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Entanglement and Null States in Quantum Gravity

A dissertation submitted in partial satisfaction
of the requirements for the degree
Doctor of Philosophy in Physics

by

John Golden Gardiner

2022

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ABSTRACT OF THE DISSERTATION

Entanglement and Null States in Quantum Gravity

by

John Golden Gardiner

Doctor of Philosophy in Physics

University of California, Los Angeles, 2022

Professor Per Kraus, Co-Chair

Professor Michael Gutperle, Co-Chair

A standard insight of the AdS/CFT correspondence is that some aspects of the geometry of a bulk state are encoded in the entanglement structure of its dual boundary state. As entanglement is not a linear quantum observable, this means that geometry in a quantum theory of gravity should likewise not be a linear observable.

This is seemingly in tension with an understanding of quantum gravity as a gravity path integral summing over geometries, which inherently treats geometry as a linear observable. The authors of [1] explore this tension in the context of a simple model of a gravity path integral where it is resolved by the appearance of null states, or in other words geometries that are equal to superpositions of other geometries. In Chapter 1 of this work, we extend this 2d topological gravity model of [1] to have as its bulk action any open/closed TQFT obeying Atiyah's axioms. We describe the Hilbert spaces of these more general theories that remain after null states have been accounted for. The holographic duals of these topological gravity models are ensembles of 1d topological theories with random dimension. These holographic interpretations of our gravity models require projecting out negative-norm states from the baby universe Hilbert space, which in [1] was achieved by adding a nonlocal boundary term to the bulk action. We describe the analogous solution in the framework of a TQFT with defects by coupling the boundaries of the gravity models to auxiliary 2d TQFTs in a non-

gravitational (i.e. fixed topology) region. The gravity model is then holographically dual to an ensemble of boundary conditions in an open/closed TQFT without gravity.

In Chapter 2 we explore linear dependencies between certain states with simple geometric duals: states made up of n copies of a thermofield double state and the states obtained from this one by permuting the n right hand sides. We derive expressions for the maximum fidelity between one such state and a linear combination of the others, and see that this fidelity approaches 1 as the number n of black holes increases. We also consider the possibility of obtaining a single thermofield double state as the partial trace of linear combinations of such states with topologies with no connection between the two untraced sides. We derive lower bounds for the fidelity between the thermofield double state and such partial traces and comment on the conceptual implications of the existence of such states.

The dissertation of John Golden Gardiner is approved.

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CHAPTER 1

2d TQFTs and baby universes

1.1 Chapter Introduction

Recent work has re-emphasized the role of the Euclidean path integral in quantum gravity. Specifically we have seen the revitalization of the old idea that quantum gravity should be described by a sum over spacetime topologies and an integration over all metrics within each topology. Making sense of such a path integral is, of course, difficult or impossible in general, so this course of action has mostly focused on two dimensions. Despite the notoriously hard problem of classifying topological manifolds in 3d, some attempts at path integrating over a subset of 3d manifolds (often Seifert manifolds) have also been made [4, 5, 6].

A helpful point of view into this sum over topologies of spacetimes is afforded by the old ideas of baby universes [7, 8, 9], where one thinks of non-trivial topologies as the emission and re-absorption of baby-universes. In Euclidean signature, this emission/re-absorption process takes the form of a spacetime wormhole.¹ Specifically, [7] considers a formalism in which the Hamiltonian of the universe contains couplings between the fundamental fields and “baby universe field” operators A_i which describe the creation and annihilation of different types of baby universes. Because couplings between the A_i and other fields appear in the Hamiltonian, tracing over the number and types of baby universes in existence (facts that are presumably unknowable to an observer in the parent universe) produces an effective evolution that is non-unitary. Crucially however, the baby universe operators A_i commute with each

¹Spacetime wormholes differ from the usual Einstein-Rosen wormholes in that the latter are topological connections between otherwise distant regions of *space*, whereas the former are additional topological connections between regions of spacetime.

other and with the Hamiltonian. This renders the non-unitarity relatively benign, as *within* any eigenspace of A_i the evolution remains unitary. This means ignorance of the number and types of baby universes does not, in fact, lead to observable decoherence. Rather, such ignorance is just operationally equivalent to ignorance of some number of coupling constants in the theory describing parent universe physics. In other words, the physics of the parent universe is described by a statistical ensemble of unitary theories. The theories in this ensemble are parametrized by the simultaneous eigenstates $|\alpha\rangle$ of the A_i , which are called alpha-states. The unknown coupling constants of parent universe physics are the eigenvalues of A_i , also called alpha-parameters.

The baby universe idea and the appearance of ensembles has a counterpart in the context of holography. In this context, where partition functions of a boundary theory are dual to gravity in the bulk, there is a question of whether the calculation of a boundary partition function with disconnected spacetime components should include bulk geometries that connect those components [10]. However natural a sum including connected topologies may otherwise be, it destroys the manifest factorization between spacetime components of the boundary partition function. That is to say, a rule including contributions from connected geometries cannot a priori be expected to give boundary partition functions satisfying $Z[M \sqcup N] = Z[M]Z[N]$ for disconnected spacetimes M and N . One way to proceed is to reinterpret the boundary partition functions appearing in this holographic context as ensemble averages of partition functions, where bulk gravity is then interpreted as dual to a statistical ensemble of boundary theories.² This leads to what you could call a new entry in the holographic dictionary:

$$\begin{array}{ccc}
 \text{ensemble of non-gravitational} & & \text{Euclidean gravity path integral} \\
 \text{boundary theories on} & \Leftrightarrow & \text{with contributions connecting} \\
 \text{disconnected boundaries} & & \text{disconnected boundary components} \\
 & & (1.1)
 \end{array}$$

²Equally, we could turn around the logic and stipulate factorization as a *criterion* for any well-behaved quantum theory of gravity, thus declaring most generic gravity path integrals to be in the so-called “swamp-land” [11].

which now has evidence from a number of lines of study, including the JT-gravity [12] and SYK model [13] correspondence [14, 15], replica wormhole calculations of the Page curve [16, 17], and recent work suggesting a possible correspondence between 3d gravity and ensembles of 2d conformal field theories [18, 4, 5, 19].

All these different ideas merge in [1], wherein the authors demonstrate, via logic reminiscent of the older baby universe picture, how ensembles of boundary theories can naturally emerge from gravity path integrals, and connect with the replica wormhole discussions of the Page curve. Important to the present work, the authors of [1] introduce and analyze a simple topological model of a gravity path integral as an explicit example of these ideas. Their gravity model is a sum over 2d surfaces weighted by a topological action. Subsequently, the authors of [20] extend this model to include surfaces with spin structures. The present work picks up from where [1], and in a certain sense [20] left off, by considering more general topological bulk theories.

The rest of the paper is organized as follows. In section 1.1.1, we review the simple topological model described in [1] which this work can be seen as generalizing. In section 1.2, we consider a gravity path integral constructed from Dijkgraaf-Witten theory in two dimensions. We find (not unexpectedly [21]) that the correlators of boundary insertion operators factorize between sectors described by irreducible representations of G . What's more, the boundary dual theory can be interpreted as a 1d topological theory with global symmetry group G whose Hilbert space is a random representation of G . Specifically, the number of times a copy of an irreducible representation of G appears in the Hilbert space is given by a random integer, with integers for different irreducible representations chosen independently. In section 1.3, we consider a gravity path integral constructed from any 2d TQFT as defined by Atiyah's axioms [22]. We show that the Hilbert space of the boundary theory is the direct sum of sectors labeled by the eigenstates of the TQFT handle operator with the number of dimensions in each sector chosen independently from a different Poisson distribution. In section 1.4 we generalize further by considering open/closed TQFTs, that is to say TQFTs

with boundary. We describe a gravity path integral constructed from open/closed TQFTs, again using Dijkgraaf-Witten as a first example. Similar to the closed case, for open/closed TQFTs we find that correlation functions factorize between sectors labeled by eigenstates of the handle operator. We discuss the interpretation of the gravity path integral in terms of a boundary ensemble theory. We encounter the same difficulty encountered by [1] with defining a theory without negative-norm baby universe states, a difficulty for which they proposed the solution of adding a nonlocal boundary term to the bulk action. We discuss this difficulty in section 1.5, reframing the solution in the language of 2d TQFTs with defects. Along the way, we introduce generalized boundary observables representing non-gravitational regions coupled to the gravity region with fluctuating topology. The gravity region then is dual to an ensemble of boundary conditions for an open/closed TQFT. Besides motivating a solution to the negative-norm states in terms of a defect line, this picture also motivates an understanding of the additional alpha-parameters associated with having end-of-the-world branes (or more general boundary conditions). In section 1.6 we discuss directions for future work, including the aim of studying more realistic models of gravity, as well as the possibility of exploring aspects of holography such as bulk reconstruction in the simple setting of topological theories. In appendix A, we review a state sum formulation of 2d Dijkgraaf-Witten theory with defects, which is sufficient to calculate all the Dijkgraaf-Witten theory results used in the paper.

1.1.1 Review of a simple gravity model

The model of [1] describes a gravity path integral built from a sum over spacetime topology. The authors consider 2d orientable spacetimes without metric or other geometric structure and an action that is (nearly) just the Euler characteristic. It is, in fact the Euler characteristic together with an additional term depending on the number of boundaries. Specifically, for a 2d manifold with genus g and n boundaries they assign the action

$$S[M_{g,n}] = S_0 \chi(M_{g,n}) + n S_\partial = S_0(2 - 2g) + n(S_\partial - S_0) , \quad (1.2)$$

where χ is the Euler characteristic $2 - 2g - n$ and S_0 and S_∂ are parameters of the theory. The gravity path integral for spacetime with fixed n number of circle boundary components then becomes

$$Z^{\text{QG}}[n \text{ boundaries}] = \sum_{\substack{M \text{ s.t. } \partial M \text{ has} \\ n \text{ components}}} \mu(M) e^{S_0 \chi(M) + S_\partial n} . \quad (1.3)$$

The measure $\mu(m)$ here is $\mu(M) = \frac{1}{\prod_g m_g!}$, where m_g is the number of components with genus g that are not connected to any boundary. This accounts for the residual gauge symmetries permuting identical connected components of M .³ We point out here that for the particular choice $S_\partial = S_0$, the number of boundaries n affects the combinatorics both in the sum over topologies and in the measure μ , but it has no effect on the action.

We should now make two comments. First, by analogy with a correlation function being computed as a sum over Feynman diagrams with fixed external legs, we notate the output of the gravity path integral (a sum over spacetimes with fixed boundaries) as a correlation function, where the fixed boundaries play the role of operator insertions. So for example the gravity path integral over manifolds with n circle boundary components will be notated

$$\langle \widehat{Z}^n \rangle \equiv Z^{\text{QG}}[n \text{ boundaries}] , \quad (1.4)$$

so that the operator \widehat{Z} simply denotes the insertion of an additional circle boundary component. Second, we point out that the action (1.2), at least with S_∂ set to zero, produces the partition function of a (particularly simple) TQFT, so that the gravity path integral is simply a sum over TQFT partition functions:

$$Z^{\text{QG}}[n \text{ boundaries}] = \sum_{\substack{M \text{ s.t. } \partial M \text{ has} \\ n \text{ components}}} \mu(M) Z^{\text{TQFT}}[M] . \quad (1.5)$$

We mention this, as our aim in this paper is to allow Z^{TQFT} to be a more general TQFT partition function.

The values of the correlators $\langle \widehat{Z}^n \rangle$ can be gathered in the generating function $\langle e^{u\widehat{Z}} \rangle$. The generating function for the connected correlators is simply $\log \langle e^{u\widehat{Z}} \rangle$, which evaluates

³Permutations that involve components with boundaries are of course not gauge redundancies.

to

$$\log \langle e^{u\widehat{Z}} \rangle = \sum_{n=0}^{\infty} \frac{u^n}{n!} \langle \widehat{Z}^n \rangle_{\text{conn}} = \lambda e^{u e^{S_{\partial}-S_0}}, \quad (1.6)$$

where $\lambda = \sum_g e^{S_0(2-2g)} = e^{2S_0}/(1 - e^{-2S_0})$ is the value of the connected vacuum correlator. From the resulting expression for $\langle e^{u\widehat{Z}} \rangle$, the correlators $\langle \widehat{Z}^n \rangle$ can be extracted

$$\langle \widehat{Z}^n \rangle = e^\lambda B_n(\lambda) e^{(S_{\partial}-S_0)n}, \quad (1.7)$$

where B_n here is the n -th Touchard (or Bell) polynomial. This can be written equivalently as

$$\langle \widehat{Z}^n \rangle = \sum_{d=0}^{\infty} \frac{\lambda^d}{d!} (d e^{S_{\partial}-S_0})^n. \quad (1.8)$$

The normalized correlators are thus

$$\langle \widehat{Z}^n \rangle / \langle \mathbf{1} \rangle = e^{-\lambda} \sum_{d=0}^{\infty} \frac{\lambda^d}{d!} (d e^{S_{\partial}-S_0})^n = \sum_{d=0}^{\infty} p_d(\lambda) (d e^{S_{\partial}-S_0})^n, \quad (1.9)$$

where $p_d(\lambda) = e^{-\lambda} \frac{\lambda^d}{d!}$ is the Poisson distribution with mean λ .

The gravitational path integral also gives a way to define the Hilbert space of quantum gravity. The construction of the Hilbert space begins by picking the vacuum, often called the Hartle-Hawking state $|\text{HH}\rangle$, to be the empty set thought of as a 1d manifold. This is the state of no boundaries. The rest of the Hilbert space is constructed by application of the \widehat{Z} operators which insert boundaries, $\widehat{Z}^n |\text{HH}\rangle = |\widehat{Z}^n\rangle$. The gravitational path integral then provides the means to calculate inner products of any two such states, thereby defining the Hilbert space.

The authors of [1] also consider the above model with the addition of so-called end-of-the-world (EofW) branes. These are boundaries on which spacetime ends, but unlike the \widehat{Z} boundaries we have discussed above, they are taken to be dynamical, in that the gravity path integral includes a sum over all configurations of such branes. By contrast the \widehat{Z} boundaries are fixed boundaries and can be considered observables of the gravity path integral. With the presence of EofW branes, on which spacetime can end, we now have, in addition to the \widehat{Z} fixed boundaries, the possibility of another type of fixed boundary. This is an interval

that is bounded on both sides by EofW branes.⁴ These intervals bounded by branes are not dynamical (only the EofW branes are taken to be dynamical). The gravity path integral now includes manifolds whose boundary components are of three different types, fixed circles (inserted by operators \widehat{Z}), circles with EofW brane boundary conditions running completely around the circle, and circles made from alternating fixed and EofW brane intervals. Given some number of fixed circle and interval boundaries, the gravity path integral will only include manifolds whose fixed boundaries match those given, but will include a sum over all the different possible ways of consistently configuring the EofW branes.

Like the circle boundary components inserted by \widehat{Z} , we can associate an operator with the inclusion of an additional fixed interval boundary, which we will call \widehat{S} . The model of [1] considers the possibility of having some number K of “flavors” of EofW branes, index by $a = 1, \dots, K$. These different types of EofW brane differ only in their label $a = 1, \dots, K$, with the rule that only EofW branes with the same label can be connected together. The fixed intervals \widehat{S} now have their endpoints labeled by the flavor of EofW brane on which they end, giving operators \widehat{S}_{ab} . Correlation functions of the gravity model include insertions of both \widehat{Z} and \widehat{S}_{ab} operators. For example:

$$\langle \text{HH} | \widehat{Z}^n \widehat{S}_{ab} \widehat{S}_{ba} \widehat{S}_{cd} \widehat{S}_{de} \widehat{S}_{ec} | \text{HH} \rangle . \quad (1.10)$$

The operators \widehat{Z} and \widehat{S}_{ab} all commute within these Euclidean path integral correlators. Given a configuration of fixed boundaries, there are many ways we can partition them into “future” and “past” boundaries, and reinterpret the gravity path integral correlator as an inner product of states in the baby universe Hilbert space. For example we can write (1.10) variously as $\langle \widehat{Z}^n \widehat{S}_{ab} \widehat{S}_{ba} | \widehat{S}_{cd} \widehat{S}_{de} \widehat{S}_{ec} \rangle$, $\langle \widehat{Z}^{n-m} \widehat{S}_{ab} | \widehat{Z}^m \widehat{S}_{ba} \widehat{S}_{cd} \widehat{S}_{de} \widehat{S}_{ec} \rangle$, $\langle \widehat{Z}^n \widehat{S}_{ab} \widehat{S}_{ba} \widehat{S}_{cd} \widehat{S}_{de} | \widehat{S}_{ec} \rangle$, etc. The baby universe Hilbert space is spanned by states of the form

$$\left| \widehat{Z}^n \prod_{a,b} \widehat{S}_{ab}^{n_{ab}} \right\rangle \equiv \widehat{Z}^n \prod_{a,b} \widehat{S}_{ab}^{n_{ab}} | \text{HH} \rangle . \quad (1.11)$$

⁴In this work, to match the language of open/closed TQFTs we will sometimes call these interval boundaries “open sector” boundaries, and we call circle boundary components like \widehat{Z} “closed sector” boundaries.

From AdS/CFT we recall that the non-normalizable asymptotic modes of fields in AdS specify sources in the boundary CFT, and a bulk path integral with given boundaries ∂M is dual to the appropriate CFT partition function on that boundary. Inspired by this, we might expect that the boundary conditions for the 2d gravity path integral correspond to partition functions of a 1d theory. Or, invoking the idea of (1.1), we might lower our expectations only slightly to include the possibility that boundary conditions in this case are dual to an average of partition functions in an ensemble of 1d theories. Indeed, the authors of [1] find a dual description of this sort for the gravity path integral (1.3). From this point of view, the correlator $\langle \widehat{Z} \rangle$ is no longer a correlator, but the average value of a partition function Z in a 1d topological theory. Each additional boundary component \widehat{Z} is another copy of this boundary partition function, so that the correlators $\langle \widehat{Z}^n \rangle$ with multiple fixed boundaries probe higher moments, $\langle Z^n \rangle$, in the ensemble distribution.

The only parameter in topological quantum mechanics is the dimension d of the Hilbert space, so an ensemble of 1d topological theories is a probability distribution for d . We immediately run into a problem, though. For a single theory within the ensemble

$$Z = \text{tr}_{\mathcal{H}} \mathbf{1} = d, \tag{1.12}$$

but (1.9) suggests that \widehat{Z} takes the value $de^{S_{\partial}-S_0}$ for nonnegative integer d . Expecting a dual boundary interpretation of \widehat{Z} as a partition function in a 1d topological theory, thus forces us to set $S_{\partial} = S_0$ in (1.9).

Alternatively (taking a perspective more in line with that taken in the rest of this paper) we can forgo adding the term S_{∂} to the boundary and instead take the holographic map to be rescaled by some factor $e^{S_{\partial}}$, so that

$$e^{S_{\partial}} \widehat{Z} \leftrightarrow \text{tr}(\mathbf{1}) = Z^{\text{boundary}}. \tag{1.13}$$

Then a choice of rescaling given by $e^{S_{\partial}} = e^{S_0}$ gives a sensible boundary dual, whose theories all have integer dimensional Hilbert spaces. This perspective has the downside, of course, of making the choice of notation \widehat{Z} for the boundary insertion operators something of a misnomer.

The boundary interpretation just introduced motivates a conceptually useful basis for our baby universe Hilbert space: the basis of eigenvectors of \widehat{Z} ,

$$\widehat{Z}|\alpha\rangle = \alpha|\alpha\rangle . \tag{1.14}$$

The eigenstates $|\alpha\rangle$, called alpha-states, are of course orthogonal: $\langle\alpha'|\alpha\rangle \sim \delta_{\alpha',\alpha}$. But they have one very special property. The boundary theories in our ensemble are characterized by the values they give to the observables \widehat{Z} , so the set of alpha states is precisely the sample space of boundary theories in our ensemble. The probability distribution over the theories in our ensemble can be extracted from the overlap between a given alpha-state and the Hartle-Hawking state:

$$p(\alpha) = \frac{\langle\text{HH}|\alpha\rangle \langle\alpha|\text{HH}\rangle}{\langle\alpha|\alpha\rangle \langle\text{HH}|\text{HH}\rangle} . \tag{1.15}$$

Whereas obtaining a sensible boundary interpretation for a theory without end-of-the-world branes necessitated only a judicious choice of rescaling (1.13) for the \widehat{Z} operators, a potentially more serious problem manifests when we include end-of-the-world branes and their attendant \widehat{S}_{ab} operators. In the 1d dual theory, the interval boundary insertions \widehat{S}_{ab} have a natural interpretation as inner products of states induced by boundary conditions a and b at the endpoints of the interval. So within a particular boundary theory in the ensemble, the operators \widehat{S}_{ab} should take as values the components of a K by K positive semidefinite Hermitian matrix M . A boundary ensemble will be given by a joint probability distribution over the dimension d and the matrix M . Viewing correlators of \widehat{Z} and \widehat{S}_{ab} operators as averages in such an ensemble, they are moments of the ensemble probability distribution. In particular, the generating function for normalized correlators, which [1] calculate to be

$$\left\langle e^{iu\widehat{Z} + \sum_{a,b}^K it_{ab}\widehat{S}_{ab}} \right\rangle / \langle\mathbb{1}\rangle = \exp\left(\lambda \left(\frac{e^{iu}}{\det(\mathbb{1} - it)}\right)^{e^{S_\partial - S_0}} - \lambda\right) , \tag{1.16}$$

should be the inverse Fourier transform of the probability density $p(d, M)$ defining the ensemble. As we have described above, the \widehat{Z} operators take values $de^{S_\partial - S_0}$ with probability

$p_\lambda(d) = e^{-\lambda} \lambda^d / d!$, giving an expansion of (1.16) as

$$\left\langle e^{iu\widehat{Z} + \sum_{a,b}^K it_{ab} \widehat{S}_{ab}} \right\rangle / \langle \mathbf{1} \rangle = \sum_{d=0}^{\infty} p_\lambda(d) e^{iude^{S_\partial - S_0}} \left\langle e^{\sum_{a,b}^K it_{ab} \widehat{S}_{ab}} \right\rangle_{Z=d e^{S_\partial - S_0}} . \quad (1.17)$$

For a given d , the residual probability distribution $p_d(M)$ over the matrices M will then be the Fourier transform of the generating function

$$\left\langle e^{\sum_{a,b}^K it_{ab} \widehat{S}_{ab}} \right\rangle_{Z=d e^{S_\partial - S_0}} = \det(\mathbf{1} - it)^{-de^{S_\partial - S_0}} . \quad (1.18)$$

Unfortunately, only for certain values does the above generating function have an inverse Fourier transform that can be interpreted as a valid probability distribution [23].⁵ Specifically, the exponent $de^{S_\partial - S_0}$ must lie in the set $\{0, 1, 2, \dots, K-1\} \cup [K-1, \infty)$, where, as a reminder, K is the number of flavors of end-of-the-world brane included in the theory. As d runs over all nonnegative integers, the factor $e^{S_\partial - S_0}$ must lie in the set $\{0, 1, 2, \dots, K-1\} \cup [K-1, \infty)$. A natural choice is to take $S_\partial = S_0$.

The above argument highlights an important point. It need not be the case that a theory without factorization has a description as an ensemble. As in the situation where $S_\partial = 0$, the correlation functions of a non-factorizing theory are not necessarily the moments of a probability distribution. Failure to have an ensemble description is linked with the presence of negative-norm states in the baby universe Hilbert space. To see this, consider a gravity theory with boundary insertion operators \widehat{Z}_i . The theories in the ensemble will be parametrized by values these operators \widehat{Z}_i take. Assume for simplicity, these values are continuous, real, and independent. Then we can formally construct the alpha-states as

$$|\alpha\rangle = \int \prod_i \left(\frac{du_i}{2\pi} \right) e^{i \sum_i u_i (\widehat{Z}_i - \alpha_i)} |\text{HH}\rangle , \quad (1.19)$$

where α_i are the values which Z_i takes in the theory described by $|\alpha\rangle$. From this, the inner

⁵In which case, it is known as the Wishart distribution.

product of two alpha states is

$$\begin{aligned}
\langle \alpha' | \alpha \rangle &= \int \prod_i \left(\frac{du'_i}{2\pi} \frac{du_i}{2\pi} \right) e^{-i \sum_i u'_i (\alpha_i - \alpha'_i)} \left\langle e^{i \sum_i u_i (\widehat{Z}_i - \alpha_i)} \right\rangle \\
&= \delta(\alpha' - \alpha) \int \prod_i \left(\frac{du_i}{2\pi} \right) e^{-i \sum_i u_i \alpha_i} \left\langle e^{i \sum_i u_i \widehat{Z}_i} \right\rangle \\
&= \delta(\alpha' - \alpha) \langle \mathbb{1} \rangle p(\alpha) .
\end{aligned} \tag{1.20}$$

The last equality comes from viewing the correlators of \widehat{Z}_i as moments of a putative probability distribution $p(\alpha)$. Viewed thus, the generating function $\left\langle e^{i \sum_i u_i \widehat{Z}_i} \right\rangle$ is simply the inverse Fourier transform of $p(\alpha)$. This equation (1.20) suggests something about theories that fail to be ensembles. If the distribution $p(\alpha)$ takes negative values, this means both that the theory does not have an ensemble description and that the baby universe Hilbert space contains negative-norm states.

Returning again to the gravity model with end of the world branes, one could complain that in order to cure the boundary interpretation we have ruined the locality of the bulk TQFT theory. Indeed, the action for $S_\partial \neq 0$ no longer depends only on the Euler characteristic of the manifold, so it is no longer consistent with cuttings and gluings of the spacetime. Alternatively, if we insist on locality, the S_∂ term can be interpreted as the contribution of local degrees of freedom associated to the boundaries, both brane and fixed. In that case, however, a question arises of whether or not we should consider additional \widehat{S} operators corresponding to these additional degrees of freedom. Doing so would be equivalent to considering the theory with $S_\partial = 0$ just with more flavors of end-of-the-world brane, so the problem would arise again. The problem seems to require that the degrees of freedom propagate, unprobed, along the boundaries. We refer the reader to the discussion of this boundary term and its meaning in [1]. We address the problem as it shows up in our case in section 1.5, where we offer a somewhat different description for these degrees of freedom, and some speculation on their meaning.

The topological action of the model described in this section is, in fact, that of a 2d TQFT with a one-dimensional (closed sector) Hilbert space. In some sense it describes the simplest possible 2d TQFT. In what follows, we will analyze gravity models built from more

complicated 2d TQFTs. We will find that much of this analysis can be reduced to that of the simpler one-dimensional TQFT.

1.2 2d Dijkgraaf-Witten theory

The simple model of a gravity path integral from [1] can be generalized to include any topological action in the bulk. As a first example, we will examine 2d Dijkgraaf-Witten theory, a topological gauge theory, as our bulk theory and construct a gravity path integral from that. We will find a dual interpretation of the gravity path integral in terms of a one-dimensional ensemble theory whose Hilbert space is a random representation of the gauge group. We also find that the correlation functions of boundary insertion operators factorize between the irreducible representations of the gauge group, similar to the results in [21]. In section 1.3 we will see that analogous features hold in the case of more general 2d TQFTs. Before describing the gravity path integral, however, we will first briefly review Dijkgraaf-Witten theory and present the results of Dijkgraaf-Witten theory in two-dimensions that will be relevant to our construction.

1.2.1 Review of Dijkgraaf-Witten theory

Dijkgraaf-Witten theory is a topological gauge theory with finite symmetry group G [24, 25]. The path integral is given as a sum over G -principal bundles on the spacetime manifold. Given a connected, manifold without boundary M , let \mathcal{C}_M be the set of G -principal bundles on M . A G -principal bundle on M can be identified with a homomorphism from the fundamental group $\pi_1(M, x)$ to the group G , where x is some chosen basepoint in M . So we will take \mathcal{C}_M to be the set $\text{Hom}(\pi_1(M, x), G)$. Identified this way, some of the principal bundles can be related to each other via residual gauge symmetries. Specifically, a gauge transformation $g \in G$ with support over all of M will act on a bundle $\phi : \pi_1(M, x) \rightarrow G$ via conjugation, like so: $\phi(\cdot) \mapsto g \phi(\cdot) g^{-1}$. The gauge invariant path integral must take these gauge symmetries into account, and is thus over the space of G -principal bundles \mathcal{C}_M ,

orbifolded by this action of G . We'll denote this orbifolded space by $\overline{\mathcal{C}_M}$.

The measure over $\overline{\mathcal{C}_M}$ induced by this orbifolding will weight bundles inversely to the size of their stabilizer subgroup under the action of G . Without any further weighting of the bundles beyond this, the sum over $\overline{\mathcal{C}_M}$ gives the “untwisted” version of Dijkgraaf-Witten theory. In that case the partition function for M is [25]

$$Z^{\text{DW}}[M] = \sum_{\phi \in \overline{\mathcal{C}_M}} \frac{1}{|\text{Stab}(\phi)|} = \frac{|\mathcal{C}_M|}{|G|}. \quad (1.21)$$

The numerator $|\mathcal{C}_M|$ is $|\text{Hom}(\pi_1(M, x), G)|$, the number of homomorphisms from $\pi_1(M, x)$ to G . When M_g is the connected, closed, oriented surface of genus g , this count is given by a result known as Mednykh's formula:

$$|\mathcal{C}_{M_g}| = |G| \sum_q \left(\frac{d_q}{|G|} \right)^{2-2g}, \quad (1.22)$$

where q labels the irreducible representations of G , and d_q are the dimensions of each irreducible representation. (See [26] and references therein.) This gives the Dijkgraaf-Witten partition function of M_g as

$$Z^{\text{DW}}[M_g] = \frac{|\mathcal{C}_{M_g}|}{|G|} = \sum_q \left(\frac{d_q}{|G|} \right)^{2-2g}. \quad (1.23)$$

The above partition function is for a closed surface. If our manifold is a surface with boundaries, the path integral is a sum over G -principal bundles that satisfy given boundary conditions. Specifically each boundary component is a circle with boundary condition given by a conjugacy class of G , representing the holonomy around that circle. Let $\mathcal{C}_{M_{g,n}}(k_1, \dots, k_n)$ denote the set of G -principal bundles on the genus g surface with n boundary components having holonomy boundary conditions k_1, \dots, k_n respectively. The path integral is again over the orbifolded space $\overline{\mathcal{C}_{M_{g,n}}(k_1, \dots, k_n)}$, and comes out to $\text{vol}(\overline{\mathcal{C}_{M_{g,n}}(k_1, \dots, k_n)}) = |\mathcal{C}_{M_{g,n}}(k_1, \dots, k_n)| / |G|$.

A G -principal bundle on a surface $M_{g,n}$ with boundaries is still a choice of homomorphism from $\pi_1(M_{g,n}, x)$ to G , but now with the restriction that it is compatible with the boundary conditions on $M_{g,n}$ in the following sense: if a path $a_i \in \pi_1(M_{g,n}, x)$ is homologous to the i -th

boundary, the bundle $\phi : \pi_1(M_{g,n}, x) \rightarrow G$ must map a_i to an element of k_i . (Note this notion of compatibility is well-defined, as all paths in π_1 homologous a_i will be conjugates of each other and hence must map to the same conjugacy class in G .) The partition function will be given by the count $|\mathcal{C}_{M_{g,n}}(k_1, \dots, k_n)|$ of such compatible homomorphisms ϕ . Mednykh's formula can be generalized to the case of a connected surface with boundaries as

$$|\mathcal{C}_{M_{g,n}}(k_1, \dots, k_n)| = |G| \sum_q \left(\frac{d_q}{|G|} \right)^{2-2g-n} \prod_i \frac{|k_i|}{|G|} \chi_q(k_i), \quad (1.24)$$

where g is the genus of the surface, k_1, \dots, k_n are the respective boundary conditions of the n boundaries, and $\chi_q(k)$ is the irreducible character q evaluated on an element in k . (See Proposition 1 in [27]. This can also be obtained by first obtaining the result for an $(n + 2g)$ -holed sphere $M_{0,n+2g}$ with given holonomies on the boundaries. This is done by counting maps from $\pi_1(M_{0,n+2g})$, the free group on $n + 2g$ generators, to G that satisfy the boundary constraints. Then one can glue $2g$ of the boundaries together in pairs by summing over boundary conditions.) The path integral on $M_{g,n}$ with boundary conditions k_1, \dots, k_n is thus

$$Z^{DW}[M_{g,n}; k_1, \dots, k_n] = \sum_q \left(\frac{d_q}{|G|} \right)^{2-2g-n} \prod_i \frac{|k_i|}{|G|} \chi_q(k_i). \quad (1.25)$$

We can interpret this path integral on a manifold with boundaries as defining a multilinear map, from the Hilbert space of n circles to \mathbb{C} , that takes the state $|k_1\rangle \otimes \dots \otimes |k_n\rangle$ to the complex number $Z^{DW}[M_{g,n}; k_1, \dots, k_n]$.

Note that the space of states on a circle is evidently spanned by states labeled by conjugacy classes. We can take the path integral $Z^{DW}[M_{0,2}; k_1, k_2]$ as defining a bilinear pairing on this Hilbert space. We get

$$(|k_1\rangle, |k_2\rangle) = Z^{DW}[M_{0,2}; k_1, k_2] = \sum_q \frac{|k_1|}{|G|} \chi_q(k_1) \frac{|k_2|}{|G|} \chi_q(k_2) = \frac{|k_1|}{|G|} \delta_{k_1, k_2^{-1}}. \quad (1.26)$$

Under this pairing the states $|k\rangle$ are evidently linearly independent, but they are not orthogonal. We can switch to a diagonal basis, $|q\rangle \equiv \sum_k \chi_q(k^{-1}) |k\rangle$, where $\chi_q(k^{-1})$ is the character for irreducible representation q evaluated at an element whose inverse is in the conjugacy class k . In this basis labeled by irreps of G , the pairing above is simply $(|q_1\rangle, |q_2\rangle) = \delta_{q_1, q_2}$,

so the $|q\rangle$ are orthogonal.⁶ The partition function for a connected, genus g surface with boundaries labeled (in this irrep basis) by q_1, \dots, q_n is simply

$$Z^{\text{DW}}[M_{g,n}; q_1, \dots, q_n] = \sum_q \left(\frac{d_q}{|G|} \right)^{2-2g-n} \delta_{qq_1 \dots q_n}. \quad (1.27)$$

Note that this partition function for a connected manifold with boundaries evaluates to zero, unless all boundaries are labeled by the same irreducible representation.

In addition to “untwisted” Dijkgraaf-Witten described above, one can also define a “twisted” generalization of Dijkgraaf-Witten by further weighting each G -principal bundle in the path integral by a $U(1)$ -valued characteristic class α of the bundle. This is equivalent to adding a term iS_α to the action satisfying $\alpha[\phi] = e^{iS_\alpha[\phi]}$. So the twisted partition function for a closed, connected manifold M is

$$Z^{\text{DW},\alpha}[M] = \sum_{\phi \in \overline{\mathcal{C}}_M} \frac{1}{|\text{Stab}(\phi)|} \alpha[\phi] = \sum_{\phi \in \overline{\mathcal{C}}_M} \frac{1}{|\text{Stab}(\phi)|} e^{iS_\alpha[\phi]}. \quad (1.28)$$

In what follows, we will consider the untwisted case before discussing in section 1.2.4 how the results are altered in the twisted case.

1.2.2 A Dijkgraaf-Witten gravity path integral

In preparation for defining a gravity path integral, we will include, in addition to the Dijkgraaf-Witten action, the topological action term $S_0\chi(M)$ of the simple theory described in [1]. This will have the effect of suppressing higher genus manifolds in an eventual sum over topology. With this addition to the action, the bulk theory partition function is given by

$$Z^{\text{bulk}}[M_{g,n}; q_1, \dots, q_n] = \sum_q \left(e^{S_0} \frac{d_q}{|G|} \right)^{2-2g-n} \delta_{qq_1 \dots q_n}. \quad (1.29)$$

In the model of [1] the action includes an additional, nonlocal term nS_∂ proportional to the number of boundaries. This can be regarded as either the contribution to the action of some

⁶If we take the $|q\rangle$ basis as a real basis, then complex conjugation induces the antiunitary map $|k\rangle \mapsto |k^{-1}\rangle$ with the interpretation of a reflection. This antiunitary composed with the pairing described above defines an inner product on the Hilbert space, under which the states $|k\rangle$ are orthogonal and the states $|q\rangle$ are orthonormal.

additional degrees of freedom living on the boundary or as a rescaling of boundary insertion operators in the gravity path integral. We won't include this term here and will discuss its meaning and inclusion in section 1.2.3. For now, we take our partition function to be that described above, which is the partition function of a TQFT; in other words, it is local, in the sense of being compatible with cutting and gluing.

A gravity path integral defined from (1.29) will take as input a boundary manifold (so, for a 2d bulk, some number of circles) with specified boundary conditions, and will output the partition function (1.29) summed over all manifolds with the given boundary conditions. Following the notation of [1], we denote the inclusion of a circle with boundary condition k by the operator $\widehat{Z}[k]$. The gravity path integral is

$$\left\langle \widehat{Z}[k_1] \cdots \widehat{Z}[k_n] \right\rangle = \sum_{\substack{M \text{ s.t. } \partial M \\ \text{is } n \text{ circles}}} \mu(M) Z^{\text{bulk}}[M; k_1, \dots, k_n]. \quad (1.30)$$

Where $\mu(M)$ is the appropriate measure, with a factor of $1/m!$ whenever M has m identical closed components.

The connected contribution to the vacuum correlator, $\lambda \equiv \langle \mathbb{1} \rangle_{\text{conn.}} = \log \langle \mathbb{1} \rangle$, is a sum over connected surfaces with no boundary, so in effect a sum over genus:

$$\lambda = \sum_g \sum_q \left(e^{S_0} \frac{d_q}{|G|} \right)^{2-2g} = \sum_q \frac{\left(\frac{e^{S_0} d_q}{|G|} \right)^2}{1 - \left(\frac{e^{S_0} d_q}{|G|} \right)^{-2}} = \sum_q \lambda_q. \quad (1.31)$$

Here we've denoted $\sum_g \left(e^{S_0} d_q / |G| \right)^{2-2g}$ by λ_q . The full vacuum correlator is correspondingly $\langle \mathbb{1} \rangle = e^\lambda = \prod_q e^{\lambda_q}$.

Calculating the correlators of boundary insertion operators will be easier in the basis labeled by irreducible representations. To that end, define operators $\widehat{Z}_q \equiv \sum_k \chi_q(k^{-1}) \widehat{Z}[k]$, corresponding to the TQFT states $|q\rangle$. We can define a generating function for the general correlator, $F(u_q) = \left\langle e^{\sum_q u_q \widehat{Z}_q} \right\rangle$ with chemical potentials u_q for the insertion of each operator \widehat{Z}_q . The logarithm of this generating function will simply be the corresponding generating function for connected correlators

$$\log F(u_q) = \sum_{\dots, n_q, \dots} \prod_q \frac{1}{n_q!} u_q^{n_q} \left\langle \prod_q \widehat{Z}_q^{n_q} \right\rangle_{\text{conn.}}. \quad (1.32)$$

Connected correlators are simple to calculate as the surfaces to be summed over in the corresponding gravity path integral are parametrized by genus. For example,

$$\begin{aligned}
\left\langle \widehat{Z}_{q_1} \cdots \widehat{Z}_{q_n} \right\rangle_{\text{conn.}} &= \sum_g Z^{\text{bulk}}[M_{g,n}; q_1, \dots, q_n] \\
&= \sum_g \sum_q \left(e^{S_0} \frac{d_q}{|G|} \right)^{2-2g-n} \delta_{qq_1 \cdots q_n} \\
&= \lambda_{q_1} \left(e^{S_0} \frac{d_{q_1}}{|G|} \right)^{-n} \delta_{q_1 \cdots q_n}.
\end{aligned} \tag{1.33}$$

Note that this connected correlator evaluates to zero unless all boundaries are labeled by the same irreducible representation. This fact simplifies the resulting expression for $\log F(u_q)$. The correlator with n_q boundaries for each type q will be zero unless no more than one of the n_q is nonzero. So the sum over all possible numbers of boundary n_q for each q reduces to a sum over just one of the n_q , followed by a sum over q . We obtain

$$\log F(u_q) = \sum_q \sum_{n_q} \frac{1}{n_q!} u_q^{n_q} \lambda_q \left(e^{S_0} \frac{d_q}{|G|} \right)^{-n_q} = \sum_q \lambda_q \exp \left(u_q \frac{|G|}{e^{S_0} d_q} \right). \tag{1.34}$$

The final result of this simplification is that the full generating function $F(u_q)$ factorizes between the different labels q :

$$F(\dots, u_q, \dots) = \prod_q e^{\lambda_q \exp(u_q |G| / e^{S_0} d_q)}. \tag{1.35}$$

From (1.35) the full correlators can be extracted. They are

$$\left\langle \prod_q \widehat{Z}_q^{n_q} \right\rangle = \prod_q e^{\lambda_q} B_{n_q}(\lambda_q) \left(\frac{|G|}{e^{S_0} d_q} \right)^{n_q}, \tag{1.36}$$

where B_m denotes the m -th Touchard, or Bell, polynomial. Note the normalized correlation functions have the property of factorizing between boundaries labeled by different irreducible representations:

$$\left\langle \prod_q \widehat{Z}_q^{n_q} \right\rangle / \langle \mathbb{1} \rangle = \prod_q \left(\left\langle \widehat{Z}_q^{n_q} \right\rangle / \langle \mathbb{1} \rangle \right), \tag{1.37}$$

whereas no such factorization holds between boundaries label by the same irrep, e.g. $\left\langle \widehat{Z}_q^{n+m} \right\rangle \approx \left\langle \widehat{Z}_q^n \right\rangle \left\langle \widehat{Z}_q^m \right\rangle$. The correlators for the operators $\widehat{Z}[k]$ can be gotten through the change of basis back from the \widehat{Z}_q operators to the $\widehat{Z}[k]$ operators, namely $\widehat{Z}[k] = \sum_q \frac{|k|}{|G|} \chi_q(k) \widehat{Z}_q$.

1.2.3 Boundary interpretation

The above result (1.36) is analogous to having multiple copies of the model presented in [1]. In fact, each factor labeled by q is equivalent to one copy of the model of [1], where $e^{S_0} \rightarrow e^{S_0 \frac{d_q}{|G|}}$. We will see that our gravity path integral with a Dijkgraaf-Witten bulk likewise has a dual interpretation as a random theory living on the boundary. This will in fact be equally true of any 2d TQFT satisfying Atiyah's axioms, as we will demonstrate in section 1.3. We present Dijkgraaf-Witten theory here as a representative example.

In a standard holographic boundary interpretation the \widehat{Z} operators would get reinterpreted as the partition functions of a one-dimensional theory. In our case, however, this is impossible because the correlators of the \widehat{Z} operators do not factorize. Instead we will look for an ensemble of one-dimensional theories and interpret the correlator $\langle \widehat{Z}[k_1] \cdots \widehat{Z}[k_n] \rangle / \langle \mathbf{1} \rangle$ as the average of the quantity $Z[k_1] \cdots Z[k_n]$. The boundary theories in our ensemble are characterized by the value they assign to each operator $\widehat{Z}[k]$, so the ensemble will be a probability distribution over the space \mathbf{C}^r , where r is the number of irreducible representations of G and where each copy of \mathbf{C} represents the values that one of the $\widehat{Z}[k]$ can take. This makes the correlators $\langle \widehat{Z}[k_1] \cdots \widehat{Z}[k_n] \rangle / \langle \mathbf{1} \rangle$ interpretable as moments of the ensemble probability distribution. The problem of finding the ensemble probability distribution from the correlators (1.36) is thus an instance of the so-called moment problem, wherein one attempts to find a probability distribution from its moments.

If we let $\vec{\alpha} \in \mathbf{C}^r$ index the theories in our ensemble, the probability distribution p_α over the theories should satisfy $\langle \widehat{Z}_{q_1} \cdots \widehat{Z}_{q_n} \rangle / \langle \mathbf{1} \rangle = \int d^r \alpha p_\alpha \alpha_1 \cdots \alpha_r$. In terms of the generating function $F(u_1, \dots, u_r)$ for the correlators,

$$F(u_1, \dots, u_r) / \langle \mathbf{1} \rangle = e^{\sum_q \lambda_q (\exp(u_q |G| / e^{S_0} d_q) - 1)} = \int d^r \alpha p_\alpha e^{\sum_q u_q \alpha_q}. \quad (1.38)$$

We can extract the function p_α by performing a Fourier transform with respect to the variables iu_q . The result is

$$p_\alpha = \prod_q \sum_{N_q=0}^{\infty} \frac{\lambda_q^{N_q}}{N_q! e^{\lambda_q}} \delta \left(\alpha_q - \frac{|G|}{e^{S_0} d_q} N_q \right). \quad (1.39)$$

In other words, each α_q takes the values $\frac{|G|}{e^{S_0 d_q}} N_q$ where the N_q are random integers drawn independently from Poisson distributions with respective means λ_q . Recall that the α_q are the values of Z_q in the different theories in our ensemble, so

$$Z_q = \frac{|G|}{e^{S_0 d_q}} N_q . \quad (1.40)$$

Switching from the Z_q to the $Z[k]$ basis gives

$$Z[k] = \sum_q \frac{|k|}{e^{S_0 d_q}} \chi_q(k) N_q . \quad (1.41)$$

Our bulk theory is topological, so its boundary dual will likewise be topological. As the $\widehat{Z}[k]$ operators represent the insertion of a boundary with holonomy k , we are tempted to interpret $Z[k]$ as a partition function in a one-dimensional topological quantum mechanics theory, with an insertion of a G -symmetry operator with conjugacy class k . That is to say, $Z[k] = \text{tr}(U(g))$, where $U(g)$ is the representation on the Hilbert space of a group element $g \in k$. In fact, this is only nearly so. Looking at (1.41) we see that $Z[k]$ has the form of a trace of an element of k in a representation that has $\frac{|k|}{e^{S_0 d_q}} N_q$ copies of the representation q , for each q . Unfortunately for this interpretation, $\frac{|k|}{e^{S_0 d_q}} N_q$ is not necessarily an integer, which it would have to be to avoid the absurdity of a theory with a fractional number of copies of a representation. One immediate fix would appear to be picking a specific value for S_0 such that $\frac{|G|}{e^{S_0 d_q}} \in \mathbb{Z}$. This is not possible though, as it would ruin the convergence of eq. (1.33) and, what's worse, would render λ_q negative, giving negative probabilities in our ensemble distribution.

On the other hand, going back to the Z_q operators, in light of their definition $Z_q = \sum_k \chi_q(k^{-1}) Z[k]$ a natural interpretation would be for Z_q to be the contribution to the partition function of states with charge q . That is to say, $Z_q = \text{tr}(P_q)$, where P_q is a projection onto states living in copies of irreducible representation q . Again, this is nearly so, but unfortunately $Z_q = \frac{|G|}{e^{S_0 d_q}} N_q$ is not an integer for all N_q . One possible solution is to identify the size of the q -sector, $\text{tr}(P_q)$, with a rescaled operator $e^{S_q} Z_q$ rather than with Z_q , where we choose e^{S_q} so that $e^{S_q} Z_q$ is an integer in every α -state. (We will discuss a possible motivation for this rescaling in section 1.5.) For now, we can interpret this rescaling as a modification

of the expected holographic map $Z_q \leftrightarrow \text{tr}\{P_q\}$ to be $e^{S_q} Z_q = \text{tr}(P_q)$. Similarly, we can to identify $\text{tr}(U(g))$, with an appropriately rescaled operator $e^{S_k} Z[k]$, rather than with $Z[k]$ as is, in order to avoid the situation of having a fractional number of copies of a representation. A choice of rescalings for the \widehat{Z}_q and the $\widehat{Z}[k]$ that avoids noninteger dimensions, that avoids noninteger copies of irreducible representations, and that respects the identity $\text{tr}(U(g)) = \sum_q \text{tr}(P_q) \chi_q(g) / d_q$ is

$$S_k = S_0 + \log(|G| / |k|) \tag{1.42}$$

$$S_q = S_0 + \log d_q . \tag{1.43}$$

These result in a consistent interpretation of the gravity model as a boundary theory with $\frac{|G|}{d_q} N_q \in \mathbb{Z}$ copies of the irreducible representation q .⁷

1.2.4 Twisted Dijkgraaf-Witten

We now turn our attention to twisted Dijkgraaf-Witten theories. After a review of the essential background, we use twisted Dijkgraaf-Witten in our gravity models, and describe their boundary interpretation.

1.2.4.1 Background

The purpose of this subsection is to describe a practical way to compute the partition function of “twisted” 2d Dijkgraaf-Witten. Here we follow the exposition in [21], and refer the reader there for details. See also [28]. We will see that the partition functions of 2d twisted Dijkgraaf-Witten are given as a sum similar to (1.25) but over projective representations of G .

As was briefly mentioned above, the different twisted theories are labeled by characteristic classes of G -principal bundles. The different characteristic classes that describe the possible twisted action terms are classified by elements of $H^2(BG, U(1))$, the second $U(1)$ -valued

⁷This is an integer by the basic result from representation theory that $|G|/d_q \in \mathbb{Z}$ for any irreducible representation.

cohomology classes of the classifying space BG [24, 25]. Specifically, given a class α in $H^2(BG, U(1))$, and viewing a G -principal bundle on a manifold M as a map ϕ from M to BG , consider the pullback of $\phi^*\alpha$ along this map. Evaluating $\phi^*\alpha$ on M gives a phase $\phi^*\alpha(M) = \alpha(\phi(M)) \in U(1)$ for the bundle ϕ . The partition function on M is a sum over inequivalent G -principal bundles ϕ weighted by these phases. See equation (1.28). For simplicity, in what follows we will only consider the particular case where $H^2(BG, U(1)) = \mathbb{Z}_N$.

One simplification afforded by restricting to the case $H^2(BG, U(1)) = \mathbb{Z}_N$ is that all the twisted Dijkgraaf-Witten actions can be described in terms of any characteristic class $\alpha_0 \in H^2(BG, U(1))$ which generates the others, so $H^2(BG, U(1)) = \mathbb{Z}_N = \{\alpha_0^k : k = 0, \dots, N-1\}$. The first part of this section is devoted to the construction of such a α_0 . Then, after eq. (1.46) we consider α_0^k with general k and show how the partition functions for general k become sums over projective representations of G .

We will relate α_0 to the failure of the bundle ϕ to lift to a \tilde{G} -principal bundle when \tilde{G} is a central extension of G by \mathbb{Z}_N . Consider a G -bundle on a surface M described in terms of transition functions g_{ij} which satisfy the triple overlap condition

$$g_{ij}g_{jk}g_{ki} = 1 . \quad (1.44)$$

Picking a lift of each transition function $g_{ij} \mapsto \tilde{g}_{ij}$, we get

$$\tilde{g}_{ij}\tilde{g}_{jk}\tilde{g}_{ki} = \tilde{c}_{ijk} \in \mathbb{Z}_N , \quad (1.45)$$

which for nontrivial c_{ijk} is a violation of the \tilde{G}_S -bundle cocycle condition. Gauge transformations and changes of lift can change \tilde{c}_{ijk} locally but in general there is a global obstruction to removing all such violations. The assignment of an element of \mathbb{Z}_N to each triple intersection, modulo gauge transformations and changes of lift, defines a 2-cocycle

$$[\omega_N] \in H^2(M, \mathbb{Z}_N) , \quad (1.46)$$

which can be paired with the 2-cycle $[M]$ to give an element $\omega = \langle [\omega_N], [M] \rangle \in \mathbb{Z}_N$. Note that the element $\omega \in \mathbb{Z}_N$ depends on the bundle ϕ . The characteristic class $\alpha_0 \in H^2(BG, U(1))$ is

then defined via $\alpha_0(\phi(M)) = e^{2\pi i\omega(\phi)/N}$. In fact, these are the phases by which the Dijkgraaf-Witten theory described by α_0 weights each bundle.

The characteristic class α_0 obtained this way depends on the choice of the central extension \tilde{G} . We would like to choose \tilde{G} so that α_0 is a generating element of $H^2(BG, U(1)) = \mathbb{Z}_N$ so that any Dijkgraaf-Witten action can be obtained as α_0^k for some k . We are also interested in \tilde{G} with the property that all irreducible representations of G , projective or linear, can be lifted to linear representations of \tilde{G} . There is always a central extension of G that satisfies these criteria, namely the Schur covering group of G . The Schur covering group \tilde{G}_S is a central extension

$$1 \rightarrow H_2(G, \mathbb{Z}) \xrightarrow{i} \tilde{G}_S \xrightarrow{\pi} G \rightarrow 1 . \quad (1.47)$$

For finite groups G ,

$$H^2(BG, U(1)) = H^2(G, U(1)) = H_2(G, \mathbb{Z}) . \quad (1.48)$$

Because we are restricting to the case $H^2(BG, U(1)) = \mathbb{Z}_N$, this means the kernel of the quotient map $\pi : \tilde{G}_S \rightarrow G$ is $H_2(G, \mathbb{Z}) = \mathbb{Z}_N$. So \tilde{G}_S is indeed a central extension of G by \mathbb{Z}_N .

For what follows it is convenient to classify the irreducible representations of \tilde{G}_S by how they represent the subgroup $\ker \pi = \mathbb{Z}_N$. Notate the generator of $\ker \pi = \mathbb{Z}_N$ by $e^{2\pi i/N}$. For any irreducible representation \tilde{q} of \tilde{G}_S , the element $e^{2\pi i/N}$ must get represented as

$$U(\tilde{q})(e^{2\pi i/N}) = e^{2\pi i k'(\tilde{q})/N} \mathbf{1} , \quad (1.49)$$

for some $k'(\tilde{q}) \in \{0, 1, \dots, N-1\}$. Refer to the value $k'(\tilde{q})$ for a given \tilde{q} as the N -ality of \tilde{q} . Notice that, of course, a choice of how to represent $e^{2\pi i/N}$ doesn't completely specify \tilde{q} , so that there are many \tilde{q} of the same N -ality. We will see that the partition function of Dijkgraaf-Witten theory with "twist" k' can be written as a sum over representations with N -ality k' .

Bringing everything together, let's calculate the partition function of twisted Dijkgraaf-Witten on a manifold with boundaries. First recall that in the untwisted case (section 1.2.2),

we were simply counting isomorphism classes of G -bundles (without weighting them with phases, i.e. $k = 0$) consistent with conjugacy classes specified on closed boundaries. A gauge transformation keeps us in the same isomorphism class and changes the boundary holonomy only up to conjugacy, so that the counting is well-defined. However, in the twisted case, although the counting is still a well-defined problem, the weighting is not well-defined, when there are boundaries of fixed conjugacy class.

On a closed manifold, a gauge transformation will not change the cohomology class ω_N defined in (1.46). This is not necessarily the case, however on a manifold with boundaries. A gauge transformation that is nonzero at the boundary can nontrivially transform ω_N :

$$\omega_N \rightarrow \omega_N + db \quad (1.50)$$

where b is a 1-cocycle. Then on a manifold with boundaries

$$\delta\omega = \langle db, [M] \rangle = \langle b, \partial[M] \rangle \quad (1.51)$$

is potentially nonzero, causing phase ambiguities in the partition function

$$e^{2\pi i k \omega(\phi)/N} \rightarrow e^{2\pi i k \omega(\phi)/N} e^{2\pi i k \delta\omega(\phi)/N} . \quad (1.52)$$

The presence of these phase ambiguities implies that twisted Dijkgraaf-Witten is not gauge invariant on manifolds with boundary, so on the boundary there has to be a theory with an 't Hooft anomaly. To have an unambiguously defined bulk partition function, we must make an ad hoc choice for those phases. We can do this by picking a lift $U_i \rightarrow \tilde{U}_i$ for the each specified boundary holonomy U_i .

We are now equipped to calculate the partition function of twisted Dijkgraaf-Witten theory on a manifold with boundaries. Remember that for a Riemann surface with genus g and n boundary components, the holonomies have to be such that

$$V_1 W_1 V_1^{-1} W_1^{-1} \cdots V_g W_g V_g^{-1} W_g^{-1} U_1 \cdots U_n = 1_G , \quad (1.53)$$

where g is the genus and U_i are the holonomies on the n boundaries of M . Trying to uplift in a straightforward way would give

$$u = \tilde{V}_1 \tilde{W}_1 \tilde{V}_1^{-1} \tilde{W}_1^{-1} \cdots \tilde{V}_g \tilde{W}_g \tilde{V}_g^{-1} \tilde{W}_g^{-1} \tilde{U}_1 \cdots \tilde{U}_n \in \mathbb{Z}_N , \quad (1.54)$$

which means that this G -bundle can't be uplifted to a \tilde{G}_S -bundle. In such a case u is in fact $u = e^{2\pi i\omega(\phi)/N}$ where ω is that defined earlier. Although this G -bundle doesn't uplift, we could excise a disk and impose the holonomy u^{-1} on this new boundary to create a \tilde{G}_S -bundle, albeit over a manifold with an extra boundary. This gives us a way to count the number of G -bundles that are weighted by the same value ω . Specifically, we get the number of G -bundles on $M_{g;U_1,\dots,U_n}$ with weight ω by (correctly) counting the number of \tilde{G}_S -bundles on $M_{g;\tilde{U}_1,\dots,\tilde{U}_n,u^{-1}}$. Counting correctly here means accounting for the fact that when a G -bundle can be uplifted there are N^{2g+n} possible upliftings. We have

$$Z_G^{(k)}[g; \tilde{U}_1, \dots, \tilde{U}_n] = \sum_{\phi} \frac{e^{2\pi i k \omega(\phi)/N}}{|\text{Stab}(\phi)|} = \sum_{\omega=0}^{N-1} e^{2\pi i k \omega/N} \frac{|\tilde{G}| Z_{\tilde{G}_S}^{(0)}[g; \tilde{U}_1, \dots, \tilde{U}_n, e^{-2\pi i \omega}]}{N^{2g+n-1}} \quad (1.55)$$

where $Z_{\tilde{G}_S}^{(0)}[g; \tilde{U}_1, \dots, \tilde{U}_n, e^{-2\pi i \omega}]$ is the partition function of untwisted Dijkgraaf-Witten theory for \tilde{G}_S on $M_{g;\tilde{U}_1,\dots,\tilde{U}_n,u^{-1}}$. Using our known expression for the partition function of untwisted Dijkgraaf-Witten theory

$$Z_{\tilde{G}_S}^{(0)}[g; \tilde{U}_1, \dots, \tilde{U}_n, u^{-1}] = \sum_{\tilde{q}} \left(\frac{d_{\tilde{q}}}{|\tilde{G}|} \right)^{2-2g-n} \prod_{i=1}^n \left(\frac{|\tilde{U}_i|}{|\tilde{G}|} \chi_{\tilde{q}}(\tilde{U}_i) \right) \frac{|u^{-1}|}{|\tilde{G}|} \chi_{\tilde{q}}(u^{-1}) \quad (1.56)$$

and substituting this in to (1.55) gives the result

$$Z_G^{(k)}[g; \tilde{U}_1, \dots, \tilde{U}_n] = \sum_{\tilde{q}} \delta_{k,k'(\tilde{q})} \left(\frac{d_{\tilde{q}}}{|\tilde{G}|} \right)^{2-2g-n} \prod_{i=1}^n \left(\frac{|\tilde{U}_i|}{|\tilde{G}|} \chi_{\tilde{q}}(\tilde{U}_i) \right) \quad (1.57)$$

after expanding $\chi_{\tilde{q}}(u^{-1}) = d_{\tilde{q}} e^{-2\pi i k'(\tilde{q})\omega/N}$ then simplifying via the identity $\sum_{\omega} e^{2\pi i \omega(k-k')/N} = N \delta_{k,k'}$.

1.2.4.2 The gravity path integral

Since the form of eq. (1.57) is very similar to the untwisted case, the calculations proceed in the same fashion. We define boundary conditions labeled by projective representation \tilde{q} by $\hat{Z}_{\tilde{q}}^{(k)} = \sum_{\tilde{k}} \chi_{\tilde{q}}(\tilde{k}^{-1}) \hat{Z}^{(k)}[\tilde{k}]$, where the sum goes over all conjugacy classes \tilde{k} of \tilde{G}_S . This definition makes sense for any irreducible representation \tilde{q} of \tilde{G}_S , but the representations of the wrong N -ality will give zero in correlators because

$$Z_G^{(k)}[M_{g,n}; \tilde{q}_1, \dots, \tilde{q}_n] = \sum_{\tilde{q}} \delta_{k,k'(\tilde{q})} \left(e^{S_0} \frac{d_{\tilde{q}}}{|\tilde{G}|} \right)^{2-2g-n} \delta_{\tilde{q}\tilde{q}_1 \dots \tilde{q}_n}, \quad (1.58)$$

as can be readily worked out from (1.57). Thus the most general correlator, one with $n_{\tilde{q}}$ insertions of each boundary with conditions \tilde{q} , is

$$\left\langle \prod_{\substack{\tilde{q} \text{ with} \\ N\text{-ality } k}} \widehat{Z}_{\tilde{q}}^{n_{\tilde{q}}} \right\rangle = \prod_{\substack{\tilde{q} \text{ with} \\ N\text{-ality } k}} e^{\lambda_{\tilde{q}}} B_{n_{\tilde{q}}}(\lambda_{\tilde{q}}) \left(\frac{|G|}{e^{S_0} d_{\tilde{q}}} \right)^{n_{\tilde{q}}}, \quad (1.59)$$

with $\lambda_{\tilde{q}}$ defined analogously to λ_q in the untwisted case. From the similarity of these correlators to those of the untwisted case, we immediately recognize the boundary dual of our twisted Dijkgraaf-Witten bulk gravity model. Namely, (after making the rescalings described in section 1.2.3) it is a 1d topological theory whose Hilbert space consists of random numbers of *projective* representations of G that have the correct N -ality. In other words, the boundary dual is an ensemble of 1d topological theories with anomalous global symmetry G .

1.3 General 2d TQFTs

In the previous section we constructed a simple gravity path integral with a bulk action of Dijkgraaf-Witten theory and found a dual interpretation as an ensemble of 1d theories on its boundary. We will now show that a general 2d TQFT bulk theory likewise leads to a model gravity path integral with similar features. To be precise, we will consider here TQFTs as defined by Atiyah's axioms [22] and over the field \mathbf{C} . Such TQFTs are fairly simple. They have finite dimensional Hilbert spaces, and, as we will discuss, can be viewed as a direct sum of theories all with Hilbert space dimension 1. Though they are simple, we speculate that the important features of our analysis will extend appropriately to 2d TQFTs more broadly defined, and perhaps even, in some form, to TQFTs in higher dimensions.

There are many excellent expositions of TQFTs;⁸ we present here only the most basic sketch for those not familiar. In the language of category theory, a 2d TQFT can be defined as a functor (with certain requirements) from the category of 2d cobordisms $\mathbf{Cob}(2)$ to the category of complex vector spaces $\mathbf{Vect}(\mathbf{C})$. This definition, again, for those not familiar

⁸See, for example [29]; or for a more abstract and more general exposition see [30].

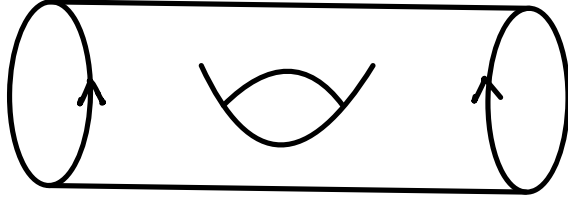


Figure 1.1: The handle creation operator: $\mathcal{H}_{S^1} \rightarrow \mathcal{H}_{S^1}$.

with this language, is a concise codification of the cutting and gluing properties that would naturally be expected of a path integral. Like a path integral, a 2d TQFT assigns to each closed 2d manifold a number, the partition function on that manifold. To a closed 1d manifold it assigns a Hilbert space. To a 2d manifold M with boundary ∂M it assigns a state in the Hilbert space associated to the boundary ∂M .⁹ These assignments are compatible in the way expected of the output of a path integral. For example, gluing two boundary components of a manifold together corresponds to a sum over matching states on the two components. As another example, if a TQFT assigns to a circle the Hilbert space \mathcal{H}_{S^1} , then it assigns to the “handle creation operator” (the manifold as seen in fig. 1.1) a unitary map $\mathcal{H}_{S^1} \rightarrow \mathcal{H}_{S^1}$. From this map and the state in \mathcal{H}_{S^1} assigned to a hemisphere, we can construct any closed manifold by gluing. This handle creation map always has positive, real eigenvalues [31]. We will denote these eigenvalues by μ_I^{-2} where $I = 1, \dots, \dim(\mathcal{H}_{S^1})$. In the eigenbasis, $|I\rangle$, of the handle creation operator, the calculation of partition functions becomes a simple matter. In particular, the partition function of a connected manifold $M_{g,n}$ with genus g and n boundaries, and states $|I_1\rangle, \dots, |I_n\rangle$ input on the boundaries respectively, is

$$Z^{\text{TQFT}}[M_{g,n}; I_1, I_2, \dots, I_n] = \sum_I \mu_I^{2-2g-n} \delta_{II_1 I_2 \dots I_n} . \quad (1.60)$$

Note this evaluates to zero if the boundary labels I_i are not all the same.

From the result (1.60), the generating function for connected correlation functions of the

⁹Or, alternatively, an object that takes as input a state and outputs a complex number.

gravity path integral is

$$\begin{aligned} \log \left\langle e^{\sum_I u_I \widehat{Z}_I} \right\rangle &= \sum_I \sum_n \frac{u_I^n}{n!} \sum_g \mu_I^{2-2g-n} \\ &= \sum_I \lambda_I e^{u_I/\mu_I} , \end{aligned} \tag{1.61}$$

where $\lambda_I \equiv \mu_I^2/(1-\mu_I^{-2})$. The full generating function of correlators is then $\left\langle \exp\left(\sum_I u_I \widehat{Z}_I\right) \right\rangle = \prod_I \exp(\lambda_I e^{u_I/\mu_I})$, from which we can extract the result

$$\left\langle \prod_I \widehat{Z}_I^{n_I} \right\rangle = \prod_I e^{\lambda_I} B_{n_I}(\lambda_I) \mu_I^{-n_I} . \tag{1.62}$$

This can be written

$$\left\langle \prod_I \widehat{Z}_I^{n_I} \right\rangle / \langle \mathbb{1} \rangle = \prod_I \sum_{N_I} p_{\lambda_I}(N_I) \left(\frac{N_I}{\mu_I} \right)^{n_I} \tag{1.63}$$

where p_{λ_I} here is the Poisson distribution with mean λ_I . The normalized correlators then have an interpretation as an average, where Z_I takes the value N_I/μ_I , and the N_I are independently chosen Poisson random integers. As we saw for the case of Dijkgraaf-Witten, the Z_I don't quite have the interpretation as partition functions of a one-dimensional topological theory. If, however, we rescale each of them by a factor of, in this case, μ_I , we do find a nice interpretation for them. Namely we have as our boundary theory a topological quantum mechanics with sectors labeled by I and the number of dimensions in each sector I given by the Poisson distribution with mean λ_I . Then $\mu_I Z_I = \text{tr}(P_I)$ where P_I is a projection onto the sector I .

1.4 2d TQFTs with boundaries

In their simple model of a gravity path integral, [1] consider the addition of end-of-the-world branes (EofW branes). These are boundaries on which bulk spacetime ends. Generalizing this construction, we will consider general boundaries for general 2d TQFTs and then describe the model gravity path integrals built out of them. As a first example we will again consider Dijkgraaf-Witten theory, but this time with the addition of EofW branes. As we will explain,

EofW branes are in some sense the simplest boundary conditions possible in the theory, but we will nonetheless see features that will hold in the more general case. These include the factorization between different sectors of the theory, as before, as well as a new difficulty with interpreting the gravity path integral as a boundary ensemble theory without an additional modification. We will discuss this difficulty and a solution to it in section 1.5.

1.4.1 End-of-the-world branes for Dijkgraaf-Witten

As a first example of an open/closed TQFT we consider Dijkgraaf-Witten theory with the addition of end-of-the-world (EofW) brane boundaries. Following the constructions of [1, 20], we allow for some number of “flavors” of otherwise identical EofW branes, which we will label by $a = 1, 2, \dots, K$. Though these K differently labeled EofW branes have identical dynamics, the number K of such branes will play an important role later. The path integral for Dijkgraaf-Witten theory with EofW branes is still defined as a sum over gauge backgrounds, but now that our spacetime manifolds have boundaries, the specification of a gauge background must include two kinds of data: the holonomies around loops (as before), as well as the parallel transports along paths that begin and end on EofW branes. Cutting a manifold with brane boundaries will result in a new, non-brane boundary. These resulting boundaries, which we will call variously “gluing” boundaries, “state” boundaries, or “Hilbert space” boundaries, are not EofW brane boundaries, but rather, correspond to a Hilbert space of states that represent the input of field configuration data on that boundary. The “gluing” boundaries, in our case, will have fixed parallel transports along them which serve as boundary conditions for the possible gauge backgrounds. More specifically, a circle gluing boundary component will be labeled by the holonomy about that circle, and an interval gluing boundary component will be labeled by an element of G representing the parallel transport across that interval. Note, that the intervals, unlike the circles, are labeled by *elements* of G , rather than conjugacy classes. This is similar to the construction of [20]. Though they consider boundary conditions for spin structures, these are analogous to our

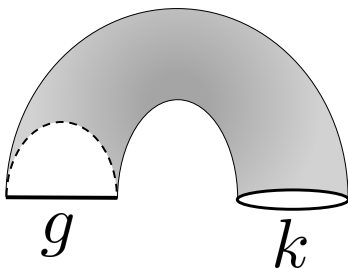


Figure 1.2: A cylinder with a mixture of end-of-the-world branes and gluing boundaries with boundary conditions. With the above boundary conditions the path integral evaluates to $\delta(k, [g^{-1}])$, where $[g^{-1}]$ is the conjugacy class of the element $g^{-1} \in G$.

$\mathcal{H}_{\text{open}} \otimes \mathcal{H}_{\text{open}} \rightarrow \mathcal{H}_{\text{open}}$, represented in diagram form by

$$\begin{array}{c} \text{---} \\ \diagup \quad \diagdown \\ \text{---} \quad \text{---} \\ g_1 \quad g_2 \end{array} = |g_2 g_1\rangle . \tag{1.65}$$

Finally, consider a cylinder where one boundary circle is a gluing boundary with fixed holonomy k and where the other boundary circle is made up of an EofW brane interval together with a gluing interval with fixed parallel transport g . (See figure 1.2.) The count of gauge bundles on this manifold is $\delta(k, [g^{-1}])$, where $[g^{-1}]$ is the conjugacy class of the element $g^{-1} \in G$. Via the pairing (1.26), this provides a map from the open sector Hilbert space $\mathcal{H}_{\text{open}}$ to the circle Hilbert space \mathcal{H}_{S^1} given by the diagram

$$\begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \\ g \end{array} = \frac{|G|}{|[g]|} |[g]\rangle , \tag{1.66}$$

where, again, $[g]$ is the conjugacy class of $g \in G$.

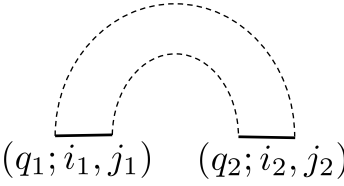
These are all the diagrams necessary to compute any partition function. The computations proceed more easily in bases for the open and closed sectors that are labeled by

irreducible representation. We define bases

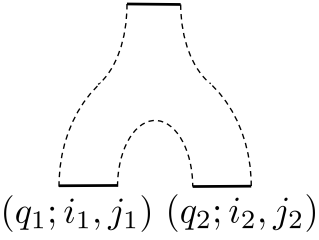
$$|q\rangle \equiv \sum_k \chi_q(k^{-1}) |k\rangle \quad (1.67)$$

$$|q; i, j\rangle \equiv \frac{d_q}{|G|} \sum_{g \in G} U_{ij}^{(q)}(g^{-1}) |g\rangle \quad (1.68)$$

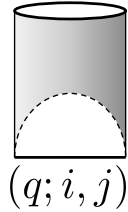
for the closed and open Hilbert spaces, respectively, where $i, j = 1, \dots, d_q$. In these bases the maps (1.64), (1.65), and (1.66) are



$$= \frac{d_{q_1}}{|G|} \delta_{q_1 q_2} \delta_{j_1 i_2} \delta_{j_2 i_1} \quad (1.69)$$

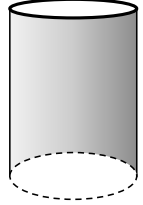


$$= \delta_{q_1 q_2} \delta_{j_1 i_2} |q_1; i_1, j_2\rangle \quad (1.70)$$

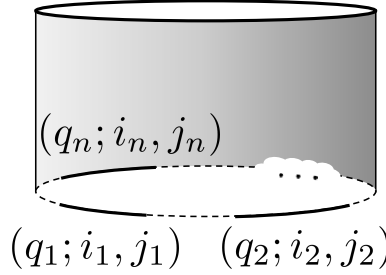


$$= \delta_{ij} |q\rangle \quad (1.71)$$

In particular, these imply the following two diagrams:



$$= \sum_q d_q |q\rangle \quad (1.72)$$



$$= \sum_q \delta_{q q_1 \dots q_n} \delta_{j_1 i_2} \delta_{j_2 i_3} \dots \delta_{j_n i_1} |q\rangle \quad (1.73)$$

with which, we can calculate any partition function.

A manifold can have boundary components of three different types: closed sector circles, circles made up of a single EofW brane, and circles made up of alternating open sector intervals and EofW brane intervals. The implications of the last two diagrams above, for the partition functions of connected manifolds, include the following four facts:

- Any connected manifold that has boundaries, of any type, labeled by different irreps, will evaluate to zero.
- Any connected manifold with a boundary made up of alternating open sector and EofW interval boundaries, where the i, j indices on the open sector labels do not match up appropriately, will also evaluate to zero. Matching appropriately means that the second index of one open sector interval equals the first index of the next open sector interval, and so on around the circle. In the case where we have more than one type of EofW brane, we also require that any indices for the species of EofW brane match similarly.
- A boundary made up of alternating open sector and EofW brane intervals, with all open sector intervals labeled by q and all the i, j indices matching appropriately, will simply contribute the same as a closed sector boundary labeled by q .

- A circular boundary made up of a single EofW brane contributes a factor of $|G|$ relative to just filling in that boundary with a disk.

Let's use these facts to obtain the partition function of a general manifold. We will consider two cases: (1) manifolds with at least one “gluing” boundary and (2) manifolds with no boundaries other than EofW brane boundaries. For the first case, let M be a surface of genus g , with n closed sector boundaries, m circle EofW brane boundaries, and ℓ boundaries made up of alternating open sector and EofW brane intervals, where all closed and open sector boundaries are labeled by q and all i, j indices are matched appropriately. Note that in this first case at least one of n or ℓ must be greater than 0. The partition function for such a manifold will be

$$Z^{\text{DW}}[M] = \left(\frac{d_q}{|G|} \right)^{2-2g-n-m-\ell} d_q^m. \quad (1.74)$$

The additional m factors of d_q come from the last fact above: filling in a boundary with a disk will contribute a factor of $d_q/|G|$, and a circular EofW brane boundary will contribute $|G|$ times that, so it will contribute an overall factor of d_q . Now consider the second case: let M be a manifold with m circular EofW brane boundaries, but no closed or open sector boundaries. Then the partition function of M is

$$Z^{\text{DW}}[M] = \sum_q \left(\frac{d_q}{|G|} \right)^{2-2g-m} d_q^m. \quad (1.75)$$

1.4.2 The gravity path integral

We are now equipped to build a gravity path integral out of Dijkgraaf-Witten with EofW branes. The gravity path integral will be a sum over all manifolds with compatible EofW brane boundaries. Specifically, we take the EofW branes to be dynamical while we take the open and closed sector “gluing” boundaries to be fixed. The open sector boundaries are intervals with a fixed label at each end designating the type of EofW brane found there, and in the gravity path integral we only allow configurations where an EofW brane of a given type attaches only to interval endpoints labeled by that type.

Operators inserting fixed circle boundaries will be denoted by $\widehat{Z}[k]$ or \widehat{Z}_q , as before. The operators inserting fixed interval boundaries will be denoted by \widehat{S}_{gab} where g specifies the fixed gauge background boundary conditions (i.e. the parallel transport $g \in G$ along the interval) and where a and b are the just mentioned labels designating the type of EofW brane allowed at the endpoints. Note that the boundary insertion operators are in one-to-one correspondence with the states of the TQFT: the \widehat{Z} operators correspond to states in the closed sector, and the \widehat{S} operators labeled with a and b correspond to states in the open sector with boundary conditions a and b . Using the change of basis (1.68) we can define the more convenient interval operators \widehat{S}_{qijab} representing the insertion of an interval with the state $|q; i, j\rangle$ fixed on it.

We will add a term $S_0\chi$ proportional to the Euler characteristic to the Dijkgraaf-Witten action, as we did before. This will have the effect of suppressing higher genus manifolds, allowing the sum over topologies to converge. The gravity path integral is the sum of the TQFT partition functions over every manifold that is compatible with the inserted boundaries with a measure that takes into account possible residual diffeomorphisms.

We now describe the calculation of correlators of the boundary insertion operators. Let m_a denote the number of circular EofW brane boundaries of type a on a connected manifold, so that $m = \sum_a m_a$. Then the connected vacuum correlator for Dijkgraaf-Witten with EofW branes is

$$\begin{aligned}
\lambda' &\equiv \log \langle \mathbf{1} \rangle = \langle \mathbf{1} \rangle_{\text{connected}} \\
&= \sum_g \sum_{m_1, \dots, m_K} \frac{1}{m_1! \cdots m_K!} \sum_q e^{S_0(2-2g-\sum_a m_a)} \left(\frac{d_q}{|G|} \right)^{2-2g-\sum_a m_a} d_q^{\sum_a m_a} \\
&= \sum_q \left(e^{S_0} \frac{d_q}{|G|} \right)^2 \frac{1}{1 - \left(e^{S_0} \frac{d_q}{|G|} \right)^{-2}} \prod_{a=1}^K \exp \left(\left(\frac{e^{S_0}}{|G|} \right)^{-1} \right) \\
&= \sum_q \lambda_q e^{K|G|e^{-S_0}} \\
&= e^{K|G|e^{-S_0}} \lambda,
\end{aligned} \tag{1.76}$$

where we have used our previous definitions (1.31) of λ_q and λ . The connected manifolds

without fixed boundary are entirely specified by their genus g and then by the number m_a of dynamical EofW brane circle boundaries they have for each type a , hence the sums over g and each m_a in the above expression. Also note the factor of $1/m_a!$ for each brane type. These are factors in the measure of the path integral and are due to the $m_a!$ large diffeomorphisms that permute the EofW brane circles (which are dynamical and hence taken to be indistinguishable). Comparing this expression for the connected vacuum correlator to that of the theory without EofW branes (1.31), we see that the effect of the circle EofW brane boundaries is simply to multiply λ by a factor.

Let's now consider the generating function for all correlators

$$F(\vec{u}, T) = \left\langle \exp \left(\sum_q u_q \widehat{Z}_q + \sum_{a,b=1}^K \sum_q \sum_{i,j=1}^{d_q} t_{qijab} \widehat{S}_{qijab} \right) \right\rangle. \quad (1.77)$$

For convenience, we will collect the chemical potentials t_{qijab} into an object T and the chemical potentials u_q into the object \vec{u} . Also, note that the i, j indices within each q sector play the same role as the EofW brane labels a, b , so to ease notation for the calculation of the above generating function we will combine i and a into a single index A and similarly j and b into B .

Following [1], and analogously to the procedure in section 1.2.2 for the theory without EofW branes, we calculate the logarithm of the generating function,

$$\begin{aligned} \log F(\vec{u}, T) &= \sum_n \sum_k \frac{1}{n!} \frac{1}{k!} \left\langle \left(\sum_q u_q \widehat{Z}_q \right)^n \left(\sum_{qAB} t_{qAB} \widehat{S}_{qAB} \right)^k \right\rangle_{\text{conn}} \\ &= \sum_q \sum_{n,k} \frac{1}{n!} u_q^n \frac{1}{k!} \sum_{A_1 B_1 \dots A_k B_k} t_{qA_1 B_1} \dots t_{qA_k B_k} \left\langle \widehat{Z}_q^n \widehat{S}_{qA_1 B_1} \widehat{S}_{qA_2 B_2} \dots \widehat{S}_{qA_k B_k} \right\rangle_{q, \text{conn}} \end{aligned} \quad (1.78)$$

where we have used the fact that a connected correlator with boundaries labeled by different irreps is always zero.

Each correlator in the above will be a sum over genus, a sum over numbers of circle EofW brane boundaries, and finally a sum over all the ways of connecting the \widehat{S} operators by EofW branes. We can express this final sum as a sum over all permutations of k elements, where k is the number of open sector interval boundaries. Say $\widehat{S}_{qA_1 B_1}, \widehat{S}_{qA_2 B_2}, \dots, \widehat{S}_{qA_k B_k}$ are the

intervals in our correlator. For a given permutation $\pi \in S(k)$ we attach the outgoing end of $\widehat{S}_{qA_i B_i}$ to the ingoing end of $\widehat{S}_{qA_{\pi(i)} B_{\pi(i)}}$ for every i . A particular way of connecting the branes will only result in a nonzero partition function if all pairs of connected endpoints have the same index. In other words, for any π the partition function will be proportional to $\delta_{B_1, A_{\pi(1)}} \delta_{B_2, A_{\pi(2)}} \cdots \delta_{B_k, A_{\pi(k)}}$. For a permutation π , denote the number of 1-cycles in π by $a_1(\pi)$, the number of 2-cycles by $a_2(\pi)$, and so on. The total number of cycles we will denote by $a(\pi) = \sum_i a_i(\pi)$, and therefore the number of alternating-type boundaries will be $\ell = a(\pi)$ for any π .

We are now ready to calculate the general correlator of n closed sector boundaries and k open sector interval boundaries. We get

$$\begin{aligned} & \langle \widehat{Z}_q^n \widehat{S}_{qA_1 B_1} \widehat{S}_{qA_2 B_2} \cdots \widehat{S}_{qA_k B_k} \rangle_{\text{conn}} \\ &= \sum_{m_1, \dots, m_K} \frac{1}{m_1! \cdots m_K!} \sum_g \sum_{\pi \in S(k)} \delta_{B_1, A_{\pi(1)}} \cdots \delta_{B_k, A_{\pi(k)}} Z^{\text{bulk}} [M_{g, n+m+\ell}; q, m] \\ &= \lambda'_q \left(e^{S_0} \frac{d_q}{|G|} \right)^{-n} \sum_{\pi \in S(k)} \delta_{B_1, A_{\pi(1)}} \cdots \delta_{B_k, A_{\pi(k)}} \left(e^{S_0} \frac{d_q}{|G|} \right)^{-a(\pi)}, \end{aligned} \quad (1.79)$$

where we have used $Z^{\text{bulk}} [M_{g, n+m+\ell}; q, m]$ to denote the partition function of a connected manifold of genus g with $n+m+\ell$ circular boundaries, n of which are closed sector boundaries labeled by q , $m = \sum_a m_a$ of which are EofW brane boundaries, and $\ell = a(\pi)$ of which are alternating interval boundaries labeled by q .

We can now plug this in to our expansion (1.78) above. We get

$$\log F(\vec{u}, T) = \sum_q \lambda'_q \exp\left(u_q \frac{|G|}{d_q e^{S_0}}\right) \sum_k \frac{1}{k!} \sum_{\pi \in S(k)} \left(e^{S_0} \frac{d_q}{|G|} \right)^{-\sum_j a_j(\pi)} \prod_j \text{tr}(T_{(q)}^j)^{a_j(\pi)}, \quad (1.80)$$

Note that the functions a_j on the permutation group only depend on the cycle structure of the permutation. So we can replace the sum over permutations with a sum over cycle structures, using the fact that there are $\frac{k!}{\prod_j a_j! j^{a_j}}$ permutations with a_j cycles of length j for each j . This results in

$$\log F(\vec{u}, T) = \sum_q \lambda'_q \exp\left(u_q \frac{|G|}{d_q e^{S_0}}\right) \prod_{j=0}^{\infty} \sum_{a_j} \frac{1}{a_j!} \left(\frac{|G|}{d_q e^{S_0}} \frac{1}{j} \text{tr}(T_{(q)}^j) \right)^{a_j}. \quad (1.81)$$

Recognizing the rightmost sum as an exponential and then using the identity $\exp\left(\sum_j \frac{1}{j} \text{tr}(T^j)\right) = \det(\mathbb{1} - T)^{-1}$, we get a final expression for the generating function:

$$F(\vec{u}, T) = \prod_q \exp\left(\lambda'_q \left(\frac{e^u}{\det(\mathbb{1} - T_{(q)})}\right)^{\frac{|G|}{d_q e^{S_0}}}\right) \quad (1.82)$$

Compare to (1.16), the analogous result for the simple model of [1].

1.4.3 Boundary Interpretation

As explained in section 1.2.3, finding an interpretation of the gravity path integral in terms of an ensemble of boundary theories reduces to an instance of the moment problem, the problem of finding a distribution given its moments. In the case of Dijkgraaf-Witten with EofW branes, the different boundary theories that could appear in our ensemble are characterized by the values that the boundary partition functions Z_q and S_{qijab} take in them. These are the alpha-parameters and the (normalized) correlators of the \widehat{Z} and \widehat{S} operators are moments probing the probability distribution over the alpha-parameters. From this point of view we obtain the following relation between the generating function (1.82) obtained above and the probability distribution $p(\alpha)$ over the alpha-parameters:

$$\prod_q \exp\left(\lambda'_q (e^u / \det(\mathbb{1} - T_{(q)}))^{|G|/d_q e^{S_0}} - \lambda'_q\right) = \int d\alpha p(\alpha) e^{\sum_q u_q \alpha_q + \sum_q \text{tr}(T_{(q)} M_{(q)})}. \quad (1.83)$$

Here the integral is over all $\alpha \in \mathbf{C}^{r+|G|K^2}$ with $\alpha = (\alpha_1, \dots, \alpha_r; M_{(1)}, \dots, M_{(r)})$ where the α_q are the values the Z_q take in a particular theory and each $M_{(q)}$ is a Kd_q by Kd_q matrix representing the value of the matrix $S_{(q)}$ in a particular theory. In principle we can extract the probability distribution $p(\alpha)$ by taking the appropriate Fourier transform of both sides of the above. In fact, we already know the probability distribution for the α_q . These come out nearly the same as in section 1.2.3. The result is that Z_q takes the value $Z_q = \frac{|G|}{e^{S_0} d_q} N_q$ where N_q is an integer drawn from the Poisson distribution with mean λ'_q .

So we are left with the task of determining the probability distribution for the matrices $M_{(q)}$. To see what that will be, we can consider the generating function for the \widehat{S} operators

but conditioned on chosen values α_q for the \widehat{Z} operators. This can read off from the result of taking the Fourier transform of both sides of 1.83 with respect to the variables iu_q . We get

$$\left\langle \exp \left(\sum_{a,b=1}^K \sum_q \sum_{i,j=1}^{d_q} t_{qijab} \widehat{S}_{qijab} \right) \right\rangle_{Z_q=\alpha_q} = \det(\mathbb{1} - T_{(q)})^{-\alpha_q}, \quad (1.84)$$

so we are left with

$$\det(\mathbb{1} - T_{(q)})^{-\alpha_q} = \int dM p(\vec{\alpha}; M) e^{\sum_q \text{tr}(T_{(q)} M_{(q)})}, \quad (1.85)$$

where dM represents an integral over all values for the matrices $M_{(1)}, \dots, M_{(r)}$. The situation is simply that of (1.18), encountered for the simple model of [1]. Namely, $\det(\mathbb{1} - T_{(q)})^{-\alpha_q}$ is a generating function of moments of a probability distribution only when the exponent α_q takes certain values [23], in this case values in the set $\{0, 1, 2, \dots, Kd_q - 1\} \cup [Kd_q - 1, \infty)$. Unfortunately, α_q takes values $\frac{|G|}{e^{S_0 d_q}} N_q$ where N_q is a nonnegative integer. For α_q to always be in the set $\{0, 1, 2, \dots, Kd_q - 1\} \cup [Kd_q - 1, \infty)$ would require that $\frac{|G|}{e^{S_0 d_q}}$ be an integer. This is not possible as it would render λ'_q negative, which is nonsensical as λ'_q is the mean of the Poisson distribution for N_q . So we are left with the problem that the gravitational path integral correlators are not the moments of some probability distribution, meaning it cannot be dual to an ensemble of boundary theories. This problem is worse than the one encountered earlier which necessitated identifying the boundary partition functions with rescaled \widehat{Z} operators. Rescaling the \widehat{Z} or \widehat{S} operators does not change the values that α_q takes.

One solution is to modify the theory by adding terms to the TQFT action proportional to the number of boundary components. This would be for all three sorts of boundary components: circular EofW branes, closed sector fixed boundaries, and alternating EofW brane and open sector intervals. Specifically we can add a term S_q in the action for each boundary component labeled by q . Retracing through the calculations of section 1.4.2 we see how an additional factor of e^{S_q} for each boundary circle modifies the generating function of correlators. First, considering the boundaries made up entirely of a single EofW brane component, we see that the connected vacuum correlator gets modified: $\lambda'_q = \lambda_q e^{K|G|e^{-S_0}} \rightarrow$

$\lambda_q e^{K|G|e^{S_q-S_0}}$. Second, a factor of e^{S_q} for circle fixed boundaries gives $u_q \rightarrow e^{S_q} u_q$. And finally, a factor of e^{S_q} for circles made of alternating brane and fixed boundaries changes the factors $\text{tr} \left(T_{(q)}^j \right)$ like $\text{tr} \left(T_{(q)}^j \right) \rightarrow e^{S_q} \text{tr} \left(T_{(q)}^j \right)$. Making these substitutions in 1.81 leads to a modified generating function of

$$F(\vec{u}, T) = \prod_q \exp \left(\lambda'_q \left(\frac{e^u}{\det(\mathbf{1} - T_{(q)})} \right)^{\frac{e^{S_q}|G|}{d_q e^{S_0}}} \right). \quad (1.86)$$

Then with a the choice of $S_q = S_0 + \ln d_q$ as in (1.43), for example, the values of α_q are integers, ensuring the generating functional (1.85) is indeed the generating function for moments of a probability distribution.

The additional S_q terms in the action can be viewed as the contributions of degrees of freedom that propagate along the boundaries. This picture is odd, however, in that these degrees of freedom must propagate along both EofW brane *and* fixed boundaries. In this solution the “fixed” boundary components are no longer fixed, but must include these dynamical degrees of freedom that propagate across them. In some sense the solution requires us to not allow the full set of \widehat{S} operators, which would otherwise allow us to completely fix the interval boundary conditions in the gravity path integral. We discuss the problem in these terms in section 1.5.

1.4.4 General open/closed TQFTs

In this section we will describe the gravity path integral obtained by choosing our bulk theory to be a general open/closed TQFT (aka a general 2d TQFT with boundaries). We’ll see that this has the same features that we found above for the specific case of Dijkgraaf-Witten theory with end-of-the-world brane boundaries.

When discussing Dijkgraaf-Witten theory above, we considered only end-of-the-world brane boundaries. These are, in the context of Dijkgraaf-Witten theory, the simplest sort of boundary compatible with gauge symmetry. In general, we could consider more complicated boundaries, and in particular for theories without gauge symmetry, there is no condition of

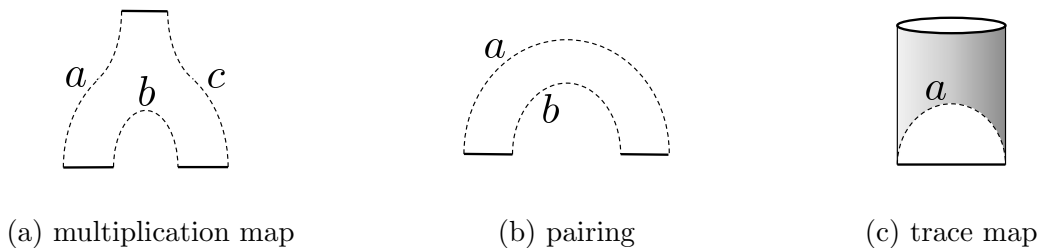



Figure 1.3: Three open/closed TQFT diagrams: (a) A map from $\mathcal{H}_{ab} \otimes \mathcal{H}_{bc}$ to \mathcal{H}_{ac} . The open sector states can be represented as matrices such that this map implements matrix multiplication. (b) The pairing between Hilbert spaces \mathcal{H}_{ab} and \mathcal{H}_{ba} . (c) The trace map, which maps \mathcal{H}_{aa} to \mathcal{H}_{S^1} . With the open sector states represented as matrices, this map implements the trace of these matrices.

gauge symmetry that boundaries must be compatible with. The only conditions that we need to require of our boundaries are that they be compatible with the possible ways of cutting and gluing spacetimes with boundaries. For a review of open/closed TQFTs and the axioms that define them we refer interested readers to [32]. Here we will simply state some of the results that apply to our case.


We'll use the index a to label the different types of boundaries present in the theory. For any two types a and b we have a corresponding open sector Hilbert space \mathcal{H}_{ab} , which is the Hilbert space associated with an interval with a boundary conditions on one end and b boundary conditions on the other. In addition to these open sectors we have the closed sector Hilbert space \mathcal{H}_{S^1} , which is the Hilbert space of states on a circle. The closed sector makes, in its own right, a 2d TQFT. So, as described in section 1.3, the partition function for the closed, connected manifold of genus g has the form $\sum_I \mu_I^{2-2g}$, for some positive, real numbers μ_I , and accordingly, the closed sector Hilbert space \mathcal{H}_{S^1} has an orthonormal basis $|I\rangle$ labeled by the index I .

The interval Hilbert space \mathcal{H}_{ab} will have the form $\mathcal{H}_{ab} \cong \bigoplus_I (\mathbb{C}^{d_{aI}} \otimes \mathbb{C}^{d_{bI}})$, where d_{aI} are integers. In fact, we can think of the states $|\psi\rangle$ in \mathcal{H}_{ab} as being direct sums of matrices $|\psi\rangle = \bigoplus_I \Psi_I$ where the ‘‘multiplication’’ diagrams as in figure 1.3a are computed via matrix

multiplication of the matrices Ψ_I . (For example, the multiplication  acting on states $|\psi\rangle \in \mathcal{H}_{ab}$ and $|\phi\rangle \in \mathcal{H}_{bc}$ produces a state in \mathcal{H}_{ac} described by $\bigoplus_I \Psi_I \Phi_I$.) We can take as a basis for \mathcal{H}_{ab} the matrices with one entry 1 and the rest zero. Denote these basis states by $|I; i, j\rangle$ where I labels the block and where indices $i = 1, \dots, d_{aI}$ and $j = 1, \dots, d_{bI}$ label the position of the nonzero entry. (To make connection to Dijkgraaf-Witten theory with end-of-the-world branes, the label I is in that case the label q running over irreducible representations, the values μ_I are $e^{S_0} d_q / |G|$, and the dimensions d_{aI} are simply d_q for all boundaries. The diagram 1.70 can be seen to be describing matrix multiplication.)

There are two additional facts we make use of. First, the partition function calculated by a strip with boundary type a on one side and boundary type b on the other (see figure 1.3b) induces a pairing between the Hilbert spaces \mathcal{H}_{ab} and \mathcal{H}_{ba} . To be entirely consistent with the cutting and gluing axioms defining an open/closed TQFT, this pairing must take the form

$$(|\psi\rangle, |\phi\rangle) = \sum_I \mu_I \text{tr}(\Psi_I \Phi_I) \quad (1.87)$$

where $|\psi\rangle \in \mathcal{H}_{ab}$ and $|\phi\rangle \in \mathcal{H}_{ba}$. Second, we have the map  : $\mathcal{H}_{aa} \rightarrow \mathcal{H}_{S^1}$ from any diagonal open sector \mathcal{H}_{aa} to the closed sector \mathcal{H}_{S^1} . This map simply implements the trace $|\psi\rangle \mapsto \sum_I \text{tr}(\Psi_I) |I\rangle$.

With the multiplication, pairing, and trace diagrams in hand we can calculate the partition function of any manifold with boundaries, and from there define a gravity path integral as a subsequent sum over all compatible manifolds with boundaries. The observables in this gravity theory are operators \widehat{Z}_I that insert a circle with state $|I\rangle \in \mathcal{H}_{S^1}$, and \widehat{S}_{abIij} which insert the interval with state $|I; i, j\rangle \in \mathcal{H}_{ab}$ on it. The correlators are given by a sum over manifolds where the boundary configurations are compatible with the boundary labels of all the insertions \widehat{S}_{abIij} . In our sum over manifolds M we include the appropriate measure $\mu(M)$. In addition to the factors for having multiple identical components without fixed boundaries, the measure $\mu(M)$ also includes a factor of $1/m_a!$ whenever a component of M has m_a circle boundaries with boundary conditions a all around the circle. These are due to

residual diffeomorphisms that permute the identical dynamical boundaries.

The calculation of correlators is analogous to the Dijkgraaf-Witten case in section 1.4.2. We simply state the results. The connected vacuum correlator is

$$\lambda' \equiv \log \langle \mathbb{1} \rangle = \langle \mathbb{1} \rangle_{\text{connected}} = \sum_I \lambda_I e^{K_I/\mu_I} = \sum_I \lambda'_I, \quad (1.88)$$

where $K_I = \sum_a d_{aI}$ count the boundary degrees of freedom, and the $\lambda_I = \mu_I^2/(1 - \mu_I^{-2})$ are as defined in section 1.3 for the corresponding closed TQFT. The generating function

$$F(u_I, t_{abIij}) = \left\langle e^{\sum_I u_I \widehat{Z}_I + \sum_{ab} \sum_{I,i,j} t_{abIij} \widehat{S}_{abIij}} \right\rangle \quad (1.89)$$

is given by

$$F(u, t) = \prod_I e^{\lambda'_I \exp(\mu_I^{-1} u_I + \mu_I^{-1} \sum_{j=1}^{\infty} \frac{1}{j} \text{tr}(T_I^j))} = \prod_I \exp\left(\lambda'_I e^{u_I/\mu_I} \det(\mathbb{1} - T_I)^{-1/\mu_I}\right), \quad (1.90)$$

where we have collected the chemical potentials t_{abIij} into K_I by K_I matrices T_I . This is of course the same form as (1.82). Similar to the special case of Dijkgraaf-Witten with end-of-the-world branes, correlators will factorize between the I sectors, and will also fail to correspond to the moments of an ensemble distribution without negative probabilities. We discuss this failure and possible solutions in the next section.

1.5 Boundaries and the ensemble problem

In order to get a sensible holographic interpretation from a gravity path integral built from a closed 2d TQFT, we were forced to introduce in a seemingly ad hoc fashion a rescaling of the \widehat{Z}_I operators. What's more, for an open/closed theory, we find that rescalings of the operators \widehat{Z} and \widehat{S} are no longer enough to land on a sensible boundary interpretation. The solution discussed in [1] and reviewed here in section 1.1.1 is to add a nonlocal boundary term $S_{\partial} = S_0$ to the action. The authors attempt to justify this in terms of a large number of additional degrees of freedom that are allowed to propagate on the boundary. We discuss the analogous solution in our case and attempt to paint a clearer picture of how these degrees of freedom fit into a local framework.

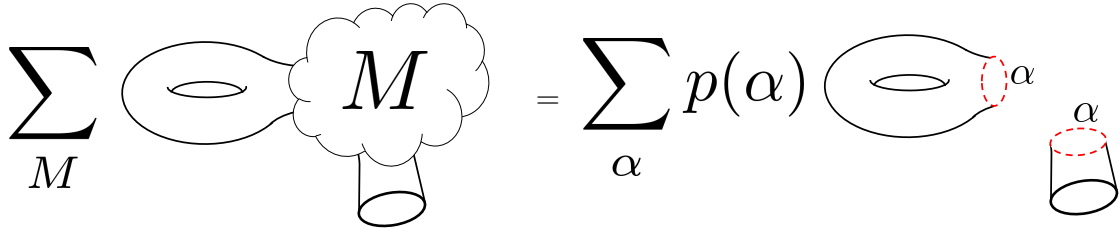


Figure 1.4: A gravitational region whose topology is summed over is holographically dual to an ensemble of boundary conditions. The above figure is a schematic illustration of a gravity correlator of nongravitational regions. The correlator is a sum over all manifolds that are compatible with the boundaries of the nongravitational regions. Equivalently, it is an average over different boundary conditions α (represented by dashed lines) that are placed on the boundaries of the nongravitational regions.

The observables in our gravity theory are in one-to-one correspondence with states in a TQFT. By definition, a TQFT is guaranteed to be compatible with cutting and gluing spacetime manifolds. In particular, upon cutting a spacetime manifold the newly created boundary is what we will call a “gluing boundary.” By this we mean this boundary corresponds to a Hilbert space of states and hence can be glued to a similar gluing boundary by appropriately summing over states on the two boundaries. So far, the operators describing observables in the theory (the \widehat{Z} and \widehat{S} operators) are operators that insert such gluing boundaries in our gravitational spacetime. This is suggestive. Specifically, it suggests the possibility of gluing the boundaries of our gravity spacetime to nongravitational TQFT path integrals. That is to say, we can consider spacetimes with designated nongravitational regions where topology is fixed, while the rest of the manifold has unspecified (i.e. summed over) topology.

Holography on such hybrid gravitational/nongravitational spacetimes will reduce the gravitational bulk region to a boundary condition on the nongravitational region. Or, more generally, as in our case, the gravitational region will be dual to an *ensemble* of boundary

conditions. Specifically, each alpha-state of the gravitational theory will correspond to a distinct boundary condition. For each type of boundary α , there are corresponding open sector Hilbert spaces for intervals that end on that boundary. For example, in a theory that already has boundaries (for example, these could be end-of-the-world branes) labeled by the index a , we will get additional open sector Hilbert spaces like $\mathcal{H}_{a\alpha}$ and $\mathcal{H}_{\alpha a}$. Even in a theory without additional boundaries, such as those considered in sections 1.2 and 1.3, upon going to the holographic dual we will have an open/closed TQFT with boundary α and the open sector $\mathcal{H}_{\alpha\alpha}$.

Taking gravity as a boundary condition provides a new perspective on the interpretational problems that necessitated rescaling the \widehat{Z} operators and that led to negative probabilities when \widehat{S} operators are added. We can see how such problems arise in this framework. Consider the correlator $\left\langle \widehat{\text{annulus}} \right\rangle = \sum_I \left\langle \widehat{Z}_I \widehat{Z}_I \right\rangle$, where I labels the closed sector states of the TQFT. This is dual to the annulus where both boundaries are the gravity boundary. This diagram is a trace that computes the dimension of the open sector Hilbert space $\mathcal{H}_{\alpha\alpha}$. Consequently, it ought to be an integer. However, from the results of section 1.3,

$$\sum_I \left\langle \widehat{Z}_I \widehat{Z}_I \right\rangle = \sum_{\dots, N_I, \dots} \prod_I \frac{\lambda_I^{N_I}}{N_I! e^{N_I}} \sum_I \frac{N_I}{\mu_I} \frac{N_I}{\mu_I}, \quad (1.91)$$

meaning the annulus with gravity boundaries evaluates to $\sum_I \frac{N_I^2}{\mu_I^2}$ where N_I is a Poisson random integer. This is not in general an integer. In fact, it cannot be an integer for all N_I if we restrict μ_I to the values that are compatible with a convergent gravity path integral or with an ensemble without negative probabilities. Likewise, if we start with a TQFT where boundary conditions are already present, similar problems arise for the “mixed” open sectors $\mathcal{H}_{\alpha a}$ and $\mathcal{H}_{a\alpha}$, where a here labels the original, nongravity, boundary conditions in the theory. An annulus with one boundary of type a and the other with gravity theory α on it, will evaluate to $\sum_I d_{aI} \frac{N_I}{\mu_I}$, where d_{aI} are integers. This, again, is not in general an integer. This implies the absurdity that the Hilbert spaces $\mathcal{H}_{\alpha\alpha}$ and $\mathcal{H}_{\alpha a}$ have noninteger dimension. One can equally view this as a breakdown of locality, in that cutting a gravity boundary α cannot be done consistently.

$$\sum_M \text{[cloud } M \text{ with strip } a, b] = \sum_\alpha p(\alpha) \text{[strip } a, b \text{ with gravity boundary } \alpha \text{ and operators } \psi_{\alpha a}, \psi_{b\alpha}]$$

Figure 1.5: A strip with a and b boundary conditions that enters a gravitational region is dual to a strip that ends with gravity boundary conditions α interpolating between a and b . At the junction between α and a boundary conditions there must be a boundary condition changing operator, which we call $\psi_{\alpha a}$. Likewise for the junction between b and α boundary conditions.

The perspective of gravity as dual to a boundary condition on a nongravitational TQFT also motivates an understanding of what the alpha-states for a theory with boundaries, like end-of-the-world branes, are. Consider a state $|I; i, j\rangle$ in an open sector \mathcal{H}_{ab} that propagates from a nongravitational region into the gravitational region. This is holographically dual to a half-disk bounded by three segments: a segment with boundary conditions a , a segment with gravity boundary conditions α , and a segment with boundary conditions b . (See figure 1.5.) At the two points on the edge of the half-disk where the boundary conditions switch, there are so-called boundary condition changing operators [33]. A boundary condition changing operator between boundaries of two types, say from a to b , is a state in the open sector Hilbert space \mathcal{H}_{ab} . (This can be seen from considering the slice surrounding the point where the boundary conditions change. As this slice ends on the boundaries with conditions a and b , the Hilbert space associated to it is \mathcal{H}_{ab} .) So in this case of a half-disk with three different boundary types, there must be specified two states, in $\mathcal{H}_{b\alpha}$ and $\mathcal{H}_{\alpha a}$ respectively, which represent the boundary condition changing operators. We take the specification of these states, call them $|\psi_{b\alpha}\rangle \in \mathcal{H}_{b\alpha}$ and $|\psi_{\alpha a}\rangle \in \mathcal{H}_{\alpha a}$, to be additional alpha parameters. That is to say, fully specifying the alpha state, and hence the boundary theory, includes not

just specifying the dimensions $d_{\alpha I}$, but also the boundary condition changing states $|\psi_{\alpha\alpha}\rangle$ that must appear when switching from the gravity path integral to its holographic dual.¹² Recalling from section 1.4.4 that the states of an open sector \mathcal{H}_{ab} can be regarded as direct sums of d_{aI} by d_{bI} matrices, we can represent the states $|\psi_{\alpha\alpha}\rangle$ as $\bigoplus_I \Psi_{(\alpha\alpha I)}$. We see that the probability distribution $p(\alpha)$ defining our ensemble should be over the nonnegative integers $d_{\alpha I}$, for each I , and the $d_{\alpha I}$ by $d_{\alpha I}$ complex matrices $\Psi_{(\alpha\alpha I)}$, for each choice of a and I .

We see an immediate problem, however. For the naive gravity path integral the dimensions $d_{\alpha I}$ are $d_{\alpha I} = N_I/\mu_I$, and hence, as explained above, not nonnegative integers for every $N_I \in \{0, 1, 2, \dots\}$. Defining the boundary theories in our ensemble then seems to entail choosing the elements of a matrix with noninteger dimension. We saw that, for a TQFT without additional boundaries, the holographic dual already has the problem that the Hilbert space $\mathcal{H}_{\alpha\alpha}$ does not exist. (It would have noninteger dimension.) This can be viewed as a violation of locality, insofar as it forbids us from making cuts in our manifolds that intersect α boundaries. With that restriction on making cuts in place, perhaps it might otherwise make sense. In the presence of additional boundaries a , however, the problems with the holographic dual theory become worse. The inability to make cuts intersecting α boundaries prevents us from relating boundary condition changing operators to states $|\psi_{\alpha\alpha}\rangle \in \mathcal{H}_{\alpha\alpha}$, as the Hilbert space $\mathcal{H}_{\alpha\alpha}$ does not exist. Instead, while such states $|\psi_{b\alpha}\rangle$ and $|\psi_{\alpha a}\rangle$ don't exist, we *do* have access to the state in \mathcal{H}_{ba} that would be their product under open sector multiplication described in section 1.4.4 (see figure 1.3a). When represented as a matrix, the entries of this state in \mathcal{H}_{ba} are simply the values that the operators \widehat{S}_{abIij} take. As we have explained, the generating function (1.90), by analogy with the arguments of [1], does not lead to a nonnegative probability distribution for the values S_{abIij} , for all values of $d_{\alpha I} = N_I/\mu_I$. Thus, even with the locality-violating restriction of disallowing cuts that intersect gravity boundaries, we are left without a consistent boundary ensemble theory.

So should we give up hope of consistently viewing the gravity path integral as dual to

¹²Reflection positivity ensures that the state $|\psi_{\alpha\alpha}\rangle$ switching α boundaries to a boundaries and the state $|\psi_{\alpha\alpha}\rangle$ in the other direction, are determined by each other. Represented as matrices they are each other's Hermitian conjugates.

an ensemble of boundary conditions? In the case of the simple model with end-of-the-world branes considered in [1], they discuss, as a solution to the negative probabilities, adding e^{S_0} degrees of freedom that propagate along the end-of-the-world branes. These degrees of freedom contribute to the existing bulk action a boundary term $S_\partial = S_0$ for every boundary in the theory. This effectively rescales the boundary insertion operators \widehat{Z} and likewise ensures that d in eq. (1.16) is an integer. Then the Fourier transform of eq. (1.18) does, indeed, give a valid probability distribution, namely the Wishart distribution.

This solution unavoidably has one of two problems, however. On the one hand, if the degrees of freedom are taken to propagate only along the end-of-the-world branes, then the open sectors \mathcal{H}_{ab} consequently have e^{2S_0} times as many states as they did before, and we have a corresponding e^{2S_0} times as many interval insertion operators \widehat{S} . As the problem of negative probabilities is, roughly speaking, the problem of having too many boundary flavors, the additional interval insertion operators \widehat{S} reintroduce the problem that adding the e^{S_0} degrees of freedom might have solved. On the other hand, if we allow the additional boundary degrees of freedom to propagate along not just the branes, but also the “gluing” boundaries, these boundaries lose their interpretation as gluing boundaries. In other words, the bulk action is now nonlocal.

The perspective of a gravitational region being dual to boundary conditions on a non-gravitational region clarifies what the problem is. Adding, in our case, μ_I degrees of freedom to each boundary type a , implements the change $d_{aI} \rightarrow d'_{aI} = \mu_I d_{aI}$. On its face, this seems to solve the problem of $\dim \mathcal{H}_{\alpha\alpha}$ being a noninteger, as $\sum_I d'_{aI} \frac{N_I}{\mu_I} = \sum_I d_{aI} N_I \in \mathbb{Z}$, but the new size of the open sector Hilbert space \mathcal{H}_{ba} is then inconsistent with a multiplication rule $\mathcal{H}_{b\alpha} \otimes \mathcal{H}_{\alpha a} \rightarrow \mathcal{H}_{ba}$ that involves matrices with $d_{bI} N_I$ and $d_{aI} N_I$ number of entries.¹³ That is to say, there are no sizes that two matrices could have such that

- the number of entries in each are $d_{bI} N_I$ and $d_{aI} N_I$ respectively, and
- their matrix product has $\mu_I d_{bI} \mu_I d_{aI}$ entries,

¹³It also does not solve the problem of noninteger $\dim \mathcal{H}_{\alpha\alpha}$.

other than their inner dimensions being N_I/μ_I , which is not an integer for all N_I . (This makes precise the statement we made above that the problem is in some sense having too many flavors of boundary.)

The difficulties with the boundary interpretation all ultimately stem from the fact that the $d_{\alpha I} = N_I/\mu_I$ are not integers for all N_I . This suggests that a solution should be something that adds μ_I additional degrees of freedom to the gravity boundary α , rather than merely adding degrees of freedom to the original boundaries a . If we substitute $d_{\alpha I} \rightarrow d'_{\alpha I} = \mu_I d_{\alpha I}$, the Hilbert spaces $\mathcal{H}_{\alpha\alpha}$, $\mathcal{H}_{a\alpha}$, etc. are all defined as well as the boundary condition changing operators. With $\mathcal{H}_{a\alpha}$ well-defined, locality would then imply that the S_{abI} matrices factorize as $S_{abI} = \Psi_{\alpha a I}^\dagger \Psi_{\alpha b I}$. Such a solution would presumably cure the problem of negative probabilities for the values of S_{abIij} .

One solution fitting these requirements involves adding a defect line separating the gravitational and nongravitational regions in our gravity path integral. We let this defect line, call it ε , have $\sum_I \mu_I$ degrees of freedom, and we take it to be coupled to the 2d TQFT so that there are μ_I degrees of freedom in each I sector. To explain what we mean by this, consider that with the addition of such a defect line our TQFT has additional open sector Hilbert spaces, namely those corresponding to intervals that cross the defect some number of times. If the interval crosses the defect ε once, the corresponding Hilbert space, call it $\mathcal{H}_{a\varepsilon b}$ will be $\mathcal{H}_{a\varepsilon b} \cong \bigoplus_I (\mathbf{C}^{d_{aI}} \otimes \mathbf{C}^{\mu_I} \otimes \mathbf{C}^{d_{bI}})$. In fact, each I sector of the Hilbert space will have an additional tensor factor of \mathbf{C}^{μ_I} for each time the interval crosses a defect line ε . Likewise, there are additional closed sector Hilbert spaces, for circles that intersect the defect line some number of times. The simplest way to define a gravity path integral with the addition of such a defect line is to consider surfaces where the defect line is placed surrounding the gravity region, so that it runs along brane boundaries within the gravity region and along the interface between the gravity and nongravity regions. To be precise, with the addition of the defect, a boundary a within the gravity region gets replaced by the combined boundary and defect $a \otimes \varepsilon$, and at a triple junction of a , ε , and $a \otimes \varepsilon$ we contract the degrees of freedom in the natural way. (See figure 1.6.) Calculating the resulting correlators of \widehat{Z}_I and \widehat{S}_{abIij} is

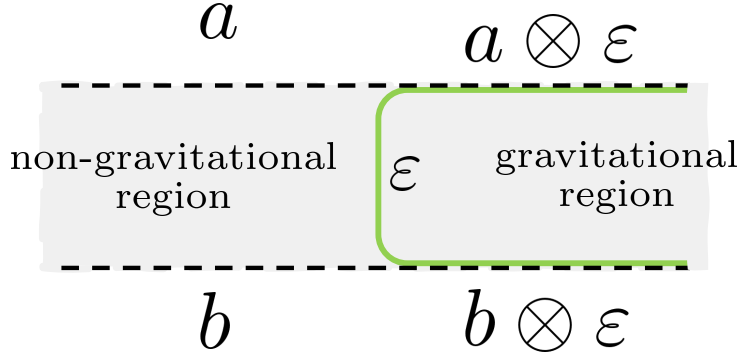


Figure 1.6: The gravitational and non-gravitational regions are separated by the defect. At the junctions where, for example, a , ε , and $a \otimes \varepsilon$ meet, we have the degrees of freedom of a and the degrees of freedom of ε propagate in the obvious way.

straightforward. For every boundary component, whether made by the \widehat{Z} , or a circle with boundary conditions a , or made from a combination of boundaries a and interval operators \widehat{S} , the TQFT action now has an additional factor of μ_I for each I sector. The result is that (1.90) is modified to

$$F(u, t) = \prod_I e^{\lambda_I e^{K_I} \exp(u_I + \sum_{j=1}^{\infty} \frac{1}{j} \text{tr}(T_I^j))} = \prod_I \exp(\lambda_I e^{K_I} e^{u_I} \det(\mathbf{1} - T_I)^{-1}), \quad (1.92)$$

This is not hard to see: each \widehat{Z}_I boundary component comes with an additional factor μ_I , effectively giving $u_I \rightarrow \mu_I u_I$; circles with boundary conditions a gets an additional factor μ_I for each I sector, giving $\lambda_I e^{K_I/\mu_I} \rightarrow \lambda_I e^{K_I}$; and finally, each boundary component made of j connected \widehat{S}_I intervals gets a factor of μ_I , giving $\text{tr}(T^j) \rightarrow \mu_I \text{tr}(T^j)$. This modified generating function for the correlators implies an ensemble of theories where $Z_I = d_{\alpha I} = N_I$ are random integers N_I , independently chosen from Poisson distributions with respective means $\lambda_I e^{K_I}$. Within the space of theories with given $Z_I = N_I$, the remaining probability distribution over the values of S_{abIij} has the generating function $\det(\mathbf{1} - T_I)^{-N_I}$ for its moments. This is the generating function for the moments of the Wishart distribution

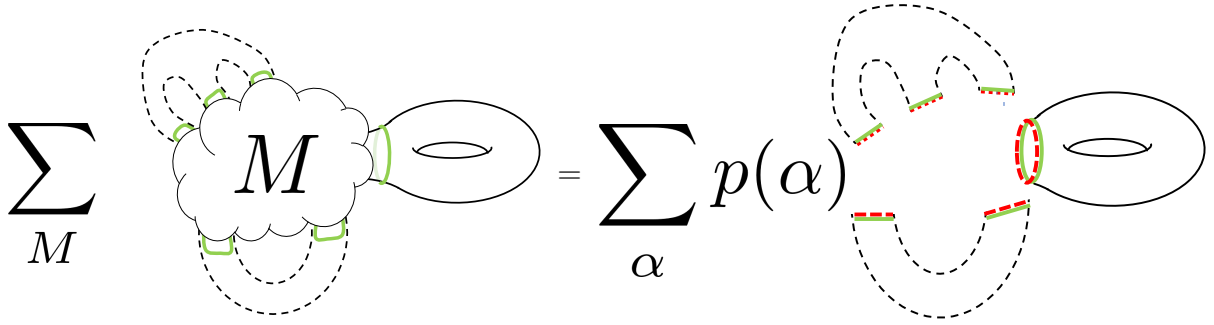


Figure 1.7: The gravity path integral where a defect line ε (green solid lines) separates the gravitational and nongravitational regions, and the boundaries within the gravity region now have boundary conditions $a \otimes \varepsilon$. This gravity region is holographically dual to an ensemble of boundary conditions α (green lines with red dashed lines) that are enhanced with the μ_I degrees of freedom from the defect, so that $d_{\alpha I} = N_I$.

[23]. Recognizing this, we can write it suggestively as

$$\det(\mathbb{1} - T_I)^{-N_I} = \pi^{-K_I N_I} \int \prod_a \left(d\Psi_{\alpha a I} d\bar{\Psi}_{\alpha a I} e^{-\text{tr}(\Psi_{\alpha a I}^\dagger \Psi_{\alpha a I})} \right) e^{\sum_{a,b} \text{tr}(T_{ab I} \Psi_{\alpha b I}^\dagger \Psi_{\alpha a I})}, \quad (1.93)$$

where by $d\Psi_{\alpha a I} d\bar{\Psi}_{\alpha a I}$ we mean integration over the $2d_{\alpha I} N_I$ -dimensional space of complex N_I by $d_{\alpha I}$ matrices $\Psi_{\alpha a I}$, and where $T_{ab I}$ is the $d_{\alpha I}$ by $d_{\beta I}$ matrix with entries $t_{ab I ij}$. We can see immediately that the operator $\hat{S}_{ab I}$ takes the matrix value $S_{ab I} = \Psi_{\alpha b I}^\dagger \Psi_{\alpha a I}$ in a given alpha-state. This is precisely consistent with the holographic picture of gravity as a boundary condition. In that case locality implies that the matrix S_{ab} should indeed factorize into the matrix multiplication of two boundary condition changing states, namely $|\psi_{b\alpha}\rangle = \bigoplus_I \Psi_{\alpha b I}^\dagger$ and $|\psi_{\alpha a}\rangle = \bigoplus_I \Psi_{\alpha a I}$. Thus we can replace the gravity region's fluctuating topology with an ensemble of boundary conditions α , as illustrated in the example of figure 1.7. The α boundary conditions are, as expected completely characterized by the dimensions $d_{\alpha I}$ and the boundary condition changing states $|\psi_{\alpha a}\rangle$, where $d_{\alpha I}$ are drawn from Poisson distributions and the entries in the matrix representation of $|\psi_{\alpha a}\rangle$ are independent complex Gaussian random variables.

There are many other ways we could choose to configure defect lines, and there may certainly be others that lead to a well-defined ensemble of boundary conditions. For example, instead of the simple junctions pictured in figure 1.6, we could allow the degrees of freedom to mix or to end at the junction. Such different setups would lead to the inclusion of additional operators in the traces $\text{tr}(T_I^j)$ in the double exponent of (1.90). The setup considered above is simply the most straightforward option, and it does, in fact, cure the problem of negative ensemble probabilities. We speculate that in a TQFT that descends from a real-istic gravity theory, a defect line separating gravitational and nongravitational regions may descend from the data that specify how the geometries of the gravity and nongravity regions are consistently glued together. For example, in the construction of [34] and subsequent papers, wherein JT gravity with CFT matter has its boundary glued to a flat region without gravity, the “boundary graviton” mode of JT gravity appears as a reparametrization of the boundary, determining how the JT gravity region is glued to the non-gravitational flat region.

The alpha-states $|\alpha\rangle$, defined similarly to (1.19), satisfy $\langle\alpha'|\alpha\rangle \sim \delta_{\alpha',\alpha} p(\alpha)$. The failure to obtain a well-defined ensemble with nonnegative probabilities is thus equivalent to the presence of negative norm states in the baby universe Hilbert space. One possible solution, then, to the problem discussed in this section would be to simply project out the negative norm states. The baby universe Hilbert space can be constructed by acting on $|\text{HH}\rangle$ with the single-boundary operators \widehat{Z} and \widehat{S} . A projection on the space of single-boundary operators would thus induce a projection on the baby universe Hilbert space. It is possible for the states projected out by such an operation to include the offending negative-norm states. In fact, the gravity model with defect separating gravity and nongravity regions is an example of precisely this. The diagram pictured in figure 1.6, when viewed from right to left, constitutes a projection.¹⁴ Namely the large Hilbert space $\mathcal{H}_{a\otimes\varepsilon, b\otimes\varepsilon}$ is mapped to the smaller Hilbert space \mathcal{H}_{ab} . The observables that non-gravitational observers have access to only include those

¹⁴The analogous map on the closed sector Hilbert space \mathcal{H}_{S^1} does not project out any states. It is simply a rescaling, though, as explained above, a crucial one for the interpretation of \widehat{Z} as a partition function in a 1d topological theory or as boundary conditions in a non-gravitational TQFT.

built from the $\dim \mathcal{H}_{ab}$ operators \widehat{S}_{abIij} , rather than the much larger number of operators that correspond to states in the $\mathcal{H}_{a \otimes \varepsilon, b \otimes \varepsilon}$ Hilbert space. Specifically, it is the defect degrees of freedom that are inaccessible.

1.6 Future directions

In our work we extended the 2d topological gravity model of [1] to a broader class of topological actions. The holographic duals of these gravity models are ensembles of 1d topological theories with random dimension. This is, in retrospect, not terribly surprising, as all 2d TQFTs are in a sense direct sums of the simplest TQFT, whose Hilbert space is one-dimensional and whose action is proportional to the Euler characteristic, like in [1].

Perhaps the most obvious limitation of the present work, then, is our restriction to TQFTs as defined by Atiyah’s axioms. In particular, TQFTs satisfying Atiyah’s axioms are always finite dimensional, so this restriction rules out many TQFTs of physical interest. These include, such TQFTs as the A- and B-models of topological string theory (both examples of the broader class of “topological conformal field theories”). Likewise, JT gravity has a description as a modified BF theory with gauge group $SL(2, \mathbb{R})$ [35], an infinite dimensional topological field theory.

Also in the spirit of working towards more realistic theories would be the extension to higher dimensional spacetimes. Several recent works attempt to make connections between a 3d bulk gravity and an ensemble of 2d CFTs on the boundary [18, 4, 5, 19]. Making sense of gravity path integrals in three dimensions, however, runs up against the difficulty that the equivalent of the genus expansion, for 3d manifolds, is not so well-behaved. Just as any 2d closed, connected, oriented manifold is the connected sum of some number of tori, any 3d closed, connected, oriented manifold can be uniquely written as the connected sum of so-called “prime” manifolds. In contrast to the simplicity of the genus expansion in 2d, the prime manifolds are infinite in number, and not completely and uniquely classified.

Besides the above extensions, there is, of course, always the possibility of considering

models of surfaces with more complicated structures like defects, foliations, or (as in [20]) spin structures.

Finally, the coupling of gravitational regions to non-gravitational TQFT regions opens other avenues for further study. We find especially interesting the question of whether a version of the black hole information paradox can be phrased in this framework, perhaps analogously to the construction in [36], or in some different way. Also of interest in this framework is the question of bulk reconstruction. For example, one could consider TQFT operators in the bulk gravity region that are appropriately “gravitationally dressed” so as to be well-defined for spacetimes without fixed topology. (A simple example of “gravitational dressing” would be to specify a fixed boundary component that the observable must remain path connected to when we allow topology to fluctuate.) Then the question arises of whether and how such bulk operators can be represented once we switch to the dual picture where gravity is an ensemble of boundary conditions.

CHAPTER 2

Null states from large superpositions of two-sided black holes

2.1 Introduction

A standard insight from holography is that some aspects of the geometry of spacetime are encoded in the entanglement of the dual holographic boundary state [37]. Entanglement is nonlinear, i.e. the entanglement of a superposition can be different from the entanglement of the terms in the superposition, so it is not a quantum observable. This leads to the observation that there cannot be a general quantum observable corresponding to geometry [38], except perhaps approximately [39]. This fact is made manifest in situations where a geometric state is equal to the superposition of states with different geometries [40, 41].

Here we provide an elementary example of such a situation. The geometries in question all consist of some number of disjoint copies of a two-sided black hole, and are distinguished from each other only by how the several left and right black hole exteriors are joined together by shared black hole interiors. With n copies of a two-sided black hole, there are $n!$ states obtained by permuting the exteriors on one side.¹ We argue on general grounds that, for a sufficiently large number n of copies, these states are not linearly independent, but can be superposed to give a null state (or at least a state that is “approximately null” in a sense we make precise). The existence of this linear dependency implies that we can write a geometry of a number of copies of a two-sided black hole as a superposition of states with the exteriors

¹Such a permutation includes a permutation of the boundaries and is thus, of course, a nontrivial diffeomorphism and not merely a gauge redundancy.

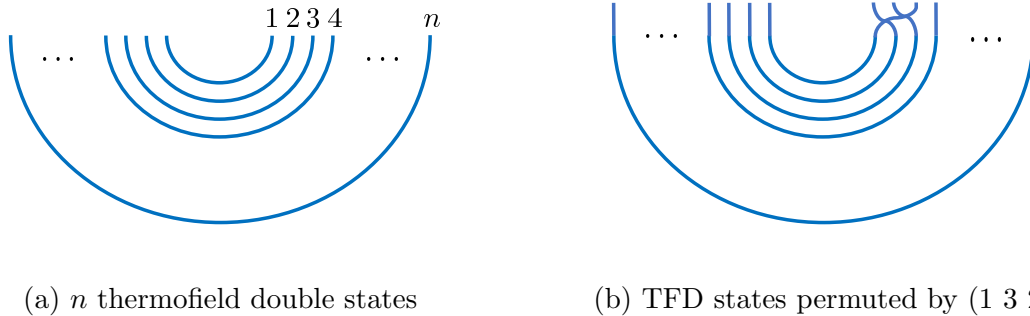


Figure 2.1: Two states: (a) A schematic diagram of the Euclidean path integral preparing n copies of a thermofield double state. (b) Diagram of n thermofield double states permuted by $(1\ 3\ 2)$. We denote this state by $|(1\ 3\ 2)\rangle$.

joined up differently.

On its face, the possibility of writing a configuration of wormholes as a superposition of wormholes connected up differently raises some conceptual questions. For example, if observers jump into opposite sides of one of n AdS-Schwarzschild black holes, they could in principle meet each other in the black hole interior. Yet our goal is to rewrite such a state of n black holes as a superposition of different geometries in most of which meeting is impossible because the observers' black holes are distinct and disconnected.

We also explore another possibility that makes the conceptual question sharper. Is it possible for a superposition of geometries, none of which connect the two observers, to be equivalent to a connected two-sided black hole geometry? More precisely, can we write a single two-sided black hole as a partial trace of a superposition of geometries that are all disconnected? In this case, were the observers to jump in, they would meet in the middle, even though this is impossible in any term in the superposition considered separately. This would imply that there is no quantum observable corresponding to the question of whether the two observers meet. We argue that such superpositions do exist approximately in the limit of large number n of black holes.

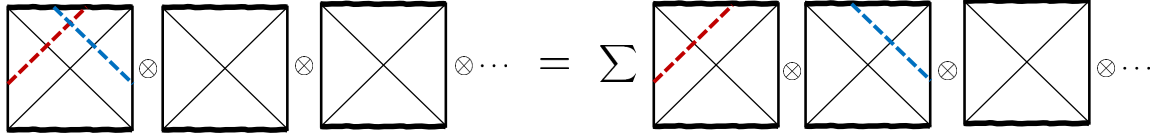


Figure 2.2: Left: Alice (red) and Bob (blue) jump into a black hole. Right: their trajectories do not intersect in any term of an equivalent superposition.

2.2 Some illustrative examples

We illustrate the idea with a very simple example involving qubits shared between two parties, Alice and Bob. Suppose Alice and Bob share 3 different Bell pairs of the form $|0\rangle|0\rangle + |1\rangle|1\rangle$. Suppose further that after Alice and Bob divvy up the qubits, Bob loses track of which of his qubits are which. In other words, some unknown permutation of Bob's three qubits occurs. There are 6 possibilities for the state shared by Alice and Bob. Explicitly, we have

$$\begin{aligned}
 |(1)\rangle &= \frac{1}{\sqrt{8}} \sum_{ijk} |i\rangle_A |j\rangle_A |k\rangle_A \otimes |i\rangle_B |j\rangle_B |k\rangle_B, \\
 |(23)\rangle &= \frac{1}{\sqrt{8}} \sum_{ijk} |i\rangle_A |j\rangle_A |k\rangle_A \otimes |i\rangle_B |k\rangle_B |j\rangle_B, \\
 |(12)\rangle &= \frac{1}{\sqrt{8}} \sum_{ijk} |i\rangle_A |j\rangle_A |k\rangle_A \otimes |j\rangle_B |i\rangle_B |k\rangle_B, \\
 |(13)\rangle &= \frac{1}{\sqrt{8}} \sum_{ijk} |i\rangle_A |j\rangle_A |k\rangle_A \otimes |j\rangle_B |k\rangle_B |i\rangle_B, \\
 |(123)\rangle &= \frac{1}{\sqrt{8}} \sum_{ijk} |i\rangle_A |j\rangle_A |k\rangle_A \otimes |k\rangle_B |i\rangle_B |j\rangle_B, \\
 |(132)\rangle &= \frac{1}{\sqrt{8}} \sum_{ijk} |i\rangle_A |j\rangle_A |k\rangle_A \otimes |k\rangle_B |j\rangle_B |i\rangle_B,
 \end{aligned}$$

labeled here by permutations. The situation is that they no longer know which of Alice's qubits are entangled with which of Bob's. To what extent are Alice and Bob able to determine which qubits are entangled with which? The task cannot be completed deterministically,

as there are nontrivial overlaps between the six states. For example, $\langle(1)|(12)\rangle = \frac{1}{2}$ and $\langle(1)|(123)\rangle = \frac{1}{4}$. An immediate consequence of this is that “which qubit is entangled with which” is not a quantum observable.

The situation is improved if Alice and Bob share entangled qudit pairs $\sum_{i=1}^N \frac{1}{\sqrt{N}} |i\rangle_A |i\rangle_B$ where N is large. Then the nontrivial overlaps between states are order $\mathcal{O}(N^{-1})$ or smaller, making the different situations approximately distinguishable. Alice and Bob can engineer joint measurements that will correctly determine which qudits are entangled with high probability. This is true even when Alice and Bob share large numbers of qudit pairs. In effect, more entanglement between the qudits helps “keep track” of which are matched with which.

An interesting situation arises when Alice and Bob share sufficiently large numbers of entangled pairs. Let n be the number of shared pairs. Then there are $n!$ states labeled by permutations, which live in a joint Hilbert space of size N^{2n} . As $n!$ grows faster than N^{2n} , for sufficiently large n the $n!$ states labeled by permutations must be linearly dependent. In fact, linear dependence happens much sooner than the above dimension-counting argument suggests. We get a nontrivial linear dependence whenever $n > N$.

More generally, suppose Alice and Bob share n qudit pairs of the form $\sum_{i=1}^N \alpha_i |i\rangle |i\rangle$. There is still a linear dependence whenever $n > N$, no matter what the coefficients α_i are. Specifically, the superposition

$$|\psi\rangle = \sum_{\pi \in S_n} \sigma(\pi) |\pi\rangle, \quad (2.1)$$

where $\sigma(\pi)$ is the sign of the permutation π , is null when $n > N$. As we will explain in more detail later, $\langle\psi|\psi\rangle$ is proportional to the sum

$$\sum_{i_1 < i_2 < \dots < i_n} \alpha_{i_1}^2 \alpha_{i_2}^2 \dots \alpha_{i_n}^2 \quad (2.2)$$

which contains no terms when $n > N$.

In summary, for sufficiently large n there is a nontrivial null state made up of superpositions of different permutation states. By symmetry under S_n , the existence of such a null state implies that the unpermuted state $|(1)\rangle$ is equivalent to some superposition of permuted states.

2.3 Shared thermofield double states

In the context of the AdS/CFT correspondence, the thermofield double state $|\text{TFD}\rangle = \sum_i e^{-\beta/2E_i} |E_i\rangle |E_i\rangle$ is dual to a two-sided Schwarzschild black hole with inverse temperature β . The density matrix for one side of the thermofield double (TFD) state describes the exterior on one side of the black hole, whereas the full state including the entanglement information between the two sides describes the full geometry including the interior of the black hole [37].

Suppose Alice and Bob share n such thermofield double states, but neither knows which of their n systems is entangled with which of the other's. This is analogous to the situation described in the previous section but now with an additional interpretation in terms of bulk geometries: Alice and Bob each have access to n black hole exteriors. Each of Alice's exteriors is connected to one of Bob's via a black hole interior, but they do not know which is connected to which. Alice and Bob are unable to perfectly determine which of the $n!$ geometries they have. What's more, as we will argue, a linear dependency emerges as n grows larger, so that any one situation can be approximately written as a linear combination of the others.

To be more precise, consider a boundary theory on a spatial manifold Σ . Let $|\text{TFD}(\beta)\rangle \in H_L \otimes H_R$ be the thermofield double state in this theory with inverse temperature β . Now consider n copies of this thermofield double, i.e. the state $|\text{TFD}(\beta)\rangle |\text{TFD}(\beta)\rangle \cdots |\text{TFD}(\beta)\rangle = |\text{TFD}(\beta)\rangle^{\otimes n}$ in the Hilbert space $H_L^{\otimes n} \otimes H_R^{\otimes n}$. We can act on this state $|\text{TFD}(\beta)\rangle^{\otimes n}$ by permuting the right boundaries, in other words by permuting the n copies of the Hilbert space H_R . There are $n!$ possible such boundary permutations, described by the symmetric group S_n . Denote the state resulting from a permutation $\pi \in S_n$ as $|\pi\rangle \equiv \pi \circ |\text{TFD}(\beta)\rangle^{\otimes n}$.

First, note that the inner product between two states, $|\pi\rangle$ and $|\tilde{\pi}\rangle$, is $\langle \tilde{\pi} | \pi \rangle = \langle \text{id} | \tilde{\pi}^{-1} \pi \rangle = \langle \tilde{\pi}^{-1} \pi \rangle$, where for notational simplicity we denote inner products $\langle \text{id} | \pi \rangle$ by $\langle \pi \rangle$. Let $Z(\beta)$ be the partition function on $\Sigma \times S^1$ where the circle has length β . Then we can evaluate $\langle \pi \rangle$ as

$$\langle \pi \rangle = \prod_{\text{cycles } i \text{ of } \pi} Z(n_i \beta) \quad (2.4)$$

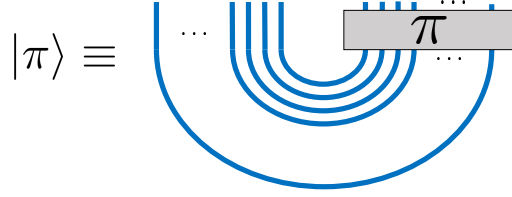


Figure 2.3: The state $|\pi\rangle$

$$\langle \text{id} | (1\ 2) \rangle = \text{Diagram} \quad (2.3)$$

Figure 2.4: The inner product $\langle \text{id} | (1\ 2) \rangle = \langle (1\ 2) \rangle$. Normalized, this is $Z(2\beta)/Z(\beta)^2$.

where n_i are the sizes of the cycles. E.g. $\sum_i n_i = n$. Note for later that $\langle \pi \rangle = \langle \pi^{-1} \rangle$.

The overlaps between these states will tend to be small. The largest overlap, $\langle \text{id} | (1\ 2) \rangle$, will be smaller than the norms $\langle \text{id} | \text{id} \rangle = Z(\beta)^n$ by a factor $Z(2\beta)/Z(\beta)^2$, which will be small on the order of the typical relative spacing $(e^{-\beta E_i} - e^{-\beta E_{i+1}})/e^{-\beta E_i} \approx \beta(E_{i+1} - E_i)$. The smallness of the overlaps implies that the question of which sides of the TFD states are connected to which is *approximately* a quantum observable, and that this could break down for large superpositions of such permutation states.² In section 2.2 we saw that for finite dimensional systems a counting argument implies linear dependence between the $n!$ permutation states for sufficiently large n . In the more general case of infinite dimensional Hilbert spaces, the counting argument no longer holds. To the extent that we can approximate a TFD state (and its dual black hole state) by truncating energies above a threshold,

²In the special case where there is a Hagedorn temperature $1/\beta_*$, exact orthogonality is achieved in the limit $\beta \rightarrow \beta_*$, as $Z(2\beta)/Z(\beta)^2$ will become zero.

we can again obtain a null state by having sufficiently large n . This suggests the possibility of obtaining a null state in the limit of large n , or a state whose difference from the null state (in a way we will make precise) goes to 0 as n goes to infinity. One goal of this work is to get a handle on such “approximate null states.”

A general superposition of n permuted thermofield doubles is $|A\rangle = \sum_{\pi} \alpha_{\pi} |\pi\rangle$ and has norm

$$\langle A|A\rangle = \sum_{\pi} \sum_{\tilde{\pi}} \bar{\alpha}_{\tilde{\pi}} \alpha_{\pi} \langle \tilde{\pi}|\pi\rangle = \sum_{\pi} \sum_{\tilde{\pi}} \bar{\alpha}_{\tilde{\pi}} \alpha_{\pi} M_{\tilde{\pi}\pi} \quad (2.5)$$

where M is the matrix with entries $M_{\tilde{\pi}\pi} = \langle \tilde{\pi}|\pi\rangle = \langle \tilde{\pi}^{-1}\pi\rangle$. Null states made up of the states $|\pi\rangle$ will correspond to eigenvectors of M with eigenvalue 0.

It isn't hard to find the eigenvalues of M , given its particular structure in terms of the group S_n . Let $U^{(q)}(\pi)$ be the irreps of S_n , labeled by q . The eigenstates of M are then the states of the form

$$|q; ij\rangle \equiv \sum_{\pi} U_{ij}^{(q)}(\pi) |\pi\rangle, \quad (2.6)$$

where $U_{ij}^{(q)}(\pi)$ is the ij entry of the matrix $U^{(q)}(\pi)$. This is simple to check using Schur orthogonality. First note that $\langle \pi\rangle$ is a class function, so it can be uniquely expanded as a sum of characters χ^q of S_n :

$$\langle \pi\rangle = \sum_q c_q \chi^q(\pi). \quad (2.7)$$

Then

$$\begin{aligned} \langle p; kl|q; ij\rangle &= \sum_{\pi} \sum_{\tilde{\pi}} \overline{U_{kl}^{(p)}(\tilde{\pi})} U_{ij}^{(q)}(\pi) \langle \tilde{\pi}^{-1}\pi\rangle \\ &= \sum_r c_r \sum_{\pi} \sum_{\tilde{\pi}} \overline{U_{kl}^{(p)}(\tilde{\pi})} U_{ij}^{(q)}(\pi) \sum_m U_{mm}^{(r)}(\pi^{-1}\tilde{\pi}) \\ &= \sum_r c_r \sum_{\pi} \sum_{\tilde{\pi}} \overline{U_{kl}^{(p)}(\tilde{\pi})} U_{ij}^{(q)}(\pi) \sum_m \sum_h U_{mh}^{(r)}(\pi^{-1}) U_{hm}^{(r)}(\tilde{\pi}) \\ &= \sum_m \sum_h \sum_r c_r \sum_{\tilde{\pi}} \overline{U_{kl}^{(p)}(\tilde{\pi})} U_{hm}^{(r)}(\tilde{\pi}) \sum_{\pi} U_{ij}^{(q)}(\pi) \overline{U_{hm}^{(r)}(\pi)} \\ &= \sum_m \sum_h \sum_r c_r \frac{n!}{d_p} \delta_{pr} \delta_{kh} \delta_{lm} \frac{n!}{d_q} \delta_{qr} \delta_{ih} \delta_{jm} \\ &= \delta_{pq} \delta_{ki} \delta_{lj} \frac{n!^2}{d_q^2} c_q, \end{aligned} \quad (2.8)$$

where d_q is the dimension of the irrep q . We used Schur orthogonality between the fourth and fifth lines, and the fact $\langle \pi \rangle = \langle \pi^{-1} \rangle$ between the first and second lines. So the states $|q; ij\rangle$ are orthogonal, with norm-squared $\frac{n!^2}{d_q^2} c_q$. They give an orthogonal basis for the space spanned by the $|\pi\rangle$ states.

This means that any null states will be precisely those $|q; ij\rangle$ for which c_q is 0. By their definition, the coefficients c_q are

$$c_q = \frac{1}{n!} \sum_{\pi} \chi^q(\pi) \langle \pi \rangle. \quad (2.9)$$

In general, for q other than the trivial representation, the values $\chi^q(\pi)$ can be either positive or negative. So we can't rule out null states a priori, even if we do not expect them exactly for finite n in an infinite dimensional Hilbert space. We will tend to assume that the c_q are nonzero.

It is worth pointing out, that the coefficients c_q are the Schur polynomials [42] (for finite dimensional systems, Schur functions more generally) in the variables $e^{-\beta E_1}, e^{-\beta E_2}, \dots$. Specifically, the Schur function $s_{\lambda}(e^{-\beta E_1}, e^{-\beta E_2}, \dots)$ for a partition λ of n is the coefficient c_q for the S_n irrep corresponding to λ . In Section 2.2 we have already used the fact that $c_{\sigma} = c_{(1,1,\dots,1)}$, the coefficient corresponding to the sign irrep σ of S_n , is the Schur polynomial $s_{(1,1,\dots,1)}$ corresponding to the partition $1 + \dots + 1 = n$, together with the fact that the Schur polynomial $s_{(1,1,\dots,1)}(x_1, x_2, \dots)$ equals the n -th elementary polynomial

$$e_n(x_1, x_2, \dots) \equiv \sum_{i_1 < \dots < i_n} x_{i_1} \cdots x_{i_n}. \quad (2.10)$$

For simplicity and clarity we will sometimes notate partitions with ‘‘exponential notation’’, so that e.g. $c_{(1,1,\dots,1)}$ is written $c_{(1^n)}$. We will find the fact

$$c_{(1^n)} = \sum_{i_1 < \dots < i_n} e^{-\beta(E_{i_1} + \dots + E_{i_n})} \quad (2.11)$$

to be of importance later. One additional fact about Schur polynomials we will use is that

$$\left(\sum_i x_i \right) s_{\lambda}(x_1, x_2, \dots) = \sum_{\mu} s_{\mu}(x_1, x_2, \dots), \quad (2.12)$$

where the sum on the RHS is over partitions μ obtainable by adding a single box to the young diagram of λ .

2.3.1 Nearly null states

On its face it does not make much sense to refer to a state as “almost” null, as any state whose norm is not zero can be normalized to have norm 1. To define a sense of “nearly null” states, or even a sequence of states whose limit is the null state we must have some additional criterion that determines a preferred normalization. A nearly null state is then one whose preferred normalization, in this sense, is small.

One outcome of having a null state, say $|a\rangle - |b\rangle = 0$, is that we can substitute $|b\rangle$ for $|a\rangle$ in expressions and get the same result. We can take an “approximate null state” $|a\rangle - |b\rangle \approx 0$ to include the criterion that replacing $|a\rangle$ with $|b\rangle$ gives us *approximately* the same result. For example, we could require that the fidelity between $|a\rangle$ and $|b\rangle$ be close to one. This is equivalent to the norm of $|a\rangle - \lambda |b\rangle$ being small relative to the norm of $|a\rangle$ for some choice of λ .

We are interested in states of the form $|\text{geometry}\rangle + \sum_i \alpha_i |\text{other geometry } i\rangle$ and the possibility $|\text{geometry}\rangle \approx -\sum_i \alpha_i |\text{other geometry } i\rangle$. So the notion of “nearly null” we will be interested in is states with norm that is small compared to the norm of a distinguished term $|\text{geometry}\rangle$ or states $|\phi\rangle$ such that the fidelity between $|\text{geometry}\rangle$ and $|\phi\rangle - |\text{geometry}\rangle$ is close to 1. This definition of course only makes sense relative to some distinguished vector or vectors. In our context we have the distinguished vector $|\text{id}\rangle = |\text{TFD}\rangle^{\otimes n}$ and the states obtained by permutations acting on this.

First we consider the possibility of writing the state of n unpermuted two-sided black holes, $|\text{id}\rangle$, as a superposition $|\psi\rangle = \sum_{\pi \neq \text{id}} \alpha_\pi |\pi\rangle$ of permuted states. We wish to find the maximum fidelity between $|\text{id}\rangle$ and $|\psi\rangle$:

$$F_{\max}(n) = \max_{|\psi\rangle} \frac{|\langle \text{id} | \psi \rangle|^2}{\langle \text{id} | \text{id} \rangle \langle \psi | \psi \rangle} \quad (2.13)$$

where the maximization is over states $|\psi\rangle$ that are linear combinations only of $|\pi\rangle$ where $\pi \neq \text{id}$. We can evaluate (2.13) by expanding in the $|q; ij\rangle$ basis. First we expand $|\psi\rangle$ as

$$|\psi\rangle = \sum_{\pi \neq \text{id}} \alpha_\pi |\pi\rangle = \sum_q \sum_{i,j=1}^{d_q} \gamma_{qij} |q; ij\rangle \quad (2.14)$$

where γ_{qij} are coefficients satisfying

$$\alpha_\pi = \sum_q \sum_{i,j=1}^{d_q} \gamma_{qij} U_{ij}^{(q)}(\pi). \quad (2.15)$$

In terms of the γ_{qij} coefficients, the condition that α_{id} be zero is given by

$$\sum_q \sum_{i=1}^{d_q} \gamma_{qii} = 0. \quad (2.16)$$

Now note that

$$\begin{aligned} \langle \text{id}|q; ij \rangle &= \sum_\pi U_{ij}^{(q)}(\pi) \langle \pi \rangle \\ &= \sum_\pi U_{ij}^{(q)}(\pi) \left(\sum_p c_p \chi_p(\pi) \right) \\ &= \sum_k \sum_p c_p \sum_\pi U_{ij}^{(q)}(\pi) U_{kk}^{(p)}(\pi) \\ &= \sum_k \sum_p c_p \delta_{i,k} \delta_{j,k} \delta_{\bar{q},p} \frac{n!}{d_p} \\ &= \delta_{ij} \frac{n!}{d_q} c_q \end{aligned} \quad (2.17)$$

where we have used Schur orthogonality as well as the fact that the c_q are real. This fact now gives us

$$\langle \text{id}|\psi \rangle = \sum_q \sum_{i=1}^{d_q} \gamma_{qii} \frac{n!}{d_q} c_q \quad (2.18)$$

as well as

$$\langle \psi|\psi \rangle = \sum_q \sum_{i,j=1}^{d_q} \overline{\gamma_{qij}} \gamma_{qij} \frac{n!^2}{d_q^2} c_q. \quad (2.19)$$

Plugging these expressions into (2.13), we get

$$F_{\max} = \max_\gamma \frac{\left| \sum_q \frac{n!}{d_q} c_q \text{tr} \gamma_q \right|^2}{Z^n \sum_q \frac{n!^2}{d_q^2} c_q \text{tr}(\gamma_q^\dagger \gamma_q)}. \quad (2.20)$$

where the maximization is subject to the constraint that $\sum_q \sum_{i=1}^{d_q} \gamma_{qii} = 0$, and γ_q denotes the d_q -by- d_q matrix whose entries are γ_{qij} . Splitting the matrix γ_q up into its trace and traceless parts, $\gamma_q = \frac{\text{tr} \gamma_q}{d_q} \mathbf{1} + X$ we see that $\text{tr}(\gamma_q^\dagger \gamma_q) = \frac{1}{d_q} \text{tr}(\gamma_q)^2 + \text{tr}(X^\dagger X)$. The traceless

degrees of freedom of γ_q do not participate in either the constraint, or the numerator of (2.20). They do add a nonnegative contribution to the denominator of (2.20), however, which can only decrease F . So, because they don't participate in the constraint, they will be 0 when F is maximized. Define

$$\lambda_q = \sqrt{Z^n \frac{n!^2 c_q}{d_q^3}} \text{tr } \gamma_q. \quad (2.21)$$

With the traceless part of γ_q set to zero the max fidelity is now

$$F_{\max} = \max_{\lambda} \frac{\left| \sum_q \sqrt{\frac{d_q c_q}{Z^n}} \lambda_q \right|^2}{\sum_q |\lambda_q|^2} \quad (2.22)$$

where the maximization is over the λ_q and the constraint can be written

$$\sum_q \sqrt{\frac{d_q^3}{Z^n n!^2 c_q}} \lambda_q = 0. \quad (2.23)$$

Written this way, F_{\max} is simply the maximum (normalized) overlap between the vectors $\vec{u} = \left(\dots, \sqrt{\frac{d_q c_q}{Z^n}}, \dots \right)$ and $\vec{\lambda} = (\dots, \lambda_q, \dots)$ with the constraint that $\vec{\lambda}$ is orthogonal to the vector $\vec{v} = \left(\dots, \sqrt{\frac{d_q^3}{Z^n n!^2 c_q}}, \dots \right)$. You can then see that the maximum is obtained by choosing $\vec{\lambda}$ in the plane defined by \vec{u} and \vec{v} and orthogonal to $\vec{\lambda}$. Where θ is the angle between \vec{u} and \vec{v} , the angle between the optimal $\vec{\lambda}$ and \vec{v} is $\frac{\pi}{2} - \theta$, making the maximum fidelity $F_{\max} = (\cos(\frac{\pi}{2} - \theta))^2 = 1 - (\cos \theta)^2$. To easily get θ :

$$\cos \theta = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \|\vec{v}\|} = \frac{\sum_q \frac{d_q^2}{Z^n n!}}{\sqrt{\sum_q \frac{d_q c_q}{Z^n}} \sqrt{\sum_q \frac{d_q^3}{Z^n n!^2 c_q}}}. \quad (2.24)$$

After simplifications from the facts $\sum_q d_q^2 = n!$ and $\sum_q d_q c_q = Z(\beta)^n$,³ this gives the maximum fidelity

$$F_{\max} = 1 - \frac{n!^2}{Z(\beta)^n \sum_q \frac{d_q^3}{c_q}}. \quad (2.25)$$

We remind the reader that the sum is over irreps q of S_n so that the sum, the dimensions d_q and the coefficients c_q all have a dependence on n .

³The second fact here can be seen by expanding the definitions of the c_q then noting that the LHS is $\frac{1}{n!} \sum_{\pi} \chi_{\text{reg}}(\pi) \langle \pi \rangle$, where $\chi_{\text{reg}}(\pi) = n! \delta_{\text{id}, \pi}$ is the regular representation.

This result gives us a criterion for having an approximate null state: is $n!^2 / \left(Z^n \sum_q d_q^3 / c_q \right)$ small? Of course determining the coefficients c_q or even getting a bound on them may be difficult in general. If any of $c_q(n)/Z^n$ go to zero as n gets large, the fidelity goes to 1. This is also consistent with the fact already seen that there is an exact null state whenever one of the c_q is zero. Further note that in the above derivation of F_{\max} we have not used any assumptions about the partition function, in particular we have not assumed either finite dimensions or a discrete spectrum.

We can go further and obtain an actual state that instantiates the above maximum fidelity. This will be the projection of \vec{u} to the subspace orthogonal to \vec{v} . So

$$\vec{\lambda}_{\max} \sim \vec{u} - \frac{\vec{v} \cdot \vec{u}}{\|\vec{v}\|^2} \vec{v}. \quad (2.26)$$

Unpacking our definition of λ_q in terms of $\text{tr}(\gamma_q)$ we get

$$\text{tr}(\gamma_q) \sim \frac{d_q^2}{Z^n n!} - \frac{d_q^3 / c_q}{Z^n \sum_p d_p^3 / c_p} \quad (2.27)$$

after some simplification. This gives the coefficients

$$\gamma_{qij} \sim \delta_{ij} \left(\frac{d_q}{Z^n n!} - \frac{d_q^2 / c_q}{Z^n \sum_p d_p^3 / c_p} \right). \quad (2.28)$$

This result, in terms of the original coefficients α_π , is

$$\begin{aligned} \alpha_\pi &\sim \sum_q \sum_{i,j=1}^{d_q} \delta_{ij} \left(\frac{d_q}{Z^n n!} - \frac{d_q^2 / c_q}{Z^n \sum_p d_p^3 / c_p} \right) U_{ij}^{(q)}(\pi) \\ &= \frac{\chi_{\text{reg}}(\pi)}{Z^n n!} - \frac{1}{Z^n} \frac{\sum_q d_q^2 \chi^q(\pi) / c_q}{\sum_p d_p^3 / c_p} \\ &\sim \delta_{\pi, \text{id}} - \frac{\sum_q d_q^2 \chi^q(\pi) / c_q}{\sum_q d_q^2 \chi^q(\text{id}) / c_q}. \end{aligned} \quad (2.29)$$

Written this way, we can see that α_{id} is indeed 0, consistent with the constraint on the optimization.

We have determined the optimal $|\psi\rangle$ up to an overall constant. To fix the constant, remember that we are interested in interpreting the state $|\text{id}\rangle - |\psi\rangle$ as a “nearly null” state.

We choose the phase of $|\psi\rangle$ so that $\langle \text{id}|\psi\rangle$ is real and positive, and we choose the norm to be close to the norm of $|\text{id}\rangle$. Taking α_π to be equal to the last line of (2.29) gives

$$\langle \text{id}|\psi\rangle = Z^n F_{\max}, \quad (2.30)$$

which is positive. And likewise

$$\langle \psi|\psi\rangle = Z^n F_{\max} \quad (2.31)$$

which for F_{\max} close to 1 as desired will be approximately $Z^n = \langle \text{id}|\text{id}\rangle$. Thus, granted F_{\max} is close to 1, we get a suitable “nearly null” state in

$$|\text{id}\rangle - |\psi\rangle = |\text{id}\rangle + \sum_{\pi} \left(\frac{\sum_q \frac{d_q^2}{c_q} \chi^q(\pi)}{\sum_q \frac{d_q^2}{c_q} \chi^q(\text{id})} - \delta_{\pi, \text{id}} \right) |\pi\rangle \quad (2.32)$$

or, written more simply,

$$|\text{nearly null}\rangle \sim \sum_{\pi \in S_n} \left(\sum_q \frac{d_q^2}{c_q} \chi^q(\pi) \right) |\pi\rangle. \quad (2.33)$$

2.3.2 Meeting in a black hole interior

In the previous Section 2.3.1 we found a candidate state $|\text{id}\rangle - \sum_{\pi \neq \text{id}} \alpha_\pi |\pi\rangle$ for the property $|\text{id}\rangle \approx \sum_{\pi \neq \text{id}} \alpha_\pi |\pi\rangle$. Suppose this were an equality. Then by combining on one side the terms $|\pi\rangle$ where $\pi(1) = 1$, we could write this in the form

$$|\text{TFD}\rangle \otimes \sum_{\tilde{\pi} \in S_{n-1}} \alpha_{\tilde{\pi}} |\tilde{\pi}\rangle = - \sum_{\substack{\pi \in S_n \\ \pi(1) \neq 1}} \alpha_\pi |\pi\rangle. \quad (2.34)$$

In a holographic context, the RHS would be a superposition of geometries, none of which connect the first black hole exterior on each side, but whose partial trace down to the first black hole is a connected black hole geometry.⁴ Two observers jumping into the first black hole could in principle meet in the middle, this is despite the fact in no term in the RHS of (2.34) has a geometry that connects the observers.

⁴There is also the possibility that $\sum_{\tilde{\pi} \in S_{n-1}} \alpha_{\tilde{\pi}} |\tilde{\pi}\rangle$ is itself null. If this is the case, we apply the same consideration to it, extracting terms that have $|\text{TFD}\rangle$ on the *second* black hole, and so on. This process will bottom out eventually, as we have fixed the coefficient of $|\text{id}\rangle$ to be 1.

We extend our search to superpositions of states $|\pi\rangle$ with $\pi(1) \neq 1$ whose partial trace down to the first copy of $H_L \otimes H_R$ is approximately the thermofield double state. This previous paragraph shows that this is a weaker condition than that considered in Section 2.3.1, and it will turn out to be easier to get good bounds on the fidelity for this case. So consider a superposition $|\psi\rangle = \sum_{\pi, \pi(1) \neq 1} \alpha_\pi |\pi\rangle$ and its partial trace $\rho_1 = \text{tr}_{2\dots n} (|\psi\rangle\langle\psi|)$. We want to maximize the fidelity

$$F = \frac{\langle \text{TFD} | \rho_1 | \text{TFD} \rangle}{\langle \text{TFD} | \text{TFD} \rangle \text{tr}(\rho_1)} = \frac{\langle \psi | (| \text{TFD} \rangle \langle \text{TFD} | \otimes \mathbf{1}_{2\dots n}) | \psi \rangle}{\langle \text{TFD} | \text{TFD} \rangle \langle \psi | \psi \rangle} \quad (2.35)$$

Like before, we will expand this in the irrep basis $|q; ij\rangle$, though this time we will consider both the basis $|q; ij\rangle_{1\dots n} = \sum_{\pi \in S_n} U_{ij}^{(q)}(\pi) |\pi\rangle_{1\dots n}$ for n copies of $H_L \otimes H_R$ and the basis $|p; k\ell\rangle_{2\dots n} = \sum_{\pi \in S_{n-1}} U_{k\ell}^{(p)}(\pi) |\pi\rangle_{2\dots n}$ for only the the last $n - 1$ copies of $H_L \otimes H_R$. We will allow ourselves to drop the subscripts when there is no ambiguity and will tend to use q for irreps of S_n and p for irreps of S_{n-1} in what follows.

Let $P = \sum_{p, k, \ell} |p; k\ell\rangle\langle p; k\ell| / \langle p; k\ell | p; k\ell \rangle$ be the projector onto the subspace spanned by the states $|\pi\rangle_{2\dots n}$ with $\pi \in S_{n-1}$. Replacing the identity with P in the expression for fidelity gives a lower bound

$$F \geq \frac{\langle \psi | (| \text{TFD} \rangle \langle \text{TFD} | \otimes P) | \psi \rangle}{\langle \text{TFD} | \text{TFD} \rangle \langle \psi | \psi \rangle}. \quad (2.36)$$

The norms $\langle p; k\ell | p; k\ell \rangle$ are $n!^2 c_p / d_p^2$. Expanding $|\psi\rangle = \sum_{q, i, j} \gamma_{qij} |q; ij\rangle$ and P , we get an expression involving inner products $(\langle \text{TFD} | \langle p; k\ell |) |q; ij\rangle$. These can be worked out. An irrep q of S_n will split when restricted to S_{n-1} into those irreps whose young diagrams are obtained by removing a single corner block from the young diagram of q . Assume we chose our bases such that the matrices $U_{ij}^{(q)}(\pi)$ for S_n irrep q are block diagonal when restricted to $\pi \in S_{n-1}$ and so that the indices k and ℓ match an appropriate subset of the indices i and j , when p is a block in the restriction of q . We have

$$\begin{aligned} (\langle \text{TFD} | \otimes \langle p; k\ell |) |q; ij\rangle &= \sum_{\tilde{\pi} \in S_{n-1}} \sum_{\pi \in S_n} \overline{U_{k\ell}^{(p)}(\tilde{\pi})} U_{ij}^{(q)}(\pi) \langle \tilde{\pi} | \pi \rangle \\ &= \sum_m \sum_{\tilde{\pi} \in S_{n-1}} \sum_{\pi \in S_n} \overline{U_{k\ell}^{(p)}(\tilde{\pi})} U_{im}^{(q)}(\tilde{\pi}) U_{mj}^{(q)}(\pi) \langle \tilde{\pi} | \pi \rangle. \end{aligned} \quad (2.37)$$

Expanding $U_{im}^{(q)}(\tilde{\pi})$ into blocks and using Schur orthogonality we obtain

$$(\langle \text{TFD} | \otimes \langle p; k\ell |) |q; ij\rangle = \frac{(n-1)!}{d_p} \delta_{p \subset q} \delta_{ki} \sum_{\pi \in S_n} U_{\ell j}^{(q)}(\pi) \langle \pi |, \quad (2.38)$$

where by $\delta_{p \subset q}$ we mean 1 when p is in the restriction of q and 0 if it is not, and δ_{ki} is meant to be understood as enforcing that the k -th index of the copy of p in q matches the i -th index of q . The remaining sum here evaluates to $\delta_{\ell j} n! c_q / d_q$, which can be seen by expanding $\langle \pi |$ into characters of S_n then using Schur orthogonality. So in all we have

$$(\langle \text{TFD} | \otimes \langle p; k\ell |) |q; ij\rangle = \delta_{p \subset q} \delta_{ki} \delta_{\ell j} \frac{(n-1)! n!}{d_p d_q} c_q. \quad (2.39)$$

After expanding $|\psi\rangle$ and P into the $|q; ij\rangle$ and $|p; k\ell\rangle$ bases respectively, our bound (2.36) becomes

$$F \geq \frac{\sum_{p,k,\ell} \sum_{q \supset p} \sum_{\tilde{q} \supset p} \overline{\gamma_{\tilde{q}k\ell}} \gamma_{qk\ell} \frac{c_{\tilde{q}} c_q}{d_{\tilde{q}} d_q} \frac{1}{c_p}}{Z(\beta) \sum_{p,k,\ell} \sum_{q \supset p} |\gamma_{qk\ell}|^2 \frac{c_q}{d_q^2}}. \quad (2.40)$$

The constraint on $|\psi\rangle$ that α_π be zero for all π such that $\pi(1) = 1$ becomes

$$0 = \sum_{qij} \gamma_{qij} U_{ij}^{(q)}(\pi) \quad \text{for } \pi \in S_{n-1}. \quad (2.41)$$

Split into p blocks this is

$$0 = \sum_{pk\ell} \sum_{q \supset p} \gamma_{qk\ell} U_{k\ell}^{(p)}(\pi) \quad \text{for } \pi \in S_{n-1}. \quad (2.42)$$

Together, these conditions are equivalent to

$$0 = \sum_{q \supset p} \gamma_{qk\ell} \quad \text{for all } p, k, \text{ and } \ell, \quad (2.43)$$

the constraint that the sum of all blocks in γ_{qij} corresponding to an S_{n-1} irrep must be the zero matrix. Define $\lambda_{pk\ell q} = \frac{\sqrt{c_q}}{d_q} \gamma_{qk\ell}$ where, for a given p , q ranges over irreps of S_n that include the S_{n-1} irrep p in their restriction. Then (2.40) becomes

$$F \geq \frac{\sum_{p,k,\ell} \frac{1}{c_p} \sum_{\tilde{q} \supset p} \sqrt{c_{\tilde{q}}} \overline{\lambda_{pk\ell \tilde{q}}} \sum_{q \supset p} \sqrt{c_q} \lambda_{pk\ell q}}{Z(\beta) \vec{\lambda} \cdot \vec{\lambda}}. \quad (2.44)$$

Given fixed contribution $a_{pk\ell}^2 = \sum_{q \supset p} |\lambda_{pk\ell q}|^2$ to the norm-squared from a given p, k, ℓ block, maximizing within that block is the problem of maximizing $\vec{u} \cdot \vec{v}$ given $\vec{w} \cdot \vec{v} = 0$ where \vec{v} has

fixed norm $a_{pk\ell}$, where $v_q = \lambda_{pk\ell q}$, $u_q = \sqrt{c_q}$, and $w_q = d_q/\sqrt{c_q}$. The maximizing \vec{v} will be proportional the projection of the \vec{u} to the subspace orthogonal to \vec{w} . The result is

$$F_{\max} \geq \max_{a_{pk\ell}} \frac{\sum_{p,k,\ell} a_{pk\ell}^2 \frac{1}{c_p} \left(\sum_{q \supset p} c_q - \frac{(\sum_{q \supset p} d_q)^2}{\sum_{q \supset p} d_q^2/c_q} \right)}{Z(\beta) \sum_{p,k,\ell} a_{pk\ell}^2}. \quad (2.45)$$

Maximizing over the $a_{pk\ell}$ simply results in

$$F_{\max} \geq \max_p \frac{1}{Z(\beta) c_p} \left(\sum_{q \supset p} c_q - \frac{(\sum_{q \supset p} d_q)^2}{\sum_{q \supset p} d_q^2/c_q} \right). \quad (2.46)$$

Using an aforementioned fact about Schur functions, $Z(\beta) c_p = \sum_{q \supset p} c_q$, this simplifies to

$$F_{\max} \geq 1 - \min_p \frac{(\sum_{q \supset p} d_q)^2}{\sum_{q \supset p} c_q \sum_{q \supset p} d_q^2/c_q}. \quad (2.47)$$

Numerical experimentation suggests that the optimal irrep p of S_{n-1} is often the sign irrep. This is not always the case however. We will encounter a counterexample in Section 2.7.

In any case, choosing the sign irrep does give a lower bound on F_{\max} . There are two irreps q of S_n whose reduction to S_{n-1} includes the sign irrep, namely the irreps corresponding to the partition $2 + 1 + \dots + 1 = n$ and to the sign irrep of S_n (whose corresponding partition is $1 + 1 + \dots + 1 = n$). The dimensions d_q of these irreps are $n - 1$ and 1 respectively. This leads to our final expression

$$F_{\max} \geq 1 - \frac{n^2}{1 + (n-1)^2 + \frac{c_{(2,1^{n-2})}}{c_{(1^n)}} + (n-1)^2 \frac{c_{(1^n)}}{c_{(2,1^{n-2})}}} \quad (2.48)$$

In examples, the coefficient $c_{(1,\dots,1)}$ can be easier to calculate than other coefficients c_q . It is thus sometimes useful to use $Z(\beta) c_{(1^{n-1})} = c_{(1^n)} + c_{(2,1^{n-2})}$ to write $c_{(2,1^{n-2})}$ in terms of the potentially easier to calculate $c_{(1^n)}$ and $c_{(1^{n-1})}$. Similarly, $c_{(n)}$ and $c_{(n-1,1)}$ may be easier to calculate in examples than other coefficients.

2.4 Harmonic oscillator

To illustrate the application of the results of Section 2.3 we consider the simple (though non-holographic) example of a harmonic oscillator. In this case, the coefficients $c_{(1^n)}$ can be

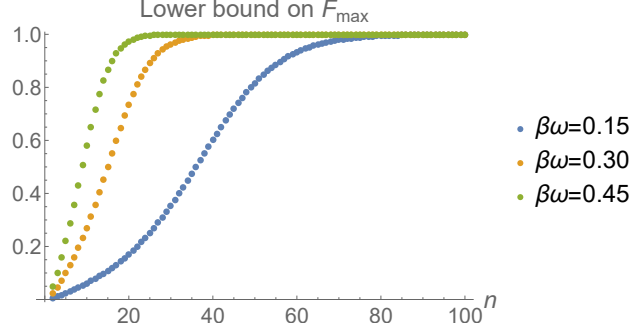


Figure 2.5: Lower bound on the maximum fidelity as a function of n , for harmonic oscillator TFD states with different values of $\beta\omega$

calculated using the recursion relation between elementary polynomials of different degree:

$$e_n(x_1, x_2, \dots) = e_n(x_2, x_3, \dots) + x_1 e_{n-1}(x_2, x_3, \dots). \quad (2.49)$$

For a harmonic oscillator with spacing ω between energy levels, this gives

$$\begin{aligned} c_{(1^n)} &= e_n(e^{-\beta\omega\frac{1}{2}}, e^{-\beta\omega\frac{3}{2}}, \dots) \\ &= e_n(e^{-\beta\omega\frac{3}{2}}, e^{-\beta\omega\frac{5}{2}}, \dots) + e^{-\beta\omega\frac{1}{2}} e_{n-1}(e^{-\beta\omega\frac{3}{2}}, e^{-\beta\omega\frac{5}{2}}, \dots) \\ &= e^{-\beta\omega n} e_n(e^{-\beta\omega\frac{1}{2}}, e^{-\beta\omega\frac{3}{2}}, \dots) + e^{-\beta\omega\frac{1}{2}} e^{-\beta\omega(n-1)} e_{n-1}(e^{-\beta\omega\frac{1}{2}}, e^{-\beta\omega\frac{3}{2}}, \dots) \\ &= e^{-\beta\omega n} c_{(1^n)} + e^{-\beta\omega(n-\frac{1}{2})} c_{(1^{n-1})}. \end{aligned} \quad (2.50)$$

So the ratio $c_{(1^{n-1})}/c_{(1^n)}$ is

$$\frac{c_{(1^{n-1})}}{c_{(1^n)}} = \frac{1 - e^{-\beta\omega n}}{e^{-\beta\omega(n-\frac{1}{2})}} = e^{\beta\omega(n-\frac{1}{2})} - e^{-\beta\omega\frac{1}{2}}. \quad (2.51)$$

With this we can calculate $c_{(2,1^{n-2})} = Z(\beta) \frac{c_{(1^{n-1})}}{c_{(1^n)}} - 1$ and apply (2.48). The lower bound on the maximum fidelity is

$$F_{\max} \geq 1 - \frac{n^2}{\frac{e^{\beta\omega n} - 1}{e^{\beta\omega} - 1} + (n-1)^2 \frac{e^{\beta\omega n} - 1}{e^{\beta\omega n} - e^{\beta\omega}}}. \quad (2.52)$$

This approaches 1 as n gets large, as the first term in the denominator dominates for large n . How quickly the fidelity approaches 1 as n increases depends on $\beta\omega$ for the system, with smaller $\beta\omega$ giving a slower increase to 1. This is general feature of other examples, that smaller spacing between energies or larger temperature both make it harder to achieve high fidelity.

2.5 The Marolf and Maxfield toy model

We now turn to a holographic example. In [1] the authors Marolf and Maxfield explore a simple toy model of a gravity path integral in 2 dimensions. Interestingly, they show that this simple bulk path integral is holographically dual to an ensemble of 1 dimensional boundary theories, namely an ensemble of topological quantum mechanics theories where the dimension of the Hilbert space is randomly taken from a Poisson distribution. In other words, a given theory in the ensemble has the N by N zero matrix as its Hamiltonian, and the ensemble is formed by choosing N from a Poisson distribution. The only parameter of the theory is the mean λ of the Poisson distribution from which we choose N .

This model is simple enough that we can calculate averages of some quantities of interest. The c_q are Schur polynomials in N variables all taking the value 1. In particular the elementary polynomial $c_{(1^n)} = e_n$ of degree n is simply $\binom{N}{n}$. Using $Zc_{(1^{n-1})} = c_{(2,1^{n-2})} + c_{(1^n)}$ (the relation (2.12)) we get

$$c_{(2,1^{n-2})} = Ze_{n-1} - e_n = N \binom{N}{n-1} - \binom{N}{n} \quad (2.53)$$

In particular the ratios $c_{(2,1^{n-2})}/c_{(1^n)}$ are

$$\frac{c_{(2,1^{n-2})}}{c_{(1^n)}} = N \frac{n!(N-n)!}{(n-1)!(N-n+1)!} - 1 = \frac{Nn}{N-n+1} - 1 = \frac{(N+1)(n-1)}{N-n+1} \quad (2.54)$$

Plugging this in to (2.48), we see the fidelity with which we can imitate a TFD state by states not connected between Alice and Bob is bounded by

$$\begin{aligned} F_{\max} &\geq 1 - \frac{n^2}{1 + (n-1)^2 + \frac{(N+1)(n-1)}{N-n+1} + (n-1)^2 \frac{N-n+1}{(N+1)(n-1)}} \\ &= \frac{n-1}{N(N-n-2)}. \end{aligned} \quad (2.55)$$

This expression is valid for $n \leq N$. The case $n > N$ can be obtained by a limit by considering Schur polynomials in n variables N of which take the value 1 and $n - N$ of which take a value ϵ which we take to zero. The result is a lower bound of 1, so for $n > N$ we obtain $F = 1$. In other words, in a topological quantum mechanics theory with Hilbert space of

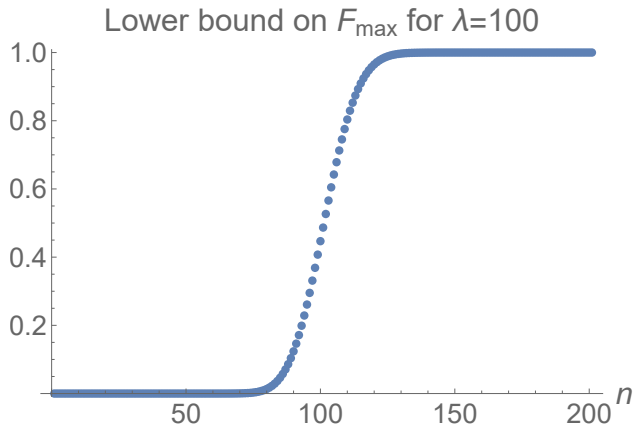


Figure 2.6: Lower bound on the maximum fidelity as a function of n , for the case $\lambda = 100$ in the Marolf-Maxfield model.

dimension N , Alice and Bob can meet using superpositions with $N + 1$ black holes. In all we get the bound

$$F_{\max} \geq \begin{cases} \frac{n-1}{N(N-n-2)} & N \geq n \\ 1 & N < n \end{cases} \quad (2.56)$$

The Poisson average of this expression,

$$\langle F_{\max} \rangle \geq e^{-\lambda} \left(\sum_{N=0}^{n-1} \frac{\lambda^N}{N!} + \sum_{N=n}^{\infty} \frac{\lambda^N}{N!} \frac{n-1}{N(N-n+2)} \right), \quad (2.57)$$

where λ is the mean of the Poisson distribution, has a complicated closed form. We can obtain a simpler bound by dropping the $N \geq n$ terms

$$\langle F_{\max} \rangle > \sum_{N=0}^{n-1} e^{-\lambda} \frac{\lambda^N}{N!} = \frac{\Gamma(n, \lambda)}{\Gamma(n)}, \quad (2.58)$$

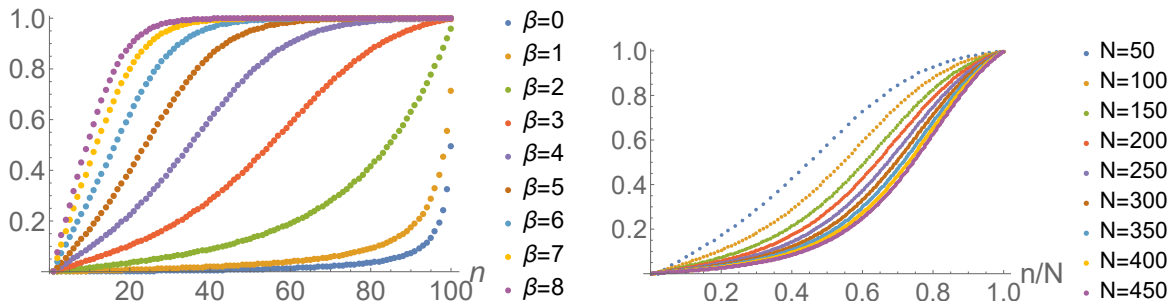
where $\Gamma(n, \lambda)$ is the incomplete gamma function. This partial sum is close to 1 when n is significantly larger than the Poisson mean λ . In fact, not only is the average fidelity bounded close to 1, the probability that the boundary theory has Hilbert space dimension less than n , and hence allows fidelity equal to 1, becomes high. To be more precise, the function $\Gamma(n, \lambda)/\Gamma(n)$ of n starts at 0 then grows to 1 as n passes λ . The case $\lambda = 100$ is plotted in figure 2.6

The path integral in the Marolf-Maxfield model is a weighted sum over 2d spacetime topologies. The model has a single parameter S_0 , which determines the factor e^{-2S_0} by which we weight each additional handle on a spacetime. This is related to the mean dimension λ by $\lambda = e^{2S_0}/(1 - e^{-2S_0})$. The name S_0 was chosen by analogy with JT gravity where the same weighting by number of handles occurs. JT gravity can be understood as a dimensional reduction of gravitational dynamics of near-extremal black holes in higher dimensions [43, 44]. In this context S_0 has the interpretation as the area of the black hole horizon and thus the entropy of the black hole. The S_0 parameter is thus the conceptual stand-in for the black hole entropy in the much simpler Marolf-Maxfield model. The S_0 parameter also has meaning in the matrix integral dual of JT gravity, where the e^{S_0} controls the density of energy levels [14]. In this sense the dependence on S_0 that we see is consistent with the general principle that closer energy spacing and higher temperature make achieving fidelity close to 1 harder.

2.6 Random Hamiltonians

As our third example we consider a system with random Hamiltonian. Much work over the past few years has explored bulk gravity theories dual to statistical *ensembles* of boundary theories. For example, JT has been shown to be dual to a double-scaled random matrix integral [14]. Here, however, we simply consider a theory with Hamiltonian taken from a Gaussian unitary ensemble, the goal being merely to ascertain typical behavior of the maximum fidelity (2.35).

We numerically calculated the bound (2.48) for various choices of β and Hilbert space dimension N . Elementary symmetric polynomials were calculated numerically using the recursion relation described in Section 2.4. (In fact, other desired Schur polynomials can be calculated accurately and efficiently from the elementary symmetric polynomials using the dual Jacobi-Trudi formula [45].) Some results are plotted in figures 2.7a and 2.7b. The n required for F_{\max} to be bounded near 1 depends on both β and N . For small β , n must be nearly N . For a given β , the required n as a fraction of N increases with N , but seems to converge as N gets large.



(a) Bound on F_{\max} as a function of n for various β , with $N = 100$

(b) Bound on F_{\max} as a function of n/N for various N , with $\beta = 3$

2.7 Black holes with $T_{HP} < T < 2T_{HP}$

Now take as our system a holographic CFT on the spatial sphere S^{d-1} . If we consider just the semiclassical approximation of the gravity contributions to the path integral, for low temperatures the partition function will be dominated by the saddlepoint corresponding to thermal AdS while for high enough temperatures the dominant saddlepoint will be a Euclidean AdS-Schwarzschild black hole [46, 47]. Taking just the background contributions of these saddlepoints we get an approximation for the partition function

$$Z(\beta) \approx \begin{cases} e^{-I[\text{Schwarzschild}]} & \beta < 1/T_{HP} \\ e^{-I[\text{AdS}]} & \beta > 1/T_{HP} \end{cases}, \quad (2.59)$$

where I is the classical gravitational action of the background and T_{HP} is the Hawking-Page transition temperature.

Recall that the inner products $\langle \pi \rangle$ equal products of factors of the form $Z(k\beta)$ where k is a positive integer. Given a fixed β_0 , the coefficients c_q thus only depend on the partition function evaluated at the points $k\beta_0$. Choose β_0 such that $\beta_0 < 1/T_{HP}$ and $2\beta_0 > 1/T_{HP}$. Note that $I[\text{AdS}]$ is of the form $e^{c\beta}$ where c is constant in β [47]. Rescaling $Z(\beta)$ by a factor $e^{c\beta}$ does not change the maximum fidelities; it is merely a change in normalization corresponding shifting the energies by a constant c . After fixing β_0 and rescaling we get the

form

$$Z(k\beta_0) \approx \begin{cases} z & k = 1 \\ 1 & k > 1, k \in \mathbf{N} \end{cases}, \quad (2.60)$$

where $z = \exp(I[\text{AdS}] - I[\text{Schwarzschild}])|_{\beta=\beta_0}$ is a value greater than 1 that depends on the choice of β_0 . This form for the values $Z(k\beta_0)$ simplifies the expressions for the inner products and allows us to calculate the coefficients $c_{(1^n)}$ and $c_{(n)}$ corresponding to the sign and trivial irreps respectively. From the expressions (2.4) for the inner products, it is straightforward to calculate generating functions

$$\sum_{n=0}^{\infty} c_{(1^n)} t^n = (1+t)e^{(z-1)t} \quad (2.61)$$

and

$$\sum_{n=0}^{\infty} c_{(n)} t^n = \frac{1}{1-t} e^{(z-1)t}, \quad (2.62)$$

from which we extract the coefficients $c_{(1^n)} = (z-1)^{n-1}(z-1+n)/n!$ and $c_{(n)} = \sum_{m=0}^n (z-1)^m/m!$. Using these values for $c_{(1^n)}$ in (2.48) gives a bound on fidelity that decreases as n gets large. The true maximum fidelity cannot, of course, decrease as n increases. This signifies either a deficiency in the bound (2.48) or a deficiency in our approximate form (2.60). Using the above values for $c_{(n)}$ in (2.47) also fails to give a bound that increases to 1; In this case, as n increases the bound approaches $1 - \frac{1}{z}$. Again, we either require a better bound on fidelity or a better estimate for the values $Z(k\beta_0)$ in order to see the fidelity increase to 1. We will see a bound that accomplishes this in the next section.

2.8 Bounds on fidelity from subgroups

Using expression (2.25) for the fidelity of writing one geometry as a superposition of the others requires knowing the coefficients c_q for the given system and temperature. This is in general difficult. In particular, considering limits in large n requires knowing the c_q for the groups S_n for arbitrary n , not likely to be simple in general. One option to proceed is to consider a subgroup of S_n in the hopes that the coefficients c_q for irreps of the subgroup are

easier to find. Let G be a subgroup of S_n , and let F_G be the maximum fidelity between $|\text{id}\rangle$ and a superposition of states $\{|\pi\rangle \mid \pi \in G, \pi \neq \text{id}\}$. Then $F_G \leq F_{\max}$, as we are maximizing over a smaller subspace of states. Following the procedure in Section 2.3.1, but for an arbitrary subgroup G of S_n , we find

$$F_G = 1 - \frac{|G|^2}{Z^n \sum_q d_q^3 / c_q}, \quad (2.63)$$

where the sum is over irreps q of G .

We'd like to choose a subgroup G that is large enough to potentially give an approximate null state, but that has an easily understandable representation structure. Consider the choice

$$G = \{(1\ 2)^{a_1} (3\ 4)^{a_2} \dots (n-1\ n)^{a_{n/2}} \mid a_i \in \{0, 1\}\}. \quad (2.64)$$

For ease of notation we can refer to an element of G by the $(n/2)$ -tuple $\vec{b} = (b_1, b_2, \dots, b_{n/2})$ with entries either 0 or 1. The group G is abelian so its irreps are very simple. Like elements of G they are labeled by $(n/2)$ -tuples with entries either 0 or 1, notated $\vec{\gamma} = (\gamma_1, \gamma_2, \dots, \gamma_{n/2})$. They are given by

$$\chi_{\vec{\gamma}}(\vec{b}) = \chi_{(\gamma_1, \dots, \gamma_{n/2})}(b_1, \dots, b_{n/2}) = (-1)^{\sum_i \gamma_i b_i}. \quad (2.65)$$

The coefficients $c_{\vec{\gamma}}$ are

$$\begin{aligned} c_{\vec{\gamma}} &= \frac{1}{2^{n/2}} \sum_{\vec{b}} \vec{\gamma}(\vec{b}) \langle \vec{b} \rangle = \frac{1}{2^{n/2}} \sum_{\vec{b}} (-1)^{\sum_i \gamma_i b_i} Z(\beta)^n \prod_{i=1}^{n/2} (Z(2\beta)/Z(\beta)^2)^{b_i} \\ &= \frac{Z^n}{2^{n/2}} \prod_{i=1}^{n/2} \left(1 + (-1)^{\gamma_i} \frac{Z(2\beta)}{Z(\beta)^2} \right). \end{aligned} \quad (2.66)$$

As G is abelian the dimension of every irrep is 1. Applying (2.63) results in

$$F_G = 1 - \left(1 - \frac{Z(2\beta)^2}{Z(\beta)^4} \right)^{n/2} \quad (2.67)$$

Note that $Z(2\beta)/Z(\beta)^2$ is less than 1. So as n gets large F_G , and hence F_{\max} approaches 1.

Given a value of β , we can ensure $F_G > 1 - \epsilon$ and hence $F_{\max} > 1 - \epsilon$, for any small $\epsilon > 0$ by choosing

$$n > \frac{2 \log \epsilon}{\log \left(1 - \frac{Z(2\beta)^2}{Z(\beta)^4} \right)}. \quad (2.68)$$

2.8.1 Dependence on Newton’s constant G_N

In cases where the partition function takes the form $Z(\beta) \sim e^{f(\beta)/G_N + \mathcal{O}(G_N)}$, the requirement (2.68) implies that $\log n$ must grow like $1/G_N$ for small G_N . Achieving the fidelity bound F_G thus requires a superposition with $e^{\mathcal{O}(1/G_N)}$ terms in it, so doubly exponential in $1/G_N$. That an approximate null state would require the superposition of a large number of states is consistent with the results of [39]. They found that the entropy is linear to leading order on superpositions of much fewer than $e^{\mathcal{O}(1/G_N)}$ geometric states, and that this linearity can break down for superpositions of on order $e^{\mathcal{O}(1/G_N)}$ states. Our null state resulting in fidelity F_G , on the other hand, requires a much larger superposition.

This suggests the possibility that a much better bound is achievable. In the Marolf-Maxfield model, for example, $Z(2\beta)/Z(\beta)^2$ is $1/N$ where N is Poisson random with mean λ . This leads to a bound $F_G(N) = 1 - (1 - 1/N)^{n/2}$, whereas we know that the true maximum fidelity is 1 as soon as $n > N$. Thus F_{\max} will be close to 1 when n is significantly greater than the average dimension λ . Taking $S_0 \sim 1/G_N$ then suggests a superposition with merely $e^{\mathcal{O}(\frac{1}{G_N} \log \frac{1}{G_N})}$ terms.

2.9 Additional discussion

As explained in the introduction, the possibility of rewriting the TFD state as a partial trace of a superposition $\sum_{\pi(1) \neq 1} \alpha_\pi |\pi\rangle$ leads to a situation where Alice and Bob could meet despite the fact that they do not meet in any term of the superposition. Can a superposition of worlds where Alice does not meet Bob really equal a world where she does? This is of course possible if there is not in fact a “meeting” quantum observable. This in turn is reasonable if geometry is itself not a quantum observable.

One thing to point out is that, while the fidelity between two states being close to 1 bounds correlators of the states to be close to each other, this bound on correlators is not uniform. That is to say, depending on the quantity one desires to measure, one may need a larger or smaller number n to get outcomes within a desired error. It’s possible that a

putative “meeting” operator might have an especially stringent requirement on n .

The question of what Alice observes must start with an identification of “Alice” within the system. It’s possible that there is more than one way to do this or that this identification isn’t linear. Perhaps the concept of an observer “Alice” is state-dependent. This might neatly solve the paradox. The paradox arises from the fact that Alice, by recording whether or not she meets Bob, seems to be recording a fact about the entanglement of the state. For an observer as usually understood, this cannot be the case, as entanglement is not linear on the space of states. But perhaps there’s nothing wrong with a state-dependent sense of “Alice” being able to measure a nonlinear observable.

Throughout this work we have assumed the duality between two-sided eternal black holes and TFD states. Two TFD states with different temperatures will have a nonzero inner product even though they correspond to different geometries: two black holes with different horizon area. If there is not a geometry quantum observable, this is not troubling. If, however, we are set on the existence of a quantum operator for geometry, we can make sense of the nonzero overlap by interpreting the TFD state as dual to a wave function over different, orthogonal (by supposition) geometries where the peak amplitude is at the black hole geometry with the appropriate temperature. Then the nonzero inner product measures fluctuations away from the peak geometry, and the smallness of

$$\frac{|\langle \text{TFD}(\beta_1) | \text{TFD}(\beta_2) \rangle|^2}{\langle \text{TFD}(\beta_1) | \text{TFD}(\beta_1) \rangle \langle \text{TFD}(\beta_2) | \text{TFD}(\beta_2) \rangle} = \frac{Z((\beta_1 + \beta_2)/2)^2}{Z(\beta_1)Z(\beta_2)} \quad (2.69)$$

for β_1, β_2 very different simply signifies that in the wavefunction dual to $|\text{TFD}(\beta_1)\rangle$ the amplitude of the black hole geometry with β_2 is small. This point of view is less able to explain the nonzero overlaps between the $|\pi\rangle$ states, however. By locality, the only geometries that should appear in the wave function dual to $|\text{TFD}\rangle \otimes |\text{TFD}\rangle$ should be geometries made of two disjoint components each of which appears in the wavefunction for $|\text{TFD}\rangle$. In particular, no geometries connecting the boundary of the first factor with the boundary of the second should ever appear. but the wave function for $(1\ 2) \circ |\text{TFD}\rangle \otimes |\text{TFD}\rangle$, two copies of the TFD state with their right sides swapped, will *only* have support over such geometries, again, by locality of the boundary theory. Why then the nonzero overlap $\langle \text{id} | (1\ 2) \rangle =$

$Z(2\beta)/Z(\beta)^2$? If we are to understand these states as wavefunctions over orthogonal states labeled by geometries then the wavefunction of $|\text{TFD}\rangle \otimes |\text{TFD}\rangle$ is not simply two copies of the wavefunction for $|\text{TFD}\rangle$.⁵ One natural conclusion is that there simply is no geometry operator and corresponding complete set of states labeled by geometry. Another natural conclusion is that different geometry states are related by null states, which amounts to the same thing.

The rewriting of a thermofield double state described in this work may be interesting in the context of traversable wormholes as described in [48] and [49]. In these cases information can travel from one side to the other via an interaction that is introduced between the boundaries. From the bulk point of view, this interaction changes the geometry to include a shockwave that makes the wormhole traversable. This bulk point of view no longer holds once we have rewritten the TFD state to be a superposition of states where the two sides are not connected. Is there a bulk explanation as to how information can traverse that makes sense within each term of the superposition?

⁵This is reminiscent of the factorization problem that generically appears in gravity path integrals that sum over geometries [10], though it is not quite the same, as the nonzero overlap $\langle \text{id} | (1 \ 2) \rangle$ in question has a single boundary spacetime component. Instead, this is a nonfactorization at the level of Hilbert spaces (codimension 1) rather than partition functions (codimension 0).

APPENDIX A

State sum formulation of Dijkgraaf-Witten theory

Dijkgraaf-Witten theory can be equivalently formulated as a lattice gauge theory on a triangulated manifold. The outputs of the theory formulated this way are so-called state-sum expressions for the partition function. In [50], this approach was generalized to DW theory with defects. Their methods can be straightforwardly generalized to provide an independent method of obtaining our open/closed DW rules (1.69)-(1.73), and at the same time provide directions for future work.

A.1 Review of state sum for DW with defects

Given a surface-curve pair $\Sigma \supset C$, we can define a refine notion of triangulation as follows.

Definition A.1.1. *A triangulation T of a surface with curve $\Sigma \supset C$ is flag-like if C is a subcomplex, and if, for every 2-simplex $\sigma \in T$, the intersection of σ and C is either a face (i.e. one vertex or one edge) or is empty.*

In other words a flag-like triangulation has to have the curve lie along faces, and only use triangles like the ones in figure A.1.

In DW theory without defects, the construction proceeds by choosing appropriate an appropriate group G whose elements will label the edges of the triangles in the triangulation. In the presence of defects, we need to pick two finite groups G , H and a space X with a right G -action and a left H -action. Physically G specifies the degrees of freedom in the bulk, H the degrees of freedom on the defect and X is a way of coupling (or not) the two together. Then we assign elements of G , H and X on edges of the triangles from figure A.1, in a

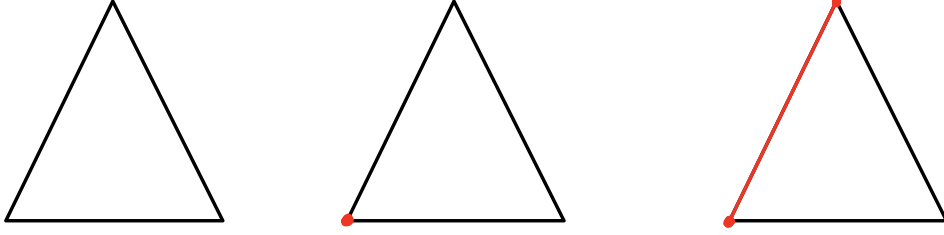


Figure A.1: Basic building blocks of a flag-like triangulation.

consistent way as shown in fig A.2.

Now we can define the partition function of DW theory.

Definition A.1.2. *Let G, H be finite groups and X a set equipped with commuting group action of G on the right and H on the left. Then, for any flag-like triangulation T of a surface-curve pair $\Sigma \supset C$, the partition function of untwisted DW theory is defined to be*

$$Z_{(H,X,G)}(\Sigma \supset C) := |G|^{-\sigma} |H|^{-c} \kappa_{(H,X,G)}(T) \quad (\text{A.1})$$

where $\kappa_{(H,X,G)}(T)$ is the number of admissible colorings of the triangulation, σ is the number of bulk vertices, and c is the number of vertices lying on the defect.

The method of [50] for defining the partition function of 2d DW with 1d defects on a closed manifold Σ begins by choosing two groups G, H and a space X on which G has a right action and H has a left action. The physical interpretation of these choices are as the dofs of the DW, the dofs on the defects, and a coupling between these dofs, respectively. Then they introduce a triangulation of Σ , that is made out of three basic triangles, see figure A.1. A triangulation obtained from these triangles they call flag-like, and then the partition function of untwisted DW is essentially the counting of possible group element assignments on each edge of the triangulation, which is the lattice equivalent of counting isomorphism classes of bundles,

$$Z_{(H,X,G)}(\Sigma \supset C) := |G|^{-|T_0^0|} |H|^{-|T_0^1|} \kappa_{(H,X,G)}(T) \quad (\text{A.2})$$

where T_n^k denote the set of n -simplices of T with k vertices on the curve, and $\kappa_{(H,X,G)}(T)$ is the number of admissible (H, X, G) -colorings of T . Then they show that this is independent

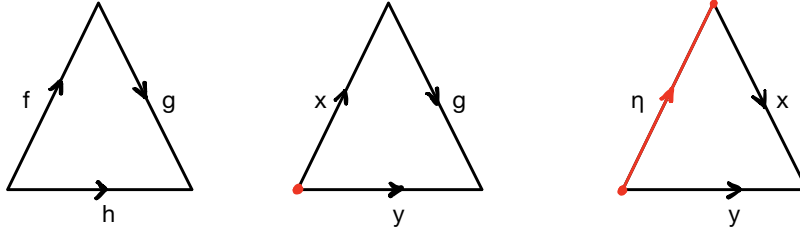


Figure A.2: Admissible colorings of a flag-like triangulation. Here $f, g, h \in G$, $x, y \in X$ and $\eta \in H$.

of the flag-like triangulation chosen, and hence a topological invariant of the surface-curve pair $\Sigma \supset C$.

A.2 Generalization and importance for our methods

Although, in [50] they restrict attention to closed manifolds Σ , their methods can be straightforwardly generalized to manifolds with fixed holonomies and/or parallel transports on their boundaries. So using (A.2), we can reproduce all of the DW TQFT rules (1.69)-(1.73). The simplest choice for X is the direct product $X = G \times H$. With this choice, we label the brane boundaries by parallel transports $(h, 1) \in X$, and since the brane boundaries are dynamical we will sum over all those h . Each non-dynamical, open boundary has to be made up by at least two different edges, since there is no 2-simplex with two defect vertices and no defect edge. Let $g \in G$ be the parallel transport specified along a non-dynamical, open boundary, then we will model this boundary by allowing a non zero H parallel transport along it, so that the total transport is $(h, g) \in X$ for some $h \in H$. Other than reproducing (1.69)-(1.73), the lattice perspective helps visualize how a (small) gauge G transformation wouldn't act on the boundary, since the brane boundaries are fixed to have parallel transport $(h, 1)$.

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