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Some results involving embedded contact homology

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Some results involving embedded contact homology

by

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Abstract

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Doctor of Philosophy in Mathematics

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This dissertation is a collection of four papers involving *embedded contact homology* (ECH). ECH is a three-manifold invariant recently defined by Michael Hutchings. The dissertation covers both technical results about ECH (for example a proof that the absolute grading in ECH is a topological invariant) and applications of ECH to Reeb dynamics and symplectic embedding problems. All four papers were written while the author was a graduate student at UC Berkeley.

To mom and dad

Wind giving presence to fragments-
Ted Berrigan

Contents

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Chapter 1

Introduction and background

1.1 Format

This dissertation is a collection of four papers

(i) *The asymptotics of ECH capacities*
(with Michael Hutchings and Vinicius Gripp)

(ii) *From one Reeb orbit to two*
(with Michael Hutchings)

(iii) *The absolute gradings on embedded contact homology and Seiberg-Witten Floer cohomology*

(iv) *Period collapse of Ehrhart quasipolynomials and symplectic embeddings of ellipsoids*
(with Aaron Kleinman)

that I wrote while a doctoral student at UC Berkeley. Each chapter contains one paper.

The papers are loosely connected by embedded contact homology, a three-manifold invariant recently defined by Michael Hutchings. The results range from applications of ECH to Reeb dynamics and symplectic embedding problems to relationships between ECH and Seiberg-Witten theory.

1.2 Background

Each chapter in this dissertation is self-contained, but we include here a brief introduction to the relevant material.

Contact manifolds and embedded contact homology

A *contact manifold* is a pair (Y, λ) where Y is a smooth oriented $2n-1$ dimensional manifold and λ is a differential one-form satisfying

$$\lambda \wedge (d\lambda)^{n-1} > 0,$$

called a *contact form*. A contact form determines a canonical vector field, R, called the *Reeb vector field*. The closed integral curves of the Reeb vector field, called *Reeb orbits*, are of considerable interest. For example, if (M, g) is any Riemanninan manifold then there is a canonical contact form on the unit cotangent bundle of M for which the Reeb orbits are equivalent to closed geodesics on M. Reeb vector fields also arise as restrictions of Hamiltonian vector fields to certain regular energy hypersurfaces. The *Weinstein conjecture*, one of the most important unsolved problems in symplectic geometry today, asserts that if Y is closed then the Reeb vector field always has at least one Reeb orbit. The dynamics of the Reeb vector field are the theme of chapter 3.

If (Y, λ) is a closed contact three-manifold then the *embedded contact homology*

$$ECH(Y, \lambda, J)$$

is defined. This is the homology of a chain complex freely generated over $\mathbb{Z}/2$ by certain finite sets of Reeb orbits, called *orbit sets*, relative to a chain complex differential that counts *ECH index one* J-holomorphic curves in the "symplectization" $\mathbb{R} \times Y$ of Y for an appropriate almost complex structure J. The ECH index is the key nontrivial feature of the definition of ECH and is discussed in chapter 4.

Isomorphism with Seiberg-Witten Floer cohomology

While in principle, ECH could depend on the choice of contact form λ or almost complex structure J, in fact it does not. ECH only depends on the three-manifold Y and is thus a *topological invariant*. This follows from an isomorphism

$$\mathcal{T} : ECH_*(Y, \lambda, J) \xrightarrow{\simeq} \widehat{HM}^{-*}(Y) \tag{1.1}$$

between ECH and the *Seiberg-Witten Floer cohomology* defined by Kronheimer and Mrowka in [32], which is known to be a topological invariant. The isomorphism (1.1) is constructed by Taubes in [40] and is discussed in some detail in chapter 4.

The isomorphism (1.1) has several important consequences. One is that $ECH(Y)$ is infinitely generated as a $\mathbb{Z}/2$ module. Since ECH is generated by Reeb orbits, this for example implies the Weinstein conjecture in dimension 3[1]. ECH admits a grading by homotopy classes of oriented 2-plane fields, and the isomorphism (1.1) implies that ECH is finitely generated in each grading.

[1]Taubes had already proven the Weinstein conjecture by similar methods before proving the isomorphism (1.1).

Symplectic embeddings

A *symplectic manifold* is a pair (M, ω) where M is a smooth manifold and ω is a closed nondegenerate differential two-form, called a *symplectic form.* A *symplectic embedding* from one symplectic manifold (M, ω_M) to another (N, ω_N) is a smooth embedding $\Psi : M \hookrightarrow N$ such that $\Psi^*\omega_N = \omega_M$. We will always denote symplectic embeddings by the symbol $\overset{s}{\hookrightarrow}$.

It seems that the question of when one symplectic manifold embeds into another is very rich. For example, define the *symplectic ellipsoid*

$$E(a,b) = \{(z_1, z_2) \in \mathbb{C}^2 | \frac{\pi|z_1|^2}{a} + \frac{\pi|z_2|^2}{b} \leq 1\}.$$

In [37], McDuff and Schlenk compute the function $c : [1, \infty) \to \mathbb{R}$ defined by

$$c(a) := \inf\{\mu : E(1,a) \overset{s}{\hookrightarrow} \mu E(1,1)\}$$

explicitly. To state their answer for $1 \leq a \leq \tau^4$ (here τ denotes the golden ratio), denote by $g_n := f_{2n-1}$ the n^{th} *odd index Fibonacci number* (so $g_1 = 1, g_2 = 2, g_3 = 5$, etc). Define $a_n = \frac{g_{n+1}^2}{g_n^2}$ and define $b_n = \frac{g_{n+2}}{g_n}$. McDuff and Schlenk show:

Theorem 1.2.1. $c(a) = \frac{a}{\sqrt{a_n}}$ *for* $a \in [a_n, b_n]$ *and* c *is constant with value* $\sqrt{a_{n+1}}$ *on the interval* $[b_n, a_{n+1}]$.

Thus, for $1 \leq a \leq \tau^4$, the graph of $c(a)$ is given by an infinite "Fibonacci staircase". In contrast, McDuff and Schlenk also show that if $a \geq \frac{17^2}{6^2}$, then the only obstruction to embedding the interior of $E(1,a)$ into a scaling of $B(1,1)$ is the volume. Part of chapter 5 is devoted to a new proof of Theorem 1.2.1.

Symplectic capacities

Symplectic embeddings are often studied by using *symplectic capacities.* A symplectic capacity is a nonnegative (possibly infinite) real number associated to any symplectic manifold of a given dimension that is monotone under symplectic embeddings and satisfies a certain scaling property. ECH can be used to define capacities of symplectic four-manifolds, called *ECH capacities.* These are defined for Liouville domains[2] in terms of the ECH of their boundary.

Chapter two examines the relationship between the ECH capacities and the classical volume

$$\mathrm{vol}(X, \omega) = \frac{1}{2}\int_X \omega \wedge \omega$$

of the manifold.

[2]A Liouville domain is a compact symplectic four-manifold (X, ω) such that ω is exact and $\omega|_{\partial X} = d\lambda$ for some contact form λ.

Chapter 2

The asymptotics of ECH capacities

Daniel Cristofaro-Gardiner, Vinicius Gripp and Michael Hutchings

Abstract: In a previous paper, the third author used embedded contact homology (ECH) of contact three-manifolds to define "ECH capacities" of four-dimensional symplectic manifolds. In the present paper we prove that for a four-dimensional Liouville domain with all ECH capacities finite, the asymptotics of the ECH capacities recover the symplectic volume. This follows from a more general theorem relating the volume of a contact three-manifold to the asymptotics of the amount of symplectic action needed to represent certain classes in ECH. The latter theorem was used by the first and third authors to show that every contact form on a closed three-manifold has at least two embedded Reeb orbits.

2.1 Introduction

Define a four-dimensional *Liouville domain*[1] to be a compact symplectic four-manifold (X, ω) with oriented boundary Y such that ω is exact on X, and there exists a contact form λ on Y with $d\lambda = \omega|_Y$. In [22], a sequence of real numbers

$$0 = c_0(X, \omega) < c_1(X, \omega) \leq c_2(X, \omega) \leq \cdots \leq \infty$$

called *ECH capacities* was defined. The definition is reviewed below in §2.1. The ECH capacities obstruct symplectic embeddings: If (X, ω) symplectically embeds into (X', ω'), then

$$c_k(X, \omega) \leq c_k(X', \omega') \tag{2.1}$$

for all k. For example, a theorem of McDuff [36], see also the survey [23], shows that ECH capacities give a sharp obstruction to symplectically embedding one four-dimensional ellipsoid into another.

[1]This definition of "Liouville domain" is slightly weaker than the usual definition, which would require that ω have a primitive λ on X which restricts to a contact form on Y.

The first goal of this paper is to prove the following theorem, relating the asymptotics of the ECH capacities to volume. This result was conjectured in [22] based on experimental evidence; it was proved in [22, §8] for star-shaped domains in $\mathbb{R}^4$ and some other examples.

Theorem 2.1.1. *[22, Conj. 1.12] Let (X, ω) be a four-dimensional Liouville domain such that $c_k(X, \omega) < \infty$ for all k. Then*

$$\lim_{k \to \infty} \frac{c_k(X, \omega)^2}{k} = 4 \operatorname{vol}(X, \omega).$$

Here the symplectic volume is defined by

$$\operatorname{vol}(X, \omega) = \frac{1}{2} \int_X \omega \wedge \omega.$$

In particular, when all ECH capacities are finite, the embedding obstruction (2.1) for large k recovers the obvious volume constraint $\operatorname{vol}(X, \omega) \leq \operatorname{vol}(X', \omega')$. As we review below, the hypothesis that $c_k(X, \omega) < \infty$ for all k is a purely topological condition on the contact structure on the boundary; for example it holds whenever X is a star-shaped domain in $\mathbb{R}^4$.

We will obtain Theorem 2.1.1 as a corollary of the more general Theorem 2.1.3 below, which also has applications to refinements of the Weinstein conjecture in Corollary 2.1.4. To state Theorem 2.1.3, we first need to review some notions from embedded contact homology (ECH). More details about ECH may be found in [20] and the references therein.

Embedded contact homology

Let Y be a closed oriented three-manifold and let λ be a contact form on Y, meaning that $\lambda \wedge d\lambda > 0$. The contact form λ determines a contact structure $\xi = \operatorname{Ker}(\lambda)$, and the Reeb vector field R characterized by $d\lambda(R, \cdot) = 0$ and $\lambda(R) = 1$. Assume that λ is nondegenerate, meaning that all Reeb orbits are nondegenerate. Fix $\Gamma \in H_1(Y)$. The *embedded contact homology* $ECH(Y, \xi, \Gamma)$ is the homology of a chain complex over $\mathbb{Z}/2$ defined as follows.

A generator of the chain complex is a finite set of pairs $\alpha = \{(\alpha_i, m_i)\}$ where the α_i are distinct embedded Reeb orbits, the m_i are positive integers, $m_i = 1$ whenever α_i is hyperbolic, and the total homology class $\sum_i m_i[\alpha_i] = \Gamma \in H_1(Y)$. To define the chain complex differential ∂ one chooses a generic almost complex structure J on $\mathbb{R} \times Y$ such that $J(\partial_s) = R$ where s denotes the $\mathbb{R}$ coordinate, $J(\xi) = \xi$ with $d\lambda(v, Jv) \geq 0$ for $v \in \xi$, and J is $\mathbb{R}$-invariant. Given another chain complex generator $\beta = \{(\beta_j, n_j)\}$, the differential coefficient $\langle \partial\alpha, \beta \rangle \in \mathbb{Z}/2$ is a mod 2 count of J-holomorphic curves in $\mathbb{R} \times Y$ that converge as currents to $\sum_i m_i \alpha_i$ as $s \to +\infty$ and to $\sum_j n_j \beta_j$ as $s \to -\infty$, and that have "ECH index" equal to 1. The definition of the ECH index is explained in [25]; all we need to know here is that the ECH index defines a relative $\mathbb{Z}/d$-grading on the chain complex, where d denotes the divisibility of $c_1(\xi) + 2\operatorname{PD}(\Gamma)$ in $H^2(Y; \mathbb{Z})$ mod torsion. It is shown in [27, §7] that $\partial^2 = 0$.

One now defines $ECH(Y,\lambda,\Gamma,J)$ to be the homology of the chain complex $ECC(Y,\lambda,\Gamma,J)$. Taubes [41] proved that if Y is connected, then there is a canonical isomorphism of relatively graded $\mathbb{Z}/2$-modules

$$ECH_*(Y,\lambda,\Gamma,J) = \widehat{HM}^{-*}(Y,\mathfrak{s}_\xi + \mathrm{PD}(\Gamma)). \tag{2.2}$$

Here $\widehat{HM}^*$ denotes the 'from' version of Seiberg-Witten Floer cohomology as defined by Kronheimer-Mrowka [32], with $\mathbb{Z}/2$ coefficients[2], and $\mathfrak{s}_\xi$ denotes the spin-c structure determined by the oriented 2-plane field ξ, see e.g. [24, Ex. 8.2]. It follows that, whether or not Y is connected, $ECH(Y,\lambda,\Gamma,J)$ depends only on Y, ξ, and Γ, and so can be denoted by $ECH_*(Y,\xi,\Gamma)$.

There is a degree -2 map

$$U : ECH_*(Y,\xi,\Gamma) \longrightarrow ECH_{*-2}(Y,\xi,\Gamma). \tag{2.3}$$

This map on homology is induced by a chain map which counts J-holomorphic curves with ECH index 2 that pass through a base point in $\mathbb{R}\times Y$. When Y is connected, the U map (3.3) does not depend on the choice of base point, and agrees under Taubes's isomorphism (2.2) with an analogous map on Seiberg-Witten Floer cohomology [44]. If Y is disconnected, then there is one U map for each component of Y.

Although ECH is a topological invariant by (2.2), it contains a distinguished class which can distinguish some contact structures. Namely, the empty set of Reeb orbits is a generator of $ECC(Y,\lambda,0,J)$; it is a cycle by the conditions on J, and so it defines a distinguished class

$$[\emptyset] \in ECH(Y,\xi,0), \tag{2.4}$$

called the *ECH contact invariant*. Under the isomorphism (2.2), the ECH contact invariant agrees with an analogous contact invariant in Seiberg-Witten Floer cohomology [44].

There is also a "filtered" version of ECH, which is sensitive to the contact form and not just the contact structure. If $\alpha = \{(\alpha_i, m_i)\}$ is a generator of the chain complex $ECC(Y,\lambda,\Gamma,J)$, its *symplectic action* is defined by

$$\mathcal{A}(\alpha) = \sum_i m_i \int_{\alpha_i} \lambda. \tag{2.5}$$

It follows from the conditions on the almost complex structure J that if the differential coefficient $\langle \partial\alpha, \beta\rangle \neq 0$ then $\mathcal{A}(\alpha) > \mathcal{A}(\beta)$. Consequently, for each $L \in \mathbb{R}$, the span of those generators α with $\mathcal{A}(\alpha) < L$ is a subcomplex, which is denoted by $ECC^L(Y,\lambda,\Gamma,J)$. The homology of this subcomplex is the *filtered ECH* which is denoted by $ECH^L(Y,\lambda,\Gamma)$. Inclusion of chain complexes induces a map

$$ECH^L(Y,\lambda,\Gamma) \longrightarrow ECH(Y,\xi,\Gamma). \tag{2.6}$$

[2] One can define ECH with integer coefficients [19, §9], and the isomorphism (2.2) also exists over $\mathbb{Z}$, as shown in [43]. However $\mathbb{Z}/2$ coefficients will suffice for this paper.

It is shown in [29, Thm. 1.3] that $ECH^L(Y,\lambda,\Gamma)$ and the map (2.6) do not depend on the almost complex structure J.

A useful way to extract invariants of the contact form out of filtered ECH is as follows. Given a nonzero class $\sigma \in ECH(Y,\xi,\Gamma)$, define

$$c_\sigma(Y,\lambda) \in \mathbb{R}$$

to be the infimum over L such that the class σ is in the image of the inclusion-induced map (2.6). So far we have been assuming that the contact form λ is nondegenerate. If λ is degenerate, one defines $c_\sigma(Y,\lambda) = \lim_{n\to\infty} c_\sigma(Y,\lambda_n)$, where $\{\lambda_n\}$ is a sequence of nondegenerate contact forms which C^0-converges to λ, cf. [22, §3.1].

ECH capacities

Let (Y,λ) be a closed contact three-manifold and assume that the ECH contact invariant (2.4) is nonzero. Given a nonnegative integer k, define $c_k(Y,\lambda)$ to be the minimum of $c_\sigma(Y,\lambda)$, where σ ranges over classes in $ECH(Y,\xi,0)$ such that $A\sigma = [\emptyset]$ whenever A is a composition of k of the U maps associated to the components of Y. If no such class σ exists, define $c_k(Y,\lambda) = \infty$. The sequence $\{c_k(Y,\lambda)\}_{k=0,1,\ldots}$ is called the *ECH spectrum* of (Y,λ).

Now let (X,ω) be a Liouville domain with boundary Y and let λ be a contact form on Y with $d\lambda = \omega|_Y$. One then defines the ECH capacities of (X,ω) in terms of the ECH spectrum of (Y,λ) by

$$c_k(X,\omega) = c_k(Y,\lambda).$$

This definition is valid because the ECH contact invariant of (Y,λ) is nonzero by [29, Thm. 1.9]. It follows from [22, Lem. 3.9] that $c_k(X,\omega)$ does not depend on the choice of contact form λ on Y with $d\lambda = \omega|_Y$.

Theorem 2.1.1 is now a consequence of the following result about the ECH spectrum:

Theorem 2.1.2. *[22, Conj. 8.1] Let (Y,λ) be a closed contact three-manifold with nonzero ECH contact invariant. If $c_k(Y,\lambda) < \infty$ for all k, then*

$$\lim_{k\to\infty} \frac{c_k(Y,\lambda)^2}{k} = 2\,\mathrm{vol}(Y,\lambda).$$

Here the contact volume is defined by

$$\mathrm{vol}(Y,\lambda) = \int_Y \lambda \wedge d\lambda. \tag{2.7}$$

Note that the hypothesis $c_k(Y,\lambda) < \infty$ just means that the ECH contact invariant is in the image of all powers of the U map when Y is connected, or all compositions of powers of the U maps when Y is disconnected. The comparison with Seiberg-Witten theory implies that this is possible only if $c_1(\xi) \in H^2(Y;\mathbb{Z})$ is torsion; see [22, Rem. 4.4(b)].

By [22, Prop. 8.4], to prove Theorem 2.1.2 it suffices to consider the case when Y is connected. Theorem 2.1.2 in this case follows from our main theorem which we now state.

The main theorem

Recall from §2.1 that if $c_1(\xi) + 2\,\mathrm{PD}(\Gamma) \in H^2(Y;\mathbb{Z})$ is torsion, then $ECH(Y,\xi,\Gamma)$ has a relative $\mathbb{Z}$-grading, and we can arbitrarily refine this to an absolute $\mathbb{Z}$-grading. The main theorem is now:

Theorem 2.1.3. *[22, Conj. 8.7] Let Y be a closed connected contact three-manifold with a contact form λ and let $\Gamma \in H_1(Y)$. Suppose that $c_1(\xi) + 2\,\mathrm{PD}(\Gamma)$ is torsion in $H^2(Y;\mathbb{Z})$, and let I be an absolute $\mathbb{Z}$-grading of $ECH(Y,\xi,\Gamma)$. Let $\{\sigma_k\}_{k\geq 1}$ be a sequence of nonzero homogeneous classes in $ECH(Y,\xi,\Gamma)$ with $\lim_{k\to\infty} I(\sigma_k) = \infty$. Then*

$$\lim_{k\to\infty} \frac{c_{\sigma_k}(Y,\lambda)^2}{I(\sigma_k)} = \mathrm{vol}(Y,\lambda). \tag{2.8}$$

The following application of Theorem 2.1.3 was obtained in [4]:

Corollary 2.1.4. *[4, Thm. 1.1] Every (possibly degenerate) contact form on a closed three-manifold has at least two embedded Reeb orbits.*

The proof of Theorem 2.1.3 has two parts. In §2.2 we show that the left hand side of (2.8) (with lim replaced by lim sup) is less than or equal to the right hand side. This is actually all that is needed for Corollary 2.1.4. In §2.3 we show that the left hand side (with lim replaced by lim inf) is greater than or equal to the right hand side. The two arguments are independent of each other and can be read in either order. The proof of the upper bound uses ingredients from Taubes's proof of the isomorphism (2.2). The proof of the lower bound uses properties of ECH cobordism maps to reduce to the case of a sphere, where (2.8) can be checked explicitly.

2.2 The upper bound

In this section we prove the upper bound half of Theorem 2.1.3:

Proposition 2.2.1. *Under the assumptions of Theorem 2.1.3,*

$$\limsup_{k\to\infty} \frac{c_{\sigma_k}(Y,\lambda)^2}{I(\sigma_k)} \leq \mathrm{vol}(Y,\lambda). \tag{2.9}$$

To prove Proposition 2.2.1, we can assume without loss of generality that λ is nondegenerate. To see this, assume that (2.9) holds for nondegenerate contact forms and suppose that λ is degenerate. We can find a sequence of functions $f_1 > f_2 > \cdots > 1$, which C^0-converges to 1, such that $f_n\lambda$ is nondegenerate for each n. It follows from the monotonicity property in [22, Lem. 4.2] that

$$c_{\sigma_k}(Y,\lambda) \leq c_{\sigma_k}(Y,f_n\lambda)$$

for every n and k. For each n, it follows from this and the inequality (2.9) for λ_n that

$$\limsup_{k\to\infty} \frac{c_{\sigma_k}(Y,\lambda)^2}{I(\sigma_k)} \le \mathrm{vol}(Y, f_n\lambda).$$

Since $\lim_{n\to\infty} \mathrm{vol}(Y, f_n\lambda) = \mathrm{vol}(Y,\lambda)$, we deduce the inequality (2.9) for λ.

Assume henceforth that λ is nondegenerate. In §2.2–§2.2 below we review some aspects of Taubes's proof of the isomorphism (2.2) and prove some related lemmas. In §2.2 we use these to prove Proposition 2.2.1.

Seiberg-Witten Floer cohomology

The proof of the isomorphism (2.2) involves perturbing the Seiberg-Witten equations on Y. To write down the Seiberg-Witten equations we first need to choose a Riemannian metric on Y. Let J be a generic almost complex structure on $\mathbb{R}\times Y$ as needed to define the ECH chain complex. The almost complex structure J determines a Riemannian metric g on Y such that the Reeb vector field R has length 1 and is orthogonal to the contact planes ξ, and

$$g(v,w) = \frac{1}{2} d\lambda(v, Jw), \qquad v,w \in \xi_y. \tag{2.10}$$

Note that this metric satisfies

$$|\lambda| = 1, \qquad *d\lambda = 2\lambda. \tag{2.11}$$

One could dispense with the factors of 2 in (4.11) and (2.11), but we are keeping them for consistency with [24] and its sequels.

Let $\mathbb{S}$ denote the spin bundle for the spin-c structure $\mathfrak{s}_\xi + \mathrm{PD}(\Gamma)$. The inputs to the Seiberg-Witten equations for this spin-c structure are a connection $\mathbb{A}$ on $\det(\mathbb{S})$ and a section ψ of $\mathbb{S}$. The spin bundle $\mathbb{S}$ splits as a direct sum

$$\mathbb{S} = E \oplus (E\otimes\xi),$$

where E and $E\otimes\xi$ are, respectively, the $+i$ and $-i$ eigenspaces of Clifford multiplication by λ. Here ξ is regarded as a complex line bundle using the metric and the orientation. A connection $\mathbb{A}$ on $\det(\mathbb{S})$ is then equivalent to a (Hermitian) connection A on E via the relation $\mathbb{A} = A_0 + 2A$, where A_0 is a distinguished connection on ξ reviewed in [45, §2.1].

For a positive real number r, consider the following version of the perturbed Seiberg-Witten equations for a connection A on E and spinor ψ:

$$\begin{aligned} *F_A &= r(\langle cl(\cdot)\psi,\psi\rangle - i\lambda) + i(*d\mu + \bar\omega) \\ D_A\psi &= 0. \end{aligned} \tag{2.12}$$

Here cl denotes Clifford multiplcation, $\bar\omega$ denotes the harmonic 1-form such that $*\bar\omega/\pi$ represents the image of $c_1(\xi)$ in $H^2(Y;\mathbb{R})$, and μ is a generic coclosed 1-form that is L^2-orthogonal to the space of harmonic 1-forms and that has "P-norm" less than 1, see [45, §2.1].

The group of *gauge transformations* $C^\infty(Y, S^1)$ acts on the space of pairs $(\mathbb{A}, \psi)$ by $g\cdot(\mathbb{A},\psi) = (\mathbb{A} - 2g^{-1}dg, g\psi)$. The quotient of the space of pairs $(\mathbb{A}, \psi)$ by the group of gauge transformations is called the *configuration space*. The set of solutions to (4.12) is invariant under gauge transformations. A solution to the Seiberg-Witten equations is called *reducible* if $\psi \equiv 0$ and *irreducible* otherwise. An irreducible solution is called *nondegenerate* if it is cut out transversely after modding out by gauge transformations, see [45, §3.1].

For fixed μ, when r is not in a certain discrete set, there are only finitely many irreducible solutions to (4.12) and these are all nondegenerate. In this case one can define the Seiberg-Witten Floer cohomology chain complex with $\mathbb{Z}/2$ coefficients, which we denote by $\widehat{CM}^*(Y, \mathfrak{s}_{\xi,\Gamma}, \lambda, J, r)$. The chain complex is generated by irreducible solutions to (4.12), along with additional generators determined by the reducible solutions. The differential counts solutions to a small abstract perturbation of the four-dimensional Seiberg-Witten equations on $\mathbb{R}\times Y$. In principle the chain complex differential may depend on the choice of abstract perturbation, but since the abstract perturbation is irrelevant to the proof of Proposition 2.2.1, we will omit it from the notation.

The grading

The chain complex $\widehat{CM}^*$ has a noncanonical absolute $\mathbb{Z}$-grading defined as follows. The linearization of the equations (4.12) modulo gauge equivalence at a pair (A, ψ), not necessarily solving the equations (4.12), defines a self-adjoint Fredholm operator $\mathcal{L}_{A,\psi}$. If (A, ψ) is a nondegenerate irreducible solution to (4.12), then the operator $\mathcal{L}_{A,\psi}$ has trivial kernel, and one defines the grading $gr(A, \psi) \in \mathbb{Z}$ to be the spectral flow from $\mathcal{L}_{A,\psi}$ to a reference self-adjoint Fredholm operator $\mathcal{L}_0$ between the same spaces with trivial kernel. The grading function gr depends on the choice of reference operator; fix one below. To describe the gradings of the remaining generators, recall that the set of reducible solutions modulo gauge equivalence is a torus $\mathbb{T}$ of dimension $b_1(Y)$. As explained in [32, §35.1], one can perturb the Seiberg-Witten equations using a Morse function

$$f : \mathbb{T} \to \mathbb{R}, \tag{2.13}$$

so that the chain complex generators arising from reducibles are identified with pairs $((A, 0), \phi)$, where $(A, 0)$ is a critical point of f and ϕ is a suitable eigenfunction of the Dirac operator D_A. The grading of each such generator is less than or equal to $gr(A, 0)$, where the latter is defined as the spectral flow to $\mathcal{L}_0$ from an appropriate perturbation of the operator $\mathcal{L}_{A,0}$.

We will need the following key result of Taubes relating the grading to the Chern-Simons functional. Fix a reference connection A_E on E. Given any other connection A on E, define the *Chern-Simons functional*

$$cs(A) = -\int_Y (A - A_E) \wedge (F_A + F_{A_E} - 2i{*}\bar{\omega}). \tag{2.14}$$

Note that this functional is gauge invariant because the spin-c structure $\mathfrak{s}_\xi + \mathrm{PD}(\Gamma)$ is assumed torsion.

Proposition 2.2.2. *[45, Prop. 5.1] There exists $K > 0$ such that for all r sufficiently large, if (A, ψ) is a nondegenerate irreducible solution to (4.12), or a reducible solution which is a critical point of (2.13), then*

$$\left| gr(A, \psi) + \frac{1}{4\pi^2} cs(A) \right| < K r^{31/16}. \tag{2.15}$$

Energy

Given a connection A on E, define the *energy*

$$\mathcal{E}(A) = i \int_Y \lambda \wedge F_A.$$

Filtered ECH has a Seiberg-Witten analogue defined using the energy functional as follows. Given a real number L, define $\widehat{CM}_L^*$ to be the submodule of $\widehat{CM}^*$ spanned by generators with energy less than $2\pi L$. It is shown in [45], as reviewed in [29, Lem. 2.3], that if r is sufficiently large, then all chain complex generators with energy less than $2\pi L$ are irreducible, and $\widehat{CM}_L^*$ is a subcomplex, whose homology we denote by $\widehat{HM}_L^*$. Moreover, as shown in [45] and reviewed in [29, Eq.(3.3)], if there are no ECH generators of action exactly L and if r is sufficiently large, then there is a canonical isomorphism of relatively graded chain complexes

$$ECC_*^L(Y, \lambda, \Gamma, J) \longrightarrow \widehat{CM}_L^{-*}(Y, \mathfrak{s}_{\xi,\Gamma}, \lambda_1, J_1, r). \tag{2.16}$$

Here (λ_1, J_1) is an "L-flat approximation" to (λ, J), which is obtained by suitably modifying (λ, J) near the Reeb orbits of action less than L; the precise definition is reviewed in [29, §3.1] and will not be needed here.

The isomorphism (2.16) is induced by a bijection on generators; the idea is that in the L-flat case[3], if r is sufficiently large, then for every ECH generator α of action less than L, there is a corresponding irreducible solution (A, ψ) to (4.12) such that the zero set of the E component of ψ is close to the Reeb orbits in α, the curvature F_A is concentrated near these Reeb orbits, and the energy of this solution is approximately $2\pi\mathcal{A}(\alpha)$.

The isomorphism of chain complexes (2.16) induces an isomorphism on homology

$$ECH_*^L(Y, \lambda, \Gamma, J) \xrightarrow{\simeq} \widehat{HM}_L^{-*}(Y, \mathfrak{s}_{\xi,\Gamma}, \lambda_1, J_1, r), \tag{2.17}$$

and inclusion of chain complexes defines a map

$$\widehat{HM}_L^{-*}(Y, \mathfrak{s}_{\xi,\Gamma}, \lambda_1, J_1, r) \longrightarrow \widehat{HM}^{-*}(Y, \mathfrak{s}_{\xi,\Gamma}). \tag{2.18}$$

Composing the above two maps gives a map

$$ECH_*^L(Y, \lambda, \Gamma, J) \longrightarrow \widehat{HM}^{-*}(Y, \mathfrak{s}_{\xi,\Gamma}). \tag{2.19}$$

The isomorphism (2.2) is the direct limit over L of the maps (2.19).

[3]In the non-L-flat case, there may be several Seiberg-Witten solutions corresponding to the same ECH generator, and/or Seiberg-Witten solutions corresponding to sets of Reeb orbits with multiplicities which are not ECH generators because they include hyperbolic orbits with multiplicity greater than one.

Volume in Seiberg-Witten theory

The volume enters into the proof of Proposition 2.2.1 in two essential ways.

The first way is as follows. It is shown in [24, §3] that for any given grading, there are no generators arising from reducibles if r is sufficiently large. That is, given an integer j, let s_j be the supremum of all values of r such that there exists a chain complex generator with grading at least $-j$ associated to a reducible solution to (4.12). Then $s_j < \infty$ for all j.

We now give an upper bound on the number s_j in terms of the volume. Fix $0 < \delta < \frac{1}{16}$. Given a positive integer j, let r_j be the largest real number such that

$$j = \frac{1}{16\pi^2} r_j^2 \operatorname{vol}(Y, \lambda) - r_j^{2-\delta}. \tag{2.20}$$

Lemma 2.2.3. *If j is sufficiently large, then $s_j < r_j$.*

Proof. Observe that $(A_r^{red}, \psi) = (A_E - \frac{1}{2} ir\lambda + i\mu, 0)$ is a solution to (4.12). Moreover, every other reducible solution is given by $(A, 0)$, where $A = A_r^{red} + \alpha$ for a closed 1-form α. It follows from (2.14) that

$$cs(A) = cs(A_r^{red}) = \frac{1}{4} r^2 \operatorname{vol}(Y, \lambda) + O(r). \tag{2.21}$$

Now suppose that j is sufficiently large that Proposition 2.2.2 is applicable to $r = r_j$, fix $r > r_j$, and suppose that $gr(A, 0) \geq -j$. Then equation (2.21) contradicts Proposition 2.2.2 if r is sufficiently large, which is the case if j is sufficiently large. □

The second essential way that volume enters into the proof of Proposition 2.2.1 is via the following a priori upper bound on the energy:

Lemma 2.2.4. *There is an r-independent constant C such that any solution (A, ψ) to (4.12) satisfies*

$$\mathcal{E}(A) \leq \frac{r}{2} \operatorname{vol}(Y, \lambda) + C. \tag{2.22}$$

Proof. This follows from [45, Eq. (2.7)], which is proved using a priori estimates on solutions to the Seiberg-Witten equations. Note that there is a factor of 1/2 in (2.22) which is not present in [45, Eq. (2.7)]. The reason is that the latter uses the Riemannian volume as defined by the metric (2.11), which is half of the contact volume (3.1) which we are using. □

Max-min families

Given a connection A on E and a section ψ of $\mathbb{S}$, define a functional

$$\mathcal{F}(A, \psi) = \frac{1}{2}(cs(A) - r\mathcal{E}(A)) + e_\mu(A) + \frac{r}{2} \int_Y \langle D_A \psi, \psi \rangle d\text{vol},$$

where
$$e_\mu(A) = i\int_Y F_A \wedge \mu.$$
Since the spin-c structure $\mathfrak{s}_\xi + \mathrm{PD}(\Gamma)$ is assumed torsion, the functional $\mathcal{F}$ is gauge invariant. The significance of the functional $\mathcal{F}$ is that the differential on the chain complex $\widehat{CM}^*$ counts solutions to abstract perturbations of the upward gradient flow equation for $\mathcal{F}$. In particular, $\mathcal{F}$ agrees with an appropriately perturbed version of the Chern-Simons-Dirac functional from [32], up to addition of an r-dependent constant, see [29, Eq. (7.2)].

A key step in Taubes's proof of the Weinstein conjecture [45] is to use a "minimax" approach to find a sequence (r_n, ψ_n, A_n), where $r_n \to \infty$ and (ψ_n, A_n) is a solution to (4.12) for $r = r_n$ with an n-independent bound on the energy. We will use a similar construction in the proof of Proposition 2.2.1.

Specifically, fix an integer j, and let s_j be the number from §2.2. Let $\hat{\sigma} \in \widehat{HM}^*(Y, \mathfrak{s}_{\xi,\Gamma})$ be a nonzero homogeneous class with grading greater than or equal to $-j$. Fix $r > s_j$ for which the chain complex $\widehat{CM}^*(Y, \mathfrak{s}_{\xi,\Gamma}, \lambda, J, r)$ is defined. Since we are using $\mathbb{Z}/2$-coefficients, any cycle representing the class $\hat{\sigma}$ has the form $\eta = \Sigma_i(A_i, \psi_i)$, where the pairs (A_i, ψ_i) are distinct gauge equivalence classes of solutions to (4.12). Define $\mathcal{F}_{\min}(\eta) = \min_i \mathcal{F}(A_i, \psi_i)$, and $\mathcal{F}_{\hat{\sigma}}(r) = \max_{[\eta]=\hat{\sigma}} \mathcal{F}_{\min}(\eta)$. Note that $\mathcal{F}_{\hat{\sigma}}(r)$ must be finite because there are only finitely many irreducible solutions to (4.12).

The construction in [41, §4.e] shows that for any such class $\hat{\sigma}$, there exists a piecewise smooth, possibly discontinuous family of solutions $(A_{\hat{\sigma}}(r), \psi_{\hat{\sigma}}(r))$ to (4.12) of the same grading as $\hat{\sigma}$ defined for $r > s_j$ such that $\mathcal{F}_{\hat{\sigma}}(r) = \mathcal{F}(A_{\hat{\sigma}}(r), \psi_{\hat{\sigma}}(r))$. Call the family $(A_{\hat{\sigma}}(r), \psi_{\hat{\sigma}}(r))_{r>s_j}$ a *max-min family* for $\hat{\sigma}$. Given such a max-min family, define $\mathcal{E}_{\hat{\sigma}}(r) = \mathcal{E}(A_{\hat{\sigma}}(r), \psi_{\hat{\sigma}}(r))$.

Lemma 2.2.5. *(a) $\mathcal{F}_{\hat{\sigma}}(r)$ is a continuous and piecewise smooth function of $r \in (s_j, \infty)$.*

(b) $\dfrac{d}{dr}\mathcal{F}_{\hat{\sigma}}(r) = -\dfrac{1}{2}\mathcal{E}_{\hat{\sigma}}(r).$

Proof. (a) follows from [41, Prop. 4.7], and (b) follows from [45, Eq. (4.6)]. □

In particular, $\mathcal{E}_{\hat{\sigma}}(r)$ does not depend on the choice of max-min family, except for a discrete set of real numbers r.

Max-min energy and min-max symplectic action

The numbers $\mathcal{E}_{\hat{\sigma}}(r)$ from §2.2 are related to the numbers $c_\sigma(Y, \lambda)$ from §2.1 as follows:

Proposition 2.2.6. *Let σ be a nonzero homogeneous class in $ECH(Y, \xi, \Gamma)$, and let $\hat{\sigma} \in \widehat{HM}^*(Y, \mathfrak{s}_{\xi,\Gamma})$ denote the class corresponding to σ under the isomorphism* (2.2)*. Then*
$$\lim_{r\to\infty} \mathcal{E}_{\hat{\sigma}}(r) = 2\pi c_\sigma(Y, \lambda).$$

The proof of Proposition 2.2.6 requires two preliminary lemmas which will also be needed later. To state the first lemma, recall from [46, Prop. 2.8] that in the case $\Gamma = 0$, if r is sufficiently large then there is a unique (up to gauge equivalence) "trivial" solution (A_{triv}, ψ_{triv}) to (4.12) such that $1 - |\psi| < 1/2$ on all of Y. If (λ, J) is L-flat with $L > 0$, then (A_{triv}, ψ_{triv}) corresponds to the empty set of Reeb orbits under the isomorphism (2.16) with $\Gamma = 0$, see the beginning of [42, §3]. Any solution not gauge equivalent to (A_{triv}, ψ_{triv}) will be called "nontrivial". Let L_0 denote one half the minimum symplectic action of a Reeb orbit.

Lemma 2.2.7. *There exists an r-independent constant c such that if r is sufficiently large, then every nontrivial solution (A, ψ) to (4.12) satisfies $\mathcal{E}(A) > 2\pi L_0$ and*

$$|cs(A)| \leq cr^{2/3}\mathcal{E}(A)^{4/3}. \tag{2.23}$$

Proof. The chain complex $ECC_*^{L_0}(Y, \lambda, \Gamma, J)$ has no generators unless $\Gamma = 0$, in which case the only generator is the empty set of Reeb orbits. In particular, the pair (λ, J) is L_0-flat. By (2.16), if r is sufficiently large then every nontrivial solution (A, ψ) to (4.12) has $\mathcal{E}(A) \geq 2\pi L_0$. Given this positive lower bound on the energy, the estimate (2.23) now follows as in [45, Eq. (4.9)]. Note that it is assumed there that $E(A) \geq 1$, but the same argument works as long as there is a positive lower bound on $E(A)$. $\square$

Now fix a positive number γ such that $\gamma < \delta/4$.

Lemma 2.2.8. *For every integer j there exists $\rho \geq 0$ such if $r \geq \rho$ and (A, ψ) is a nontrivial irreducible solution to (4.12) of grading $-j$, then*

$$|cs(A)| \leq r^{1-\gamma}\mathcal{E}(A). \tag{2.24}$$

Proof. Fix j. Let (A, ψ) be a nontrivial solution to (4.12) of grading $-j$ with

$$|cs(A)| > r^{1-\gamma}\mathcal{E}(A). \tag{2.25}$$

By Lemma 2.2.7, if r is sufficiently large then

$$|cs(A)| \leq cr^{2/3}\mathcal{E}(A)^{4/3}. \tag{2.26}$$

Combining (2.25) with (2.26), we conclude that $\mathcal{E}(A) \geq c^{-3}r^{1-3\gamma}$. Using (2.25) again, it follows that

$$|cs(A)| > c^{-3}r^{2-4\gamma}.$$

But this contradicts Proposition 2.2.2 when r is sufficiently large with respect to j, since $\delta > 4\gamma$. $\square$

Proof of Proposition 2.2.6. Choose $L_0 > c_\sigma(Y, \lambda)$ and let (λ_1, J_1) be an L_0-flat approximation to (λ, J). For r large, define $f_1(r)$ to be the infimum over L such that the class $\hat{\sigma}$ is in the image of the map (2.18). We first claim that

$$\lim_{r \to \infty} (f_1(r) - c_\sigma(Y, \lambda)) = 0. \tag{2.27}$$

This holds because for every $L \leq L_0$ which is not the symplectic action of an ECH generator, in particular $L \neq c_\sigma(Y, \lambda)$, if r is sufficiently large that the isomorphism (2.17) is defined, then the class $\hat{\sigma}$ is in the image of the map (2.18) if and only if $L > c_\sigma(Y, \lambda)$.

Next define $f(r)$ for r large to be the infimum over L such that the class $\hat{\sigma}$ is in the image of the inclusion-induced map

$$\widehat{HM}^*_L(Y, \mathfrak{s}_{\xi,\Gamma}, \lambda, J, r) \to \widehat{HM}^*(Y, \mathfrak{s}_{\xi,\Gamma}). \tag{2.28}$$

It follows from [29, Lem. 3.4(c)] that

$$\lim_{r\to\infty} (f(r) - f_1(r)) = 0. \tag{2.29}$$

By (2.27) and (2.29), to complete the proof of Proposition 2.2.6 it is enough to show that

$$\lim_{r\to\infty} (\mathcal{E}_{\hat{\sigma}}(r) - 2\pi f(r)) = 0. \tag{2.30}$$

To prepare for the proof of (2.30), assume that r is sufficiently large so that Lemma 2.2.7 is applicable and Lemma 2.2.8 is applicable to $j = -gr(\hat{\sigma})$. Also assume that r is sufficiently large so that all nontrivial Seiberg-Witten solutions in grading $gr(\hat{\sigma})$ are irreducible and have positive energy. Let (A, ψ) be a nontrivial solution in grading $gr(\hat{\sigma})$. Then

$$\mathcal{F}(A, \psi) = \frac{1}{2}(cs(A) - r\mathcal{E}(A)) + e_\mu(A).$$

By [45, Eq. (4.2)] and Lemma 2.2.7, we have

$$|e_\mu(A)| \leq \kappa\mathcal{E}(A) \tag{2.31}$$

where κ is an r-independent constant. The above and Lemma 2.2.8 imply that

$$(1 - r^{-\gamma} - 2\kappa r^{-1})\mathcal{E}(A) \leq \frac{-2}{r}\mathcal{F}(A, \psi) \leq (1 + r^{-\gamma} + 2\kappa r^{-1})\mathcal{E}(A). \tag{2.32}$$

Also, it follows from the construction of the trivial solution in [46] that

$$\lim_{r\to\infty} \mathcal{E}(A_{triv}) = \lim_{r\to\infty} \frac{\mathcal{F}(A_{triv}, \psi_{triv})}{r} = 0. \tag{2.33}$$

Now (2.30) can be deduced easily from (2.32) and (2.33). The details are as follows. Fix $\varepsilon > 0$ and suppose that r is sufficiently large as in the above paragraph. By the definition of $f(r)$, the class $\hat{\sigma}$ is in the image of the map (2.28) for $L = f(r) + \varepsilon$. Also, if r is sufficiently large, then by (2.32) and (2.33), and the fact that L has an upper bound when r is large by (2.27) and (2.29), if $\eta = \sum_i (A_i, \psi_i)$ is a cycle in $\widehat{CM}_L$ representing the class $\hat{\sigma}$, then $-2\mathcal{F}(A_i, \psi_i)/r < 2\pi(L + \varepsilon)$ for each i. Consequently $-2\mathcal{F}_{\hat{\sigma}}(r)/r < 2\pi(L + \varepsilon)$. By (2.32) and (2.33) again, if r is sufficiently large then $\mathcal{E}_{\hat{\sigma}}(r) < 2\pi(L + 2\varepsilon)$, which means that $\mathcal{E}_{\hat{\sigma}}(r) < f(r) + 3\varepsilon$.

By similar reasoning, if $\mathcal{E}_{\hat{\sigma}}(r) < f(r) - \varepsilon$, then if r is sufficiently large, the class $\hat{\sigma}$ is in the image of the map (2.28) for $L = f(r) - \varepsilon/2$, which contradicts the definition of $f(r)$. □

Proof of the upper bound

Proof of Proposition 2.2.1. The proof has six steps.

Step 1: Setup. If $\sigma \in ECH_*(Y,\xi,\Gamma)$ is a nonzero homogeneous class, let $\hat{\sigma} \in \widehat{HM}^*(Y,\mathfrak{s}_{\xi,\Gamma})$ denote the corresponding class in Seiberg-Witten Floer cohomology via the isomorphism (2.2). We can choose the absolute grading I on $ECH(Y,\xi,\Gamma)$ so that the Seiberg-Witten grading of $\hat{\sigma}$ is $-I(\sigma)$ for all σ. For Steps 1–5, fix such a class σ and write $j = I(\sigma)$. We will obtain an upper bound on $c_\sigma(Y,\lambda)$ in terms of j when j is sufficiently large, see (2.45) below.

To start, we always assume that j is sufficiently large so that $j > 0$, the number r_j defined in (2.20) satisfies $r_j \geq 1$, Proposition 2.2.2 and Lemma 2.2.7 are applicable to $r \geq r_j$, Lemma 2.2.3 is applicable so that $r_j > s_j$, and the trivial solution (A_{triv}, ψ_{triv}) does not have grading $-j$.

Fix a max-min family $(A_{\hat{\sigma}}(r), \psi_{\hat{\sigma}}(r))_{r>s_j}$ for $\hat{\sigma}$ as in §2.2. For $r > s_j$ define

$$\begin{aligned} \mathcal{E}(r) &= \mathcal{E}_{\hat{\sigma}}(r) = \mathcal{E}(A_{\hat{\sigma}}(r)), \\ cs(r) &= cs(A_{\hat{\sigma}}(r)), \\ v(r) &= -\frac{2\mathcal{F}_{\hat{\sigma}}(r)}{r} = \mathcal{E}(r) - \frac{cs(r)}{r} - \frac{2e_\mu(A_{\hat{\sigma}}(r))}{r}. \end{aligned} \tag{2.34}$$

It follows from Lemma 2.2.5 that $v(r)$ is continuous and piecewise smooth, and

$$\frac{dv}{dr} = \frac{cs(r)}{r^2}. \tag{2.35}$$

By Proposition 2.2.2 we have the key estimate

$$\left| -j + \frac{1}{4\pi^2} cs(r) \right| < K r^{2-\delta} \tag{2.36}$$

whenever $r \geq r_j$. Here we are using the fact that Lemma 2.2.3 is applicable, so that the solution $(A_{\hat{\sigma}}(r), \psi_{\hat{\sigma}}(r))$ is irreducible, so that $gr(A_{\hat{\sigma}}(r), \psi_{\hat{\sigma}}(r)) = -j$.

Define a number $\overline{r} = \overline{r}_{\hat{\sigma}}$ as follows. We know from Lemma 2.2.8 that if r is sufficiently large then

$$|cs(r)| \leq r^{1-\gamma}\mathcal{E}(r). \tag{2.37}$$

If (2.37) holds for all $r \geq r_j$, define $\overline{r} = r_j$. Otherwise define $\overline{r}$ to be the supremum of the set of r for which (2.37) does not hold.

Step 2. We now show that

$$\limsup_{r \geq \bar{r}} \mathcal{E}(r) \leq v(\bar{r}) g(\bar{r}), \tag{2.38}$$

where

$$g(r) = \exp\left(\frac{r^{-\gamma}}{\gamma\left(1 - r^{-\gamma} - 2\kappa r^{-1}\right)} \right), \tag{2.39}$$

and κ is the constant in (2.31). Here and below we assume that j is sufficiently large so that $1 - r_j^{-\gamma} - 2\kappa r_j^{-1} > 0$.

To prove (2.38), assume that $r \geq \bar{r}$. Then by (2.34), (2.37), and (2.31), as in (2.32), we have

$$\mathcal{E}(r) \leq \frac{1}{1 - r^{-\gamma} - 2\kappa r^{-1}} v(r). \tag{2.40}$$

Also $v(r) > 0$, since $r \geq 1$. By (2.35), (2.37) and (4.13) we have

$$\frac{dv(r)}{dr} \leq r^{-1-\gamma}\mathcal{E}(r) \leq \frac{r^{-1-\gamma}}{1 - r^{-\gamma} - 2\kappa r^{-1}} v(r) \leq \frac{r^{-1-\gamma}}{1 - \bar{r}^{-\gamma} - 2\kappa \bar{r}^{-1}} v(r).$$

Dividing this inequality by $v(r)$ and integrating from $\bar{r}$ to r gives

$$\ln\left(\frac{v(r)}{v(\bar{r})}\right) \leq \frac{1}{\gamma\left(1 - \bar{r}^{-\gamma} - 2\kappa\bar{r}^{-1}\right)}\left(\bar{r}^{-\gamma} - r^{-\gamma}\right) < \frac{\bar{r}^{-\gamma}}{\gamma\left(1 - \bar{r}^{-\gamma} - 2\kappa\bar{r}^{-1}\right)}.$$

Therefore

$$v(r) < v(\bar{r})g(\bar{r}).$$

Together with (4.13), this proves (2.38).

Step 3. We claim now that

$$v(\bar{r}) \leq \frac{1}{2} r_j \operatorname{vol}(Y, \lambda) + C_1 \bar{r}^{1-\delta}. \tag{2.41}$$

Here and below, $C_1, C_2 \ldots$ denote positive constants which do not depend on $\hat{\sigma}$ or r, and which we do not need to know anything more about.

To prove (2.41), use (2.35) and (2.36) to obtain

$$\frac{dv}{dr} \leq \frac{4\pi^2(j + K r^{2-\delta})}{r^2}.$$

Integrating this inequality from r_j to $\bar{r}$ and using $j > 0$, we deduce that

$$\begin{aligned} v(\bar{r}) - v(r_j) &\leq \frac{4\pi^2 j}{r_j} - \frac{4\pi^2 j}{\bar{r}} + \frac{4\pi^2 K(\bar{r}^{1-\delta} - r_j^{1-\delta})}{1-\delta} \\ &\leq \frac{4\pi^2 j}{r_j} + C_2 \bar{r}^{1-\delta}. \end{aligned} \tag{2.42}$$

Also, by (2.34), (2.36), (2.31), and Lemma 2.2.4, we have

$$\begin{aligned} v(r_j) &\leq \frac{1}{2} r_j \operatorname{vol}(Y, \lambda) + C + \frac{4\pi^2(-j + K r_j^{2-\delta}) + 2\kappa}{r_j} \\ &\leq \frac{1}{2} r_j \operatorname{vol}(Y, \lambda) - \frac{4\pi^2 j}{r_j} + C_3 r_j^{1-\delta}. \end{aligned} \tag{2.43}$$

Adding (2.42) and (2.43) gives (2.41).

Step 4. We claim now that if j is sufficiently large then

$$\bar{r} \leq C_4 r_j^{\frac{1}{1-2\gamma}}. \tag{2.44}$$

To prove this, by the definition of $\bar{r}$, if $\bar{r} > r_j$ then there exists a number r slightly smaller than $\bar{r}$ such that $|cs(r)| > r^{1-\gamma}\mathcal{E}(r)$. It then follows from Lemma 2.2.7 that

$$r^{1-\gamma}\mathcal{E}(r) < cr^{2/3}\mathcal{E}(r)^{4/3}.$$

Therefore

$$r^{2-4\gamma} \leq c^3 r^{1-\gamma}\mathcal{E}(r) \leq c^3|cs(r)|.$$

By (2.36) and the definition of r_j in (2.20), we have

$$c^3|cs(r)| \leq C_5 r_j^2 + C_6 r^{2-\delta}.$$

Combining the above two inequalities and using the fact that r can be arbitrarily close to $\bar{r}$, we obtain

$$\bar{r}^{2-4\gamma} \leq C_5 r_j^2 + C_6 \bar{r}^{2-\delta}.$$

Since $\delta > 4\gamma$ and $\bar{r} > r_j \to \infty$ as $j \to \infty$, if j is sufficiently large then

$$C_6 \bar{r}^{2-\delta} \leq \frac{1}{2}\bar{r}^{2-4\gamma}.$$

Combining the above two inequalities proves (2.44).

Assume henceforth that j is sufficiently large so that (2.44) holds.

Step 5. We claim now that

$$c_\sigma(Y,\lambda) \leq \frac{1}{4\pi} r_j \operatorname{vol}(Y,\lambda) g(\bar{r}) + C_7 r_j^{1-\nu}, \tag{2.45}$$

where $\nu = 1 - \frac{1-\delta}{1-2\gamma} > 0$.

To prove (2.45), insert (2.44) into (2.41) to obtain

$$v(\bar{r}) \leq \frac{1}{2} r_j \operatorname{vol}(Y,\lambda) + C_8 r_j^{1-\nu}.$$

The above inequality and (2.38) imply that

$$\begin{aligned}\limsup_{r\to\infty} \mathcal{E}(r) &\leq \left(\frac{1}{2} r_j \operatorname{vol}(Y,\lambda) + C_8 r_j^{1-\nu}\right) g(\bar{r}) \\ &\leq \frac{1}{2} r_j \operatorname{vol}(Y,\lambda) g(\bar{r}) + C_9 r_j^{1-\nu}.\end{aligned}$$

It follows from this and Proposition 2.2.6 that (2.45) holds.

Step 6. We now complete the proof of Proposition 2.2.1 by applying (2.45) to the sequence $\{\sigma_k\}$ and taking the limit as $k \to \infty$.

Let $j_k = I(\sigma_k)$ and $\bar{r}_k = \bar{r}_{\hat{\sigma}_k}$. It then follows from (2.45) and the definition of the numbers r_{j_k} in (2.20) that for every k sufficiently large,

$$\begin{aligned} \frac{c_{\sigma_k}(Y,\lambda)^2}{I(\sigma_k)} &\le \frac{(16\pi^2)^{-1} r_{j_k}^2 \operatorname{vol}(Y,\lambda)^2 g(\bar{r}_k)^2 + C_{10} r_{j_k}^{2-\nu}}{(16\pi^2)^{-1} r_{j_k}^2 \operatorname{vol}(Y,\lambda) - r_{j_k}^{2-\delta}} \\ &= \frac{\operatorname{vol}(Y,\lambda) g(\bar{r}_k)^2 + C_{11} r_{j_k}^{-\nu}}{1 - C_{12} r_{j_k}^{-\delta}}. \end{aligned} \tag{2.46}$$

By hypothesis, as $k \to \infty$ we have $j_k \to \infty$, and hence $\bar{r}_k > r_{j_k} \to \infty$. It then follows from (2.39) that $\lim_{k\to\infty} g(\bar{r}_k) = 1$. Putting all this into the above inequality proves (2.9). □

2.3 The lower bound

In this last section we prove the following proposition, which is the lower bound half of Theorem 2.1.3:

Proposition 2.3.1. *Under the assumptions of Theorem 2.1.3,*

$$\liminf_{k\to\infty} \frac{c_{\sigma_k}(Y,\lambda)^2}{I(\sigma_k)} \ge \operatorname{vol}(Y,\lambda). \tag{2.47}$$

In §2.3 we review some aspects of ECH cobordism maps, and in §2.3 we use these to prove Proposition 2.3.1.

ECH cobordism maps

Let (Y_+,λ_+) and (Y_-,λ_-) be closed oriented three-manifolds, not necessarily connected, with nondegenerate contact forms. Following [22], define a "weakly exact symplectic cobordism" from (Y_+,λ_+) to (Y_-,λ_-) to be a compact symplectic four-manifold (X,ω) with boundary $\partial X = Y_+ - Y_-$, such that the symplectic form ω is exact on X, and $\omega|_{Y_\pm} = d\lambda_\pm$.

It is shown in [22, Thm. 2.3], by a slight modification of [29, Thm. 1.9], that a weakly exact symplectic cobordism as above induces a map

$$\Phi^L(X,\omega) : ECH^L(Y_+,\lambda_+,0) \longrightarrow ECH^L(Y_-,\lambda_-,0)$$

for each $L \in \mathbb{R}$, defined by counting solutions to the Seiberg-Witten equations, perturbed using ω, on a "completion" of X.

More generally, let $A \in H_2(X,\partial X)$, and write $\partial A = \Gamma_+ - \Gamma_-$ where $\Gamma_\pm \in H_1(Y_\pm)$. Suppose that ω has a primitive on X which agrees with $\lambda_\pm$ on each component of $Y_\pm$ for which the corresponding component of $\Gamma_\pm$ is nonzero. Then the same argument constructs a map

$$\Phi^L(X,\omega,A) : ECH^L(Y_+,\lambda_+,\Gamma_+) \longrightarrow ECH^L(Y_-,\lambda_-,\Gamma_-), \tag{2.48}$$

defined by counting solutions to the Seiberg-Witten equations in the spin-c structure corresponding to A. As in [22, Thm. 2.3(a)], there is a well-defined direct limit map

$$\Phi(X,\omega,A) = \lim_{L\to\infty} \Phi^L(X,\omega,A) : ECH(Y_+,\xi_+,\Gamma_+) \longrightarrow ECH(Y_-,\xi_-,\Gamma_-), \tag{2.49}$$

where $\xi_\pm = \operatorname{Ker}(\lambda_\pm)$.

The relevance of the map (2.49) for Proposition 2.3.1 is that given a class $\sigma_+ \in ECH(Y_+,\xi_+,\Gamma_+)$, if $\sigma_- = \Phi(X,\omega,A)\sigma_+$, then

$$c_{\sigma_+}(Y_+,\lambda_+) \ge c_{\sigma_-}(Y_-,\lambda_-). \tag{2.50}$$

The inequality (2.50) follows directly from (2.49) and the definition of $c_{\sigma_\pm}$ in §2.1, cf. [22, Lem. 4.2]. Here we interpret $c_\sigma = -\infty$ if $\sigma = 0$. By a limiting argument as in [22, Prop. 3.6], the inequality (2.50) also holds if the contact forms $\lambda_\pm$ are allowed to be degenerate.

The map (2.48) is a special case of the construction in [21] of maps on ECH induced by general strong symplectic cobordisms. Without the assumption on the primitive of ω, these maps can shift the symplectic action filtration, but the limiting map (2.49) is still defined.

For computations we will need four properties of the map (2.49). First, if $X = ([a,b]\times Y, d(e^s\lambda))$ is a trivial cobordism from $(Y,e^b\lambda)$ to $(Y,e^a\lambda)$, where s denotes the $[a,b]$ coordinate, then

$$\Phi(X,\omega,[a,b]\times\Gamma) = \mathrm{id}_{ECH(Y,\xi,\Gamma)}\,. \tag{2.51}$$

This follows for example from [29, Cor. 5.8].

Second, suppose that (X,ω) is the composition of strong symplectic cobordisms (X_+,ω_+) from (Y_+,λ_+) to (Y_0,λ_0) and (X_-,ω_-) from (Y_0,λ_0) to (Y_-,λ_-). Let $\Gamma_0 \in H_1(Y_0)$ and let $A_\pm \in H_2(X_\pm,\partial_\pm X_\pm)$ be classes with $\partial A_+ = \Gamma_+ - \Gamma_0$ and $\partial A_- = \Gamma_0 - \Gamma_-$. Then

$$\Phi(X_-,\omega_-,A_-)\circ\Phi(X_+,\omega_+,A_+) = \sum_{A|_{X_\pm}=A_\pm} \Phi(X,\omega,A). \tag{2.52}$$

This is proved the same way as the composition property in [29, Thm. 1.9].

Third, if X is connected and $Y_\pm$ are both nonempty, then

$$\Phi(X,\omega,A)\circ U_+ = U_-\circ\Phi(X,\omega,A), \tag{2.53}$$

where $U_\pm$ can be the U map associated to any of the components of $Y_\pm$. This is proved as in [22, Thm. 2.3(d)].

Fourth, since we are using coefficients in the field $\mathbb{Z}/2$, it follows from the definitions that the ECH of a disjoint union is given by the tensor product

$$ECH((Y,\xi)\sqcup(Y',\xi'),\Gamma\oplus\Gamma') = ECH(Y,\xi,\Gamma)\otimes ECH(Y',\xi',\Gamma'). \tag{2.54}$$

If (X,ω) is a strong symplectic cobordism from (Y_+,λ_+) to (Y_-,λ_-), and if (X',ω') is a strong symplectic cobordism from (Y'_+,λ'_+) to (Y'_-,λ'_-), then it follows from the construction of the cobordism map that the disjoint union of the cobordisms induces the tensor product of the cobordism maps:

$$\Phi((X,\omega)\sqcup(X',\omega'),A\oplus A')=\Phi(X,\omega,A)\otimes\Phi(X',\omega',A'). \tag{2.55}$$

Proof of the lower bound

Proof of Proposition 2.3.1. The proof has four steps.

Step 1. We can assume without loss of generality that

$$U\sigma_{k+1}=\sigma_k \tag{2.56}$$

for each $k\geq 1$. To see this, note that by the isomorphism (2.2) of ECH with Seiberg-Witten Floer cohomology, together with properties of the latter proved in [32, Lemmas 22.3.3, 33.3.9], we know that if the grading $*$ is sufficiently large, then $ECH_*(Y,\xi,\Gamma)$ is finitely generated and

$$U: ECH_*(Y,\xi,\Gamma)\longrightarrow ECH_{*-2}(Y,\xi,\Gamma)$$

is an isomorphism. Hence there is a finite collection of sequences satisfying (2.56) such that every nonzero homogeneous class in $ECH(Y,\xi,\Gamma)$ of sufficiently large grading is contained in one of these sequences (recall that we are using $\mathbb{Z}/2$ coefficients). Thus it is enough to prove (2.47) for a sequence satisfying (2.56). Furthermore, in this case (2.47) is equivalent to

$$\liminf_{k\to\infty}\frac{c_{\sigma_k}(Y,\lambda)^2}{k}\geq 2\operatorname{vol}(Y,\lambda). \tag{2.57}$$

Step 2. When (Y,λ) is the boundary of a Liouville domain, the lower bound (2.57) was proved for a particular sequence $\{\sigma_k\}$ satisfying (2.56) in [22, Prop. 8.6(a)]. We now set up a modified version of this argument.

Fix $a>0$ and consider the symplectic manifold

$$([-a,0]\times Y,\omega=d(e^s\lambda))$$

where s denotes the $[-a,0]$ coordinate. The idea is that if a is large, then $([-a,0]\times Y,\omega)$ is "almost" a Liouville domain whose boundary is (Y,λ).

Fix $\varepsilon>0$. We adopt the notation that if $r>0$, then $B(r)$ denotes the closed ball

$$B(r)=\{z\in\mathbb{C}^2 \mid \pi|z|^2\leq r\}.$$

Choose disjoint symplectic embeddings

$$\{\varphi_i: B(r_i)\to[-a,0]\times Y\}_{i=1,\dots,N}$$

such that $([-a,0]\times Y)\setminus \sqcup_i \varphi_i(B(r_i))$ has symplectic volume less than ε. Let

$$X = ([-a,0]\times Y)\setminus \bigsqcup_{i=1}^{N} \operatorname{int}(\varphi_i(B(r_i))).$$

Then (X,ω) is a weakly exact symplectic cobordism from (Y,λ) to $(Y,e^{-a}\lambda)\sqcup\bigsqcup_{i=1}^{N}\partial B(r_i)$. Here we can take the contact form on $B(r_i)$ to be the restriction of the 1-form $\frac{1}{2}\sum_{k=1}^{2}(x_k dy_k - y_k dx_k)$ on $\mathbb{R}^4$; we omit this from the notation. Note that there is a canonical isomorphism

$$H_2(X,\partial X) = H_1(Y).$$

The symplectic form ω on X has a primitive $e^s\lambda$ which restricts to the contact forms on the convex boundary (Y,λ) and on the component $(Y,e^{-a}\lambda)$ of the concave boundary. Hence, as explained in §2.3, we have a well-defined map

$$\Phi = \Phi(X,\omega,\Gamma) : ECH(Y,\xi,\Gamma) \longrightarrow ECH\left((Y,\xi)\sqcup\bigsqcup_{i=1}^{n}\partial B(r_i),(\Gamma,0,\ldots,0)\right) \tag{2.58}$$

which satisfies (2.50). By (2.54), the target of this map is

$$ECH\left((Y,\xi)\sqcup\bigsqcup_{i=1}^{n}\partial B(r_i),(\Gamma,0,\ldots,0)\right) = ECH(Y,\xi,\Gamma)\otimes\bigotimes_{i=1}^{n} ECH(\partial B(r_i)).$$

Let U_0 denote the U map on the left hand side associated to the component Y, and let U_i denote the U map on the left hand side associated to the component $\partial B(r_i)$. Note that U_0 or U_i acts on the right hand side as the tensor product of the U map on the appropriate factor with the identity on the other factors. By (2.53) we have

$$\Phi(U_0\sigma) = U_i\Phi(\sigma) \tag{2.59}$$

for all $\sigma\in ECH(Y,\xi,\Gamma)$ and for all $i=0,\ldots,N$.

Step 3. We now give an explicit formula for the cobordism map Φ in (2.58).

Recall that $ECH(\partial B(r_i))$ has a basis $\{\zeta_k\}_{k\geq 0}$ where $\zeta_0 = [\emptyset]$ and $U_i\zeta_{k+1} = \zeta_k$. This follows either from the computation of the Seiberg-Witten Floer homology of S^3 in [32], or from direct calculations in ECH, most of which are explained in [30, Ex. 4.2]. We can now state the formula for Φ:

Lemma 2.3.2. *For any class $\sigma\in ECH(Y,\xi,\Gamma)$, we have*

$$\Phi(\sigma) = \sum_{k\geq 0}\sum_{k_1+\ldots+k_N=k} U_0^k\sigma\otimes\zeta_{k_1}\otimes\cdots\otimes\zeta_{k_N}.$$

Note that the sum on the right is finite because the map U_0 decreases symplectic action.

Proof of Lemma 2.3.2. Given σ, we can expand $\Phi(\sigma)$ as

$$\Phi(\sigma) = \sum_{k_1,\ldots,k_N \geq 0} \sigma_{k_1,\ldots,k_N} \otimes \zeta_{k_1} \otimes \cdots \otimes \zeta_{k_N} \tag{2.60}$$

where $\sigma_{k_1,\ldots,k_N} \in ECH(Y,\xi,\Gamma)$. We need to show that

$$\sigma_{k_1,\ldots,k_N} = U_0^{k_1+\cdots+k_N}\sigma. \tag{2.61}$$

We will prove by induction on $k = k_1 + \cdots + k_N$ that equation (2.61) holds for all σ.

To prove (2.61) when $k = 0$, let X' denote the disjoint union of the trivial cobordism $([-a-1,a] \times Y, d(e^s\lambda))$ and the balls $B(r_i)$. Then the composition $X' \circ X$ is the trivial cobordism $([-a-1,0] \times Y, d(e^s\lambda))$ from (Y, e^λ) to $(Y, e^{-a-1}\lambda)$. Now each ball $B(r_i)$ induces a cobordism map

$$\Phi_{B(r_i)} : ECH(\partial B(r_i)) \longrightarrow \mathbb{Z}/2$$

as in (2.49). By (2.55) and (2.51) we have

$$\Phi(X',\Gamma) = \mathrm{id}_{ECH(Y,\xi,\Gamma)} \otimes \Phi_{B(r_1)} \otimes \cdots \otimes \Phi_{B(r_N)}.$$

It then follows from (2.51) and the composition property (2.52) that

$$\begin{aligned} \sigma &= (\Phi(X',\Gamma) \circ \Phi)(\sigma)) \\ &= \sum_{k_1,\ldots,k_N \geq 0} \sigma_{k_1,\ldots,k_N} \prod_{i=1}^{N} \Phi_{B(r_i)}(\zeta_{k_i}). \end{aligned}$$

Now $\Phi_{B(r_i)}$ sends ζ_0 to 1 by [22, Thm. 2.3(b)], and ζ_m to 0 for all $m > 0$ by grading considerations (the corresponding moduli space of Seiberg-Witten solutions in the completed cobordism has dimension $2m$). Therefore $\sigma = \sigma_{0,\ldots,0}$ as desired.

Next let $k > 0$ and suppose that (2.61) holds for smaller values of k. To prove (2.61), we can assume without loss of generality that $k_1 > 0$. Applying U_1 to equation (2.60) and then using equation (2.59) with $i = 1$, we obtain

$$\sigma_{k_1,\ldots,k_N} = (U_0\sigma)_{k_1-1,k_2,\ldots,k_N}.$$

By inductive hypothesis,

$$(U_0\sigma)_{k_1-1,k_2,\ldots,k_N} = U_0^{k-1}(U_0\sigma).$$

The above two equations imply (2.61), completing the proof of Lemma 2.3.2. □

Step 4. We now complete the proof of Proposition 2.3.1. Let $\{\sigma_k\}_{k\geq 1}$ be a sequence in $ECH(Y,\xi,\Gamma)$ satisfying (2.56). By (2.50) we have

$$c_{\sigma_k}(Y,\lambda) \geq c_{\Phi(\sigma_k)}\left((Y, e^{-a}\lambda) \sqcup \bigsqcup_{i=1}^{N} \partial B(r_i)\right).$$

By Lemma 2.3.2 and [22, Eq. (5.6)], we have

$$c_{\Phi(\sigma_k)}\left((Y, e^{-a}\lambda) \sqcup \bigsqcup_{i=1}^{N} \partial B(r_i)\right) = \max_{U^{k'}\sigma_k \neq 0} \max_{k_1+\cdots+k_N=k'} \left(c_{U_0^{k'}\sigma_k}(Y, e^{-a}\lambda) + \sum_{i=1}^{N} c_{\zeta_{k_i}}(\partial B(r_i))\right).$$

Since $U^{k-1}\sigma_k = \sigma_1 \neq 0$, it follows from the above equation and inequality that

$$c_{\sigma_k}(Y, \lambda) \geq \max_{k_1+\cdots+k_N=k-1} \sum_{i=1}^{N} c_{\zeta_{k_i}}(\partial B(r_i)). \tag{2.62}$$

Now recall from [22] that Theorem 2.1.3 holds for $B(r)$. In detail, we know from [22, Cor. 1.3] that

$$c_{\zeta_k}(\partial B(r)) = dr$$

where d is the unique nonnegative integer such that

$$\frac{d^2+d}{2} \leq k \leq \frac{d^2+3d}{2}.$$

Consequently,

$$\lim_{k\to\infty} \frac{c_{\zeta_k}(\partial B(r))^2}{k} = 2r^2 = 4\,\mathrm{vol}(B(r)). \tag{2.63}$$

It follows from (2.62) and (2.63) and the elementary calculation in [22, Prop. 8.4] that

$$\liminf_{k\to\infty} \frac{c_{\sigma_k}(Y, \lambda)^2}{k} \geq 4 \sum_{i=1}^{N} \mathrm{vol}(B(r_i)). \tag{2.64}$$

By the construction in Step 2,

$$\begin{aligned} \sum_{i=1}^{N} \mathrm{vol}(B(r_i)) &\geq \mathrm{vol}([-a, 0] \times Y, d(e^s\lambda)) - \varepsilon \\ &= \frac{1-e^{-a}}{2} \mathrm{vol}(Y, \lambda) - \varepsilon. \end{aligned} \tag{2.65}$$

Since $a > 0$ can be arbitrarily large and $\varepsilon > 0$ can be arbitrarily small, (2.64) and (2.65) imply (2.57). This completes the proof of Proposition 2.3.1. $\square$

Chapter 3

From one Reeb orbit to two

Daniel Cristofaro-Gardiner and Michael Hutchings

Abstract: We show that every (possibly degenerate) contact form on a closed three-manifold has at least two embedded Reeb orbits. We also show that if there are only finitely many embedded Reeb orbits, then their symplectic actions are not all integer multiples of a single real number, and if there are exactly two embedded Reeb orbits, then the product of their symplectic actions is less than or equal to the contact volume of the manifold. Our proofs use a relation between the contact volume and the asymptotics of the amount of symplectic action needed to represent certain classes in embedded contact homology, recently proved by the authors and V. Gripp.

3.1 Statement of results

Let Y be a closed oriented three-manifold. Recall that a *contact form* on Y is a 1-form λ on Y such that $\lambda \wedge d\lambda > 0$. A contact form λ determines the *contact structure* $\xi := \operatorname{Ker}(\lambda)$, and the *Reeb vector field* R characterized by $d\lambda(R, \cdot) = 0$ and $\lambda(R) = 1$. A *Reeb orbit* is a closed orbit of the vector field R, i.e. a map $\gamma : \mathbb{R}/T\mathbb{Z} \to Y$ for some $T > 0$ such that $\gamma'(t) = R(\gamma(t))$, modulo reparametrization. The Reeb orbit γ is *nondegenerate* if the linearized Reeb flow along γ does not have 1 as an eigenvalue, and the contact form λ is called nondegenerate if all Reeb orbits are nondegenerate.

The three-dimensional Weinstein conjecture, proved by Taubes [45], asserts that any contact form on a closed three-manifold has at least one Reeb orbit. It is interesting to try to improve the lower bound on the number of Reeb orbits. In fact, it seems that the only known examples of contact forms on closed three-manifolds with only finitely many embedded Reeb orbits are certain contact forms on lens spaces with exactly two embedded Reeb orbits. (Here we consider S^3 to be a lens space.) It is shown in [30, Thm. 1.3] that any nondegenerate contact form on a closed three-manifold Y has at least two embedded Reeb orbits; and if Y is not a lens space, then there are at least three embedded Reeb orbits. The main theorem of the present paper asserts that one can prove the existence of at least two embedded Reeb orbits without the nondegeneracy assumption:

Theorem 3.1.1. *Every (possibly degenerate) contact form on a closed three-manifold has at least two embedded Reeb orbits.*

For example, Theorem 3.1.1 has the following implication for Hamiltonian dynamics. Recall that if Y is a hypersurface in a symplectic manifold (X, ω), then the *characteristic foliation* on Y is the rank one foliation $L_Y := \mathrm{Ker}(\omega|_{TY})$, and a *closed characteristic* in Y is an embedded loop in Y tangent to L_Y. If Y is a regular level set of a smooth function $H : X \to \mathbb{R}$, then closed characteristics on Y are the same as unparametrized embedded closed orbits of the Hamiltonian vector field X_H on Y. Now consider $X = \mathbb{R}^4$ with the standard symplectic form $\omega = \sum_{i=1}^{2} dx_i\, dy_i$. If Y is a star-shaped hypersurface in $\mathbb{R}^4$, meaning that it is tranverse to the radial vector field, then

$$\lambda = \frac{1}{2}\sum(x_i dy_i - y_i dx_i)$$

restricts to a contact form on Y (giving the tight contact structure), and the unparametrized embedded Reeb orbits are the same as the closed characteristics. Thus Theorem 3.1.1 implies the following:

Corollary 3.1.2. *Any smooth compact star-shaped hypersurface in $\mathbb{R}^4$ has at least two closed characteristics.*

There are a number of previous results related to Corollary 3.1.2. Hofer-Wysocki-Zehnder showed in [17, Thm. 1.1] that any strictly convex hypersurface in $\mathbb{R}^4$ has either two or infinitely many closed characteristics, and in [16, Cor. 1.10] that any nondegenerate contact form on S^3 giving the tight contact structure has either two or infinitely many embedded Reeb orbits, provided that all stable and unstable manifolds of the hyperbolic periodic orbits intersect transversally. More recently, Long [33] has shown that any symmetric, compact star-shaped hypersurface in $\mathbb{R}^4$ has at least two closed characteristics. And in higher dimensions, Wang [48] has shown that there are at least $\left\lfloor \frac{n+1}{2} \right\rfloor + 1$ closed characteristics on every compact strictly convex hypersurface Σ in $\mathbb{R}^{2n}$. It has long been conjectured that there are at least n closed characteristics on every compact convex hypersurface in $\mathbb{R}^{2n}$; for example, almost the same conjecture appears in [6, Conj. 1].

The method used to prove Theorem 3.1.1 yields a slightly more general result. To state it, if γ is a Reeb orbit, define its *symplectic action* by

$$\mathcal{A}(\gamma) := \int_\gamma \lambda.$$

We then have:

Theorem 3.1.3. *Let (Y, λ) be a closed contact three-manifold having only finitely many embedded Reeb orbits $\gamma_1, \ldots, \gamma_m$. Then their symplectic actions $\mathcal{A}(\gamma_1), \ldots, \mathcal{A}(\gamma_m)$ are not all integer multiples of a single real number.*

Remark 3.1.4. *If λ has infinitely many embedded Reeb orbits, then their symplectic actions can all be integer multiples of a single real number, for example in a prequantization space, or in an ellipsoid $(\frac{|z_1|^2}{a_1} + \frac{|z_2|^2}{a_2} = 1) \subset \mathbb{C}^2$ with a_1/a_2 rational. Theorem 3.1.3 (and its proof) does extend to contact forms with infinitely many embedded Reeb orbits if they are isolated in the free loop space.*

To state one more result, if λ is a contact form on a closed oriented three-manifold Y, define the *volume* of (Y, λ) by

$$\mathrm{vol}(Y, \lambda) := \int_Y \lambda \wedge d\lambda. \tag{3.1}$$

One can ask whether there exists a Reeb orbit with an upper bound on its symplectic action in terms of the volume of (Y, λ). One might also expect that in most cases there are at least three embedded Reeb orbits. The following theorem asserts that at least one of these two statements always holds:

Theorem 3.1.5. *Let (Y, λ) be a closed contact three-manifold. Then either:*

- *λ has at least three embedded Reeb orbits, or*
- *λ has exactly two embedded Reeb orbits, and their symplectic actions T, T' satisfy $TT' \leq \mathrm{vol}(Y, \lambda)$.*

3.2 Embedded contact homology and volume

To prepare for the proofs of Theorem 3.1.1, 3.1.3, and 3.1.5, we need to recall some notions from embedded contact homology (ECH). For more about ECH, see [20] and the references therein.

Definition of embedded contact homology

If λ is nondegenerate, then for each $\Gamma \in H_1(Y)$ the *embedded contact homology* with $\mathbb{Z}/2$ coefficients, which we denote by $ECH_*(Y, \lambda, \Gamma)$, is defined. (ECH can actually be defined over $\mathbb{Z}$, see [19], but $\mathbb{Z}/2$ coefficients are sufficient for the applications in this paper). This is the homology of a chain complex $ECC(Y, \lambda, \Gamma, J)$ generated by finite sets $\alpha = \{(\alpha_i, m_i)\}$ such that each α_i is a Reeb orbit, $m_i = 1$ if α_i is hyperbolic, and

$$\sum_i m_i[\alpha_i] = \Gamma \in H_1(Y).$$

Here a Reeb orbit γ is called *hyperbolic* if the linearized Reeb flow around γ has real eigenvalues. We use the notation $[\alpha]$ to denote the homology class $\sum_i m_i[\alpha_i] \in H_1(Y)$. The J that enters into the chain complex is an $\mathbb{R}$-invariant almost complex structure on $\mathbb{R}\times Y$ that sends the two-plane field ξ to itself, rotating it positively with respect to $d\lambda$, and satisfies $J(\partial_s) = R$, where s denotes the $\mathbb{R}$ coordinate on $\mathbb{R}\times Y$. The chain complex differential ∂ counts certain mostly embedded J-holomorphic curves in $\mathbb{R}\times Y$. Specifically, if α and β are two chain complex generators, then the differential coefficient $\langle\partial\alpha, \beta\rangle \in \mathbb{Z}/2$ is a count of J-holomorphic curves in $\mathbb{R}\times Y$, modulo translation of the $\mathbb{R}$ coordinate, that are asymptotic as currents to $\mathbb{R}\times\alpha$ as $s \to \infty$ and to $\mathbb{R}\times\beta$ as $s \to -\infty$. The curves are required to have *ECH index* 1. The ECH index is a certain function of the relative homology class of the curve, explained e.g. in [25]; we do not need to recall the definition here. If J is generic, then ∂ is well-defined and $\partial^2 = 0$, as shown in [27, 19].

The ECH index induces a relative $\mathbb{Z}/d$-grading on $ECH_*(Y, \lambda, \Gamma)$, where d denotes the divisibility of $c_1(\xi) + 2\,\mathrm{PD}(\Gamma)$ in $H^2(Y)$ mod torsion, see [25, §2.8]. Here $\mathrm{PD}(\Gamma)$ denotes the Poincare dual of Γ.

The isomorphism with Seiberg-Witten Floer cohomology

Although a priori the homology of the chain complex $ECC(Y, \lambda, \Gamma, J)$ might depend on J, in fact it does not. This follows from a theorem of Taubes [**e4**, 41, 42, 43] asserting that when Y is connected, there is a canonical isomorphism between embedded contact homology and a version of Seiberg-Witten Floer cohomology. The precise statement is that there is a canonical isomorphism of relatively graded $\mathbb{Z}/2$-modules

$$ECH_*(Y, \lambda, \Gamma) \simeq \widehat{HM}^{-*}(Y, \mathfrak{s}_\xi + PD(\Gamma)), \tag{3.2}$$

where $\mathfrak{s}_\xi$ is the spin-c structure determined by the oriented two-plane field ξ, see e.g. [32, Lem. 28.1.1]. (The isomorphism also holds over $\mathbb{Z}$.) In particular, there is a well-defined relatively graded $\mathbb{Z}/2$-module $ECH(Y, \xi, \Gamma)$. By summing over all $\Gamma \in H_1(Y)$, one also obtains a well-defined relatively graded $\mathbb{Z}/2$-module $ECH(Y, \xi)$.

Filtered ECH

If $\alpha = \{(\alpha_i, m_i)\}$ is a generator of the ECH chain complex, define the *symplectic action* of α by

$$\mathcal{A}(\alpha) := \sum_i m_i \mathcal{A}(\alpha_i) = \sum_i m_i \int_{\alpha_i} \lambda.$$

It follows from the conditions on J that the ECH differential decreases the symplectic action. Hence, for any real number L, one can define the *filtered ECH*, denoted by $ECH^L(Y, \lambda, \Gamma)$, to be the homology of the subcomplex of ECC spanned by generators with action strictly less than L.

It is shown in [29, Thm. 1.3] that $ECH^L(Y,\lambda,\Gamma)$ does not depend on the choice of generic J required to define the chain complex differential. On the other hand, $ECH^L(Y,\lambda,\Gamma)$, for fixed Y and Γ, does depend on the contact form λ and not just on the contact structure ξ.

As in the previous section, one can remove the homology class Γ from the notation by summing over all possible homology classes. Denote the resulting relatively graded $\mathbb{Z}/2$ module by $ECH^L(Y,\lambda)$.

The U map

If Y is connected, there is a degree -2 map

$$U : ECH(Y,\lambda,\Gamma) \to ECH(Y,\lambda,\Gamma). \tag{3.3}$$

It is induced by a chain map U_z which is defined similarly to the differential ∂, but instead of counting ECH index 1 curves modulo translation, it counts J-holomorphic curves of ECH index 2 passing through $(0,z) \in \mathbb{R}\times Y$, where z is a base point in Y which is not contained in any Reeb orbit, and J is suitably generic. The connectedness of Y implies that the induced map (3.3) does not depend on z. (When Y is disconnected there is one U map for each component.) For details see [30, §2.5].

There is an analogous U map on Seiberg-Witten Floer cohomology, and it is shown in [44, Thm. 1.1] that this agrees with the U map on ECH under the isomorphism (3.2).

Minimum symplectic action needed to represent a class

Let $0 \neq \sigma \in ECH(Y,\xi)$. We now recall from [22] the definition of a real number $c_\sigma(Y,\lambda)$, which roughly speaking is the minimum symplectic action needed to represent the class σ.

If λ is nondegenerate, then $c_\sigma(Y,\lambda)$ is the infimum over L such that σ is in the image of the inclusion-induced map $ECH^L(Y,\lambda) \to ECH(Y,\xi)$. Note that for any J as needed to define the chain complex $ECC(Y,\lambda,J)$, there exists a cycle θ in the chain complex representing the class σ, such that every chain complex generator α that appears in θ satisfies $\mathcal{A}(\alpha) \leq c_\sigma(Y,\lambda)$, and $c_\sigma(Y,\lambda)$ is the smallest number with this property. We call a cycle θ as above an *action-minimizing representative* of σ.

If λ is degenerate, one defines

$$c_\sigma(Y,\lambda) = \lim_{n\to\infty} c_\sigma(Y,f_n\lambda), \tag{3.4}$$

where $f_n : Y \to \mathbb{R}$ are smooth functions such that $f_n\lambda$ is nondegenerate and $\lim_{n\to\infty} f_n = 1$ in the C^0 topology.

The numbers $c_\sigma(Y,\lambda)$ then satisfy the following axioms:

(Monotonicity) If $f : Y \to \mathbb{R}$ is a smooth function with $f > 1$, then $c_\sigma(Y,\lambda) \leq c_\sigma(Y,f\lambda)$.

(Scaling) If $\kappa > 0$ is a constant then $c_\sigma(Y,\kappa\lambda) = \kappa c_\sigma(Y,\lambda)$.

(Continuity) If $f_n : Y \to \mathbb{R}$ are smooth functions with $\lim_{n\to\infty} f_n = 1$ in the C^0 topology, then $\lim_{n\to\infty} c_\sigma(Y, f_n\lambda) = c_\sigma(Y, \lambda)$.

To see that (3.4) is well-defined and to prove the above axioms, one can first show that the Monotonicity and Scaling axioms hold for nondegenerate contact forms, see [22, §4]. It then follows from this that the definition (3.4) does not depend on the sequence $\{f_n\}$, and that the Monotonicity, Scaling, and Continuity axioms hold without any nondegeneracy assumption.

Asymptotics and volume

In [5], the following result was established relating the asymptotics of the numbers $c_\sigma(Y, \lambda)$ to the contact volume (3.1). If $\Gamma \in H_1(Y)$ is such that $c_1(\xi) + 2PD(\Gamma) \in H^2(Y; \mathbb{Z})$ is torsion, then we know from §3.2 that $ECC(Y, \xi, \Gamma)$ has a relative $\mathbb{Z}$-grading. Choose any normalization of this to an absolute $\mathbb{Z}$-grading, and denote the grading of a generator x by $I(x) \in \mathbb{Z}$. We then have:

Theorem 3.2.1. *[5, Thm. 1.3] Let (Y, λ) be a closed connected contact three-manifold, let $\Gamma \in H_1(Y)$, suppose that $c_1(\xi) + 2PD(\Gamma) \in H^2(Y, \mathbb{Z})$ is torsion, and choose an absolute $\mathbb{Z}$-grading as above on $ECH(Y, \xi, \Gamma)$. Let $\{\sigma_k\}_{k=1,2,\ldots}$ be a sequence of nonzero homogeneous elements of $ECH(Y, \xi, \Gamma)$ satisfying $\lim_{k\to\infty} I(\sigma_k) = \infty$. Then*

$$\lim_{k\to\infty} \frac{c_{\sigma_k}(Y, \lambda)^2}{I(\sigma_k)} = \mathrm{vol}(Y, \lambda). \tag{3.5}$$

To prove Theorem 3.1.1 and Theorem 3.1.3, we just need the following weaker result:

Corollary 3.2.2. *Let (Y, λ) be a closed connected contact three-manifold. Then there exist nonzero classes $\{\sigma_k\}_{k\geq 1}$ in $ECH(Y, \xi)$ such that*

$$U\sigma_{k+1} = \sigma_k \tag{3.6}$$

for all $k \geq 1$, and

$$\lim_{k\to\infty} \frac{c_{\sigma_k}(Y, \lambda)}{k} = 0. \tag{3.7}$$

.

Proof. We can always find a class $\Gamma \in H_1(Y)$ such that $c_1(\xi) + 2\,\mathrm{PD}(\Gamma) \in H^2(Y; \mathbb{Z})$ is torsion. It follows from the isomorphism (3.2) of $ECH(Y, \xi, \Gamma)$ with Seiberg-Witten Floer cohomology, together with known properties of the latter [32, Lem. 33.3.9, Cor. 35.1.4] that there exists a sequence $\{\sigma_k\}_{k\geq 1}$ of nonzero homogeneous elements of $ECH(Y, \xi, \Gamma)$ satisfying (3.6). Since the U map has degree -2, we have $I(\sigma_{k+1}) = I(\sigma_k) + 2$. Hence, Theorem 3.2.1 applies to give (3.5), which then implies (3.7). □

Remark 3.2.3. *The analysis in [5] is not required for Corollary 1.2, because it was already shown in [22] that Theorem 3.2.1 holds for any contact form on S^3 giving the tight contact structure. In particular, it follows from [22, Rmk. 3.3, Prop. 4.5] that Theorem 3.2.1 holds for the boundary of an ellipsoid in $\mathbb{R}^4$, and it then follows from [22, Prop. 8.6(b)] that Theorem 3.2.1 holds for any other contact form giving the same contact structure.*

3.3 The key lemma

The key to the proofs of Theorems 3.1.1, 3.1.3, and 3.1.5 is the following:

Lemma 3.3.1. *Let Y be a closed connected three-manifold and let λ be a (possibly degenerate) contact form on Y with kernel ξ. Assume that λ has only finitely many embedded Reeb orbits $\gamma_1, \ldots, \gamma_m$. Then:*

(a) If $0 \neq \sigma \in ECH(Y, \xi)$, then $c_\sigma(Y, \lambda)$ is a nonnegative integer linear combination of $\mathcal{A}(\gamma_1), \ldots, \mathcal{A}(\gamma_m)$.

(b) If $\sigma \in ECH(Y, \xi)$ and $U\sigma \neq 0$, then $c_{U\sigma}(Y, \lambda) < c_\sigma(Y, \lambda)$.

Proof. Fix a nonzero class $\sigma \in ECH(Y, \xi)$ and write $L = c_\sigma(Y, \lambda)$. Choose open tubular neighborhoods N_i of the Reeb orbits γ_i whose closures are disjoint, and let $N = \bigcup_{i=1}^m N_i$. Fix a point $z \in Y \setminus \overline{N}$ for use in defining the U map. By shrinking the tubular neighborhoods N_i if necessary, we may assume that:

(i) If γ is a Reeb trajectory intersecting both z and $\overline{N}$ then $\int_\gamma \lambda \geq L + 3$.

Next, choose a sequence of smooth functions $\{f_n : Y \to \mathbb{R}^{>0}\}$ such that:

(ii) $f_n|_{Y \setminus N} \equiv 1$,

(iii) The contact form $f_n\lambda$ is nondegenerate,

(iv) $\lim_{n\to\infty} f_n = 1$ in the C^1 topology, and

(v) Every Reeb orbit of $f_n\lambda$ with symplectic action less than $L+1$ is contained in some N_i, and has symplectic action within $1/n$ of an integer multiple of $\mathcal{A}(\gamma_i)$.

(The reason we can obtain condition (v) is that otherwise there would be a sequence f_n such that each $f_n\lambda$ has a Reeb orbit of action less than $L+1$ not contained in N, or a Reeb orbit in N_i of action $< L+1$ whose action is not within ε of an integer multiple of $\mathcal{A}(\gamma_i)$ for some n-independent $\varepsilon > 0$. Then a subsequence of these Reeb orbits would converge to a Reeb orbit of λ which could not be a multiple of one of the Reeb orbits γ_i.)

It follows from conditions (iii) and (v) that $c_\sigma(Y, f_n\lambda)$ is within m/n of an integer linear combination of $\mathcal{A}(\gamma_1), \ldots, \mathcal{A}(\gamma_n)$. Assertion (a) of the lemma now follows from condition (iv) and the Continuity axiom for c_σ.

To prove (b), continue to fix the above data, and assume that $U\sigma \neq 0$. For each n, choose a generic almost complex structure J_n on $\mathbb{R} \times Y$ as needed to define the filtered ECH chain complex $ECC^{L+1}(Y, f_n\lambda, J_n)$ and the chain map U_z on it. Specifically, we need J_n to satisfy the genericity conditions listed in the first paragraph of [19, §10], for J_n-holomorphic curves counted by ∂ or U_z whose positive ends have total action less than $L+1$. These conditions on J_n can all be achieved by perturbing J_n near the Reeb orbits of action less than $L+1$. So by condition (v) above, we can arrange that the almost complex structures J_n agree with a fixed almost complex structure J_0 on $\mathbb{R} \times (Y \setminus N)$.

We know from the proof of (a) that if n is sufficiently large then $c_\sigma(Y, f_n\lambda) < L+1$, so we can choose an action-minimizing representative θ_n of σ in $ECC^{L+1}(Y, f_n\lambda)$.

Claim. There exists $\delta > 0$ and a positive integer n_0 such that if $n \geq n_0$ and C_n is a J_n-holomorphic curve counted by $U_z\theta_n$, then $\int_{C_n} d(f_n\lambda) \geq \delta$.

The Claim implies (b), because it implies that if $n \geq n_0$ then $c_{U\sigma}(Y, f_n\lambda) \leq c_\sigma(Y, f_n\lambda) - \delta$, and so by the Continuity axiom $c_{U\sigma}(Y, \lambda) \leq c_\sigma(Y, \lambda) - \delta$.

Proof of Claim: Recall that the conditions on J_n imply that if C_n is any J_n-holomorphic curve, then $d(f_n\lambda)$ is pointwise nonnegative on C_n, with equality only where the tangent space to C_n is the span of the $\mathbb{R}$ direction and the Reeb direction (or where C_n is singular, but this does not happen for curves counted by $U_z\theta_n$). In particular, $\int_{C_n} d(f_n\lambda) \geq 0$. Consequently, if the Claim is false, then we can find an increasing sequence $\{n_i\}_{i\geq 1}$ of positive integers, and for each i a J_{n_i}-holomorphic curve C_{n_i} counted by $U_z\theta_{n_i}$, such that $\lim_{i\to\infty} \int_{C_{n_i}} d(f_{n_i}\lambda) = 0$.

We now use the following proposition, which is a special case of a result of Taubes [47, Prop. 3.3]:

Proposition 3.3.2. *Let (X, ω) be a compact symplectic 4-manifold with boundary with a compatible almost complex structure J. Let $\{C_i\}_{i\in\mathbb{N}}$ be a sequence of compact J-holomorphic curves in X with boundary contained in ∂X, and suppose that there exists $E > 0$ such that $\int_{C_i} \omega < E$ for all i. Then one can pass to a subsequence such that:*

(Convergence as currents) The curves $\{C_i\}$ converge weakly as currents to a compact J-holomorphic curve C_0 with boundary in ∂X such that $\int_{C_0} \omega \leq E$, and

(Pointwise convergence)

$$\lim_{i\to\infty} \left(\sup_{x\in C_{i^*}} \mathrm{dist}(x, C_0) + \sup_{x\in C_0} \mathrm{dist}(x, C_{i^*}) \right) = 0.$$

We apply the above proposition to the intersections of the holomorphic curves C_{n_i} with $X = [-1,1] \times (Y \setminus N)$, with the symplectic form $\omega = d(e^s\lambda)$. To see why we have the necessary upper bound on ω to apply the proposition, given i, choose $s_+ \in [1,2]$ and $s_- \in [-2,-1]$ such that C_{n_i} is tranverse to $\{s_\pm\} \cap Y$. Then since $d(e^s f_{n_i}\lambda)$ and $d(f_{n_i}\lambda)$ are pointwise nonnegative on C_{n_i}, we have an upper bound

$$\begin{aligned}\int_{C_{n_i}\cap([-1,1]\times(Y\setminus N))}\omega &\leq \int_{C_{n_i}\cap([s_-,s_+]\times Y)} d(e^s f_{n_i}\lambda)\\ &= e^{s_+}\int_{C_{n_i}\cap(\{k\}\times Y)} f_{n_i}\lambda - e^{s_-}\int_{C_{n_i}\cap(\{-k\}\times Y)} f_{n_i}\lambda\\ &< e^2(L+1).\end{aligned}$$

So we can pass to a subsequence such that $C_{n_i}\cap([-1,1]\times(Y\setminus N))$ converges in the sense of Proposition 3.3.2 to a (possibly multiply covered) J_0-holomorphic curve C_0 in $[-1,1]\times(Y\setminus N)$. By the "pointwise convergence" condition, the curve C_0 contains the point $(0,z)$, since each C_{n_i} does.

Since C_0 is J_0-holomorphic, it follows that $d\lambda$ is pointwise nonnegative on C_0, with equality only where C_0 is singular or the tangent space of C_0 is the span of the $\mathbb{R}$ direction and the Reeb direction. In particular,

$$\int_{C_0} d\lambda \geq 0. \tag{3.8}$$

In fact, the inequality (3.8) must be strict. Otherwise C_0, regarded as a current, is invariant under translation of the $[-1,1]$ coordinate on $[-1,1]\times(Y\setminus N)$. It follows that $C_0\cap(\{0\}\times(Y\setminus N))$ is tangent to the Reeb vector field for λ. In particular, $C_0\cap(\{0\}\times(Y\setminus N))$, regarded as a subset of Y, contains a Reeb trajectory for λ passing through z with endpoints on $\partial\overline{N}$. So by (i) above,

$$\int_{C_0\cap(\{0\}\times(Y\setminus N))}\lambda \geq L+3.$$

By the convergence of currents above, it follows that

$$\int_{C_{n_i}\cap(\{s\}\times(Y\setminus N))} f_{n_i}\lambda \geq L+2 \tag{3.9}$$

whenever i is sufficiently large and $s\in[-1,1]$ is such that C_{n_i} is transverse to $\{s\}\times Y$. When this transversality holds, we orient $C_{n_i}\cap(\{s\}\times Y)$, regarded as a submanifold, by the "$\mathbb{R}$-direction first" convention. The conditions on J_{n_i} imply that $f_{n_i}\lambda$ is pointwise nonnegative on this oriented one-manifold, so it follows from (3.9) that

$$\int_{C_{n_i}\cap(\{s\}\times Y)} f_{n_i}\lambda \geq L+2. \tag{3.10}$$

But this is impossible, because the left hand side of (3.10) must be less than or equal to the maximum symplectic action of a generator in θ_{n_i}, which is less than $L+1$. This contradiction proves that the inequality (3.8) is strict.

Given this, let $\delta = \frac{1}{2}\int_{C_0} d\lambda > 0$. It follows from the convergence of currents that if i is sufficiently large then

$$\begin{aligned}\int_{C_{n_i}} d(f_{n_i}\lambda) &\geq \int_{C_{n_i}\cap([-1,1]\times(Y\setminus N))} d(f_{n_i}\lambda)\\ &= \int_{C_{n_i}\cap([-1,1]\times(Y\setminus N))} d\lambda\\ &\geq \int_{C_0} d\lambda \ - \delta\\ &= \delta.\end{aligned}$$

This contradicts our assumption that $\lim_{i\to\infty}\int_{C_{n_i}} d(f_{n_i}\lambda) = 0$ and thus completes the proof of the Claim, and with it Lemma 3.3.1. □

Remark 3.3.3. *In the above argument we can not quote the SFT compactness theorem from [9], because that result assumes both a genus bound (which one does not have in ECH) as well as nondegeneracy of the contact form. This is why we use Taubes's approach via currents. Although this is only applicable in four dimensions, if one has a genus bound then one can cite [10] for similar arguments in higher dimensions.*

3.4 Proofs of theorems

Proof of Theorem 3.1.1. This follows from Theorem 3.1.3. □

Proof of Theorem 3.1.3. Suppose that λ has only finitely many embedded Reeb orbits and suppose that their symplectic actions are all integer multiples of a single real number $T > 0$. Let $\{\sigma_k\}_{k\geq 1}$ be any sequence satisfying (3.6). Then by Lemma 3.3.1, we have $c_{\sigma_k}(Y,\lambda) = n_k T$ where $\{n_k\}_{k\geq 1}$ is a strictly increasing sequence of nonnegative integers. It follows that

$$\liminf_{k\to\infty} \frac{c_{\sigma_k}(Y,\lambda)}{k} \geq T, \tag{3.11}$$

so that (3.7) cannot hold. This contradicts Corollary 3.2.2. □

Proof of Theorem 3.1.5. Suppose there are fewer than three embedded Reeb orbits. We know from Theorem 3.1.1 that Y is connected and there are exactly two embedded Reeb orbits; denote their symplectic actions by T and T'.

Let $\{\sigma_k\}_{k\geq 1}$ be a sequence of homogeneous classes satisfying (3.6). By Lemma 3.3.1, we have $c_{\sigma_k}(Y,\lambda) = n_k T + n'_k T'$ where n_k and n'_k are nonnegative integers such that $n_{k+1}T + n'_{k+1}T' > n_k T + n'_k T'$. It follows from this that

$$\lim_{k\to\infty} \frac{c_{\sigma_k}(Y,\lambda)^2}{k} \geq 2TT'. \tag{3.12}$$

To see this, note that if we fix k and write $L = c_{\sigma_k}(Y,\lambda) = n_k T + n_{k'} T'$, then k is less than or equal to the number of pairs of nonnegative integers (x,y) with $xT + yT' \le L$, which is the number of lattice points in the triangle enclosed by the line $Tx + T'y = L$ and the x and y axes, which is $L^2/(2TT') + O(L)$, compare [22, §3.3]. On the other hand, since the U map has degree -2, we have

$$\lim_{k\to\infty} \frac{I(\sigma_k)}{k} = 2. \tag{3.13}$$

Putting (3.12) and (3.13) into (3.5) gives $\operatorname{vol}(Y,\lambda) \ge TT'$. □

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Chapter 4

The absolute gradings on embedded contact homology and Seiberg-Witten Floer cohomology

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Abstract: Let Y be a closed connected contact 3-manifold. In [40], Taubes defines an isomorphism between the embedded contact homology (ECH) of Y and its Seiberg-Witten Floer cohomology. Both the ECH of Y and the Seiberg-Witten Floer cohomology of Y admit absolute gradings by homotopy classes of oriented two-plane fields. We show that Taubes' isomorphism preserves these gradings. To do this, we prove another result relating the expected dimension of any component of the Seiberg-Witten moduli space over a completed connected symplectic cobordism to the ECH index of a corresponding homology class.

4.1 Introduction

Let Y be a closed connected oriented 3-manifold. A *contact form* on Y is a 1-form λ such that $\lambda \wedge d\lambda > 0$. A contact form determines the *Reeb vector field* R by the equations

$$d\lambda(R, \cdot) = 0, \quad \lambda(R) = 1,$$

and an oriented 2-plane field $\xi := \mathrm{Ker}(\lambda)$, called the *contact structure* for α. A *Reeb orbit* is a map $\gamma : \mathbb{R}/T\mathbb{Z}$ for some $T > 0$ such that $\gamma'(t) = R(\gamma(t))$. A Reeb orbit γ is called *nondegenerate* if for some y on the image of γ the linearized flow along γ restricted to ξ_y does not have 1 as an eigenvalue. If γ is nondegenerate and the eigenvalues of the linearized flow are real then γ is called *hyperbolic*; otherwise, γ is called *elliptic*. A contact form is called *nondegenerate* if all of its Reeb orbits are nondegenerate.

If λ is nondegenerate and $\Gamma \in H_1(Y)$, then the *embedded contact homology* $ECH(Y, \lambda, \Gamma)$ of Y is defined. This is the homology of a chain complex freely generated over $\mathbb{Z}/2$[1] by certain finite sets of Reeb orbits, called *orbit sets*, with respect to a differential that counts certain mostly embedded J-holomorphic curves in the symplectization of Y. In [40], Taubes defines an isomorphism between ECH and the *Seiberg-Witten Floer cohomology* defined by Kronheimer and Mrowka in [32]. Specifically, Taubes shows [41, Theorem 1] that there is a canonical isomorphism of relatively graded $\mathbb{Z}/2$ modules

$$\mathcal{T} : ECH_*(Y, \lambda, \Gamma) \simeq \widehat{HM}^{-*}(Y, \mathfrak{s}_\xi + \mathrm{PD}(\Gamma)), \tag{4.1}$$

where $\mathfrak{s}_\xi$ is a certain spin^c structure determined by ξ, see [24, §8], PD(Γ) denotes the Poincare dual of Γ, and $\widehat{HM}^{-*}$ denotes the relatively graded module $\widehat{HM}^*$ with the grading reversed.

Both embedded contact homology and Seiberg-Witten Floer cohomology admit absolute gradings by homotopy classes of oriented 2-plane fields, see [25] and [32]. The main theorem of this paper asserts that the map $\mathcal{T}$ preserves this extra structure. To be explicit, denote the direct sum of $ECH(Y, \lambda, \Gamma)$ over all Γ by $ECH(Y, \lambda)$, and denote the direct sum of $\widehat{HM}^{-*}(Y, \mathfrak{s})$ over all isomorphism classes of spin^c structures on Y by $\widehat{HM}^{-*}(Y)$. Let j be a homotopy class of oriented 2-plane fields on Y, and denote by $ECH_j(Y, \lambda)$ and $\widehat{HM}^j(Y)$ the submodules with grading j of $ECH(Y, \lambda)$ and $\widehat{HM}^{-*}(Y)$ respectively. We show:

Theorem 4.1.1. *The map $\mathcal{T}$ restricts to an isomorphism*

$$ECH_j(Y, \lambda) \simeq \widehat{HM}^j(Y). \tag{4.2}$$

Theorem 4.1.1 implies that the absolutely graded $\mathbb{Z}/2$-module $ECH(Y, \lambda)$ is a topological invariant. Theorem 4.1.1 follows from another result of potentially independent interest relating the expected dimension of any component of the Seiberg-Witten moduli space over a completed connected symplectic cobordism to the ECH index of a corresponding homology class, see Theorem 4.5.1 below for the precise statement.

4.2 Embedded contact homology

We begin by reviewing those aspects of embedded contact homology that are relevant to the proofs of Theorem 4.1.1 and Theorem 4.5.1.

Definition of embedded contact homology

We will first review the definition of embedded contact homology. Define $ECC(Y, \lambda, \Gamma, J)$ to be the chain complex generated over $\mathbb{Z}/2$ by finite sets $\alpha = \{(\alpha_i, m_i)\}$ such that each α_i

[1] Embedded contact homology can also be defined over $\mathbb{Z}$, see [19, §9].

is a Reeb orbit, $m_i = 1$ if α_i is hyperbolic, and

$$\sum_i m_i[\alpha_i] = \Gamma \in H_1(Y).$$

An $\mathbb{R}$-invariant almost complex structure J is called *admissible* if J sends the two-plane field ξ to itself, rotating it positively with respect to $d\lambda$, and satisfies $J(\partial_s) = R$, where s denotes the $\mathbb{R}$ coordinate on $\mathbb{R}\times Y$. The ECH chain complex differential ∂_{ECH} counts certain J-holomorphic curves in $\mathbb{R}\times Y$ for an admissible J. Specifically, if α and β are two chain complex generators, then the coefficient $\langle \partial\alpha, \beta\rangle \in \mathbb{Z}/2$ is a count of J-holomorphic curves in $\mathbb{R}\times Y$, modulo translation in the $\mathbb{R}$ coordinate, that are asymptotic as currents to $\mathbb{R}\times\alpha$ as $s\to\infty$ and to $\mathbb{R}\times\beta$ as $s\to-\infty$ and which have *ECH index* 1. The ECH index, a certain function of the relative homology class of the curve, will be reviewed in §4.2. If J is generic, then ∂ is well-defined and $\partial^2 = 0$, see [27] and [19].

Define $ECH(Y,\lambda,\Gamma)$ to be the homology of this chain complex. A priori, this might depend on J, but by the canonical isomorphism (4.1) it does not. The ECH index induces a relative $\mathbb{Z}/p$ grading on $ECH(Y,\lambda,\Gamma)$, as reviewed in §4.2, where p denotes the divisibility of $c_1(\xi) + 2\,\mathrm{PD}(\Gamma)$ in $H^2(Y)$ mod torsion.

The absolute grading on ECH

The relative $\mathbb{Z}/p$ grading on ECH can be refined to an absolute grading by homotopy classes of oriented 2-plane fields. We now review this construction. For a review of homotopy classes of oriented 2-plane fields, see [25, §3.1] (in particular, note that we follow the sign convention for the $\mathbb{Z}$-action on the set of homotopy classes of oriented 2-plane fields in [25, §3.1] by demanding that the isomorphism $\pi_3(S^2)\simeq\mathbb{Z}$ that sends the Hopf fibration to $+1$ is an isomorphism of $\mathbb{Z}$-sets).

Recall first that a link $\mathcal{L}$ in Y is *transversal* if $\mathcal{L}$ is transverse to the contact plane field at every point. Let $\mathcal{L}$ be a transversal link and orient $\mathcal{L}$ so that it intersects the contact plane field positively. A framing of $\mathcal{L}$ is equivalent to a homotopy class of symplectic trivializations of $\xi|_{\mathcal{L}}$. Given a transversal link $\mathcal{L}$ with framing τ, we can define a homotopy class of 2-plane fields which we will denote by $P_\tau(\mathcal{L})$.

To do this, begin by taking a tubular neighborhood N of $\mathcal{L}$. On N, choose disjoint tubular neighborhoods N_K for each component K of the link and choose coordinates $\varphi_K : N_K \xrightarrow{\simeq} S^1\times D^2$ such that φ_K sends K to $S^1\times\{0\}$ and $d\varphi_K$ sends $\xi|_K$ to $0\times\mathbb{R}^2$ compatibly with τ; extend this trivialization to a trivialization of the tangent bundle such that the contact plane field is identified with $\{0\}\times\mathbb{R}^2$ and the Reeb vector field is identified with $(1,0,0)$ at each point. Next, choose a vector field P such that on $S^1\times\{z\in D^2 \mid |z|>1/2\}$, the vector field P intersects ξ positively, on $S^1\times\{z\in D^2\mid |z|<1/2\}$ the vector field P intersects ξ negatively, and on $S^1\times\{z\in D^2\mid |z|=1/2\}$, the vector field P is given according to the above trivialization by

$$P(t, e^{i\theta}/2) := (0, e^{-i\theta}). \tag{4.3}$$

A homotopy class of vector fields determines a homotopy class of 2-plane fields. On N, define $P_\tau(\mathcal{L})$ to be the 2-plane field determined by this vector field. On $Y \setminus N$, set $P_\tau(\mathcal{L})$ equal to ξ. This uniquely determines the homotopy class of $P_\tau(\mathcal{L})$.

Remark 4.2.1. *To compare the above construction to a perhaps more familiar one, note that if instead of requiring* (4.3)*, we require that*

$$P(t, e^{i\theta}/2) := (0, e^{i\theta}),$$

then the homotopy class of the resulting 2*-plane field corresponds to the contact structure obtained from* ξ *via a* Lutz twist *along* $\mathcal{L}$ *as defined for example in [13]. In particular, the resulting homotopy class of* 2*-plane field does not depend on the framing* τ*. In our case, the homotopy class does depend on the framing: if* τ' *is another trivialization, then*

$$P_\tau(\mathcal{L}) - P_{\tau'}(\mathcal{L}) \equiv 2(\tau - \tau') \mod d(c_1(\xi) + 2\,\mathrm{PD}([\mathcal{L}])).$$

This is explained in [25, §3.3].

To associate a homotopy class of two-plane fields to an orbit set $\alpha = \{(\alpha_i, m_i)\}$, first choose trivializations $\tau = \{\tau_i\}$ of ξ over each α_i. Next, choose disjoint tubular neighborhoods N_i of the α_i. Finally, in each N_i choose a braid ζ_i with m_i strands around each α_i (this means that ζ_i is an oriented link in N_i such that the projection of ζ_i to α_i is a degree m orientation preserving submersion), and define $\mathcal{L}$ to be the union of these braids, with the framing induced by τ. Define $I_{ECH}(\alpha)$ by the formula

$$I_{ECH}(\alpha) := P_\tau(\mathcal{L}) - \sum_i w_{\tau_i}(\zeta_i) + \mu_\tau(\alpha), \tag{4.4}$$

where $w_{\tau_i}(\zeta_i)$ is the writhe of the link ζ_i with respect to τ_i as defined in [25, §2.6], and $\mu_\tau(\alpha)$ is a certain sum of Conley-Zehnder index terms associated to α, see [25, §2.8] for the precise definitions.

It is shown in [25, Lem. 3.7] that $I_{ECH}(\alpha)$ is well-defined. The homotopy class of 2-plane fields $I_{ECH}(\alpha)$ is the absolute grading of the generator α.

Symplectic cobordisms and the ECH index

The proof of Theorem 4.1.1 and the statement of Theorem 4.5.1 both involve the ECH index. We now briefly review this construction.

Let (Y_+, λ_+) and (Y_-, λ_-) be closed contact 3-manifolds. A (connected) *symplectic cobordism* from Y_+ to Y_- is a connected compact symplectic 4-manifold (X, ω) such that $\partial X = -Y_- \sqcup Y_+$ and $\omega|_{Y_\pm} = d\lambda_\pm$. Given a symplectic cobordism, it is a standard fact that one can always find neighborhoods $N_\pm$ of $Y_\pm$ in X such that (N_+, ω) and (N_-, ω) are symplectomorphic to $((-\varepsilon, 0] \times Y_+, d(e^s\lambda_+))$ and $([0, \varepsilon) \times Y_-, d(e^s\lambda_-))$ respectively. We can therefore attach cylindrical ends to (X, ω) to obtain a non-compact symplectic manifold $\overline{X}$ called the *symplectic completion of* X. Specifically, define $E_+ := [0, \infty) \times Y_+$ and

$E_- := (-\infty, 0] \times Y_-$. Then $(\overline{X}, \omega)$ is the symplectic manifold obtained by gluing $E_\pm$ to $Y_\pm$ via the above identifications.

Let X be a symplectic cobordism from Y_+ to Y_-. If $\alpha^+ = \{(\alpha_i^+, m_i^+)\}$ is an orbit set in Y_+ and $\alpha^- = \{(\alpha_j^-, m_j^-)\}$ is an orbit set in Y_- such that $[\alpha^+]$ and $[\alpha^-]$ represent the same class in $H_1(\overline{X})$, define $H_2(\overline{X}, \alpha^+, \alpha_-)$ to be the set of relative homology classes of 2-chains in $\overline{X}$ such that

$$\partial Z = \sum_i m_i^+ \{1\} \times \alpha_i^+ - \sum_j m_j^- \{-1\} \times \alpha_j^- .$$

Here, two 2-chains are equivalent if and only if their difference is the boundary of a 3-chain.

Let τ be a homotopy class of symplectic trivializations τ_i^+ of the restriction of $\xi_+ = \mathrm{Ker}(\lambda_+)$ to α_i^+ and τ_j^- of the restriction of $\xi_- = \mathrm{Ker}(\lambda_-)$ to α_j^-. Let $Z \in H_2(\overline{X}, \alpha^+, \alpha_-)$. Define the ECH index, $I_{ECH}(Z)$ by the formula

$$I_{ECH}(Z) := c_\tau(Z) + Q_\tau(Z) + \mu_\tau(\alpha^+) - \mu_\tau(\alpha^-), \tag{4.5}$$

where $c_\tau(Z)$ and $Q_\tau(Z)$ are respectively the *relative first Chern class* and the *relative intersection pairing* of Z with respect to the trivialization τ, as defined in [25, §4.2]. As explained in [25, §4.2], the ECH index does not depend on τ.

In the case where $(\overline{X}, \omega) = (\mathbb{R} \times Y, d(e^s\lambda))$, the ECH index induces a relative $\mathbb{Z}/p$ grading on $ECH_*(Y, \lambda, \Gamma)$. This is explained (for example) in [25, §2.8].

4.3 Seiberg-Witten Floer cohomology

We now review those aspects of Seiberg-Witten Floer cohomology that are relevant to the proofs of our main theorems. For more details, see [32].

Basic terminology

Let Y be a closed oriented Riemannian 3-manifold. A *spin*^{c} structure on Y is a unitary rank-2 complex vector bundle $\mathbb{S} \to Y$ with a *Clifford multiplication*,

$$\rho : TY \to \mathrm{Hom}(\mathbb{S}, \mathbb{S}).$$

The Clifford multiplication is required to identify TY isometrically with the subbundle of traceless skew-adjoint endomorphisms equipped with the inner product $(a, b) \to \frac{1}{2}(a^*b)$. It is also required to respect orientation, by which we mean that if e_i is an oriented frame then $\rho(e_1)\rho(e_2)\rho(e_3) = 1$. Spin^c structures exist over any closed oriented Riemannian 3-manifold and the set of isomorphism classes of spin^c structures is an affine space over $H^2(Y, \mathbb{Z})$. A *spinor* is a smooth section of $\mathbb{S}$. A unitary connection $\mathbb{A}$ on $\mathbb{S}$ is called *spin*^{c} if parallel transport via $\mathbb{A}$ is compatible with the Clifford multiplication. The set of spin^c connections is an affine space over the space of imaginary valued 1-forms. Associated to a spin^c structure is the *determinant* line bundle $\det(\mathbb{S})$. This is the line bundle $\Lambda^2\mathbb{S}$. If $\mathbb{A}$ is a spin^c connection, we denote by $\mathbb{A}^t$ the induced connection on $\Lambda^2\mathbb{S}$. A spin^c connection is equivalent to a Hermitian connection on $\Lambda^2\mathbb{S}$. Given a spin^c connection $\mathbb{A}$, define the *Dirac operator* $D_{\mathbb{A}}$ to be the composition

$$\Gamma(Y,\mathbb{S}) \xrightarrow{\nabla_A} \Gamma(Y, T^*X \otimes \mathbb{S}) \xrightarrow{\rho} \Gamma(Y,\mathbb{S}).$$

Here, the Clifford multiplication ρ by 1-forms is defined by the isomorphism between vector fields and 1-forms induced by the metric.

Over a closed oriented Riemannian 4-manifold X, a spin^c structure $\mathfrak{s}_X$ is again a unitary complex vector bundle $\mathbb{S}$, this time of rank 4, together with a Clifford multiplication $\rho : TY \to \mathrm{Hom}(\mathbb{S},\mathbb{S})$. The requirements for ρ to be a Clifford multiplication are similar to the requirements for the three-manifold case. Spin^c structures also exist over any 4-manifold, and the set of isomorphism classes of spin^c structures is again an affine space over $H^2(X,\mathbb{Z})$. This is all explained in [32, §1.1]. Clifford multiplication extends to k-forms by the rule

$$\rho(\alpha \wedge \beta) = \frac{1}{2}(\rho(\alpha)\rho(\beta) + (-1)^{\deg(\alpha)\deg(\beta)}\rho(\beta)\rho(\alpha)),$$

and over a 4-manifold Clifford multiplication by the volume form induces an important decomposition of $\mathbb{S}$ into two orthogonal rank-2 complex vector bundles, S^+ and S^-, where S^+ is defined to be the -1 eigenspace of Clifford multiplication by the volume form. In the 4-dimensional case, a spinor is again defined to be a section of $\mathbb{S}$, and a spin^c connection is again defined by requiring that Clifford multiplication be parallel. The connection on $\Lambda^2 S^+$ induced by a spin^c connection $\mathbb{A}$ is denoted by $\mathbb{A}^t$. As in the three-dimensional case, the space of spin^c connections on $\mathfrak{s}_X$ is an affine space over iT^*X.

The definition of the Dirac operator $D_{\mathbb{A}}$ for a spin^c structure over a 4-manifold is completely analogous to the definition in the three-dimensional case. Over a 4-manifold, the Dirac operator interchanges sections of S^+ and S^- and hence we have a decomposition $D_{\mathbb{A}} = D_{\mathbb{A}^+} + D_{\mathbb{A}^-}$ where

$$D_{\mathbb{A}^+} : \Gamma(S^+) \to \Gamma(S^-),$$

and

$$D_{\mathbb{A}^-} : \Gamma(S^-) \to \Gamma(S^+).$$

In dimensions three or four, an automorphism of a spin^c structure $(\mathbb{S},\rho)$ is a bundle isomorphism of $\mathbb{S}$ that is compatible with ρ. This is the same as a map from the underlying manifold into S^1. We call the set of maps from the underlying manifold to S^1 the *gauge group* and we call elements of this group *gauge transformations*. If M is a 3-manifold or a 4-manifold and $\mathfrak{s}$ is a spin^c structure over M, denote by $\mathcal{C}(Y,\mathfrak{s})$ the space of pairs $(\mathbb{A},\Psi)$ such that $\mathbb{A}$ is a spin^c connection and Ψ is a spinor. We call such a pair a *configuration* and call $\mathcal{C}$ the *configuration space*. The gauge group acts on $\mathcal{C}$ by

$$g \cdot (\mathbb{A},\Psi) := (\mathbb{A} - 2g^{-1}dg, g\Psi).$$

The three-dimensional Seiberg-Witten equations

We will now introduce the three-dimensional Seiberg-Witen equations. Let Y be a closed oriented Riemannian 3-manifold with spin^c structure $\mathfrak{s} = (\mathbb{S}, \rho)$. Fix an exact 2-form μ on Y. The *three-dimensional Seiberg-Witten equations with perturbation* are the equations for a configuration $(\mathbb{A}, \Psi)$ given by

$$\begin{aligned} D_{\mathbb{A}}\Psi &= 0, \\ *F_{\mathbb{A}^t} &= \langle \rho(\cdot)\Psi, \Psi\rangle + i * \mu. \end{aligned} \tag{4.6}$$

Here, $F_{\mathbb{A}^t}$ denotes the curvature of $\mathbb{A}^t$. Fix a reference spin^c connection $\mathbb{A}_0$. Solutions of (4.6) are equivalent to critical points of the *perturbed Chern-Simons-Dirac functional*. This is the map $\mathcal{F} : \mathcal{C}(Y, \mathfrak{s}) \to \mathbb{R}$ defined by

$$\mathcal{F}(\mathbb{A}, \varphi) = -\frac{1}{8}\int_Y (\mathbb{A}^t - \mathbb{A}_0^t) \wedge (F_{\mathbb{A}^t} + F_{\mathbb{A}_0^t} - 2i\mu) + \frac{1}{2}\int_Y \langle D_{\mathbb{A}}\varphi, \varphi\rangle d\,\mathrm{vol}\,. \tag{4.7}$$

While the functional $\mathcal{F}$ is not in general gauge invariant, the gauge group acts on solutions to (4.6).

Floer homology

We now briefly review the details of the construction of the Seiberg-Witten Floer cohomology groups, which are related to the formal Morse homology of the functional $\mathcal{F}$. Call a solution to (4.6) *reducible* if $\Psi = 0$ and call it irreducible otherwise. The Seiberg-Witten Floer cohomology chain complex $\widehat{CM}^*(Y, \mathfrak{s})$ can be decomposed into submodules

$$\widehat{CM}^*(Y, \mathfrak{s}) = \widehat{CM}^*_{irr}(Y, \mathfrak{s}) \oplus \widehat{CM}^*_{red}(Y, \mathfrak{s}),$$

where $\widehat{CM}^*_{irr}$ is the free $\mathbb{Z}/2$-module generated by gauge equivalence classes of irreducible solutions to (4.6) after choosing μ generically so that these solutions are cut out transversely, and $\widehat{CM}^*_{red}$ is another term involving the reducible solutions. Only the irreducible component of this chain complex is relevant to the construction of the map $\mathcal{T}$ from (4.1), so we will not review the definition of $\widehat{CM}^*_{red}$ here.

The part of the chain complex differential ∂ mapping the irreducible component to itself counts gauge equivalence classes of smooth one-parameter families of pairs $(\mathbb{A}(s), \Psi(s))$ that solve the equations

$$\begin{aligned} \frac{\partial}{\partial s}\Psi(s) &= -D_{\mathbb{A}(s)}\Psi(s), \\ \frac{\partial}{\partial s}\mathbb{A}(s) &= - * F_{\mathbb{A}(s)} + \langle cl(\cdot)\Psi, \Psi\rangle + i * \mu, \\ \lim_{s\to\pm\infty}(\mathbb{A}(s), \Psi(s)) &= (\mathbb{A}_\pm, \Psi_\pm), \end{aligned} \tag{4.8}$$

where $(\mathbb{A}_\pm, \Psi_\pm)$ are solutions to (4.6). These are equations for the downward gradient flow of the functional (4.7) with respect to the metric on $\mathcal{C}$ induced by the Hermitian inner product on $\mathbb{S}$ and 1/4 of the L^2 inner product on iT^*Y. Solutions to (4.8) are called *instantons.* If $\mathfrak{c}_\pm$ are two irreducible solutions to (4.6), then the coefficient of $\mathfrak{c}_-$ in the differential of $\mathfrak{c}_+$ is a signed count of gauge equivalence classes of "index one" instantons from $\mathfrak{c}_-$ to $\mathfrak{c}_+$, modulo translation in the s coordinate, after making "abstract perturbations" to (4.6) and (4.8) to obtain transversality of the relevant moduli spaces.

"Abstract perturbations" are described in [32, Ch. 11] and play little role in the proof of Theorem 4.1.1. The "index" is the local expected dimension of the moduli space of instantons modulo gauge equivalence. The index induces a relative $\mathbb{Z}/p$ grading on the chain complex such that the differential increases the grading by 1, see [29, §2.1]. Here, p is equal to the divisibility of $c_1(\mathfrak{s})$ in $H^2(Y, \mathbb{Z})$ mod torstion.

The absolute grading of a critical point

As is the case for embedded contact homology, the relative grading for $\widehat{HM}^{-*}(Y, \mathfrak{s})$ can be refined to an absolute grading. To explain Kronheimer and Mrowka's construction, we need to introduce the *four-dimensional Seiberg-Witten equations.* If X is any (possibly non-compact) spin^c 4-manifold, the four-dimensional Seiberg-Witten equations (with perturbation) on X for a configuration $(\mathbb{A}, \Psi)$ is the system

$$\begin{aligned} \frac{1}{2}\rho(F^+_{\mathbb{A}^t}) + \mathfrak{p}(\mathbb{A}, \Psi) - (\Psi\Psi^*)_0 &= 0 \\ D^+_{\mathbb{A}}\Psi &= 0. \end{aligned} \tag{4.9}$$

Here, $F^+_{\mathbb{A}^t}$ denotes the self-dual part of the curvature 2-form, $(\Psi\Psi^*)_0$ denotes the traceless component of $\Psi\Psi^*$, and $\mathfrak{p}(\mathbb{A}, \psi)$ denotes a gauge invariant perturbation term, see [32, §24.1]. When $X = \mathbb{R} \times Y$, the system (4.9) is equivalent to the system (4.8) for an appropriate spin^c structure, see [32, §4.3]. The action of the gauge group on $\mathcal{C}$ induces an action on solutions of (4.9).

To prove Theorem 4.1.1, we only need to know the definition of the absolute grading for irreducible solutions to (4.6) that are nondegenerate i.e. cut out transversely (see [32, Def. 12.1.1] for the precise definition). So let $\mathfrak{c}$ be such a solution and let X be any compact connected oriented Riemannian 4-manifold with oriented boundary Y extending the spin^c structure $\mathfrak{s}$ via a spin^c structure $\mathfrak{s}_X$. Assume that the Riemannian metric on X is such that X contains an isometric copy of $I \times Y$ for some interval $I = (-C, 0]$, with ∂X identified with $\{0\} \times Y$. We can therefore *attach a cylindrical end to* X i.e. glue in a copy of the cylinder $[0, \infty) \times Y$ to X to get a non-compact 4-manifold $\overline{X}$ with spin^c structure $\mathfrak{s}_{\overline{X}}$ extending the spin^c structure on $\overline{X}$ via a translation invariant spin^c structure on the end.

Denote the moduli space of gauge equivalence classes of configurations for the spin^c structure $\mathfrak{s}_{\overline{X}}$ that are asymptotic (as in [32, §13.1]) to $\mathfrak{c}$ on the cylindrical end of $\overline{X}$ by $\mathcal{B}(\overline{X}, \mathfrak{s}_{\overline{X}}, \mathfrak{c})$ and denote the gauge equivalence classes of solutions to (4.9) that are asymptotic to $\mathfrak{c}$ on the cylindrical end of $\overline{X}$ by $M(\overline{X}, \mathfrak{s}_{\overline{X}}, \mathfrak{c})$. Here, the perturbation term to (4.9) is constructed from the perturbation to (4.6), see [32, §24.1]. Denote by $\mathcal{B}(\overline{X}, \mathfrak{c})$ and by $M(\overline{X}, \mathfrak{c})$ the union of $\mathcal{B}(\overline{X}, \mathfrak{s}_{\overline{X}}, \mathfrak{c})$ and $M(\overline{X}, \mathfrak{s}_{\overline{X}}, \mathfrak{c})$ respectively over all spin^c structures $s_{\overline{X}}$ on $\overline{X}$ extending $\mathfrak{s}$.

In general, the space $M(\overline{X}, \mathfrak{c})$ can contain multiple connected components. These are parametrized by $\pi_0(\mathcal{B}(\overline{X}, \mathfrak{c}))$, which is an affine space over $H^2(X, \partial X, \mathbb{Z})$. Let z be an element of $\pi_0(\mathcal{B}(\overline{X}, \mathfrak{c}))$. Following [32, Defn. 24.4.5], we now define an integer $gr_z(X, \mathfrak{c})$ which is the expected dimension of the component of $M(\overline{X}, \mathfrak{c})$ corresponding to z. If $(\mathbb{A}, \Psi)$ is any element of $\mathcal{B}(\overline{X}, \mathfrak{c})$, define the operator

$$D^{\overline{X}}_{\mathbb{A},\Psi} : L^2_1(iT^*\overline{X}) \oplus L^2_1(S^+) \to L^2(i\mathbb{R}) \oplus L^2(isu(S^+)) \oplus L^2(S^-)$$

by

$$D^{\overline{X}}_{\mathbb{A},\Psi}(a, \varphi) = (-d^*a + iIm(\Psi^*\varphi), \frac{1}{2}\rho(d^+a) - (\Psi\varphi^* + \varphi\Psi^*)_0, D^+_{\mathbb{A}}\varphi + \rho(a)\Psi), \tag{4.10}$$

where $L^2_1(iT^*\overline{X}), L^2_1(S^+), L^2(i\mathbb{R}), L^2(isu(S^+))$, and $L^2(S^-)$ denote Sobolev completions of the space of compactly supported smooth sections of these bundles over $\overline{X}$, see [32, §13], and d^+a denotes the self-dual component of da. This is the linearization of the unperturbed 4-dimensional Seiberg-Witten equations with a gauge fixing term. As explained in [**taubes1**] and [**taubes3**], when $\mathfrak{c}$ is irreducible and nondegenerate the operator $D^{\overline{X}}_{\mathbb{A},\Psi}$ is Fredholm. The integer $gr_z(X, \mathfrak{c})$ is by definition the index of $D^{\overline{X}}_{\mathbb{A},\Psi}$ for $(\mathbb{A}, \Psi)$ a lift of the gauge equivalence class of an element in the component of $\mathcal{B}(\overline{X}, \mathfrak{c})$ corresponding to z. As explained in [32, §24], $gr_z(X, \mathfrak{c})$ can be defined for reducible solutions as well. We call $gr_z(X, \mathfrak{c})$ the *Seiberg-Witten index.*

If φ_0 is any section of $\mathbb{S}^+|_{\partial X}$, denote by $e(\mathbb{S}^+, \varphi_0) \in H^4(X, \partial X; \mathbb{Z})$ the *relative Euler class* of $\mathbb{S}^+$ relative to φ_0. To define the absolute grading, choose a nowhere-zero section φ_0 of $\mathbb{S}^+|_{\partial X}$ such that $e(\mathbb{S}^+, \varphi_0)[X, \partial X] = gr_z(X; \mathfrak{c})$. The pair $(\mathbb{S}^+|_{\partial X}, \varphi_0)$ is a spin^c structure on Y equipped with a non-zero section, so we can apply the following basic lemma [32, Lem. 28.1.1]:

Lemma 4.3.1. *On an oriented Riemannian* 3*-manifold* Y*, there is a one-to-one correspondence between oriented* 2*-plane fields* ξ *and isomorphism classes of pairs* $(\mathfrak{s}, \varphi)$ *consisting of a spin*^{c} *structure and a unit-length spinor* φ*.*

By [32, Prop. 28.2.2], the isomorphism class of $(\mathbb{S}, \varphi_0)$ depends only on $Y, \mathfrak{s}$, and $\mathfrak{c}$, and so the bijection of Lemma 4.3.1 induces a well-defined grading by homotopy classes of oriented 2-plane fields, which we denote by I_{SW}. This refines the relative grading on $\widehat{HM}^{-*}(Y, \mathfrak{s})$, see [32, §28]. The absolute grading can be defined for reducible critical points as well, see [32, §28].

Remark 4.3.2. *Our sign convention (as explained in §4.2) for the* $\mathbb{Z}$*-action on the set of homotopy classes of* 2*-plane fields is opposite the sign convention in [32, §28]. This is because* *the grading defined by Kronheimer and Mrowka refines the relative grading on* $\widehat{HM}^*$, *while our grading refines the relative grading on* $\widehat{HM}^{-*}$.

4.4 Taubes' isomorphism

This section very briefly summarizes Taubes' isomorphism between embedded contact homology and Seiberg-Witten Floer cohomology. For more details, see [40].

Taubes' equations

Let (Y, λ) be a contact manifold. A choice of admissible almost complex structure J induces a metric g on Y by requiring that the Reeb vector field R has length 1, is orthogonal to the contact planes ξ, and

$$g(v, w) = \frac{1}{2} d\lambda(v, Jw), \qquad v, w \in \xi_y. \tag{4.11}$$

Let $\mathbb{S}$ be the spin bundle for the spin^c structure $\mathfrak{s}_\xi + \mathrm{PD}(\Gamma)$. Clifford multiplication by λ gives a decomposition

$$\mathbb{S} = E \oplus (E \otimes \xi),$$

where E and $E \otimes \xi$ are, respectively, the $+i$ and $-i$ eigenspaces of Clifford multiplication by λ. Here ξ is regarded as a complex line bundle.

Connections on $\det \mathbb{S}$ can therefore be written as $\mathbb{A}_0 + 2\mathbb{A}$ where $\mathbb{A}_0$ is a certain fixed connection on ξ, as reviewed in [24, §2.a], and $\mathbb{A}$ is a connection on E. We can therefore regard a connection on E as a connection on $\det \mathbb{S}$. With this in mind, consider the system of equations for a connection $\mathbb{A}$ on E and a spinor ψ given by

$$\begin{aligned} *F_{\mathbb{A}} &= r(\langle \rho(\cdot)\psi, \psi\rangle - i\lambda) + i(*d\mu + \bar{\omega}) \\ D_{\mathbb{A}}\psi &= 0. \end{aligned} \tag{4.12}$$

Here, $\bar{\omega}$ denotes the harmonic 1-form such that $*\frac{\bar{\omega}}{\pi}$ represents the image of $c_1(\xi)$ in $H^2(Y; \mathbb{R})$, r is a positive real number, and μ is a suitably generic coclosed 1-form that is L^2-orthogonal to the space of harmonic 1-forms and that has "P-norm" less than 1. The P-norm controls the derivatives of μ to all orders, see [29, §2.2]. This is a a special case of (4.6) where we have also rescaled the spinor by $\sqrt{r}$.

If μ is generic, then all of the irreducible solutions to (4.12) are nondegenerate. One can also make additional small perturbations to the equations so that the moduli spaces needed to define the chain complex differential are all cut out transversely. Moreover, in any fixed grading, if r is sufficiently large, these additional perturbations can be chosen such that only irreducible solutions to this perturbed version of (4.12) contribute to the Seiberg-Witten cohomology chain complex in that grading, see [24, Prop. 3.5]. By [29, §2.1], these perturbations can be chosen to vanish to any given order on the irreducible solutions to (4.12), so that the irreducible solutions to (4.12) and the solutions to this perturbed version of (4.12) are the same.

Taubes' proof

The basic idea behind the isomorphism (4.1) is that as r gets very large, the zero set of the E component of the spinor for solutions of (4.12) converges (as a current) to an ECH chain complex generator, and the symplectic action of this chain complex generator is very close to 2π times the "energy" of the solution.

To state this precisely, recall that if $\alpha = \{(\alpha_i, m_i)\}$ is a generator of the ECH chain complex, the *symplectic action* of α is the number

$$\mathcal{A}(\alpha) := \sum_i m_i \int_{\alpha_i} \lambda.$$

Because of the conditions on J, the ECH chain complex differential decreases the symplectic action. Hence, for any real number L, we can define *filtered ECH*, $ECH^L(Y, \lambda, \Gamma)$, to be the homology of the subcomplex of the ECH chain complex spanned by generators with action strictly less than L.

Given a configuration $(\mathbb{A}, \Psi)$, define the *energy*

$$E(\mathbb{A}) := i \int_Y \lambda \wedge F_A, \tag{4.13}$$

and define $\widehat{CM}^*_L(Y, \mathfrak{s}, \lambda, r)$ to be the submodule of $\widehat{CM}^*_{irr}$ generated by irreducible solutions $(\mathbb{A}, \Psi)$ to (4.6) (perturbed as in §4.4) with energy less than $2\pi L$. If r is sufficiently large, and λ has no orbit set of action exactly L, then one can show [29, Lem. 2.3] that all of the solutions to (4.12) with energy less than $2\pi L$ are irreducible and the chain complex differential for $\widehat{CM}^*(Y, \mathfrak{s}, \lambda, r)$ maps $\widehat{CM}^*_L(Y, \mathfrak{s}, \lambda, r)$ to itself.

The key fact ([29, Prop. 3.1]) needed for the proof of (4.1) is that if r is sufficiently large and (λ, J) is "L-flat", then for any $\Gamma \in H_1(Y)$, there is a canonical bijection between the set of generators of $\widehat{CM}^{-*}_L(Y, \mathfrak{s}_\xi + \mathrm{PD}(\Gamma); \lambda, r)$ and the set of admissible orbit sets in the homology class Γ of length less than L. This induces an isomorphism of relatively graded chain complexes

$$ECC^L_*(Y, \lambda, \Gamma) \xrightarrow{\simeq} \widehat{CM}^{-*}_L(Y, \mathfrak{s}_\xi + \mathrm{PD}(\Gamma); \lambda, r), \tag{4.14}$$

which, as explained in [29, §3], induces the isomorphism $\mathcal{T}$ between $ECH(Y, \lambda, \Gamma)$ and $\widehat{HM}^{-*}(Y, \mathfrak{s}_{\xi+\mathrm{PD}(\Gamma)})$. Roughly speaking, the bijection between chain complex generators is given by constructing an approximate solution to (4.12) for large r from an ECH chain complex generator by using the "vortex equations", see [41], and then using perturbation theory to get an actual solution to (4.12).

The L-flat condition is a condition on the form of λ and J in tubular neighborhoods of those Reeb orbits with action less than L. In the case where (λ, J) is not L-flat, one can take an *L-flat approximation* of λ: a pair (λ, J) of nondegenerate contact form and admissible almost complex structure can always be approximated by an L-flat pair (λ_1, J_1)

without changing the Reeb orbits or the lengths of the orbits with action less than L, and this identification induces an isomorphism of chain complexes

$$ECC_*^L(Y, \lambda, \Gamma; J) \xrightarrow{\simeq} ECC_*^L(Y, \lambda_1, \Gamma; J_1). \tag{4.15}$$

This is all explained in [29, §3].

4.5 Proof of theorems

The Seiberg-Witten index in a symplectic cobordism

To prove Theorem 4.1.1, we will first prove another theorem relating the expected dimension of any component of the Seiberg-Witten moduli space over a symplectic cobordism to the ECH index of a corresponding relative homology class.

To be specific, let (X, ω) be a connected symplectic cobordism from (Y_1, λ_1) to (Y_2, λ_2) as in §4.2, and denote by $\overline{X}$ the symplectic completion of X. Let J be an admissible almost complex structure on $\overline{X}$, and let g be the Riemannian metric induced by ω and J. Let α_1 be an orbit set on Y_1 and let α_2 be an orbit set on Y_2. Assume that the contact forms λ_1 and λ_2 are "L-flat", where L is some constant greater than the symplectic action of either α_1 or α_2. Recall that the canonical isomorphism (4.14) is induced from a canonical bijection between the set of generators of $\widehat{CM}_L^{-*}(Y, \mathfrak{s}_\xi + \mathrm{PD}(\Gamma); \lambda, r)$ and the set of admissible orbit sets in the homology class Γ of length less than L, and denote by c_{α_1} and c_{α_2} the elements corresponding to α_1 and α_2 respectively under this bijection. By [43, §2.a], if r is sufficiently large, then c_{α_1} and c_{α_2} are both nondegenerate and belong to the irreducible component of the chain complex $\widehat{CM}^*$.

Let $\mathfrak{s}_{Y_1}$ and $\mathfrak{s}_{Y_2}$ denote the spin^c structures on Y_1 and Y_2 corresponding to c_{α_1} and c_{α_2} respectively. Then $c_{\alpha_1}, c_{\alpha_2}, \mathfrak{s}_{Y_1}$, and $\mathfrak{s}_{Y_2}$ induce a spin^c structure $\mathfrak{s}_Y$ and configuration $\mathfrak{c}$ on $Y = Y_1 \cup -Y_2$. Recall the space $\mathcal{B}(\overline{X}, \mathfrak{c})$ from §4.3, and let $(\mathbb{A}, \Psi)$ be an element of $\mathcal{B}(\overline{X}, \mathfrak{c})$. The configuration $(\mathbb{A}, \Psi)$ determines a spin^c structure $\mathfrak{s}_{\mathbb{A},\Psi}$ over $\overline{X}$. As before, denote by S^+ the -1 eigenspace of Clifford multiplication by the volume form on the spin^c structure $\mathfrak{s}_{\mathbb{A},\Psi}$. Since $\overline{X}$ is symplectic, we can write

$$S^+ = E \oplus (E \otimes K^{-1}),$$

where K^{-1} denotes the inverse of the canonical bundle and E and $E \otimes K^{-1}$ are, respectively, the $-2i$ and $+2i$ eigenspaces of Clifford multiplication by the symplectic form. This is reviewed, for example, in [26, §4.2]. We can then write the spinor

$$\Psi = (\alpha, \beta)$$

according to this decomposition, where $(\mathbb{A}, \Psi)$ now denotes a specific lift of its gauge equivalence class. Assume that $(\mathbb{A}, \Psi)$ is such that α intersects the zero section transversally. Hence, $\alpha^{-1}(0)$ is an embedded (real) surface. Denote this surface by $C_{\mathbb{A},\Psi}$.

Recall that, as reviewed in §4.4, as r gets very large, the zero sets of c_{α_1} and c_{α_2} converge as currents to α_1 and α_2, respectively. By taking orientation-preserving diffeomorphisms $[0,\infty) \simeq [0, 1-\varepsilon)$ and $(-\infty, 0] \simeq (-1+\varepsilon, 0]$ to identify

$$\overline{X} \simeq ((-1+\varepsilon, 0] \times Y_-) \cup_{Y_-} X \cup_{Y_+} ((([0, 1-\varepsilon) \times Y_+).$$

and composing the closure of the image of $C_{\mathbb{A},\Psi}$ in the latter with cobordisms to the Reeb orbits in the orbit sets α_1 and α_2, the curve $C_{\mathbb{A},\Psi}$ defines an element $Z_{\mathbb{A},\Psi} \in H_2(\overline{X}, \alpha_1, \alpha_2)$. We can relate $I_{ECH}(Z_{\mathbb{A},\Psi})$ to the expected dimension of the corresponding Seiberg-Witten moduli space, as the following theorem shows:

Theorem 4.5.1. *Let $z \in \pi_0(\mathcal{B}(\overline{X}, \mathfrak{c}))$ and represent z by a configuration $(\mathbb{A}, \Psi)$ over $\overline{X}$. The integer $gr_z(X, \mathfrak{c})$ is equal to $I_{ECH}(Z_{\mathbb{A},\Psi})$.*

Proof. Our method of proof closely tracks the argument due to Taubes in [43, §2.b]. The basic approach is to change the triple $(\overline{X}, J, \omega)$ into a new triple $(\tilde{X}, \tilde{J}, \tilde{\omega})$ (with $\tilde{\omega}$ nondegenerate but not necessarily symplectic) in which the homology class $Z_{\mathbb{A},\Psi}$ induces a homology class $\tilde{Z}_{\mathbb{A},\Psi}$ with a $\tilde{J}$-holomorphic representative with ends of a particularly nice form. An argument due to Taubes then generalizes without difficulty to allow us to compute the ECH index of $\tilde{Z}_{\mathbb{A},\Psi}$, and it is straightforward to relate this index to the ECH index of $Z_{\mathbb{A},\Psi}$. The details are given in three steps.

Step 1. First, choose a representative C_z of the homology class of $Z_{\mathbb{A},\Psi}$ with no compact components and with ends of the special form described in [43, §2.b.1]. In particular, the requirements from [43, §2.b.1] imply that the ends of C_z are asymptotic to the orbit set α_1 at $+\infty$, asymptotic to the orbit set α_2 at $-\infty$, and converge exponentially fast. We can then find a pair $(\tilde{J}, \tilde{\omega})$, where $\tilde{J}$ is an almost complex structure on a neighborhood of C_z such that C_z is $\tilde{J}$-holomorphic and $\tilde{\omega}$ is a (not necessarily closed) self-dual 2-form on $\overline{X}$ with transverse zero locus whose restriction to C_z is compatible with $\tilde{J}$. We can assume that the pair $(\tilde{J}, \tilde{\omega})$ satisfies the analogues of the additional technical conditions required in [43, §2.b.2]. Note that these conditions force $\tilde{\omega}$ to converge exponentially fast to $ds \wedge \lambda_\pm + *\lambda_\pm$ as the norm of the $\mathbb{R}$-coordinate s on each cylindrical end tends to infinity.

Denote the zero locus of the 2-form $\tilde{\omega}$ by B. Note that B consists of a finite number of disjoint embedded circles which are also disjoint from C_z. Let T denote a tubular neighborhood of B that is disjoint from C_z. We can assume that B has the special description given in [43, §2.b.2], so that we can copy the argument in [43, §2.b.3] to modify the manifold $\overline{X}$ and the metric on $\overline{X}$ in T to obtain a new Riemannian manifold $\tilde{X}$, obtained by surgery along T, such that $\tilde{\omega}$ extends to a nonvanishing self-dual 2-form on $\tilde{X}$ (which we also denote by $\tilde{\omega}$) and such that the spin^c structure on $\overline{X} - T$ extends to a spin^c structure on $\tilde{X}$.

Now denote the canonical bundle on $(\tilde{X}, \tilde{\omega})$ by $\tilde{K}^{-1}$. The self-dual part of the spinor bundle for the spin^c structure on $\tilde{X}$ splits as $E \oplus E\tilde{K}^{-1}$ with respect to Clifford multiplication by $\tilde{\omega}$. It will be important to understand the relationship between $\tilde{K}$ and K explicitly. To do this, recall that there is a canonical spin^c structure on $\overline{X}$ with self-dual component $\mathbb{C} \oplus \mathbb{C}K^{-1}$.

Denote the $+i|\tilde{\omega}|$ eigenspace of Clifford multiplication by $\tilde{\omega}$ on the self-dual component of this spin^c stucture over $\overline{X} \setminus B$ by L. Then, as explained in [43, §4.b], we have

$$\tilde{K} = L^2 K.$$

This description for L ensures that we can choose t_1, t_2 such that $Y_1 \times \{t_1\}$ and $Y_2 \times \{t_2\}$ are both in $\overline{X} - T$ and the restriction of L to $Y_1 \times [t_1, \infty)$ and $Y_2 \times (-\infty, t_2]$ is canonically isomorphic to the trivial bundle.

Step 2. We can now copy the construction from [43, §2.b.6] to construct a particular irreducible configuration $(\mathbb{A}_s, \Psi_s)$ for our spin^c structure over $\tilde{X}$ with large $|s|$ limit gauge equivalent to $\mathfrak{c}$. Let k_L denote the relative first Chern class of L evaluated on C_z, relative to the section 1 on $Y_1 \times \{t_1\}$ and $Y_2 \times \{t_2\}$. The significance of the configuration $(\mathbb{A}_s, \Psi_s)$ is given by the following proposition:

Proposition 4.5.2. *The index of $D^{\tilde{X}}_{\mathbb{A}_s,\Psi_s}$ is equal to $I_{ECH}(Z_{\mathbb{A},\Psi}) - 2k_L$.*

Proof. This is proved (in different notation) in [43, §2c]. In this section, Taubes is working over a manifold which arises via surgery on the symplectization of a contact 3-manifold Y, but his argument also holds in the slightly greater generality we require, see Remark 4.5.3 below. □

Remark 4.5.3. *It is worth summarizing Taubes' argument from [43, §2c], since this is the key step in the proof of Theorem 4.5.1. This will also clarify why his argument holds in the greater generality we are demanding.*

To motivate Taubes' argument, we need to review how Taubes in [42] constructs a Seiberg-Witten instanton with the appropriate asymptotics from a curve counted by the ECH chain complex differential. Recall from §4.4 that the bijection between chain complex generators that induces the isomorphism (4.14) *is given by using solutions to the vortex equations to construct approximate solutions to Taubes' perturbed Seiberg-Witten equations and then using perturbation theory. To construct an instanton from an ECH index one J-holomorphic curve, Taubes again uses the vortex equations to construct an approximate solution and uses perturbation theory to produce an instanton.*

This approximate solution is essentially the configuration $(\mathbb{A}_s, \Psi_s)$. To construct an instanton, Taubes considers a family of deformations of $(\mathbb{A}_s, \Psi_s)$ parametrized by a certain Banach space

$$\mathcal{K} \hookrightarrow L^2_1(iT^*\overline{X}) \oplus L^2_1(S^+),$$

where the $\hookrightarrow$ means that the map is an injection (in fact, it can be made nearly isometric after putting the norm described in [43, Equation 2.63] on $L^2_1(iT^*\overline{X}) \oplus L^2_1(S^+)$). The space $\mathcal{K}$ is also constructed using the vortex equations. Taubes then shows that constructing an instanton by perturbing $(\mathbb{A}_s, \Psi_s)$ is equivalent to solving the projection of the relevant PDE onto another Banach space

$$\mathcal{L} \hookrightarrow L^2(i\mathbb{R}) \oplus L^2(isu(S^+)) \oplus L^2(S^-),$$

see [42, §7], which Taubes then solves by using the contraction mapping theorem. The basic idea behind Taubes' method for the index computation in [43, §2.c] is to decompose the operator $D^{\tilde{X}}_{\mathbb{A}_s,\Psi_s}$ to get an operator,

$$\Delta : \mathcal{K} \to \mathcal{L}.$$

Taubes shows that the index of $D^{\tilde{X}}_{\mathbb{A}_s,\Psi_s}$ is equal to the index of Δ, and the kernel and cokernel of the operator Δ can both be described explicitly, see [43, §2.c.3]. At any rate, for our purposes, the key point is that all the relevant analysis takes place local to the curve C_z, hence the generalization to a cobordism with cylindrical ends.

Step 3. We now complete the proof by comparing the index of $D^{\tilde{X}}_{\mathbb{A}_s,\Psi_s}$ to the index of $D^{\overline{X}}_{\mathbb{A},\Psi}$.

Denote the component of $\overline{X}$ bounded by $Y_1 \times \{t_1\}$ and $Y_2 \times \{t_2\}$ by M and denote the corresponding component of $\tilde{X}$ by $\tilde{M}$. Glue M to $\tilde{M}$ (reversing the orientation on $\tilde{M}$) along their common boundary to obtain a closed spin^c 4-manifold $(S, \mathfrak{s}_S)$. Let $(\mathbb{A}_S, \Psi_S)$ be a configuration on $(S, \mathfrak{s}_S)$. The additivity of gr under gluing (e.g. as explained in [32]) implies that

$$\operatorname{ind}(D^{\overline{X}}_{\mathbb{A},\Psi}) = \operatorname{ind}(D^{\tilde{X}}_{\mathbb{A}_s,\Psi_s}) + \operatorname{ind}(D^{S}_{\mathbb{A}_S,\Psi_S}). \tag{4.16}$$

It is a simple matter to compute the index of $\operatorname{ind}(D^{S}_{\mathbb{A}_S,\Psi_S})$. Indeed, by [32, Thm. 1.4.1], we have

$$\operatorname{ind}(D^{S}_{\mathbb{A}_S,\Psi_S}) = \frac{1}{4}(c_1(S^+)^2[S] - 2\chi(S) - 3\sigma(S)), \tag{4.17}$$

where σ denotes the signature of S, and by [32, Lem. 28.2.3] we also know that

$$(c_2(S^+) - \frac{1}{4}c_1(S^+)^2)[S] = -\frac{1}{4}(2\chi(S) + 3\sigma(S)). \tag{4.18}$$

Combining these two equations gives

$$\operatorname{ind}(D^{S}_{\mathbb{A}_S,\Psi_S}) = c_2(S^+)[S]. \tag{4.19}$$

We therefore have

$$\begin{aligned} \operatorname{ind}(D^{S}_{\mathbb{A}_S,\Psi_S}) &= 2(c_1(E) \cup c_1(L))[M] \\ &= 2k_L. \end{aligned} \tag{4.20}$$

The result now follows by combining Proposition 4.5.2, (4.16), and (4.20). □

A concave symplectic filling

Our strategy for proving Theorem 4.1.1 is to apply Theorem 4.5.1 to an appropriate cobordism. To produce this cobordism, let $\Gamma \in H_1(Y)$ and fix an orbit set $\alpha \in ECC(Y, \lambda, \Gamma)$.

Recall from [7, Thm. 2.5] that any smooth knot can be C^0 approximated by a Legendrian knot. Thus, we can choose a Legendrian knot $\mathcal{K}$ which represents the class Γ.

Recall now the concept of Legendrian surgery. This is reviewed, for example, in [8]. Recall also from [28, §1.6] that if $\mathcal{K}$ is a Legendrian knot in (Y, λ), then one can perform a Legendrian surgery along $\mathcal{K}$ to obtain another contact 3-manifold (Y', λ') such that there exists a symplectic cobordism from (Y, λ) to (Y', λ') obtained by attaching a 2-handle along a tubular neighborhood of $\mathcal{K}$. Recall that a *concave symplectic filling* of (Y, ξ) is a symplectic cobordism from (Y, ξ) to the empty set. Concerning concave symplectic fillings, Etnyre and Honda prove [8, Thm. 1.3] that any contact 3-manifold has infinitely many concave symplectic fillings.

Given an orbit set α, we can therefore combine these results to define a manifold X_α by first performing Legendrian surgery on Y along $\mathcal{K}$ to obtain another contact 3-manifold and then composing the resulting symplectic cobordism with a concave symplectic filling. In the next section, we will apply Theorem 4.5.1 to X_α.

Proof of main theorem

To prove Theorem 4.1.1, we will assume that the contact form is L-flat and show that the canonical bijection (4.14) preserves the absolute gradings. This will prove the theorem for any contact form λ, since the isomorphism (4.15) preserves the absolute grading. So, assume that the contact form is L-flat, let $\alpha \in ECC^L(Y, \lambda, \Gamma)$ be an orbit set, and denote by $\mathfrak{c}_\alpha$ the element corresponding to α under the canonical bijection between the set of generators of $\widehat{CM}_L^{-*}(Y, \mathfrak{s}_\xi + \mathrm{PD}(\Gamma); \lambda, r)$ and the set of admissible orbit sets in the homology class Γ of length less than L.

Recall from §4.2 that the ECH absolute grading is given by

$$I_{ECH}(\alpha) := P_\tau(\mathcal{L}) - \sum_i w_{\tau_i}(\zeta_i) + \mu_\tau(\alpha), \tag{4.21}$$

where $w_{\tau_i}(\zeta_i)$ is the writhe of a braid ζ_i around α_i with m_i strands, $\mu_\tau(\alpha)$ is a certain sum of Conley-Zehnder index terms associated to α, and $\mathcal{L}$ is the union of the ζ_i. To relate $I_{ECH}(\alpha)$ to $I_{SW}(\mathfrak{c}_\alpha)$, begin by recalling the symplectic manifold X_α defined in the previous section. Let $\overline{X}_\alpha$ denote the manifold X_α with cylindrical ends attached. Recall that the homotopy class of two plane fields $P_\tau(\mathcal{L})$ determines a spin^c structure $\mathfrak{s}(P_\tau(\mathcal{L}))$. By [25, Thm 3.1(b)],

$$\mathfrak{s}(P_\tau(\mathcal{L})) = \mathfrak{s}_\xi + \mathrm{PD}([\alpha]).$$

Remember that $[\alpha]$ vanishes in $H_1(X_\alpha)$. Since $\mathfrak{s}_\xi$ extends to a spin^c structure on $\overline{X}_\alpha$, it follows that $\mathfrak{s}(P_\tau(\mathcal{L}))$ does as well.

To simplify the notation, denote the "plus" summand of the spin bundle for the extension of $\mathfrak{s}(P_\tau(\mathcal{L}))$ to X_α by S^+_α and denote $\mathfrak{s}(P_\tau(\mathcal{L}))$ by $\mathfrak{s}_\alpha$. Recall from §4.3 that $I_{SW}(\mathfrak{c}_\alpha)$ is the homotopy class of two-plane fields corresponding to $(\mathfrak{s}_\alpha, \varphi_0)$, where φ_0 is a section of $S^+_\alpha|_Y$ satisfying

$$e(S^+_\alpha, \varphi_0)[X_\alpha, \partial X_\alpha] = gr_z(X_\alpha; \mathfrak{c}_\alpha), \tag{4.22}$$

and z is any element of $\pi_0(\mathcal{B}(\overline{X}_\alpha, \mathfrak{c}_\alpha))$. For φ any section of $S^+_\alpha|_Y$, denote by $\tilde{e}(S^+_\alpha, \varphi) \in \mathbb{Z}$ the *relative Euler number* $e(S^+_\alpha, \varphi)[X_\alpha, \partial X_\alpha]$. Recall that the set of homotopy classes of 2-plane fields in a given spin^c structure has a $\mathbb{Z}$-action. This induces an action on the second component of isomorphism classes of pairs $(\mathfrak{s}_\alpha, \varphi)$, where φ is a nowhere zero section. With respect to this $\mathbb{Z}$-action, the relative Euler number satisfies:

$$\tilde{e}((S^+_\alpha, \varphi) + a) = \tilde{e}(S^+_\alpha, \varphi) - a. \tag{4.23}$$

In particular, it follows from (4.22) and (4.23) that

$$I_{SW}(\mathfrak{c}_\alpha) = (\mathfrak{s}_\alpha, \varphi) + \tilde{e}(S^+_\alpha, \varphi) - gr_z(X_\alpha; \mathfrak{c}_\alpha), \tag{4.24}$$

where φ is any section.

To relate (4.24) to (4.21), let $\varphi_{\mathcal{L}}$ be such that $(\mathfrak{s}_\alpha, \varphi_{\mathcal{L}}) = P_\tau(\mathcal{L})$. Let Ψ be a section of S^+_α extending $\varphi_{\mathcal{L}}$ and transverse to the zero section, and write $S^+_\alpha = E \oplus (E \otimes K^{-1})$ over X_α. Write $\Psi = (\gamma, \tilde{\gamma})$ with respect to this decomposition. The zero set of γ defines an embedded real surface in X_α, which we will denote by $C_{\mathcal{L}}$. Composing $C_{\mathcal{L}}$ with a cobordism to the Reeb orbits in α determines a homology class $Z_{\mathcal{L}} \in H_2(\overline{X}, \emptyset, \alpha)$. We can now apply Theorem 4.5.1 to choose $z \in \pi_0(\mathcal{B}(\overline{X}_\alpha, \mathfrak{c}_\alpha))$ such that

$$I_{ECH}(Z_{\mathcal{L}}) = gr_z(X_\alpha, \mathfrak{c}_\alpha). \tag{4.25}$$

By (4.24) and (5.16), we therefore have

$$I_{SW}(\mathfrak{c}_\alpha) = (\mathfrak{s}_\alpha, \varphi_{\mathcal{L}}) + \tilde{e}(S^+_\alpha, \varphi_{\mathcal{L}}) - I_{ECH}(Z_{\mathcal{L}}). \tag{4.26}$$

By the definition of $\varphi_{\mathcal{L}}$, $P_\tau(\mathcal{L}) = (\mathfrak{s}_\alpha, \varphi_{\mathcal{L}})$. To complete the proof, we therefore just need to show that

$$\tilde{e}(S^+_\alpha, \varphi_{\mathcal{L}}) = -\sum_i \omega_{\tau_i}(\zeta_i) + \mu_\tau(\alpha) + I_{ECH}(Z_{\mathcal{L}}). \tag{4.27}$$

This computation is easiest if we choose a particular representative of the isomorphism class of $(\mathfrak{s}_\alpha, \varphi_{\mathcal{L}})$, since this determines the boundary of the curve $C_{\mathcal{L}}$. Call a representative of the isomorphism class of $(\mathfrak{s}_\alpha, \varphi_{\mathcal{L}})$ *$\mathcal{L}$-compatible* if the boundary of $C_{\mathcal{L}}$ is $\mathcal{L}$. Let N denote the normal bundle of $C_{\mathcal{L}}$. Given an $\mathcal{L}$-compatible representative, projection induces a canonical isomorphism between $\xi|_{\partial C_{\mathcal{L}}}$ and $N|_{\partial C_{\mathcal{L}}}$ and the trivialization τ induces a trivialization of N over $\partial C_{\mathcal{L}}$. Remembering that $K^{-1}|_Y = \xi$, we can therefore follow [25] and define $c_1(N, \tau)$ (resp. $c_1(K^{-1}|_{C_{\mathcal{L}}}, \tau)$) to be a signed count of the zeroes of a generic section of N (resp. $K^{-1}|_{C_{\mathcal{L}}}$) extending a nonzero section over $\partial C_{\mathcal{L}}$ that has winding number 0 with respect to τ.

We now have the following lemma:

Lemma 4.5.4. *There exists an $\mathcal{L}$-compatible representative for the isomorphism class of $(\mathfrak{s}_\alpha, \varphi_{\mathcal{L}})$ and a choice of Ψ extending φ_L for which*

$$\tilde{e}(S^+_\alpha, \varphi_{\mathcal{L}}) = c_1(N|_{C_{\mathcal{L}}}, \tau) + c_1(K^{-1}|_{C_{\mathcal{L}}}, \tau).$$

Proof. The number $\tilde{e}(S^+, \varphi)$ is a signed count of the zeroes of Ψ. A signed zero of Ψ is precisely a signed zero of $\tilde{\gamma}$ over $C_{\mathcal{L}}$. Now observe that $d\gamma$ induces an isomorphism $N \xrightarrow{\simeq} E|_{C_{\mathcal{L}}}$, and hence the trivialization of N over $\partial C_{\mathcal{L}}$ induces a trivialization of E over $\partial C_{\mathcal{L}}$. We will arrange it so that

$$\tilde{\gamma} = e \otimes k, \tag{4.28}$$

where e is a section of $E|_{C_{\mathcal{L}}}$, k is a section of $K^{-1}|_{C_{\mathcal{L}}}$, and $e|_{\partial C_{\mathcal{L}}}$ and $k|_{\partial C_{\mathcal{L}}}$ both having winding number 0 with respect to τ. The lemma will then follow after a sign check.

To arrange for (4.28), we need to analyze the boundary of $C_{\mathcal{L}}$. Begin by letting U_j be a tubular neighborhood of one of the components for one of the ζ_i; assume that U_j is small enough so that U_j does not contain any other components of any of the ζ_i. Recall from §4.2 that there is a trivialization of TU_j extending the trivialization τ such that the Reeb vector field is always given by $\langle 1, 0, 0\rangle$ and ξ is given by $\{0\} \oplus \mathbb{C}$ according to this trivialization. Recall from §4.4 the definition of the Riemannian metric determined by the contact form and the almost complex structure. By choosing a new representative of the homotopy class of τ if necessary, we can ensure that the Riemannian metric is given by the standard dot product in this trivialization.

We will now choose a $\mathcal{L}$-compatible representative for $P_\tau(\mathcal{L})$. Let (t, r, θ) denote coordinates on U_j, and use the above trivialization to regard a vector field over U_j as a function with values in $\mathbb{R} \oplus \mathbb{R}^2$. Define a vector field P_j in (t, r, θ) coordinates by

$$P_j(t, re^{i\theta}) = (-cos(\pi r), sin(\pi r)cos(\theta), -sin(\pi r)sin(\theta)), \tag{4.29}$$

and extend the P_j by the Reeb vector field to a vector field P on Y. Because the P_j satisfy the conditions described in §4.2, the 2-plane field $\tilde{\xi}$ corresponding to P represents the homotopy class of $P_\tau(\mathcal{L})$.

We then have $S^+_\alpha = \mathbb{C} \oplus \tilde{\xi}$ with $\varphi = (1, 0)$. Take $\tilde{\xi}$ to be the orthogonal complement of P. Remember that E is by definition the $+i$ eigenspace of Clifford multiplication by the Reeb field and EK^{-1} is the $-i$ eigenspace. To prove the lemma, we therefore need to understand the Clifford multiplication ρ. Recall from the proof of [32, Lem. 28.1.1] that the Clifford multiplication is determined by requiring that $\mathbb{C}$ is the $+i$ eigenspace of Clifford multiplication by P, $\tilde{\xi}$ is the $-i$ eigenspace, and, for any vector v orthogonal to P, $\rho(v)(\varphi) = (0, v)$.

In particular, away from the U_j, the E component of φ is everywhere nonzero. The boundary of $C_{\mathcal{L}}$ is therefore contained in the union of the U_j. Restrict to a single U_j. To understand the components of φ in an eigenbasis for $\rho(R)$, it is convenient to define the vector field:

$$\tilde{P}_j(t, r, \theta) = (sin(\pi r), cos(\pi r)cos(\theta), -cos(\pi r)sin(\theta)).$$

Observe that $\tilde{P}_j$ and P_j are orthogonal, and moreover

$$\langle 1, 0, 0\rangle = -cos(\pi r)P_j + sin(\pi r)\tilde{P}_j.$$

Because $\tilde{P}_j$ is orthogonal to P_j, $\tilde{P}_j$ also defines a section of $\tilde{\xi}$ over U_j. We can therefore view $\{\varphi, (0, \tilde{P}_j)\}$ as a frame for S^+_α over U_j, and in this frame, Clifford multiplication by the Reeb vector field is given by

$$\rho(R) = \begin{pmatrix} -icos(\pi r) & -sin(\pi r) \\ sin(\pi r) & icos(\pi r) \end{pmatrix}. \tag{4.30}$$

Observe first of all that $\varphi = (1, 0)$ is in the $-i$ eigenspace of $\rho(R)$ precisely when $r = 0$. This implies that the boundary of $\partial C_{\mathcal{L}}$ is $\mathcal{L}$. Since $\rho(R)$ does not depend on t, we can arrange for (4.28) with $e|_{\partial C_{\mathcal{L}}}$ and $k|_{\partial C_{\mathcal{L}}}$ in fact constant with respect to τ. The lemma now follows. □

We can now show (4.27), completing the proof of Theorem 4.1.1. Hutchings' argument from [27, Prop. 3.1] gives

$$c_1(N, \tau) = -\omega_\tau(\mathcal{L}) + Q_\tau(Z_\alpha), \tag{4.31}$$

and we also know that

$$c_\tau(Z_\alpha) = c_1(K^{-1}|_{C_\alpha}, \tau). \tag{4.32}$$

Equation (4.27) now follows by choosing an $\mathcal{L}$-compatible representative and then applying Lemma 4.5.4, (4.31), (4.32), and the definition of I_{ECH}. This completes the proof of Theorem 4.1.1.

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Chapter 5

Period collapse of Ehrhart quasipolynomials, infinite staircases, and symplectic embeddings of ellipsoids

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Abstract: We use the recently defined "embedded contact homology" (ECH) capacities due to Hutchings to give new proofs of McDuff and Schlenk's "Fibonacci staircase" from [37] and Frenkel and Müller's "Pell staircase" from [11], and we prove that another infinite staircase occurs when computing when a four-dimensional symplectic ellipsoid embeds into a scaling of $E(1, \frac{3}{2})$. Our proofs give examples of a combinatorial phenomenon of independent interest called "period collapse".

5.1 Introduction

Statement of results

Beginning with the work of Gromov [14], there has been considerable interest in understanding when one symplectic manifold embeds into another [2] [38] [31] [15] [3]. Recently, McDuff has proven a powerful theorem concerning when one four-dimensional *symplectic ellipsoid*

$$E(a,b) = \{(z_1, z_2) \in \mathbb{C}^2 | \frac{\pi|z_1|^2}{a} + \frac{\pi|z_2|^2}{b}\}^1$$

[1]Here, the symplectic form is given by restricting the standard form $\omega = \Sigma_{i=i}^2 dx_i dy_i$ on $\mathbb{R}^4 = \mathbb{C}^2$.

embeds into another. To explain McDuff's result, let $c_k(E(a,b))$ be the $(k+1)^{st}$ smallest element in the matrix of numbers

$$(am+bn)_{m,n\in\mathbb{Z}_{\geq 0}}.$$

The number $c_k(E(a,b))$ is the k^{th} *embedded contact homology* capacity of $E(a,b)$. We can now state McDuff's theorem, which was originally conjectured by Hofer:

Theorem 5.1.1. *(Hofer conjecture)*

$$\operatorname{Int} E(a,b) \overset{s}{\hookrightarrow} E(c,d),$$

if and only if

$$c_k(E(a,b)) \leq c_k(E(c,d)),$$

for all k.

Despite its strength, there have been very few applications of Theorem 5.1.1. The aim of this present work is to understand an interesting "infinite staircase" phenomenon originally discovered by McDuff and Schlenk in [37] from the point of view of Theorem 5.1.1. In [37], McDuff and Schlenk found that embeddings of a four-dimensional ellipsoid $E(1,a)$ into a ball are controlled for $1 \leq a \leq \frac{7+3\sqrt{5}}{2}$ by an infinite staircase determined by the odd index Fibonacci numbers. In [11], Frenkel and Müller discovered a similar phenomenon involving the Pell numbers in studying embeddings of $E(1,a)$ into a scaling of $E(1,2)$[2]. Our results show that there is a surprising relationship between these staircases and a purely combinatorial phenomenon concerning Ehrhart quasipolynomials of rational polytopes called "period collapse". By exploiting this phenomenon, we give new proofs of these staircases and we show that another infinite staircase appears when considering embeddings into scalings of $E(1,\frac{3}{2})$.

To state our results in a unfied way, and to review the precise statement of McDuff-Schlenk and Frenkel-Müller's infinite staircases, define the function

$$c(a,b) := \inf\{\mu : E(1,a) \overset{s}{\hookrightarrow} E(\mu, b\mu)\}. \tag{5.1}$$

By scaling, the function $c(a,b)$ completely determines when one four-dimensional symplectic ellipsoid embeds into another. Define also for any pair of positive integers (k,l) the sequence $r(k,l)_n$ given by $r(k,l)_0 = 1, r(k,l)_1 = 1$ and

$$r(k,l)_{2n+1} = \frac{k+l+1}{k} r(k,l)_{2n} - r(k,l)_{2n-1}, \tag{5.2}$$

$$r(k,l)_{2n} = \frac{k+l+1}{l} r(k,l)_{2n-1} - r(k,l)_{2n-2}. \tag{5.3}$$

[2]In fact, McDuff and Schlenk determine for all a exactly when $E(1,a)$ embeds into a ball and Frenkel and Müller also determine when $E(1,a)$ embeds into a scaling of $E(1,2)$ for all a. It is an interesting problem which we do not address here to deduce their results directly from Theorem 5.1.1.

For example, the $r(1,1)_n$ are the odd index Fibonacci numbers familiar from the work of McDuff and Schenk [37] and the $r(2,1)_n$ are related to the Pell numbers. The $r(k,l)_n$ need not be integers, although in a moment we will restrict to pairs (k,l) for which they are. Also define for $n \geq 0$ the sequences

$$a(k,l)_n := \begin{cases} \dfrac{kr(k,l)_{n+1}^2}{lr(k,l)_n^2} & \text{if } n \text{ is even,} \\ \dfrac{lr(k,l)_{n+1}^2}{kr(k,l)_n^2} & \text{if } n \text{ is odd,} \end{cases}$$

and

$$b(k,l)_n := \frac{r(k,l)_{n+2}}{r(k,l)_n}.$$

Finally, define the positive real number

$$\phi(k,l) = \frac{k}{l}\left(\frac{k+l+1+\sqrt{(k+l+1)^2-4kl}}{2k}\right)^2.$$

Now assume that k and l both divide $k+l+1$. This implies in particular that the $r(k,l)_n$ are all integers and strongly constrains the possibilities for k and l. Indeed, if $k \geq l$ then it is easy to see that k and l both divide $k+l+1$ if and only if

$$(k,l) \in \{(1,1),(2,1),(3,2)\}. \tag{5.4}$$

We always have

$$a(k,l)_0 < b(k,l)_0 < a(k,l)_1 < b(k,l)_1 < \ldots < \phi(k,l).$$

The "staircase" theorem concerning symplectic embeddings that is the main subject of this paper shows that the $a(k,l)_n$ and $b(k,l)_n$ are the endpoints of the steps of an infinite staircase (see Figure 1):

Theorem 5.1.2. *Assume that $(k,l) \in \{(1,1),(2,1),(3,2)\}$. Then for a in the interval $[1,\phi(k,l)]$,*

$$c(a,\frac{k}{l}) = \begin{cases} 1 & \text{if } a \in [1,\frac{k}{l}], \\ \frac{a}{\sqrt{\frac{k}{l}a(k,l)_{n+1}}} & \text{if } a \in [a(k,l)_n, b(k,l)_n], \\ \sqrt{\frac{l}{k}a(k,l)_{n+1}} & \text{if } a \in [b(k,l)_n, a(k,l)_{n+1}]. \end{cases}$$

For $(k,l)=(1,1)$, this is a restatement of [37, Thm. 1.1.i] and for $(k,l)=(2,1)$ this is a restatement of [11, Thm. 1.1.i]. We will very briefly compare our approach to Theorem 5.1.2 with the approaches of McDuff-Schlenk and Frenkel-Müller in §5.1.

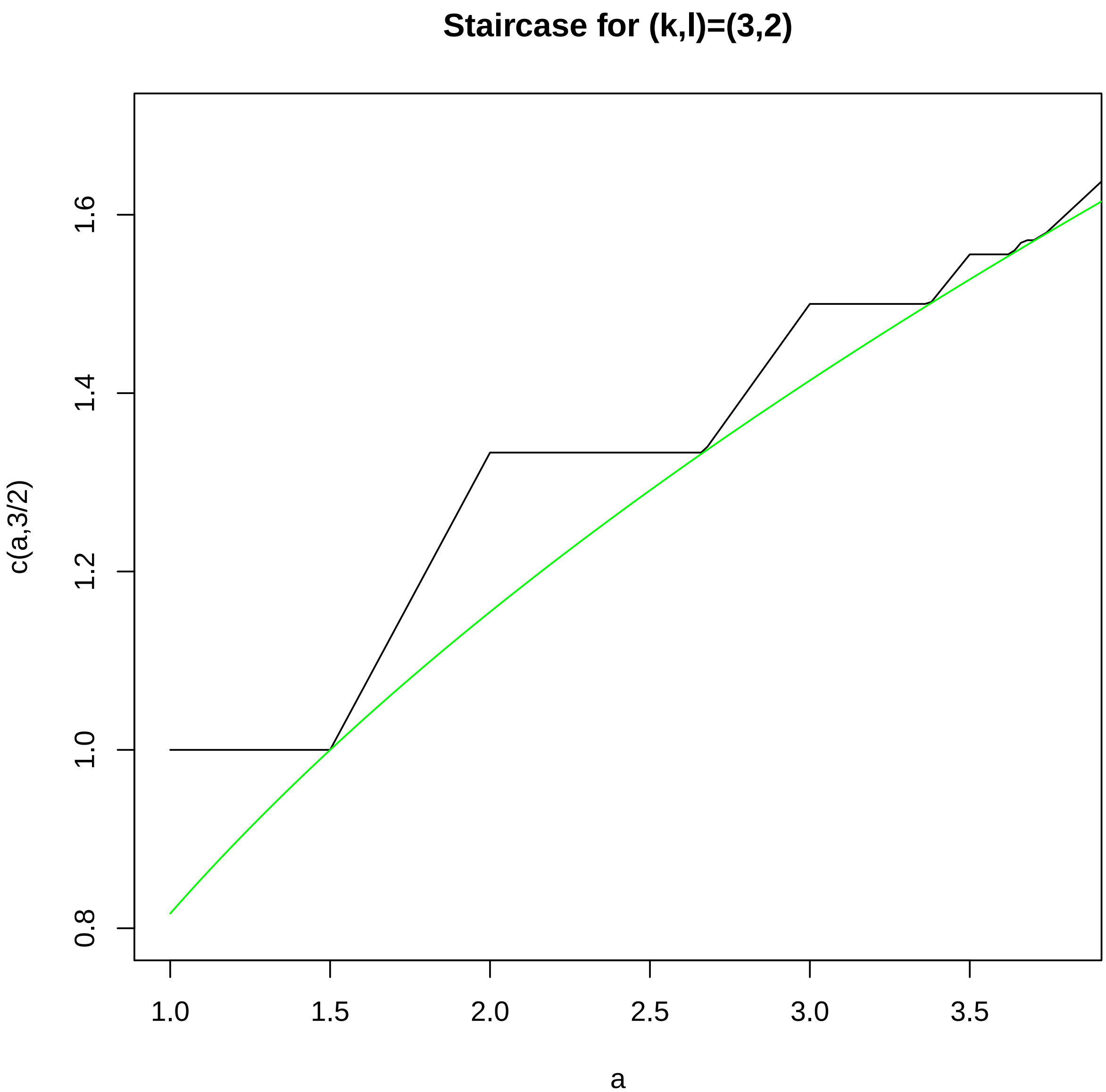

Figure 5.1: Staircase for $(k, l) = (3, 2)$

To explain the relationship between Theorem 5.1.2 and Ehrhart theory, recall that if $\mathcal{P}$ is any rational d-dimensional polytope and t is a positive integer, then the counting function

$$L_{\mathcal{P}}(t) := \#(\mathcal{P} \cap \frac{1}{t}\mathbb{Z}^d)$$

is called the *Ehrhart quasipolynomial* of $\mathcal{T}$. The *denominator* $\mathcal{D}(\mathcal{P})$ is the smallest $\mathcal{D} \in \mathbb{Z}_{>0}$ such that the vertices of $\mathcal{D} \cdot \mathcal{P}$ are integral. The significance of the denominator comes from the fact that the Ehrhart quasipolynomial is always a degree d polynomial in t with periodic coefficients of period $\mathcal{D}(\mathcal{P})$.

Following [18], we say that two polytopes $\mathcal{P}_1$ and $\mathcal{P}_2$ are *Ehrhart equivalent* if

$$L_{\mathcal{P}_1}(t) = L_{\mathcal{P}_2}(t)$$

for all positive integers t. The idea behind our proof of Theorem 5.1.2 is to establish non-trivial Ehrhart equivalences within certain families of triangles. Specifically, for positive real numbers u and v, denote the triangle with vertices $(0,0), (0,u)$ and $(v,0)$ by $\mathcal{T}_{u,v}$. As explained in §5.1, it is natural in view of Theorem 5.1.1 to ask when the triangles $\mathcal{T}_{\frac{q}{kp},\frac{p}{lq}}$ and $\mathcal{T}_{\frac{1}{k},\frac{1}{l}}$ for positive integers p and q are Ehrhart equivalent. If kp^2 and lq^2 are relatively prime, then there is a simple number theoretic obstruction to finding an Ehrhart equivalence between these two triangles, namely that the quadruple (k,l,p,q) satisfies the equation

$$kp^2 - (k+l+1)pq + lq^2 = -1. \tag{5.5}$$

The equation (5.5) comes from equating the linear terms of the corresponding Ehrhart quasipolynomials and its short derivation will be presented in §5.2. If k and l both divide $k+l+1$, then it turns out that (5.5) is the *only* obstruction to finding an Ehrhart equivalence:

Theorem 5.1.3. *Let k and l satisfy* (5.4) *and assume that $k \geq l$ and kp^2 and lq^2 are relatively prime. Then*

$$\mathcal{T}_{\frac{q}{kp},\frac{p}{lq}} \quad \text{and} \quad \mathcal{T}_{\frac{1}{k},\frac{1}{l}}$$

are Ehrhart equivalent if and only if (k,l,p,q) satisfies (5.5).

Remark 5.1.4. *Theorem 5.1.3 can be made very explicit. In particular, in §5.4, we show that the solutions of* (5.5) *are precisely the pairs $(p,q) = (r(k,l)_{2n\pm1}, r(k,l)_{2n})$ when* (5.4) *holds.*

Because of Theorem 5.1.3, we call triangles $\mathcal{T}_{\frac{q}{kp},\frac{p}{lq}}$ for which k and l divide $k+l+1$ and (k,l,p,q) satisfies (5.5) *perfect triangles* (we will show in §5.2 that if (k,l,p,q) satisfies (5.5) and (k,l) satisfies (5.4) then kp^2 and lq^2 are automatically relatively prime). The "if" direction of Theorem 5.1.3 is the key nontrivial result needed for our proof of Theorem 5.1.2. We will prove the "only if" direction as well since it is of independent combinatorial interest.

Remark 5.1.5. *It is interesting to ask whether the very strong condition that k and l both divide $k+l+1$ in Theorem 5.1.3 can be replaced by something weaker. Indeed, it is because of the strength of this assumption that we are only able to use our methods to produce three infinite staircases.*

Without some extra condition on k and l, Theorem 5.1.3 certainly does not hold. For example, for $(k,l)=(3,1)$ there are many examples of triangles $\mathcal{T}_{\frac{q}{kp},\frac{p}{lq}}$ that are not Ehrhart equivalent to $\mathcal{T}_{\frac{1}{k},\frac{1}{l}}$ even if (k,l,p,q) satisfies (5.5). Moreover, preliminary experimental evidence indicates that this condition on k and l can not be replaced by any *weaker condition in Theorem 5.1.3. For example, one can check with the aid of a computer that for $1\le k,l\le 50$, $c(\phi(k,l),\frac{k}{l})$ is not equal to the volume constraint unless k and l both divide $k+l+1$. This is significant because if Theorem 5.1.3 held for some other (k,l), we would expect an infinite staicrase in the graph of $c(a,\frac{k}{l})$ intersecting the volume curve at $a=\phi(k,l)$.*

Theorem 5.1.3 gives examples of a purely combinatorial phenomenon of independent interest. If $\mathcal{P}$ is any rational polytope, then the minimal period of $L_{\mathcal{P}}(t)$ is called the *period* of $\mathcal{P}$. Recently, see for example [18] [34] [49], there has been considerable interest in understanding when the period of $\mathcal{P}$ is less than the denominator of $\mathcal{P}$. This phenomenon is called *period collapse*. For example, in [34, Thm. 2.2] Woods and McAllister construct a d-dimensional rational polytope whose Ehrhart quasipolynomial has period s for all dimensions d, denominators $\mathcal{D}$, and periods s dividing $\mathcal{D}$. Because if k and l are relatively prime the denominator of $\mathcal{T}_{\frac{1}{k},\frac{1}{l}}$ is kl, Theorem 5.1.3 naturally gives examples of triangles whose periods are much smaller than their denominators, and for $(k,l)\in\{(1,1),(2,1)\}$ the proof of Theorem 5.1.3 actually explains how to classify all such triangles for which period collapse occurs. Specifically, we show:

Theorem 5.1.6. *Assume that p and q are relatively prime.*

(i) If $(k,l)\in\{(1,1),(2,1)\}$, then the Ehrhart quasipolynomial of $\mathcal{T}_{\frac{q}{p},\frac{p}{q}}$ has period kl if and only if for some n

$$(p,q)=(r(k,l)_{2n\pm1},r(k,l)_{2n})\quad\text{or}\quad(p,q)=(lr(k,l)_{2n},kr(k,l)_{2n\pm1}).$$

(ii) The Ehrhart quasipolynomial of $\mathcal{T}_{\frac{q}{3p},\frac{p}{2q}}$ has period 6 if

$$(p,q)=(r(3,2)_{2n\pm1},r(3,2)_{2n})\quad\text{or}\quad(p,q)=(2r(3,2)_{2n},3r(2,1)_{2n\pm1}).$$

It is interesting to ask whether other infinite staircases appear in the graph of $c(a,b)$. Work of Frenkel and Schlenk [12] implies that $c(a,4)$ is equal to the volume obstruction except on finitely many intervals for which it is linear, and it is suspected by both the authors and Schlenk [39] that the graph of $c(a,k)$ never contains an infinite staircase for integer $k>3$. In fact, computer experimentation suggests that it is possible that the three infinite staircases in Theorem 5.1.2 are the only examples of infinite staircases in the graph of $c(a, b)$, although this certainly demands further study. A simple characterization of period collapse for the family $\mathcal{T}_{u,v}$ along the lines of Theorem 5.1.6 is also open.

We close this section by noting that for many toric 4-manifolds, there are purely combinatorial formulas for the ECH capacities in terms of the image of (a suitably chosen) moment map, see [22, Thm. 1.11]. It is possible that Ehrhart theory could be helpful for better understanding these obstructions. It is also our hope that Ehrhart theory can be used to understand other features of the function $c(a, b)$.

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ECH capacities

In this section we briefly summarize the definition of the embedded contact homology capacities, see [22] for a complete treatment.

Let Y be a closed oriented 3-manifold. A *contact form* on Y is a one-form λ such that $\lambda \wedge d\lambda > 0$. A contact form determines a canonical vector field R called the *Reeb vector field*. The closed integral curves of the Reeb vector field, called *Reeb orbits*, are of considerable interest. If Y is equipped with a contact form λ then the *embedded contact homology* of Y, denoted $ECH(Y, \lambda)$, is defined. This is the homology of a chain complex freely generated over $\mathbb{Z}/2$ by certain finite sets of Reeb orbits, called *orbit sets*, relative to a chain complex differential which counts “ECH index“ 1 J-holomorphic curves in the symplecization of Y for an “admissible” J. We will not define the ECH index here; for details, and for more about ECH, see [20]. In [40], it is shown that $ECH(Y)$ always has infinite rank as a $\mathbb{Z}/2$ module.

In fact, by [40] $ECH(Y, \lambda)$ only depends on the three-manifold Y and not on the contact form λ. Thus, ECH is a topological invariant. Nevertheless, ECH can still be used to define invariants of the contact form λ. One such invariant which will be used to define the ECH capacities is the *ECH spectrum* of (Y, λ). The ECH spectrum of Y is a sequence of nonnegative real numbers

$$c_0(Y, \lambda) \leq c_1(Y, \lambda) \leq \ldots \leq \infty,$$

defined by computing the minimum symplectic action required to represent certain special classes in ECH. It is defined in [22]. The significance of the ECH spectrum from the point of view of symplectic capacities is that if (X, ω) is a “weakly exact” symplectic cobordism from (Y_+, λ_+) to (Y_-, λ_-) then

$$c_k(Y_+, \lambda_+) \geq c_k(Y_-, \lambda_-). \tag{5.6}$$

See [22] for the definition of weakly exact.

Define a *Liouville domain* to be a compact symplectic 4-manifold (X, ω) such that ω is exact and $\omega|_{\partial X} = d\lambda$ for some contact form λ. For example, any symplectic ellipsoid $E(a, b)$ is a Liouville domain. If (X, ω) is a Liouville domain, then define

$$c_k(X, \omega) := c_k(\partial X, \lambda)$$

where λ is any contact form on ∂X with $\omega|_{\partial X} = d\lambda$. By [22, Lem. 3.9], this does not depend on the choice of λ. The equation (5.6) is the key fact needed to show that the ECH capacities are monotone under embeddings, see [22, Thm. 1.1].

ECH capacities can be defined for any symplectic 4-manifold (and in fact for any subset of a symplectic 4-manifold), see [22]. Since in this paper we are only concerned with symplectic ellipsoids, we will not introduce the definition here.

Ehrhart quasipolynomials and embeddings of ellipsoids

In this section we explain the connection between Ehrhart quasipolynomials and embeddings of ellipsoids.

For positive real numbers a, b and t, let

$$N(a, b; t) = \#\{i : c_i(E(a, b)) \leq t\}.$$

By Theorem 5.1.1, $E(a, b) \overset{s}{\hookrightarrow} E(c, d)$ if and only if $N(a, b; t) \geq N(c, d; t)$ for all real numbers t. Now assume that c and d are both integers. Since $c_k(E(a, b))$ is always an integer, we know that $N(a, b; t) \geq N(c, d; t)$ for all real t if and only if $N(a, b; t) \geq N(c, d; t)$ for positive integer t. Since for integer t we know that $N(a, b; t) = L_{\mathcal{T}_{\frac{1}{a}, \frac{1}{b}}}(t)$, we have proven the following claim:

Claim 5.1.7. *Let c and d be integers. Then $E(a, b) \overset{s}{\hookrightarrow} E(c, d)$ if and only if*

$$L_{\mathcal{T}_{\frac{1}{a}, \frac{1}{b}}}(t) \geq L_{\mathcal{T}_{\frac{1}{c}, \frac{1}{d}}}(t) \quad \forall t \in \mathbb{Z}_{>0}.$$

Note that by scaling, to determine when one rational ellipsoid embeds into another we can in fact assume that both are integral.

Comparison with the methods of McDuff-Schlenk and Frenkel-Müller

This section very briefly summarizes the methods from [37] and [11].

In [35], McDuff shows that if $a \geq 1$ is rational then there is a finite sequence of rational numbers

$$\mathbf{w}(a) = (a_1, ..., a_n)$$

such that $E(1, a)$ embeds into a ball $B(\mu)$ if and only if the disjoint union of balls

$$\sqcup_i B(a_i)$$

embeds into $B(\mu)$.

The sequence $\mathbf{w}(a)$ is called a *weight expansion* for a and can be derived purely combinatorially from a in terms of its continued fraction expansion, see [37, §2.2]. McDuff and Schlenk show that for rational a an ellipsoid $E(1,a)$ embeds into the interior of a ball $B^4(\mu)$ if and only if

(i) $\mu^2 > \Sigma_i w_i^2$,

(ii) $d\mu > \Sigma_i m_i w_i \qquad \forall (d, \mathbf{m}) \in \mathcal{E}$,

where w_i denotes the i^{th} coordinate in the weight expansion, m_i denotes the i^{th} coordinate of the vector $\mathbf{m}$, and $\mathcal{E}$ is a set of nonnegative integers $(d; m_1, \ldots, m_n)$ such that

$$\Sigma_i m_i = 3d - 1 \qquad \Sigma m_i^2 = d^2 + 1,$$

and such that $(d, \mathbf{m})$ can be reduced to $(0; -1, 0, \ldots, 0)$ be a sequence of standard *Cremona moves*. A Cremona move is a certain linear transformation that can be applied to an ordered tuple $(d; \mathbf{m})$, see [37, Defn. 1.2.11] for the definition.

To prove the Fibonacci staircase in Theorem 5.1.2 using these methods, the key step is to show that for $g_n = r(1,1)_n$, both of the tuples

$$(g_{n+1}; g_n \mathbf{w}(b_n)), \qquad (g_n g_{n+1}; (g_n^2(\mathbf{w}(a_n), 1))),$$

are in $\mathcal{E}$.

5.2 Preliminaries

We begin by developing the combinatorial machinery that will be used in the proof of Theorem 5.1.3.

Ehrhart quasipolynomials and Fourier-Dedekind sums

We first observe that for triangles of the form

$$\mathcal{T} = \left\{ (x,y) \in \mathbb{R}^2 : x \geq \frac{a}{d}, y \geq \frac{b}{d}, ex + fy \leq r \right\},$$

one can compute the Ehrhart quasipolynomial in terms of "Fourier-Dedekind" sums by using an appropriate generating function. This is explained in [1, §2]. To prove Theorem 5.1.3, we will only need to consider the special case where $a = b = 0$, $e = kp^2$, $f = lq^2$, and $r = pq$, for p, q, k, and l positive integers with kp^2 and lq^2 relatively prime. In this case, [1, Thm.

2.10] gives:

$$\begin{aligned}L_{\mathcal{T}}(t) &= \frac{1}{2klpq}t^2 + \frac{1}{2}\left(\frac{q}{kp} + \frac{p}{lq} + \frac{1}{klpq}\right)t \\ &+ \frac{1}{4}\left(1 + \frac{1}{kp^2} + \frac{1}{lq^2}\right) + \frac{1}{12}\left(\frac{kp^2}{lq^2} + \frac{lq^2}{kp^2} + \frac{1}{klp^2q^2}\right) \\ &+ s_{-tpq}(lq^2, 1; kp^2) + s_{-tpq}(kp^2, 1; lq^2),\end{aligned} \tag{5.7}$$

where s_n denotes the *Fourier-Dedekind sum*

$$s_n(a_1, a_2; b) = \frac{1}{b}\sum_{k=1}^{b-1} \frac{\xi_b^{kn}}{(1 - \xi_b^{a_1 k})(1 - \xi_b^{a_2 k})}.$$

Here and in the following sections, ξ_b denotes the primitive b^{th} root of unity $\xi_b = e^{\frac{2\pi i}{b}}$

Diophantine equations

To establish the Ehrhart equivalences asserted by Theorem 5.1.3, recall that we need to study triangles $\mathcal{T}_{\frac{q}{kp}, \frac{p}{lq}}$ with kp^2 and lq^2 relatively prime for which (p, q, k, l) satisfy the diophantine equation

$$kp^2 - (k + l + 1)pq + lq^2 + 1 = 0.$$

As mentioned in the introduction, this equation is natural to consider in view of the Ehrhart equivalences asserted by Theorem 5.1.3, since by (5.7), (5.5) is exactly the condition needed for the coefficient of t for each corresponding pair of Ehrhart quasipolynomials to agree. In §5.4 we completely classify the solutions of (5.5) when (k, l) satisfies (5.4).

Convolutions

In view of (5.7), the hard part of Theorem 5.1.3 is evaluating the sum

$$s_{-tpq}(kp^2, 1; lq^2) + s_{-tpq}(lq^2, 1; kp^2). \tag{5.8}$$

For (k, l, p, q) satisfying (5.5), first rewrite the sum

$$s_{-tpq}(kp^2, 1; lq^2) = \frac{1}{lq^2}\sum_{j=1}^{lq^2-1} \frac{\xi_{lq^2}^{-tjpq}}{(1 - \xi_{lq^2}^{jkp^2})(1 - \xi_{lq^2}^{j})}$$

by writing $j = ilq + u$. This gives

$$\begin{aligned}&\frac{1}{lq^2}\sum_{i=1}^{q-1} \frac{\xi_{lq^2}^{-tpq^2il}}{(1 - \xi_{lq^2}^{kp^2ilq})(1 - \xi_{lq^2}^{ilq})} + \frac{1}{lq^2}\sum_{u=1}^{lq-1}\left(\xi_{lq^2}^{-tpqu}\sum_{i=0}^{q-1} \frac{1}{(1 - \xi_{lq^2}^{kp^2(ilq+u)})(1 - \xi_{lq^2}^{ilq+u})}\right) \\ =&\frac{1}{lq^2}\sum_{i=1}^{q-1} \frac{1}{(1 - \xi_q^{-i})(1 - \xi_q^{i})} + \frac{1}{lq^2}\sum_{u=1}^{lq-1}\left(\xi_{lq^2}^{-tpqu}\sum_{i=0}^{q-1} \frac{1}{(1 - \xi_{lq^2}^{ukp^2 - ilq})(1 - \xi_{lq^2}^{u+ilq})}\right),\end{aligned} \tag{5.9}$$

where the last line of (5.9) follows by (5.5). The significance of (5.9) is that the inner sum in the last line is a *convolution* and can be evaluated explicitly for the (k, l, p, q) we are interested in by using the Fourier transform. This will be explained in §5.2.

The sum

$$s_{-tpq}(lq^2, 1; kp^2) = \frac{1}{kp^2} \sum_{j=1}^{kp^2-1} \frac{\xi_{kp^2}^{-tjpq}}{(1-\xi_{kp^2}^{jlq^2})(1-\xi_{kp^2}^{j})}$$

can be similarly simplified by writing $j = ikp + u$ and applying (5.5) to obtain

$$\frac{1}{kp^2} \sum_{i=1}^{p-1} \frac{1}{(1-\xi_p^{-i})(1-\xi_p^{i})} + \frac{1}{kp^2} \sum_{u=1}^{kp-1} \left(\xi_{kp^2}^{-tpqu} \sum_{i=0}^{p-1} \frac{1}{(1-\xi_{kp^2}^{ulq^2-ikp})(1-\xi_{kp^2}^{u+ikp})} \right), \tag{5.10}$$

whose inner sum is also a convolution.

Fourier transform

To evaluate the convolutions in (5.9) and (5.10), we will apply the following general lemma:

Lemma 5.2.1. *Let a_1, a_2, b and c be integers such that b divides neither a_1 nor a_2. Then*

$$\frac{1}{c} \sum_{k=0}^{c-1} \frac{1}{(1-\xi_{bc}^{a_1+kb})(1-\xi_{bc}^{a_2-kb})} = \frac{\gamma}{(1-\xi_{bc}^{a_1 c})(1-\xi_{bc}^{a_2 c})}, \tag{5.11}$$

where

$$\gamma = \begin{cases} \frac{1-\xi_{bc}^{(a_1+a_2)c}}{1-\xi_{bc}^{a_1+a_2}} & \text{if } bc \not| a_1 + a_2, \\ c & \text{if } bc | a_1 + a_2. \end{cases}$$

We will be most interested in the lemma when in addition b divides $a_1 + a_2$ but bc does not divide $a_1 + a_2$. In this case, Lemma 5.2.1 implies that the sum in (5.11) is equal to 0.

Proof. Our proof is given in three steps.

Step 1. This step summarizes the inputs from finite Fourier analysis.

If f is a function with period b, recall that its *Fourier transform* is the function

$$\hat{f}(n) = \frac{1}{b} \sum_{k=0}^{b-1} f(k) \xi_b^{-kn}.$$

The convolution of two periodic functions f, g with period b is given by

$$(f * g)(n) = \sum_{m=0}^{b-1} f(n-m) g(m).$$

A version of the *convolution theorem* [1, Thm. 7.10] for the Fourier transform says that

$$(f * g)(n) = b \sum_{k=0}^{b-1} \hat{f}(k)\hat{g}(k)\xi_b^{kn}. \tag{5.12}$$

Step 2. We can compute the Fourier transform of the family of functions that are relevant to the proof of Lemma 5.2.1 explicitly:

Claim 5.2.2. *Fix positive integers a, b and c such that b does not divide a. Let f_a be the periodic function of period c given by*

$$f_a(n) := \frac{1}{1 - \xi_{bc}^{a+bn}}.$$

Then for integer $0 \le n \le c-1$,

$$\hat{f}_a(n) = \frac{\xi_{bc}^{an}}{1 - \xi_{bc}^{ac}}.$$

Proof. For $n \ge 0$, we have

$$\begin{aligned}
\hat{f}_a(n) &= \frac{1}{c} \sum_{k=0}^{c-1} \frac{\xi_{bc}^{-knb}}{1 - \xi_{bc}^{a+kb}} \\
&= \frac{\xi_{bc}^{an}}{c} \sum_{k=0}^{c-1} \frac{\xi_{bc}^{-(a+kb)n}}{1 - \xi_{bc}^{a+kb}} \\
&= \frac{\xi_{bc}^{an}}{c} \sum_{k=0}^{c-1} \left(\frac{1}{1 - \xi_{bc}^{a+kb}} - \frac{1 - \xi_{bc}^{-(a+kb)n}}{1 - \xi_{bc}^{a+kb}} \right) \\
&= \frac{\xi_{bc}^{an}}{c} \sum_{k=0}^{c-1} \left(\frac{1}{1 - \xi_{bc}^{a+kb}} + \sum_{m=1}^{n} \xi_{bc}^{-(a+kb)m} \right).
\end{aligned}$$

We can break the last line up into two sums and interchange the order of summation in the last sum to get

$$\frac{\xi_{bc}^{an}}{c} \sum_{k=0}^{c-1} \frac{1}{1 - \xi_{bc}^{a+kb}} + \frac{1}{c} \sum_{m=1}^{n} \left(\xi_{bc}^{a(n-m)} \sum_{k=0}^{c-1} \xi_{bc}^{-kbm} \right). \tag{5.13}$$

The innermost sum on the right hand side of (5.13) is 0 if c does not divide m. Since $m \le n$, when $0 \le n \le c-1$ the sum always vanishes and

$$\hat{f}_a(n) = \frac{\xi_{bc}^{an}}{c} \sum_{k=0}^{c-1} \frac{1}{1 - \xi_{bc}^{a+kb}}. \tag{5.14}$$

To simplify the summation, let $z_k = \frac{1}{1-\xi_{bc}^{a+kb}}$ and note that the z_k are the roots of the degree-c polynomial $(z-1)^c = \xi_{bc}^{ac} z^c$. Hence

$$(z-1)^c - \xi_{bc}^{ac} z^c = (1-\xi_{bc}^{ac}) \prod_{k=0}^{c-1} (z - z_k). \tag{5.15}$$

Equating the coefficient of z^{c-1} on each side of (5.15) gives

$$\sum_{k=0}^{c-1} \frac{1}{1-\xi_{bc}^{a+kb}} = \frac{c}{1-\xi_{bc}^{ac}}, \tag{5.16}$$

and the claim now follows by combining (5.14) and (5.16). □

Step 3. The sum in (5.11) is $(f_{a_1} * f_{a_2})(0)$, so by (5.12) and Claim 5.2.2,

$$\begin{aligned}(f_{a_1} * f_{a_2})(0) &= c \sum_{k=0}^{c-1} \left(\frac{\xi_{bc}^{a_1 k}}{1-\xi_{bc}^{a_1 c}} \right) \left(\frac{\xi_{bc}^{a_2 k}}{1-\xi_{bc}^{a_2 c}} \right) \\ &= \frac{c}{(1-\xi_{bc}^{a_1 c})(1-\xi_{bc}^{a_2 c})} \sum_{k=0}^{c-1} \xi_{bc}^{(a_1+a_2)k}.\end{aligned}$$

The sum in the last line evaluates to c if $bc | a_1 + a_2$, and otherwise we have

$$\sum_{k=0}^{c-1} \xi_{bc}^{(a_1+a_2)k} = \frac{1-\xi_{bc}^{(a_1+a_2)c}}{1-\xi_{bc}^{a_1+a_2}}.$$

This completes the proof. □

Reciprocity

The final tool that we need for the proof of Theorem 5.1.3 is the following *reciprocity* statement from [1, Thm. 8.8]:

Lemma 5.2.3. *(Rademacher Reciprocity:) Let $n = 1, 2, \ldots, (a+b+c)-1$. Then*

$$\begin{aligned}&s_n(a,b;c) + s_n(c,a;b) + s_n(b,c;a) = \\ &\qquad -\frac{n^2}{2abc} + \frac{n}{2}\left(\frac{1}{ab} + \frac{1}{ca} + \frac{1}{bc}\right) - \frac{1}{12}\left(\frac{3}{a} + \frac{3}{b} + \frac{3}{c} + \frac{a}{bc} + \frac{b}{ca} + \frac{c}{ab}\right).\end{aligned}$$

For $n = 0$, there is another reciprocity statement that we will also use [1, Cor. 8.7]:

Lemma 5.2.4.

$$\begin{aligned}&s_0(a,b;c) + s_0(c,a;b) + s_0(b,c;a) = \\ &\qquad 1 - \frac{1}{12}\left(\frac{3}{a} + \frac{3}{b} + \frac{3}{c} + \frac{a}{bc} + \frac{b}{ca} + \frac{c}{ab}\right).\end{aligned}$$

5.3 Proof of Theorem 5.1.3

We will now apply the machinery from the previous section to prove Theorem 5.1.3.

Recall that a triangle $\mathcal{T}_{\frac{q}{kp},\frac{p}{lq}}$ is *perfect* if k and l both divide $k+l+1$ and (k,l,p,q) satisfies (5.5). We want to show that if kp^2 and lq^2 are relatively prime, then $\mathcal{T}_{\frac{q}{kp},\frac{p}{lq}}$ is Ehrhart equivalent to $\mathcal{T}_{\frac{1}{k},\frac{1}{l}}$ if and only if $\mathcal{T}_{\frac{q}{kp},\frac{p}{lq}}$ is perfect. As explained in §5.2, the "only if" direction of this statement is clear. We can therefore assume that $\mathcal{T}_{\frac{q}{kp},\frac{p}{lq}}$ is perfect and prove the desired Ehrhart equivalence.

Proof. The proof follows in five steps. Throughout we assume that $\mathcal{T}_{\frac{q}{kp},\frac{p}{lq}}$ is perfect and kp^2 and lq^2 are relatively prime.

Step 1. It is clear by (5.7) that for the claimed equivalence, the coefficient of the t^2 terms are equal. By (5.5), the coefficients of t for each claimed equivalence are equal as well. We therefore only need to consider the other terms of each Ehrhart quasipolynomial.

Step 2. To evaluate the relevant Fourier-Dedekind sums, the following elementary fact will be useful:

Claim 5.3.1. *q is relatively prime to $\frac{k+l+1}{l}$ and p is relatively prime to $\frac{k+l+1}{k}$.*

Proof. Since (k,l) satisfies (5.4) we can argue case by case. If $(k,l)=(1,1)$, then Claim 5.3.1 follows by reducing (5.5) mod 3. If $(k,l)=(2,1)$, then reducing (5.5) mod 8 shows that p and $\frac{k+l+1}{k}$ are relatively prime, and reducing (5.5) mod 4 shows that q and $\frac{k+l+1}{q}$ are relatively prime. Finally, if $(k,l)=(3,2)$ then reducing (5.5) mod 3 shows that q and $\frac{k+l+1}{l}$ are relatively prime, and reducing (5.5) mod 2 shows that p and $\frac{k+l+1}{k}$ are relatively prime. $\square$

Step 3. We now begin the computation of the Fourier-Dedekind sums. By (5.5), (5.9), and (5.10) we have

$$s_{-tpq}(kp^2,1;lq^2)+s_{-tpq}(lq^2,1;kp^2)=$$

$$\begin{aligned}&\frac{1}{lq^2}\sum_{i=1}^{q-1}\frac{1}{(1-\xi_q^{-i})(1-\xi_q^{i})}+\frac{1}{lq^2}\sum_{u=1}^{lq-1}\left(\xi_{lq^2}^{-tpqu}\sum_{i=0}^{q-1}\frac{1}{(1-\xi_{lq^2}^{ukp^2-ilq})(1-\xi_{lq^2}^{u+ilq})}\right)+\\&\frac{1}{kp^2}\sum_{i=1}^{p-1}\frac{1}{(1-\xi_p^{-i})(1-\xi_p^{i})}+\frac{1}{kp^2}\sum_{u=1}^{kp-1}\left(\xi_{kp^2}^{-tpqu}\sum_{i=0}^{p-1}\frac{1}{(1-\xi_{kp^2}^{ulq^2-ikp})(1-\xi_{kp^2}^{u+ikp})}\right).\end{aligned}\tag{5.17}$$

By (5.5) we always have $q|u(kp^2+1)$, but by Claim 5.3.1, $lq^2|u(kp^2+1)$ if and only if $q|u$. So applying Lemma 5.2.1 with $b=lq, c=q, a_1=ukp^2$, and $a_2=u$ gives

$$\sum_{i=0}^{q-1}\frac{1}{(1-\xi_{lq^2}^{ukp^2-ilq})(1-\xi_{lq^2}^{u+ilq})}=0\tag{5.18}$$

unless $q|u$, in which case

$$\sum_{i=0}^{q-1} \frac{1}{(1-\xi_{lq^2}^{ukp^2-ilq})(1-\xi_{lq^2}^{u+ilq})} = \frac{q^2}{(1-\xi_{lq}^{kp^2u})(1-\xi_{lq}^{u})}. \tag{5.19}$$

Since our argument is symmetric in (k,p) and (l,q), we also have that

$$\sum_{i=0}^{p-1} \frac{1}{(1-\xi_{kp^2}^{ulq^2-kip})(1-\xi_{kp^2}^{u+kip})} = 0, \tag{5.20}$$

unless $p|u$, in which case

$$\sum_{i=0}^{p-1} \frac{1}{(1-\xi_{kp^2}^{ulq^2-kip})(1-\xi_{kp^2}^{u+kip})} = \frac{p^2}{(1-\xi_{kp}^{lq^2u})(1-\xi_{kp}^{u})}. \tag{5.21}$$

Now $kp^2 \equiv -1 \pmod{l}$ and $lq^2 \equiv -1 \pmod{k}$ by (5.5), so combining (5.18), (5.19), (5.20) and (5.21) with (5.17) gives

$$\begin{gathered} s_{-tpq}(kp^2,1;lq^2) + s_{-tpq}(lq^2,1;kp^2) = \\ \frac{1}{lq^2}\sum_{i=1}^{q-1} \frac{1}{(1-\xi_q^{-i})(1-\xi_q^{i})} + \frac{1}{kp^2}\sum_{i=1}^{p-1} \frac{1}{(1-\xi_p^{-i})(1-\xi_p^{i})} + \\ \frac{1}{l}\sum_{i=1}^{l-1} \frac{\xi_l^{-tip}}{(1-\xi_l^{i})(1-\xi_l^{-i})} + \frac{1}{k}\sum_{i=1}^{k-1} \frac{\xi_k^{-tiq}}{(1-\xi_k^{i})(1-\xi_k^{-i})}. \end{gathered} \tag{5.22}$$

Step 3. By (5.7), we must show

$$\begin{aligned} &\frac{1}{4}\left(1+\frac{1}{kp^2}+\frac{1}{lq^2}\right) + \frac{1}{12}\left(\frac{kp^2}{lq^2}+\frac{lq^2}{kp^2}+\frac{1}{klp^2q^2}\right) + s_{tpq}(lq^2,1;kp^2) + s_{tpq}(kp^2,1;lq^2) \\ &\quad = \frac{1}{4}\left(1+\frac{1}{k}+\frac{1}{l}\right) + \frac{1}{12}\left(\frac{k}{l}+\frac{l}{k}+\frac{1}{kl}\right) + s_t(l,1;k) + s_t(k,1;l). \end{aligned} \tag{5.23}$$

for all $t \le 0$. The right hand side of (5.23) is periodic in t with period kl, and by (5.22), the left hand side is as well. For $(k,l) = (3,2)$, by (5.22) the values at $t = 5$ and $t = 1$ are the same, as are the values at $t = 4$ and $t = 2$. Thus, when $(k,l) = (1,1)$ we can assume $t = 0$, when $(k,l) = (2,1)$ we can assume $t = 0, 1$, and when $(k,l) = (3,2)$ we can assume $0 \le t \le 3$.

When $t = 0$, we can apply Lemma 5.2.4 to both sides of (5.23) to get the desired equality.. For other t, we can apply Rademacher reciprocity to evaluate $s_{tpq}(kp^2,1;lq^2)+s_{tpq}(lq^2,1;kp^2)$

as long as $0 < tpq < kp^2 + lq^2$. This holds for all p, q when $0 < t < 2\sqrt{kl}$, and since we can always assume t lies in this range by the previous paragraph, for these t Lemma 5.2.3 gives

$$\begin{aligned} s_{tpq}(kp^2, 1; lq^2) + s_{tpq}(lq^2, 1; kp^2) \\ = -\frac{1}{12}\left(\frac{3}{kp^2} + \frac{3}{lq^2} + 3 + \frac{kp^2}{lq^2} + \frac{lq^2}{kp^2} + \frac{1}{klp^2q^2}\right) \\ - \frac{t^2}{2kl} + \frac{t}{2}\left(\frac{1}{klpq} + \frac{q}{kp} + \frac{p}{lq}\right). \end{aligned} \tag{5.24}$$

Since (5.24) also holds for $(p, q) = (1, 1)$, and because

$$\frac{pq}{2}\left(\frac{1}{klp^2q^2} + \frac{1}{kp^2} + \frac{1}{lq^2}\right) = \frac{1}{2}\left(\frac{1}{kl} + \frac{1}{k} + \frac{1}{l}\right) \tag{5.25}$$

by (5.5), Theorem 5.1.3 follows in this case as well by (5.7).

□

5.4 Classification of solutions to (5.5)

In this section we prove the following:

Proposition 5.4.1. *Suppose (k, l) satisfies (5.4). Then (p, q) is a solution to (5.5) if and only if $(p, q) = (r(k, l)_{2n\pm1}, r(k, l)_{2n})$ for some n.*

Remark 5.4.2. *To prove Theorem 5.1.3, we only need the "if" direction of Proposition 5.4.1. When $(k, l) \in \{(1, 1), (2, 1)\}$, the "only if" direction will be used in the proof of Theorem 5.1.6. We include the $(k, l) = (3, 2)$ case here for completeness in view of Theorem 5.1.3, see Remark 5.1.4.*

Proof. Fix k, l satisfying (5.4) and consider the pair of congruence relations

$$ka^2 \equiv -1 \pmod{lb}, \qquad lb^2 \equiv -1 \pmod{ka}. \tag{5.26}$$

Since k and l both divide $k + l + 1$, if (p, q) satisfies (5.5) then $(a, b) = (p, q)$ is a solution to (5.26). We will show that the converse holds, so classifying solutions of (5.26) is the same as classifying solutions of (5.5), and we will then show that the solutions to (5.26) are precisely those (a, b) of the form $(r(k, l)_{2n\pm1}, r(k, l)_{2n})$.

We first solve (5.26). The key observation needed is that if (p, q) is a solution of (5.26), then

$$p' := \frac{lq^2 + 1}{kp}, \qquad q' := \frac{kp^2 + 1}{lq}$$

are integers and (p', q) and (p, q') are also solutions to (5.26). Motivated by this, define the involutions

$$\sigma : (p, q) \to (\frac{lq^2 + 1}{kp}, q), \qquad \tau : (p, q) \to (p, \frac{kp^2 + 1}{lq}).$$

We claim that if $(p,q) \neq (1,1)$ then either σ or τ decreases a coordinate. Suppose that $p \leq p'$ and $q \leq q'$. Then $|kp^2 - lq^2| \leq 1$. If $kp^2 = lq^2$ then $(k,l,p,q) = (1,1,1,1)$, if $kp^2 = lq^2 + 1$ then $lq|2$ so $(l,q) \in \{(1,1),(1,2),(2,1)\}$, and if $kp^2 = lq^2 - 1$ then $(k,p) \in \{(1,1),(1,2),(2,1)\}$. By examining each of these cases separately we see that if $k \geq l$, $p' \geq p$, $q' \geq q$, then

$$(k,l,p,q) \in \{(1,1,1,1),(2,1,1,1),(3,2,1,1),(5,1,1,2)\}.$$

In particular, if we assume in addition that (k,l) satisfies (5.4), then $(p,q) = (1,1)$.

Now define the sequence $s(k,l)_n$ by $s(k,l)_0 = s(k,l)_1 = 1$,

$$s_{2n+1} = \frac{ls(k,l)_{2n}^2 + 1}{ks(k,l)_{2n-1}}, \qquad s(k,l)_{2n} = \frac{ks(k,l)_{2n-1}^2 + 1}{ls(k,l)_{2n-2}}. \tag{5.27}$$

If (p,q) satisfies (5.26) then $(p,q) = (s(k,l)_{2n\pm1}, s(k,l)_{2n})$ for some n. This follows by induction after applying either σ or τ. Another induction using (5.27) shows that $(s(k,l)_{2n\pm1}, s(k,l)_{2n})$ satisfies (5.5). Thus, the solutions of (5.5) and (5.26) are the same.

To see that the r_n and the s_n are the same, first note that an induction using (5.2) and (5.3) shows that

$$kr_{2n+1}^2 - (k+l+1)r_{2n+1}r_{2n} + lr_{2n}^2 = -1, \tag{5.28}$$

$$kr_{2n-1}^2 - (k+l+1)r_{2n-1}r_{2n} + lr_{2n}^2 = -1. \tag{5.29}$$

We can then apply a final induction using the recurrence relations (5.27), (5.3) and (5.2). □

5.5 Proof of Theorem 5.1.2

There are several basic properties of $c(a, \frac{k}{l})$ that significantly simplify the proof of Theorem 5.1.2:

Lemma 5.5.1. *(cf. [37, Lem. 1.1.1]) Fix k and l. Then the function $c(a, \frac{k}{l})$ satisfies:*

(i) (Continuity) $c(a, \frac{k}{l})$ is a continuous function of a.

(ii) (Monotonicity) $c(a, \frac{k}{l})$ is a monotonically nondecreasing function of a.

(iii) (Subscaling) $c(\lambda a, \frac{k}{l}) \leq \lambda c(a, \frac{k}{l})$ when $\lambda > 1$.

Proof. Statements (i) and (ii) are clear. Statement (iii) follows because we have

$$E(1,a) \subset \sqrt{\lambda}E(1,a)$$

for any $\lambda > 1$, and we know that

$$\sqrt{\lambda}E(c, c\frac{k}{l}) = E(\lambda c, \lambda c\frac{k}{l}).$$

□

Our strategy for proving Theorem 5.1.2 is to calculate $c(a(k,l)_n, \frac{k}{l})$, bound $c(b(k,l)_n, \frac{k}{l})$ from below, and apply Lemma 5.5.1.

Calculating $c(a(k,l)_n, \frac{k}{l})$

We first claim that $c(a(k,l)_n, \frac{k}{l})$ is always equal to the volume obstruction. This is where Theorem 5.1.3 is relevant to the proof of Theorem 5.1.2. To simplify the notation, we will now let a_n, b_n, and r_n denote $a(k,l)_n, b(k,l)_n$, and $r(k,l)_n$ for fixed (k,l) satisying (5.4).

By definition, $\frac{a_{2n}l}{k} = \frac{r_{2n+1}^2}{r_{2n}^2}$ and $\frac{a_{2n+1}l}{k} = \frac{l^2 r_{2n+2}^2}{k^2 r_{2n+1}^2}$. To show that

$$c(a_{2n}, \frac{k}{l}) = \sqrt{\frac{a_{2n}l}{k}} = \frac{r_{2n+1}}{r_{2n}}, \tag{5.30}$$

$$c(a_{2n+1}, \frac{k}{l}) = \sqrt{\frac{a_{2n+1}l}{k}} = \frac{l r_{2n+2}}{k r_{2n+1}}, \tag{5.31}$$

it suffices by Claim 5.1.7 to show that

$$L_{\mathcal{T}_{\frac{q}{kp}, \frac{p}{lq}}}(t) \geq L_{\mathcal{T}_{\frac{1}{k}, \frac{1}{l}}}(t) \tag{5.32}$$

when $(p,q) = (r_{2n\pm 1}, r_{2n})$.

By induction, (5.2) and (5.3) show that that r_{2n+1} and r_{2n} are relatively prime. Since $kl | k+l+1$ for (k,l) satisfying (5.4), induction also shows that $k \not| r_{2n}$, $l \not| r_{2n+1}$. Then (5.32) follows from (5.28), (5.29) and Theorem 5.1.3.

Calculating $c(b(k,l)_n, \frac{k}{l})$

Continue to let a_n, b_n, and r_n denote $a(k,l)_n, b(k,l)_n$, and $r(k,l)_n$ for fixed (k,l) satisying (5.4). By Lemma 5.5.1 and (5.32), to prove Theorem 5.1.2 it remains to show that

$$c(b_n, \frac{k}{l}) \geq \sqrt{\frac{l a_{n+1}}{k}} = \begin{cases} \frac{r_{n+2}}{r_{n+1}} & n \text{ odd}, \\ \frac{l}{k}\frac{r_{n+2}}{r_{n+1}} & n \text{ even}. \end{cases}.$$

We will show that for the index

$$f_n := \frac{r_{n+2}r_n + r_{n+2} + r_n - 1}{2}, \tag{5.33}$$

we have

$$c_{f_n}(E(1, b_n)) = r_{n+2}, \tag{5.34}$$

$$c_{f_n}(E(1, \frac{k}{l})) = \begin{cases} r_{n+1} & n \text{ odd}, \\ \frac{k}{l} r_{n+1} & n \text{ even}. \end{cases} \tag{5.35}$$

We begin with the proof of (5.34). We have

$$\begin{aligned}
\max_m\{m : c_m(E(1,b_n)) \le r_{n+2}\} &= -1 + \sum_{i=0}^{r_{n+2}} \left(\left\lfloor \frac{i}{b_n} \right\rfloor + 1 \right) \\
&= r_{n+2} + r_n + \sum_{i=0}^{r_{n+2}-1} \left\lfloor \frac{ir_n}{r_{n+2}} \right\rfloor \\
&= \frac{(r_{n+2}+1)(r_n+1)}{2},
\end{aligned}$$

where the last line follows from the well-known identity

$$\sum_{i=0}^{q-1} \left\lfloor \frac{ip}{q} \right\rfloor = \frac{(p-1)(q-1)}{2}$$

for $(p,q)=1$. The fact that $\gcd(r_{n+2}, r_n) = 1$ follows from an induction using (5.2) and (5.3). Since

$$\#\{m : c_m(E(1,b_n)) = r_{n+2}\} = 2,$$

we have that $c_{f_n}(E(1,b_n)) = c_{f_{n+1}}(E(1,b_n))$, hence (5.34) follows.

We next prove (5.35). We have that

$$\begin{aligned}
f_{2n} &= \frac{r_{2n+2}r_{2n} + r_{2n+2} + r_{2n} - 1}{2} \\
&= \frac{1}{2}((\frac{k+l+1}{l} r_{2n+1} - r_{2n})(r_{2n}+1) + r_{2n} - 1) \qquad (5.36)\\
&= \frac{kr_{2n+1}^2 + (k+l+1)r_{2n+1} - (l-1)}{2l},
\end{aligned}$$

where the second line follows from (5.3) and the last from (5.28). Similarly, by (5.2) and (5.29),

$$f_{2n-1} = \frac{lr_{2n}^2 + (k+l+1)r_{2n} - (k-1)}{2k}. \qquad (5.37)$$

By (5.7),

$$\begin{aligned}
\max_m\{m : &c_m(E(1, \frac{k}{l})) \le r_{2n}\} = L_{\mathcal{T}_{k,l}}(lr) - 1 \\
&= \frac{lr_{2n}^2 + (k+l+1)r_{2n}}{2k} + \frac{1}{4}\left(1 + \frac{1}{k} + \frac{1}{l}\right) \\
&+ \frac{1}{12}\left(\frac{k}{l} + \frac{l}{k} + \frac{1}{kl}\right) + s_{-lr_{2n}}(l,1;k) + s_{-lr_{2n}}(k,1;l) - 1.
\end{aligned}$$

For n even, this is equal to f_{2n-1} if

$$\frac{k+1}{2k} = \frac{1}{4}\left(1 + \frac{1}{k} + \frac{1}{l}\right) + \frac{1}{12}\left(\frac{k}{l} + \frac{l}{k} + \frac{1}{kl}\right) + s_{-lr_{2n}}(l,1;k) + s_{-lr_{2n}}(k,1;l). \qquad (5.38)$$

Similarly, (5.35) holds for n odd if

$$\frac{l+1}{2l} = \frac{1}{4}\left(1+\frac{1}{k}+\frac{1}{l}\right)+\frac{1}{12}\left(\frac{k}{l}+\frac{l}{k}+\frac{1}{kl}\right)+s_{-kr_{2n+1}}(l,1;k)+s_{-kr_{2n+1}}(k,1;l). \quad (5.39)$$

Induction on (5.2) and (5.3) shows $2 \not| r(2,1)_{2n}$, $3 \not| r(3,2)_{2n}$ and $2 \not| r(3,2)_{2n+1}$. Direct computation then shows (5.38) and (5.39) hold for each (k,l) satisfying (5.4). This completes the proof of Theorem 5.1.2.

5.6 Proof of Theorem 5.1.6

We conclude by proving Theorem 5.1.6. Assume throughout that (k,l) satisfies (5.4), and continue the notation of the previous section by letting r_n denote $r(k,l)_n$.

We first prove the "if" statements of Theorem 5.1.6. If $(p,q) = (r_{2n\pm1}, r_{2n})$ then, as explained in §5.4, kp^2 and lq^2 are relatively prime and (k,l,p,q) satisfies (5.5). Thus, by Theorem 5.1.3 $\mathcal{T}_{\frac{q}{kp},\frac{p}{ql}}$ is Ehrhart equivalent to $\mathcal{T}_{\frac{1}{k},\frac{1}{l}}$ and so $\mathcal{T}_{\frac{q}{kp},\frac{p}{ql}}$ has period kl. Similarly, if $(p,q) = (lr(k,l)_{2n}, kr(k,l)_{2n\pm1})$ then for

$$(p',q') := (\frac{p}{l}, \frac{q}{k})$$

kq'^2 and lp'^2 are relatively prime and (k,l,q',p') satisfies (5.5). Hence, by Theorem 5.1.3, $\mathcal{T}_{\frac{p'}{kq'},\frac{q'}{lp'}}$ is Ehrhart equivalent to $\mathcal{T}_{\frac{1}{k},\frac{1}{l}}$. Thus, $\mathcal{T}_{\frac{q}{kp},\frac{p}{ql}}$ has period kl, since $\mathcal{T}_{\frac{q}{kp},\frac{p}{ql}}$ is Ehrhart equivalent to $\mathcal{T}_{\frac{p'}{kq'},\frac{q'}{lp'}}$.

Assume now in addition that $(k,l) \in \{(1,1),(2,1)\}$. We now complete the proof of Theorem 5.1.3 by proving the "only if" statements. If kp^2 and lq^2 are relatively prime and $\mathcal{T}_{\frac{q}{kp},\frac{p}{lq}}$ has period kl, then by (5.7) we must have

$$s_{klpq}(kp^2,1;lq^2)+s_{klpq}(lq^2,1;kp^2) = s_0(kp^2,1;lq^2)+s_0(lq^2,1;kp^2). \quad (5.40)$$

We know in addition that $klpq \leq kp^2 + lq^2$. Hence, we can apply Lemma 5.2.3 and Lemma 5.2.4 to (5.40) to conclude that (k,l,p,q) satisfies (5.5), so the "only if" direction of Theorem 5.1.6 follows by Proposition 5.4.1.

If kp^2 and lq^2 are not relatively prime, then we must have $(k,l) = (2,1)$ and it must also be the case that q is divisible by 2 and p is not divisible by 2. Define $q' := \frac{q}{2}$. We know that $2q'^2$ and p^2 are relatively prime. Moreover, $\mathcal{T}_{\frac{q}{2p},\frac{p}{q}}$ is Ehrhart equivalent to $\mathcal{T}_{\frac{p}{2q'},\frac{q'}{p}}$. If $\mathcal{T}_{\frac{p}{2q'},\frac{q'}{p}}$ has period 2 then by (5.7) we must have

$$s_{2pq'}(2p^2,1;q'^2)+s_{2pq'}(q'^2,1;2p^2) = s_0(2p^2,1;q'^2)+s_0(q'^2,1;2p^2). \quad (5.41)$$

Since $2pq' \leq 2p^2 + q'^2$, we can apply Lemma 5.2.3 and Lemma 5.2.4 to (5.41) to conclude that $(2,1,q',p)$ satisfies (5.5). Theorem 5.1.6 again then follows by Proposition 5.4.1.

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