omega_k^{-}) \right] \ X$$

$$\sum_{\alpha,\nu} \frac{2n_\alpha n_\nu q_\alpha q_\nu}{\pi} \int d\underset{\sim}{v}_1 \int d\underset{\sim}{v}_2 \frac{g_{\alpha\nu}(\underset{\sim}{k}, \underset{\sim}{v}_1, \underset{\sim}{v}_2, t=0)}{(\underset{\sim}{k}\cdot\underset{\sim}{v}_1 - \Omega_k - i|\gamma_k|)(\underset{\sim}{k}\cdot\underset{\sim}{v}_2 - \Omega_k + i|\gamma_k|)} \quad . \tag{V-113}$$

The surface in k space $\gamma_k = 0$ is defined as the limit $\gamma_k \to 0$ <u>from either side</u> ($\gamma_k < 0$ or $\gamma_k > 0$) .

We return to the proof of the H theorem, and consider all terms of the kinetic equation that involve $\underset{\sim}{k}\cdot \frac{df}{d\underset{\sim}{v}}$. We define the operator R_2.

$$R_2 = \sum_\mu \frac{n_\mu q_\mu^2}{m_\mu^2} \int d\underset{\sim}{v} \int \frac{d\underset{\sim}{k}\left(\underset{\sim}{k}\cdot \frac{df_\mu}{d\underset{\sim}{v}}\right)^2}{k^4 f_\mu(\underset{\sim}{v})} \sum_\nu \frac{2n_\nu q_\nu^2}{\pi} \int d\underset{\sim}{v}_1 f_\nu(\underset{\sim}{v}_1) \quad . \tag{V-114}$$

and consider γ_k as small, so that we may write

$$\frac{\delta(\underset{\sim}{k}\cdot\underset{\sim}{v}_1 - \omega_k^+)}{\epsilon(-\underset{\sim}{k}, -\underset{\sim}{k}\cdot\underset{\sim}{v}_1)} \simeq \frac{\delta(\underset{\sim}{k}\cdot\underset{\sim}{v}_1 - \Omega_k^+)}{\left.\frac{d\epsilon}{d\omega}\right|_{\omega_{-k}} (-2i\gamma_k)} \quad . \tag{V-115}$$

$$\frac{\delta(\underset{\sim}{k}\cdot\underset{\sim}{v}_1 + \omega_{-k}^+)}{\epsilon(\underset{\sim}{k}, \underset{\sim}{k}\cdot\underset{\sim}{v}_1)} \simeq \frac{\delta(\underset{\sim}{k}\cdot\underset{\sim}{v}_1 + \Omega_k^+)}{\left.\frac{d\epsilon}{d\omega}\right|_{\omega_{-k}} (-2i\gamma_k)} \quad . \tag{V-116}$$

The contribution to $\frac{dH}{dt}$ from the terms we now consider is given by

$$\frac{dH^{II}}{dt} = - R_2 \left[\pi\delta(\underset{\sim}{k}\cdot\underset{\sim}{v} - \underset{\sim}{k}\cdot\underset{\sim}{v}_1) \left\{ \frac{1}{|\epsilon(\underset{\sim}{k}, \underset{\sim}{k}\cdot\underset{\sim}{v}_1)|^2} - \frac{\pi\delta(\underset{\sim}{k}\cdot\underset{\sim}{v}_1 - \Omega_k^{+})}{\left|\frac{d\epsilon}{d\omega}(\omega_k)\right|^2 \gamma_k} - \right.\right.$$

$$\left. \frac{\pi\delta(\underset{\sim}{k}\cdot\underset{\sim}{v}_1 + \Omega_{-k}^{+})}{\left|\frac{d\epsilon}{d\omega}(\omega_k)\right|^2 \gamma_k} \right\} + \frac{e^{2\gamma_k t}}{\left|\frac{d\epsilon}{d\omega}(\omega_k)\right|^2} \left\{ \frac{1}{i(\underset{\sim}{k}\cdot\underset{\sim}{v} - \omega_k)|\underset{\sim}{k}\cdot\underset{\sim}{v}_1 - \omega_k|^2} - \right.$$

$$\frac{\pi i\delta(\bar{\underset{\sim}{k}}\cdot\underset{\sim}{v}_1 + \omega_{-k}^{-})}{\gamma_k(\underset{\sim}{k}\cdot\underset{\sim}{v} - \omega_k)} - \frac{\pi i\delta(\bar{\underset{\sim}{k}}\cdot\underset{\sim}{v}_1 - \omega_k^{-})}{\gamma_k(\underset{\sim}{k}\cdot\underset{\sim}{v} - \underset{\sim}{k}\cdot\underset{\sim}{v}_1 - i\rho)} - \frac{2\pi\delta(\bar{\underset{\sim}{k}}\cdot\underset{\sim}{v} - \omega_k^{-})}{(\underset{\sim}{k}\cdot\underset{\sim}{v}_1 + \omega_{-k})(\underset{\sim}{k}\cdot\underset{\sim}{v} - \underset{\sim}{k}\cdot\underset{\sim}{v}_1 - i\rho)} +$$

$$\left.\left. \frac{2\pi^2}{\gamma_k} \left[\delta(\bar{\underset{\sim}{k}}\cdot\bar{\underset{\sim}{v}}_1 + \omega_{-k}^{-})\delta(\bar{\underset{\sim}{k}}\cdot\underset{\sim}{v} - \omega_k^{-}) + \delta(\bar{\underset{\sim}{k}}\cdot\bar{\underset{\sim}{v}}_1 - \omega_k^{-})\delta(\bar{\underset{\sim}{k}}\cdot\underset{\sim}{v} - \omega_k^{-}) \right] \right\} \right] . \qquad \text{(V-117)}$$

The expression may be broken up into several parts: a non resonant part and a resonant part involving both the stable and unstable volume in k space. We consider first the unstable volume in k space. In this case only four terms appear

$$\frac{dH^{II}}{dt} = - R_2 \left[\frac{\pi\delta(\underset{\sim}{k}\cdot\underset{\sim}{v} - \underset{\sim}{k}\cdot\underset{\sim}{v}_1)}{|\epsilon(k, k\cdot v_1|^2} - \frac{\pi^2\delta(\underset{\sim}{k}\cdot\underset{\sim}{v} - \underset{\sim}{k}\cdot\underset{\sim}{v}_1)\delta(\bar{\underset{\sim}{k}}\cdot\underset{\sim}{v} - \omega_k^{+})}{\left|\frac{d\epsilon}{d\omega}(\omega_k)\right|^2 \gamma_k} - \right.$$

$$\left. \frac{\pi^2 \delta(\underset{\sim}{k}\cdot\underset{\sim}{v} - \underset{\sim}{k}\cdot\underset{\sim}{v}_1)}{\left|\frac{d\epsilon}{d\omega}(\omega_k)\right|^2 \gamma_k} + \frac{\gamma_k\, e^{2\gamma_k t}}{|\underset{\sim}{k}\cdot\underset{\sim}{v} - \Omega_k|^2 \left|\frac{d\epsilon}{d\omega}(\omega_k)\right|^2 |\underset{\sim}{k}\cdot\underset{\sim}{v}_1 - \omega_k|^2} \right] . \tag{V-118}$$

In the resonance approximation we have

$$\frac{dH^{II}}{dt} = - R_2 \left[\frac{\delta(\underset{\sim}{k}\cdot\underset{\sim}{v} - \underset{\sim}{k}\cdot\underset{\sim}{v}_1)}{\left|\frac{d\epsilon}{d\omega}(\omega_k)\right|^2} \delta(\underset{\sim}{k}\cdot\underset{\sim}{v}_1 - \Omega_k^{+}) \quad \frac{e^{2\gamma_k t} - 1}{\gamma_k} \right] . \tag{V-119}$$

The operator R_2 is positive definite, while $e^{2\gamma_k t} - 1$ is positive for k in the unstable volume of k space. Thus we find $\frac{dH^{II}}{dt} < 0$.

When k is in the stable volume of k space we use approximations V-109 and V-111 on equation V-117 to write

$$\frac{dH^{II}}{dt} = - R_2 \left[\frac{\pi\delta(\underset{\sim}{k}\cdot\underset{\sim}{v} - \underset{\sim}{k}\cdot\underset{\sim}{v}_1)}{|\epsilon(\underset{\sim}{k}, \underset{\sim}{k}\cdot\underset{\sim}{v}_1)|^2} + \frac{e^{2\gamma_k t}}{\left|\frac{d\epsilon}{d\omega}(\omega_k)\right|^2} \left\{ \frac{1}{i(\underset{\sim}{k}\cdot\underset{\sim}{v} - \Omega_k + i\gamma_k)|\underset{\sim}{k}\cdot\underset{\sim}{v}_1 - \omega_k|^2} \right. \right.$$

$$\left. \left. - \frac{\pi i\delta(\underset{\sim}{k}\cdot\underset{\sim}{v}_1 + \omega_{-k}^{-})}{\gamma_k(\underset{\sim}{k}\cdot\underset{\sim}{v} - \Omega_k + i\gamma_k)} - \frac{\pi i\delta(\underset{\sim}{k}\cdot\underset{\sim}{v}_1 - \omega_k^{-})}{\gamma_k(\underset{\sim}{k}\cdot\underset{\sim}{v} - \Omega_k + i\gamma_k)} \right\} \right] . \tag{V-120}$$

We take the real part of the expression, and use the resonance approximation to calculate

$$\frac{dH^{II}}{dt} = - R_2 \left[\frac{\delta(\underset{\approx}{k}\cdot\underset{\approx}{v} - \underset{\approx}{k}\cdot\underset{\approx}{v}_1)}{\left|\frac{d\epsilon}{d\omega}(\omega_k)\right|^2} \delta(\underset{\approx}{k}\cdot\underset{\approx}{v} - \Omega_k^{-}) \left\{ \frac{e^{2\gamma_k t} - 1}{\gamma_k} \right\} \right] . \qquad \text{(V-121)}$$

Since γ_k is negative in the stable volume of k space, we again find $\frac{dH^{II}}{dt} < 0$.

We have thus far considered only the region of k space in which γ_k is small. There is also the region in which γ_k is large. (It must, therefore, be negative). Here the only significant terms is that involving $1/|\epsilon(\underset{\sim}{k}, \underset{\sim}{k}\cdot\underset{\sim}{v}_1|^2$. Lenard has shown that this term, plus the contribution from the final part of the collision term (also involving $1/|\epsilon(\underset{\sim}{k}, \underset{\sim}{k}\cdot\underset{\sim}{v}_1)|^2$ yield $\frac{dH}{dt} \leq 0$, with equality when the system is in equilibrium.

A final proof is necessary. One term involving $1/|\epsilon|^2$ has been split into two regions in our proof. Therefore we must split the remaining term into two parts. We must still consider the small γ_k contribution from this final term. The term in question yields the following contribution to $\frac{dH}{dt}$.

$$\frac{dH^{III}}{dt} = \int \frac{d\underset{\sim}{k}}{8\pi^3}\, i \sum_\mu \frac{\omega_\mu^2}{k^2} \int d\underset{\sim}{v}\; \underset{\sim}{k}\cdot \frac{df_\mu}{d\underset{\sim}{v}} \left[\frac{1}{\epsilon(-\underset{\sim}{k}, -\underset{\sim}{k}\cdot\underset{\sim}{v})} - \frac{2\pi i \delta(\underset{\sim}{k}\cdot\underset{\sim}{v} + \omega_{-k}^{+})}{\left.\frac{d\epsilon}{d\omega}\right|_{\omega_{-k}}} \right] . \qquad \text{(V-122)}$$

Since γ_k is small we expand ϵ about a zero, and keep only the first term.

$$\frac{dH^{III}}{dt} = -\int \frac{d\underset{\sim}{k}}{8\pi^3}\, i \sum_\mu \frac{\omega_\mu^2}{k^2 \frac{d\epsilon}{d\omega}\Big|_{\omega_k}} \int d\underset{\sim}{v}\, \underset{\sim}{k} \cdot \frac{df_\mu}{d\underset{\sim}{v}} \left[\frac{1}{\underset{\sim}{k}\cdot\underset{\sim}{v} + \omega_{-k}} + 2\pi i \delta(\bar{\underset{\sim}{k}}\cdot\underset{\sim}{v} + \omega_{-k}^{+}) \right] . \tag{V-123}$$

We note the fact that $\frac{d\epsilon}{d\omega}\Big|_{\omega_{-k}}$ is real for ω_{-k} on the real axis (the resonance approximation) and take the real part of V-123 to find

$$\frac{dH^F}{dt} = \int_{\gamma_{k\sim 0}} \frac{d\underset{\sim}{k}}{8\pi^3} \frac{1}{\frac{d\epsilon}{d\omega}\Big|_{\omega_{-k}}} \left\{ \pi \sum \frac{\omega_\mu^2}{k^2} \int \underset{\sim}{k} \cdot \frac{df_\mu}{d\underset{\sim}{v}} \left[\delta(\underset{\sim}{k}\cdot\underset{\sim}{v} + \Omega_k^{+}) - \delta(\underset{\sim}{k}\cdot\underset{\sim}{v} + \Omega_k^{-}) \right] \right\} d\underset{\sim}{v} =$$

$$\int_{\gamma_{k\sim 0}} \frac{d\underset{\sim}{k}}{8\pi^3} \frac{1}{\frac{d\epsilon}{d\omega}\Big|_{\omega_{-k}}} \left| \text{Imag } \epsilon(-\underset{\sim}{k}, -\Omega_k) \right| = \int_{\gamma_{k\sim 0}} \frac{d\underset{\sim}{k}}{8\pi^3} |\gamma_k| \quad . \tag{V-124}$$

In the resonance approximation this term is zero. The fact that it is zero only in the resonance approximation is not significant. Other terms in the kinetic equation give $\frac{dH}{dt} < 0$, regardless of approximation.

5. The Simplified Kinetic Equation

The approximation methods we have used here may be applied to the kinetic equation itself to yield a collision term which has all contours of integration on the "axis" of real variables. This more tractable, but slightly less accurate equation is given by

$$\frac{df_\mu}{dt} = -\frac{q_\mu^2}{m_\mu^2}\frac{d}{d\underset{\sim}{v}}\cdot\int\frac{d\underset{\sim}{k}}{2\pi^2}\frac{\underset{\sim}{k}}{k^2}f_\mu(\underset{\sim}{v})\frac{|\text{Imag }\epsilon(\underset{\sim}{k},\ \underset{\sim}{k}\cdot\underset{\sim}{v})|}{|\epsilon(\underset{\sim}{k},\ \underset{\sim}{k}\cdot\underset{\sim}{v})|^2} + \frac{q_\mu^2}{m_\mu^2}\frac{d}{d\underset{\sim}{v}}\cdot$$

$$\int\frac{d\underset{\sim}{k}\ \underset{\sim}{k}\ \underset{\sim}{k}\cdot\frac{df_\mu}{d\underset{\sim}{v}}}{k^4}\sum_\nu\frac{2n_\nu q_\nu^2}{\pi}\int d\underset{\sim}{v}_1 f_\nu(\underset{\sim}{v}_1)\ X\left[\frac{\pi\delta(\underset{\sim}{k}\cdot\underset{\sim}{v} - \underset{\sim}{k}\cdot\underset{\sim}{v}_1)}{|\epsilon(\underset{\sim}{k},\ \underset{\sim}{k}\cdot\underset{\sim}{v}_1)|^2}\Theta(\underset{\sim}{k}\cdot\underset{\sim}{v}_1) + \right.$$

$$\left.\left\{\frac{1}{|\underset{\sim}{k}\cdot\underset{\sim}{v} - \omega_k|^2} + \frac{2\pi\delta(\underset{\sim}{k}\cdot\underset{\sim}{v} - \Omega_k^-)}{\gamma_k}\right\}\frac{e^{2\gamma_k t}\ \pi}{\left|\frac{d\epsilon}{d\omega}(\omega_k)\right|^2}\delta(\underset{\sim}{k}\cdot\underset{\sim}{v}_1 - \Omega_k)\right] +$$

$$\frac{q_\mu^2}{m_\mu^2}\frac{d}{d\underset{\sim}{v}}\cdot\int\frac{d\underset{\sim}{k}\ \underset{\sim}{k}\ \underset{\sim}{k}\cdot\frac{df_\mu}{d\underset{\sim}{v}}\ e^{2\gamma_k t}}{k^4\left|\frac{d\epsilon}{d\omega}(\omega_k)\right|^2}\left\{\frac{\gamma_k}{|\underset{\sim}{k}\cdot\underset{\sim}{v} - \omega_k|^2} + 2\pi\delta(\underset{\sim}{k}\cdot\underset{\sim}{v} - \Omega_k^-)\right\}\ X$$

$$\sum_{\alpha,\nu}\frac{2n_\alpha n_\nu q_\alpha q_\nu}{\pi}\int d\underset{\sim}{v}_1\int\frac{d\underset{\sim}{v}_2\quad g_{\alpha\nu}(\underset{\sim}{k},\ \underset{\sim}{v}_1,\ \underset{\sim}{v}_2,\ t = 0\ )}{(\Omega_k - \underset{\sim}{k}\cdot\underset{\sim}{v}_1 - i(\gamma_k|)(\Omega_k - \underset{\sim}{k}\cdot\underset{\sim}{v}_2 + i|\gamma_k|)}\ . \qquad \text{(V-125)}$$

We have defined the function

$$\Theta(\underset{\sim}{k}\cdot\underset{\sim}{v}_1) = \begin{cases} +1 & \text{Imag } \epsilon(\underset{\sim}{k}, \underset{\sim}{k}\cdot\underset{\sim}{v}_1) > 0 \quad \text{(stable)} \\ -1 & \text{Imag } \epsilon(\underset{\sim}{k}, \underset{\sim}{k}\cdot\underset{\sim}{v}_1) < 0 \quad \text{(unstable)} \end{cases} \tag{V-126}$$

The value of the integrand on the hypersurface where Imag $\epsilon(\underset{\sim}{k}, \underset{\sim}{k}\cdot\underset{\sim}{v}_1) = 0$ is given by the (continuous) limit from either side.

If we neglect the effect of higher order terms in the dielectric function, we obtain the generalization of the Lenard-Balescu equation:

$$\frac{df_\mu}{dt} = \frac{q_\mu^2}{m_\mu}\frac{d}{d\underset{\sim}{v}}\cdot\int d\underset{\sim}{k}\int d\underset{\sim}{v}_1 \frac{\underset{\sim}{k}\,\underset{\sim}{k}}{k^4}\sum_\nu \frac{2n_\nu q_\nu^2}{\pi}\left[\frac{\pi\delta(\underset{\sim}{k}\cdot\underset{\sim}{v} - \underset{\sim}{k}\cdot\underset{\sim}{v}_1)}{|\epsilon(\underset{\sim}{k}, \underset{\sim}{k}\cdot\underset{\sim}{v}_1)|^2} \times \right.$$

$$\left\{\frac{1}{m_\mu}\frac{df_\mu}{d\underset{\sim}{v}} f_\nu(\underset{\sim}{v}_1) - \frac{1}{m_\nu} f_\mu(\underset{\sim}{v})\frac{df_\nu}{d\underset{\sim}{v}_1}\right\}\Theta(\underset{\sim}{k}\cdot\underset{\sim}{v}_1) + \left\{\frac{1}{|\underset{\sim}{k}\cdot\underset{\sim}{v} - \omega_k|^2} + \right.$$

$$\left.\left.\frac{2\pi\delta(\underset{\sim}{k}\cdot\underset{\sim}{v} - \Omega_k^-)}{\gamma_k}\right\}\frac{e^{2\gamma_k t}\,\pi}{\left|\frac{d\epsilon}{d\omega}(\omega_k)\right|^2}\delta(\underset{\sim}{k}\cdot\underset{\sim}{v}_1 - \Omega_k)\right] + \frac{q_\mu^2}{m_\mu^2}\frac{d}{d\underset{\sim}{v}}\cdot$$

$$\int\frac{d\underset{\sim}{k}\,\underset{\sim}{k}\,\underset{\sim}{k}\cdot\frac{df_\mu}{d\underset{\sim}{v}}\,e^{2\gamma_k t}}{k^4\left|\frac{d\epsilon}{d\omega}(\omega_k)\right|^2}\left\{\frac{\gamma_k}{|\omega_k - \underset{\sim}{k}\cdot\underset{\sim}{v}|^2} + 2\pi\delta(\underset{\sim}{k}\cdot\underset{\sim}{v} - \Omega_k^-)\right\} \times$$

$$\sum_{\alpha,\nu} \frac{2n_\alpha n_\nu q_\alpha q_\nu}{\pi} \int d\underset{\sim}{v}_1 \int d\underset{\sim}{v}_2 \quad \frac{g_{\alpha\nu}(\underset{\sim}{k}, \underset{\sim}{v}_1, \underset{\sim}{v}_2, t = 0)}{(\Omega_k - \underset{\sim}{k}\cdot\underset{\sim}{v}_1 - i|\gamma_k|)(\Omega_k - \underset{\sim}{k}\cdot\underset{\sim}{v}_2 + i|\gamma_k|)} . \tag{V-127}$$

Equations V-126 and V-127 contain all the essential features of the original kinetic equation.

In view of its relative simplicity, equation V-127 is greatly preferable to the original kinetic equation (equation V-57), and we shall limit further discussion to the properties of equation V-127. At a cost of small approximations we have obtained a relatively tractable result. Thus we have overcome practical objection to equation V-57.

C. The Long Time ($t > t_{ad}$) Kinetic Equation

1. Derivation of the Equation

There remains a theoretical objection to equation V-127. Suppose we consider the factor $e^{2\gamma_k t}$, which occurs in the equation. For γ_k positive this factor represents an exponential growth of collective effects in the plasma. We now observe the long time behavior of this factor. ($t > t_{ad}$). By virtue of the H theorem γ_k decreases, and then becomes negative, until the collective effects (coming from the zeros of ϵ) die away. Thus over very long periods of time the factor $e^{2\gamma_k t}$ becomes infinitesmal. Although this is quite reasonable, there is another feature that is not immediately apparent. Since γ_k generally decreases with time, the factor $e^{2\gamma_k t}$ may decrease with time, despite the fact that γ_k is positive.

In particular as γ_k goes through zero we have $\frac{d}{dt}(e^{2\gamma_k t}) =$ $\left(2\gamma_k + t\,\frac{d\gamma_k}{dt}\right)e^{2\gamma_k t}\Big|_{\gamma_k=0} = 2t\,\frac{d\gamma_k}{dt} = 0$; This is incorrect, as the collective effects should continue to grow as long as γ_k is positive. We now propose to remedy this defect.

Equation V-127 is a valid <u>kinetic</u> equation for $0 < t < t_{ad}$, but it does not yield correct behavior for $t > t_{ad}$, because the adiabatic hypothesis breaks down. How may we correct this? The answer is straightforward. We start the system off at $t = 0$, then permit it to evolve forward to time $t_1 \lesssim t_{ad}$. By this time the adiabatic hypothesis is beginning to cause difficulty. Accordingly we adjust $f(\underset{\sim}{v})$ to its current value $f(\underset{\sim}{v}, t_1)$ and allow the system to evolve forward in time for another time step $t \lesssim t_{ad}$. By continuing this procedure we may follow the system as it evolves through times much greater than t_{ad}. The calculation is conceptually simple, but not very elegant. It is formally and theoretically better to adjust $f(\underset{\sim}{v}, t)$ continuously (if infinitesmally) while the system evolves. The calculation is straightforward, and is based on the proof given earlier that the collision term is invariant under translation of time.

The initial value term and the f dependent terms are functionally independent, so we shall consider them separately. It is also convenient to treat the case of k in the unstable volume of k space. The extension to the stable volume of k space may be taken for granted now.

We shall consider the initial value term first. With k in the unstable volume of k space we have

$$\langle \delta f_\mu(v)\delta \underset{\sim}{E}|t\rangle_I = \frac{q_\mu}{m_\mu}\int \frac{d\underset{\sim}{k}\, \underset{\sim}{k}\, \underset{\sim}{k}\cdot \frac{df_\mu}{d\underset{\sim}{v}} \exp[2\gamma_k(t - t_o)]}{k^4\left|\frac{d\epsilon}{d\omega}(\omega_k\right|^2 i(\underset{\sim}{k}\cdot\underset{\sim}{v} - \omega_k)} \sum_{\alpha,\nu} \frac{2n_\alpha n_\nu q_\alpha q_\nu}{\pi} \quad X$$

$$\int d\underset{\sim}{v}_1 \int \frac{d\underset{\sim}{v}_2 \quad g_{\alpha\nu}(\underset{\sim}{k}, \underset{\sim}{v}_1, \underset{\sim}{v}_2, t = 0)}{(\Omega_k - \underset{\sim}{k}\cdot\underset{\sim}{v}_1 - i\gamma_k)(\Omega_k - \underset{\sim}{k}\cdot\underset{\sim}{v}_2 + i\gamma_k)} \quad . \tag{V-128}$$

where we have set the time origin at t_o for generality. We are concerned with the fact that $f(v)$ and γ_k do not remain constant with time. We now omit all factors of equation V-128 which are not relevant to the argument by defining an operator D which contains these factors. We have, at time t_1

$$\langle \delta(f_\mu(v)\delta \underset{\sim}{E}|t_1\rangle_I = D \frac{q_\mu}{m_\mu} \frac{\underset{\sim}{k}}{k^2}\cdot\frac{df_\mu(t_o)}{d\underset{\sim}{v}} \; \frac{\exp[2\gamma_k(t_o)(t_1 - t_o)]}{i(\underset{\sim}{k}\cdot\underset{\sim}{v} - \omega_k(t_o)\;)} \quad X$$

$$\frac{\sum_\nu 4\pi n_\nu q_\nu}{\left.\frac{d\epsilon}{d\omega}\right|_{\omega_k}(t_o)} \int \frac{d\underset{\sim}{v}_1}{\omega_k(t_o) - \underset{\sim}{k}\cdot\underset{\sim}{v}_1} \; \langle \delta f_\nu(\underset{\sim}{v}_1)\delta\underset{\sim}{E}|t_o\rangle \quad . \tag{V-129}$$

We now use equation V-129 as the initial value term to carry the system forward from time t_1 to time t_2 .

$$\langle \delta f_\mu(\underset{\sim}{v})\delta \underset{\sim}{E}|t_2\rangle_I = D\,\frac{q_\mu}{m_\mu}\,\frac{\underset{\sim}{k}}{k^2}\cdot\frac{df_\mu(t_1)}{d\underset{\sim}{v}}\;\frac{\exp\left[2\gamma_k(t_1)(t_2 - t_1)\right]}{i(\underset{\sim}{k}\cdot\underset{\sim}{v} - \omega_k(t_1))}\;X$$

$$\frac{\sum_\nu 4\pi n_\nu q_\nu}{\left.\frac{d\epsilon}{d\omega}\right|_{\omega_k}(t_1)}\int\frac{d\underset{\sim}{v}_1}{\omega_k(t_o)-\underset{\sim}{k}\cdot\underset{\sim}{v}_1}\;\langle \delta f_\nu(\underset{\sim}{v}_1)\delta E|t_1\rangle_I \tag{V-130}$$

Using the fact that

$$\sum_\nu \frac{4\pi n_\nu q_\nu^2}{m_\nu}\int\frac{d\underset{\sim}{v}_1\,\frac{\underset{\sim}{k}}{k^2}\cdot\frac{df_\nu}{d\underset{\sim}{v}_1}}{(\underset{\sim}{k}\cdot\underset{\sim}{v}_1 - \omega_k)^2} = -\left.\frac{d\epsilon}{d\omega}\right|_{\omega_k} \quad . \tag{V-131}$$

We have

$$\langle \delta f_\mu(\underset{\sim}{v})\delta \underset{\sim}{E}|t_2\rangle_I = D\,\frac{q_\mu}{m_\mu}\,\frac{\underset{\sim}{k}}{k^2}\,\frac{df_\mu(t_1)}{d\underset{\sim}{v}}\;\frac{\exp\left[2\gamma_k(t_1)(t_2 - t_1) + 2\gamma_k(t_o)(t_1 - t_o)\right]}{i(\underset{\sim}{k}\cdot\underset{\sim}{v} - \omega_k(t_1))}$$

$$X\;\frac{\sum_\nu 4\pi n_\nu q_\nu}{\left.\frac{d\epsilon}{d\omega}\right|_{\omega_k}(t_1)}\int\frac{d\underset{\sim}{v}_1}{\omega_k(t_o) - \underset{\sim}{k}\cdot\underset{\sim}{v}_1}\;\langle \delta f_\nu(\underset{\sim}{v}_1)\delta \underset{\sim}{E}|t_o\rangle_I \quad . \tag{V-132}$$

Using this value for $\langle \delta f_\mu(\underset{\sim}{v})\delta \underset{\sim}{E}|t_2\rangle$ as the initial value term we compute $\langle \delta f_\mu(\underset{\sim}{v})\delta \underset{\sim}{E}|t_3\rangle$, and find

$$\langle \delta f_\mu(\underset{\sim}{v})\delta \underset{\sim}{E} | t_3 \rangle = D \frac{q_\mu}{m_\mu} \frac{\underset{\sim}{k}}{k^2} \cdot \frac{df_\mu(t_2)}{d\underset{\sim}{v}} \frac{1}{i\left[\underset{\sim}{k}\cdot\underset{\sim}{v} - \omega_k(t_2)\right]} \quad X$$

$$\exp\left[2\gamma_k(t_2)(t_3 - t_2) + 2\gamma_k(t_1)(t_2 - t_1) + 2\gamma_k(t_o)(t_1 - t_o)\right] \quad X$$

$$\sum_\nu \frac{4\pi n_\nu q_\nu}{\left.\frac{d\epsilon}{d\omega}\right|_{\omega_k}(t_2)} \int \frac{d\underset{\sim}{v}_1}{\omega_k(t_o) - \underset{\sim}{k}\cdot\underset{\sim}{v}_1} \langle \delta f_\nu(\underset{\sim}{v}_1)\, \delta\underset{\sim}{E} | t_o \rangle \quad . \tag{V-133}$$

The result is apparent by now. With $t_{i+1} - t_i$ infinitesmal, we have

$$\sum_{i=0}^{n-1} \gamma_k(t_i)(t_{i+1} - t_i) = \int_o^t \gamma_k(\tau)d\tau \quad . \tag{V-134}$$

and the contribution to the kinetic equation from the initial value term is given by (we include the effect of damped modes)

$$\langle \delta f_\mu \delta E | t \rangle = \int \frac{d\underset{\sim}{k}\, \underset{\sim}{k}\, \underset{\sim}{k}\cdot \frac{df_\mu}{d\underset{\sim}{v}} \exp\left[\int_o^t \gamma_k(\tau)d\tau\right]}{k^4 \left|\frac{d\epsilon}{d\omega}(\omega_k)\right|^2} \left\{ \frac{\gamma_k}{|\omega_k - \underset{\sim}{k}\cdot\underset{\sim}{v}|^2} + \right.$$

$$\left. 2\pi\delta(\underset{\sim}{k}\cdot\underset{\sim}{v} - \Omega_k^-) \right\} \sum_{\alpha,\nu} \frac{2n_\alpha n_\nu q_\alpha q_\nu}{\pi} \int d\underset{\sim}{v}_1 \int \frac{d\underset{\sim}{v}_2 \quad g_{\alpha\nu}(\underset{\sim}{k}, \underset{\sim}{v}_1, \underset{\sim}{v}_2, t = 0)}{(\Omega_k - \underset{\sim}{k}\cdot\underset{\sim}{v}_1 - i|\gamma_k|)(\Omega_k - \underset{\sim}{k}\cdot\underset{\sim}{v}_2 + i|\gamma_k|)} \tag{V-135}$$

The term now has the expected form, in that it continues to grow as long as γ_k is positive. We also note in passing that in each velocity integration we have made the approximation $\frac{1}{\omega_k(t_{i+1}) - \underset{\sim}{k}\cdot\underset{\sim}{v}_1} \rightarrow \frac{1}{\omega_k(t_i) - \underset{\sim}{k}\cdot\underset{\sim}{v}_1}$. This leads to an error of order $1/\omega_k t_{ad}$ which we neglect.

In the preceding we made no use of the fact that the kinetic equation (and hence the initial value term) is invariant under translation of the origin of time. The cancelation of the factors $\left.\frac{d\epsilon}{d\omega}\right|_{\omega_k}$ was carried out explicitly, in order to demonstrate the formal method in detail. The procedure is now established, and we skip these steps here, for the demonstration of invariance permits us to do so. This shortens the ensuing calculation considerably.

We now wish to establish the form of the terms in the kinetic equation which have explicit time dependence $e^{2\gamma_k t}$, yet depend only on f. We define an appropriate operator D' , and write these terms as

$$J(t_1) = D' \frac{\exp\left[2\gamma_k(t_o)(t_1 - t_o)\right]}{|\omega_k - \underset{\sim}{k}\cdot\underset{\sim}{v}|^2} \int d\underset{\sim}{v}_1 \, f_\nu(\underset{\sim}{v}_1, t_o)\delta(\Omega_k - \underset{\sim}{k}\cdot\underset{\sim}{v}_1) . \qquad \text{(V-136)}$$

Again we have set the origin of time at t_o and chosen k in the unstable volume of k space. We now use the collision term at time t_1 as the initial value term and permit the system to evolve to time t_2 .

$$J(t_2) = D' \frac{\exp\left[2\gamma_k(t_1)(t_2 - t_1)\right]}{|\omega_k - \underset{\sim}{k}\cdot\underset{\sim}{v}|^2} \int d\underset{\sim}{v}_1 f_\nu(\underset{\sim}{v}_1, t_1)\delta(\Omega_k - \underset{\sim}{k}\cdot\underset{\sim}{v}_1) +$$

$$D' \exp\left[2\gamma_k(t_1)(t_2 - t_1)\right] \int d\underset{\sim}{v}_1 f_\nu(v_1, t_o) \left[\frac{\exp\left[2\gamma_k(t_o)(t_1 - t_o)\right] - 1}{|\omega_k - \underset{\sim}{k}\cdot\underset{\sim}{v}|^2}\right]. \tag{V-137}$$

The form is correct, because if $f(\underset{\sim}{v}_1, t_o) = f(\underset{\sim}{v}_1, t_1)$, we find that equation V-137 gives the same $J(t_2)$ as when we set $t_o \Rightarrow t_2$ in equation V-136. In fact $f(\underset{\sim}{v}_1, t)$ changes slowly with time, and we are calculating the effect of this change. For $t_1 - t_o$ small, we have

$$J(t_2) = D' \frac{\exp\left[2\gamma_k(t_1)(t_2 - t_1)\right]}{|\omega_k - \underset{\sim}{k}\cdot\underset{\sim}{v}|^2} \int d\underset{\sim}{v}_1 f_\nu(\underset{\sim}{v}_1, t)\delta(\Omega_k - \underset{\sim}{k}\cdot\underset{\sim}{v}_1) +$$

$$D' \exp\left[2\gamma_k(t_1)(t_2 - t_1)\right] \int d\underset{\sim}{v}_1 f_\nu(\underset{\sim}{v}_1, t_o) \frac{2\gamma_k(t_o)(t_1 - t_o)}{|\omega_k - \underset{\sim}{k}\cdot\underset{\sim}{v}|^2} . \tag{V-138}$$

We now use $J(t_2)$ as the initial value term to carry the system forward to time t_3 .

$$J(t_3) = D' \frac{\exp\left[2\gamma_k(t_2)(t_3 - t_2)\right]}{|\omega_k - \underset{\sim}{k}\cdot\underset{\sim}{v}|^2} \int d\underset{\sim}{v}_1 f_\nu(\underset{\sim}{v}_1, t_2)\delta(\Omega_k - \underset{\sim}{k}\cdot\underset{\sim}{v}_1) +$$

$$D' \exp\left[2\gamma_k(t_2)(t_3 - t_2)\right] \left[J(t_2) - \int \frac{d\underset{\sim}{v}_1 f_\nu(\underset{\sim}{v}_1, t_1)\delta(\Omega_k - \underset{\sim}{k}\cdot\underset{\sim}{v}_1)}{|\omega_k - \underset{\sim}{k}\cdot\underset{\sim}{v}|^2} \right] =$$

$$D' \frac{\exp\left[2\gamma_k(t_2)(t_3 - t_2)\right]}{|\omega_k - \underset{\sim}{k}\cdot\underset{\sim}{v}|^2} \int d\underset{\sim}{v}_1 f_\nu(\underset{\sim}{v}_1, t_2)\delta(\Omega_k - \underset{\sim}{k}\cdot\underset{\sim}{v}_1) +$$

$$D' \exp\left[2\gamma_k(t_2)(t_3 - t_2)\right] \int d\underset{\sim}{v}_1 f_\nu(\underset{\sim}{v}_1, t_1)\delta(\Omega_k - \underset{\sim}{k}\cdot\underset{\sim}{v}_1) \frac{2\gamma_k(t_1)(t_2 - t_1)}{|\omega_k - \underset{\sim}{k}\cdot\underset{\sim}{v}|^2} +$$

$$D' \exp\left[2\gamma_k(t_2)(t_3 - t_2) + 2\gamma_k(t_1)(t_2 - t_1)\right] \int d\underset{\sim}{v}_1 f_\nu(\underset{\sim}{v}_1, t_o)\delta(\Omega_k - k\cdot v_1) \times$$

$$\frac{2\gamma_k(t_o)(t_1 - t_o)}{|\omega_k - \underset{\sim}{k}\cdot\underset{\sim}{v}|^2} . \qquad \text{(V-139)}$$

The generalization is straightforward.

$$J(t_{n+1}) = D' \exp\left[2\gamma_k(t_n)(t_{n+1} - t_n)\right] \int d\underset{\sim}{v}_1 f_\nu(\underset{\sim}{v}_1, t_n)\delta(\Omega_k - \underset{\sim}{k}\cdot\underset{\sim}{v}_1) +$$

$$D' \sum_{i=0}^{n-1} \frac{2\gamma_k(t_i)(t_{i+1} - t_i)}{|\omega_k - \underset{\sim}{k}\cdot\underset{\sim}{v}|^2} \int d\underset{\sim}{v}_1 f_\nu(\underset{\sim}{v}_1, t_i)\delta(\Omega_k - \underset{\sim}{k}\cdot\underset{\sim}{v}_1) \times$$

$$\exp\left[2\sum_{j=i+1}^{n}\gamma_k(t_j)(t_{j+1}-t_j)\right] \quad . \tag{V-140}$$

Passing to the integral form, we find

$$J(t) = D'\int d\underset{\sim}{v}_1 f_\nu(\underset{\sim}{v}_1,\ t)\delta(\Omega_k - \underset{\sim}{k}\cdot\underset{\sim}{v}_1) + \int_0^t dt'\exp\left[2\int_{t'}^{t}\gamma_k(\tau)d\tau\right] \ X$$

$$\frac{2\gamma_k(t')}{|\omega_k - \underset{\sim}{k}\cdot\underset{\sim}{v}|^2}\int d\underset{\sim}{v}_1 f_\nu(\underset{\sim}{v}_1,\ t')\delta(\Omega_k - \underset{\sim}{k}\cdot\underset{\sim}{v}_1) \quad . \tag{V-141}$$

Using this result, plus the result of equation V-135, we obtain the kinetic equation for an unstable plasma.

$$\frac{df_\mu}{dt} = \frac{q_\mu^{\ 2}}{m_\mu}\ \frac{d}{d\underset{\sim}{v}}\cdot\int d\underset{\sim}{k}\int d\underset{\sim}{v}_1\ \frac{\underset{\sim}{k}\,\underset{\sim}{k}}{k^4}\ \sum_\nu \frac{2n_\nu q_\nu^{\ 2}}{\pi}\left[\frac{\pi\delta(\underset{\sim}{k}\cdot\underset{\sim}{v} - \underset{\sim}{k}\cdot\underset{\sim}{v}_1)}{|\epsilon(\underset{\sim}{k},\ \underset{\sim}{k}\cdot\underset{\sim}{v}_1\ |^2}\ \Theta(\underset{\sim}{k}\cdot\underset{\sim}{v}_1)\ X\right.$$

$$\left\{\frac{1}{m_\mu}\frac{df_\mu}{d\underset{\sim}{v}}\ f_\nu(\underset{\sim}{v}_1) - \frac{1}{m_\nu}\ f_\mu(\underset{\sim}{v})\ \frac{df_\nu}{d\underset{\sim}{v}_1}\right\} + \left\{\frac{1}{|\underset{\sim}{k}\cdot\underset{\sim}{v} - \omega_k|^2} + \right.$$

$$\left.\left.\frac{2\pi\delta(\underset{\sim}{k}\cdot\underset{\sim}{v} - \Omega_k^{\ -})}{\gamma_k}\right\}\ \frac{1}{m_\mu}\frac{df_\mu}{d\underset{\sim}{v}}\ f_\nu(\underset{\sim}{v}_1)\ \frac{\pi\delta(\underset{\sim}{k}\cdot\underset{\sim}{v}_1 - \Omega_k)}{\left|\frac{d\epsilon}{d\omega}(\omega_k)\right|^2}\right] + \frac{q_\mu^{\ 2}}{m_\mu^{\ 2}}\ \frac{d}{d\underset{\sim}{v}}$$

$$\int \frac{d\underset{\sim}{k}\,\underset{\sim}{k}\,\underset{\sim}{k}\cdot\frac{df_\mu}{d\underset{\sim}{v}}}{k^4\left|\frac{d\epsilon}{d\omega}(\omega_k)\right|^2} \sum_\nu 4\pi n_\nu q_\nu^2 \int_o^t dt' \exp\left[2\int_{t'}^t \gamma_k(\tau)d\tau\right]\left\{\frac{\gamma_k(t')}{|\omega_k - \underset{\sim}{k}\cdot\underset{\sim}{v}|^2} +\right.$$

$$\left.2\pi\delta(\underset{\sim}{k}\cdot\underset{\sim}{v} - \Omega_k^-)\right\} \int d\underset{\sim}{v}_1 f_\nu(\underset{\sim}{v}_1, t')\delta(\Omega_k - \underset{\sim}{k}\cdot\underset{\sim}{v}_1) + \frac{q_\mu^2}{m_\mu^2} \frac{d}{d\underset{\sim}{v}} \cdot$$

$$\int \frac{d\underset{\sim}{k}\,\underset{\sim}{k}\,\underset{\sim}{k}\cdot\frac{df_\mu}{d\underset{\sim}{v}} \exp\left[2\int_o^t \gamma_k(\tau)d\tau\right]}{k^4\left|\frac{d\epsilon}{d\omega}(\omega_k)\right|^2} \left\{\frac{\gamma_k}{|\omega_k - \underset{\sim}{k}\cdot\underset{\sim}{v}|^2} + 2\pi\delta(\underset{\sim}{k}\cdot\underset{\sim}{v} - \Omega_k^-)\right\} X$$

$$\sum_{\alpha,\nu} \frac{2n_\alpha n_\nu q_\alpha q_\nu}{\pi} \int d\underset{\sim}{v}_1 \int \frac{d\underset{\sim}{v}_2 \quad g_{\alpha\nu}(\underset{\sim}{k}, \underset{\sim}{v}_1, \underset{\sim}{v}_2, t = 0)}{(\Omega_k - \underset{\sim}{k}\cdot\underset{\sim}{v}_1 - i|\gamma_k|)(\Omega_k - \underset{\sim}{k}\cdot\underset{\sim}{v}_2 + i|\gamma_k|)} \quad . \qquad \text{(V-142)}$$

The equation remains valid as $f(\underset{\sim}{v})$ changes in time. It is straightforward to verify that the equation reduces to the Lenard-Balescu equation after long times. The system first stabilizes itself (γ_k) becomes negative for all modes), after which the effects coming from the zeros of ϵ die away.

The equation is non-Markoffian in two respects:

1. It contains explicitly the effect of the initial correlation function $g(t = 0)$. This initial value term may continue to affect the system for long times.

2. The equation contains a time integral over the previous state of the system. This represents the fact that the particles of the system continually create inhomogeneities, which may grow in time in the same fashion as those already present in the system through $g(t = 0)$. In the usual nomenclature the particles emit waves.

We shall not carry out the demonstration that the equation conserves number density, momentum and energy, and leads to an H theorem. This is unnecessary, since we have previously demonstrated that these results are true at each instant for equation V-127. The manipulations of the last few pages do not affect these results.

A final comment is in order. It is straightforward to demonstrate that the electrostatic energy is given by

$$\langle \delta E^2 \rangle = \sum_\nu \frac{2n_\nu q_\nu^2}{\pi} \int \frac{d\underset{\sim}{k}\, \underset{\sim}{k}\, \underset{\sim}{k}}{k^4} \int d\underset{\sim}{v}_1 f_\nu(v_1) \left[\frac{\Theta(\underset{\sim}{k}\cdot\underset{\sim}{v}_1)}{|\epsilon(\underset{\sim}{k}, \underset{\sim}{k}\cdot\underset{\sim}{v}_1)|^2} + \frac{\pi\delta(k\cdot v_1 - \Omega_k)}{\gamma_k \left|\frac{d\epsilon}{d\omega}(\omega_k)\right|^2} \right] + \sum_{\alpha,\nu} 4n_\nu q_\nu^2 \int \frac{d\underset{\sim}{k}\, \underset{\sim}{k}\, \underset{\sim}{k}}{k^4} \int_0^t dt' \exp\left[2\int_{t'}^t \gamma_k(\tau)d\tau\right] \times$$

$$\int d\underset{\sim}{v}_1 f_\nu(\underset{\sim}{v}_1, t') \frac{\delta(\underset{\sim}{k}\cdot\underset{\sim}{v}_1 - \Omega_k)}{\left|\frac{d\epsilon}{d\omega}(\omega_k)\right|^2} + \sum_{\alpha,\nu} \frac{2n_\alpha n_\nu q_\alpha q_\nu}{\pi} \times$$

$$\int \frac{d\underset{\sim}{k}\, \underset{\sim}{k}\, \underset{\sim}{k} \;\; \exp\left[2\int_0^t \gamma_k(\tau)d\tau\right]}{k^4\left|\frac{d\epsilon}{d\omega}(\omega_k)\right|^2} \int d\underset{\sim}{v}_1 \int \frac{d\underset{\sim}{v}_2 \;\; g_{\alpha\nu}(\underset{\sim}{k}, \underset{\sim}{v}_1, \underset{\sim}{v}_2, t = 0)}{(\underset{\sim}{k}\cdot\underset{\sim}{v}_1 - \Omega_k - i|\gamma_k|)(\underset{\sim}{k}\cdot\underset{\sim}{v}_2 - \Omega_k + i|\gamma_k|)} .$$

(V-143)

where the result is now valid for long times. If we take the time derivative of V-143 , we find

$$\frac{d\langle\delta\underset{\sim}{E}^2\rangle}{dt} = \sum_\nu 4n_\nu q_\nu^2 \int \frac{d\underset{\sim}{k}\;\underset{\sim}{k}\;\underset{\sim}{k}}{k^4\left|\frac{d\epsilon}{d\omega}(\omega_k)\right|^2} \int d\underset{\sim}{v}_1 f_\nu(\underset{\sim}{v}_1, t)\delta(\underset{\sim}{k}\cdot\underset{\sim}{v}_1 - \Omega_k) +$$

$$\sum_\nu 8n_\nu q_\nu^2 \int \frac{d\underset{\sim}{k}\;\underset{\sim}{k}\;\underset{\sim}{k}\;\; \gamma_k(t)}{k^4\left|\frac{d\epsilon}{d\omega}(\omega_k)\right|^2} \int_0^t dt' \;\exp\left[2\int_{t'}^t \gamma_k(\tau)d\tau\right] \int d\underset{\sim}{v}_1 f_\nu(\underset{\sim}{v}_1, t') \;\; X$$

$$\delta(\underset{\sim}{k}\cdot\underset{\sim}{v}_1 - \Omega_k) + \sum_{\alpha,\nu} \frac{4\, n_\alpha n_\nu q_\alpha q_\nu}{\pi} \int \frac{d\underset{\sim}{k}\;\underset{\sim}{k}\;\underset{\sim}{k}\;\; \gamma_k(t)}{k^4\left|\frac{d\epsilon}{d\omega}(\omega_k)\right|^2} \int d\underset{\sim}{v}_1 \quad X$$

$$\int \frac{d\underset{\sim}{v}_2 \;\; g_{\alpha\nu}(\underset{\sim}{k}, \underset{\sim}{v}_1, \underset{\sim}{v}_2, t = 0) \;\; \exp\left[2\int_0^t \gamma_k(\tau)d\tau\right]}{(\underset{\sim}{k}\cdot\underset{\sim}{v}_1 - \Omega_k - i|\gamma_k|)(\underset{\sim}{k}\cdot\underset{\sim}{v}_2 - \Omega_k + i|\gamma_k|)} . \qquad \text{(V-144)}$$

Note the fact that the electrostatic energy continues to grow even when $\gamma_k = 0$. Thus the collision term continues to extract energy from the particles even after a marginally stable state has been reached.

2. The Effect of a Uniform Magnetic Field

We now extend the kinetic equation to include the effect of a uniform magnetic field. Since the procedure has been demonstrated in detail in earlier pages, we simply carry out the steps which are different: we must recalculate the P operator.

The principal effect of the magnetic field (in terms of analysis) is to complicate considerably the formal expressions which we must handle. In order to keep the results relatively tractable we shall consider the case in which the plasma distribution function, $f(v, t)$ is independent of the angle about the magnetic field (ϕ). In the more general case we simply expand $f(v_\perp, v_z, \phi, t) \rightarrow \sum_n f_\mu(v_\perp, v_z, t)e^{in\phi}$ and repeat the same analysis.

The initial steps are taken largely from Bernstein[49] and Rostoker[50]. As the signs of changes appear explicitly at many points in the analysis, we adopt the notation $q_\mu = |q_\mu|$ for the respective charges and indicate signs explicitly in the equations. Similarly we define the cyclotron frequency $\omega_\mu = \left|\frac{q_\mu B}{m_\mu c}\right|$.

The natural coordinate system of the problem is cylindrical coordinates, with the axis (z) along $\underset{\sim}{B}$. We choose an arbitrary axis perpendicular to $\underset{\sim}{B}$ as the zero of the azimuthal angles, and write $\underset{\sim}{v} = (v_\perp, v_z, \phi)$, $\underset{\sim}{k} = (k_\perp, k_z, \alpha)$. The P operator is obtained by considering the linearized Vlasov operator acting on an arbitrary function, which we shall call δf.

$$\left(\frac{d}{dt} + \underset{\sim}{v} \cdot \nabla \right) \delta f_\mu \pm \frac{q_\mu}{m_\mu} \frac{\underset{\sim}{v} \times \underset{\sim}{B}}{c} \cdot \frac{d\delta f_\mu}{d\underset{\sim}{v}} \pm \frac{q_\mu}{m_\mu} \delta \underset{\sim}{E} \cdot \frac{df_\mu}{d\underset{\sim}{v}} = 0 \quad . \tag{V-145}$$

We are ignoring electromagnetic effects, so that $\nabla \times \delta \underset{\sim}{E} = 0$. We Fourier transform in space and Laplace transform in time, expressing quantities in the coordinate system already chosen.

$$\left[- i\omega + ik_z v_z + ik_\perp v_\perp \cos(\phi - \alpha) \right] \delta f_\mu \mp \omega_\mu \frac{d\delta f_\mu}{d\phi} =$$

$$\delta f_\mu(\underset{\sim}{k}, \underset{\sim}{v}, t = 0) \mp \frac{q_\mu}{m_\mu} \frac{(\underset{\sim}{k} \cdot \delta \underset{\sim}{E})}{k^2} \left[k_z \frac{df_\mu}{dv_z} + k_\perp \cos(\phi - \alpha) \frac{df_\mu}{dv_\perp} \right] \quad . \tag{V-146}$$

We use the integrating factor

$$\exp \left[\pm \int_c^\phi \omega_\mu^{-1} \left\{ -i\omega + ik_z v_z + ik_\perp v_\perp \cos(\phi' - \alpha) \right\} d\phi' \right] \quad . \tag{V-147}$$

and the fact that ω has a positive imaginary part to produce the result

$$\delta f_\mu(\underset{\sim}{k}, \underset{\sim}{v}) = \mp \frac{1}{\omega_\mu} \int_{\pm i\infty}^{\phi} d\phi'' \quad \exp. \left[\int_{\phi''}^{\phi} \omega_\mu^{-1} \left\{ -i\omega + ik_z v_z + \right. \right.$$

$$\left. \left. ik_\perp v_\perp \cos(\phi' - \alpha) \right\} d\phi' \right] \left[\delta f_\mu(k, v_\perp, v_z, \phi'', t = 0) \mp \frac{q_\mu}{m_\mu} \frac{(\underset{\sim}{k} \cdot \delta \underset{\sim}{E})}{k^2} \times \right.$$

$$\left\{k_z \frac{df_\mu}{dv_z} + k_\perp \cos(\phi'' - \alpha) \frac{df_\mu}{dv_z}\right\}\Bigg] \quad . \tag{V-148}$$

By using the expansion of $\exp\left[iz \sin \phi\right]$ in terms of Bessel functions[51]

$$\exp\left[iz \sin \phi\right] = \sum_{n=-\infty}^{\infty} e^{in\phi} J_n(z) \quad . \tag{V-149}$$

and Poisson's equation

$$i\underset{\sim}{k} \cdot \delta \underset{\sim}{E} (\underset{\sim}{k}, \omega) = \sum_\mu \pm 4\pi n_\mu q_\mu \int d\underset{\sim}{v} \; \delta f_\mu(\underset{\sim}{k}, \underset{\sim}{v}, \omega) \quad . \tag{V-150}$$

we readily find δf_μ and $\delta \underset{\sim}{E}$.

$$\delta f_\mu = \pm \frac{1}{\omega_\mu} \int_{\pm i\infty}^{\phi} d\phi'' \exp\left[\pm \omega_\mu^{-1} \left\{(- i\omega + ik_z v_z)(\phi - \phi'') + \right.\right.$$

$$\left.\left. ik_\perp v_\perp \sin(\phi - \alpha)\right\}\right] \sum_n e^{in(\phi - \alpha)} J_n \left(\frac{k_\perp v_\perp}{\omega_\mu}\right) \delta f_\mu(\underset{\sim}{k}, v_\perp , v_z , \phi'', t = 0)$$

$$\mp \frac{q_\mu}{m_\mu} \sum_n J_n \left(\frac{k_\perp v_\perp}{\omega_\mu}\right) \frac{\exp\left[\pm i\omega_\mu^{-1} k_\perp v_\perp \sin(\phi - \alpha) \mp in(\phi - \alpha)\right]}{\epsilon(\underset{\sim}{k}, \omega) \left[i\omega + i(\underset{\sim}{k} \cdot \underset{\sim}{v})_{\mu n}\right]} \quad X$$

$$\left(\frac{i\underset{\sim}{k}}{k^2} \cdot \frac{d}{d\underset{\sim}{v}}\right)_{\mu n} f_\mu(\underset{\sim}{v}) \sum_\nu \frac{4\pi n_\nu q_\nu}{\omega_\nu} \int d\underset{\sim}{v}' \int_{\pm i\infty}^{\phi'} d\phi'' \sum_m \exp\left[\pm \omega_\nu^{-1} \right. \quad X$$

$$\left\{(i\omega + ik_z v_z')(\phi' - \phi'') + ik_\perp v_\perp'(\phi' - \alpha)\right\} \mp im(\phi' - \alpha)\Big] \times$$

$$\frac{J_m\left(\frac{k_\perp v_\perp'}{\omega_\nu}\right) \delta f_\nu\ (\underset{\sim}{k},\ v_\perp',\ v_z',\ \phi'',\ t = 0)}{\left[-\ i\omega + i(\underset{\sim}{k}\cdot\underset{\sim}{v}')_{\nu m}\right]} \qquad \text{(V-151)}$$

$$\delta\underset{\sim}{E}(\underset{\sim}{k},\ \omega) = \sum_\mu \frac{4\pi n_\mu q_\mu\left(\frac{i\underset{\sim}{k}}{k^2}\right)}{\omega_\mu\ \epsilon(\underset{\sim}{k},\ \omega)} \int d\underset{\sim}{v} \int_{\pm i\infty}^{\phi} d\phi'' \ \sum_n J_n\left(\frac{k_\perp v_\perp}{\omega_\mu}\right) \times$$

$$\exp\left[\pm\ \omega_\mu^{-1} \left\{(-\ i\omega + ik_z v_z)(\phi - \phi'') + ik_\perp v_\perp \sin(\phi - \alpha)\right\} \mp in(\phi'' - \alpha)\right] \times$$

$$\frac{\delta f_\mu(\underset{\sim}{k},\ v_\perp\ ,\ v_z,\ \phi''\ ,\ t = 0)}{\left[-\ i\omega + i(\underset{\sim}{k}\cdot\underset{\sim}{v})_{\mu n}\right]} \ . \qquad \text{(V-152)}$$

where we have defined

$$(\underset{\sim}{k}\cdot\underset{\sim}{v})_{\mu n} = k_z v_z + n\omega_\mu \ . \qquad \text{(V-153)}$$

$$\left(\underset{\sim}{k}\cdot \frac{d}{d\underset{\sim}{v}}\right)_{\mu n} = k_z \ \frac{d}{dv_z} + \frac{n\omega_\mu}{v_\perp} \ \frac{d}{dv_\perp} \ . \qquad \text{(V-154)}$$

and the dielectric function is given by

$$\epsilon(\underset{\sim}{k}, \omega) = 1 + \Sigma \frac{\omega_\mu^2}{k^2} \int d\underset{\sim}{v} \sum_n \frac{J_n^2\left(\frac{k_\perp v_\perp}{\omega_\mu}\right)\left(\underset{\sim}{k} \cdot \frac{d}{d\underset{\sim}{v}}\right)_{\mu n} f_\mu(\underset{\sim}{v})}{\omega - (\underset{\sim}{k}\cdot\underset{\sim}{v})_{\mu n}} . \qquad \text{(V-155)}$$

The collision integral which we wish to calculate is given by

$$\frac{d}{d\underset{\sim}{v}} \cdot \langle \delta f_\mu \delta \underset{\sim}{E} \rangle = \int \frac{d\underset{\sim}{k}}{(2\pi)^3} \frac{d}{d\underset{\sim}{v}} \cdot \int \frac{d\omega_1}{2\pi} e^{-i\omega_1 t} \int \frac{d\omega_2}{2\pi} e^{-i\omega_2 t} \quad X$$

$$\langle \delta f_\mu(\underset{\sim}{k}, \underset{\sim}{v}, \omega_1) \delta \underset{\sim}{E}(-\underset{\sim}{k}, \omega_2) \rangle . \qquad \text{(V-156)}$$

The procedure is now extremely straightforward, but extremely tedious. We simply list the next (elementary) operations, in order to save pages of explicit calculation.

1) Using V-151 and V-152 we write down the formal expression for $\langle \delta f_\mu(\underset{\sim}{k}, \underset{\sim}{v}, \omega_1) \delta \underset{\sim}{E}(-\underset{\sim}{k}, \omega_2) \rangle$. Since the distribution function is independent of ϕ we may carry out the indefinite ϕ'' integrals. (We also assume $g(t = 0)$ independent of ϕ'').

2) We then make use of the fact that $\langle \delta f_\alpha(\underset{\sim}{k}, \underset{\sim}{v}, t = 0) \delta f_\beta(-\underset{\sim}{k}, \underset{\sim}{v}_1, t = 0) \rangle = \frac{\delta_{\alpha\beta}}{n_\alpha} \delta(\underset{\sim}{v} - \underset{\sim}{v}_1) f_\alpha(\underset{\sim}{v}_1) + g(0)$ to carry out an integration over $\underset{\sim}{v}_1$. Note the fact that $\underset{\sim}{k} = (k_\perp, k_z, \alpha)$ implies $-\underset{\sim}{k} = (k_\perp, -k_z, \alpha + \pi)$. This causes cancelation of terms in the exponentials, and leads to sums of Bessel functions of the form $\sum_n J_n^2\left(\frac{k_\perp v_\perp}{\omega_\alpha}\right)$.

3) We then carry out the angular integrations over the initial value term. Again we are lead to sums of squares of Bessel functions. In these terms and those mentioned above we shall leave the integration implicit. Thus $\int_0^{2\pi} d\phi' \, e^{i(m-n)\phi'} = 2\pi\delta_{mn} = \int_0^{2\pi} d\phi' \, \delta_{mn}$, where δ_{mn} is the Kronecker delta, which acts on the sums of Bessel functions.

4) There remains the angular dependence of $\underset{\sim}{v}$ where $\underset{\sim}{v}$ is the independent variable of the equation. We now include the velocity derivative which is included in the collision term.

$$\frac{df_\mu}{dt} \pm \frac{q_\mu}{m_\mu} \frac{\underset{\sim}{v} \times \underset{\sim}{B}}{c} \cdot \frac{df_\mu}{d\underset{\sim}{v}} = \pm \frac{d}{d\underset{\sim}{v}} \cdot \left\langle \frac{q_\mu}{m_\mu} \delta f_\mu \delta \underset{\sim}{E} \right\rangle \quad . \tag{V-157}$$

Using the fact that

$$\underset{\sim}{k} \cdot \frac{d}{d\underset{\sim}{v}} = \frac{k_\perp}{v_\perp} \sin(\alpha - \phi)\frac{d}{d\phi} + k_\perp \cos(\alpha - \phi) \frac{d}{dv_\perp} + k_z \frac{d}{dv_z} \quad . \tag{V-158}$$

we may perform the α integration (implicitly). We must use the fact that

$$J_{n+1}(z) + J_{n-1}(z) = \frac{2n}{z} J_n(z) \tag{V-159}$$

5) We then invert the Laplace transforms, and evaluate them in exxentially the same way that we analyzed the case $B = 0$. There is a single exception: we displace contours of v_{1z}, v_{2z} and k_z in order to produce damping of the contours. Thus all poles which we pick up in the

various integrals will result from deformations of the respective z components. The field B makes z a "preferred" direction. Of course we may have to displace perpendicular components of the respective integrals on occasion, in order to make contours damp out sufficiently rapidly. These terms will not appear in the result.

We now write down the kinetic equation for a plasma in a magnetic field which is valid for short time $(t < t_{ad})$. We drop the explicit signs of the charges, leaving both cyclotron frequencies positive.

$$\frac{df_\mu}{dt} + \frac{q_\mu}{m_\mu} \frac{\underset{\sim}{v}}{c} \times \underset{\sim}{B} \cdot \frac{df_\mu}{d\underset{\sim}{v}} = \frac{q_\mu^2}{m_\mu} \int \frac{d\underset{\sim}{k}}{k^4} \sum_n \left(\underset{\sim}{k} \cdot \frac{d}{d\underset{\sim}{v}}\right)_{\mu n} \sum_\nu 2n_\nu q_\nu^2 \quad X$$

$$\int d\underset{\sim}{v}_1 \sum_m J_n^2 \left(\frac{k_\perp v_\perp}{\omega_\mu}\right) J_m^2 \left(\frac{k_\perp v_{1\perp}}{\omega_\nu}\right) \quad X \left[\left\{ \frac{\delta\left[(\underset{\sim}{k}\cdot\underset{\sim}{v})_{\mu n} - (\underset{\sim}{k}\cdot\underset{\sim}{v}_1)_{\nu m}\right]}{\left|\epsilon\left[\underset{\sim}{k},\ (\underset{\sim}{k}\cdot\underset{\sim}{v}_1)_{\nu m}\right]\right|^2} \right.\right. \quad X$$

$$\left[\frac{1}{m_\mu}\left(\underset{\sim}{k}\cdot\frac{d}{d\underset{\sim}{v}}\right)_{\mu n} - \frac{1}{m_\nu}\left(\underset{\sim}{k}\cdot\frac{d}{d\underset{\sim}{v}_1}\right)_{\nu m}\right] f_\mu(\underset{\sim}{v}) f_\nu(\underset{\sim}{v}_1) \quad \Theta\left[(\underset{\sim}{k}\cdot\underset{\sim}{v}_1)_{\nu m}\right] \quad +$$

$$\frac{e^{2\gamma_k t}\,\delta\left[(\underset{\sim}{k}\cdot\underset{\sim}{v}_1)_{\nu m} - \Omega_k\right]}{m_\mu\left|\frac{d\epsilon}{d\omega}(\omega_k)\right|^2} \left[\frac{1}{|\omega_k - (\underset{\sim}{k}\cdot\underset{\sim}{v})_{\mu n}|^2} + \frac{2\pi\delta\left[(\underset{\sim}{k}\cdot\underset{\sim}{v})_{\mu n} - \Omega_k^-\right]}{\gamma_k}\right] X$$

$$\left.\left.\left(\underset{\sim}{k}\cdot\frac{d}{d\underset{\sim}{v}}\right)_{\mu n} f_\mu(\underset{\sim}{v}) f_\nu(\underset{\sim}{v}_1)\right\}\right] + \frac{q_\mu^2}{m_\mu^2} \sum_n \int \frac{d\underset{\sim}{k}}{k^4} \left(\underset{\sim}{k}\cdot\frac{d}{d\underset{\sim}{v}}\right)_{\mu n} \quad X$$

$$\left\{ \frac{\gamma_k}{|\omega_k - k\cdot v)_{\mu n}|^2} + 2\pi\delta\left[(\underset{\sim}{k}\cdot\underset{\sim}{v})_{\mu n} - \Omega_k^{-} \right] \right\} X$$

$$\frac{e^{2\gamma_k t} \; J_n^{\,2}\left(\frac{k_\perp v_\perp}{\omega_\mu}\right)\left(\underset{\sim}{k}\cdot\frac{d}{d\underset{\sim}{v}}\right)_{\mu n} f_\mu(\underset{\sim}{v})}{\left|\frac{d\epsilon}{d\omega}(\omega_k)\right|^2} \sum_{\alpha,\nu} \frac{2n_\alpha n_\nu q_\alpha q_\nu}{\pi} \int d\underset{\sim}{v}_1 \quad X$$

$$\int \frac{d\underset{\sim}{v}_2 \sum_m \sum_p J_m^{\,2}\left(\frac{k_\perp v_1}{\omega_\nu}\right) J_p^{\,2}\left(\frac{k_\perp v_2}{\omega_\alpha}\right) g_{\nu\alpha}(\underset{\sim}{k}, \underset{\sim}{v}_1, \underset{\sim}{v}_2, t = 0)}{(\Omega_k - (\underset{\sim}{k}\cdot\underset{\sim}{v}_1)_{\nu m} + i|\gamma_k|)(\Omega_k - (\underset{\sim}{k}\cdot\underset{\sim}{v}_2)_{\alpha p} - i|\gamma_k|)} \; . \qquad \text{(V-160)}$$

As before the function Θ is given by

$$\Theta\left[(\underset{\sim}{k}\cdot\underset{\sim}{v}_1)_{\nu m}\right] = +1 \quad \text{Imag } \epsilon\left[\underset{\sim}{k}, \; (\underset{\sim}{k}\cdot\underset{\sim}{v}_1)_{\nu m}\right] > 0 \quad \text{(stable)}$$

$$= -1 \quad \text{Imag } \epsilon\left[\underset{\sim}{k}, \; (\underset{\sim}{k}\cdot\underset{\sim}{v}_1)_{\nu m}\right] < 0 \quad \text{(unstable)}$$

(V-161)

The calculation to make the equation valid for long time is the same as in the case given previously. The general equation for a uniform plasma in a uniform magnetic field is given by

$$\frac{df_\mu}{dt} + \frac{q_\mu}{m_\mu} \frac{\underset{\sim}{v}}{c} \times B \cdot \frac{df_\mu}{d\underset{\sim}{v}} = \frac{q_\mu^2}{m_\mu} \int \frac{d\underset{\sim}{k}}{k^4} \sum_n \left(\underset{\sim}{k} \cdot \frac{d}{d\underset{\sim}{v}}\right)_{\mu n} \sum_\nu 2 n_\nu q_\nu^2 \quad X$$

$$\int d\underset{\sim}{v}_1 \sum_m J_n^2\left(\frac{k_\perp v_\perp}{\omega_\mu}\right) J_m^2\left(\frac{k_\perp v_{1\perp}}{\omega_\nu}\right) \quad X \left[\left\{ \frac{\delta\left[(\underset{\sim}{k}\cdot\underset{\sim}{v})_{\mu n} - (\underset{\sim}{k}\cdot\underset{\sim}{v}_1)_{\nu m}\right]}{\left|\epsilon(\underset{\sim}{k}, \left[\underset{\sim}{k}\cdot\underset{\sim}{v}_1\right]_{\nu m})\right|^2} \quad X \right.\right.$$

$$\left[\frac{1}{m_\mu}\left(\underset{\sim}{k} \cdot \frac{d}{d\underset{\sim}{v}}\right)_{\mu n} - \frac{1}{m_\nu}\left(\underset{\sim}{k} \cdot \frac{d}{d\underset{\sim}{v}_1}\right)_{\nu m}\right] f_\mu(\underset{\sim}{v}) f_\nu(\underset{\sim}{v}_1)\, \Theta\left(\left[\underset{\sim}{k}\cdot\underset{\sim}{v}_1\right]_{\nu m}\right) +$$

$$\frac{\delta\left[(\underset{\sim}{k}\cdot\underset{\sim}{v}_1)_{\nu m} - \Omega_k\right]}{m_\mu\left|\frac{d\epsilon}{d\omega}(\omega_k)\right|^2}\left[\frac{1}{\left|\omega_k - (\underset{\sim}{k}\cdot\underset{\sim}{v})_{\mu n}\right|^2} + \frac{2\pi\delta\left[(\underset{\sim}{k}\cdot\underset{\sim}{v})_{\mu n} - \Omega_k^-\right]}{\gamma_k}\right] \quad X.$$

$$\left.\left.\left(\underset{\sim}{k}\cdot \frac{d}{d\underset{\sim}{v}}\right)_{\mu n} f_\mu(\underset{\sim}{v}) f_\nu(\underset{\sim}{v}_1)\right\}\right] + \frac{q_\mu^2}{m_\mu^2} \int \frac{d\underset{\sim}{k}}{k^4} \sum_n \left(\underset{\sim}{k}\cdot \frac{d}{d\underset{\sim}{v}}\right)_{\mu n} \sum_\nu 4 n_\nu q_\nu^2 \int d\underset{\sim}{v}_1 \quad X$$

$$\sum_m J_n^2\left(\frac{k_\perp v_\perp}{\omega_\mu}\right) J_m^2\left(\frac{k_\perp v_{1\perp}}{\omega_\nu}\right) \int_o^t dt' \frac{\delta\left((\underset{\sim}{k}\cdot\underset{\sim}{v}_1)_{\nu m} - \Omega_k\right)}{\left|\frac{d\epsilon}{d\omega}(\omega_k)\right|^2} \exp\left[2\int_{t'}^t \gamma_k(\tau)\, d\tau\right] \quad X$$

$$\left[\frac{\gamma_k}{|\omega_k - (k\cdot v)_{\mu n}|^2} + 2\pi\delta\left[(k\cdot v)_{\mu n} - \Omega_k^-\right]\right]\left(\underset{\sim}{k}\cdot\frac{d}{d\underset{\sim}{v}}\right)_{\mu n} f_\mu(\underset{\sim}{v}, t) \; X$$

$$f_\nu(\underset{\sim}{v}_1, t') + \frac{q_\mu^2}{m_\mu^2}\sum_n\int\frac{d\underset{\sim}{k}}{k^4}\left(\underset{\sim}{k}\cdot\frac{d}{d\underset{\sim}{v}}\right)_{\mu n} X \left[\frac{\gamma_k}{|\omega_k - \underset{\sim}{k}\cdot\underset{\sim}{v})_{\mu n}|^2} + \right.$$

$$\left. 2\pi\delta\left[(\underset{\sim}{k}\cdot\underset{\sim}{v})_{\mu n} - \Omega_k^-\right]\right] X \; \exp\left[2\int_0^t\gamma_k(\tau)d\tau\right] J_n^2\left(\frac{k_\perp v_\perp}{\omega_\mu}\right)\left(\underset{\sim}{k}\cdot\frac{d}{d\underset{\sim}{v}}\right)_{\mu n} f_\mu(\underset{\sim}{v}) \; X$$

$$\sum_{\alpha,\nu}\frac{2n_\alpha n_\nu q_\alpha q_\nu}{\pi\left|\frac{d\epsilon}{d\omega}(\omega_k)\right|^2}\int d\underset{\sim}{v}_1\int d\underset{\sim}{v}_2 \frac{\sum_m\sum_p J_m^2\left(\frac{k_\perp v_{1\perp}}{\omega_\nu}\right)J_p^2\left(\frac{k_\perp v_{2\perp}}{\omega_\alpha}\right) g_{\alpha\nu}(\underset{\sim}{k}, \underset{\sim}{v}_1, \underset{\sim}{v}_2, t=0)}{(\Omega_k - (\underset{\sim}{k}\cdot\underset{\sim}{v}_1)_{\nu m} + i|\gamma_k|)(\Omega_k - (\underset{\sim}{k}\cdot\underset{\sim}{v}_2)_{\alpha p} - i|\gamma_k|)}.$$

(V-162)

D. Criticism of the Quasilinear Theory

In view of the earlier section devoted to quasilinear theory (II-D) we may write out a corrected quasilinear equation without further explanation. The necessary analysis has been carried out in the preceding pages. We have

$$\frac{df_\mu^{\,o}}{dt} = \frac{q_\mu^{\,2}}{m_\mu^{\,2}} \frac{d}{d\underset{\sim}{v}} \cdot \sum_{\underset{\approx}{k}} \underset{\sim}{E}^k(t = 0)\underset{\sim}{E}^{-k}(t = 0) \; \frac{df_\mu^{\,o}}{d\underset{\sim}{v}} \, x \; e^{\int_o^t \gamma_k(\tau)d\tau} \quad X$$

$$\left\{ \frac{\gamma_k}{|\omega_k - \underset{\approx}{k}\cdot\underset{\sim}{v}|^2} + 2\pi\delta(\underset{\approx}{k}\cdot\underset{\sim}{v} - \Omega_k^{\,-}) \right\} . \qquad (V\text{-}163)$$

This equation is better than equation II-116 in three respects:

1) The expression is mathematically well defined. (The surface in k space $\gamma_k = 0$ is to be taken as the limiting value from either side).

2) The equation treats the stable modes as well as the unstable modes.

3) The exponential time dependence is given correctly. The time dependence given here is sometimes derived, but the author does not consider these derivations satisfactory. The problem is to give a derivation which is valid for damped modes also.

Despite these improvements equation V-163 must be considered defective. The reasons have appeared in the derivation of the kinetic equation, and we simply make them explicit here. The difficulties are the following:

1. It is not possible to assign a size to the collision term (involving g) in an unstable plasma. The physical reason is that the Bogoluibov hypothesis breaks down, and g does not relax to become a functional of f . Thus the neglect of g is essentially arbitrary.

2. Even if we set $g(t = 0)$ equal to zero, we cannot expect to keep it small for great lengths of time. As noted earlier (page 172) the collision term involving g continues to grow while the system approaches marginal stability ($\gamma \sim 0$). This is significant, for the quasilinear term involving $\exp\left[\int_0^t \gamma_k d\tau\right]$ stops growing. We shall return to this point shortly.

3. In general $f_o(\underset{\sim}{v}, t)$ is not a slowly varying function for small times $t \lesssim \frac{1}{\omega_p}$. This is because a general distribution $f(\underset{\sim}{r}, \underset{\sim}{v}_1, t = 0)$ will evolve rapidly for small times. The volume average of such a rapidly varying function is in general <u>not</u> a slowly varying function. The author does not know of a general way of ensuring that $f(\underset{\sim}{r}, \underset{\sim}{v}, t = 0)$ will evolve in such a way that $f^o(\underset{\sim}{v}, t)$ will be a slowly varying function for small times.

In view of these criticisms it seems more reasonable to consider the inhomogeneity of the system as statistical. In this way we can treat equation V-163 as a special case of the general kinetic equation. The quasilinear equation represents the case in which the initial value term (now called $g(t = 0)$) dominates the evolution of the system for early periods of time. For longer periods of time the f dependent terms determine the behavior of the system. Note in particular that there is no asymptotic state, in the sense of quasilinear theory, for the collision term continues to grow after the system reaches a state or marginal stability.

E. Discussion of Approximations

We consider now the basic assumptions and approximations used in setting up the theory. We proceed by qualitative arguments - a quantitative calculation would amount to an extension of the theory.

The most powerful (and severe) approximation used in this work was the truncation of the infinite set of coupled equations describing the evolution of the system. Two rather distinct regimes are possible, depending on the relative importance of different terms in the kinetic equation.

1. The terms with implicit time dependence dominate those with explicit time dependence $(\exp\left[\int \gamma d\tau\right])$. In this case the usual plasma expansion in $\frac{1}{\Lambda}$ is valid, and the inclusion of higher order effects (h of the BBGKY hierarchy) leads to minor corrections. We have obtained an estimate, valid to order $1/\ell n\Lambda$ for part of these corrections. We have calculated the effect of these corrections on the dielectric function of the plasma, while omitting the correction to the motion of the particles. Thus the resultant equation is not more accurate than if these effects were excluded altogether. On the other hand we have established the correct asymptotic form of the equation. After a few collision times $\left(t \gtrsim \frac{\Lambda}{\omega_p \ell n\Lambda}\right)$ the Lenard-Balescu is the correct equation. If we do not include the correction to the dielectric function coming from higher order effects, we must keep all terms of the kinetic equation. Pure Landau damping acts so slowly that we must keep all terms with explicit time dependence for an indefinite time. The inclusion of collisional damping causes the collective modes to die out on the collisional time scale.

2. The terms with explicit time dependence dominate the kinetic equation. As discussed earlier this tends to be a local effect, with the time dependent terms most important in driving an unstable volume of velocity space. We may consider our truncation of the infinite set of equations in the following light. There is no general criterion for throwing away any higher order terms. All higher order terms may be specified arbitrarily at $t = 0$. Furthermore the long wavelength effects arising from these terms do not die away rapidly $(t \sim \frac{1}{\omega_p})$, and in the case of an unstable plasma they grow. Thus our neglect of h restricts the validity of our equation to the simplest case - the case in which the usual plasma ordering remains valid. Note that the expansion parameter does not have to be $\frac{1}{\Lambda}$. We shall consider this point further when we discuss the significance of the equation.

A second approximation used in this work is the adiabatic hypothesis. We hold f fixed while calculating the behavior of the collision term, and then permit f and g to evolve together on the long $(t > t_{ad})$ time scale. We trust it is clear that this approximation is very good in most cases. The "normal" collision term becomes a functional of f in a time of order $\frac{1}{\omega_p}$. This "normal" collision term causes f to change on a time of order $\frac{\Lambda}{\omega_p \ell n\Lambda}$. The quantitative error introduced by the adiabatic hypothesis is far below other inadequacies of the equation; e.g., the large cutoff. Note that the adiabatic hypothesis does tend to break down when the collective effects (involving $\exp\left[\int \gamma d\tau\right]$) become very large. The collective effect may become large enough to cause f to vary on the time scale $\frac{1}{\omega_p}$. However in this case the neglect of higher order effects (h, etc.) is not justified, and our equation simply does not apply.

F. Significance of the Kinetic Equation for a Uniform Plasma

We now discuss the significance of equation V-142 , with regard to other work on the subject, and also with regard to possible experimental verification.

1. We first compare our equation to the Lenard-Balescu equation. Accordingly we restrict our attention to stable systems. For this case our result is a substantial vindication of the ordinary Fokker-Planck equation. For a stable system, or a marginally stable system, the Fokker-Planck equation will give a satisfactory description of the time evolution of $f(\underset{\sim}{v}, t)$. This is in marked contrast to the Lenard-Balescu equation, which contains a divergent integral for marginally stable systems, yet predicts essentially the same behavior[52] as the Fokker-Planck equation for a system which is sufficiently stable. Thus the ordinary Fokker-Planck equation may be regarded as preferable to the Lenard-Balescu equation. We must qualify this statement slightly, for in general the contribution to the kinetic equation from the initial value term $g(t = 0)$ may be large, even though the system is stable. In this case neither the Lenard-Balescu nor the Fokker-Planck equation is adequate.

2. In the general case of the unstable plasma, equation V-142 is qualitatively different from others which have been derived, so we confine the discussion to general comments.

An outstanding difficulty with the kinetic equation is the appearance of terms dependent on the initial value of the correlation function $g(t = 0)$. However desirable the goal of an equation for f involving only f itself, this simply cannot be done. In general one might consider a truly statistical treatment, in which one would calculate $g(t = 0)$, give $f(t = 0)$, by direct averaging over Γ space. On the other hand it is not clear how one might choose the ensemble in Γ space to correspond to a given physical system. The value of $g(t = 0)$ is certainly dependent on the manner in which the system was made unstable. A possible choice will be mentioned when we consider the possiblity of experimental verification.

One feature of the kinetic equation has apparently not been recognized. This is the fact that the collision term continues to grow as the system approaches marginal stability. This is in marked contrast to the usual "quasilinear type" collision terms, which stop growing as the system approaches marginal stability, because $\exp\left[\int_0^t \gamma d\tau\right] \to$ (constant) as $\gamma(\tau) \to 0$. The significance of this difference is considerable. Quasilinear theorists are forced to calculate perturbation corrections (mode coupling) to their lowest order equation, for the lowest order equation only drives the system to a marginally stable state. The mode coupling terms are needed to drive the system on to positive stability. On the other hand the collision term of equation V-142 does not lead to an asymptotic state which is marginally stable. Thus one is not forced to invoke mode coupling (h) in order to produce positive stability. Again it is not clear how unstable a system one may permit, while neglecting mode coupling. The conflict is the result is assumptions regarding two different variables. In the case of quasilinear theory one considers the effect of g as given by an ordinary collision frequency, say $\frac{\omega_p \ell n \Lambda}{\Lambda}$. Since Λ is typically of the order of 10^6 , one passes to the limit $\Lambda = \infty$, and neglects g altogether. We trust that is is clear this is incorrect, for g does not have an effective collision frequency in an unstable system. On the other hand one may construct theories for $\Lambda = \infty$, granted their physical application is limited.

However, there is another set of quantitities which one may choose arbitrarily. The initial values $g(t = 0)$, $h(t = 0)$, etc. may be considered very large at $t = 0$. In that case the initial value terms (including $h(t = 0)$) may drive the system to positive stability before the terms of equation V-142 become relevant. A single comment is in order - one may <u>not</u> start the initial value terms off at thermal level $\sim \frac{1}{\Lambda}$, and neglect those growing terms of equation V-142 which depend only on f .

The entire situation is confused by the fact that many authors apparently believe that the Vlasov equation drives a system toward equilibrium, despite the fact that $\frac{dH}{dt} = 0$ for a spatially uniform system described by the Vlasov equation. (H is given by the usual $\int f \ell n f$). The domain of validity of equation V-142 versus the quasilinear theories which include mode coupling, is still to be demonstrated.

3. We consider now the possibility of experimental verification of equation V-142. From earlier remarks we conclude that there is little

point in trying to verify the equation for the case of a stable plasma. We cannot exceed the accuracy of the ordinary Fokker-Planck equation until the short range behavior is cleared up satisfactorily. Along this line Kihara et al[53,54,55] have done calcultions on transport coefficients.

The problem of an unstable plasma is rather different. The buildup and decay of energy in collective fluctuations may well be observable in several interesting cases:

a) A double stream (or a single stream into a background plasma) experiment.

b) Plasma in a strong electric field. Here our equation is not strictly valid. However if the electric field is not too strong the only significant effect on the collision term will be a modification of the growth and damping rate of the coherent fluctuations. (A perturbative corrective proportional to $\langle E \rangle$ should be added to the dielectric function. This gives the dielectric function a preferred direction tending to destabilize electron fluctuations traveling in the direction of electron drift, and ion wave fluctuations traveling in the direction of ion drift).

In either case we are faced with a difficulty: what do we choose for the initial value term? The most plausible suggestion is a "sudden" approximation. We take a stable (possibly equilibrium) plasma, and suddenly shoot in a beam, or turn on an electric field. By sudden we mean in a time short compared to the time for collective fluctuations to grow or decay appreciably, according to the Vlasov equation. We may hope that the initial value term is given by the equilibrium value, so that the collective fluctuations will build up from a "thermal" level.

The hypothesis seems plausible, though it lacks theoretical justification. We need a short time equation which can describe the brief interval during which the beam (or the field) is turned on. It is quite possible that g may change very rapidly as the experiment begins. Nevertheless the sudden approximation seems worth a try.

G. Conclusion

We have used the elegant formal methods of Dupree, and the computational procedures suggested by Landau, to derive a kinetic equation for a homogeneous coulomb plasma. The result is free from mathematical difficulties, valid for both stable and unstable plasmas, and has the expected form in the asymptotic (large time) limit. The equation has been shown to satisfy the usual physical laws which are demanded by basic physical concepts. The result is then generalized to include the effect of a uniform magnetic field.

The essential limitation of the work is the truncation of the infinite set of equations which describe the plasma motion. In the case of the stable plasma this truncation is ordinarily an excellent quantitative approximation. For the highly unstable plasma the approximation is not always justified. The region of validity is determined by the effective size of h, the three particle correlation function.

We have corrected several formal difficulties with the usual quasilinear equation, while raising a number of questions about the significance of the entire theory. It is apparent that the basic assumptions of quasilinear theory need to be considered in the light of the kinetic equation described above.

A collision term valid to order $1/\ln\Lambda$ has been obtained for small amplitude waves in a uniform plasma. This result generalizes the ordinary Fokker-Planck equation from the domain $0 \leq \omega << \omega_p$, $0 \leq k << k_d$ to the domain $0 \leq \omega << \Lambda\omega_p$, $0 \leq k \gtrsim k_d$.

ACKNOWLEDGEMENTS

The author wishes to thank his advisor, Dr. Allan Kaufman, for discussions concerning this work and for helpful criticism of the text. Conversations with Dr. David Sachs are much appreciated.

This work was supported by the Atomic Energy Commission, and by the National Science Foundation.

APPENDIX A

In this section we demonstrate the equivalence in physical content of the BBGKY hierarchy and the set of equations developed by Dupree and Klimontovich (KD). The two schemes are obviously not identical; however, by simple mathematical operations we can go from one to the other.

The BBGKY hierarchy consists of equations for $\mathfrak{F}_n(X_1, X_2 \cdots X_n)$, where $\mathfrak{F}_n$ is the probability distribution for n <u>different</u> particles, and $X_i = \{\vec{r}_i, \vec{v}_i\}$. We now relate these functions to similar quantities which may be constructed according to the Dupree scheme:

$$f(X_1) = \langle F(X_1) \rangle \equiv \langle \frac{1}{n} \sum_i \delta(X_1 - X_i) \rangle = \mathfrak{F}_1(X_1) . \quad \text{(A-1)}$$

Thus the first terms are identical. For the second term

$$F_2(X_1, X_2) = \langle F(X_1) F(X_2) \rangle$$

$$= \langle \frac{1}{n^2} \sum_i \delta(X_1 - X_i) \sum_j \delta(X_2 - X_j) \rangle$$

$$= \langle \frac{1}{n^2} \sum_{i=j} \delta(X_1 - X_i) \delta(X_2 - X_j) \rangle + \langle \frac{1}{n} \sum_i \delta(X_1 - X_i) \rangle \frac{1}{n} \delta(X_1 - X_2) \rangle$$

$$= \mathfrak{F}_2 (X_1, X_2) + \mathfrak{F}_1(X_1) \Delta(X_1, X_2) \quad \text{(A-2)}$$

where we have defined a new function $\Delta(X_1, X_2) = \frac{1}{n} \delta(X_1 - X_2)$.

It is convenient to devise a notation which recognizes the possible identity of particles in different spaces $\{X_i, X_j\}$. We adopt the following:

1. The symbol $\overbrace{AB}$ between two functions A and B means that the particles involved are to be taken as identical.
2. In an expression containing this symbol no other functions are to be taken as identical unless they are appropriately marked by their own $\overbrace{\quad}$.
3. We shall use curly brackets to indicate an expression with all particles different.

Thus

$$F(X_1)\,F(X_2) \;=\; \{F(X_1)\,F(X_2)\} + F(\overbrace{X_1)\,F(X_2}) \quad . \qquad \text{(A-3)}$$

We proceed directly to the three-body function

$$
\begin{aligned}
F_3(X_1,X_2,X_3) &= \langle F(X_1)\,F(X_2)\,F(X_3)\rangle \\
&= \langle\{F(X_1)\,F(X_2)\,F(X_3)\}\rangle + \langle F(\overbrace{X_1)\,F(X_2})\,F(X_3)\rangle \\
&+ \langle F(\overbrace{X_1)F(X_2)F(X_3})\rangle + \langle F(X_1)F(\overbrace{X_2)F(X_3})\rangle + \langle F(\overbrace{X_1)F(}\overbrace{X_2)F(X_3})\rangle \\
&= \mathscr{F}_3(X_1,X_2,X_3) + \mathscr{F}_2(X_1,X_3)\,\Delta(X_1,X_2) \\
&+ \mathscr{F}_2(X_1,X_2)\,\Delta(X_1,X_3) + \mathscr{F}_2(X_1,X_2)\,\Delta(X_2,X_3) + f(X_1) \\
&\times \;\Delta(X_1,X_2)\,\Delta(X_2,X_3) \quad . \qquad \text{(A-4)}
\end{aligned}
$$

By now the generalization is obvious.

$$F_n(X_1,X_2,\cdots X_n) = \langle F(X_1)\,F(X_2)\cdots F(X_n)\rangle$$

$$= \mathcal{F}_n(X_1,X_2,\cdots X_n) + \sum_{\alpha\neq\beta} \mathcal{F}_{n-1}(X_1,X_2\cdots X_{\alpha-1},X_{\alpha+1}\cdots X_\beta\cdots X_n)$$

$$\times\Delta(X_\alpha,X_\beta)$$

$$+ \sum \cdots + \mathcal{F}_1(X_1)\,\Delta(X_1,X_2)\cdots\Delta(X_{n-1},X_n)\ . \qquad \text{(A-5)}$$

It is clear that the functions $\mathcal{F}_n$ differ from those of the BBGKY in that they are singular. The singularities result from allowing a given particle to appear in different spaces $\{X_i\}$, $\{X_j\}$.

Since the Dupree-Klimontovich system involves equation for $\langle\delta f(X_1)\,\delta f(X_2)\cdots\delta f(X_n)\rangle$ we now consider the relation of these quantities to those of the BBGKY hierarchy: f,g,h$\cdots$. Since $F(X_i) = f(X_i) + \delta f(X_i)$

$$F_2(X_1,X_2) = \langle[f(X_1)+\delta f(X_1)]\,[f(X_2)+\delta f(X_2)]\rangle = f(X_1)f(X_2)+\langle\delta f(X_1)\delta f(X_2)\rangle\ . \qquad \text{(A-6)}$$

The terms $f\langle\delta f\rangle$ vanish because $\langle\delta f\rangle = 0$. Comparing with equation A-2, and using the fact that $\mathcal{F}_2(X_1,X_2) = f(X_1)f(X_2) + g(X_1,X_2)$ we have

$$\langle\delta f(X_1)\delta f(X_2)\rangle = g(X_1,X_2) + f(X_1)\,\Delta(X_1,X_2)\ . \qquad \text{(A-7)}$$

For the next quantity $\langle \delta f(X_1)\, \delta f(X_2)\, \delta f(X_3)\rangle$, we calculate

$$\langle F(X_1)F(X_2)F(X_3)\rangle = f(X_1)f(X_2)f(X_3) + f(X_1)\langle \delta f(X_2)\delta f(X_3)\rangle$$

$$+ f(X_2)\, \langle \delta f(X_1)\delta f(X_3)\rangle + f(X_3)\, \langle \delta f(X_1)\delta f(X_2)\rangle$$

$$+ \langle \delta f(X_1)\, \delta f(X_2)\, \delta f(X_3)\rangle \quad . \qquad \text{(A-8)}$$

We use A-7 for $\langle \delta f \delta f\rangle$, and compare with A-3 to find

$$\langle \delta f(X_1)\delta f(X_2)\delta f(X_3)\rangle = h(X_1,X_2,X_3) + g(X_1,X_2)\, \Delta(X_2,X_3)$$

$$+ g(X_1,X_3)\, \Delta(X_1,X_2) + g(X_2,X_3)\, \Delta(X_1,X_3)$$

$$+ f(X_1)\, \Delta(X_1,X_2)\, \Delta(X_2,X_3) \; . \qquad \text{(A-9)}$$

The notation is simplified if we define H_n, a function of n sets of coordinates $\{X_1, X_2, \cdots X_n\}$, as the sum of all ways of connecting these coordinates by lines so that each coordinate is connected at least once. The lines may represent correlation or a Δ function. If we use a jagged line for correlations, and a straight line for connection by a Δ function, we have

$$H_1(X_1) \Rightarrow \begin{array}{c} X_1 \\ \cdot \\ f(X_1) \end{array}$$

$$H_2(X_1,X_2) \Rightarrow \begin{array}{cc} X_1 & X_2 \\ \multicolumn{2}{c}{g(X_1,X_2)} \end{array} + \begin{array}{cc} X_1 & X_2 \\ \multicolumn{2}{c}{f(X_1)\,\triangle(X_1,X_2)} \end{array}$$

$$H_3(X_1,X_2,X_3) \Rightarrow \begin{array}{ccc} & X_1 & \\ X_2 & & X_3 \\ \multicolumn{3}{c}{h(X_1,X_2,X_3)} \end{array} + \begin{array}{ccc} & X_1 & \\ X_2 & & X_3 \\ \multicolumn{3}{c}{g(X_1,X_2)\triangle(X_2,X_3)} \end{array} + \begin{array}{ccc} & X_1 & \\ X_2 & & X_3 \\ \multicolumn{3}{c}{g(X_1,X_3)\triangle(X_1,X_2)} \end{array}$$

$$+ \begin{array}{ccc} & X_1 & \\ X_2 & & X_3 \\ \multicolumn{3}{c}{g(X_2,X_3)\triangle(X_1,X_3)} \end{array} + \begin{array}{ccc} & X_1 & \\ X_2 & & X_3 \\ \multicolumn{3}{c}{f(X_1)\triangle(X_1,X_2)\triangle(X_1,X_3)} \end{array} \quad . \qquad \text{(A-10)}$$

Comparing with equations A-7 and A-9, we see that $\langle \delta f(X_1)\delta f(X_2)\rangle = H_2(X_1,X_2)$ and $\langle \delta f(X_1)\delta f(X_2)\delta f(X_3)\rangle = H_3(X_1,X_2,X_3)$. These first two terms are a little misleading however. We write

$$F_n(X_1,X_2\cdots X_n) = \langle (F + \delta f)(F + \delta f)\cdots(F + \delta f)\rangle$$

$$= \sum_{\substack{\text{all} \\ \text{combinations}}} ffff\cdots f\ \langle \delta f\delta f\cdots\delta f\rangle$$

$$= \sum_{\substack{\text{all} \\ \text{combinations}}} H_1H_1\cdots H_1\ \langle \delta f\delta f\cdots\delta f\rangle$$

$$= \sum_{\substack{a+b+c\cdots=n \\ \text{each coordinate appears once.}}} H_a(X_i,X_j,\cdots)H_bH_c\cdots \qquad \text{(A-11)}$$

Where the cluster expansion is the same as that customarily used in the BBGKY hierarchy except that Δ functions may also be used for linking coordinates.

To find $\langle \delta f(X_1) \cdots \delta f(X_n) \rangle$, we eliminate all terms involving H_1 from each side; therefore,

$$\langle \delta f(X_1)\, \delta f(X_2) \cdots \delta f(X_n) \rangle = \sum_{\substack{\text{cluster} \\ a,b,c \cdots > 1}} H_a H_b H_c \cdots \qquad \text{(A-12)}$$

Having established the relation between the functions of the two systems, we turn to the equations for these quantities. Equivalence to all orders could presumably be demonstrated by the use of efficient notation and a great deal of labor. It is hard to imagine a less enlightening task. We spare the reader (and the author) by confining ourselves to the equivalence between the equations for f and g of the BBGKY hierarchy, and those for f and $\langle \delta f \delta f \rangle$ of the Dupree-Klimontovich hierarchy.

The BBGKY equations for f and g of a single species have been written out previously. (Section II-B). We consider now the Dupree equations. The equation for f is :

$$\left(\frac{d}{dt} + \underset{\sim}{v}_1 \cdot \nabla_1\right) f(X_1) + \frac{q}{m} \langle \underset{\sim}{E} \rangle \cdot \frac{df}{d\underset{\sim}{v}_1} + \frac{q}{m} \frac{d}{d\underset{\sim}{v}_1} \cdot \langle \{ \delta \underset{\sim}{E} \delta f \} \rangle = 0 \; . \qquad \text{(A-13)}$$

Recall that the curly brackets $\{\ \}$ mean that the particles of δf and δE may not be identical.

In the notation of BBGKY

$$q\langle E\rangle = -n\int dX_3\, \nabla_1 \phi_{13}\, f(X_3)$$

$$q\langle\{\delta \underset{\sim}{E} \delta f\rangle = -n\int \nabla_1\, \phi_{13}\, \langle\{\delta f(X_1)\delta f(X_3)\}\rangle\, dX_3\,. \qquad \text{(A-14)}$$

Since $\langle\{\delta f\delta f\}\rangle = g$ we have

$$\left(\frac{d}{dt} + \underset{\sim}{v}_1\cdot\nabla_1\right)f(X_1) - \frac{n}{m}\int\nabla_1\phi_{13}\, f(X_3)dX_3\cdot\frac{df(X_1)}{d\underset{\sim}{v}_1} - \frac{n}{m}\int\nabla_1\phi_{13}\cdot\frac{d}{dv_1}$$

$$\times\, g(X_1,X_3)\, dX_3 = 0\,. \qquad \text{(A-15)}$$

so that A-13 is also the first equation of the BBGKY .

The second Dupree equation, like all higher equations is constructed by using the equation for the fluctuation quantities:

$$\left(\frac{d}{dt} + \underset{\sim}{v}_1\cdot\nabla_1\right)\delta f(X_1) + \frac{q}{m}\langle\underset{\sim}{E}\rangle\cdot\frac{d\delta f}{d\underset{\sim}{v}_1} + \frac{q}{m}\,\delta\underset{\sim}{E}\cdot\frac{df}{d\underset{\sim}{v}_1} =$$

$$\frac{q}{m}\,\frac{d}{d\underset{\sim}{v}_1}\cdot[\langle\{\delta\underset{\sim}{E}\delta f\}\rangle - \{\delta\underset{\sim}{E}\delta f\}]\quad. \qquad \text{(A-16)}$$

We create an equation for $\langle\delta f(X_1)\,\delta f(X_2)\rangle$ by multiplying A-16 by $\delta f(X_2)$, writing a similar equation with subscripts 1 and 2 interchanged, adding and averaging. Note that $\langle\delta f(X_1)\,\delta f(X_2)\rangle$ contains a singular term $f(X_1)\Delta(X_1,X_2)$. Since it is convenient to write functions to the left and right of differential operators, we use the notation $\frac{d}{dt}\left(f(X_1)\Delta(X_1,X_2)\right) = \Delta(X_1,X_2)\,\frac{df(X_1)}{dt} + \Delta(X_1,X_2)\,\frac{df(X_2)}{dt}$, etc. The equation we consider is:

$$\overset{(1)}{\left(\frac{d}{dt} + \underset{\sim}{v}_1\cdot\nabla_1 + \underset{\sim}{v}_2\cdot\nabla_2\right)\langle\delta f(X_1)\delta f(X_2)\rangle} + \frac{q}{m}\overset{(2)}{\left[\langle\underset{\sim}{E}\rangle\cdot\frac{d}{d\underset{\sim}{v}_1}\langle\delta f(X_1)\delta f(X_2)\rangle + (1\leftrightarrow 2)\right]}$$

$$+ \frac{q}{m}\overset{(3)}{\left[\langle\delta\underset{\sim}{E}\delta f(X_2)\rangle\cdot\frac{df}{d\underset{\sim}{v}_1} + (1\leftrightarrow 2)\right]}$$

$$+ \frac{q}{m}\overset{(4)}{\left[\langle\{\delta\underset{\sim}{E}\cdot\frac{d\delta f}{d\underset{\sim}{v}_1}\}\,\delta f(X_2)\rangle + (1\leftrightarrow 2)\right]} = 0 \quad . \qquad \text{(A-17)}$$

We consider each numbered term separately:

1 $$\left(\frac{d}{dt} + \underset{\sim}{v}_1\cdot\nabla_1 + \underset{\sim}{v}_2\cdot\nabla_2\right)\langle\delta f\delta f\rangle = \left(\frac{d}{dt} + \underset{\sim}{v}_1\cdot\nabla_1 + \underset{\sim}{v}_2\cdot\nabla_2\right)\left[g(X_1,X_2) + f(X_1)\,\nabla(X_1,X_2)\right],$$

2 $$\frac{q}{m}\langle\underset{\sim}{E}\rangle\cdot\frac{d}{d\underset{\sim}{v}_1}\langle\delta f\delta f\rangle = -\frac{n}{m}\int dX_3\,\nabla_1\phi_{13}\,f(X_3)\cdot\frac{d}{d\underset{\sim}{v}_1}\,g(X_1,X_2)$$
$$-\frac{n}{m}\int dX_3\,\nabla_1\phi_{13}\,f(X_3)\,\nabla(X_1,X_2)\,\frac{df(X_1)}{d\underset{\sim}{v}_1}\quad ,$$

3 $$\frac{q}{m}\langle\delta\underset{\sim}{E}\delta f(X_2)\rangle\cdot\frac{df}{d\underset{\sim}{v}_1} = -\frac{n}{m}\int dX_3\,\nabla_1\phi_{13}\cdot\frac{df}{d\underset{\sim}{v}_1}\,g(X_1,X_3)$$
$$-\frac{n}{m}\int dX_3\,\nabla_1\phi_{13}\cdot\frac{df}{d\underset{\sim}{v}_1}\,f(X_2)\,\Delta(X_2,X_3)$$
$$= -\frac{n}{m}\int dX_3\,\nabla_1\phi_{13}\cdot\frac{df}{d\underset{\sim}{v}_1}\,g(X_1,X_3) - \frac{1}{m}\nabla_1\phi_{12}\cdot\frac{df}{d\underset{\sim}{v}_1}\,f(X_2)\quad . \qquad \text{(A-18)}$$

In term 4 we must remember there is no self force on a particle.

Thus

$$4 \quad \frac{q}{m}\langle\{\delta \underset{\sim}{E}\cdot\frac{d}{d\underset{\sim}{v}_1}\,\delta f(X_1)\}\,\delta f(X_2)\rangle = \frac{q}{m}\langle\{\delta \underset{\sim}{E}\cdot\frac{d}{d\underset{\sim}{v}_1}\,\delta f(X_1)\,\delta f(X_2)\}\rangle$$

$$+\ \frac{q}{m}\langle\delta \overline{\underset{\sim}{E}\cdot\frac{d}{d\underset{\sim}{v}_1}\,\delta f(X_1)\,\delta f}(X_2)\rangle + \frac{q}{m}\langle\delta \underset{\sim}{E}\cdot\frac{d}{d\underset{\sim}{v}_1}\,\delta f(\overline{X_1)\,\delta f(X_2})\rangle$$

$$= -\frac{n}{m}\int dX_3\,\nabla_1\phi_{13}\cdot\frac{d}{d\underset{\sim}{v}_1}\;h(X_1,X_2,X_3) - \frac{1}{m}\int dX_3\nabla_1\phi_{13}\cdot\frac{d}{d\underset{\sim}{v}_1}\;g(X_1,X_3)\Delta(X_1,X_2)$$

$$-\frac{n}{m}\nabla_1\phi_{12}\cdot\frac{d}{d\underset{\sim}{v}_1}\;g(X_1,X_2)\ . \qquad \text{(A-19)}$$

We substitute these quantities into equation A-17. The singular terms may be eliminated by using the first equation (which is the same in either system), A-15. Multiply A-15 by $\Delta(1,2)$ take $1 \leftrightarrow 2$, and add. When the resulting equation is subtracted from A-17, thereby eliminating all singular terms, the result is the equation for g of the BBGKY hierarchy.

As a final step we consider the ordering, i.e., estimation of relative order of magnitude of terms, used by Dupree. Note that Dupree does not order the equation for δf ! In particular $\delta f \delta E$ may <u>not</u> be treated as higher order than δf in equation A-16. The only quantities which may be ordered are average quantities, such as $\langle\delta f\delta f\rangle$.

Dupree orders equations in terms of the quantities H_n previously defined. As noted $f(X_1) = H_1(X_1)$; $\langle\delta f(X_1)\delta f(X_2) = H_2(X_1,X_2)$; $\langle\delta f(X_1)\delta f(X_2)\delta f(X_3)\rangle = H_3$ while further terms are more complicated. Dupree orders these functions by treating H_n as order η^{n-1} , where η is small. Thus $f = \mathcal{O}(1)$, $\langle\delta f\delta f\rangle = \mathcal{O}(\eta)$, $\langle\delta f\delta f\delta f\rangle = \mathcal{O}(\eta^2)$. We demonstrate that this ordering is equivalent to that used in the BBGKY hierarchy. The singular terms cause no difficulty and are considered as order one while $\frac{1}{n}$ is order η .

In equation A-13 for $f(X_1)$ the lowest order $\mathcal{O}(1)$ terms represent the Vlasov equation, the correction $\mathcal{O}(\eta)$ is the collision term. This agrees with the ordering of the hierarchy, where g is treated as small compared to ff --see equation A-15. Equation A-17 is an equation for $H_2(X_1, X_2) = \mathcal{O}(\eta)$ involving two terms in H_3 , with several of the Δ functions omitted. When a term involving H_3 is written out explicitly (see equation A-19) one finds a singular term, and two non-singular terms. The former is the higher order $\mathcal{O}(\eta)$ term in the equation for $f(X_1)\,\Delta(X_1, X_2)$. The remaining two terms are those customarily dropped in the BBGKY equation for as being higher order.

It is hardly surprising that the sets of equations and approximation methods are the same, since they relate to a common physical process. Sometimes one set of equations is more convenient to work with, sometimes the other. It is best to be familiar with both.

APPENDIX B. Integrals over K

From the analysis of the collision term for small amplitude waves in a plasma (page 23) and for fluctuations in a plasma (page 95) we obtain the following integrals:

$$\underset{\approx}{\mathcal{J}}_1 = \frac{1}{\pi} \int \frac{d\underset{\sim}{K}\underset{\sim}{K}\underset{\sim}{K}}{K^4} \delta(\omega - \underset{\sim}{k}\cdot\underset{\sim}{v} + \underset{\sim}{K}\cdot[\underset{\sim}{v} - \underset{\sim}{v}_1]) \tag{B-1}$$

$$\underset{\approx}{\mathcal{J}}_2 = \frac{1}{\pi} \int \frac{d\underset{\sim}{K}\underset{\sim}{K}(\underset{\sim}{K}-\underset{\sim}{k})}{k^2(K-k)^2} \delta(\omega - \underset{\sim}{k}\cdot\underset{\sim}{v} + K\cdot[\underset{\sim}{v} - \underset{\sim}{v}_1]) \quad . \tag{B-2}$$

We consider $\mathcal{J}_1$ first and set $\omega - \underset{\sim}{k}\cdot\underset{\sim}{v} = a$ and $\underset{\sim}{v}_1 - \underset{\sim}{v} = \underset{\sim}{b}$. The k integration runs from $|k| = k_d$ to $|k| = k_0$. The only vector quantity available is $\underset{\sim}{b}$, so that we may write

$$\underset{\approx}{\mathcal{J}}_1 = A \underset{\approx}{I} + B \underset{\sim}{b}\underset{\sim}{b} \tag{B-3}$$

where $\underset{\approx}{I}$ is the unit tensor and A and B are to be determined. We dot equation B-3 twice with $\underset{\sim}{b}$ to find

$$A\, b^2 + B\, b^4 = \frac{1}{\pi} \int \frac{d\underset{\sim}{K}}{K^4} \delta(a - \underset{\sim}{K}\cdot\underset{\sim}{b}) (\underset{\sim}{K}\cdot\underset{\sim}{b})^2$$

$$\approx 0 \quad . \tag{B-4}$$

The $\underset{\sim}{K}$ integration converges rapidly for large K , so that the resulting terms are order $1/\ell n\Lambda$ compared to those we shall keep. This error is of the same order as that produced by the ordinary cutoffs.

We now take the trace of B-3, and use the result of B-4 ($a = -b^2 B$) to find

$$A = \frac{1}{2\pi} \int \frac{d\underset{\sim}{K}}{K^2} \, \delta(a - \underset{\sim}{K} \cdot \underset{\sim}{b}) \, . \tag{B-5}$$

We introduce cylindrical coordinates with the axis parallel to $\underset{\sim}{b}$. The angular integration yields 2π, while the integration over $K \parallel b$ yields three values, depending on the relative magnitudes of a and b. For $\frac{a}{b} < k_d$ we have

$$A = \frac{1}{b} \int_{[k_d^2 - (a/b)^2]^{\frac{1}{2}}}^{[k_0^2 - (a/b)^2]^{\frac{1}{2}}} k_\perp \, dk_\perp \, [k_\perp^2 + (a/b)^2]^{-1} = \frac{1}{b} \, \ell n \Lambda \, . \tag{B-6}$$

For $k_d < \frac{a}{b} < k_0$ we have

$$A = \int_0^{[k_0^2 - (a/b)^2]^{\frac{1}{2}}} k_\perp \, dk_\perp \, [k_\perp^2 + (a/b)^2]^{-1} = \frac{1}{b} \, \ell n \, \frac{k_0 |b|}{|a|} \, . \tag{B-7}$$

For $\frac{a}{b} > k_0$ the region of integration vanishes and we find $a = 0$. The final result is that given in the text

$$\underset{\approx}{\mathcal{J}}_1 = \frac{1}{|\underset{\sim}{v}_1 - \underset{\sim}{v}|} \ell n\Lambda \left[\underset{\approx}{I} - \frac{(\underset{\sim}{v}_1 - \underset{\sim}{v})(\underset{\sim}{v}_1 - \underset{\sim}{v})}{|\underset{\sim}{v}_1 - \underset{\sim}{v}|^2} \right] \quad \text{for} \quad k_d > \frac{|\omega - \underset{\sim}{k}\cdot\underset{\sim}{v}|}{|\underset{\sim}{v}_1 - \underset{\sim}{v}|}$$

$$= \frac{1}{|\underset{\sim}{v}_1 - \underset{\sim}{v}|} \ell n\left(\frac{k_0 |\underset{\sim}{v}_1 - \underset{\sim}{v}|}{|\omega - \underset{\sim}{k}\cdot\underset{\sim}{v}|} \right) \left[\underset{\approx}{I} - \frac{(\underset{\sim}{v}_1 - \underset{\sim}{v})(\underset{\sim}{v}_1 - \underset{\sim}{v})}{|\underset{\sim}{v}_1 - \underset{\sim}{v}|^2} \right]$$

$$\text{for} \quad k_d < \frac{|\omega - \underset{\sim}{k}\cdot\underset{\sim}{v}|}{|\underset{\sim}{v}_1 - \underset{\sim}{v}|} < k_0$$

$$= 0 \quad \text{for} \quad \frac{|\omega - \underset{\sim}{k}\cdot\underset{\sim}{v}|}{|\underset{\sim}{v}_1 - \underset{\sim}{v}|} > k_0 \quad . \tag{B-8}$$

We now consider $\underset{\approx}{\mathcal{J}}_2$. Because $k < k_d \ll k_0$ we could simply neglect the k dependence outside the delta function, in which case we would have $\underset{\approx}{\mathcal{J}}_1 = \underset{\approx}{\mathcal{J}}_2$. Instead we displace the origin by the vector $\underset{\sim}{k}$, then neglect the k dependence outside the delta function.

$$\underset{\approx}{\mathcal{J}}_2 \simeq \frac{1}{\pi} \int \frac{d\underset{\approx}{KKK}}{K^4} \delta(\omega - \underset{\sim}{k}\cdot\underset{\sim}{v}_1 + \underset{\sim}{K} \cdot [v_1 - \underset{\sim}{v}]) \quad . \tag{B-9}$$

We make this choice so that symmetry is preserved between $\underset{\sim}{v}$ and $\underset{\sim}{v}_1$ in the collision term. This has the effect of yielding exact conservation laws for $f_1(\underset{\sim}{k}, \underset{\sim}{v}, \omega)$. Of course this is not necessary, for the collision term is only approximate. It may perfectly well lead to small errors in the conservation laws, and elsewhere. $\underset{\approx}{\mathcal{J}}_2$ may be evaluated now by the methods used in obtaining $\underset{\approx}{\mathcal{J}}_1$. The difference between $\underset{\approx}{\mathcal{J}}_1$ and $\underset{\approx}{\mathcal{J}}_2$ is of order $1/\ell n\Lambda$, and hence is not significant.

FOOTNOTES AND REFERENCES

1. R. Balescu, Phys. Fluids, 3, 52 (1960).
2. A. Lenard, Ann. Phys., 3, 390 (1960).
3. R. Balescu, Statistical Mechanics of Charged Particles, (Interscience, New York, 1963), Appendix 9 .
4. T. H. Dupree, Phys. Fluids, 6, 1714 (1963).
5. D. C. Montgomery and D. A. Tidman, Plasma Kinetic Theory, (McGraw Hill, New York, 1964), Chapter 2.
6. L. Landau, J. Phys. (U.S.S.R.) 10, 25 (1946).
7. S. Chandrasekhar, Rev. Mod. Phys. 15, 31 (1943).
8. M. Rosenbluth, W. M. MacDonald, and D. L. Judd, Phys. Rev. 107, 1 (1957).
9. E. Frieman and D. Book, Phys. Fluids 6, 1700 (1963).
10. J. Weinstock, Phys. Rev. 133, 3A, A673, (1964).
11. T. Kihara and O. Aono, Journal Phys. Soc. Japan 18, 837 (1963).
12. N. Bogoliubov, Studies in Statistical Mechanics, V.1, (North Holland Publishing Co., Amsterdam, 1962).
13. M. Born and H. A. Green, A General Theory of Liquids, (Cambridge University Press, London, 1949).
14. H. Green, The Molecular Theory of Fluids, (Interscience, New York, 1952).
15. J. G. Kirkwood, Journal Chem. Phys. 14, 180 (1946) and 15, 72 (1947).
16. J. Yvon, Actualities Scientifiques et Industrielles, Hermann and Cie, Paris (1935).
17. R. Guernsey, Phys. Fluids 5, 322, (1962).

18. N. Rostoker, Phys. Fluids 7, 479, 491 (1963).

19. E. Frieman and P. Rutherford, Phys. Fluids 6, 1139 (1963).

20. E. Frieman and P. Rutherford, Ann. Phys. 28, 142 (1964).

21. E. Frieman, Journal of Math. Phys. 4, 410 (1963).

22. H. Berk, Phys. Fluids 17, 257 (1964).

23. V. Silin, JETP 11, 1277 (1960).

24. Y. L. Klimontovich, JETP 6, 753 (1958).

25. Y. L. Klimontovich, JETP 10, 524 (1960).

26. T. Apostol, Mathematical Analysis, (Addison Wesley, Reading, Mass. 1959) p. 472.

27. T. Apostol, Ibid, p. 473.

28. R. Aamodt and W. Drummond, Phys. Fluids 7, 1817 (1964).

29. E. Frieman, S. Bodner, and P. Rutherford, Phys. Fluids 6, 1299 (1963).

30. B. Kadomtsev, Plasma Turbulence, (Academic Press, London, 1965), Chapter I.

31. E. Frieman and P. Rutherford, Ann. Phys. 24, 142 (1964).

32. G. Backus, J. Math. Phys. 1, 178 (1960).

33. G. Sandri, Ann. Phys. 24, 332 (1963).

34. G. Sandri, Ibid., p 335, 351 .

35. D. C. Montgomery and D. A. Tidman, Ibid, Chapter 3 .

36. D. C. Montgomery and D. A. Tidman, Ibid, p. 45 .

37. D. C. Montgomery and D. A. Tidman, Ibid, Appendix A .

38. R. Guernsey, Ph.D. Thesis, University of Michigan, (1960).

39. D. C. Montgomery and D. A. Tidman, Ibid., p. 264.

40. S. Misawa, Phys. Rev. Letters 13, 337 (1964) .

41. N. Rostoker and T. O'Neil, Phys. Fluids 8, 1109 (1965).

42. E. Frieman, Journal of Math Phys. 4, 410 (1963).

43. N. Van Kampen, Physica 21, 949 (1955).

44. T. H. Stix, The Theory of Plasma Waves, (McGraw Hill, New York, 1962), p. 137.

45. E. Frieman and P. Rutherford, Phys. Fluids 6, 1139 (1963).

46. R. Balescu, Ibid, p. 431.

47. K. Nishikawa and Y. Osaka, Progress in Theoretical Phys. 33, 402 (1965).

48. B. Abraham, J. Math. Physics 6, 630 (1965).

49. I. Bernstein, Phys. Rev. 109, 90 (1958).

50. N. Rostoker, Phys. Fluids 3, 922 (1960).

51. P. Morse and H. Feshbach, Methods of Theoretical Physics, (McGraw Hill, New York, 1953), p. 620.

52. A. Dolinsky, Phys. Fluids 8, 436 (1965).

53. T. Kihara, O. Aono, and Y. Itakawa, J. Phys. Soc. Japan 18, 1043 (1963).

54. Y. Itakawa, J. Phys. Soc. of Japan 18, 1499 (1963).

55. N. Honda, J. Phys. Soc. of Japan 19, 1935 (1964).