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By

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Abstract

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The principal-agent problem is a classic problem in economics, in which the principal seeks an optimal way to delegate a task to an agent that has private information or hidden action. A general continuous-time stochastic control problem based on the moral hazard problem in Sannikov (2008) is considered, with more general retirement cost and structure. In the problem, a risk-neutral principal tries to determine an optimal contract to compensate a risk-averse agent for exerting costly and hidden effort over an infinite time horizon. The compensation is based on observable output, which has a drift component equal to the hidden effort and a noise component driven by a Brownian motion.

In this thesis, a rigorous mathematical formulation is posed for the problem, which is modeled as a combined optimal stopping and control problem. Conditions are given on how a solution to the control problem could be implemented as a contract in the principal-agent framework with moral hazard. Our formulation allows for general continuous retirement profit functions, subject to an upper bound by the first-best profit. The optimal profit function is studied and proved to be concave and continuous. It is shown that the optimal profit function is the unique viscosity solution of the Hamilton-Jacobi-Bellman (HJB) equation.

Contents

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Chapter 1

Introduction

1.1 Principal-agent problem

A principal-agent problem arises when a *principal* delegates a task to an *agent*, often with private information known only to the agent. The goal for the principal is to identify an optimal contract that is based on the observable output rather than the unobservable private types or actions.

Specifically, we are interested in the case where the principal's payoff is affected by the private actions of the agent and some noise. It is the noise that makes the agent's private actions unobservable to the principal. In the economic literature, this class of principal-agent problems (where the agent has a private action) is known as *moral hazard*, whereas the case of an agent with private type is known as *adverse selection*. The case with perfect information, usually used as a benchmark for comparison, is referred as *risk-sharing*. A comprehensive introduction to the motivation and the broader literature of principal-agent problems can be found in Laffont and Martimort (2002) and Bolton and Dewatripont (2005).

For the case of a moral hazard problem, the goal is usually to maximize the principal's profit, and the contract is designed to induce the agent to take the actions preferred by principal.

1.1.1 Moral hazard problems in real life

One of the earliest example of principal-agent problems is that of sharecropping, where a landlord (as the principal) is to have the landlord's tenant perform agricultural work on his land. The tenant's effort in farming is not perfectly reflected in the harvest because of variability in the climate and other factors. To give the tenant incentives to work hard, the tenant often receives a portion of the crop as his reward.

In the modern-day society, examples of principal-agent problem arise frequently in management. An intuitive example is the relationship between a manager and his or her worker. The manager, due to time or technology constraint, cannot observe the actual effort exerted by the worker, and can only pay the worker based on some practical measure of productivity, usually the number of output units produced or the quantity of tasks performed. This performance measure is only a noisy outcome of the effort exerted and does not perfectly reflect the effort. In the literature, some studies have focussed on the optimal compensation package for a CEO of a company (Edmans et al., 2011), where the principal and agent are respectively the shareholders and the CEO.

Another example, closer to the financial industry, is the management of investor's portfolio by an investment manager. Investors may not have the time or expertise to find out whether the manager has been managing their portfolio in their best interests, as the actual return of the portfolio is again only a noisy signal of the effort of the manager, subject to variable market conditions and uncertainty in stock returns. It is then important to determine an appropriate compensation scheme to induce effort from the manager.

Other examples include financial contracting, where banks lend money to businesses that may be misreporting the potential return and risk of their projects (DeMarzo and Fishman, 2007), and retailer selection, where the buyer needs to choose retailers for sale of products.

1.2 An example of moral hazard

In this section, we illustrate the idea of moral hazard in an example with a binary effort and only two possible levels of production output. The relatively simple mathematical setting allows us to focus on some important economic concepts commonly found in moral hazard problems, as well as common ideas in formulating such problems. This example originally came from the example in Chapter 4 of Laffont and Martimort (2002) and has been rewritten and simplified to suit our later discussion.

In this example, the principal delegates a task to the agent. The output of the task takes two possible values $\bar{q}$ and $\underline{q}$, with $\bar{q} > \underline{q}$.

The output is stochastic and its distribution is affected by the agent's effort. The agent can choose between two possible levels of effort, 0 and 1, corresponding to no effort and a positive effort. The actual action of the agent is not observable to the principal. The output is $\bar{q}$ with probability p_0 if the agent makes zero effort, and p_1 if the agent makes effort 1, with $p_0 < p_1$. The agent incurs a cost of c_0 if he makes no effort, and a cost of c_1 if he makes effort, where $c_0 < c_1$. We normalized c_0 to be zero but retain the notation for convenience.

	high output $\bar{q}$	low output $\underline{q}$
no effort	p_0	$1 - p_0$
effort	p_1	$1 - p_1$

Figure 1.1: Probability of different output levels given the effort level

The principal would like to design a contract that pays the agent and induces him to work hard. Since the principal cannot observe the agent's actual effort, the compensation can only depend on the output. Specifically, the principal can decide on a contract $(\bar{m}, \underline{m})$, where $\bar{m}$ is paid to the agent if the output is $\bar{q}$ and $\underline{m}$ is paid to the agent if the output is $\underline{q}$.

At the beginning, the principal decides on the contract and announces it to the agent. The agent then decides whether to accept or reject the contract. Next, the agent chooses to make effort or not. Finally, the output is realized and the agent is paid correspondingly.

The agent's utility from the compensation is represented by a utility function u. If the agent receives an amount of m for compensation, his utility from the compensation is $u(m)$. We assume u is increasing and concave, where the concavity reflects the agent's risk aversion.

Given a contract $(\bar{m}, \underline{m})$, the agent compares his expected payoffs from the contract while making zero or positive effort, to decide if he should enter the contract and if so, whether to make positive effort.

If he enters the contract and makes no effort, he gets $p_0 u(\bar{m}) + (1 - p_0)u(\underline{m})$, while if he enters the contract and makes positive effort, he gets $p_1 u(\bar{m}) + (1 - p_1)u(\underline{m}) - c_1$. If the agent chooses not to enter the contract, he gets his *reservation utility*, the value of his outside opportunities when not participating in the contract. We normalize the reservation utility to zero.

Obviously, if the agent will earn less than his reservation utility whether or not he makes effort, then he will not enter the contract. Therefore, for the agent to enter the contract and make an effort, we need

$$p_1 u(\bar{m}) + (1 - p_1)u(\underline{m}) \geq c_1, \tag{1.1}$$

which ensures that his expected payoff is at least as good as not entering the contract, and

$$p_1 u(\bar{m}) + (1 - p_1)u(\underline{m}) - c_1 \geq p_0 u(\bar{m}) + (1 - p_0)u(\underline{m}),$$

i.e.

$$(p_1 - p_0)(u(\bar{m}) - u(\underline{m})) \geq c_1. \tag{1.2}$$

The second constraint ensures that the agent is at least as good to make positive effort than to make no effort. This gives him the incentives to make effort. We call (1.1) the *individual-rationality* constraint and (1.2) the *incentive-compatibility* constraint. Note that when equality holds in a constraint, the agent is indifferent between the two options and it is common in the literature to assume that the agent will do choose an action preferred by the principal in that case.

Now that we know what is required to induce effort from the agent, we look at the principal's problem. If the principal induces an effort from the agent via a contract $(\bar{m}, \underline{m})$, she would receive $p_1(\bar{q} - \bar{m}) + (1 - p_1)(\underline{q} - \underline{m})$.

Thus, the principal's optimal profit from a contract inducing positive effort from the agent is given by

$$V_1 = \max_{(\bar{m}, \underline{m}) \in \mathbb{R}^2} p_1(\bar{q} - \bar{m}) + (1 - p_1)(\underline{q} - \underline{m}) \tag{1.3}$$

such that (1.1) and (1.2) holds.

While a positive effort from the agent improves production output, the principal does not necessarily benefit from choosing a contract that induces positive effort from the agent over one that does not, because it is costly to induce effort from the agent. If the principal wishes not to induce effort from the agent, she can set $\bar{m} = \underline{m} = 0$ to give the agent his reservation utility and the principal will receive $p_0\bar{q} + (1 - p_0)\underline{q}$.

Combining the two cases, the principal will choose to induce an effort, with a contract that gives the maximum in (1.3) and receive V_1 if $V_1 \geq p_0\bar{q} + (1 - p_0)\underline{q}$. Otherwise, the principal will not induce an effort from the agent, and she will receive $p_0\bar{q} + (1 - p_0)\underline{q}$.

Before we go to the solution for the problem in (1.3), we look at the complete information case for a benchmark for comparison.

1.2.1 First-best case

A common benchmark for the moral hazard is to consider the *first-best* case where the principal can perfectly observes, and therefore directly contracts on, the effort of the agent. Let e the effort chosen by the principal and $(\bar{m}, \underline{m})$ be the compensation given to the agent when the output is $\bar{q}$ and $\underline{q}$ respectively. The first-best problem is written as

$$V_{FB} = \max_{e \in \{0,1\} (\bar{m}, \underline{m}) \in \mathbb{R}^2} p_e(\bar{q} - \bar{m}) + (1 - p_e)(\underline{q} - \underline{m})$$

s.t.

$$p_e u(\bar{m}) + (1 - p_e) u(\underline{m}) - c_e \geq 0.$$

It is easy to see that the constraint will be binding by the monotonicity of u.

To solve the complete information problem when the principal chooses to induce effort, i.e. $e = 1$, we let λ be the Lagrange multiplier of the constraint, and the first-order condition gives

$$-p_1 + \lambda p_1 u'(\bar{m}) = 0$$
$$(1 - p_1) + \lambda(1 - p_1) u'(\underline{m}) = 0,$$

which implies that $u'(\bar{m}) = u'(\underline{m}) = 1/\lambda$. This means that the payment to the agent does not depend on the production output level, i.e. the agent obtains *full insurance* from the principal. This is because the agent is risk-averse, i.e. u is concave, and it is more costly for the principal to provide the same utility with a payment that varies with the production level. Together with the constraint

$$p_1 u(\bar{m}) + (1 - p_1) u(\underline{m}) = c_1,$$

we have $\bar{m} = \underline{m} = u^{-1}(c_1)$. If the principal does not induce effort, he pays $\bar{m} = \underline{m} = u^{-1}(c_0) = 0$ to the agent.

The principal gets

$$p_e \bar{q} + (1 - p_e)\underline{q} - u^{-1}(c_e)$$

if she chooses $e \in \{0, 1\}$ for the agent. Therefore, the principal will choose to induce effort if and only if

$$(p_1 - p_0)(\bar{q} - \underline{q}) \geq u^{-1}(c_1), \tag{1.4}$$

i.e. if and only if the expected gain from the effort is not less than the first-best cost of inducing effort.

1.2.2 Second-best case and inefficiency from risk-aversion

Going back to the moral hazard case, the solution to the problem in (1.3) can be obtained by rewriting the optimization problem as a convex one and applying the Karush-Kuhn-Tucker (KKT) conditions. The details can be found in Laffont and Martimort (2002, Section 4.4.1) and we simply state the solution here:

$$\bar{m} = u^{-1}\left(c_1 + (1-p_1)\frac{c_1}{p_1-p_0}\right) \underline{m} = u^{-1}\left(c_1 - p_1\frac{c_1}{p_1-p_0}\right). \tag{1.5}$$

We call this the *second-best* solution.

Compared to the first-best case, where the agent is paid $u^{-1}(c_1)$ regardless of the production output level, in this case with moral hazard, the agent is rewarded (by getting paid more than $u^{-1}(c_1)$) when the output is high, and is punished when the output is low. By Jensen's inequality and the concavity of u, the expected payment to the agent is

$$\begin{aligned} &p_1 u^{-1}\left(c_1 + (1-p_1)\frac{c_1}{p_1-p_0}\right) + (1-p_1)u^{-1}\left(c_1 - p_1\frac{c_1}{p_1-p_0}\right) \\ &\quad \geq u^{-1}\left(p_1\left(c_1 + (1-p_1)\frac{c_1}{p_1-p_0}\right) + (1-p_1)\left(c_1 - p_1\frac{c_1}{p_1-p_0}\right)\right) \\ &\quad = u^{-1}(c_1). \end{aligned} \tag{1.6}$$

This implies that the principal is paying a *risk premium* to the risk-averse agent compared to the first-best case, to compensate him for bearing the risk.

Regarding the optimal effort level, the principal will choose to induce effort if and only if

$$\begin{aligned} &(p_1-p_0)(\bar{q}-\underline{q}) \\ &\quad \geq p_1 u^{-1}\left(c_1 + (1-p_1)\frac{c_1}{p_1-p_0}\right) + (1-p_1)u^{-1}\left(c_1 - p_1\frac{c_1}{p_1-p_0}\right). \end{aligned}$$

Noting (1.6) again and comparing to condition (1.4), for which the principal induces effort under complete information, we see that the first-best effort is always at least as good as the second-best effort. If the agent is risk-neutral, i.e. u is linear, then the two conditions coincide. Otherwise, there would be cases where the principal would prefer not to induce effort under moral hazard while the first-best solution is to induce effort. This shows the inefficiency caused by moral hazard and the risk-aversion of the agent.

1.2.3 Limited liability

In the second-best solution in (1.5), the agent's punishment in case the production level is low could be negative, meaning that the principal would fine the agent in that case. In reality, the agent may have a *limited liability* that prohibits the principal to fine him, or at least limits the amount that he could be fined to some $l > 0$. In such case, we would have additional constraints

$$\bar{m} \geq -l, \quad \underline{m} \geq -l.$$

This limited liability creates inefficiency in the solution even when the agent is risk-neutral. A detailed analysis can be found in Laffont and Martimort (2002), but the main idea we learn here is that limited liability may cause efficiency and is interesting to consider in models.

1.3 Continuous-time principal-agent problems

1.3.1 Single-period, multi-period and continuous-time models

The moral hazard example in Section 1.2 has two possible levels of production and a binary effort, but a model could become much more complex to reflect real-world settings. Both the possible levels of production and that of effort can become more than two, or even to be picked from a continuum. For example, Grossman and Hart (1983) posed a single-period principal-agent problem with finite number of possible production levels, where the agent can chooses his effort from a compact subset of a finite dimensional Euclidean space.

An additional dimension that we have not yet considered is the time horizon of a contract. It is possible that an agent works for the principal for multiple periods and the possibility of a long-term contract may be considered. In particular, in a long-term contract, the compensation in a certain period could depend on the production level of the current periods as well as the historical production levels. Radner (1985) studies a repeated principal-agent games with discounting, which is extend the one-period problem to multiple time periods. Fudenberg et al. (1990) considered a multi-period principal-agent model and discussed the advantages of long-term contracts over short-term contracts. They gave conditions for which efficient long-term contracts could be implemented as a sequence of short-term contracts when the agent can save and borrow. These multi-period studies are considered the study of dynamic contracting in discrete time.

1.3.2 Literature review in continuous-time models

As the frequency of compensation and measurement of output level increases, the situation would become closer and closer to measuring the output and paying the agent *continuously*. This gives rise to the idea of continuous-time principal-agent problems. In continuous-time principal-agent problems, as its name suggests, time takes value from a continuum instead of discrete periods.

On one hand, this moves the long-term contracting problem to the realm of optimization problems over stochastic processes instead of finite-dimensional vectors and makes the problem more mathematically challenging. On the other hand, continuous-time principal-agent framework has an advantage over discrete-time multi-period ones in that there are well-developed mathematical techniques in stochastic analysis and control that allow us to characterize optimal contracts by differential equations.

The theoretical literature in continuous-time principal-agent problem started with the seminal paper of Holmstrom and Milgrom (1987) (HM hereafter). In the paper, they proposed a continuous-time principal-agent problem with moral hazard, first motivated by an approximation of a discrete-time multi-period model. In their model, both the principal and the agent have exponential utilities and it was found that the optimal contract is linear. In the following, we use the HM model to illustrate the common components of continuous-time principal-agent problems.

A key part of the principal-agent problem is the relationship between the output and the controls of the agent. In HM, the dynamics of the output process is

$$dX_t = u_t dt + dB_t,$$

where the noise B_t is a standard n-dimensional Brownian motion, and the drift of the process u_t, is controlled by the agent. The process u_t could depend on the path of X up to time t. The control u_t is usually interpreted as the agent's action or effort. In more general models, the drift could be a more function of the control u_t and the path of X up to time t. The time horizon of the problem is the unit time interval, i.e. $t \in [0, 1]$.

In order to exert the control, the agent would incur a cost for the control u_t, the rate of which is given by a function $c(u_t)$. The agent is then compensated for his effort.

The difference in a risk-sharing and a moral hazard problem lies in the information asymmetry. In the risk-sharing setting, the principal can observe the control u_t of the agent, making the control contractible and allowing the principal to choose explicitly what control the agent use. In the moral hazard case, i.e. the case of HM, the principal cannot observe what u_t is, and can only specify a contract that pays depending on the output process X.

In HM's model, the agent receives $s(X_\cdot|_{[0,1]})$ at time 1. The agent has an exponential utility function

$$U_A(y) = -\exp(-ry), \quad r > 0$$

which gives him expected payoff

$$E\left[U_A(s(X_\cdot|_{[0,1]}) - \int_0^1 c(u_t)dt\right].$$

The principal also has an exponential utility function

$$U_P(y) = -\exp(-Ry), \quad R > 0$$

that gives her expected payoff

$$E\left[U_P(\sum_{i=1}^{n} X_1^i - s(X_\cdot|_{[0,1]}))\right].$$

The principal's problem involves choosing the recommended action $\{u_t\}_{0\leq t\leq 1}$ for the agent and a incentive-compatible sharing rule s to maximize her expected payoffs. HM showed that the solution corresponds to the solution of a static problem where the principal is constrained to choose the sharing rule to be a linear function of the $Z_i(1)$'s and the agent can choose a constant μ once and for all at time zero. In other words, they showed the optimality of a linear contract in the setting.

1.3.3 Risk-sharing

The setting with complete information, i.e. the principal and the agent share the same information, is called risk-sharing in the literature. This is relevant because such models usually serve as a benchmark for understanding and comparing the moral hazard models. Risk-sharing models are also directly applicable, for example in the case of portfolio management (see Cvitanič et al., 2006, Example 2.1).

Following Holmstrom and Milgrom (1987)'s moral hazard model, Müller (1998) considered the first-best case with no hidden action. The model has exponential utilities and the optimality contract was also linear. Another paper of his (Müller, 2000) showed how this can be approximated by revising the control at discrete times. Duffie et al. (1994) considered multiple-agent continuous-time setting with stochastic differential utility and characterized the Pareto-efficient allocations. Dumas et al. (2000) studied efficient allocations with multiple agents who have recursive, non-time-additive utility functions.

Cadenillas et al. (2007) considers a first-best problem with control affecting both the drift and volatility of the output process linearly. The work was then followed up by Cvitanič et al. (2006), which considers a similar problem with more general dynamics in the output process. The agent is compensated at a terminal time. The controlled output process has the dynamics

$$dX_t = f(t, X_t, u_t, v_t)dt + v_t dZ_t, \tag{1.7}$$

where $\{Z_t\}_{t\geq 0}$ is a d-dimensional Brownian Motion on a probability space $(\Omega, \mathcal{F}, P)$ and $\mathbf{F} := \{\mathcal{F}_t\}_{t\leq T}$ is its augmented filtration on the interval $[0, T]$. (X, u, v) take values in $\mathbb{R} \times \mathbb{R}^m \times \mathbb{R}^d$, and f is a function taking values in $\mathbb{R}$, possibly random and is $\mathbf{F}$-adapted.

Their model for the dynamics is quite general and the goal is to determine the compensation $C_T = F(\omega, X)$ given to the agent at time T. The paper uses the stochastic maximum principle for analysis and gives necessary conditions for optimality. They found that the optimal contract is proportional to the difference between the terminal output value and some stochastic benchmark, thus justifying the use of linear contracts in practical cases.

1.3.4 Moral hazard

The moral hazard problem, as we have already seen in Section 1.2, describes the scenario when the actual effort of the agent cannot be observed. In practice, this is due to the principal's lack of expertise or a high cost in monitoring the actual effort. The information asymmetry lies in the principal's inability to observe the agent's actions.

As already mentioned, Holmstrom and Milgrom (1987) posed a contracting problem with moral hazard in continuous-time with exponential utilities and the optimal contract is found to be linear. The model was later extended by Schättler and Sung (1993) using martingale methods from stochastic control, still having exponential utilities.

More recent literature in continuous-time principal-agent problems varies in the generality of models and the methodologies are roughly divided into those using the stochastic maximum principle, and the others using the dynamic programming and the Hamilton-Jacobi-Bellman (HJB) equation. General models with a lump-sum payment have been studied in Cvitanič et al. (2009, 2008), Cadenillas et al. (2007), Cvitanič and Zhang (2007).

The study in Cvitanič et al. (2009) gives a general model for moral hazard problem where the agent is compensated by a lump-sum payment at the terminal time. Their model allows the noise of the output process to be driven by a one-dimensional Brownian motion, and allows more general utility functions and cost functions. Mathematically, let $\{Z_t\}_{t\geq 0}$ be a standard Brownian Motion on a probability space $(\Omega, \mathcal{F}, P)$ and $\mathbf{F}^W = \{\mathcal{F}_t^W\}_{t\leq T}$ be its augmented filtration on the interval $[0, T]$. The dynamics of the output process X is then given by

$$dX_t = u_t v_t dt + v_t dZ_t,$$

where u and v are $\mathbf{F}^W$-adapted processes that may be controlled by the agent. For example, in the portfolio management example, u_t and v_t could represent the excess return and volatility of the portfolio respectively.

The information asymmetry lies in that the principal cannot observe u or the underlying Brownian motion Z. The agent is compensated by $C_T = F(X.)$ at the terminal time T. This is a lump sum payment, as opposed to the alternative of having a continuous stream of payments, or a combination of both. (In Section 1.4, we will look at another model that incorporates continuous compensation.)

Since v can be verified by the principal, a contract is given in the form of (F, v). The agent chooses the control u to maximize his expected utility

$$E[U_A(F(X^{u,v}), G_T^{u,v})]$$

where the accumulated cost of the agent is

$$G_t^{u,v} = \int_0^t g(s, X_s, u_s, v_s) ds.$$

The principal maximizes her utility

$$E[U_P(X_T^{u,v} - F(X.))]$$

over all implementable contracts (F, v) and corresponding agent's optimal efforts u, such that the participation constraint holds:

$$V_1(F, v) \geq R.$$

With certain smoothness assumptions and concave utility functions, the study in Cvitanič et al. (2009) gives the necessary conditions for optimality of the problem via the stochastic maximum principle. They also solve fully the case of quadratic cost and separable utility functions, giving the optimal contract as the solution of a deterministic equation and the optimal effort of the agent as a linear backward stochastic differential equation (BSDE).

Along the same line of research is the study of Cvitanič et al. (2008), which consider the second-best problem with exit options, where the payment time is a stopping time instead of a fixed time.

There are also models where compensation is not limited to a lump-sum payment at the terminal time. The model in Williams (2009) studies compensation with both continuous consumption and a terminal compensation with drift control. On the other hand, a particularly interesting model is that of Sannikov (2008) (henceforth simply referred as "Sannikov"), which considers an infinite-horizon moral hazard problem with a stream of consumption as compensation and effort possibly varying over time and taking values from a continuum. This model will be a main focus of my research and the Section 1.4 reviews Sannikov's model and results. Cvitanič et al. (2013) extended Sannikov's model by considering a principal-agent problem with both moral hazard and adverse selection.

Williams (2009) considered a dynamic moral hazard models with hidden actions and possibly hidden states. DeMarzo and Sannikov (2006) considered a similar optimal contracting problem with binary action and a no-shirking condition. The work is followed by Zhu (2013) which relaxed the no-shirking condition, resulting in contracts with shirking phases, both for relaxation as reward and suspension as punishment.

Other related studies include He (2009) where the size of a firm follows a geometric Brownian motion with the drift controlled by the agent, taking again two possible values, with a liquidation time in the model, and Biais et al. (2010), where a risk-neutral agent acts to reduce the likelihood of large but relatively infrequent losses, and the agent is incentivized by a contract consisting of downsizing, investment and liquidation decisions dependent on the agent's performance. There is also the work of He (2011) which addresses a model that allow the agent to have privates savings, under output process with Poisson noises. Sannikov (2013) also discusses discrete and continuous-time approaches to optimal contracts, with reference to the results in Sannikov (2008) and discussion on approximately optimal contracts and applications to corporate finance.

1.4 Sannikov's model

In this section, we describe the framework in Sannikov (2008) and his results.

The significance of Sannikov's model lies in the fact that his framework gave an almost-explicit solution while maintaining generality of the model. While some other general models (Cvitanič et al., 2006, 2009, 2008) have characterized solutions by the stochastic maximum principle and consider only lump-sum payment at the terminal time, Sannikov's model considered consumption payment over an infinite horizon and characterized the optimal value function as a solution to an ordinary differential equation with a smooth-pasting condition. There are also other studies (DeMarzo and Sannikov, 2006, Zhu, 2013) that employ a framework and methodology similar to Sannikov's model to study models with more specific assumptions. In comparison, Sannikov's model allows for flexibility in the cost and utility functions, requiring only mild conditions on them, and considers a consumption process as compensation instead of a lump-sum payment. It also allows the set of possible effort to be a continuum instead of restricting to binary action. There is also technical significance in his work, as he gave an almost-explicit solution to a highly non-trivial control problem. The control problem for his principal-agent problem is essentially a two-stage optimization problem (carried out by the principal and the agent) and he reduces it to a single optimization problem with the incentive-compatibility constraint, which is still a highly non-trivial problem to solve.

In the following, we describe Sannikov's framework. Afterwards, we will point out what is missing in the model and why it is worth further investigation.

1.4.1 The model

The benchmark model in Sannikov's paper considers a setting where a risk-averse agent works for a risk-neutral principal for an infinite time horizon. The principal and the agent discount future utility at a common rate r. The agent chooses action $\{A_t\}_{t\geq 0}$, which affects the output $\{X_t\}_{t\geq 0}$ for the principal with dynamics

$$dX_t = A_t dt + \sigma dZ_t, \tag{1.8}$$

where $Z = \{Z_t, \mathcal{F}_t;\ 0 \leq t < \infty\}$ is a standard Brownian Motion that drives the noise in the output process. The agent's choice of A_t is understood in a weak sense, i.e. the agent chooses the distribution of the output in the sense of selecting the drift process A_t, with which Girsanov theorem is applied for a change of measure to $\mathbf{P}^A$. We show more details in Section 3.1 when we introduce the formulation of the problem studied in this thesis.

The action $\{A_t\}_{t\geq 0}$ to be chosen by the agent is progressively measurable with respect to the filtration $\mathcal{F}_t = \sigma(X_s, 0 \leq s \leq t)$ [1]. For action $\{A_t\}_{t\geq 0}$, the agent incurs a cost of effort at rate $h(A_t)$ at each time $t \geq 0$, in the same unit as his utility of consumption. The set $\mathcal{A}$ is compact with smallest element 0. The function $h : \mathcal{A} \to \mathbb{R}$ is assumed to be continuous, increasing and convex. It is also normalized so that $h(0) = 0$ and we assume that there is $\gamma_0 > 0$ such that $h(a) \geq \gamma_0 a$ for all $a \in \mathcal{A}$.

The principal compensates the agent by a stream of consumption $\{C_t\}$. The consumption needs to be nonnegative, and thus C_t takes value in $[0, \infty)$. This reflects the limited liability of the agent. The value of C_t can depend on the path of $\{X_s\}$ up to time t. The agent cannot save and his utility function $u : [0, \infty) \to [0, \infty)$ is assumed to be increasing and concave. $u(0)$ is normalized to zero and $u \in C^2$. Moreover, it is assumed that $u'(c) \to 0$ as $c \to \infty$. This assumption creates the *income effect* that causes permanent retirement in the optimal case (see Sannikov, 2008, P.965).

[1] The action process can intuitively be chosen to be progressively measurable with respect to $\{\mathcal{F}_t^Z\}$. However, in order to use the weak formulation, where the choice of effort is understood as a choice of distribution of the output, we need to invoke the Girsanov theorem, which requires the drift to be progressively measurable with respect to $\{\mathcal{F}_t^X\}$. This agrees with the setting and methodology used in Sannikov's proofs in the appendix of Sannikov (2008). The remarks in (Cvitanič et al., 2008, Remark 2.3) discuss some difference between the strong and weak formulation

The agent seeks to maximize

$$E^A \left[\int_0^\infty e^{-rt}(u(C_t) - h(A_t))dt \right],$$

over all possible effort A, where E^A denotes expectation under the probability measure $\mathbf{P}^A$. The principal is assumed to be risk-neutral, and so he maximizes

$$E^A \left[\int_0^\infty e^{-rt} dX_t - \int_0^\infty e^{-rt} C_t dt \right] = E^A \left[\int_0^\infty e^{-rt}(A_t - C_t)dt \right],$$

over all possible compensation contracts, with A being chosen by the agent.

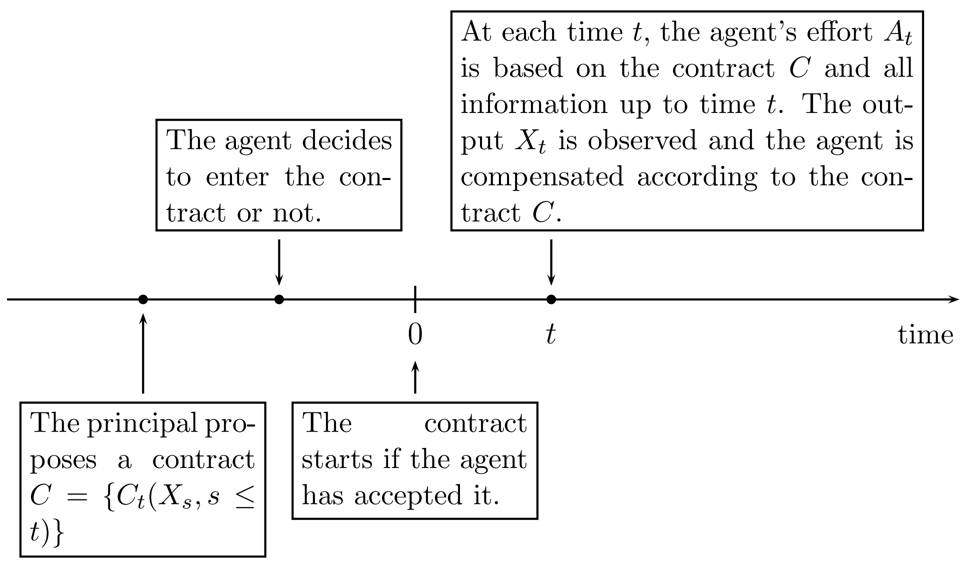

Figure 1.2: Sequence of events

The sequence of events are as follows: The principal first determines a contract $C = \{C_t(X_s, s \leq t)\}$ and proposes it to the agent. The agent then decides whether he should enter the contract or not. If the agent accepts the contract, then it starts at time 0. At each time $t > 0$, the agent's effort A_t could be based on his information up to time t as well as the contract C, i.e. how the compensation depends on the output. The output X_t is observed and the agent is compensated according to the contract C. This sequence of events is depicted in Figure 1.2. Note that since the principal commits to her contract and compensates the agent according to the contract and the output path X, she makes no further decision after the contract starts. The agent's choice of A_t at each time t can therefore be viewed as a one-off decision to choose a $\{\mathcal{F}_t\}$-progressively measurable process A.

1.4.2 The compensation and recommended effort pair

Given the principal-agent setting, the intuitive formulation is that the principal will offer a compensation at time t depends solely on the path of X up to time t, i.e. historical output up to time t. In other words, the compensation process C is progressively measurable with respect to $\{\mathcal{F}_t^X\}$. For any such compensation contract, the agent chooses some $\{F_t^X\}$-progressively measurable process A that maximizes his expected utility. The principal does not observe the choice of A directly. However, since the cost function and utility function are common knowledge, the principal knows the set of all A that is optimal for the agent given this compensation contract. In cases where the process A is not uniquely determined, we assume that we can choose arbitrarily one such A. This is due to the common assumption in the literature that the agent will choose what the principal prefers if there is more than one optimal choice for the agent, since he will have no incentives to deviate from that choice.

We summarize the idea as follows:

Formulation 1: Compensation based on output. The principal first chooses and announces a contract that specifies the compensation depending on the historical output path, i.e. the compensation C is progressively measurable with respect to $\{\mathcal{F}_t^X\}$. Knowing this contract, the agent chooses his action A which maximizes his expected payoff. The process A is restricted to be progressively measurable with respect to $\{\mathcal{F}_t^X\}$.

This formulation is cast as a two-step sequential game. Sequential games of simpler payoffs are frequently tackled by the means of backward induction. However, in this case, this leads to a two-stage optimization problem, both stages of which involves maximizing over a set of processes.

Sannikov elegantly transformed this two-stage problem into a single control problem that seeks to determine an optimal pair of compensation and recommended effort, denoted by (C, A). The first step involves adding in the recommended effort into the optimization problem. This is common in the principal-agent literature and, for example, has also appeared in the discrete-time example in Section 1.2.

Mathematically, the compensation process $C = \{C_t\}_{t\geq 0}$ is progressively measurable with respect to $\{\mathcal{F}_t^X\}$. It specifies how the agent is compensated based on the realized output process, and applies whether or not the agent is following the recommended effort. The recommended effort $A = \{A_t\}_{t\geq 0}$, on the other hand, is the effort process that the principal recommends the agent to choose. A_t is restricted to be progressively measurable with respect to $\{\mathcal{F}_t^X\}$.

In words, we allow the principal to specify a *recommended effort* along with the compensation contract, with the additional constraint that the pair must be *incentive-compatible*, i.e. the recommended effort process maximizes the agent's payoffs under the compensation contract and so the agent has the incentives to follow the recommended effort. It is important to note that the compensation C in the pair, being progressively measurable with respect to $\{\mathcal{F}_t^X\}$, not only prescribes the compensation when the agent follows the recommended effort, but it also prescribes how the agent is compensated when the agent deviates from the recommended effort. It is only then that we can consider the incentive-compatibility. The incentive-compatibility of the pair means that the agent will be no better if he chooses an alternative action.

Since the agent has the freedom to chooses A_t in reality, and the set of compensation-recommended-effort (C, A) pair that the principal can choose is restricted by the incentive-compatibility constraint

$$E^A\left[\int_0^\infty e^{-rt}(u(C_t) - h(A_t))dt\right] \geq E^{A'}\left[\int_0^\infty e^{-rt}(u(C_t) - h(A'_t))dt\right]$$

for any other effort $\{A'_t\}$. If this constraint holds, then the agent will indeed be willing to follow the principal's recommended effort.

To ensure that the agent will accept the contract in the first place, we need an individual rationality constraint and give the agent, in expectation, at least his reservation utility w_0. Mathematically, it is easier to consider the problem with exactly w_0 given to the agent rather than at least w_0.

This can be summarized as follows:

Formulation 2: Compensation along with recommended effort. The principal chooses and announces a pair of "incentive-compatible" compensation and recommended effort (C, A). Both the compensation C and recommended effort A are progressively measurable with respect to $\{\mathcal{F}_t^X\}$. The pair (C, A) must satisfy the incentive-compatibility constraint (1.9c) and individual rationality constraint (1.9b).

The principal therefore has an optimization problem of the form

$$\sup E^A \left[\int_0^\infty e^{-rt}(A_t - C_t)dt \right] \tag{1.9a}$$

over all (C, A) such that both C and A are progressively measurable with respect to $\{\mathcal{F}_t^X\}$ and

$$E^A \left[\int_0^\infty e^{-rt}(u(C_t) - h(A_t))dt \right] = w_0 \tag{1.9b}$$

$$E^A \left[\int_0^\infty e^{-rt}(u(C_t) - h(A_t))dt \right] \geq E^{A'} \left[\int_0^\infty e^{-rt}(u(C_t) - h(A'_t))dt \right] \tag{1.9c}$$

for any other effort $\{A'_t\}$.

We remark that in the actual process of the principal proposing the contract, announcing C alone is sufficient, unless there is a tie in the agent's choice of effort, so that at least two processes A and A' are both optimal for the agent and the principal wishes to specify which one she prefers. However, in formulating the optimization problem for the principal, since it is needed to predict what the agent will choose, the effort process A necessarily enters the formulation. The freedom of the agent in choosing A is then reflected in the formulation by the incentive-compatibility constraint, which mandates that the principal chooses only among the (C, A) pairs such that the recommended effort must be optimal from the agent's perspective given the compensation contract. Therefore, while in the formulation above it seems that the principal can optimize over various possible effort processes A at her will, the incentive-compatibility constraint forbids her to choose any effort process that, when paired with the choice of C, is against the interest of the agent. This explains why the effort A enters the formulation of the optimization problem even though it cannot be contracted nor observed.

1.4.3 The continuation value and incentive-compatibility

The optimization we formulated in (1.9) is not easy to solve. The difficulty lies in that the expectations in both the objective function and the constraints are taken under a probability measure that depends on A, part of what we are optimizing over. Sannikov's work is remarkable in that he reduced the problem to a stochastic control problem with the dynamics of the continuation value of the agent. The resulting stochastic control problem resembles standard stochastic control problems (see Section 2). In particular, he rewrote the incentive-compatibility constraint by an additional process Y_t.

To gain an understanding of the problem reduction Sannikov did, we will review some of his definitions and results, along with explanation of the motivation and arguments behind them.

In Sannikov's paper, the continuation value of the agent at any time t given a contract (C, A) refers to the agent's expected future discounted payoffs from the contract if he follows the recommended effort A. It reflects the agent's valuation of his average future payoff at time t. Mathematically, given the compensation-recommended-effort pair (C, A), The continuation value of the agent is defined to be

$$W_t(C, A) = E^A \left[\int_t^\infty e^{-r(s-t)} (u(C_s) - h(A_s)) ds | \mathcal{F}_t \right], \tag{1.10}$$

which represents the expected payoff for the agent given information at time t and discounted to time t. In particular, $W_0 = w_0$ represents the initial continuation value for the agent.

With any continuation value w at some time t, one way to give this value to the agent is to *permanently retire* the agent from time t, i.e. to give the agent a constant consumption of c such that $u(c) = w$ and allow him to exert zero effort. This gives the *retirement profit* $F_0(w) = -u^{-1}(w)$.

Sannikov's definition and use of the agent's continuation value should have been motivated by similar notions of promised future expected utility in discrete-time models (see Spear and Srivastava, 1987). The continuation value is a key variable in the problem because it is all that the agent cares about. In fact, it is shown by Sannikov that under some conditions the optimal compensation and effort are Markovian in the continuation value process.

At this point, while $W_t(C, A)$ is well-defined, it is not immediately clear how it changes over time with the output process X. The first important technical result from Sannikov, therefore, involves a representation of $W_t(C, A)$ to illustrate its dynamics. The proofs for Propositions 1.1 and 1.2 are in the appendix of Sannikov (2008) and are restated in our Appendix A for reader's reference, as these two propositions play an important role to understanding the subject of this thesis.

Proposition 1.1 (Representation)**.** *There exists a progressively measurable process* $Y = \{Y_t, \mathcal{F}_t; 0 \leq t < \infty\}$ *in* $\mathcal{L}^*$ [2] *such that*

$$W_t(C,A) = W_0(C,A) + \int_0^t r(W_s(C,A) - u(C_s) + h(A_s))ds + \int_0^t r\sigma Y_s(dX_s - A_s ds) \tag{1.11}$$

for every $t \in [0, \infty)$.

There are a few points to note. First, $Y = Y(C,A)$ depends on (C,A) and it represents the sensitivity of the agent's continuation value to the output. Secondly, if the agent follows the recommended effort A, then $dX_s - A_s ds = dZ_s$ and so the last integral is done with respect to dZ_s. This suggests the consideration of $\{W_t\}$ as a solution to a stochastic differential equation of the form (1.11) but driven by a Brownian motion Z_t, i.e.

$$dW_t = r(W_t - u(C_t) + h(A_t))dt + r\sigma Y_t dZ_t. \tag{1.12}$$

This is indeed what was done in his paper. Instead of considering all pairs of (C,A) processes that are progressively measurable with respect to $\{\mathcal{F}_t^X\}$ and satisfy constraints (1.9b) and (1.9c), Sannikov recast the incentive-compatibility constraint and made Y a decision variable that we can optimize over as well, subject to a simpler constraint that is due to the following proposition.

This proposition characterizes the incentive compatibility of a given contract (C,A).

Proposition 1.2 (Incentive compatibility)**.** *For a given strategy* A*, let* Y_t *be the process from the Proposition 1.1 that represents* $W_t(C,A)$*. Then* A *is optimal (for the agent) if and only if*

$$\forall a \in \mathcal{A}, \quad Y_t A_t - h(A_t) \geq Y_t a - h(a), \quad 0 \leq t < \infty \tag{1.13}$$

almost everywhere.

The elegance of Proposition 1.2 is that for (C,A) to be incentive-compatible, the necessary and sufficient condition that the corresponding process $Y = Y(C,A)$ need to satisfy at each time t involves only quantities at time t. This is called the one-shot deviation principle in Sannikov (2013).

[2] A process Y_t is in $\mathcal{L}^*$ if $E\left[\int_0^t Y_s^2 ds\right] < \infty$ for $t \in (0, \infty)$

Sannikov also argued that it would be optimal for the principal to choose contracts such that $Y_t(C, A) = \gamma(A_t)$. The argument is an economic one. The process Y_t can be understood as the sensitivity of the agent's continuation value to the output process, and therefore carries risk to the agent. Since the agent is risk-averse, as characterized by his concave utility function, it is costly for the risk-neutral principal to impose risk on the agent. In other words, it is more costly for the principal to give the agent the same expected utility if the payoffs were risky. From the principal's perspective, therefore, she should avoid imposing unnecessary risk to the agent and use only the minimum risk level that is sufficient to induce the desired effort from the agent. This suggests that the principal should set Y_t to be minimum level that induces the effort, i.e. $\gamma(A_t)$.

1.4.4 Sannikov's HJB equation and his results

After working on the continuation value and the condition for incentive-compatibility, Sannikov proceeded to give the HJB equation that turns out to characterize the optimal solution.

He first stated an HJB equation in the form

$$rF(W) = \max_{a>0,c} r(a-c)+F'(W)r(W-u(c)+h(a))+\frac{F''(W)}{2}r^2\gamma(a)^2\sigma^2, \quad (1.14)$$

but mostly worked with the rewritten equation in the form of

$$F''(W) = \min_{a>0,c} \frac{F(W) - a + c - F'(W)(W - u(c) + h(a))}{r\gamma(a)^2\sigma^2/2}, \quad (1.15)$$

which he described as "suitable for computation". His existence and uniqueness results were established based on the second form (1.15) rather than the first one (1.14).

The additional boundary and smooth-pasting conditions that he used to compute the optimal contract are

$$F(0) = 0, F(w_{gp}) = F_0(w_{gp}) \text{ and } F'(w_{gp}) = F_0'(w_{gp}), \quad (1.16)$$

at some point $w_{gp} \geq 0$.

Understanding Sannikov's HJB equation

To understand how the HJB equation is related to the original problem, we first note that the HJB equation (1.14) corresponds to an optimal control problem of the form

$$\sup_{(C,A)} E^A \left[\int_0^\tau e^{-rt}(A_t - C_t)dt + F_0(W_\tau) \right]$$

s.t.

$$dW_t = r(W_t - u(C_t) + h(A_t))dt + r\sigma\gamma(A_t)dZ_t; \; W_0 = w_0$$

where τ is the first exit time of W_t from the interval $(0, w^*_{gp})$.

The appearance of γ in the problem is due to the economic argument that asserts $Y_t = \gamma(A_t)$. To remove the effects of that heuristic, a related problem would be an optimal stopping and control problem of the form

$$\sup_{(C,A),\tau\leq\infty} E^A \left[\int_0^{\tau\wedge\tau_{(0,w^*_{gp})}} e^{-rt}(A_t - C_t)dt + F_0(W_{\tau\wedge\tau_{(0,w^*_{gp})}}) \right] \tag{1.17a}$$

s.t.

$$Y_t A_t - h(A_t) \geq Y_t a - h(a), \quad \forall a \in \mathcal{A}, \; 0 \leq t < \infty \tag{1.17b}$$

$$dW_t = r(W_t - u(C_t) + h(A_t))dt + r\sigma Y_t dZ_t; \; W_0 = w_0 \tag{1.17c}$$

where $\tau_{(0,w^*_{gp})}$ is the first exit time of W_t from the interval $(0, w^*_{gp})$.

There are two major conceptual changes from the previous problem in (1.9) to this reduced problem. First, we now consider W_t as the controlled process driven by a Brownian motion Z_t. The individual rationality constraint becomes embedded in the initial value of W_t. Secondly, the incentive-compatibility constraint has transformed into a restriction on Y_t, the sensitivity of the agent's continuation value to the output, which has now become part of the control as well.

How is a solution (C, A) of the original problem (1.9) related to the new problem? Suppose (C, A) is a feasible solution of (1.9). By Proposition 1.1, there is some $Y(C, A) = \{Y_t(C, A)\}$ as the representation of the martingale such that the dynamics of W_t as stated in (1.11) holds for the continuation value $W_t = W_t(C, A)$. We write $Y_t = Y_t(C, A)$.

Now, the individual rationality constraint (1.9b) implies that $W_0 = w_0$. On the other hand, the incentive-compatibility condition in its original form

(1.9c), as asserted in Proposition 1.2, ensures that the one-shot incentive-compatibility condition (1.17b) holds. Moreover, $X_t - \int_0^t A_s ds =: Z_t^A$ is a Brownian motion under $\mathbf{P}^A$. This almost gives a feasible solution to the new problem (1.17), except for the difference in the filtration that these processes are based on. If (C, A, Y) are also progressively measurable with respect to $\{\mathcal{F}_t^{Z^A}\}$, then we know that $(\Omega, \mathcal{F}, \mathbf{P}^A, Z^A, (C, A, Y))$ with $\tau = \infty$, where $\Omega = C[0, \infty)$ and $\mathcal{F} = \lim_{t\uparrow\infty} \mathcal{F}_t^{Z^A}$ is a weak solution to the problem (1.17).

If, furthermore, such (C, A) invokes permanent retirement upon the continuation value hitting a particular point w_{gp} or 0, then the objective value evaluated for (C, A) in (1.9a) will also be the same as that evaluated in (1.17a) for (C, A, Y) and w_{gp}.

On the other hand, more importantly, when can we build a solution to the original problem (1.9) if we have a solution to problem (1.17)? Given a solution (C, A, Y, τ) of the problem (1.17), we can define

$$\tilde{C}_t = \begin{cases} C_t & t \leq \tau \wedge \tau_{(0,w_{gp})} \\ -F_0(W_{\tau\wedge\tau_{(0,w_{gp})}}) & \text{otherwise} \end{cases}$$

$$A_t = \begin{cases} A_t & t \leq \tau \wedge \tau_{(0,w_{gp})} \\ 0 & \text{otherwise} \end{cases}$$

and we can show that $W_t(C, A) = W_t$, and $Y_t(C, A) = Y_t$ by the uniqueness of SDE (1.12) and the uniqueness of Y in Proposition 1.1. However, such (C, A) is progressively measurable with respect to $\{\mathcal{F}_t^Z\}$, and is not necessarily progressively measurable with respect to $\{\mathcal{F}_t^X\}$. We would like to have some Z that is progressively measurable with respect to $\{\mathcal{F}_t^X\}$, is a Brownian motion under $\mathbf{P}^A$ and satisfies

$$dX_t = A_t dt + \sigma dZ_t.$$

The reason that this is not trivial is because A above is progressively measurable with respect to $\{\mathcal{F}_t^Z\}$ and depends on Z. As a result, the existence and uniqueness of Z is not immediately clear. However, if the optimal control is Markovian in W_t and satisfies certain Lipschitz conditions , then we know such Z exists and is unique (see Proposition 3.1). This Z is understood as a perceived noise process under recommended action A, and coincides with the true noise process observable by the agent only if the agent actually follows A. When such Z exists and is unique, then a solution in problem (1.17) will indeed leads to a solution in problem (1.9).

In this problem, the supremum is taken over the set of all (C, A, Y) that is progressively measurable with respect to $\{\mathcal{F}_t^Z\}$ instead of $\{\mathcal{F}_t^X\}$, where Z is understood as the perceived noise process $Z_t = X_t - \int_0^t A_s ds$. On one hand, this is great because the reduced problem is no longer dependent on the dynamics of X, and the controlled process therefore remains one-dimensional. On the other hand, this is creating a gap between original problem and the reduced problem, in that the control derived from the new problem, being progressively measurable with respect to $\{\mathcal{F}_t^Z\}$, may not be progressively measurable with respect to $\{\mathcal{F}_t^X\}$ and hence is not always implementable in Formulation 1.

If we further include the restriction that the principal will only use impose the minimum risk needed to the agent to induce any effort, i.e. Y_t must be equal to $\gamma(A_t)$, we get the following formulation:

Formulation 3: Compensation and recommended effort, until retirement based on continuation value. The principal chooses and announces a pair of "incentive-compatible" compensation and recommended effort (C, A) (until τ), both of which are progressively measurable with respect to $\{\mathcal{F}_t^Z\}$, as well as a high retirement point $\bar{w}_{gp}$. The continuation value process of the agent is defined according to (1.12) with $Y_t = \gamma(A_t)$. This ensures incentive-compatibility of the contract. When the continuation value hits 0 or $\bar{w}_{gp}$ at time τ, the agent is retired permanently with a consumption $-F_0(w)$ corresponding to the continuation value w at retirement.

Sannikov's result

After giving the HJB equation, Sannikov first proved existence and uniqueness of a smooth solution to equation (1.15) with a point w_{gp} as determined by the smooth-pasting condition. He then proceeded to do verification and showed that for initial value $w \in [0, w_{gp}]$, the optimal profit is equal to this solution of the HJB equation. He showed that there is always an optimal contract with permanent retirement (i.e. no temporary retirement), i.e. the contract recommends the agent to take a nonzero action until the retirement point, after which the agent always make zero effort.

It was also showed separately that, for $w > w_{gp}^*$ (where w_{gp}^* is such that $1/u'(u^{-1}(w_{gp}^*)) = 1/h'(0)$), permanent retirement with constant payment is the optimal contract for the principal with profit $F_0(W)$. Figure 1.3 illustrates Sannikov's results.

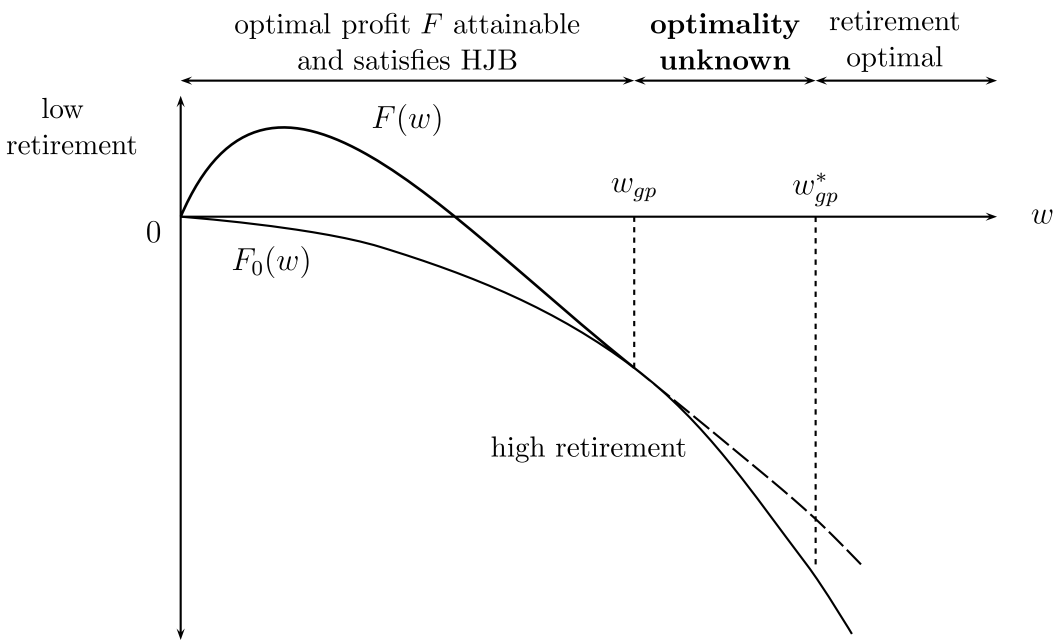

Figure 1.3: Illustration of Sannikov's result

1.4.5 Motivation for further investigation

Sannikov's work in his principal-agent problem has been significant. In particular, there is a remarkable problem reduction, as we explained earlier, from Formulation 1, to his final results, which is most closely related to Formulation 3. This was the key to his elegant representation of the optimal function as the solution of the HJB equation. However, there are several unanswered questions in his study that are worth further investigation.

First, in the last step of the problem reduction, the solution of the HJB equation does not automatically give an incentive-compatible contract that can be implemented. In particular, this requires the recommended effort derived from the optimal solution of the control problem to be progressively measurable with respect to $\{\mathcal{F}_t^X\}$, but this has not been proved in his paper.

Secondly, in terms of the optimal solution, Sannikov proved the existence and uniqueness of a solution to the HJB equation, and verified that it is the optimal value function when the continuation value w is less than the higher retirement point w_{gp}^*. However, the optimality of the solution for all possible w is incomplete. In fact, he proved optimality of his proposed contract when the continuation value is less than or equal to the high retirement point $w_{gp} < w_{gp}^*$, and the optimality of retirement when the continuation value is greater than w_{gp}^*. However, in the region where $w \in (w_{gp}, w_{gp}^*)$, the solution in Sannikov was only shown to be an upper bound and the retirement profit F_0 a lower bound. Although economic intuition suggests that retirement may be optimal in that region, he did not prove it and it is unclear whether it is indeed so. This is because that it was not clear whether the HJB inequality holds for F_0 in that region. This gives us motivation to study the model with a different methodology.

It is also worth noting that Sannikov's HJB equation as in (1.15) is different from the ones we will use (see equation (4.1)). Sannikov's HJB equation has been rewritten from (1.14) in a form "suitable for computation" (Sannikov, 2008, p.963). However, the equivalence of the two equations depends on two assumptions that were only proved in verification but not a priori. The first is the concavity of the solution, which justified the choice of $Y_t = \gamma(A_t)$. The second is the result that it is optimal to have no temporary retirement (i.e. $a > 0$ before retirement), which explains the $a > 0$ condition in the set of possible a in equation (1.15) and allowed for the rewriting from the original form with maximization to minimization since $\gamma(a)$ is nonzero for $a > 0$. These results were not proved before verification in Sannikov, and are dependent on the particular form of solution of the HJB equation. Thus Sannikov's HJB equation (1.15) is intrinsically different from the HJB equation of the control problem, i.e. the one we use.

Other than the HJB equation, the methodology used in Sannikov is also applicable only to their specific setting and cannot be extended easily upon modifications of the model. The development of the results depended on finding a smooth solution to the HJB equation, which succeeds only if the optimal value function is indeed smooth. Moreover, finding the optimal profit in the range (w_{gp}, w_{gp}^*) with this methodology is difficult, and has not been done in his work. If we do believe that retirement is optimal for the interval, the verification would amount to verifying that F_0 satisfies

$$\sup_{a \in \mathcal{A}, c \geq 0} \{r(a-c) + rF_0'(w)(w - u(c) + h(a)) + \frac{1}{2}F_0''(w)r^2y^2\sigma^2\} - rF_0(w) \leq 0$$

for all $w \in (w_{gp}, w_{gp}^*)$. This is particularly difficult because w_{gp} is implicitly determined by the smooth-pasting condition.

1.5 Outline of the thesis

Chapter 2 reviews some basics and background in stochastic control and the notion of viscosity solutions. In Chapter 3, we give the formulation of our problem and explain the problem reduction and conditions under which it is valid. In Chapter 4, we study the viscosity solution of the HJB equation of our control problem and show that the optimal profit function is the unique viscosity solution. Chapter 5 is focussed on the discussion of various additional contractual possibilities and a comparison of our results to Sannikov's results on modeling additional contractual possibilities. Chapter 6 gives a summary of our contribution and discussions on applications and future research.

Chapter 2

Review of Stochastic Control

Stochastic control involves optimizing a given objective function by controlling the dynamics of a system over time subject to some random noise. In continuous-time stochastic control, both the choice of control and the output dynamics are in the form of stochastic processes, and the random noise is often modeled by a Brownian motion.

In this chapter, we review some basics in stochastic control, which will hopefully help the reader understand more on our approach in dealing with our principal-agent problem. Yong and Zhou (1999) and Fleming and Soner (2006) give comprehensive expositions of stochastic controls with multiple-dimensional dynamics and the various approaches to tackle the problem. In this section, we focus our review on a one-dimensional stochastic control problem. - Consider the controlled stochastic differential equation

$$\begin{aligned} dX_t &= b(t, X_t, u_t)dt + \sigma(t, X_t, u_t)dZ_t \\ X_0 &= x_0 \in \mathbb{R}, \end{aligned} \tag{2.1}$$

where $b : [0, T] \times \mathbb{R} \times U \to \mathbb{R}$ is the drift of the process, $\sigma : [0, T] \times \mathbb{R} \times U \to \mathbb{R}$ is the volatility of the process and Z_t is a standard Brownian motion. The terminal time $T > 0$ is fixed. We call u the *control* since it can be chosen by the decision maker and influences the dynamics of the system. U is the set of possible values of the control u_t at each time t. As the decision maker can choose the control at time t based only on the information up to time t, the control u is restricted to be from the set

$$\mathcal{U}[0, T] := \{u : [0, T] \times \Omega \to U | u \text{ is } \{\mathcal{F}_t\}\text{-adapted}\}.$$

Any $u \in \mathcal{U}[0, T]$ is then called a feasible control.

The decision makers optimizes the choice of u to minimize cost. For the finite-horizon problem, it is usual to have a cost functional of the form

$$J(u) = E\left[\int_0^T f(t, X_t, u_t)dt + h(X_T)\right],$$

where f represents the running cost component and h is the terminal cost.

In the strong formulation of the stochastic control problem, given a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbf{P})$ our goal is to find

$$\inf_{u\in\mathcal{U}[0,T]} J(u)$$

subject to the system dynamics (2.1).

There are two common ways to solve a stochastic control problem. One approach is to obtain a set of necessary conditions for the optimal solution via the stochastic maximum principle. The stochastic maximum principle is a version of the Pontryagin's maximum principle (Pontryagin et al., 1962), and gives necessary conditions for the optimal solution of a control problem, where one is to find adapted solutions for the first-order and second-order adjoint equations, which are backward stochastic differential equations (BSDE). This approaches the strong formulation of the problem directly, and details can be found in standard references, like Yong and Zhou (1999). Many control problems have been studied using this methodology and it is especially common when the time horizon is finite. Some studies of continuous-time principal problems have also been studied using the stochastic maximum principle (see for example Cvitanič et al., 2006, Cadenillas et al., 2007, Cvitanič et al., 2008, 2009).

The other common and powerful approach to solving stochastic control problems is the method of dynamic programming. This involves considering a family of stochastic control problems with the same drift and volatility functions and cost functions, but different initial times and initial states, and then relating the optimal value function of these problems by the dynamic programming principle (DPP), from which the HJB equation is usually derived to characterize the value function. To prepare for our discussion on dynamic programming, we need to first consider the *weak formulation* of the stochastic control problem. The definition below comes from Yong and Zhou (1999).

2.1 Weak formulation

Let $T > 0$ be the terminal time. For any initial time and state $(s, y) \in [0, T) \times \mathbb{R}$, the corresponding control problem has dynamics

$$\begin{aligned} dX_t &= b(t, X_t, u_t)dt + \sigma(t, X_t, u_t)dZ_t \quad t \in [s, T], \\ X_s &= y \end{aligned} \tag{2.2}$$

with the cost functional

$$J(s, y; u) = E\left[\int_s^T f(t, x_t, u_t)dt + h(x_T).\right] \tag{2.3}$$

Then for fixed $s \in [0, T)$, the feasible control is denoted by the $\mathcal{U}^w[s, T]$ and consists of all 5-tuples $(\Omega, \mathcal{F}, \mathbf{P}, Z, u)$ such that

1. $(\Omega, \mathcal{F}, \mathbf{P})$ is a complete probability space.
2. $\{Z_t\}_{t \geq s}$ is a standard Brownian motion defined on $(\Omega, \mathcal{F}, \mathbf{P})$ over $[s, T]$, with $W_s = 0$ a.s., and the filtration $\mathcal{F}_t^s = \sigma\{Z_r, s \leq r \leq t\}$ augmented by all the $\mathbf{P}$-null sets in $\mathcal{F}$.
3. $u : [s, T] \times \Omega \to \mathbb{R} \to U$ is an $\{\mathcal{F}_t^s\}_{t \geq s}$-adapted process on $(\Omega, \mathcal{F}, \mathbf{P})$.
4. Under u, for any $y \in \mathbf{R}$ equation (2.2) admits a unique solution on $(\Omega, \mathcal{F}, \{\mathcal{F}_t^s\}_{t \geq s})$. Denote this solution by $\{X_t(u, s, y)\}_{t \geq s}$.
5. $f(., X, u)$ is $\{\mathcal{F}^s\}_{t \geq 0}$-adapted with

   $$E \int_0^T |f(t, X_t, u_t)|dt < \infty$$

   and $h(x_T)$ is $\{\mathcal{F}_T^s\}$-measurable with

   $$E|h(X_T)| < \infty.$$

The 5-tuple is called (weakly) admissible for the control problem. When the probability space $(\Omega, \mathcal{F}, \mathbf{P})$ is clear, we just say u is (weakly) admissible. The goal of the stochastic control problem with parameter (s, y) is to minimize the cost (2.3) over all admissible 5-tuples $(\Omega, \mathcal{F}, \mathbf{P}, Z, u)$.

2.2 Dynamic programming principle and the HJB equation

Now we look at the dynamic programming principle (DPP), which relates the value function of the family of stochastic control problems with different initial times and states. This is important in characterizing the value function by the solution of the HJB equation.

The idea of dynamic programming principle, called Bellman's principle of optimality, first appeared in Bellman (1952), where he considered cases where "the problem of determining an optimal sequence operations may be reduced to that of determining an optimal first operation". The idea was then applied to deterministic control problems of multiple discrete time points. Bellman's principle of optimality was later used in continuous-time stochastic control problems with a dynamics, probably first in Kushner (1962). The principle states that for a small change in time Δt, the optimal value from the control problem is equivalent to that of the minimum of the running cost in the small time period $[t, t+\Delta t)$, plus the optimal value from time $t+\Delta t$ onwards based on the evolved state.

The dynamic programming principle we introduce below is a form of the Bellman's principle of optimality based on considering the optimal action from the starting time s to a stopping time τ.

For the weak formulation of the problem we have defined, we let $v(s, y)$ the be optimal value function for the problem with initial time and state (s, y), i.e.

$$v(s,y) = \inf_{(\Omega,\mathcal{F},\mathbf{P},Z,u)\in\mathcal{U}^w[s,T]} J(u; s, y). \tag{2.4}$$

Then the dynamic programming principle (DPP) states that

$$v(s,y) = \inf_{u\in\mathcal{U}^w[s,T]} E\left\{\int_s^\tau f(t, X_t(u,s,y), u_t)dt + v(\tau, X_\tau(u,s,y))\right\}$$

for any stopping time $\tau \geq s$.

The idea behind the DPP is depicted in Figure 2.1, namely, for any optimal policy starting with initial value y and time s, the remaining policy beyond a stopping time τ must also be optimal for initial state $X_\tau(u, s, y)$ and initial time τ. Otherwise, we could deviate to an optimal path after τ and perform better than the original optimal path, contradicting its optimality.

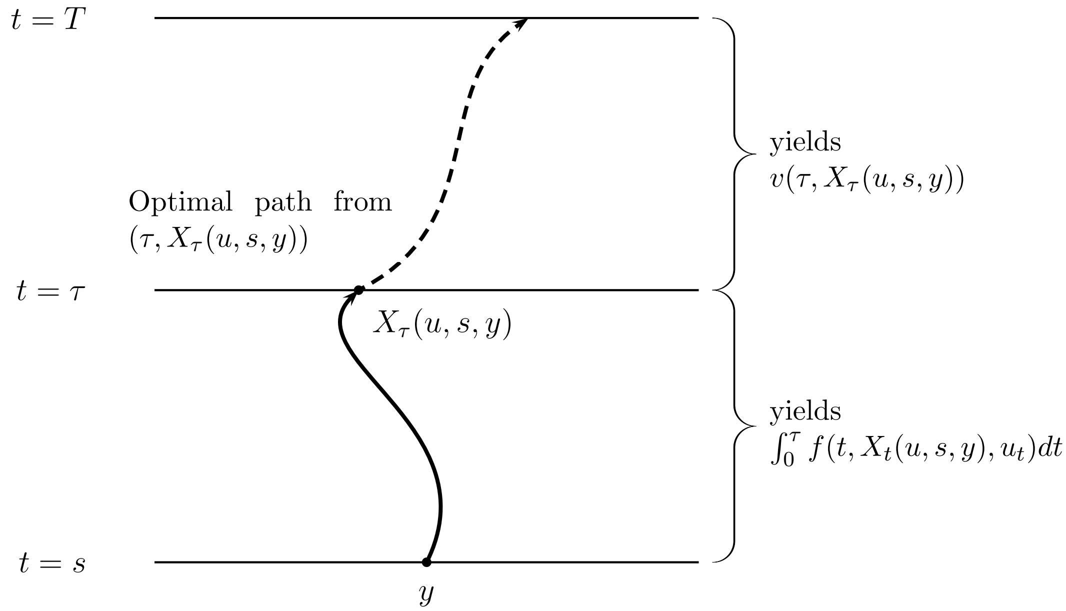

Figure 2.1: Idea of the DPP: The part of an optimal path beyond a stopping time τ must also be an optimal path.

The DPP has two parts, the first part is

$$v(s,y) \le E\left\{\int_s^\tau f(t, X_t(u,s,y), u_t)dt + v(\tau, X_\tau(u,s,y))\right\}.$$

for all $u \in \mathcal{U}^w[s,T]$ and stopping time τ. This means that for any control u, the objective value obtained by following u until τ and then following an optimal policy from τ cannot be better than $v(s,y)$, the optimal value at the beginning.

The other part is to prove the existence of u^ϵ such that

$$v(s,y) + \epsilon \ge E\left\{\int_s^\tau f(t, X_t(u^\epsilon,s,y), u_t)dt + v(\tau, X_\tau(u^\epsilon,s,y))\right\}$$

for any $\epsilon > 0$. This is basically the assertion that an optimal path must have its subpath beyond τ being also optimal. In proving this rigorously, one needs to take care of the technical details of measurability and that the infimum may not be attainable in some cases. The usual way is therefore to construct an ϵ-optimal contract and then take limit as ϵ goes to zero. Uniform continuity of the value function is often used to simplify the construction by using a countable selection argument.

Here we refer to two standard reference books in stochastic control for the conditions under which the DPP is proved. Fleming and Soner (2006, see Theorem IV.7.1) gave a proof for the DPP with the conditions that U is compact, the terminal payoff $h = 0$[1], that b and σ are continuous, $b(.,.,u)$ and $\sigma(.,.,u)$ are continuously differentiable for all $u \in U$, and that for suitable C_1, C_2, C_3, we have

$$|b_t| + |b_x| \le C_1, |\sigma_t| + |\sigma_x| \le C_1$$

$$|b(t,0,u)| + |\sigma(t,x,u)| \le C_2,$$
$$|f(t,x,u)| \le C_3(1 + |x|^k).$$

A proof for the DPP with the stopping time τ being replaced by non-random time t can also be found in Yong and Zhou (1999, see Section 3.2), based on the following conditions:

[1]They also pointed out that this is not very restrictive because the problem can be reformulated under the restriction $h = 0$ when the terminal payoff h is smooth enough

(S1) (U, d) is a Polish space[2].

(S2) The maps b, σ, f, h are uniformly continuous and Lipschitz in x, and $b(t,0,u)$, $\sigma(t,0,u)$, $f(t,0,u)$ and $h(0)$ are bounded by a positive constant L.

We remark that in the two sets of conditions given above, the Lipschitz conditions on b and σ are used to ensure that the SDE governing the process $\{X_t\}$ has unique weak solutions and so $\{X_t\}$ is well-defined. The continuity of b, σ and f are needed in constructing an ϵ-optimal control when proving one direction of the DPP.

The DPP is the key in establishing the relationship between the value function and the Hamilton-Jacobi-Bellman (HJB) equation, which has the form

$$\sup_{u \in U}\{-f(t,y,u) + \mathcal{A}^u v(t,y)\} = 0, \tag{2.5}$$

where

$$\mathcal{A}^u v = \frac{\partial v}{\partial t}(t,y) + b(y,u)\frac{\partial v}{\partial x}(t,y) + \frac{1}{2}\sigma^2(y,u)\frac{\partial^2 v}{\partial x^2}(t,y).$$

The HJB equation is in general a nonlinear second-order partial differential equation. The boundary condition is given by h, the terminal cost function.

Formally, from the DPP, we have

$$\inf_{u \in \mathcal{U}^w[s,T]} \frac{E\left\{\int_s^\tau f(t, X_t(u,s,y), u_t)dt + v(\tau, X_\tau(u,s,y))\right\} - v(s,y)}{\tau - s} = 0.$$

If v is smooth and we take $\tau \downarrow s$, then we have

$$\inf_{u \in U}\{f(t,y,u_t) - \mathcal{A}^u v(t,y)\} = 0,$$

which is equivalent to (2.5). Then boundary conditions based on the terminal payoff function h can be used together with the HJB equation to solve for the value function v.

While this derivation is only formal, it can be made rigorous under assumptions on the smoothness of v. This gives one of the common ways in which the HJB equation is used: One first identifies a candidate for the value function, usually to prove existence and uniqueness of a classical solution to the HJB equation, and then proves that it is indeed the optimal solution to

[2]A Polish space is a complete separable metric space.

the control problem by establishing a verification theorem. The verification part is usually done by first checking that the solution of the HJB equation is an upper bound of the value function, and then finding optimal controls that attains this upper bound. (see for example Fleming and Soner, 2006, Section IV.3). This methodology requires C^2 smoothness of the value function almost everywhere so that the classical solution to the HJB equation is well-defined.

On the other hand, there are problems for which the HJB equation does not have a classical solution. In those cases, the value function of the control problem could still exist but lacks enough smoothness to satisfy the HJB equation int he classical sense. It is then common to consider viscosity solution of the HJB equation and links it to the value function of the control problem. Viscosity solution is a notion of weak solutions to partial differential equations, and is reviewed next in Section 2.3.

2.2.1 Remarks on the infinite horizon case

While the general problem given in (2.4) has a fixed finite time horizon, it is also common to consider time-homogeneous problems with an infinite time horizon. In that case, b, σ and f would be independent of time, and a discounting factor is present in the payoff.

For an initial state of y, the dynamics becomes

$$\begin{aligned} dX_t &= b(X_t, u_t)dt + \sigma(X_t, u_t)dZ_t \\ X_0 &= y \in \mathbb{R} \end{aligned} \tag{2.6}$$

where $b : \mathbb{R} \times U \to \mathbb{R}$, $\sigma : \mathbb{R} \times U \to \mathbb{R}$, Z_t is a standard Brownian motion. And the cost functional is

$$J(u; y) = E\left[\int_0^\infty e^{-rt} f(X_t, u_t)dt\right],$$

where $X_0 = y$. Fleming and Soner (2006, see Chapter 3.9, P.139) gives a formulation specific to the infinite-horizon problem.

The methodology and results related to the infinite-horizon problem is very similar to the finite-horizon case. The major difference is in the time homogeneity of the new problem, as b, σ and f are all independent of time and there is also no terminal time. Letting

$$v(y) = \inf_u J(u; y)$$

be the optimal value function, the DPP would be

$$v(y) = \inf_u E \left\{ \int_0^\tau e^{-rt} f(X_t, u_t) dt + e^{-r\tau} v(X_\tau) \right\}$$

for any stopping time $\tau \leq T$.

Then the HJB equation has the form

$$\inf_{u \in U} \{ f^u(y) - \mathcal{L}^u v(y) \} = rv(y),$$

where

$$\mathcal{L}^u v = b(y, u) v'(y) + \frac{1}{2} \sigma^2(y, u) v''(y).$$

It is worth noting that the HJB equation changes from a parabolic equation to an elliptic one, i.e. there is no term on the derivative of v with respect to t in this case. Instead, a term $rv(y)$ is present because of the discounting factor.

Together with boundary conditions specifying values of v at the boundary of U, the HJB equation allows us to characterize the value function in a way similar to the finite-horizon case.

2.2.2 Combined stochastic control and optimal stopping

Stochastic control problems could come with a additional choice of stopping time. In this case, we say that we have a combined stochastic control and optimal stopping problem. The choice of the stopping time is often the choice of a terminal or liquidation time. For example, an optimal consumption problem where the wealth owner has the option of liquidating all the stocks and retire permanently has been studied in Chancelier et al. (2002).These combined problems are usually posed in a time-homogeneous setting.

Using the same dynamics for $\{X_t\}$ in (2.6), the cost functional of the combined stochastic control and optimal stopping problem would be

$$J(u, \tau; y) = E \left[\int_0^\tau e^{-rt} f(X_t, u_t) dt + \mathbf{1}_{\tau < \infty} e^{-r\tau} h(X_\tau) \right],$$

where $X_0 = y$ and $h(X_\tau)$ is the terminal payoff when you stop at state X_τ.

The optimal value function would then be defined as

$$v(x_0) = \inf_{u,\tau} J(u, \tau; x_0).$$

The optimal stopping part of the problem mandates the value function to be at least that of the stopping payoffs, i.e.

$$v(x) \geq h(x) \quad \text{for all } x.$$

Because of that, the value function v is characterized by an Hamilton-Jacobi-Bellman variational inequality rather than an equation, which can be written as

$$\max\{rv(y) - \inf_{u \in U}\{f^u(y) - \mathcal{L}^u v(y)\}, h(y) - v(y)\} = 0,$$

where

$$\mathcal{L}^u v = b(y, u)v'(y) + \frac{1}{2}\sigma^2(y, u)v''(y).$$

This methodology has been used in previous studies Chancelier et al. (2002) and the relationship between the HJB variational inequality and the combined stochastic control and optimal stopping problem has been described in Bensoussan and Lions (1982, see Section 4.4.2).

Similar to the cases without optimal stopping, in some cases, it is possible to establish a verification theorem and find smooth solutions to the variational inequalities. In other cases, viscosity solutions are again used to characterize the value function. For example, in Chancelier et al. (2002), they have provided both a verification theorem and also proved that the value function is a unique viscosity solutions.

2.3 Viscosity solutions

The notion of viscosity solution was first introduced by Crandall and Lions (1984) for first-order partial differential equations and then in Lions (1983a,b) for second-order partial differential equations. The comparison theorem for fully nonlinear second order elliptic PDEs on bounded domains was proved in Jensen et al. (1988).

The viscosity solution is a notion of weak solution for the partial differential equation that has a lower smoothness requirement than classical solutions. This is particularly useful when the equation may have non-smooth solutions.

A detailed guide to viscosity solutions of second order PDE can be found in Crandall et al. (1992). This serves as an important theoretical reference to viscosity solution, with results on its uniqueness and comparison theorems. Examples and theory of using viscosity solutions in stochastic control problems can be found in Yong and Zhou (1999), Fleming and Soner (2006).

The following definition comes from Crandall et al. (1992).

Consider a partial differential equation of the form

$$F(x, u, Du, D^2u) = 0$$

with $F : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \times \mathcal{S}(N) \to \mathbb{R}$, where $\mathcal{S}(N)$ is the set of symmetric $N \times N$ matrices.

Such an equation may or may not have a smooth solution and this gives rise to the definition of a viscosity solution.

F is assumed to be proper, i.e.

$$F(x, r, p, X) \leq F(x, s, p, X) \quad \text{whenever } r \leq s \tag{2.7}$$

and

$$F(x, r, p, X) \leq F(x, r, p, Y) \quad \text{whenever } Y \leq X \tag{2.8}$$

where $r, s \in \mathbb{R}$, $x, p \in \mathbf{R}^N$, $X, Y \in \mathcal{S}(N)$. Condition (2.8) is called "degenerate elliptic".

There are two definitions for viscosity solutions. The first uses superjets and subjets, which is based on the following inequality regarding an estimate of u:

$$u(x) \leq u(\hat{x}) + \langle p, x - \hat{x} \rangle + \frac{1}{2}\langle X(x - \hat{x}), x - \hat{x} \rangle + o(|x - \hat{x}|^2) \tag{2.9}$$

Definition 2.1. For $u : \mathcal{O} \to \mathcal{R}$ and $\hat{x} \in \mathcal{O}$, the second-order superjet of u at $\hat{x}$ is

$$J^{2,+}_{\mathcal{O}} u(\hat{x}) := \{(p, X) : 2.9 \text{ holds as } \mathcal{O} \ni x \to \hat{x}\}.$$

The second-order subjet of u at $\hat{x}$ is $J^{2,-}_{\mathcal{O}} u(\hat{x}) = -J^{2,+}_{\mathcal{O}}(-u)(\hat{x})$.

Let USC($\mathcal{O}$) (resp. LSC($\mathcal{O}$)) denote the set of upper (resp. lower) continuous functions $u : \mathcal{O} \to \mathbb{R}$.

Definition 2.2. Let F satisfy (2.7) and (2.8) and $\mathcal{O} \subset \mathbb{R}^N$. A viscosity subsolution of $F = 0$ on $\mathcal{O}$ is a function $u \in \text{USC}(\mathcal{O})$ such that

$$F(x, u(x), p, X) \leq 0 \quad \text{for all } x \in \mathcal{O} \text{ and } (p, X) \in J^{2,+}_{\mathcal{O}} u(x).$$

Similarly, a viscosity supersolution of $F = 0$ on $\mathcal{O}$ is a function $u \in \mathrm{LSC}(\mathcal{O})$ such that

$$F(x, u(x), p, X) \geq 0 \quad \text{for all } x \in \mathcal{O} \text{ and } (p, X) \in J^{2,-}_{\mathcal{O}} u(x).$$

Finally, u is a viscosity solution of $F = 0$ in $\mathcal{O}$ if it is both a viscosity subsolution and a viscosity supersolution of $F = 0$ in $\mathcal{O}$.

Equivalently, we could also have (Crandall et al., 1992, P.11)

Definition 2.3. Let F satisfy (2.7) and (2.8) and $\mathcal{O} \subset \mathbb{R}^N$. A viscosity subsolution of $F = 0$ on $\mathcal{O}$ is a function $u \in \mathrm{USC}(\mathcal{O})$ such that for any $\hat{x} \in \mathcal{O}$ and $\varphi \in C^2$ such that $u - \varphi$ has a local maximum at $\hat{x}$,

$$F(\hat{x}, u(\hat{x}), D\varphi(\hat{x}), D^2\varphi(\hat{x})) \leq 0.$$

Similarly, a viscosity supersolution of $F = 0$ on $\mathcal{O}$ is a function $u \in \mathrm{LSC}(\mathcal{O})$ such that for any $\hat{x} \in \mathcal{O}$ and $\varphi \in C^2$ such that $u - \varphi$ has a local minimum at $\hat{x}$,

$$F(\hat{x}, u(\hat{x}), D\varphi(\hat{x}), D^2\varphi(\hat{x})) \geq 0.$$

Finally, u is a viscosity solution of $F = 0$ in $\mathcal{O}$ if it is both a viscosity subsolution and a viscosity supersolution of $F = 0$ in $\mathcal{O}$.

We will use the second definition in this thesis.

The notion of viscosity solution is often used in stochastic control problems. In particular, they are used in studying HJB equations of control problems, especially when the equation does not have a smooth solution or when we do not know that a priori. The approach usually starts by proving that the value function of the control problem is a viscosity solution, and then establishes uniqueness of the viscosity solutions. This would mean the equivalence of the viscosity solution of the HJB equation to the value function of the problem. For the problem we stated in (2.4), we have the following:

Theorem 2.1 (Yong and Zhou (1999), Theorem 5.2)**.** *Let (S1) and (S2) hold. The value function v is a viscosity solution of the HJB equation (2.5).*

We will use the viscosity solution approach to study the HJB equation of a continuous-time principal-agent problem based on Sannikov's one. In particular, the uniqueness result of viscosity solutions will become useful in our work. In the following, we restate a comparison theorem from Crandall et al. (1992):

Theorem 2.2 (Crandall et al. (1992), Theorem 3.3)**.** *Let Ω be a bounded open subset of $\mathbb{R}^N$, $F \in C(\Omega \times \mathbb{R} \times \mathbb{R}^N \times \mathcal{S}(N))$ be proper and satisfy the following: There exists $\gamma > 0$ such that*

$$\gamma(r-s) \leq F(x,r,p,X)-F(x,s,p,X) \quad \textit{for } r \geq s,\ (x,p,X) \in \bar{\Omega}\times\mathbb{R}^N\times\mathcal{S}(N) \tag{2.10}$$

and there is a function $\omega : [0,\infty] \to [0,\infty]$ that satisfies $\omega(0+) = 0$ such that

$$F(y,r,\alpha(x-y),Y) - F(x,r,\alpha(x-y),X) \leq \omega(\alpha|x-y|^2 + |x-y|) \tag{2.11}$$

whenever

$$x, y \in \Omega, r \in \mathbb{R}, X, Y \in \mathcal{S}(N)$$

and

$$-3\alpha \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \leq \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \leq 3\alpha \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}.$$

Let u, an upper semi-continuous function over $\bar{\Omega}$, be a viscosity subsolution and v, a lower semi-continuous function over $\bar{\Omega}$, be a viscosity supersolution of $F = 0$ in Ω and $u \leq v$ on $\partial\Omega$. Then $u \leq v$ in $\bar{\Omega}$.

Chapter 3

Our Formulation

3.1 Problem setting

In our principal-agent problem, the principal is concerned about a certain output process that we will define as follows.

Consider $(\Omega^X, \mathcal{F}, \{\mathcal{F}_t^X\}, \mathbf{P}^0)$ be a filtered probability space such that $\Omega^X = C[0, \infty)$,

$$X_t = \sigma Z_t,$$

$\{Z_t\}$ is a standard Brownian motion on $(\mathbf{P}^0, \{\mathcal{F}_t\})$, where $\{\mathcal{F}_t^X\}$ is the filtration generated by X augmented by all null sets in $\mathcal{F}$. X is the output process that the principal obtains benefits on, and $\mathbf{P}_0$ represents the probability measure of how the process evolves when the agent makes zero effort.

Now we consider the case when the agent makes effort, represented by a stochastic process A. We require A to be progressively measurable with respect to $\{\mathcal{F}_t^X\}$ and that A_t takes values from $\mathcal{A} = [0, \bar{a}]$. Given such A, define

$$Z_t^A = \frac{X_t - \int_0^t A_s ds}{\sigma} = Z_t - \int_0^t \frac{A_s}{\sigma} ds$$

for $0 \leq t < \infty$. We can apply a change of measure via Girsanov theorem. By Karatzas and Shreve (1991, Corollary 3.5.2, P.192), there is a unique probability measure $\mathbf{P}^A$ such that $\{Z_t^A, \mathcal{F}_t^X, t \geq 0\}$ is a standard Brownian motion on $(\Omega, \mathcal{F}_\infty^X, \mathbf{P}^A)$. Since A is progressively measurable with respect to $\{\mathcal{F}_t^X\}$, Z^A is also progressively measurable with respect to $\{\mathcal{F}_t^X\}$. Therefore $\mathcal{F}_t^{Z^A} \subset \mathcal{F}_t^X$ for any t.

We interpret Z^A as the underlying noise process in the output. The agent's choice of effort is interpreted as a choice of the probability distribution of X. Thus, by changing the distribution of the output, the agent is effectively choosing the drift of the output process X. Readers may want to note that Cvitanič et al. (2009, see Section 2.2.2) also use this kind of weak formulation, where the probability measure is being controlled, and they made some remarks on the difference between a strong and weak formulation.

The triple $(X, Z^A, \mathbf{P}^A)$ is therefore a weak solution to the SDE

$$dX_t = A_t dt + \sigma dZ_t^A.$$

The principal compensates the agent for his work by a flow of compensation $C = \{C_t\}_{t \geq 0}$, which is progressively measurable with respect to $\{\mathcal{F}_t^X\}$. The principal therefore obtains

$$E^A \left[\int_0^\infty re^{-rt}(dX_t - C_t dt) \right] = E^A \left[\int_0^\infty re^{-rt}(A_t - C_t) dt \right],$$

while the agent gets

$$E^A \left[\int_0^\infty re^{-rt}(u(C_t) - h(A_t)) dt \right]$$

from his compensation net of the disutility from the effort, where E^A denotes the expectation under probability measure $\mathbf{P}^A$. The payoffs are normalized with the discount rate r to put the total payoffs in the same scale as the flow payoffs[1]. It is important to note that the principal's payoff depends on the agent's choice of A.

We impose assumptions on the set of possible effort, the utility function and the cost function for the agent.

A1 The set of possible effort at each time is $\mathcal{A} = [0, \bar{a}]$, with $0 < \bar{a} < \infty$.

A2 The utility function $u : [0, \bar{c}] \to \mathbb{R}^+$ is strictly increasing and concave with $u(0) = 0$ and a positive left-derivative of u at $\bar{c}$, i.e. $u'_-(\bar{c}) > 0$.

[1] The same normalization has been done in Sannikov (2008). The benefits, for example, include having the retirement profit equal to $F_0(w) = -u^{-1}(w)$ instead of some constant multiplied by $u^{-1}(w)$.

A3 The cost function for the agent's effort $h : \mathcal{A} \to \mathbb{R}$ is strictly increasing and convex with $h(0) = 0$. Moreover, the right-derivative of h at 0, denoted by $\gamma_0 := h'_+(0)$, is positive. This implies $h(a) \geq \gamma_0 a$ for all $a \in \mathcal{A}$ and $\gamma_0 > 0$. The cost is a disutility measured in the same unit as the agent's utility.

3.2 Formulation of our optimal stopping and control problem

In this section, we state our formulation of the control problem, which is a combined stochastic control and optimal stopping problem based on the one-dimensional dynamics of the agent's continuation value process. Compared to Sannikov's formulation, we impose some additional restrictions on the principal's choice of contracts. In particular, we model explicitly the permanent retirement time of the agent as a stopping time, and required the sensitivity of the agent's continuation value to be at least γ_0 prior to permanent retirement.

As reviewed in Section 1.4.4, Sannikov (2008) defined the agent's continuation value of a pair (C, A) at time t to be the expectation of his future discounted utility from the compensation less the cost of effort, i.e.

$$E^A\left[\int_t^\infty e^{-r(s-t)}(u(C_s) - h(A_s))ds|\mathcal{F}_t^X\right].$$

He also provided a sufficient and necessary condition for the incentive-compatibility in terms of the sensitivity of the continuation value to the output.

We follow a similar notion of continuation value, but will explicitly specify the sensitivity process Y and retirement time τ. To do so, we first define the set of incentive-compatible action-sensitivity pair to be

$$\Gamma := \{(a, y) : a \in \mathcal{A},\, y \geq \gamma_0 \text{ and } \forall a' \in \mathcal{A}\ ya - h(a) \geq ya' - h(a')\}.$$

Note that Γ is closed since h is continuous.

For initial continuation value $w \in (0, \bar{w})$, let $\mathcal{U}(w)$ denote the set of 7-tuple $(\Omega, \mathcal{F}, \mathbf{P}, Z, C, A, Y, \tau)$ such that

U1 $(\Omega, \mathcal{F}, \mathbf{P})$ is a complete probability space, with $\Omega = C([0, \infty); \mathbb{R})$.

U2 $\{Z_t\}_{t\geq 0}$ is the canonical process on Ω, i.e. $Z_t(\omega) = \omega(t)$, and a standard Brownian motion under $\mathbf{P}$. We let $\mathcal{F}_t^Z = \sigma\{Z_s, 0 \leq s \leq t\}$.

U3 The compensation process $C = \{C_t\}_{t\geq 0}$, the recommended effort process $A = \{A_t\}_{t\geq 0}$ and the sensitivity process $Y = \{Y_t\}_{t\geq 0}$ are progressively measurable with respect to $\{\mathcal{F}_t^Z\}$.

U4 For all t, we have

$$C_t \in [0, \bar{c}].$$

U5 We have

$$Y_t = \gamma(A_t) \quad 0 \leq t \leq \infty \quad \text{almost everywhere},$$

where $\gamma(a) = \min\{y : (a, y) \in \Gamma\}$ [2] . This ensures incentive-compatibility with the minimum risk posed to the agent.

U6 The retirement time τ is a $\{\mathcal{F}_t^Z\}$-stopping time taking values in $[0, \infty]$.

U7 The stochastic differential equation

$$dW_t = r(W_t - u(C_t) + h(A_t))dt + r\sigma Y_t dZ_t; \quad W_0 = w \tag{3.1}$$

admits a unique solution W such that

$$W_t \in (0, \bar{w}) \quad \forall t < \tau,$$

where $\bar{w}$ is

$$\bar{w} = \int_0^\infty re^{-rt}u(\bar{c})dt = u(\bar{c}).$$

We will call $\mathcal{U}(w)$ the set of admissible control in our problem with initial value w. When there is no ambiguity on $(\Omega, \mathcal{F}, P, Z)$, we will simply write $(C, A, Y, \tau) \in \mathcal{U}(w)$. The components in the quadruple (C, A, Y, τ) are interpreted as the following: C is the compensation process, A is the recommended effort, Y is the sensitivity of the agent's continuation value to the output, and the stopping time τ represents the time when the agent is permanently retired. Throughout this paper, we refer to the solution W above as W_t, or $W_t(C, A, Y; w)$ when explicit references to the processes (C, A, Y) or the

[2]Note that $\gamma(0) = \gamma_0$ and our definition differs slightly from that in Sannikov (2008) in that we restrict y to be at least γ_0.

initial value w is needed. The process W is the continuation value of the agent.

It is important to note that we are using the filtration $\{\mathcal{F}_t^Z\}$ instead of $\{\mathcal{F}_t^X\}$ in the above formulation. We address this gap in Section 3.3 and explain how we construct a contract that can be implemented based on the observation of $\{X_t\}$ only, from the $\{\mathcal{F}_t^Z\}$-progressively measurable controls we obtain from a solution of our control problem.

When an agent with continuation value w is permanently retired, it is optimal for the principal to give him a constant flow of consumption $-F_0(w) = u^{-1}(c)$ to deliver the continuation value w. This delivers to the agent a value of w with a minimum cost for the principal. In reality, there could, however, as proposed in Sannikov (2008), be other contractual possibilities that allow the principal to retire the agent with profit $\tilde{F}_0(w)$ when the agent has continuation value. We therefore allow a more general retirement function $\tilde{F}_0$ in our formulation, and require $\tilde{F}_0$ to satisfy:

A4 $\tilde{F}_0$ is continuous over $[0, \bar{w}]$ and we have

$$F_0(w) \leq \tilde{F}_0(w) \leq \bar{F}(w),$$

for all $w \in [0, \bar{w}]$, where $\bar{F}$ is defined[3] as

$$\bar{F}(w) = \max_{\substack{u(c)-h(a)=w, \\ a\in\mathcal{A},\ c\in[0,\bar{c}]}} (a - c).$$

Assumption (A4) implies that $\tilde{F}_0(\bar{w}) = F_0(\bar{w})$ since the functions F_0 and $\bar{F}$ coincide at $\bar{w}$.

Suppose the agent is retired at a stopping time τ, then the principal's payoff can be written as

$$J(C, A, Y, \tau; w) = E\left[\int_0^\tau re^{-rt}(A_t - C_t)dt + e^{-r\tau}\tilde{F}_0(W_\tau)\right],$$

where $W_t = W_t(C, A, Y; w)$, and $\tilde{F}_0(w)$ is the retirement profit when the continuation value is W_τ.

[3]The function $\bar{F}$ is understood as the first-best profit and more discussion on $\bar{F}$ can be found in Section 4.3. We are requiring that $\tilde{F}_0(w) \leq \bar{F}(w)$ because for continuation value that violates this condition, it would be optimal to invoke immediate permanent retirement.

The optimal profit function is

$$F(w) = \sup_{(C,A,Y,\tau)\in\mathcal{U}(w)} J(C, A, Y, \tau; w). \tag{3.2}$$

This is the maximal profit for the principal when the initial value for the agent is w. In the usual terminology in stochastic control problems, this is also called the value function.

If we take $\tilde{F}_0(w) = F_0(w) = -u^{-1}(w)$ for all w, this reduces to the benchmark case described in Sannikov (2008), where there are no other contractual possibilities and the principal is tied to the agent forever.

3.3 Our problem reduction

At this point, we would like to compare our formulation with the initial formulation of Sannikov's problem in (1.9) to show the extent to which the two problems are equivalent and the assumptions made in establishing the equivalence.

In this section, we will consider the benchmark case of our formulation where $\tilde{F}_0(w) = F_0(w) = -u^{-1}(w)$ for all w, i.e. there are no other contractual possibilities, and compare it with the formulation in Sannikov (2008).

Conceptually, a few things have been done in the problem reduction:

1. The recommended effort was added as a decision process for the principal and the incentive-compatibility constraint was added to ensure that the agent would comply.

2. The continuation process W_t is represented as a stochastic differential equation in terms of Z^A, where Z^A is a standard Brownian motion when the agent chooses A (which is in turn enforced by the incentive-compatibility constraints).

3. The compensation process can now equivalently be understood as a process progressively measurable with respect to Z^A. This is because given A (which is progressively measurable with respect to X), we can compute Z^A from X or X from Z_A.

4. The recommended effort process is changed from a decision process progressively measurable with respect to $\{\mathcal{F}_t^X\}$ to a decision process

progressively measurable with respect to $\{\mathcal{F}_t^{Z^A}\}$. There is a gap in this step and it is explained in Proposition 3.1. The dependence on the dynamics of X is now dropped and is considered entirely through Z^A.

5. We explicitly model the permanent retirement time as a stopping time and prohibit temporary retirement, where the principal allows the agent to suspend effort and be unaffected by the actual output for a period of time. (However, we allow the effort to be zero if the sensitivity of the agent's continuation value to the output level, Y_t, remains at least γ_0.)

In the five points above for our formulation, the first four steps have been implicitly taken in Sannikov's model as well, but have not be thoroughly explained. The fifth point is what we added to our formulation to make the analysis rigorous. A contract with an explicit permanent retirement time can be rewritten into a contract in Sannikov's model.

The reason that we prohibit temporary retirement is analytical. This ensures that the control problem we have is nondegenerate and allows for regularity results for the viscosity solution. In comparison, Sannikov (2008) makes similar steps in his analysis by considering an HJB equation that assumes a nonzero effort before permanent retirement, and argued after obtaining the solution that temporary retirement is not needed for the optimal policy.

For the rest of this section, we will explain the conditions under which the original problem and final formulation are equivalent.

3.3.1 Representation of stochastic processes

We will need a lemma concerning the representation of stochastic processes as a function of the continuous process, when the former stochastic process is progressively measurable to the filtration generated by the latter.

We look at a known result from Yong and Zhou (1999) regarding the representation. We first define a few notations:

$$\begin{aligned}
\mathbf{W}^m[0,T] &:= C([0,T];\mathbb{R}^m) \\
\mathbf{W}_t^m[0,T] &:= \{\zeta(.\wedge t)|\zeta(.)\in \mathbf{W}^m[0,T]\}, \quad \forall t\in[0,T], \\
\mathcal{B}_t(\mathbf{W}^m[0,T]) &:= \sigma(\mathcal{B}(\mathbf{W}_t^m[0,T])), \quad \forall t\in[0,T], \\
\mathcal{B}_{t+}(\mathbf{W}^m[0,T]) &:= \cap_{s>t}\mathcal{B}_s(\mathbf{W}^m[0,T]), \quad \forall t\in[0,T].
\end{aligned}$$

Lemma 3.1 (Yong and Zhou (1999), Theorem 2.10)**.** *Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a complete probability space and (U, d) a Polish space. Let $\xi : [0,T] \times \Omega \to \mathbb{R}^m$ be a continuous process and $\mathcal{F}_t^{\xi} = \sigma(\xi(s) : 0 \le s \le t)$. Then $\varphi : [0,T] \times \Omega \to U$ is progressively measurable with respect to $\{\mathcal{F}_t^{\xi}\}$ if and only if there exists an $\eta : [0,T] \times \mathbf{W}^m[0,T] \to U$ progressively measurable with respect to $\{\mathcal{B}_{t+}(\mathbf{W}^m[0,T])\}_{t \ge 0}$ such that*

$$\varphi(t, \omega) = \eta(t, \xi(. \wedge t, \omega)), \quad \mathbf{P}\text{-}a.s. \;\; \omega \in \Omega, \, \forall t \in [0,T].$$

Since our control problem has an infinite time horizon, we need a modified version of Lemma 3.1 with the processes defined on the infinite horizon instead. The idea and treatment is very similar to the finite-horizon case, but the proofs are included for completeness since we have not been able to find it elsewhere.

We define

$$\mathbf{W}^m := C([0,\infty); \mathbb{R}^m)$$

and consider the metric

$$\hat{\rho}(\zeta, \hat{\zeta}) = \sum_{j \ge 1} 2^{-j} [|\zeta - \hat{\zeta}|_{C([0,j];\mathbb{R}^m)} \wedge 1] \quad \forall \zeta, \hat{\zeta} \in \mathbf{W}^m,$$

with

$$|\zeta - \hat{\zeta}|_{C([0,j];\mathbb{R}^m)} = \sup_{t \in [0,j]} |\zeta(t) - \hat{\zeta}(t)|.$$

We also define

$$\mathbf{W}_t^m := \{\zeta(. \wedge t) | \zeta(.) \in \mathbf{W}^m\}, \quad \forall t \ge 0,$$

$$\mathcal{B}_t(\mathbf{W}^m) := \sigma(\mathcal{B}(\mathbf{W}_t^m)), \quad \forall t \ge 0,$$

$$\mathcal{B}_{t+}(\mathbf{W}^m) := \cap_{s>t} \mathcal{B}_s(\mathbf{W}^m), \quad \forall t \ge 0.$$

We will adapt the proof of Lemma 3.1 to the case of infinite horizon. For the "only if" direction, we will only show the case where $U = \mathbb{R}$ and Ω is the canonical space for ξ.

Lemma 3.2. *Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a complete probability space and (U, d) a Polish space. Let $\xi : [0, \infty) \times \Omega \to \mathbb{R}^m$ be a continuous process and $\mathcal{F}_t^{\xi} = \sigma(\xi(s) : 0 \le s \le t)$. Then $\varphi : [0, \infty) \times \Omega \to U$ is progressively measurable with respect* *to* $\{\mathcal{F}_t^\xi\}$ *if there exists an* $\eta : [0,\infty) \times \mathbf{W}^m \to U$ *progressively measurable with respect to* $\{\mathcal{B}_{t+}(\mathbf{W}^m)\}_{t\geq 0}$ *such that*

$$\varphi(t,\omega) = \eta(t, \xi(. \wedge t, \omega)), \quad \mathbf{P}\text{-}a.s. \;\; \omega \in \Omega, \, \forall t \geq 0. \tag{3.3}$$

Proof. We would like to show that $\varphi : [0,\infty) \times \Omega \to U$ is progressively measurable with respect to $\{\mathcal{F}_t^\xi\}$. For each $t > 0$, we would like to show that the map $(s,\omega) \mapsto \varphi(s,\omega) : [0,t] \times \Omega \to U$ is $\mathcal{F}_t^\xi$-measurable. We consider only $s \in [0,t]$ below.

We know that η is a measurable map from $([0,t]\times\mathbf{W}^m, \mathcal{B}[0,t]\times\mathcal{B}_{t+}(\mathbf{W}^m))$ to $(U, \mathcal{B}(U))$-measurable and $(s,\omega) \mapsto (s, \xi(. \wedge s))$ is a measurable map from $([0,t]\times\Omega, \mathcal{B}[0,t]\times\mathcal{F}_t^\xi)$ to $([0,t]\times\mathbf{W}_t^m, \mathcal{B}[0,t]\times\mathcal{B}_t(\mathbf{W}^m))$. Since $\mathbf{W}_t^m \subset \mathbf{W}^m$ and $\mathcal{B}_t(\mathbf{W}^m) \subset \mathcal{B}_{t+}(\mathbf{W}^m)$, for any set $A \in \mathcal{B}_{t+}(\mathbf{W}^m)$,

$$(\xi(. \wedge s))^{-1}(A) = (\xi(. \wedge s))^{-1}(\mathbf{W}_t^m \cap A) \in \mathcal{F}_t^\xi,$$

since $\mathbf{W}_t^m \cap A \in \mathcal{B}_t(\mathbf{W}^m)$. Thus $(s,\omega) \mapsto (s, \xi(. \wedge s))$ is a measurable map from $([0,t]\times\Omega, \mathcal{B}[0,t]\times\mathcal{F}_t)$ to $([0,t]\times\mathbf{W}^m, \mathcal{B}[0,t]\times\mathcal{B}_{t+}(\mathbf{W}^m))$ as well. Then φ is a measurable map from $(\Omega, \mathcal{F}_t^\xi)$ to $(U, \mathcal{B}(U))$. Since this is true for any $t > 0$, we have the desired result. □

We only prove the converse for a canonical space with ξ being the canonical process.

Lemma 3.3. *Let* $(\Omega, \mathcal{F}, \mathbf{P})$ *be a complete probability space with* $\Omega = \mathbf{W}^m$*. Let* $\xi : [0,\infty) \times \Omega \to \mathbb{R}^m$ *be the canonical process on* Ω*, i.e.*

$$\xi(t,\omega) = \omega(t)$$

for all $\omega \in \Omega$*, and* $\mathcal{F}_t^\xi = \sigma(\xi(s) : 0 \leq s \leq t)$*. Then* $\varphi : [0,\infty) \times \Omega \to \mathbb{R}$ *is progressively measurable with respect to* $\{\mathcal{F}_t^\xi\}$ *only if there exists an* $\eta : [0,\infty) \times \mathbf{W}^m \to \mathbb{R}$ *progressively measurable with respect to* $\{\mathcal{B}_{t+}(\mathbf{W}^m)\}_{t\geq 0}$ *such that*

$$\varphi(t,\omega) = \eta(t, \xi(. \wedge t, \omega)), \quad \mathbf{P}\text{-}a.s. \;\; \omega \in \Omega, \, \forall t \geq 0. \tag{3.3}$$

To prove Lemma 3.3, we need two lemmas. These are similar to the results in Yong and Zhou (1999, P.18–19, Lemma 2.11 and 2.12) except for the time horizon. The first lemma is also related to Lemma 2.17 in the same book.

For any $s > 0$, define $\mathbf{C_s}$ to be the set of all Borel cylinders in $\mathbf{W}_s^m$, i.e. all sets of the forms

$$B = \{\zeta \in \mathbf{W}_s^m | (\zeta(t_1), \zeta(t_2), \ldots, \zeta(t_j) \in E\},$$

where $0 \le t_1 < t_2 < \ldots < t_j < \infty$ and $E \in \mathcal{B}(\mathbb{R}^{jm})$.

Lemma 3.4. *For any $s > 0$, the σ-field $\sigma(\mathbf{C}_s)$ generated by $\mathbf{C}_s$ coincides with the Borel σ-field $\mathcal{B}_s(\mathbf{W}^m)$.*

Proof. Let $0 \le t_1 < t_2 < \ldots < t_j < \infty$ be given. Since we are concerned with Borel cylinders in $\mathbf{W}_s^m$, we can assume without loss of generality that $t_j \le s$. Define the map $\mathcal{T} : \mathbf{W}_s^m \to \mathbb{R}^{jm}$ such that

$$\mathcal{T}(\zeta) = (\zeta(t_1), \zeta(t_2), \ldots, \zeta(t_j)), \quad \forall \zeta \in \mathbf{W}_s^m.$$

First we show that $\mathcal{T}$ is continuous. For any $0 < \epsilon < 1$ and $\zeta_1, \zeta_2 \in \mathbf{W}_s^m$,

$$\begin{aligned} \hat{\rho}(\zeta_1, \zeta_2) \le 2^{-(\lceil s \rceil - 1)}\epsilon \quad &\Longrightarrow \quad |\zeta_1 - \zeta_2|_{C([0,\lceil s \rceil];\mathbb{R}^m)} \wedge 1 \le \epsilon \\ &\Longrightarrow \quad |\zeta_1 - \zeta_2|_{C([0,\lceil s \rceil];\mathbb{R}^m)} \le \epsilon \end{aligned}$$

since $\epsilon < 1$. That implies that

$$\|\mathcal{T}(\zeta_1) - \mathcal{T}(\zeta_2)\|_\infty = \sup_{i=1,2,\ldots,j} |\zeta_1(t_i) - \zeta_2(t_i)| \le |\zeta_1 - \zeta_2|_{C([0,\lceil s \rceil];\mathbb{R}^m)} \le \epsilon.$$

Thus $\mathcal{T}$ is continuous. Consequently, for all $E \in \mathcal{B}(\mathbb{R}^{jm})$, we have

$$\mathcal{T}^{-1}(E) \in \mathcal{B}_s(\mathbf{W}^m).$$

Since $\mathbf{C}_s$ is the collection of all sets in the form of $\mathcal{T}^{-1}(E)$ for some $E \in \mathcal{B}(\mathbb{R}^{jm})$, the above implies that $\mathbf{C}_s \subset \mathcal{B}_s(\mathbf{W}^m)$.

Now, we prove that $\mathcal{B}_s(\mathbf{W}^m) \subset \mathbf{C}_s$. First we note that for any $\zeta \in \mathbf{W}_s^m$, we have

$$\zeta(t) = \zeta(s) \quad \forall t \ge s.$$

So, we would have

$$|\zeta - \zeta_0|_{C([0,i];\mathbb{R}^m)} = |\zeta - \zeta_0|_{C([0,\lceil s \rceil];\mathbb{R}^m)}$$

for any integer $i \geq s$. So for any $\zeta \in \mathbf{W}_s^m$, we have

$$\hat{\rho}(\zeta, \zeta_0) = \sum_{j=1}^{\lceil s \rceil - 1} 2^{-j}[|\zeta - \zeta_0|_{C([0,j];\mathbb{R}^m)} \wedge 1] + 2^{-\lceil s \rceil - 1}[|\zeta - \zeta_0|_{C([0,\lceil s \rceil];\mathbb{R}^m)} \wedge 1].$$

For any $\zeta_0 \in \mathbf{W}_s^m$ and $\epsilon > 0$, let

$$E(\epsilon) = \left\{ y \in \mathbb{R}^{\lceil s \rceil} \mid \sum_{j=1}^{\lceil s \rceil - 1} 2^{-j}(y_j \wedge 1) + 2^{-\lceil s \rceil - 1}(y_{\lceil s \rceil} \wedge 1) \leq \epsilon \right\} \in \mathcal{B}(\mathbb{R}^m).$$

we have

$$\{\zeta \in \mathbf{W}_s^m \mid \hat{\rho}(\zeta, \zeta_0) \leq \epsilon\} = \bigcap_{\substack{(r_1, r_2, \ldots, r_{\lceil s \rceil}) \in \mathbb{Q}^{\lceil s \rceil} \\ r_i \in [0,i] \forall i \in \{1,2,\ldots,\lceil s \rceil\}}} \{\zeta \in \mathbf{W}_s^m \mid (|\zeta(r_1) - \zeta_0(r_1)|, \ldots, |\zeta(r_{\lceil s \rceil}) - \zeta_0(r_{\lceil s \rceil})|) \in E(\epsilon)\}. \tag{3.4}$$

We then see that

$$\{\zeta \in \mathbf{W}_s^m \mid \hat{\rho}(\zeta, \zeta_0) \leq \epsilon\} \in \sigma(\mathbf{C}_s),$$

because the set of all $\lceil s \rceil$-tuples of rational numbers are countable and

$$\{\zeta \in \mathbf{W}_s^m \mid (|\zeta(r_1) - \zeta_0(r_1)|, \ldots, |\zeta(r_{\lceil s \rceil}) - \zeta_0(r_{\lceil s \rceil})|) \in E(\epsilon)\}$$

is a Borel cylinder in $\mathbf{W}_s^m$. Now all sets in the form of the left-hand-side of (3.4), for all $\zeta_0 \in \mathbf{W}_s^m$ generates $\mathcal{B}_s(\mathbf{W}^m)$, and so we have

$$\mathcal{B}_s(\mathbf{W}^m) \subset \sigma(\mathbf{C}_s)$$

$\square$

Lemma 3.5. *Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a complete probability space and $\xi : [0, \infty) \times \Omega \to \mathbb{R}^m$ a continuous process. Then there exists an $\Omega_0 \in \mathcal{F}$ with $\mathbf{P}(\Omega_0) = 1$ such that $\xi : \Omega_0 \to \mathbf{W}^m$ and for any $s \geq 0$,*

$$\Omega_0 \cap \mathcal{F}_s^\xi = \Omega_0 \cap \xi^{-1}(\mathcal{B}_s(\mathbf{W}^m)).$$

If $\Omega = \mathbf{W}^m$, we can choose $\Omega_0 = \Omega$.

Proof. This proof is essentially the same as Lemma 2.12 in Yong and Zhou (1999), except that the space changes from $\mathbf{W}^m[0,T]$ to $\mathbf{W}^m$.

Since ξ is a continuous process, there is a set $N \in \mathcal{F}$ with $P(N) = 0$ such that $\xi(.,\omega) \in \mathbf{W}^m$ for all $\omega \in \Omega \backslash N$. We take $\Omega_0 = \Omega \backslash N \in \mathcal{F}$. If $\Omega = \mathbf{W}^m$, ξ is continuous for all $\omega \in \Omega$, and so we can choose $\Omega_0 = \Omega$.

Fix $t \in [0,s]$ and $E \in \mathcal{B}(\mathbb{R}^m)$, then

$$B_t := \{\zeta \in \mathbf{W}^m | \zeta(t) \in E\} \in \mathbf{C}^s.$$

For any $\omega \in \Omega_0$, we have

$$\begin{aligned} \omega \in \xi^{-1}(B_t) &\Longleftrightarrow \xi(.,\omega) \in B_t \\ &\Longleftrightarrow \xi(t,\omega) \in E \\ &\Longleftrightarrow \omega \in \xi^{-1}(t,)(E). \end{aligned}$$

Thus $\Omega_0 \cap \xi(t,.)^{-1}(E) = \Omega_0 \cap \xi^{-1}(B_t)$. Now,

$$\begin{aligned} \Omega_0 \cap \mathcal{F}_s^\xi &= \Omega_0 \cap \sigma(\{\xi(t,.)^{-1}(E) \mid t \in [0,s], E \in \mathcal{B}(\mathbb{R}^m)\}) \\ &= \Omega_0 \cap \sigma(\{\xi^{-1}(B_t) \mid t \in [0,s], E \in \mathcal{B}(\mathbb{R}^m)\}) \\ &= \Omega_0 \cap \xi^{-1}(\sigma(B_t) \mid t \in [0,s], E \in \mathcal{B}(\mathbb{R}^m)\})) \\ &= \Omega_0 \cap \xi^{-1}(\sigma(\mathbf{C}_s)). \end{aligned}$$

On the other hand, by Lemma 3.4, we have

$$\xi^{-1}(\mathcal{B}_s(\mathbf{W}^m)) = \xi^{-1}(\sigma(\mathbf{C}_s)).$$

The desired result then follows. □

Proof of Lemma 3.3. For $s \geq 0$, define

$$\theta^s(t,\omega) = (t \wedge s, \xi(. \wedge s, \omega)) : [0,\infty) \times \Omega \to [0,s] \times \mathbf{W}_s^m.$$

By Lemma 3.4, by considering θ^s as a function from $[0,\infty) \times \Omega$ to $\mathbf{W}^{m+1}$, we know

$$\mathcal{B}[0,s] \times \mathcal{F}_s^\xi = \sigma(\theta^s).$$

On the other hand, $(t,\omega) \mapsto \varphi(t \wedge s, \omega)$ is $(\mathcal{B}[0,s] \times \mathcal{F}_s^\xi)/\mathcal{B}(\mathbb{R})$-measurable. This means that the map $(t,\omega) \mapsto \varphi(t \wedge s, \omega)$ is $\sigma(\theta^s)/\mathcal{B}(\mathbb{R})$-measurable. By a Theorem 1.1.7 in Yong and Zhou (1999, P.5), there exists a measurable map $\eta_s : ([0,s] \times \mathbf{W}_s^m, \mathcal{B}[0,s] \times \mathcal{B}_s(\mathbf{W}^m)) \to \mathbb{R}$ such that

$$\varphi(t \wedge s, \omega) = \eta_s(\theta^s(t,\omega)) = \eta_s(t \wedge s, \xi(. \wedge s, \omega)) \quad \forall \omega \in \Omega, t \geq 0.$$

Now, for any $i \geq 1$, let $0 = t_0^i < t_1^i < t_2^i < \ldots$ be a partition of $[0, \infty)$ with $\max_{j \geq 1}(t_j^i - t_{j-1}^i) \to 0$ as $i \to \infty$ and define

$$\eta^i(t, \zeta) = \eta_0(0, \zeta(. \wedge 0))I_{\{0\}}(t) + \sum_{j \geq 1} \eta_{t_j^i}(t, \zeta(. \wedge t_j^i))I_{(t_{j-1}^i, t_j^i]}(t)$$

for all $(t, \zeta) \in [0, \infty) \times \mathbf{W}^m$. Note that the sum is finite because for fixed i, the indicator $I_{(t_{j-1}^i, t_j^i]}(t)$ is only nonzero for one value of j for each t.

Now, for any i and any $t > 0$, there is exactly one j such that $t_{j-1}^i < t \leq t_j^i$. Then

$$\eta^i(t, \xi(. \wedge t_j^i, \omega)) = \eta_{t_j^i}(t, \xi(. \wedge t_j^i, \omega)) = \varphi(t, \omega), \tag{3.5}$$

for all $\omega \in \Omega$. Now, we define

$$\eta(t, \zeta) = \limsup_{i \to \infty} \eta^i(t, \zeta). \tag{3.6}$$

The limit superior is taken point-wise for each $(t, \zeta) \in [0, \infty) \times \mathbf{W}^m$. We check that η is progressively measurable with respect to $\mathcal{B}_{t+}(\mathbf{W}^m)$. To do so, for fixed $t \geq 0$, let $j_{i,t}$ be such that $t_{j_{i,t}-1}^i < t \leq t_{j_{i,t}}^i$. For all $s \in [0, t]$,

$$\eta^i(s, \zeta) = \eta_0(0, \zeta(. \wedge 0))I_{\{0\}}(s) + \sum_{j=1}^{j_{i,t}} \eta_{t_j^i}(s, \zeta(. \wedge t_j^i))I_{(t_{j-1}^i, t_j^i]}(s)$$

since $s \leq t \leq t_{j_{i,t}}^i$. This shows that η^i restricted to $s \in [0, t]$ is $\mathcal{B}[0, t] \times \mathcal{B}_{t_{j_{i,t}}^i}(\mathbf{W}^m)/\mathcal{B}(\mathbb{R})$-measurable.

For $a \in \mathbb{R}$,

$$\begin{aligned} \eta^{-1}([a, \infty)) =& \{(t, \zeta) \in [0, \infty) \times \mathbf{W}^m \mid \lim_{i \to \infty} \sup_{k \geq i} \eta^k(t, \zeta) \geq a\} \\ &= \cap_{i \geq 1} \{(t, \zeta) \in [0, \infty) \times \mathbf{W}^m \mid \sup_{k \geq i} \eta^k(t, \zeta) \geq a\}. \end{aligned}$$

Now, note that

$$\sup_{k \geq i} \eta^k(t, \zeta) \in \mathcal{B}[0, t] \times \mathcal{B}_{r(i)}(\mathbf{W}^n)/\mathcal{B}(\mathbb{R})$$

with $r(i) = \sup_{k \geq i} t_{j_{k,t}}^k$, and $r(i) \to t$ as $i \to 0$.

This implies that $\eta^{-1}([a, \infty)) \in \mathcal{B}[0, t] \times \mathcal{B}_{t+}(\mathbf{W}^n)$. Since the collection of all intervals in the form of $[a, \infty)$ generates $\mathcal{B}(\mathbb{R})$, the map $(s, \omega) \mapsto \eta(s, \omega)$ is

$\mathcal{B}[0,t] \times \mathcal{B}_{t+}(\mathbf{W}^m)/\mathcal{B}(U)$-measurable. As the argument holds for any $t \geq 0$, we see that η is progressively measurable with respect to $\mathcal{B}_{t+}(\mathbf{W}^m)$.

Now we will show that η satisfies equation (3.3).

For fixed t and $\omega \in \Omega$, we know that $\xi(. \wedge t, \omega) \in \mathbf{W}^m = \xi(\Omega)$, so there exist ω' such that

$$\xi(., \omega') = \xi(. \wedge t, \omega). \tag{3.7}$$

We also know

$$\xi(., \omega') = \xi(. \wedge s, \omega') \quad \forall s \geq t, \tag{3.8}$$

since $\xi(., \omega')$ is constant over $[t, \infty)$.

Then if we consider again $j_{i,t}$ as defined above, we have by (3.5)

$$\eta^i(t, \xi(. \wedge t^i_{j_{i,t})}, \omega')) = \varphi(t, \omega'). \tag{3.9}$$

Combining (3.7), (3.8) and (3.9), we have

$$\varphi(t, \omega') = \xi(. \wedge t^i_{j_{i,t}}, \omega') = \xi(., \omega') = \eta^i(t, \xi(. \wedge t, \omega)).$$

By the definition of η^i in (3.6), we have

$$\eta(t, \xi(. \wedge t, \omega)) = \limsup_{i \to \infty} \eta^i(t, \xi(. \wedge t, \omega)) = \varphi(t, \omega'). \tag{3.10}$$

It remains to show that $\varphi(t, \omega) = \varphi(t, \omega')$. Since φ is progressively measurable with respect to $\{\mathcal{F}^\xi_t\}$, we know that $\varphi(t, .)$ is $\mathcal{F}^\xi_t$-measurable, where $\mathcal{F}^\xi_t$ is generated by sets of the form $\xi(s, .)^{-1}(E)$, for $E \in \mathcal{B}(\mathbb{R}^m)$ and $s \in [0, t]$. Each of these sets must contain either both or none of ω and ω' because we have $\xi(s, \omega) = \xi(s, \omega')$ for all $s \in [0, t]$. This implies that for any $A \in \mathcal{F}^\xi_t$, we have either

$$\omega \in A, \omega' \in A,$$

or

$$\omega \notin A, \omega' \notin A.$$

Now, let $A = [\varphi(t, .)]^{-1}(\{\varphi(t, \omega)\})$. We see that $A \in \mathcal{F}^\xi_t$ and $\omega \in A$. It follows that we must have

$$\omega' \in A = [\varphi(t, .)]^{-1}(\{\varphi(t, \omega)\}).$$

This means that

$$\varphi(t, \omega') = \varphi(t, \omega). \tag{3.11}$$

Together with (3.10), this yields equation (3.3). □

3.3.2 Constructing a contract

Now we will explain the conditions under which a solution of our control problem could be used to construct a contract for the original problem.

We first introduce a condition that will be used for the construction. The following conditions says that, given the output process $\{X_t\}$ and how the recommended effort on Z, we could infer an perceived noise process $\{\hat{Z}_t\}$ such that the output process is the path resulting from the recommended action (depending on $\{\hat{Z}_t\}$ as if it were $\{Z_t\}$) and the perceived noise process $\{\hat{Z}_t\}$.

Condition 3.1. Given $(\Omega, \mathcal{F}, \mathbf{P}, Z, C, A, Y, \tau) \in \mathcal{U}(w)$, let φ^A the representation, according to Lemma 3.3, such that

$$A_t(\omega) = \varphi^A(t, Z(. \wedge t, \omega)) \quad \mathbf{P}\text{-a.s.} \;\; \omega \in \Omega, \, \forall t \in [0, \infty).$$

Then this condition requires that there is a unique $\{\hat{Z}_t\}$ that is progressively measurable with respect to $\{\mathcal{F}_t^X\}$, such that

$$X_t = \int_0^{t\wedge\tau} \varphi^A(s, \hat{Z}_{.\wedge s})ds + \sigma \hat{Z}_t \quad \mathbf{P}^{A'}\text{-a.s.} \tag{3.12}$$

for any A' that is progressively measurable with respect to $\{\mathcal{F}_t^X\}$ and A'_t takes values from $[0, \bar{a}]$ at each time $t < \infty$.

Note that in the above condition, when $A' = 0$, X is a $\mathbf{P}^0$-Brownian motion and the process $\{\hat{Z}_t\}$ is a strong solution to the stochastic differential equation (3.12) with a non-Markovian drift and the Brownian motion X.

In the following, we try to construct a contract that is based on the output $\{X_t\}$, from a solution of the control problem in our formulation. Suppose $(\Omega, \mathcal{F}, P, Z, C, A, Y, \tau) \in \mathcal{U}(w)$ satisfies Condition 3.1. Then we will construct a compensation-effort pair $(\tilde{C}, \tilde{A})$, where both $\tilde{C}$ and $\tilde{A}$ are progressively measurable with respect to $\{\mathcal{F}_t^X\}$, such that under this contract it is optimal for the agent to choose A. Also, we have $Y_t(\tilde{C}, \tilde{A}) = Y_t$ and $\tilde{A}_t = 0$, $\tilde{Y}_t = 0$ for $t \geq \tau$. Note that $\mathbf{P}^{A'}$ is the unique probability measure defined at the beginning of Section 3.1 corresponding to the process A'.

Construction of the contract

We are given that $(\Omega, \mathcal{F}, P, Z, C, A, Y, \tau) \in \mathcal{U}(w)$ satisfies Condition 3.1. We would like to find corresponding controls defined on the probability space

$(\Omega^X, \mathcal{F}^X, \{\mathcal{F}_t^X\}, \mathbf{P}^0)$. Let $\{\hat{Z}_t\}$ be the process specified in Condition 3.1. By Lemma 3.3, we have φ^C, φ^A and φ^Y such that

$$A_t(\omega) = \varphi^A(t, Z(. \wedge t, \omega)) \quad \mathbf{P}\text{-a.s. } \omega \in \Omega, \forall t \in [0, \infty),$$

$$C_t(\omega) = \varphi^C(t, Z(. \wedge t, \omega)) \quad \mathbf{P}\text{-a.s. } \omega \in \Omega, \forall t \in [0, \infty).$$

$$Y_t(\omega) = \varphi^Y(t, Z(. \wedge t, \omega)) \quad \mathbf{P}\text{-a.s. } \omega \in \Omega, \forall t \in [0, \infty).$$

and

$$\mathbf{1}_{\tau \le t}(\omega) = \varphi^\tau(t, Z(. \wedge t, \omega)) \quad \mathbf{P}\text{-a.s. } \omega \in \Omega, \forall t \in [0, \infty).$$

Then we can define stopping time $\tilde{\tau}$ to be a $\{\mathcal{F}_t^X\}$-stopping time by setting

$$\mathbf{1}_{\tilde{\tau} \le t}(\omega) = \varphi^\tau(t, \tilde{Z}(. \wedge t, \omega))$$

for all $t > 0$ and $\omega \in \Omega$. We can also define

$$\tilde{C}_t = \begin{cases} \varphi^C(t, \hat{Z}(. \wedge t)), & 0 \le t \le T \wedge \tilde{\tau} \\ -F_0(\tilde{W}_{\tilde{\tau}}), & \tilde{\tau} < t \le T, \end{cases}$$

$$\tilde{A}_t = \begin{cases} \varphi^A(t, \hat{Z}(. \wedge t)) & 0 \le t \le T \wedge \tilde{\tau} \\ 0 & \tilde{\tau} < t \le T, \end{cases}$$

and

$$\tilde{Y}_t = \begin{cases} \varphi^Y(t, \hat{Z}(. \wedge t)) & 0 \le t \le T \wedge \tilde{\tau} \\ 0 & \tilde{\tau} < t \le T, \end{cases}$$

where $\tilde{W}_t := W_t(\tilde{C}, \tilde{A}, \tilde{Y}; w)$. Since $\hat{Z}$ is progressively measurable with respect to $\{\mathcal{F}_t^X\}$, we see that, by Lemma 3.2, $\tilde{C}$, $\tilde{A}$ and $\tilde{Y}$ are all progressively measurable with respect to $\{\mathcal{F}_t^X\}$. Now that $\tilde{A}$ is well-defined. We could apply the change of probability measure as mentioned in Section 3.1 to get the probability measure $\mathbf{P}^{\tilde{A}}$.

The agent's utility from the contract

Obviously, $(\tilde{W}, \tilde{Z})$ is a weak solution to the stochastic differential equation

$$d\tilde{W}_t = r(\tilde{W}_t - u(\tilde{C}_t) + h(\tilde{A}_t))dt + r\sigma \tilde{Y}_t d\tilde{Z}_t; \quad \tilde{W}_0 = w.$$

The utility that the agent derives from the contract by choosing action $\tilde{A}$ is

$$E^{\tilde{A}}\left[\int_0^\tau re^{-rt}(u(\tilde{C}_t) - h(\tilde{A}_t))dt + e^{-r\tau}F_0(\tilde{W}_\tau)\right] = w,$$

where the equality is obtained by applying Ito's lemma on $\tilde{W}_\tau$ and noting that $\{\tilde{W}_t\}$ is bounded.

Incentive-compatibility of the contract

From the fact that

$$d\tilde{W}_t = r(\tilde{W}_t - u(\tilde{C}_t) + h(\tilde{A}_t))dt + r\sigma\tilde{Y}_t d\tilde{Z}_t; \quad \tilde{W}_0 = w,$$

we can see immediately, by the uniqueness of the representation in $Y_t(\tilde{C}, \tilde{A})$ by Proposition 1.1, that $Y_t(\tilde{C}, \tilde{A}) = \tilde{Y}_t$ for almost all t, almost surely. Our construction of $\tilde{A}$ and $\tilde{Y}$ was based on A, Y, where (A_t, Y_t) takes values from Γ. This ensures that $(\tilde{A}, \tilde{Y}) \in \Gamma$ almost surely for $t > 0$ a.e. By Proposition 1.2, it follows that the contract $(\tilde{C}, \tilde{A})$ is incentive-compatible.

3.3.3 The perceived noise process for a Markovian contract

The above shows that a solution in our formulation leads to a feasible contract in the original setting in (1.9), under Condition 3.1. We are particularly interested in checking the validity of this condition in Sannikov's solution to the problem, where the policy is Markovian in the continuation value W_t.

In the following, we show that if a policy is Markovian in the continuation value W_t, then under some assumptions, we can find a unique corresponding continuation value process $\{W_t\}$ and perceived noise process $\{\hat{Z}_t\}$ that is progressively measurable with respect to $\mathcal{F}_t^X$.

Proposition 3.1. *Given* $0 < w_{gp} \leq \bar{w}$, $a^* : [0, w_{gp}] \to [0, \bar{a}]$, $c^* : [0, w_{gp}] \to [0, \bar{c}]$, *with the assumptions*

1. $a(0) = a(\bar{w}) = 0, \quad c(0) = 0, \quad c(w_{gp}) = u^{-1}(w_{gp})$.
2. $h(a(.))$, $u(c(.))$, $\gamma(a(.))$ *and* $\gamma(a(.))a(.)$ *are Lipschitz over* $[0, w^{gp}]$.

Suppose Z_t *is a Brownian motion and given*

$$X_t = \int_0^t A_s ds + \sigma Z_t, \tag{3.13}$$

where A_t *takes values from* $[0, \bar{a}]$ *and is progressively measurable with respect to* $\{\mathcal{F}_t^X\}$. *Then there is a unique pair* $(W_t, \hat{Z}_t)$, *progressively measurable with* *respect to* $\mathcal{F}_t^X$*, such that*

$$X_t = \int_0^{t\wedge\tau} a^*(W_s)ds + \sigma\hat{Z}_t, \tag{3.14}$$

$$W_t = w_0 + \int_0^{t\wedge\tau} r(W_s - u(c^*(W_s)) + h(a^*(W_s)))ds + \int_0^{t\wedge\tau} r\sigma\gamma(a^*(W_s))d\hat{Z}_s, \tag{3.15}$$

with $\tau = \inf\{t > 0 : W_t \notin (0, w_{gp})\}$.

Proof. Combining equations (3.13), (3.14) and (3.15), we have

$$\begin{aligned} W_t = w_0 &+ \int_0^{t\wedge\tau} r(W_s - u(c^*(W_s)) + h(a^*(W_s)))ds \\ &+ \int_0^{t\wedge\tau} r\gamma(a^*(W_s))(A_s ds - a^*(W_s)ds + \sigma dZ_s). \end{aligned} \tag{3.16}$$

Rearranging we have,

$$W_t = w_0 + \int_0^{t\wedge\tau} (b(W_s) + \hat{\sigma}(W_s)A_s/\sigma)ds + \int_0^{t\wedge\tau} \hat{\sigma}(W_s)dZ_s, \tag{3.17}$$

where we extend the drift and volatility such that

$$\begin{aligned} b(w) = r[w &- u(c^*((w \wedge w^{gp})^+)) \\ &+ h(a^*((w \wedge w^{gp})^+)) - \gamma(a^*((w \wedge w^{gp})^+))a^*((w \wedge w^{gp})^+)] \end{aligned}$$

and

$$\hat{\sigma}(w) = r\sigma\gamma(a((w \wedge w^{gp})^+)).$$

We can see that b and $\hat{\sigma}$ are both Lipschitz and bounded due to our assumptions. Also, $\gamma(a(.))a(.)$ is Lipschitz as

$$\begin{aligned} &|\gamma(a(w_1))a(w_1) - \gamma(a(w_2))a(w_2)| \leq \\ &\qquad |\gamma(a(w_1))a(w_1) - \gamma(a(w_2))a(w_1) + \gamma(a(w_2))(a(w_1) - a(w_2)| \end{aligned}$$

Fix $T > 0$. We then consider

$$W_t = w_0 + \int_0^{t} (b(W_s) + \hat{\sigma}(W_s)A_s/\sigma)ds + \int_0^{t} \hat{\sigma}(W_s)dZ_s, \tag{3.18}$$

for all $0 \leq t \leq T$.

Noting that $A_s \in [0, \bar{a}]$, we obtain uniform Lipschitz condition on the drift over all $\omega \in \Omega$, since

$$\begin{aligned}|b(w_1)+&\hat{\sigma}(w_1)A_s/\sigma - b(w_2) + \hat{\sigma}(w_2)A_s/\sigma| \\ &\leq |b(w_1) - b(w_2)| + (\bar{a}/\sigma)|\hat{\sigma}(w_1) - \hat{\sigma}(w_2)| \leq L|w_1 - w_2|\end{aligned}$$

where L is independent of ω.

Then, applying Theorem 6.16 of Yong and Zhou (1999, P.49), equation (3.18) admits a unique solution W up to time T.

Now that W_t is uniquely defined up to time T, we have for any $t \leq T$,

$$W_{t\wedge\tau} = W_t \mathbf{1}_{t\leq\tau} + W_\tau \mathbf{1}_{t\geq\tau}$$

is well-defined. Since T is arbitrary, together with the uniqueness of W, we see that $W_{t\wedge\tau}$ is in fact uniquely defined for any $0 \leq t < \infty$. (Note that we do not require $W_{t\wedge\tau}$ to converge as t approaches infinity.) Once $\{W_t\}$ is well-defined, $\{\hat{Z}_t\}$ is also well-defined by equation (3.14). □

Note that the first condition in Proposition 3.1 means permanent retirement at the points 0 and w^{gp}.

While the proof above makes use of the unobserved processes A and Z, the processes W and $\hat{Z}$ can be obtained without observing A and Z since equation (3.14) and (3.15) only involve X directly but not A and Z. This means that after setting the recommended effort and the compensation based on some Z, the principal can indeed infer a perceived noise process $\hat{Z}$, on which the compensation to the agent is based. In particular, the principal can do this by an equidistant discrete-time Euler approximation of the equation

$$W_t = w_0 + \int_0^t b(W_s)ds + \int_0^t \hat{\sigma}(W_s)/\sigma \; dX_s. \tag{3.19}$$

It should be noted that this $\hat{Z}$ will not coincide with the real noise process Z that the agent observes if the agent chooses not to follow the recommended effort. However, this process is well-defined and so is the agreed compensation under the policy designated by $a^*(.)$ and $c^*(.)$.

The following proposition shows the convergence of the Euler approximation scheme, and therefore also shows that the principal can implement the contract without actually observing the hidden effort A and real noise process Z:

Proposition 3.2. *For any fixed time $T > 0$, for all $0 \le t \le T$, define $\hat{W}_t^\delta = W_{\lfloor t/\delta \rfloor}^\delta$, where*

$$W_{n+1}^\delta = W_n^\delta + b(W_n^\delta)\delta + \hat{\sigma}(W_n^\delta)(X_{n\delta} - X_{(n-1)\delta})/\sigma, \quad W_0^\delta = w_0, \quad (3.20)$$

which does not require the observation of A and Z. For any fixed $T > 0$, the process $\{\hat{W}_t^\delta\}_{0 \le t \le T}$ converges to $\{W_t\}_{0 \le t \le T}$ strongly as δ goes to zero, i.e.

$$\lim_{\delta \downarrow 0} E(|W_T - \hat{W}_T^\delta|) = 0.$$

Note that the approximation is limited to a finite time horizon, because a principal will only be able to use an approximation at a time T and would only have the information up to time T to approximate the perceived noise process $\hat{Z}_t$ (also only up to time T).

Proof. Our proof is similar to the proof of Theorem 9.6.2 in Kloeden and Platen (1992, P.324–326), but adapted to the fact that we do not directly observe the Brownian motion.

For $0 \le t \le T$, we set

$$Z_t = \sup_{0 \le s \le t} E(|\hat{W}_s^\delta - W_s|^2)$$

and note that, letting $n_s = \lfloor s/\delta \rfloor$ for all s,

$$
\begin{aligned}
Z_t = &\sup_{0\le s\le t} E\left(\left|\sum_{n=0}^{n_s-1}(W^\delta_{n+1} - W^\delta_n) - \int_0^s b(W_r)dr - \int_0^s \hat{\sigma}(W_r)/\sigma dX_r\right|^2\right) \\
= &\sup_{0\le s\le t} E\left(\left|\int_0^{n_s\delta}(b(W^\delta_{n_r}) - b(W_r))dr + \int_0^{n_s\delta}(\hat{\sigma}(W^\delta_{n_r}) - \hat{\sigma}(W_r))/\sigma dX_r\right.\right. \\
&\left.\left. - \int_{n_s\delta}^s b(W_r)dr - \int_{n_s\delta}^s \hat{\sigma}(W_r)/\sigma dX_r\right|^2\right) \\
\le &C \sup_{0\le s\le t}\left\{E\left(\left|\int_0^{n_s\delta}(b(W^\delta_{n_r}) - b(W_r))dr\right|^2\right)\right. \\
&+ E\left(\left|\int_0^{n_s\delta}(\hat{\sigma}(W^\delta_{n_r}) - \hat{\sigma}(W_r))/\sigma dX_r\right|^2\right) \\
&\left.+E\left(\left|\int_{n_s\delta}^s b(W_r)dr\right|^2\right) + E\left(\left|\int_{n_s\delta}^s \hat{\sigma}(W_r)/\sigma dX_r\right|^2\right)\right\}
\end{aligned}
$$

Now, looking at the individual terms, we have the first term being

$$
\begin{aligned}
E\left(\left|\int_0^{n_s\delta}(b(W^\delta_{n_r}) - b(W_r))dr\right|^2\right) \le &E\left(\left|\int_0^{n_s\delta} K(W^\delta_{n_r} - W_r)dr\right|^2\right) \\
\le &TE\left(\int_0^{n_s\delta} K^2(W^\delta_{n_r} - W_r)^2 dr\right) \qquad (3.21) \\
\le &K^2 T\int_0^{n_s\delta} Z_r dr.
\end{aligned}
$$

The second term is

$$\begin{aligned}
&E\left(\left|\int_0^{n_s\delta}(\hat{\sigma}(W_{n_r}^{\delta})-\hat{\sigma}(W_r))/\sigma dX_r\right|^2\right)\\
\leq &E\left(\left|\int_0^{n_s\delta}K|W_{n_r}^{\delta}-\hat{\sigma}(W_r)|dX_r\right|^2\right)\\
\leq &2E\left(\left|\int_0^{n_s\delta}K|W_{n_r}^{\delta}-\hat{\sigma}(W_r)|\bar{a}dr\right|^2\right)+2E\left(\left|\int_0^{n_s\delta}K|W_{n_r}^{\delta}-\hat{\sigma}(W_r)|\sigma dZ_r\right|^2\right)\\
\leq &2sE\left(\int_0^{n_s\delta}K^2|W_{n_r}^{\delta}-\hat{\sigma}(W_r)|^2\bar{a}^2dr\right)+2E\left(\int_0^{n_s\delta}K^2|W_{n_r}^{\delta}-\hat{\sigma}(W_r)|^2\sigma^2dr\right)\\
\leq &K'(1+T)\int_0^{n_s\delta}Z_rdr.
\end{aligned} \tag{3.22}$$

The third term is

$$E\left(\left|\int_{n_s\delta}^{s}b(W_r)dr\right|^2\right)\leq\delta(r(\bar{w}-u(\bar{c})+h(\bar{a})-\gamma(\bar{a})\bar{a})^2\leq K_1^2\delta. \tag{3.23}$$

The fourth term is

$$\begin{aligned}
E\left(\left|\int_{n_s\delta}^{s}\hat{\sigma}(W_r)/\sigma dX_r\right|^2\right)\leq &E\left(\left|\int_{n_s\delta}^{s}\hat{\sigma}(W_r)A_rdr+\int_{n_s\delta}^{s}\hat{\sigma}(W_r)dZ_r\right|^2\right)\\
\leq &2(\delta\sup_{w\in[0,\bar{w}]}\hat{\sigma}(w)\bar{a})^2+2E\left[\int_{n_s\delta}^{s}(\hat{\sigma}(W_r))^2dr\right]\\
\leq &2(\delta\sup_{w\in[0,\bar{w}]}\hat{\sigma}(w)\bar{a})^2+2\delta\left(\sup_{w\in[0,\bar{w}]}\hat{\sigma}(w)\right)^2\\
\leq &K'(\delta^2+\delta).
\end{aligned} \tag{3.24}$$

Combining equations (3.21) – (3.24), we have

$$\begin{aligned}
Z_t&\leq C_1\sup_{0\leq s\leq t}\{(K'(1+T)+K^2T)\int_0^{n_s\delta}Z_rdr+K_1^2\delta+K'(\delta^2+\delta)\}\\
&=C_2\int_0^t Z_rdr+C_3(\delta+\delta^2).
\end{aligned}$$

By Gronwall's inequality, we have

$$Z_t \leq C_4(\delta + \delta^2).$$

Finally,

$$E(|\hat{W}_T^\delta - W_T|) \leq \sqrt{E(|\hat{W}_T^\delta - W_T|^2)} = \sqrt{Z_T} \leq \sqrt{C_4(\delta + \delta^2)},$$

which goes to zero as $\delta \downarrow 0$. $\square$

Chapter 4

Our Viscosity Solution Approach

In this chapter, we look at the HJB equation of our control problem and consider its viscosity solution. The goal is characterize the optimal profit of our control problem as the viscosity solution of the HJB equation. The motivation of our approach, as compared to the work in Sannikov (2008), is to develop a methodology to study the problem that does not rely on the solution being C^2 almost everywhere and thus is more easily extensible to other cases. We will first state the HJB equation and its boundary conditions, and then define viscosity solutions for the equation.

To simplify our notations and allow easy reference to existing theorems, define

$$H_{a,c,y}(w,p,\alpha) = r(a-c) + rp(w - u(c) + h(a)) + \frac{\alpha}{2} r^2 y^2 \sigma^2$$

$$\mathcal{H}(w,p,\alpha) = \sup_{a \in \mathcal{A}, c} H_{a,c,\gamma(a)}(w,p,\alpha).$$

Now consider the HJB equation

$$\max\{\mathcal{H}(w, F'(w), F''(w)) - rF(w),\ \tilde{F}_0(w) - F(w)\} = 0, \quad w \in [0, \bar{w}] \quad (4.1)$$

with

$$F(0) = \tilde{F}_0(0) \text{ and } F(\bar{w}) = \tilde{F}_0(\bar{w}). \quad (4.2)$$

Our HJB equation differs from the HJB equation proposed in Sannikov (2008) in a few aspects. First, recall that as reviewed in Section 1.4, the HJB equation first proposed by Sannikov was

$$rF(W) = \max_{a>0,c} r(a-c)+F'(W)r(W-u(c)+h(a))+\frac{F''(W)}{2}r^2\gamma(a)^2\sigma^2, \quad (1.14)$$

which is equivalent to saying $\mathcal{H}(w, F'(w), F''(w)) = 0$ for $w \in [0, w_{gp}]$. This is the same as the first part of our equation (4.1). However, Sannikov mostly worked with the equation

$$F''(W) = \min_{a>0,c} \frac{F(W) - a + c - F'(W)(W - u(c) + h(a))}{r\gamma(a)^2\sigma^2/2}, \quad (1.15)$$

which is equivalent to (1.14) because $\gamma(a) > 0$ for $a > 0$. In comparison, while we do not require that the effort is always nonzero, we impose a minimum sensitivity $\gamma(0) = \gamma_0$ to avoid degeneracy in the problem.

A greater difference between our HJB equation and Sannikov's one lies in the boundary conditions. Sannikov's boundary and smooth-pasting conditions were:

$$F(0) = 0, F(w_{gp}) = F_0(w_{gp}) \text{ and } F'(w_{gp}) = F_0'(w_{gp}). \quad (1.16)$$

This imposes a structural assumption on the solution that the continuation region of w, i.e. the set of continuation value where it is not optimal for the principal to permanently retire the agent, is an interval $(0, w_{gp})$. With the benchmark model in Sannikov (2008), this is plausible in an economic sense, but was not proved a priori to the setup of the framework and the verification of the solution to the ODE as the optimal profit function. When we allow the retirement function $\tilde{F}_0$ to be something other than $F_0(w) = -u^{-1}(w)$, it becomes possible that the continuation region is not connected. See Section 5.2.1 for more discussion.

In comparison, the approach we have in setting up our HJB equation is that we require (formally) $\mathcal{H}(w, F'(w), F''(w)) \leq 0$ and $\tilde{F}_0(w) \leq F(w)$ at all times, and that one of the two inequalities holds with equality. This loosely translates to requiring that either the principal continues in an optimal way, or she retires the agent permanently, and she must choose the better of the two. We do require permanent retirement at continuation value 0 and $\bar{w}$, because those are the minimum and maximum continuation value attainable for the agent, and can only be attained by subjecting the agent to no more risk. Our HJB equation would potentially allow for cases where the continuation region of w may not be in the form of one open interval $(0, w_{gp})$.

Sannikov's proof of optimality of the solution of the HJB equation as the optimal profit relied on the fact that the solution is C^1 over these regions and C^2 in the interior of each of these. In general, however, if the model is changed slightly with addition of other elements, solutions may not have enough smoothness and Sannikov's methodology may not be extensible, so we introduce the notion of viscosity solutions to our problem. (See Section 2.3 for a review of viscosity solutions.) In the following, we give the definition for viscosity subsolution, supersolution and solution for our problem. These are consistent with the definitions we have reviewed, but additionally requires the boundary conditions to hold.

Definition 4.1. We say that an upper-semicontinuous function $F : [0, \bar{w}] \to \mathbb{R}$ is a *viscosity subsolution* of equations (4.1)–(4.2) if

(i) $F(0) = \tilde{F}_0(0)$ and $F(\bar{w}) = \tilde{F}_0(\bar{w})$, and

(ii) for every $\varphi \in C^2((0, \bar{w}))$ and $w_0 \in (0, \bar{w})$ such that $\varphi - F$ attains a local minimum at w_0 with $\varphi(w_0) = F(w_0)$, we have

$$\max\{\mathcal{H}(w_0, \varphi'(w_0), \varphi''(w_0)) - r\varphi(w_0), \tilde{F}_0(w_0) - \varphi(w_0)\} \geq 0. \quad (4.3)$$

Definition 4.2. We say that a lower-semicontinuous function $F : [0, \bar{w}] \to \mathbb{R}$ is a *viscosity supersolution* of equations (4.1)–(4.2) if

(i) $F(0) = \tilde{F}_0(0)$ and $F(\bar{w}) = \tilde{F}_0(\bar{w})$, and

(ii) for every $\varphi \in C^2((0, \bar{w}))$ and $w_0 \in (0, \bar{w})$ such that $\varphi - F$ attains a local maximum at w_0 with $\varphi(w_0) = F(w_0)$, we have

$$\max\{\mathcal{H}(w_0, \varphi'(w_0), \varphi''(w_0)) - r\varphi(w_0), \tilde{F}_0(w_0) - \varphi(w_0)\} \leq 0. \quad (4.4)$$

Definition 4.3. We say that a continuous function $F : [0, \bar{w}] \to \mathbb{R}$ is a *viscosity solution* of equations (4.1)–(4.2) if F is both a viscosity subsolution and supersolution.

4.1 Main theorem

Recall that F is defined in (3.2) as

$$F(w) = \sup_{(C,A,Y,\tau) \in \mathcal{U}(w)} J(C, A, Y, \tau; w),$$

where $\mathcal{U}(w)$ is the set of admissible control satisfying (**U1**) – (**U7**).

Under assumptions (A1)–(A4) (stated in Sections 3.1 and 3.2), we prove that F is concave and continuous in $[0, \bar{w}]$. More importantly, we have:

Theorem 4.1. *F is a viscosity solution of (4.1)–(4.2).*

Theorem 4.2. *The viscosity solution to (4.1)–(4.2) is unique.*

The rest of the chapter is dedicated to the development and proof of these results.

4.2 Preliminary

First, we look at the boundedness of F. Note that

$$\begin{aligned} F(W) &= \sup_{(C,A,\tau,Y)\in\mathcal{U}(w)} E\left[\int_0^\tau re^{-rt}(A_t - C_t) + e^{-r\tau}\tilde{F}_0(W_\tau)\right] \\ &\leq E\left[\int_0^\infty re^{-rt}(\bar{a} - 0)\right] + \sup_{w\in[0,\bar{w}]} \tilde{F}_0(w) = \bar{a} + \sup_{w\in[0,\bar{w}]} \tilde{F}_0(w) \end{aligned}$$

as $A_t \leq \bar{a}$, $C_t \geq 0$ and $\sup_{w\in[0,\bar{w}]} \tilde{F}_0(w) < \infty$ as $\tilde{F}_0$ is continuous. On the other hand, by setting $\tau = 0$, we see

$$F(w) \geq \tilde{F}_0(w) \geq F_0(w) \geq F_0(\bar{w}) = -\bar{c}.$$

Thus F is bounded.

4.3 First-best profit

The first-best profit represents the maximum expected profit that can be achieved if the principal can control directly the consumption and effort of the agent. This has also been considered briefly in Section 5 of Sannikov (2008). This function is useful as it provides an upper bound for the optimal profit and helps us obtain concavity of the optimal profit function F.

This is also where we need the assumption that $\mathcal{A} = [0, \bar{a}]$ here.

For $-h(\bar{a}) \le w \le u(\bar{c})$, define

$$\bar{F}(w) = \max_{\substack{u(c)-h(a)=w,\\ a\in\mathcal{A},\ c\in[0,\bar{c}]}} (a-c)$$
$$= \max_{\substack{\upsilon-\eta=w,\\ h^{-1}(\eta)\in\mathcal{A},\ \upsilon\in[0,\bar{w}]}} (h^{-1}(\eta) - u^{-1}(\upsilon)).$$

Note that $\bar{F}$ is well-defined for any $w \in [-h(\bar{a}), u(\bar{c})]$ because the feasible set is nonempty (e.g. $c = u^{-1}(w)$ and $a = 0$ is feasible for nonnegative w, while $c = 0$ and $a = h^{-1}(-w)$ is feasible for negative w). The second representation is the maximization of a continuous function over a compact set, which ensures that the supremum exists and is attainable. Note that $\bar{F}(\bar{w}) = -\bar{c}$, and we see that $\bar{F}(w) \ge F_0(w)$ for $w \in [0, \bar{w}]$ by considering $a = 0$ and $c = u^{-1}(w)$. This implies that $\bar{F}(0) \ge 0$.

We call $\bar{F}$ the *first-best profit*, because $\bar{F}(w)$ represents the profit of the principal when she can perfectly control the agent's consumption and effort and decides to give the agent initial value w. This is not immediate from the definition of F; however, the fact that $\bar{F}$ is concave (as we show in Lemma 4.2) proves so. It is worth noting that $\bar{F}(0)$ could be positive, reflecting that the difference in the utility function of the agent and principal can create positive profit for principal even when the agent receives nothing, in the case when the principal can control the agent's actions directly.

Lemma 4.1. *$\bar{F}$ is decreasing.*

Proof. Consider some fixed $-h(\bar{a}) < w \le u(\bar{c})$ and a, c such that $\bar{F}(w) = a - c$ with $u(c) - h(a) = w$. Now consider any w' such that $-h(\bar{a}) \le w' < w$, and would like to show that $\bar{F}(w') > \bar{F}(w)$. If $u(0) - h(a) = -h(a) \le w'$, then we could let c' be such that

$$u(c') = u(c) - (w - w'),$$

and so $a - c' > a - c$. This implies

$$\bar{F}(w') = \max_{u(c)-h(a)=w', a\in\mathcal{A}, c\in[0,\bar{c}]} (a-c) \ge a - c' > a - c = \bar{F}(w).$$

If $-h(a) > w'$, we could set $c = 0$ and $a' = h^{-1}(-w') > h^{-1}(u(c) - w) = a$. So $a - c > a' - c$. This implies

$$\bar{F}(w') = \max_{u(c)-h(a)=w', a\in\mathcal{A}, c\in[0,\bar{c}]} (a-c) \ge a' - c > a - c = \bar{F}(w).$$

□

Lemma 4.2. *$\bar{F}$ is concave over $[-h(\bar{a}), u(\bar{c})]$.*

The proof explicitly makes use of the assumption that $\mathcal{A} = [0, \bar{a}]$, which implies that $\mathcal{A}$ is a convex set.

Proof. Recall that

$$\bar{F}(w) = \max_{\substack{\upsilon - \eta = w, \\ h^{-1}(\eta) \in \mathcal{A},\ \upsilon \in [0, \bar{w}]}} (h^{-1}(\eta) - u^{-1}(\upsilon)).$$

Suppose $\bar{F}(w_1) = h^{-1}(\eta_1) - u^{-1}(\upsilon_1)$ with $\upsilon_1 - \eta_1 = w_1$ and $\bar{F}(w_2) = h^{-1}(\eta_2) - u^{-1}(\upsilon_2)$ with $\upsilon_2 - \eta_2 = w_2$.

Then, for $w = \lambda w_1 + (1-\lambda) w_2$ with $\lambda \in (0, 1)$, consider $\upsilon = \lambda \upsilon_1 + (1-\lambda)\upsilon_2$ and $\eta = \lambda \eta_1 + (1-\lambda)\eta_2$. Obviously $\upsilon - \eta = \lambda w_1 + (1-\lambda) w_2 = w$, and $h^{-1}(\eta) \in \mathcal{A}$ and $\upsilon \in [0, \bar{w}]$. And so

$$\begin{aligned}
\bar{F}(w) &\geq h^{-1}(\eta) - u^{-1}(\upsilon) \\
&= h^{-1}(\lambda \eta_1 + (1-\lambda)\eta_2) - u^{-1}(\lambda \upsilon_1 + (1-\lambda)\upsilon_2) \\
&\geq \lambda h^{-1}(\eta_1) + (1-\lambda) h^{-1}(\eta_2) - \lambda u^{-1}(\upsilon_1) - (1-\lambda) u^{-1}(\upsilon_2) \\
&= \lambda \bar{F}(w_1) + (1-\lambda)\bar{F}(w_2),
\end{aligned}$$

where the last inequality follows from concavity of h^{-1} and $-u^{-1}$. Thus $\bar{F}$ is concave over $[-h(\bar{a}), u(\bar{c})]$. □

The next proposition shows that the $\bar{F}$ is an upper bound for F.

Proposition 4.1. *We have $F(w) \leq \bar{F}(w)$ for all $w \in [0, \bar{w}]$.*

Proof. First note that

$$J(C, A, Y, \tau; w) = E\left[\int_0^\tau r e^{-rt}(A_t - C_t)dt + e^{-r\tau}\tilde{F}_0(W_\tau)\right].$$

We define

$$\xi_t = \begin{cases} u(C_t) - h(A_t) & t < \tau \\ W_\tau & t \geq \tau. \end{cases}$$

Note that ξ_t is bounded in the interval $[-h(\bar{a}), \bar{w}]$. We have

$$\begin{aligned} J(C, A, Y, \tau; w) &\leq E\left[\int_0^\tau re^{-rt}\bar{F}(\xi_t)dt + e^{-r\tau}\tilde{F}_0(\xi_t)\right] \\ &\leq E\left[\int_0^\tau re^{-rt}\bar{F}(\xi_t)dt + e^{-r\tau}\bar{F}(\xi_t)\right] \\ &= E\left[\int_0^\infty re^{-rt}\bar{F}(\xi_t)dt\right]. \end{aligned}$$

Then noting that $\bar{F}$ is concave, we have

$$\begin{aligned} J(C, A, Y, \tau; w) &\leq E\left[\int_0^\infty re^{-rt}\bar{F}(\xi_t)dt\right] \\ &= \int_0^\infty re^{-rt}E\left[\bar{F}(\xi_t)\right]dt \quad \text{by Fubini's Theorem} \\ &\leq \int_0^\infty re^{-rt}\bar{F}(E\left[\xi_t\right])dt \quad \text{by Jensen's inequality} \\ &\leq \bar{F}\left(\int_0^\infty re^{-rt}E\left[\xi_t\right]dt\right) \quad \text{by Jensen's inequality} \\ &= \bar{F}\left(E\left[\int_0^\infty re^{-rt}\xi_t dt\right]\right) \\ &= \bar{F}(w). \end{aligned}$$

The last equality comes from the fact that

$$e^{-r\tau}W_\tau = w - \int_0^\tau e^{-rt}(u(C_t) - h(A_t))dt + \int_0^\tau e^{-rt}r\sigma Y_t dZ_t,$$

which implies that

$$E\left[\int_0^\infty re^{-rt}\xi_t dt\right] = w.$$

Taking supremum over all $(C, A, \tau, w) \in \mathcal{U}(w)$, we have $F(w) \leq \bar{F}(w)$.

$\square$

4.4 Concavity and continuity

In this section, we will establish the concavity and continuity of F over$[0, \bar{w}]$. To simplify our notation, we will use the operator θ^t defined as follows:

Definition 4.4. For each $t \in [0, \infty)$, $\theta^t : C[0, \infty) \to C[0, \infty)$ is defined as

$$\theta^t(x.) = (x.|_{[t,\infty)} - x_t).$$

We will first establish concavity of F over $[0, \bar{w}]$.

Lemma 4.3. *F is concave over $[0, \bar{w}]$.*

We first need the following lemma:

Lemma 4.4. *For fixed $0 \le w_1 < w < w_2 \le \bar{w}$, suppose we are given a bounded function $\hat{Y} : (w_1, w_2) \to [\gamma_0, \infty)$ and a process W_t such that*

$$dW_t = r\sigma\hat{Y}(W_t)dZ_t; \quad W_0 = w,$$

where Z_t is a standard Brownian motion. Let

$$\tau = \inf\{t > 0 : W_t \notin (w_1, w_2)\}.$$

Define

$$G_t = e^{rt}\left[\int_t^\tau e^{-ru}\bar{F}(W_u)du + e^{-r\tau}F(W_\tau)\right].$$

Then $E[G_t|\mathcal{F}_t]$ is a supermartingale with respect to $\{\mathcal{F}_t\}_{t\ge0}$.

Proof of Lemma 4.4. First, note that we can write

$$G_t = e^{rt}\int_t^\infty e^{-ru}\tilde{F}(W_{u\wedge\tau})du,$$

where

$$\tilde{F}(w) = \bar{F}(w)\mathbf{1}_{w\in(w_1,w_2)} + F(w)\mathbf{1}_{w\notin(w_1,w_2)}.$$

Note that $\tilde{F}$ is concave over $[w_1, w_2]$ since $\bar{F}$ is concave and $F \le \bar{F}$ at w_1 and w_2. Now

$$E[G_t|\mathcal{F}_t] \le E[e^{rt}\int_t^\infty \bar{F}(W_u)du|\mathcal{F}_t] \le \bar{F}(W_t).$$

Since $\bar{F}(W)$ is continuous over $[0, \bar{w}]$ and thus bounded, we have $E[|E[G_t|\mathcal{F}_t]|] < \infty$. Next we show the supermartingale property. By Ito's Lemma, we have

$$\begin{aligned} dG_t &= \left[re^{rt}\cdot\int_t^\infty e^{-ru}\tilde{F}(W_{u\wedge\tau})du - e^{rt}\cdot e^{-rt}\tilde{F}(W_{t\wedge\tau})\right]dt \\ &= e^{rt}\left[\int_t^\infty re^{-ru}\tilde{F}(W_{u\wedge\tau})du - e^{-rt}\tilde{F}(W_{t\wedge\tau})\right]dt. \end{aligned}$$

So

$$G_{s'} - G_s = \int_s^{s'} e^{rt} \left[\int_t^{\infty} re^{-ru} \tilde{F}(W_{u\wedge\tau}) du - e^{-rt} \tilde{F}(W_{t\wedge\tau}) \right] dt$$

$$\begin{aligned}
E[G_{s'} - G_s|\mathcal{F}_s] =& E\left[\int_s^{s'} e^{rt} \left[\int_t^{\infty} re^{-ru} \tilde{F}(W_{u\wedge\tau}) du - e^{-rt} \tilde{F}(W_{t\wedge\tau}) \right] dt |\mathcal{F}_s \right] \\
=& \int_s^{s'} e^{rt} E\left[\int_t^{\infty} re^{-ru} \tilde{F}(W_{u\wedge\tau}) du - \tilde{F}(W_{t\wedge\tau}) \middle| \mathcal{F}_s \right] dt \\
=& \int_s^{s'} e^{rt} E\left[e^{-rt} E\left[\int_0^{\infty} re^{-ru} \tilde{F}(W_{u\wedge\tau}) du - \tilde{F}(W_{t\wedge\tau}) \middle| \mathcal{F}_t \right] |\mathcal{F}_s \right] dt \\
\leq& 0.
\end{aligned}$$

The last inequality follows because $W_{t\wedge\tau}$ is a martingale and $\tilde{F}$ is concave over $[w_1, w_2]$. Finally, the supermartingale property follows because for $s < s'$,

$$E[E[G_{s'}|\mathcal{F}_{s'}]|\mathcal{F}_s] - G_s = E[G_{s'} - G_s|\mathcal{F}_s] \leq 0.$$

□

Proof of Lemma 4.3. Consider fixed $0 \leq w_1 < w < w_2 \leq \bar{w}$ such that $w = \lambda w_1 + (1-\lambda) w_2$, $\lambda \in (0,1)$. Let $\hat{a}(w)$ and $\hat{c}(w)$ be such that

$$\bar{F}(w) = \hat{a}(w) - \hat{c}(w)$$

with $u(\hat{c}(w)) - h(\hat{a}(w)) = w$, $\hat{a}(w) \in \mathcal{A}$ and $\hat{c}(w) \in [0, \bar{c}]$.

Now starting with initial continuation value w, we consider the control that uses $A_t = \hat{a}(W_t)$, $C_t = \hat{c}(W_t)$ and $Y_t = \hat{Y}(W_t) = \gamma(\hat{a}(W_t)) \geq \gamma_0 > 0$ until we hit w_1 or w_2 at τ, after which we adopt an ϵ-optimal strategy at w_1 and w_2 respectively. Mathematically, for $i = 1, 2$, we select $(C^i, A^i, Y^i, \tau^i) \in \mathcal{U}(w_i)$ such that

$$J(C^i, A^i, Y^i, \tau^i; w_i) \geq F(w_i) - \epsilon.$$

Then we define

$$\tau = \inf\{t > 0 : W_t \notin (w_1, w_2)\},$$

where

$$dW_t = r(\underbrace{W_t - u(\hat{c}(W_t)) - h(\hat{a}(W_t))}_{=0})dt + r\sigma\hat{Y}_t(W_t)dZ_t = r\sigma\hat{Y}_t(W_t)dZ_t.$$

Note that $\hat{Y}(w) = \gamma(\hat{a}(w)) \in [\gamma_0, \gamma(\bar{a})]$ with $\gamma_0 > 0$ for all $w \in [0, \bar{w}]$.

Consider

$$(\tilde{C}_s, \tilde{A}_s, \tilde{Y}_s) = \begin{cases} (\hat{c}(W_s), \hat{a}(W_s), \hat{Y}(W_s)) & s < \tau \\ (C^i, A^i, Y^i) \circ \theta^\tau & s \geq \tau \text{ and } W_\tau = w_i. \end{cases}$$

and $\tilde{\tau} = \tau + \mathbf{1}_{W_\tau = w_1} \tau^1 \circ \theta^\tau + \mathbf{1}_{W_\tau = w_2} \tau^2 \circ \theta^\tau$. The existence of a weak solution $\{W_t, Z_t\}_{t \geq 0}$ is guaranteed by nondegeneracy and local integrability condition as the volatility is bounded below by a positive constant and bounded above as well (see Karatzas and Shreve, 1991, Theorem 5.15). Thus we see that $(\tilde{C}, \tilde{A}, \tilde{Y}, \tilde{\tau}) \in \mathcal{U}(w)$. The profit is

$$F(w) \geq J(\tilde{C}, \tilde{A}, \tilde{Y}, \tilde{\tau}; w) = E\left[\int_0^\tau e^{-ru} \bar{F}(W_u) du + e^{-r\tau}(F(W_\tau) - \epsilon)\right].$$

Since $\bar{F}$ and F are both bounded, we can let $\epsilon \downarrow 0$ and see that

$$F(w) \geq E\left[\int_0^\tau e^{-ru} \bar{F}(W_u) du + e^{-r\tau} F(W_\tau)\right] = E[G_0], \tag{4.5}$$

where G_t is as defined in Lemma 4.4, and $\{E[G_t|\mathcal{F}_t]\}_{t \geq 0}$ is a supermartingale with respect to $\{\mathcal{F}_t\}_{t \geq 0}$. Since $\hat{Y}(W_t)$ is bounded below by $\gamma_0 > 0$, $P(\tau < \infty) = 1$. Moreover, $|G_\tau| \leq \max(|F(w_1)|, |F(w_2)|)$ is bounded. By the optional sampling theorem, we obtain

$$E[G_0] \geq E[G_\tau]. \tag{4.6}$$

Since $W_{t \wedge \tau}$ is a martingale and $P(\tau < \infty) = 1$, we have

$$w = E[W_0] = E[W_\tau] = P(W_\tau = w_1) w_1 + (1 - P(W_\tau = w_1)) w_2,$$

giving

$$P(W_\tau = w_1) = \frac{w_2 - w}{w_2 - w_1} = \lambda.$$

So

$$E[G_\tau] = \lambda F(w_1) + (1 - \lambda) F(w_2). \tag{4.7}$$

Finally, combining (4.5), (4.6) and (4.7),

$$F(w) \geq E[G_0] \geq E[G_\tau] = \lambda F(w_1) + (1 - \lambda) F(w_2).$$

□

The concavity of F over $[0, \bar{w}]$ immediately leads to the continuity over the interior of the interval, i.e. $(0, \bar{w})$. To see left-continuity at the right end-point $\bar{w}$, recall that

$$F_0(w) \leq \tilde{F}_0(w) \leq F(w) \leq \bar{F}(w)$$

for all $w \in (0, \bar{w})$. Noting that $F_0(\bar{w}) = \bar{F}(\bar{w})$ and taking limit as $w \uparrow \bar{w}$ shows that

$$\lim_{w \uparrow \bar{w}} F(w) = F_0(\bar{w}) = F(\bar{w}) = \bar{F}(\bar{w}).$$

For $0 < x < \bar{w}$, we define

$$\beta^+(x) := \inf_{k \geq 1, w \in [0, \bar{w} - kx]} \frac{\tilde{F}_0(w) - \tilde{F}_0(w + kx)}{k}$$

and

$$\beta^-(x) := \inf_{k \geq 1, w \in [kx, \bar{w}]} \frac{\tilde{F}_0(w) - \tilde{F}_0(w - kx)}{k}.$$

Both infimums are well-defined because $\tilde{F}_0$ is bounded.

Lemma 4.5. *We have* $\lim_{x \downarrow 0} \beta^+(x) = 0$ *and* $\lim_{x \downarrow 0} \beta^-(x) = 0$.

Proof. For $0 < x < 1$, we have

$$\begin{aligned}\left| \inf_{\substack{k > x^{-1/2}, \\ w \in [0, \bar{w} - kx]}} \frac{\tilde{F}_0(w) - \tilde{F}_0(w + kx)}{k} \right| &\leq x^{1/2} \sup_{\substack{k' > 1, \\ w \in [0, \bar{w} - k' x^{1/2}]}} |\tilde{F}_0(w) - \tilde{F}_0(w + k' x^{1/2})| \\ &\leq x^{1/2} \sup_{w, w' \in [0, \bar{w}]} |\tilde{F}_0(w) - \tilde{F}_0(w')|,\end{aligned}$$

while

$$\begin{aligned}\left| \inf_{\substack{1 \leq k \leq x^{-1/2}, \\ w \in [0, \bar{w} - kx]}} \frac{\tilde{F}_0(w) - \tilde{F}_0(w + kx)}{k} \right| &\leq \sup_{\substack{x^{1/2} \leq k' \leq 1, \\ w \in [0, \bar{w} - k' x^{1/2}]}} |\tilde{F}_0(w) - \tilde{F}_0(w + k' x^{1/2})| \\ &\leq \sup_{\substack{w, w' \in [0, \bar{w}], \\ w' - w < x^{1/2}}} |\tilde{F}_0(w) - \tilde{F}_0(w')|.\end{aligned}$$

So

$$
\begin{aligned}
&|\beta^+(x)| \\
&\le \max\left(x^{1/2} \sup_{w,w'\in[0,\bar{w}]} |\tilde{F}_0(w) - \tilde{F}_0(w')|, \sup_{\substack{w,w'\in[0,\bar{w}], \\ w'-w<x^{1/2}}} |\tilde{F}_0(w) - \tilde{F}_0(w')| \right) \\
&\to 0 \quad \text{as } x \downarrow 0,
\end{aligned}
$$

noting that $\tilde{F}_0$ is continuous and bounded. The same argument applied to the function $\tilde{F}_0(\bar{w} - .)$ can be used to show $\beta^-(x) \to 0$ as $x \downarrow 0$. □

Now we prove the following lemma, which is used multiple times in proving right-continuity, establishing uniform continuity and in proving the DPP.

Lemma 4.6. *For $w_1, w_2 \in (0, \bar{w})$ and $w_1 \neq w_2$, and $(C, A, Y, \tau) \in \mathcal{U}(w_2)$, let*

$$\tau^{w_1} = \inf\{t > 0 : W_t(C, A, Y; w_1) \notin (0, \bar{w})\}.$$

Then we have $(C, A, Y, \tau^{w_1} \wedge \tau) \in \mathcal{U}(w_1)$. We also have

(a). If $w_1 < w_2$, then

$$
\begin{aligned}
J(C, A, Y, \tau^{w_1} \wedge \tau; w_1) \ge\ & J(C, A, Y, \tau; w_2) \\
& + \min(F(0) - F(w_2 - w_1), \beta^+(w_2 - w_1)).
\end{aligned}
\tag{4.8}
$$

(b). If $w_2 < w_1$, then

$$
\begin{aligned}
J(C, A, Y, \tau^{w_1} \wedge \tau; w_1) \ge\ & J(C, A, Y, \tau; w_2) \\
& + \min(F(\bar{w}) - F(\bar{w} - (w_1 - w_2)), \beta^-(w_1 - w_2)).
\end{aligned}
\tag{4.9}
$$

Remark 4.1. If $\tilde{F}_0$ is concave, then we know that the second term in the minimum in (4.8) and (4.9) can be dropped for simplicity. This is because for $w_2 > w_1$,

$$\begin{aligned}\beta^+(w_2 - w_1) = & \inf_{k\geq 1, w\in[0,\bar{w}]}[\tilde{F}_0(w) - \tilde{F}_0(w + k(w_2 - w_1))]/k \\ = & \inf_{k\geq 1, w\in[0,\bar{w}]}[\tilde{F}_0(w) - \tilde{F}_0(w)\cdot(k-1)/k - \tilde{F}_0(w + k(w_2 - w_1))/k] \\ = & \inf_{w\in[0,\bar{w}]}[\tilde{F}_0(w) - \tilde{F}_0(w - (w_2 - w_1))] \\ = & \tilde{F}_0(0) - \tilde{F}_0(w_2 - w_1) \\ \geq & F(0) - F(w_2 - w_1),\end{aligned}$$

where the third and the fourth equalities follow from the concavity of F; for $w_1 < w_2$,

$$\begin{aligned}\beta^-(w_1 - w_2) = & \inf_{k\geq 1, w\in[0,\bar{w}]}[\tilde{F}_0(w) - \tilde{F}_0(w - k(w_1 - w_2))]/k \\ = & \inf_{k\geq 1, w\in[0,\bar{w}]}[\tilde{F}_0(w) - \tilde{F}_0(w)\cdot(k-1)/k - \tilde{F}_0(w - k(w_1 - w_2))/k] \\ = & \inf_{w\in[0,\bar{w}]}[\tilde{F}_0(w) - \tilde{F}_0(w - (w_1 - w_2))] \\ \geq & \tilde{F}_0(\bar{w}) - \tilde{F}_0(\bar{w} - (w_1 - w_2)) \\ \geq & F(\bar{w}) - F(\bar{w} - (w_1 - w_2)),\end{aligned}$$

where the third and fourth equalities follow again from the concavity of F.

Proof of Lemma 4.6. For simplicity write $\tilde{\tau} = \tau^{w_1} \wedge \tau$. We are given that $(C, A, Y, \tau) \in \mathcal{U}(w_2)$. For $i = 1, 2$, let $W_t^i := W_t(C, A, Y; w_i)$, i.e.

$$dW_t^i = r(W_t^i - u(C_t) + h(A_t))dt + r\sigma Y_t dZ_t; \quad W_0^i = w_i.$$

Notice that we let the processes W_t^1 and W_t^2 be driven by the same Brownian motion Z_t.

First we see for $t \leq \tilde{\tau}$,

$$e^{-rt}W_t^i = W_0^i - \int_0^t re^{-rs}(u(C_s) - h(A_s))ds + \int_0^t r\sigma Y_s dZ_s. \tag{4.10}$$

Considering the difference of the above equation with $i = 1, 2$, we see that

$$W_t^2 - W_t^1 = e^{rt}(w_2 - w_1) \tag{4.11}$$

for $t \leq \tilde{\tau}$. The definition of $\tilde{\tau}$ guarantees that $W_t^1 \in (0, \bar{w})$ for $t < \tilde{\tau}$, and the existence of weak solution $\{W_t^1, Z_t\}_{t \geq 0}$ follows immediately from (4.11). Thus $(C, A, Y, \tilde{\tau}) \in \mathcal{U}(w_1)$. For the profit,

$$
\begin{aligned}
& J(C,A,Y,\tilde{\tau};w_1) - J(C,A,Y,\tau;w_2) \\
=& E[\int_0^{\tilde{\tau}} re^{-rt}(A_t - C_t)dt + e^{-r\tilde{\tau}}\tilde{F}_0(W_{\tilde{\tau}}^1)] - E[\int_0^{\tau} re^{-rt}(A_t - C_t)dt + e^{-r\tau}\tilde{F}_0(W_\tau^2)] \\
=& E\left[e^{-r\tilde{\tau}}\tilde{F}_0(W_{\tilde{\tau}}^1) - \int_{\tilde{\tau}}^{\tau} re^{-rt}(A_t - C_t)dt - e^{-r\tau}\tilde{F}_0(W_\tau^2)\right] \\
=& E\left[e^{-r\tilde{\tau}}\mathbf{1}_{\tilde{\tau}<\tau}[\tilde{F}_0(W_{\tilde{\tau}}^1) - \int_{\tilde{\tau}}^{\tau} re^{-rt}(A_t - C_t)dt - e^{-r\tau}\tilde{F}_0(W_\tau^2)]\right. \\
& \left. + e^{-r\tilde{\tau}}\mathbf{1}_{\tilde{\tau}=\tau}[\tilde{F}_0(W_{\tilde{\tau}}^1) - \tilde{F}_0(W_{\tilde{\tau}}^2)]\right] \\
\geq& E\left[e^{-r\tilde{\tau}}\mathbf{1}_{\tilde{\tau}<\tau}[F(W_{\tilde{\tau}}^1) - F(W_{\tilde{\tau}}^2)] + e^{-r\tilde{\tau}}\mathbf{1}_{\tilde{\tau}=\tau}[\tilde{F}_0(W_{\tilde{\tau}}^1) - \tilde{F}_0(W_{\tilde{\tau}}^2)]\right] \\
=& E\left[e^{-r\tilde{\tau}}\mathbf{1}_{\tilde{\tau}<\tau}[F(W_{\tilde{\tau}}^1) - F(W_{\tilde{\tau}}^1 + e^{r\tilde{\tau}}(w_2 - w_1))]\right. \\
& \left. + e^{-r\tilde{\tau}}\mathbf{1}_{\tilde{\tau}=\tau}[\tilde{F}_0(W_{\tilde{\tau}}^1) - \tilde{F}_0(W_{\tilde{\tau}}^1 + e^{r\tilde{\tau}}(w_2 - w_1))]\right].
\end{aligned}
$$

If $w_2 - w_1 > 0$, we have $W_{\tilde{\tau}}^1 = 0$ whenever $\tilde{\tau} < \tau$, and so

$$
\begin{aligned}
J(C,A,Y,\tilde{\tau};w_1) - J(C,A,Y,\tau;w_2) & \\
\geq & E\left[e^{-r\tilde{\tau}}\mathbf{1}_{\tilde{\tau}<\tau}[F(0) - F(e^{r\tilde{\tau}}(w_2 - w_1))]\right. \\
& \left. + e^{-r\tilde{\tau}}\mathbf{1}_{\tilde{\tau}=\tau}[\tilde{F}_0(W_{\tilde{\tau}}^1) - \tilde{F}_0(W_{\tilde{\tau}}^1 + e^{r\tilde{\tau}}(w_2 - w_1))]\right] \\
\geq & \min(F(0) - F(w_2 - w_1), \\
& \inf_{t>0, w\in[0,\bar{w}]} e^{-rt}[\tilde{F}_0(w) - \tilde{F}_0(w + e^{rt}(w_2 - w_1))]) \\
= & \min(F(0) - F(w_2 - w_1), \beta^+(w_2 - w_1)).
\end{aligned}
$$

If $w_2 - w_1 < 0$, we have $W^1_{\tilde{\tau}} = \bar{w}$ whenever $\tilde{\tau} < \tau$, and so

$$\begin{aligned}
J(C, A, Y, \tilde{\tau}; w_1) - J(C, A, Y, \tau; w_2) \\
&\geq E\Big[e^{-r\tilde{\tau}}\mathbf{1}_{\tilde{\tau}<\tau}[F(\bar{w}) - F(\bar{w} - e^{r\tilde{\tau}}(w_1 - w_2))] \\
&\quad +e^{-r\tilde{\tau}}\mathbf{1}_{\tilde{\tau}=\tau}[\tilde{F}_0(W^1_{\tilde{\tau}}) - \tilde{F}_0(W^1_{\tilde{\tau}} - e^{r\tilde{\tau}}(w_1 - w_2))]\Big] \\
&\geq \min(F(\bar{w}) - F(\bar{w} - (w_1 - w_2)), \\
&\quad \inf_{t>0, w\in[0,\bar{w}]} e^{-rt}[\tilde{F}_0(w) - \tilde{F}_0(w - e^{rt}(w_1 - w_2))]) \\
&= \min(F(\bar{w}) - F(\bar{w} - (w_1 - w_2)), \beta^-(w_1 - w_2)).
\end{aligned}$$

The second inequalities in both cases follow from the fact that F is concave. This proves the desired result. □

Taking the supremum over all (C, A, Y, τ) in $\mathcal{U}(w_2)$, we obtain the following corollary.

Corollary 4.1. *(a). For $0 < w_1 < w_2 < \bar{w}$, we have*

$$F(w_1) \geq F(w_2) + \min(F(0) - F(w_2 - w_1), \beta^+(w_2 - w_1)).$$

(b). For $0 < w_2 < w_1 < \bar{w}$, we have

$$F(w_1) \geq F(w_2) + \min(F(\bar{w}) - F(\bar{w} - (w_1 - w_2)), \beta^-(w_1 - w_2)).$$

Now, to show the right-continuity at zero, we first quote a result from Borodin and Salminen (2002), P.310, formulae 3.2.8 (1), which can be rewritten as:

Lemma 4.7. *Let $a \leq x \leq b$, $\mu > 0$, $B^{(\mu)}_t = \mu t + B_t$, where B_t is a standard Brownian motion with $B_0 = x$, and define*

$$\tau = \inf\{t > 0 : B^{(\mu)}_t \notin (a, b)\}.$$

Then for any $r > 0$, we have

$$E\left[e^{-r\tau}\mathbf{1}_{B^{(\mu)}_\tau = a}\right] = e^{\mu(a-x)}\frac{\sinh(-\sqrt{\mu^2 + 2r^2}(b - x))}{\sinh(-\sqrt{\mu^2 + 2r^2}(b - a))}. \tag{4.12}$$

Lemma 4.8. *F is right-continuous at zero.*

Proof. Since we know F is concave and continuous over $(0, \bar{w}]$, we know that the right-hand-side limit of F at 0 exists and is at least $\tilde{F}_0(0) \geq 0$ (as $F \geq \tilde{F}_0$ and $\tilde{F}_0(0) \geq F_0(0) = 0$), i.e.

$$\tilde{F}_0(0) \leq \lim_{w \downarrow 0} F(w) < \infty.$$

By way of contradiction, suppose F is not right-continuous at 0, i.e. for some η

$$\lim_{w \downarrow 0} F(w) = \tilde{F}_0(0) + \eta > \tilde{F}_0(0) = F(0).$$

Let $\zeta = \eta/4$. By the continuity of F and $\tilde{F}_0$ over $(0, \bar{w})$, we can pick $w_0 < \bar{w}/2$ such that

$$\tilde{F}_0(0) + \eta + \zeta \geq F(w) \geq \tilde{F}_0(0) + \eta - \zeta \quad \forall w \in (0, 2w_0)$$

and

$$\tilde{F}_0(0) - \zeta \leq \tilde{F}_0(w) \leq \tilde{F}_0(0) + \zeta \quad \forall w \in (0, 2w_0).$$

Fix $\epsilon > 0$ and take $\delta > 0$ such that

$$|F(w) - F(w_0)| < \epsilon \quad \text{for all } w \in [w_0 - \delta, w_0].$$

Let $(C, A, Y, \tau) \in \mathcal{U}(w_0 - \delta)$ be an ϵ-optimal contract at $w_0 - \delta$, i.e.

$$J(C, A, Y, \tau; w_0 - \delta) \geq F(w_0 - \delta) - \epsilon \geq F(w_0) - 2\epsilon.$$

Define $\tau^{w_0} = \inf\{t > 0 : W_t(C, A, Y, w_0) \notin (0, \bar{w})\}$. By Lemma 4.6, we know that $(C, A, Y, \tau \wedge \tau^{w_0}) \in \mathcal{U}(w_0)$ and

$$J(C, A, Y, \tau \wedge \tau^{w_0}; w_0) \geq J(C, A, Y, \tau; w_0 - \delta) + \min(F(\bar{w}) - F(\bar{w} - \delta), \beta^-(\delta)).$$

We can next modify this contract in a way so that, in cases where $\tau < \tau^{w_0}$ and $W_\tau \leq 2w_0$, instead of retiring the agent, we follow some ϵ-optimal contract after τ.

Mathematically, let D_j be disjoint intervals such that

$$\cup_j D_j = (0, 2w_0]$$

$$w_j := \inf\{D_j\} > 0 \text{ and } w_j \in D_j$$

$$|D_j| < \epsilon.$$

For each j, let $(C^j, A^j, Y^j, \tau^j) \in \mathcal{U}(w_j)$ such that

$$J(C^j, A^j, Y^j, \tau^j; w_j) \geq \tilde{F}_0(0) + \eta - 2\zeta.$$

Defining $\tau^{w,j} = \inf\{t > 0 : W_t(C^j, A^j, Y^j, w) \notin (0, \bar{w})\}$ and using again Lemma 4.6, we have

$$J(C^j, A^j, Y^j, \tau^j; w) \geq \eta - 2\zeta + \min(F(\bar{w}) - F(\bar{w} - (w - w_j)), \beta^-(w - w_j)).$$

Then we define

$$(\tilde{C}_t, \tilde{A}_t, \tilde{Y}_t) = \begin{cases} (C_t, A_t, Y_t) & t < \tau \text{ or } W_\tau > 2w_0 \\ (C^j_{t-\tau}, A^j_{t-\tau}, Y^j_{t-\tau}) \circ \theta^\tau & t \geq \tau \text{ and } W_\tau \in D_j \end{cases}$$

$$\tilde{\tau} = \mathbf{1}_{\tau^{w_0} < \tau \text{ or } W_\tau > 2w_0} (\tau^{w_0} \wedge \tau) + \mathbf{1}_{\tau^{w_0} > \tau} \left(\sum_j \mathbf{1}_{W_\tau \in D_j} [\tau + [(\tau^j \wedge \tau^{W_\tau, j}) \circ \theta^\tau]] \right).$$

Then we can see

$$\begin{aligned}
& J(\tilde{C}, \tilde{A}, \tilde{Y}, \tilde{\tau}; w_0) \\
&= J(C, A, Y, \tau^{w_0} \wedge \tau; w_0) \\
&\quad + E\left[\mathbf{1}_{\tau^{w_0} > \tau} e^{-r\tau} \sum_j \mathbf{1}_{W_\tau \in D_j} [J(C^j, A^j, Y^j, \tau^j \wedge \tau^{W_\tau, j}; W_\tau) - \tilde{F}_0(W_\tau)] \right] \\
&\geq J(C, A, Y, \tau^{w_0} \wedge \tau; w_0) \\
&\quad + E\left[\mathbf{1}_{\tau^{w_0} > \tau} e^{-r\tau} \sum_j \mathbf{1}_{W_\tau \in D_j} \cdot \Big(\tilde{F}_0(0) + \eta - 2\zeta \right. \\
&\qquad \left. + \min(F(\bar{w}) - F(\bar{w} - (W_\tau - w_j)), \beta^-(W_\tau - w_j)) - \tilde{F}_0(0) - \zeta \Big) \right] \\
&\qquad \text{since } \tilde{F}_0(w) \leq \tilde{F}_0(0) + \zeta \text{ for } w \in (0, 2w_0) \\
&\geq F(w_0) - 2\epsilon + \min(F(\bar{w}) - F(\bar{w} - \delta), \beta^-(\delta)) \\
&\quad + (\eta - 3\zeta) E\left[\mathbf{1}_{\tau^{w_0} > \tau} e^{-r\tau} \mathbf{1}_{W_\tau \leq 2w_0} \right] + \inf_{w \in [0,\epsilon]} \{\min(F(\bar{w}) - F(\bar{w} - w), \beta^-(w))\}]
\end{aligned}$$

which implies

$$\begin{aligned}
F(w_0) \geq & F(w_0) - 2\epsilon + \min(F(\bar{w}) - F(\bar{w} - \delta), \beta^-(\delta)) \\
& + (\eta - 3\zeta) \cdot E\left[\mathbf{1}_{\tau < \tau^{w_0}} e^{-r\tau} \mathbf{1}_{W_\tau \leq 2w_0} \right] \\
& + \inf_{w \in [0,\epsilon]} \{\min(F(\bar{w}) - F(\bar{w} - w), \beta^-(w))\}.
\end{aligned} \tag{4.13}$$

We would like to show that as we take the limit of (4.13) as ϵ goes to zero, and the term with expectation is uniformly bounded below by a positive constant. If that is true, since ϵ is arbitrary, this would imply that

$$F(w_0) \geq F(w_0) + (\eta - 3\zeta) \cdot k(w_0),$$

where $k(w_0) > 0$, which leads to a contradiction.

Now we want to show the existence of $k(w_0) > 0$ such that

$$E\left[\mathbf{1}_{\tau \leq \tau^{w_0}} e^{-r\tau} \mathbf{1}_{W_\tau \leq 2w_0}\right] \geq k(w_0).$$

First, we apply a time-change (see Karatzas and Shreve, 1991, Theorem 3.4.6).

Let $M_t = \int_0^t r\sigma Y_t dZ_t$. Note that $\langle M \rangle_t = \int_0^t r^2\sigma^2 Y_t^2 dt \geq r^2\sigma^2\gamma_0^2 t$ and $\lim_{t\to\infty}\langle M\rangle_t = \infty$. Define, for each $0 \leq s \leq \infty$, the stopping time

$$T(s) = \inf\{t \geq 0 : \langle M_t \rangle > s\} \leq s/(r^2\sigma^2\gamma_0^2). \tag{4.14}$$

The time-changed process $\bar{Z}_t := M_{T(s)}$ is a standard Brownian motion and we have a.s. $M_t = \bar{Z}_{\langle M_t \rangle}$ for $0 \leq t < \infty$.

Next, consider

$$\bar{W}_t = w_0 + \frac{r(\bar{w} + h(\bar{a}))t}{r^2\sigma^2\gamma_0^2} + \bar{Z}_t,\ t > 0.$$

In comparison,

$$\begin{aligned} W_t &= w_0 + \int_0^t r(W_s - u(C_s) + h(A_s))ds + \int_0^t r\sigma Y_s dZ_s \\ &= w_0 + \int_0^t r(W_s - u(C_s) + h(A_s))ds + M_t \\ &\leq w_0 + \int_0^t r(\bar{w} + h(\bar{a}))ds + \bar{Z}_{\langle M_t \rangle} \quad \text{a.s.} \end{aligned}$$

And so,

$$\begin{aligned} W_{T(t)} &\leq w_0 + \int_0^{T(t)} r(\bar{w} + h(\bar{a}))ds + \bar{Z}_t \\ &\leq w_0 + \int_0^{t/(r^2\sigma^2\gamma_0^2)} r(\bar{w} + h(\bar{a}))ds + \bar{Z}_t \\ &\leq w_0 + \frac{r(\bar{w} + h(\bar{a}))t}{r^2\sigma^2\gamma_0^2} + \bar{Z}_t \leq \bar{W}_t \quad \text{a.s..} \end{aligned}$$

Thus we have

$$W_{T(t)} \leq \bar{W}_t \tag{4.15}$$

a.s. for all t.

Let

$$\nu_{2w_0} = \inf\{t > 0 : \bar{W}_t \geq 2w_0\} \quad \text{and} \quad \tau_{2w_0} = \inf\{t > 0 : W_t \geq 2w_0\}.$$

Likewise, define

$$\nu_\delta = \inf\{t > 0 : \bar{W}_t \leq \delta\} \quad \text{and} \quad \tau_\delta = \inf\{t > 0 : W_t \leq \delta\}.$$

Recall that $W_0 = \bar{W}_0$ and by (4.15), we have $W_{T(t)} \leq \bar{W}_t$. Therefore,

$$\begin{aligned} \{\tau_{2w_0} \leq t\} \quad &\Longrightarrow \quad W_s \geq 2w_0 \text{ for some } s \leq t \\ &\Longrightarrow \quad \bar{W}_{\langle M \rangle_s} \geq 2w_0 \text{ for some } s \leq t \\ &\Longrightarrow \quad \nu_{2w_0} \leq \langle M \rangle_t \\ &\Longrightarrow \quad T(\nu_{2w_0}) \leq t, \end{aligned}$$

which means

$$\tau_{2w_0} \geq T(\nu_{2w_0}). \tag{4.16}$$

Similarly, we have

$$\tau_\delta \leq T(\nu_\delta). \tag{4.17}$$

Also, we must have $\tau \leq \tau_\delta$. This is because we have $(C, A, Y, \tau) \in \mathcal{U}((w_0 - \delta))$ and therefore

$$W_t(C, A, Y, w_0 - \delta) \in (0, \bar{w}) \text{ for } t \leq \tau,$$

and so as in the proof of Lemma 4.6

$$W_t(C, A, Y, w_0) - W_t(C, A, Y, w_0 - \delta) = e^{rt}\delta.$$

If $\tau > \tau_\delta$, it would mean that $W_t(C, A, Y, w_0) \leq \delta$ for some $t < \tau$, which in turn implies $W_t(C, A, Y, w_0 - \delta) \leq \delta - e^{rt}(\delta) \leq 0$, giving a contradiction.

We take the following steps to derive the bound we needed. First, since the process must hit $2w_0$ first before hitting $\bar{w}$, we have

$$\tau_\delta \leq \tau_{2w_0} \quad \Longrightarrow \quad \tau_\delta \leq \tau^{w_0},$$

and moreover

$$\tau_\delta < \tau_{2w_0} \quad \Longrightarrow \quad W_\tau < 2w_0 \quad \text{since } \tau \leq \tau_\delta.$$

These together give

$$E\left[\mathbf{1}_{\tau\le\tau^{w_0}}e^{-r\tau}\mathbf{1}_{W_{\tau\le 2w_0}}\right]\ge E\left[e^{-r\tau_\delta}\mathbf{1}_{\tau_\delta\le\tau_{2w_0}}\right].$$

Applying (4.16) and (4.17), and then applying (4.14), we have further

$$\begin{aligned}E\left[e^{-r\tau_\delta}\mathbf{1}_{\tau_\delta\le\tau_{2w_0}}\right]\ge& E\left[e^{-rT(\nu_\delta)}\mathbf{1}_{\nu_\delta\le\nu_{2w_0}}\right]\\ \ge& E\left[e^{-rT(\nu_0)}\mathbf{1}_{\nu_0\le\nu_{2w_0}}\right]\\ \ge& E\left[e^{-r\nu_0/(r^2\sigma^2\gamma_0^2)}\mathbf{1}_{\nu_0\le\nu_{2w_0}}\right].\end{aligned}$$

This gives

$$E\left[\mathbf{1}_{\tau\le\tau^{w_0}}e^{-r\tau}\mathbf{1}_{W_{\tau\le 2w_0}}\right]\ge E\left[e^{-r\nu_0/(r^2\sigma^2\gamma_0^2)}\mathbf{1}_{\nu_0\le\nu_{2w_0}}\right].$$

Letting $r'=\frac{r}{r^2\sigma^2\gamma_0^2}>0$ and applying Lemma 4.7 with $a=\delta, x=w_0, b=2w_0$ and $\mu=r'(\bar{w}+h(\bar{a}))$, we know that $E[e^{-r'\nu_0}\mathbf{1}_{v_0\le\nu_{2w_0}}]$ is positive number that depends only on w_0, independent of $w_0-\delta$, C, A, Y and τ. We write $k(w_0)=E[e^{-r'\nu_0}\mathbf{1}_{v_0\le\nu_{2w_0}}]>0$.

Finally, we take limit of (4.13) as δ goes to zero. This would imply

$$F(w_0)\ge F(w_0)-2\epsilon+(\eta-3\zeta)\cdot k(w_0)-\sup_{w\in[0,\epsilon]}\{F(\bar{w}-w)-F(\bar{w}))].$$

Since $\epsilon>0$ is arbitrary and F is left-continuous at $\bar{w}$, we have

$$F(w_0)\ge F(w_0)+(\eta-3\zeta)\cdot k(w_0),$$

which leads to a contradiction. This implies that F is right-continuous at zero. □

Corollary 4.2. *F is uniformly continuous in $[0,\bar{w}]$.*

This follows immediately from Corollary 4.1 and Lemma 4.8 since the last term on the right-hand-side of both inequalities represents the difference in F for a interval from 0 and $\bar{w}$.

4.5 Dynamic programming principle

Proposition 4.2 (Dynamic programming principle). *For any $\{\mathcal{F}_t^Z\}$-stopping time τ, we have*

$$F(w) = \sup_{(C,A,Y,\nu)\in\mathcal{U}(w)} E[\int_0^{\tau\wedge\nu} re^{-rs}(A_s - C_s)ds + \mathbf{1}_{\tau\leq\nu}e^{-r(\tau\wedge\nu)}F(W_{\tau\wedge\nu}) + \mathbf{1}_{\nu<\tau}e^{-r(\tau\wedge\nu)}\tilde{F}_0(W_{\tau\wedge\nu})], \tag{4.18}$$

where $W_t = W_t(C, A, Y, w)$.

It should be noted that, since $\tilde{F}_0 \leq F$, the DPP also implies

$$F(w) = \sup_{(C,A,Y,\nu)\in\mathcal{U}(w)} E[\int_0^{\tau\wedge\nu} re^{-rs}(A_s - C_s)ds + e^{-r(\tau\wedge\nu)}F(W_{\tau\wedge\nu})] \tag{4.19}$$

for any stopping time τ. It is obvious that the left-hand-side is less than the right-hand-side given equation (4.18). For the other direction, it can be proved by taking $\tau' = \min(\tau, \nu)$ and then applying the DPP.

Our proof of the DPP is very similar to standard proofs except that we need to consider incentive-compatibility of the contracts. If you agree with the standard DPP, then this one is similar with a trick. The idea is that we can take some ϵ-optimal contract from time t onwards based on the value of W_t (which evolves according to (C, A) in time $0 \leq s < t$). The standard way (see for example Yong and Zhou, 1999, Theorem 4.3.3) is to consider partition D_j of $[0, \bar{w}]$ such that length of each $D_j < \delta$. For each D_j, we pick a w_j in D_j and let (C^j, A^j) be the corresponding ϵ-optimal control. The problem here is that (C^j, A^j, Y^j, τ^j) may not be feasible for other $w \in D_j$. However, we can make use of Lemma 4.6 to build feasible contracts for each point $w \in D_j$ and provide bounds for the profit.

We need the following definition and lemma:

Definition 4.5. For fixed $t > 0$ and $(y.) \in C[0, \infty)$, define the mapping $\Lambda^{t,y.} : C[0, \infty) \to C[0, \infty)$ as

$$\Lambda^{t,y.}(x.) = (x_{(.\vee t)-t} - x_0 + y_{.\wedge t})$$

Lemma 4.9. *For any* $(C, A, Y, \nu) \in \mathcal{U}(w)$ *for some* $w \in [0, \bar{w}]$ *and stopping time* $\tau \leq \nu$*, we have*

$$J(C, A, Y, \nu; w) = E\left[\int_0^\tau re^{-rt}(A_t - C_t)dt + e^{-r\tau}J(C^\omega, A^\omega, Y^\omega, \nu^\omega; W_\tau(\omega))\right] \tag{4.20}$$

where $(C^\omega, A^\omega, Y^\omega, \nu^\omega)$ *are defined as follows. For* $t > 0$ *and* $(y.) \in C[0, \infty)$*,*

$$\begin{aligned} C_s^{t,y\cdot}(z.) &= C_{s+t}(\Lambda^{t,y\cdot}(z.)), \\ A_s^{t,y\cdot}(z.) &= A_{s+t}(\Lambda^{t,y\cdot}(z.)), \\ Y_s^{t,y\cdot}(z.) &= Y_{s+t}(\Lambda^{t,y\cdot}(z.)), \\ \nu^{t,y\cdot}(z.) &= \nu(\Lambda^{t,y\cdot}(z.)) - t \end{aligned}$$

And for $\omega \in \Omega$*,*

$$C^\omega = C^{\tau(\omega), Z(.,\omega)}, \quad A^\omega = A^{\tau(\omega), Z(.,\omega)},$$
$$Y^\omega = Y^{\tau(\omega), Z(.,\omega)}, \quad \nu^\omega = \nu^{\tau(\omega), Z(.,\omega)},$$

and we have

$$(C^\omega, A^\omega, Y^\omega, \nu^\omega) \in \mathcal{U}(W_\tau(\omega)),$$

where $W_t = W_t(C, A, Y; w)$*.*

Proof of lemma. We have

$$\begin{aligned} &J(C, A, Y, \nu; w) \\ =&E\left[\int_0^\nu re^{-rs}(A_s - C_s)ds + e^{-r\nu}\tilde{F}_0(W_\nu)\right] \\ =&E\left[\int_0^\tau re^{-rs}(A_s - C_s)ds + \int_\tau^\nu re^{-rs}(A_s - C_s)ds + e^{-r\nu}\tilde{F}_0(W_\nu)\right] \\ =&E\Bigg[\int_0^\tau re^{-rs}(A_s - C_s)ds \\ &\quad + e^{-r\tau}\left(\int_\tau^\nu re^{-r(s-\tau)}(A_s - C_s)ds + e^{-r(\nu-\tau)}\tilde{F}_0(W_\nu)\right)\Bigg] \\ =&E\Bigg[\int_0^\tau re^{-rs}(A_s - C_s)ds \\ &\quad + e^{-r\tau}E\left[\int_0^{\nu-\tau} re^{-rs}(A_{\tau+s} - C_{\tau+s})ds + e^{-r(\nu-\tau)}\tilde{F}_0(W_{\nu-\tau})|\mathcal{F}_\tau\right]\Bigg]. \end{aligned}$$

In order to establish (4.20), we would like to show that $(P|\mathcal{F}_\tau) - a.s.$, we have

$$E\left[\int_0^{\nu-\tau} re^{-rs}(A_{\tau+s} - C_{\tau+s})ds + e^{-r(\nu-\tau)}\tilde{F}_0(W_{\nu-\tau})\middle| \mathcal{F}_\tau\right](\omega)$$
$$= J(C^\omega, A^\omega, Y^\omega, \nu^\omega; W_\tau(\omega))$$

By Theorem 1.3.4 of Stroock and Varadhan (1979), we know that under the regular conditional probability distribution of P given $\mathcal{F}_\tau$, i.e. $P|\mathcal{F}_\tau$, $Z_s, 0 \le s \le \tau$ is the same with probability one, i.e. there is $\tilde{z}(., \omega)$ such that

$$P(\{\omega : Z_s = \tilde{z}(s,\omega) \quad \forall\, 0 \le s \le \tau\}|\mathcal{F}_\tau) = 1,$$

and also $\theta^\tau(Z(.,\omega) - Z_\tau(\omega))$ is a Brownian motion under $P|F_\tau$. By the uniqueness of the weak solution to (3.1) guaranteed by assumption (U7), we know $W_\tau|\mathcal{F}_\tau$ is with probability one equal to a constant as well, i.e.

$$P(W_\tau = \tilde{w}(\omega)|\mathcal{F}_\tau) = 1$$

for some $\tilde{w}(.)$ for all ω. Note that $\tau(\omega)$ and $Z_\tau(\omega)$ are both $\mathcal{F}_\tau$-measurable. Since $Z_\tau(\omega)$ is with probability one constant under $P|\mathcal{F}_\tau$, we have $(C^\omega, A^\omega, Y^\omega, \nu^\omega) \in \mathcal{U}(\tilde{w}(\omega)) = \mathcal{U}(W_\tau(\omega))$ for almost all ω. Moreover,

$$\begin{aligned}
&E\left[\int_0^{\nu-\tau} re^{-rs}(A_{\tau+s} - C_{\tau+s})ds + e^{-r(\nu-\tau)}\tilde{F}_0(W_{\nu-\tau})|\mathcal{F}_\tau\right](\omega)\\
=&E\left[\int_0^{\nu-\tau} re^{-rs}[A_{\tau+s}(Z(.\wedge(\tau+s),\omega)) - C_{\tau+s}(Z(.\wedge(\tau+s),\omega))]ds\right.\\
&\left.+e^{-r(\nu-\tau)}\tilde{F}_0(W_{\nu-\tau}(\omega))|\mathcal{F}_\tau\right]\\
=&E\left[\int_0^{\nu^{\tau(\omega),Z(.\wedge\tau,\omega)}} re^{-rs}[A_s^{\tau(\omega),Z(.\wedge\tau,\omega)} - C_s^{\tau(\omega),Z(.\wedge\tau,\omega)}]ds\right.\\
&\left.+e^{-r(\nu^{\tau(\omega),Z(.\wedge\tau,\omega)})}\tilde{F}_0(W_{\nu^{\tau(\omega),Z(.\wedge\tau,\omega)}})|\mathcal{F}_\tau\right]\\
=&E\left[\int_0^{\nu^{t,z.}} re^{-rs}(A_s^{t,z.} - C_s^{t,z.})ds + e^{-r\nu}\tilde{F}_0(W_{\nu^{t,z.}}(C^{t,z.}, A^{t,z.}, Y^{t,z.}; w))\right]\Bigg|_{t=\tau(\omega),z.=Z(.,\omega)}\\
=&J(C^{\tau(\omega),Z(.,\omega)}, A^{\tau(\omega),Z(.,\omega)}, Y^{\tau(\omega),Z(.,\omega)}, \nu^{\tau(\omega),Z(.,\omega)}; W_\tau(\omega))\\
=&J(C^\omega, A^\omega, Y^\omega, \nu^\omega; W_\tau(\omega)).
\end{aligned}$$

This establishes (4.20) and completes the proof.

□

Proof of DPP. First, we would like to show

$$F(w) \leq \sup_{(C,A,Y,\nu)\in\mathcal{U}(w)} E[\int_0^{\tau\wedge\nu} re^{-rs}(A_s - C_s)ds + \mathbf{1}_{\tau\leq\nu}e^{-r(\tau\wedge\nu)}F(W_{\tau\wedge\nu}) + \mathbf{1}_{\nu<\tau}e^{-r(\tau\wedge\nu)}\tilde{F}_0(W_{\tau\wedge\nu})], \tag{4.21}$$

For any $(C, A, Y, \nu) \in \mathcal{U}(w)$, since $\tau \wedge \nu \leq \nu$, by Lemma 4.9 we have

$$\begin{aligned}
&J(C, A, Y, \nu; w) \\
&= E\left[\int_0^{\tau\wedge\nu} re^{-rt}(A_t - C_t)dt + e^{-r(\tau\wedge\nu)}J(C^\omega, A^\omega, Y^\omega, \nu^\omega; W_{\tau\wedge\nu})\right] \\
&\leq E\left[\int_0^{\tau\wedge\nu} re^{-rt}(A_t - C_t)dt + \mathbf{1}_{\tau\leq\nu}e^{-r(\tau\wedge\nu)}F(W_{\tau\wedge\nu}) + \mathbf{1}_{\nu<\tau}e^{-r(\tau\wedge\nu)}\tilde{F}_0(W_{\tau\wedge\nu})\right].
\end{aligned}$$

We then obtain (4.21) by taking supremum over $\mathcal{U}(w)$.

Now, we would like to show

$$F(w) \geq \sup_{(C,A,Y,\nu)\in\mathcal{U}(w)} E\left[\int_0^{\tau\wedge\nu} re^{-rs}(A_s - C_s)ds + e^{-r(\tau\wedge\nu)}F(W_{\tau\wedge\nu})\right],$$

which would imply our desired inequality since $F \geq \tilde{F}_0$.

For $\epsilon > 0$, we pick $\delta > 0$ such that $|\beta^-(x)| < \epsilon$ and $|F(\bar{w}) - F(\bar{w} - x)| < \epsilon$ for $x \in [0, \delta]$. We let D_j $(j \geq 1)$ be disjoint intervals such that $\cup_j D_j = [0, \bar{w}]$ and

$$diam(D_j) < \delta \tag{4.22}$$

$$F(w) - F(\hat{w}) < \epsilon \quad \forall w,\, \hat{w} \in D_j \tag{4.23}$$

$$\inf\{D_j\} > 0 \text{ unless } D_j = \{0\}, \text{ and } \inf\{D_j\} \in D_j \tag{4.24}$$

and

$$\begin{aligned}
D_j &= \{0\} \quad \text{whenever } 0 \in D_j \\
D_j &= \{\bar{w}\} \quad \text{whenever } \bar{w} \in D_j
\end{aligned}$$

so that the endpoints 0 and $\bar{w}$ are in intervals of their own. We can always choose such D_j's because F is continuous, concave and bounded over $(0, \bar{w})$. To do so, we can consider

$$w_i^1 = i\delta,\; i = 0, 1, 2, \ldots, n := \lfloor \bar{w}/\delta \rfloor; \quad w_{n+1}^1 = \bar{w}.$$

$$w_0^2 = \bar{w},\ w_i^2 = \inf\{w \in (0, \bar{w}) : |F(w) - F(w_{i-1}^2)| \le \epsilon/2\} \text{ for } i = 1, 2, 3, \dots$$

$$w_0^3 = \bar{w},\ w_i^3 = \bar{w}/2^i,\ i = 1, 2, 3, \dots$$

Note that w_i^2's are well-defined because the left-derivative at $\bar{w}$ is finite as

$$0 \ge F'_-(\bar{w}) \ge \bar{F}'_-(\bar{w}) > \min(-h'(0), -1/u'(\bar{c})).$$

Then we can let $\bar{w} = w_0 > w_1 > w_2 > w_3 > \dots,\ w_i \in (0, \bar{w}]$ such that

$$\{w_i\}_{i \ge 0} = (\{w_i^1\}_{i=1,2,\dots,n} \cup \{w_i^2\}_{i \le 0} \cup \{w_i^3\}_{i \le 0}) \backslash \{0\}.$$

Then the intervals D_j can be defined as $D_j = [w_{j+1}, w_j)$. Adding the sets $\{0\}$ and $\{\bar{w}\}$ completes our construction. The definition of w_i^1, w_i^2 and w_i^3 ensures, respectively, that conditions (4.22), (4.23) and (4.24) hold for the constructed D_j's.

For each j, define $w_j = \inf\{D_j\} \in D_j$, and then there is an ϵ-optimal control $(C^j, A^j, Y^j, \nu^j) \in \mathcal{U}(w_j)$ such that

$$J(C^j, A^j, Y^j, \nu^j; w_j) \ge F(w_j) - \epsilon.$$

Given the stopping time τ and the initial continuation value w with a contract $(C, A, Y, \nu) \in \mathcal{U}(w)$, with $W_t := W_t(C, A, Y; w)$, we consider the contract $(\tilde{C}, \tilde{A}, \tilde{Y}, \tilde{\nu})$ where

$$(\tilde{C}_s, \tilde{A}_s, \tilde{Y}_s) = \begin{cases} (C_s, A_s, Y_s) & 0 \le s \le \tau \wedge \nu \\ ((C^j, A^j, Y^j)_{s-t_j} \circ \theta^{\tau \wedge \nu} & W_{\tau \wedge \nu} \in D_j, s \ge t_j \end{cases}$$

$$\tilde{\nu} = (\tau \wedge \nu) + \left(\sum_j \mathbf{1}_{W_{\tau \wedge \nu} \in D_j} (\nu^j \wedge \tau^{W_{\tau \wedge \nu}, j}) \circ \theta^{\tau \wedge \nu} \right),$$

where

$$\tau^{w', j} = \inf\{t > 0 : W_t(C^j, A^j, Y^j, w') \notin (0, \bar{w})\}.$$

Note that $(\tilde{C}, \tilde{A}, \tilde{Y})$ is progressively measurable with respect to $\mathcal{F}_t$ and $\tilde{\nu}$ is a $\{\mathcal{F}_t\}$-stopping time. Furthermore, we can construct a weak solution (W_t, Z_t) given $(\tilde{C}, \tilde{A}, \tilde{Y})$, similar to the argument given in Borkar (1989, P.57). This ensures $(\tilde{C}, \tilde{A}, \tilde{Y}, \tilde{\nu}) \in \mathcal{U}(w)$.

Recall that for any $w' \in D_j$, Lemma 4.6 states that we have $(C^j, A^j, Y^j, \nu^j \wedge \tau^{w',j}) \in \mathcal{U}(w')$ and provides bounds for the corresponding profit. In particular, if $w' = w_j$, then the profit is identical; otherwise $w' > w_j$ and $w_j = \inf\{D_j\}$ by the construction of D_j, and the profit is

$$\begin{aligned} J(C^j, A^j, Y^j, &\nu^j \wedge \tau^{w',j}; w') \\ &\geq J(C^j, A^j, Y^j, \nu^j; w_j) - [F(\bar{w} - (w - w_j)) - F(\bar{w})] \\ &\geq F(w_j) - \epsilon + \min(F(\bar{w}) - F(\bar{w} - (w' - w_j)), \beta^-(w' - w_j)) \\ &\geq F(w_j) - 2\epsilon. \end{aligned} \tag{4.25}$$

We then have

$$\begin{aligned} &F(w) \geq J(\tilde{C}, \tilde{A}, \tilde{Y}, \tilde{\nu} \wedge \tau) \\ &= E\left[\int_0^{\tau\wedge\nu} re^{-rs}(A_s - C_s)ds \right. \\ &\quad \left. + e^{-r(\tau\wedge\nu)} \sum_j 1_{W_{\nu\wedge\tau} \in D_j} J(C^j, A^j, Y^j, \nu \wedge \tau^{W_{\nu\wedge\tau},j}; W_{\nu\wedge\tau})\right] \\ &\geq E\left[\int_0^{\tau\wedge\nu} re^{-rs}(A_s - C_s)ds + e^{-r(\tau\wedge\nu)} \sum_j 1_{W_{\nu\wedge\tau} \in D_j}(F(w_j) - 2\epsilon)\right] \quad \text{by (4.25)} \\ &\geq E\left[\int_0^{\tau\wedge\nu} re^{-rs}(A_s - C_s)ds + e^{-r(\tau\wedge\nu)}(F(W_{\tau\wedge\nu}) - 3\epsilon)\right] \quad \text{by (4.23).} \end{aligned}$$

Letting $\epsilon \downarrow 0$ and noting that $F \geq \tilde{F}_0$, we get the desired result. □

4.6 Optimal profit as the viscosity solution

In this section, we will prove Theorem 4.1, which states that the optimal profit function to our principal-agent problem is a viscosity solution of (4.1)–(4.2).

Proof of Theorem 4.1. **Viscosity supersolution:**
To show that F is a viscosity supersolution, consider $\varphi \in C^2((0, \bar{w}))$, $w_0 \in (0, \bar{w})$ such that $\varphi - F$ attains a local maximum at w_0 with $\varphi(w_0) = F(w_0)$.

Setting $\tau = 0$, we have

$$F(w_0) \geq J(C, A, Y, 0) = \tilde{F}_0(w_0) = \varphi(w_0),$$

i.e. $\tilde{F}_0(w_0) - \varphi(w_0) \leq 0$. Next we want to show that

$$\mathcal{H}(w_0, \varphi'(w_0), \varphi''(w_0)) - r\varphi(w_0) \leq 0. \tag{4.26}$$

Let ϵ be such that $0 < \epsilon < \min(w_0, \bar{w} - w_0)$ and

$$\varphi(w) \leq F(w) \quad \forall w \in [w_0 - \epsilon, w_0 + \epsilon].$$

For each $a \in \mathcal{A}$ and $c \in [0, \bar{c}]$, we consider the contract $(C, A, Y, \tau) \in \mathcal{U}(w_0)$ with

$$C_t = c,\ A_t = a,\ Y_t = \gamma(A_t)$$

and

$$\tau = \inf\{t > 0 : W_t \notin (w_0 - \epsilon, w_0 + \epsilon)\},$$

with

$$W_t = W_t(C, A, Y, w_0).$$

Note that $E[\tau] > 0$ since $w_0 \in (w_0 - \epsilon, w_0 + \epsilon)$. Then for any $t > 0$, we must have by Proposition 4.2 and equation (4.19),

$$F(w_0) \geq E\left[\int_0^{t\wedge\tau} e^{-rs}(A_s - C_s)ds + e^{-r(t\wedge\tau)}F(W_{t\wedge\tau})\right]$$

i.e.

$$E[F(w_0) - e^{-r(t\wedge\tau)}F(W_{t\wedge\tau})] \geq E\left[\int_0^{t\wedge\tau} e^{-rs}(A_s - C_s)ds\right]. \tag{4.27}$$

For $s > 0$ small enough,

$$0 \geq \frac{E\{F(w_0) - \varphi(w_0) - e^{-r(s\wedge\tau)}[F(W_{s\wedge\tau}) - \varphi(W_{s\wedge\tau})]\}}{E[s\wedge\tau]}$$

since $F(w_0) - \varphi(w_0) = 0$ and $F \geq \varphi$ for all $w \in (w_0 - \epsilon, w_0 + \epsilon)$. Then together with (4.27)

$$0 \geq \frac{E\{\int_0^{s\wedge\tau} e^{-rt}(A_t - C_t)dt - \varphi(w_0) + e^{-r(s\wedge\tau)}\varphi(W_{s\wedge\tau})\}}{E[s\wedge\tau]}.$$

Taking limit as $s \to 0$ we have

$$(a - c) - r\varphi(w_0) + r\varphi'(w_0)(w_0 - u(c) + h(a)) + \frac{1}{2}\varphi''(w_0)r^2\gamma(a)^2\sigma^2 \leq 0.$$

This holds for all $0 \leq c \leq \bar{c}, a \in \mathcal{A}$ and so taking supremum we have

$$\mathcal{H}(w, \varphi'(w_0), \varphi''(w_0)) - r\varphi(w_0) \leq 0.$$

Viscosity subsolution:
Suppose F is not a viscosity subsolution. Then there is $w_0 \in (0, \bar{w})$, $0 < \epsilon < \min(w_0, \bar{w} - w_0)$, $\varphi \in C^2$ with $\varphi(w_0) = F(w_0)$, $\varphi \geq F$ s.t. for all $w \in [w_0 - \epsilon, w_0 + \epsilon]$, by continuity

$$\mathcal{H}(w, \varphi'(w), \varphi''(w)) - r\varphi(w) \leq -v \tag{4.28}$$

$$\tilde{F}_0(w) \leq \varphi(w) - v \tag{4.29}$$

where $v > 0$.

In the following, we write

$$\begin{aligned} \mathcal{L}^{(a,c,y)}\varphi(w) &= r\varphi'(w)(w - u(c) + h(a)) + \frac{\varphi''(w)}{2} r^2\sigma^2 y^2 \\ &= H_{a,c,y}(w, \varphi'(w), \varphi''(w)) - r(a - c). \end{aligned} \tag{4.30}$$

Now, for any contract $(C, A, Y, \tau) \in \mathcal{U}(w_0)$, let

$$\tau_\epsilon = \inf\{t > 0 : W_t \notin (w_0 - \epsilon, w_0 + \epsilon)\}.$$

Applying Ito's Lemma to $e^{-rt}\varphi(W_t)$, we have

$$\begin{aligned} &E[e^{-r(\tau_\epsilon \wedge \tau)}\varphi(W_{\tau_\epsilon \wedge \tau})] \\ &\quad = \varphi(w_0) + E\left[\int_0^{\tau_\epsilon \wedge \tau} e^{-rt}(-r\varphi(W_t) + \mathcal{L}^{(A_t, C_t, \gamma(A_t))}\varphi(W_t))dt\right] \\ &\quad \leq \varphi(w_0) + E\left[\int_0^{\tau_\epsilon \wedge \tau} e^{-rt}(-r(A_t - C_t) - v)dt\right] \quad \text{by (4.28) and (4.30)} \\ &\quad = \varphi(w_0) - rE[\int_0^{\tau_\epsilon \wedge \tau} e^{-rt}(A_t - C_t)dt] - vE[\int_0^{\tau_\epsilon \wedge \tau} e^{-rt}dt] \end{aligned}$$

where $A_t \in \mathcal{A}$ for each t and ω. Rearranging this and noting that $\varphi(w_0) =$

$F(w_0)$, $\varphi \geq F$ in $[w_0 - \epsilon, w_0 + \epsilon]$ and (4.29) gives

$$\begin{aligned}
&E[\mathbf{1}_{\tau<\tau_\epsilon} e^{-r(\tau_\epsilon \wedge \tau)} \tilde{F}_0(W_{\tau_\epsilon \wedge \tau})] + E[\mathbf{1}_{\tau_\epsilon \leq \tau} e^{-r(\tau_\epsilon \wedge \tau)} F(W_{\tau_\epsilon \wedge \tau})] \\
&\quad + rE[\int_0^{\tau_\epsilon \wedge \tau} e^{-rt}(A_t - C_t)dt] \\
&\leq F(w_0) - vE[\int_0^{\tau_\epsilon \wedge \tau} e^{-rt} dt] - vE[\mathbf{1}_{\tau<\tau_\epsilon} e^{-r(\tau_\epsilon \wedge \tau)}] \\
&= F(w_0) - vE[\int_0^{\tau_\epsilon} e^{-rt} dt] - vE[\mathbf{1}_{\tau<\tau_\epsilon} e^{-r(\tau_\epsilon \vee \tau)}] \\
&\leq F(w_0) - vE[\int_0^{\tau_\epsilon} e^{-rt} dt].
\end{aligned}$$

Next we claim that $E[\int_0^{\tau_\epsilon} e^{-rt}dt] = (1 - E[e^{-r\tau_\epsilon}])/r > g_0 > 0$ for $(C, A, Y, \nu) \in \mathcal{U}(w_0)$. To do this we consider the function

$$\psi(w) = G_0 \cdot ((w - w_0)^2 - \epsilon^2)$$

where $G_0 > 0$. Note that $\psi \in C^2$, $\psi(w_0) = -G_0\epsilon^2$ and $\psi(w_0 - \epsilon) = \psi(w_0 + \epsilon) = 0$. Applying Ito's lemma to $e^{-rt}\psi(W_t)$, we have

$$E[e^{-r\tau_\epsilon}\psi(W_{\tau_\epsilon})] = \psi(w_0) + E[\int_0^{\tau_\epsilon} e^{-rt}(-r\psi + \mathcal{L}^{(C_t,A_t,Y_t)}\psi)(W_t)dt]. \quad (4.31)$$

Note that C_t takes values in $[0, \bar{c}]$ and that $Y_t = \gamma(A_t)$ is bounded so that $0 \leq \gamma(A_t) \leq \bar{y}$. Next we want to show that there is $G_0 > 0$ such that the second term on the right-hand side is at most $E[\int_0^{\tau_\epsilon} e^{-rt}dt]$ for all $(C, A, Y, \tau) \in \mathcal{U}(w)$. Indeed, we claim the existence of $G_0 > 0$ such that

$$-r\psi + \mathcal{L}^{(c,a,y)}\psi \leq 1$$

for all $c \in [0, \bar{c}], a \in \mathcal{A}$. For $w \in (w_0 - \epsilon, w_0 + \epsilon)$,

$$\begin{aligned}
&- r\psi(w) + \mathcal{L}^{(c,a,y)}\psi(w) \\
&= \left[-r + 2r(w - w_0)(G_0((w - w_0)^2 - \epsilon^2) - u(c) + h(a)) + r^2y^2\sigma^2\right] \\
&= G_0\left[-r((w - w_0)^2 - \epsilon^2) + 2r(w - w_0)[G_0((w - w_0)^2 - \epsilon^2) - u(c) + h(a)]\right. \\
&\qquad \left. + r^2y^2\sigma^2\right] \\
&\leq G_0\left[r\epsilon^2 + 2r(w - w_0)[G_0((w - w_0)^2 - \epsilon^2) - u(c) + h(a)] + r^2\bar{y}^2\sigma^2\right] \\
&\leq G_0[r\epsilon^2 + \max\left(2r\epsilon h(\bar{a}), -2r\epsilon[G_0(-\epsilon^2) - u(\bar{c})]\right) + r^2\bar{y}^2\sigma^2] \\
&= G_0[r\epsilon^2 + 2r\epsilon \max\left(h(\bar{a}), G_0\epsilon^2 + u(\bar{c})\right) + r^2\bar{y}^2\sigma^2] \\
&\leq 1
\end{aligned}$$

for sufficiently small $G_0 > 0$. This, together with (4.31), gives

$$E\left[\int_0^{\tau_\epsilon} e^{-rt}dt\right] \geq E[e^{-r\tau_\epsilon}\psi(W_{\tau_\epsilon})] - \psi(w_0) = G_0\epsilon^2 =: g_0 > 0.$$

Finally, taking supremum in (**??**) and using Proposition 4.2, we have

$$F(w_0) \leq F(w_0) - vg_0,$$

which leads to a contradiction. □

4.7 Uniqueness of viscosity solution

We have proved that the optimal profit function is a viscosity solution for (4.1) – (4.2). This would not be very helpful if the equations actually have multiple viscosity solutions. Therefore, we proceed to show the uniqueness of the viscosity solution.

In the following, we will appeal to Theorem 2.2 to obtain uniqueness of the viscosity solution for (4.1).

Lemma 4.10. *Let $G(w,z,p,\alpha) = -\max\{\mathcal{H}(w,p,\alpha) - rz, \tilde{F}_0(w) - z\}$. Then G satisfies the requirement of F in Theorem 2.2, i.e.*

1. *G is proper, i.e. it satisfies (2.7) and (2.8).*
2. *There exists $\gamma > 0$ such that (2.10) is satisfied, i.e.*

$$\gamma(z' - z) \leq G(w, z', p, \alpha) - G(w, z, p, \alpha)$$

   for $z' \geq z,\ (w,p,\alpha) \in \bar{\Omega} \times \mathbb{R} \times (\mathbb{R}^+ \cup \{0\})$.

3. *There is a function $\omega : [0,\infty] \to [0,\infty]$ that satisfies $\omega(0+) = 0$ such that (2.11) is satisfied, i.e.*

$$\begin{aligned} G(w_2, r, \alpha(w_1 - w_2), \beta_2) - G(w_1, r, \alpha(w_1 - w_2), \beta_1) \\ \leq \omega(\alpha|w_1 - w_2|^2 + |w_1 - w_2|) \end{aligned} \tag{4.32}$$

   for every $w_1, w_2 \in [0, \bar{w}], r \in \mathbb{R}, \alpha \in \mathbb{R}, \beta_1 \geq 0, \beta_2 \geq 0$ such that

$$-3\alpha \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \leq \begin{pmatrix} \beta_1 & 0 \\ 0 & -\beta_2 \end{pmatrix} \leq 3\alpha \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}.$$

Proof. First we note that

$$
\begin{aligned}
&G(w,z,p,\alpha) - G(w,z',p,\alpha) \\
=&\begin{cases} G(w,z,p,\alpha) + \mathcal{H}(w,p,\alpha) - rz' & \mathcal{H}(w,p,\alpha) - rz' \geq \tilde{F}_0(w) - z' \\ G(w,z,p,\alpha) + \tilde{F}_0(w) - z' & \text{otherwise} \end{cases} \\
\leq&\begin{cases} -\mathcal{H}(w,p,\alpha) - rz + \mathcal{H}(w,p,\alpha) - rz' & \mathcal{H}(w,p,\alpha) - rz' \geq \tilde{F}_0(w) - z' \\ -\tilde{F}_0(w) + z + \tilde{F}_0(w) - z' & \text{otherwise} \end{cases} \\
\leq& \min(1,r)(z - z').
\end{aligned}
$$

This shows (2.7). Setting $\gamma = \min(1,r) > 0$, condition 2 is also satisfied.

Second, we note that if $\mathcal{H}(w,p,\alpha') - rz < \tilde{F}_0(w) - z$,

$$
\begin{aligned}
G(w,z,p,\alpha) - G(w,z,p,\alpha') =& G(w,z,p,\alpha) + \tilde{F}_0(w) - z \\
\leq& -\tilde{F}_0(w) + z + \tilde{F}_0(w) - z' = 0.
\end{aligned}
$$

Alternatively, if $\mathcal{H}(w,p,\alpha') - rz \geq \tilde{F}_0(w) - z$,

$$
\begin{aligned}
G(w,z,p,\alpha) - G(w,z,p,\alpha') =& G(w,z,p,\alpha) + \mathcal{H}(w,p,\alpha') - rz \\
\leq& -\mathcal{H}(w,p,\alpha) - rz + \mathcal{H}(w,p,\alpha') - rz' \\
=& -\mathcal{H}(w,p,\alpha) + \mathcal{H}(w,p,\alpha')
\end{aligned}
$$

But selecting $a \in \mathcal{A}, c \in [0,\bar{c}]$ such that

$$
\mathcal{H}(w,p,\alpha') = H_{a,c,\gamma(a)}(w,p,\alpha'),
$$

we have

$$
\begin{aligned}
-\mathcal{H}(w,p,\alpha) + \mathcal{H}(w,p,\alpha') &= -\mathcal{H}(w,p,\alpha) + H_{a,c,\gamma(a)}(w,p,\alpha') \\
&\leq -H_{a,c,\gamma(a)}(w,p,\alpha) + H_{a,c,\gamma(a)}(w,p,\alpha') \\
&= -\frac{\alpha - \alpha'}{2} r^2 \gamma(a)^2 \sigma^2 \\
&\leq 0
\end{aligned}
$$

whenever $\alpha' \leq \alpha$. This proves (2.8).

To show condition 3, we consider two cases.

Case 1: $\mathcal{H}(w_1, \alpha(w_1 - w_2), \beta_1) - rz \geq \tilde{F}_0(w_1) - z$. We have

$$\begin{aligned}
&G(w_2, z, \alpha(w_1 - w_2), \beta_2) - G(w_1, z, \alpha(w_1 - w_2), \beta_1) \\
=&G(w_2, z, \alpha(w_1 - w_2), \beta_2) + \mathcal{H}(w_1, \alpha(w_1 - w_2), \beta_1) - rz \\
\leq& - \mathcal{H}(w_2, \alpha(w_1 - w_2), \beta_2) + rz + \mathcal{H}(w_1, \alpha(w_1 - w_2), \beta_1) - rz \\
=& - \mathcal{H}(w_2, \alpha(w_1 - w_2), \beta_2) + \mathcal{H}(w_1, \alpha(w_1 - w_2), \beta_1)
\end{aligned}$$

Recall

$$\begin{aligned}
\mathcal{H}(w, p, \beta) &= \sup_{a \in \mathcal{A}, c} H_{a,c,\gamma(a)}(w, p, \beta) \\
&= \sup_{a \in \mathcal{A}, c} \left\{ r(a - c) + rp(w - u(c) + h(a)) + \frac{\beta}{2} r^2 \gamma(a)^2 \sigma^2 \right\}.
\end{aligned}$$

Then we have

$$\begin{aligned}
&\mathcal{H}(w_1, \alpha(w_1 - w_2), \beta_1) - \mathcal{H}(w_2, \alpha(w_1 - w_2), \beta_2) \\
\leq& \sup_{a \in \mathcal{A}, c} \left\{ H_{a,c,\gamma(a)}(w_1, \alpha(w_1 - w_2), \beta_1) - H_{a,c,\gamma(a)}(w_2, \alpha(w_1 - w_2), \beta_2) \right\} \\
=& \sup_{a \in \mathcal{A}, c} \left\{ r\alpha(w_1 - w_2)(w_1 - w_2) + \frac{\beta_1}{2} r^2 \gamma(a)^2 \sigma^2 - \frac{\beta_2}{2} r^2 \gamma(a)^2 \sigma^2 \right\} \\
=& r\alpha |w_1 - w_1|^2 + \sup_{a \in \mathcal{A}, c} \left\{ \frac{\beta_1}{2} r^2 \gamma(a)^2 \sigma^2 - \frac{\beta_2}{2} r^2 \gamma(a)^2 \sigma^2 \right\} \\
\leq& r\alpha |w_2 - w_1|^2 + 0
\end{aligned}$$

since

$$\begin{pmatrix} \beta_1 & 0 \\ 0 & -\beta_2 \end{pmatrix} \leq 3\alpha \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}$$

implies

$$\beta_1 z^2 - \beta_2 z^2 \leq 3\alpha(z^2 + z^2 - 2z^2) = 0.$$

Case 2: $\mathcal{H}(w_1, \alpha(w_1 - w_2), \beta_1) - rz < \tilde{F}_0(w_1) - z$. We have

$$\begin{aligned}
G(w_2, z, \alpha(w_1 - w_2), \beta_2) - G(w_1, z, \alpha(w_1 - w_2), \beta_1)& \\
=&G(w_2, z, \alpha(w_1 - w_2), \beta_2) + \tilde{F}_0(w_1) - z \\
\leq& - \tilde{F}_0(w_2) + z + \tilde{F}_0(w_1) - z \\
=&\tilde{F}_0(w_1) - \tilde{F}_0(w_2)
\end{aligned}$$

At this point, we recall the assumption that $\tilde{F}_0$ is continuous over the closed and bounded interval $[0, \bar{w}]$. This implies that $\tilde{F}_0$ is also uniformly continuous, and there is an increasing function $\tilde{\omega} : [0, \infty] \to [0, \infty]$ such that $\tilde{\omega}(0+) = 0$ and

$$\tilde{F}_0(w_1) - \tilde{F}_0(w_2) \leq \tilde{\omega}(|w_1 - w_2|).$$

Combining the two cases, we can set $\omega(v) = rv + \tilde{\omega}(v)$ and have the desired result. □

The following proposition then follows immediately from Theorem 2.2 and Lemma 4.10.

Proposition 4.3 (Comparison)**.** *Suppose* $r > 0$ *and* $\underline{F}$ *a viscosity subsolution and* $\bar{F}$ *a viscosity supersolution to (4.1)–(4.2) with* $\underline{F} \leq \bar{F}$ *at 0 and* $\bar{w}$*. Then we have* $\underline{F} \leq \bar{F}$ *in* $[0, \bar{w}]$*.*

At this point, Theorem 4.2, i.e. the uniqueness of viscosity solutions for the equations (4.1)–(4.2), follows immediately. If we have two viscosity solutions F and $\hat{F}$ that satisfy (4.2), they are both supersolutions and subsolutions, and we also have $F(0) = \hat{F}(0)$ and $F(\bar{w}) = \hat{F}(\bar{w})$. Applying the above proposition twice proves that $F = \hat{F}$ in $[0, \bar{w}]$.

4.8 Smoothness of the profit function

Proposition 4.4. *Suppose* $\tilde{F}_0$ *is* C^1*. Then the optimal profit function,* F*, is* C^1*.*

Proof. Since we have proved in Lemma 4.3 that F is concave, we know that the left-derivative $F'_-(w_0)$ and the right-derivative $F'_+(w_0)$ exist at each point $w_0 \in (0, \bar{w})$ with $F'_+(w_0) \leq F'_-(w_0)$.

By way of contradiction, suppose that F is not C^1 and so at some point $w_0 \in (0, \bar{w})$, we have $F'_+(w_0) < F'_-(w_0)$.

If $F(w_0) = \tilde{F}_0(w)$, the requirement that $\tilde{F}_0(w) \leq F(w)$ for all $w \in [0, \bar{w}]$ would imply that $\tilde{F}_0$, a C^1 function, touches F at w_0 from below. That would mean $F'_+(w_0) = F'_-(w_0) = \tilde{F}'_0(w_0)$ and therefore a contradiction.

Next, we consider the case when $F(w_0) > \tilde{F}_0(w)$. We fix some q in $(F'_+(w_0), F'_-(w_0))$ and consider the function

$$\varphi_\epsilon(w) = F(w_0) + q(w - w_0) - \frac{1}{2\epsilon}(w - w_0)^2$$

for $\epsilon > 0$. We can see that

$$\varphi_\epsilon \in C^2, \quad \varphi_\epsilon(w_0) = F(w_0), \quad \varphi'_\epsilon(w_0) = q, \quad \varphi''_\epsilon(w_0) = -1/\epsilon,$$

Since $F'_+(w_0) < q < F'_-(w_0)$, $\varphi_\epsilon - F$ attains a local minimum at w_0. By Theorem 4.1, we have

$$\max\{\mathcal{H}(w_0, \varphi'_\epsilon(w_0), \varphi''_\epsilon(w_0)) - r\varphi_\epsilon(w_0), \tilde{F}_0(w_0) - \varphi_\epsilon(w_0)\} \geq 0.$$

Since we are considering the case $\tilde{F}_0(w_0) < F(w_0) = \varphi_\epsilon(w_0)$, the above implies that $\mathcal{H}(w_0, \varphi'_\epsilon(w_0), \varphi''_\epsilon(w_0)) - r\varphi_\epsilon(w_0) \geq 0$, i.e.

$$\sup_{\substack{a\in\mathcal{A}, c\in[0,\bar{c}] \\ y=\gamma(a)}} \left\{ r(a-c) + rq(w_0 - u(c) + h(a)) - \frac{1}{\epsilon} r^2 y^2 \sigma^2 - F(w_0) \right\} \geq 0.$$

However, note that q is fixed, and we know

$$r(a-c) + rq(w_0 - u(c) + h(a)) - F(w_0) \leq r(\bar{a}) + rq(w_0 - 0 + h(\bar{a})) - F(w_0)$$

is finite. Since $y \geq \gamma_0 > 0$, taking $\epsilon \to \infty$ leads to a contradiction.

□

Chapter 5

Additional Contractual Possibilities

Sannikov (2008) considered some additional contractual possibilities, where an agent could be retired in a different way, that gives the principal a higher retirement profit than F_0. This includes the case where the agent could choose to quit, could be replaced, or could be promoted. The effect is modelled is the modified retirement function, as well as a modified range for the continuation value.

In our formulation in Chapter 3, we have explicitly allowed for a retirement function $\tilde{F}_0 \geq F_0$ that could be different from F_0, but we have required in assumption (A4) that such functions be a continuous function over $[0, \bar{w}]$ and that $\tilde{F}_0 \leq \bar{F}$, where $\bar{F}$ is the first-best profit (ignoring the retirement function).

In this chapter, we will review some examples of $\tilde{F}_0$ in these additional contractual possibilities, compare our technical results with Sannikov's results in these cases, and discuss possible relaxation of our requirements on $\tilde{F}_0$.

5.1 Sannikov's results

We begin by reviewing Sannikov (2008)'s results on additional contractual possibilities and the three examples given in his work.

Sannikov modeled the additional contractual possibilities by adding a choice of stopping time τ when the agent is retired and he receives the continuation value W_τ, giving the principal a profit of $\tilde{F}_0(W_\tau)$. His problem essentially becomes a combined stochastic control and optimal stopping problem, which is also the basis of our formulation in Chapter 3. The function $\tilde{F}_0$ is assumed to an upper semi-continuous function that satisfies $\tilde{F}_0(\tilde{w}) = 0$, and $\tilde{F}_0(w) \geq F_0(w)$ for all $w \in [\tilde{w}, \infty)$, with equality when w is sufficiently large.

Sannikov then considered the cases where the continuation region is connected and can be represented as (w_L, w_H), where w_L and w_H are the low and high retirement points respectively. The same HJB equation (1.15) with modified boundary conditions and smooth-pasting conditions based on $\tilde{F}_0$ are considered. He established a verification theorem for the case when a concave solution satisfying these conditions and the equation exists. In those cases, the corresponding optimal controls are identified, and the solution of the HJB equation is shown to be equal to the optimal profit function in the continuation region. However, he did not provide conditions nor results on the existence of the solution to the HJB equation in this case. This is probably due to the technical difficulty arising from a possibly complicated form of $\tilde{F}_0$.

Three particular forms of $\tilde{F}_0$ was considered in Sannikov's work, and we briefly summarize them as follows:

Quitting: When the agent can choose to quit working and receive a value of $\tilde{w} \geq 0$ from the new employment, this can be modelled as:

$$\tilde{F}_0(w) = \begin{cases} 0 & w = \tilde{w} \\ F_0(w) & w \geq \tilde{w} \end{cases}.$$

Note that w is at least $\tilde{w}$ because the agent will not stay in the employment when his continuation value falls below $\tilde{w}$. Moreover, this $\tilde{F}_0$ is discontinuous at $\tilde{w}$.

Replacement: When the principal has the option of replacing the agent and she obtains a value D from the new agent (which depends on the negotiation of the new employment and search cost), then the principal has a profit of $\tilde{F}_0(w) = F_0(w) + D$ for all $w > 0$.

Promotion: In this case, the principal has a choice to promote the agent to a higher position by providing training of cost $K \geq 0$ to him. The training will increase his productivity from a to θa, for $\theta > 1$, and his outside option from 0 to $W_p \geq 0$. With this modified cost of effort and outside option, the principal's profit function from a trained agent, denoted $F_p(w)$ can be considered as the solution for the case with quitting. Then, for an untrained agent, the principal could decide on training (as a means to retire the current stage of the agent and take him to the new stage of his career) with $\tilde{F}_0$ representing the profit from promotion or retiring the agent. Then we would have $\tilde{F}_0(w) = \max(F_0(w), F_p(w) - K)$. When the stopping time τ is hit, the agent is promoted if $F_p(W_\tau) - K \geq F_0(w)$, and retired otherwise.

5.2 Our results for additional contractual possibilities

Our formulation in Chapter 3 explicitly allows for a retirement function $\tilde{F}_0$ that is continuous over $[0, \bar{w}]$ and satisfies $F_0 \leq \tilde{F}_0 \leq \bar{F}$, where $\bar{F}$ is the first-best profit of the benchmark model.

As explained, the advantage of our methodology is that we characterize the optimal profit function as the unique viscosity solution of the HJB equation (4.1) without assuming, a priori, the existence of a smooth solution to the HJB equation or that the continuation region is connected. In comparison, since $\tilde{F}_0$ could take many different forms, it would be hard to establish the existence of smooth solutions to the HJB equation to apply the verification theorem in Sannikov (2008).

5.2.1 Disconnected continuation region

While the form of the continuation region $[0, w_{gp}]$ is intuitive under the benchmark model, there are indeed cases of a disconnected continuation region for general $\tilde{F}_0$. In the following, we will illustrate how the continuation region for the principal could be disconnected.

Our example comes from the results and insights from Chakrabarty and Guo (2012). In their paper, it was shown that for an optimal stopping problem with stopping payoff g and value function V, the optimal stopping problem with stopping payoff h satisfying $g \leq h \leq V$ will have the same value function. In particular, they gave an example where h is constructed to be the maximum of $n \geq 2$ tangents of V.

Proposition 5.1. *Let F be the optimal profit function of our problem with retirement function $\tilde{F}_0$. Let F^{G_0} be the optimal profit function of our problem with retirement function G_0 that satisfies assumption (A4) (and all other parameters, utility function and cost function being the same). Suppose $\tilde{F}_0 \leq G_0 \leq F$. Then we have $F^{G_0} = F$.*

Proof. First, we see that $F \leq F^{G_0}$ because for $(C, A, Y, \tau) \in \mathcal{U}(w)$ (noting that the admissible set of control is independent of the retirement function $\tilde{F}_0$), we have

$$\begin{aligned} J(C, A, Y, \tau; w) &= E\left[\int_0^\tau re^{-rt}(A_t - C_t)dt + e^{-r\tau}\tilde{F}_0(W_\tau)\right] \\ &= E\left[\int_0^\tau re^{-rt}(A_t - C_t)dt + e^{-r\tau}\tilde{G}_0(W_\tau)\right] \\ &\leq F^{G_0}(w). \end{aligned}$$

Taking supremum over all $(C, A, Y, \tau) \in \mathcal{U}(w)$ yields $F \leq F^{G_0}$.

Next, we apply the DPP for the problem with $\tilde{F}_0$ and see that for any $(C, A, Y, \nu) \in \mathcal{U}(w)$ and stopping time τ, we have by the DPP

$$\begin{aligned} F(w) &\geq E[\int_0^{\tau\wedge\nu} re^{-rs}(A_s - C_s)ds + e^{-r(\tau\wedge\nu)}F(W_{\tau\wedge\nu})] \\ &\geq E[\int_0^{\tau\wedge\nu} re^{-rs}(A_s - C_s)ds + e^{-r(\tau\wedge\nu)}G_0(W_{\tau\wedge\nu})]. \end{aligned}$$

since $G_0 \leq F$. Now, we could simply take $\tau = \nu$, and this will lead to

$$F(w) \geq E[\int_0^\nu re^{-rs}(A_s - C_s)ds + e^{-r\nu}G_0(W_\nu)].$$

Taking supremum over all $(C, A, Y, \nu) \in \mathcal{U}(w)$, we obtain $F(w) \geq F^{G_0}(w)$ for all $w \in [0, \bar{w}]$. □

Now, suppose we have a retirement function $\tilde{F}_0$ with optimal profit function F such that $F > \tilde{F}_0$ for $w \in [0, w_{gp}]$. Then we could find another retirement function such that the continuation region is disconnected.

Let $0 = w_0 < w_1 < w_2 < \ldots < w_{3n+1} = w_{gp}$, we could define G as follows: For $w \in [w_{3j}, w_{3j+1}]$, $j = 1, 2, \ldots, n$, we let $G(w) = F(w)$; for $w \in [w_{3j+1}, w_{3j+2}]$, $j = 1, 2, \ldots, n-1$, we let

$$G(w) = \frac{w_{3j+2} - w}{w_{3j+2} - w_{3j+1}}F(w_{3j+1}) + \frac{w - w_{3j+1}}{w_{3j+2} - w_{3j+1}}\tilde{F}_0(w_{3j+2});$$

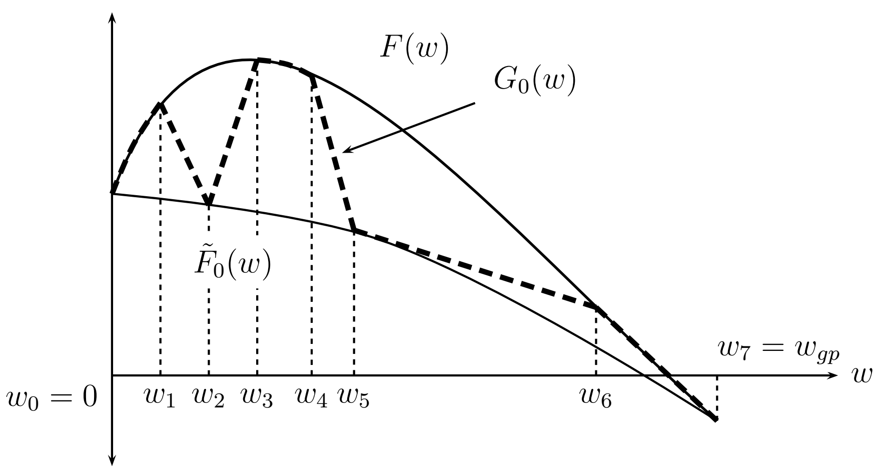

Figure 5.1: Illustration of the example with disconnected continuation region. In this case, the continuation region for the problem with retirement function G_0 is $(w_1, w_3) \cup (w_4, w_6)$, which is disconnected.

for $w \in [w_{3j+2}, w_{3j+3}],\ j = 1, 2, \ldots, n-1$, we let

$$G(w) = \frac{w_{3j+3} - w}{w_{3j+3} - w_{3j+2}} \tilde{F}_0(w_{3j+2}) + \frac{w - w_{3j+2}}{w_{3j+3} - w_{3j+2}} F(w_{3j+3}).$$

So we see that G coincides with F in every one of the three intervals, and is linear in each of the two next intervals such that G coincides with $\tilde{F}_0$ at the point connecting the two intervals. Since F is concave and $\tilde{F}_0 < F$ in $(0, w_{gp})$, we know that any lines connecting F and $\tilde{F}_0$ in $[0, w_{gp}]$ lies below F. So $F \geq G$.

Now, we let $G_0(w) = \max(G(w), \tilde{F}_0(w))$ for each $w \in [0, w_{gp}]$. We would have

$$F \geq G_0 \geq \tilde{F}_0(w)$$

over $[0, w_{gp}]$. Then $F^{G_0} = F$ by Proposition 5.1. See Figure 5.1 for a graphical illustration of G_0 (in a case where $G \geq \tilde{F}_0$ and so $G = G_0$).

Now, since $F^{G_0} = F = G_0$ for all $w_{3j} \leq w \leq w_{3j+1},\ j = 1, 2, \ldots, n-1$, it is optimal to retire the agent at those points. On the other hand, $F^{G_0} = F > G_0$ for all $w_{3j+1} \leq w \leq w_{3j+3},\ j = 1, 2, 3, \ldots, n-1$, making it optimal to continue at those points. We then see that the continuation region is disconnected.

While our example has been specifically constructed to show a case of a disconnected continuation region and seems unrealistic, the example suggests that there could be cases where a more naturally-arising retirement function could give a disconnected region. This is true especially because we do not restrict the retirement function $\tilde{F}_0$ to be concave. For example, suppose the principal has two mutually exclusive promotion opportunities for the agent. Then the retirement function is a maximum of the profit from the two opportunities and F_0. This could give rise to a retirement function of a similar shape as a smoothed version of G_0 in Figure 5.1.

5.2.2 A modified range for the continuation value

We have set up the range of continuation value to be $[0, \bar{w}]$ in our problem. At both of the endpoints, permanent retirement is the only way to deliver the corresponding value. It is possible consider the range of continuation value $[w_L, w_H]$ respectively, with $0 < w_L < w_H < \bar{w}$, and mandate permanent retirement at these points. In this case, we will require that $\tilde{F}_0 : [w_L, w_H] \to \mathbb{R}$ satisfies

$$F_0(w) \leq \tilde{F}_0(w) \leq \bar{F}(w)$$

for all $w \in [w_L, w_H]$. We will also require $\tilde{F}_0(w_H) = \bar{F}(w_H)$ for the right-continuity argument to go through.

Our analysis does not depend on the fact that $F_0(0) = 0$, and so similar results follow for the case with the range $[w_L, w_H]$. This is, for example, useful in the case of quitting that we reviewed in Section 5.1.

5.2.3 Upper-semicontinuous retirement function

If we relax the continuity requirement of $\tilde{F}_0$, and instead require that $\bar{F} \geq \tilde{F}_0 \geq F_0$ and $\tilde{F}_0$ is upper-semicontinuous. We will still have the result that the corresponding optimal profit function F is concave (as in Lemma 4.3), but the result on right-continuity at 0 and our work on the HJB equation and its viscosity solution do not follow immediately.

Instead, since we know that F is concave, we could consider the minimum concave function that is greater than $\tilde{F}_0$, i.e.

$$G_0(w) = \inf\{G(w) \mid G \text{ is continuous and concave and } G \geq \tilde{F}_0 \text{ over } [0, \bar{w}]\}.$$

Note that $G_0(w)$ is continuous and concave and $G_0 \geq \tilde{F}_0$ over $[0, \bar{w}]$. Since F is also a concave function that is greater than $\tilde{F}_0$, we know that $F \geq G_0$.

If we consider the control problem with retirement function G_0 and denote the corresponding optimal profit function by F^{G_0}, then we can argue that the two problems have the same optimal profit function:

Proposition 5.2. *$F^{G_0} = F$ for all $w \in [0, \bar{w}]$.*

Proof. The fact that $F \leq F^{G_0}$ is proved in the same way as Proposition 5.1, since the argument does not require continuity of G_0.

To proceed to prove $F \geq F^{G_0}$, we first note that, since we know that $G_0 \leq F \leq F^{G_0}$ and that G_0 and F^{G_0} coincides and are both right-continuous at $w = 0$, we know that F is also right-continuous at zero. Then the DPP will be valid for the control problem with G_0 as well, because our proof does not depend on the continuity of F_0, as long as we have the uniform continuity of F. Then the same argument used in Proposition 5.1 can be applied. So we have $F = F^{G_0}$. □

5.2.4 Retirement profit exceeding the first-best bound

We provide some discussion on the case where the retirement profit is not upper-bounded by the first-best profit function.

Our analysis relies on the assumption that $\tilde{F}_0 \leq \bar{F}$. In an economic sense, this assumption means that the principal could potentially profit more from the production output from the agent's work, than from retiring the agent. Technically, our proof of concavity of F, and therefore its continuity, depended on this assumption. Without this assumption, it is unclear whether the optimal profit function F is concave.

However, that could be cases where a retirement function $\tilde{F}$ violating the assumption, i.e. $\tilde{F}_0(w)$ could be potentially greater than $\bar{F}$ within $[0, \bar{w}$, be appropriate in modeling the real-world. In particular, the case of potential promotion in Sannikov (2008) gives us such an example. When the agent is promoted, its productivity increases. It therefore makes sense that the principal could profit more from promoting the agent (which means "retiring" in our model) than from working with the agent at the lower productivity.

To generalize our analysis, we would need to establish a proof of continuity of F over $[0, w]$ in order to establish the DPP and apply the viscosity approach. This could potentially be interesting in studying multiple sequential promotions of an agent.

Chapter 6

Concluding remarks

In this thesis, we have posed a rigorous formulation of a combined optimal stopping and control problem for a continuous-time principal-agent problem first proposed by Sannikov (2008). We provide conditions under which a solution of the formulated problem could be implemented as a contract in the original setting. Our formulation also allows for general retirement functions that are continuous and bounded by the first-best profit.

We study the optimal profit function via the notion of viscosity solutions. We show that the optimal profit function is the unique viscosity solution of the HJB equation associated with the combined optimal stopping and stochastic control problem. It is also shown that the optimal profit function is continuous and concave, and is C^1 when the retirement function $\tilde{F}_0$ is C^1.

The main contribution of the work in this thesis is that we provide a rigorous formulation of the problem and the viscosity solution approach that we use is easily extensible. Our analysis of the problem does not, a priori, assume smoothness and concavity of the optimal profit function, nor impose any structure on the retirement policy (like assuming only one low- and one high-retirement points). The analysis therefore applies to cases where the HJB equation does not have a smooth solution. Moreover, our analysis applies to the general continuous retirement function $\tilde{F}_0$ that are bounded by the first-best profit $\bar{F}$. The case where $\tilde{F}_0$ is upper-semicontinuous can be studied similarly by considering the minimum concave function greater than $\tilde{F}_0$.

Our model is generally applicable to cases where dynamic contracting is needed with private actions on the agent's side. This includes, for example, the compensation for an executive with expertise, where it is costly to monitor the actions of the executive. Also, our model could tackle cases where an agent has multiple termination options, like retirement, quitting for another job and multiple promotion prospects. These options could be modeled in the retirement function $\tilde{F}_0$ in our model, and our analysis holds even in cases where the HJB equation has a smooth solution. The multiple termination options possible could result in a complicated form of retirement function $\tilde{F}_0$. Our model then has a relative advantage in handling this, because our approach does not require an assumption on the HJB equation having a smooth solution nor a retirement structure with only one low- and high-retirement points.

One possible direction for future research is to look at retirement functions $\tilde{F}_0$ that are not bounded by the first-best profit $\bar{F}$. In such cases, it is not clear whether the optimal profit function F will be concave and this poses challenges to our analysis. Such cases will allow us to study, for example, cases with multiple levels of promotions where the promoted agent could bring in profit that is potentially higher than the first-best profit at the agent's original productivity.

It would also be interesting to look into different models for the dynamics of the output process of the principal-agent problem. For example, the output process could incorporate Poisson jumps, with the rate of the jumps also controlled by the agent. Alternatively, the constant volatility in the output process could be replaced by a regime-switching volatility. Our current model does not apply directly to these cases, but our methodology and approach could possibly be extended. This is especially because our approach does not rely on a priori assumptions on the smoothness and concavity of the optimal profit function.

Bibliography

R. Bellman. On the theory of dynamic programming. *Proceedings of the National Academy of Sciences*, 38:716–719, 1952.

A. Bensoussan and J.-L. Lions. *Applications of variational inequalities in stochastic control*. North-Holland Pub. Co., Amsterdam, 1982.

B. Biais, T. Mariotti, J.-C. Rochet, and S. Villeneuve. Large risks, limited liability, and dynamic moral hazard. *Econometrica*, 78(1):73–118, 2010.

P. Bolton and M. Dewatripont. *Contract Theory*. MIT Press, Cambridge, 2005.

V. S. Borkar. *Optimal control of diffusion processes*. Longman Group UK Limited, 1989.

A. N. Borodin and P. Salminen. *Handbook of Brownian Motion – Facts and Formulae*. Birkhauser Verlag, Basel - Boston - Berlin, 2002.

A. Cadenillas, J. Cvitanič, X. Wan, and J. Zhang. Optimal risk-sharing with effort and project choice. *J. of Economic Theory*, 133:403–440, 2007.

A. Chakrabarty and X. Guo. Optimal stopping times with different information levels and with time uncertainty. In T. Zhang and X. Zhou, editors, *Stochastic analysis and applications to finance*, volume 13 of *Interdisciplinary Mathematical Sciences*, pages 19–38. World Scientific, 2012. ISBN 978-981-4383-57-8.

J.-P. Chancelier, B. Oksendal, and A. Sulem. Combined stochastic control and optimal stopping, and application to numerical approximation of combined stochastic and impulse control. *Tr. Mat. Inst. Steklova*, 237:149–172, 2002.

M. Crandall and P.-L. Lions. Viscosity solutions of Hamilton-Jacobi equations. *Trans. A.M.S.*, 277:1–42, 1984.

M. G. Crandall, H. Ishii, and P.-L. Lions. User's guide to viscosity solutions of second order partial differential equations. *Bulletin of The American Mathematical Society*, 27:1–67, 1992.

J. Cvitanič and J. Zhang. Optimal compensation with adverse selection and dynamic actions. *Mathematics and Financial Economics*, 1(1):21–55, 2007. URL http://dx.doi.org/10.1007/s11579-007-0002-2.

J. Cvitanič, X. Wan, and J. Zhang. Optimal contracts in continuous-time models. *J. of Applied Mathematics and Stochastic Analysis*, 2006(Article ID 95203):1–27, 2006.

J. Cvitanič, X. Wan, and J. Zhang. Principal-agent problems with exit options. *The B.E. Journal in Theoretical Economics*, 8(1):Article 23, 2008.

J. Cvitanič, X. Wan, and J. Zhang. Optimal compensation with hidden action and lump-sum payment in a continuous-time model. *Applied Mathematics and Optimization*, 59(1):99–146, 2009. URL http://dx.doi.org/10.1007/s00245-008-9050-0.

J. Cvitanič, X. Wan, and H. Yang. Dynamics of contract design with screening. *Management Science*, 59:1229–1244, 2013.

P. M. DeMarzo and M. J. Fishman. Optimal long-term financial contracting. *Review of Financial Studies*, 20(6):2079 – 2128, 2007.

P. M. DeMarzo and Y. Sannikov. Optimal security design and dynamic capital structure in a continuous-time agency model. *The Journal of Finance*, 61(6), 2006.

D. Duffie, P.-Y. Geoffard, and C. Skiadas. Efficient and equilibrium allocations with stochastic differential utility. *Journal of Mathematical Economics*, 23(2):133 – 146, 1994. doi: 10.1016/0304-4068(94)90002-7. URL http://www.sciencedirect.com/science/article/pii/0304406894900027.

B. Dumas, R. Uppal, and T. Wang. Efficient intertemporal allocations with recursive utility. *Journal of Economic Theory*, 93(2):240 – 259, 2000.

A. Edmans, X. Gabaix, T. Sadzik, and Y. Sannikov. Dynamic CEO compensation. *Journal of Finance. Forthcoming.*, 2011.

W. H. Fleming and H. M. Soner. *Controlled Markov Processes and Viscosity Solutions*. Springer, 2006.

D. Fudenberg, B. Holmstrom, and P. Milgrom. Short-term contracts and long-term agency relationships. *Journal of Economic Theory*, 51:1–31, 1990.

S. J. Grossman and O. D. Hart. An analysis of the principal-agent problem. *Econometrica*, 51(1):7–45, 1983.

Z. He. Optimal executive compensation when firm size follows geometric Brownian motion. *Review of Financial Studies*, 22(2):859–892, 2009.

Z. He. Dynamic compensation contracts with private savings. *Review of Financial Studies. Forthcoming.*, 2011.

B. Holmstrom and P. Milgrom. Aggregation and linearity in the provision of intertemporal incentives. *Econometrica*, 55(2):303–328, 1987. URL `http://www.jstor.org/stable/1913238`.

R. Jensen, P.-L. Lions, and P. E. Souganidis. A uniqueness result for viscosity solutions of second order fully nonlinear partial differential equations. *Proceedings of the American Mathematical Society*, 102(4):975–978, 1988.

I. Karatzas and S. E. Shreve. *Brownian motion and stochastic calculus*. Springer, New York, 1991.

P. E. Kloeden and E. Platen. *Numerical Solution of Stochastic Differential Equations*. Springer-Verlag Berlin Heidelberg, 1992.

H. J. Kushner. Optimal stochastic control. *IRE Transactions on Automatic Control*, AC-7(5):120–122, 1962.

J. Laffont and D. Martimort. *The Theory of Incentives: The Principal-agent Model*. Princeton University Press, Princeton, 2002.

P.-L. Lions. Optimal control of diffusion processes and Hamilton-Jacobi-Bellman equations part 1: The dynamics programming principle and application. *Communications in Partial Differential Equations*, 8(10):1101–1174, 1983a.

P.-L. Lions. Optimal control of diffusion processes and Hamilton-Jacobi-Bellman equations part 2: Viscosity solutions and uniqueness. *Communications in Partial Differential Equations*, 8(11):1229–1276, 1983b.

H. M. Müller. The first-best sharing rule in the continuous-time principalagent problem with exponential utility. *Journal of Economic Theory*, 79(2): 276 – 280, 1998. doi: 10.1006/jeth.1997.2381.

H. M. Müller. Asymptotic efficiency in dynamic principal-agent problems. *Journal of Economic Theory*, 91(2):292 – 301, 2000. doi: 10.1006/jeth.1999.2623.

L. S. Pontryagin, V. G. Boltyanskii, R. V. Gamkrelidze, and E. F. Mishchenko. *The Mathematical Theory of Optimal Processes*. Interscience Pub., New York, 1962.

R. Radner. Repeated principal-agent games with discounting. *Econometrica*, 53(5):1173–1198, 1985.

Y. Sannikov. A continuous-time version of the principal-agent problem. *Review of Economic Studies*, 75:957–984, 2008.

Y. Sannikov. Contracts: The theory of dynamic principal-agent relationships and the continuous-time approach. In D. Acemoglu, M. Arellano, and E. Dekel, editors, *Advances in Economics and Econometrics*, volume 1, pages 89–124. Cambridge, 2013. ISBN 978-1-107-01604-0.

H. Schättler and J. Sung. The first-order approach to the continuous-time principalagent problem with exponential utility. *Journal of Economic Theory*, 61(2):331 – 371, 1993. ISSN 0022-0531. doi: http://dx.doi.org/10.1006/jeth.1993.1072. URL `http://www.sciencedirect.com/science/article/pii/S0022053183710720`.

S. E. Spear and S. Srivastava. On repeated moral hazard with discounting. *Review of Economic Studies*, 54(4):599–617, 1987. URL `http://www.jstor.org/stable/2297484`.

D. W. Stroock and S. R. S. Varadhan. *Multidimensional diffusion processes*. Springer-Verlag New York, 1979.

N. Williams. On dynamic principal-agent problems in continuous time. *Working Paper.*, 2009.

J. Yong and X. Y. Zhou. *Stochastic controls: Hamiltonian systems and HJB equations*. Springer-Verlag New York, 1999.

J. Y. Zhu. Optimal contracts with shirking. *The Review of Economic Studies*, 80(2):812–839, 2013.

Appendix A

Proofs of Sannikov's propositions

Since the two propositions from Sannikov (2008) play an important role in understanding our formulation and model, the proofs are extracted from his paper for the readers' reference, with a minor change in notations[1] and additional comments to assist reading.

A.1 Representation of the agent's continuation value

Recall that the continuation value in Sannikov (2008) is defined as

$$W_t(C, A) = E^A \left[\int_t^\infty e^{-r(s-t)}(u(C_s) - h(A_s))ds | \mathcal{F}_t \right]. \tag{1.10}$$

The following is the proof for Sannikov's proposition on representing the dynamics of the agent's continuation value in the form of a stochastic differential equation.

Proof of Proposition 1.1. The agent's total payoff from a compensation-effort pair (C, A) given the information at time t is

$$V_t = r \int_0^t e^{-rs}(u(C_s) - h(A_s))ds + e^{-rt}W_t(C, A), \tag{A.1}$$

[1]The most visible change is that we use $\mathbf{P}^A$ for the probability measure induced by the agent's action A

which is a $\mathbf{P}^A$-martingale. Assuming the filtration $\{\mathcal{F}_t\}$ satisfies the usual conditions, the $\mathbf{P}^A$-martingale V must have a right-continuous with left limits modification by Theorem 1.3.13 in Karatzas and Shreve (1991, P.16). Then by the martingale representation theorem (Karatzas and Shreve, 1991, P.182, Theorem 3.4.15), we get the representation[2]

$$V_t = V_0 + r \int_0^t e^{-rs} \sigma Y_s dZ_s^A, \quad 0 \le t < \infty, \tag{A.2}$$

where

$$Z_t^A = \frac{1}{\sigma}\left(X_t - \int_0^t A_s ds\right)$$

is a $\mathbf{P}^A$-Brownian motion and the factor $re^{-rt}\sigma$ that multipies Y_t is a convenient rescaling. Differentiating (A.1) and (A.2) with respect to t, we find that

$$\begin{aligned} dV_t &= re^{-rt}\sigma Y_t dZ_t^A \\ &= re^{-rt}(u(C_t) - h(A_t))dt + d(e^{-rt}W_t(C, A)) \\ &= re^{-rt}(u(C_t) - h(A_t))dt - re^{-rt}W_t(C, A)dt + e^{-rt}dW_t(C, A), \end{aligned}$$

which impiles

$$dW_t(C, A) = r(W_t(C, A) - u(C_t) + h(A_t))dt + r\sigma Y_t dZ_t^A.$$

This proves Proposition 1.1. □

A.2 Incentive compatibility

The following is the proof for Sannikov's proposition on the condition for incentive-compatibility of (C, A).

Proof of Proposition 1.2. Consider an arbitrary alternative strategy A^*. Define[3]

$$\hat{V}_t = r \int_0^t e^{-rs}(u(C_s) - h(A_s^*))ds + e^{-rt}W_t(C, A),$$

[2]This martingale representation theorem requires that the martingale is square-integrable. In this proposition, this will be satisfied, for example, when C takes value from a bounded set $[0, \bar{c}]$. The theorem also guarantees that, in this context, that if any other $\tilde{Y}_t$ satisfies the same representation, then $\int_0^\infty e^{-rt}|Y_t - \tilde{Y}_t|^2 dt = 0$ a.s.

[3]It is useful at this point to remember that $W_t(C, A)$ does not depend on the value of A from 0 to t.

to be the expected total payoff for the agent given the information at time t, if he has incurred the cost of effort from strategy A^* up to time t, and plans to follow the strategy A after time t. To identify the drift of the process $\hat{V}_t$ under the probability measure $\mathbf{P}^{A^*}$, we note that

$$\begin{aligned} d\hat{V}_t &= re^{-rt}(u(C_t) - h(A_t^*))dt + d(e^{-rt}W_t(C, A)) \\ &= re^{-rt}(u(C_t) - h(A_t^*))dt - re^{-rt}(u(C_t) - h(A_t))dt + re^{-rt}Y_t dZ_t^A \\ &= re^{-rt}(h(A_t) - h(A_t^*) + Y_t(A_t^* - A_t))dt + re^{-rt}Y_t \sigma dZ_t^{A^*}, \end{aligned}$$

where the Brownian motion under $\mathbf{P}^A$ and $\mathbf{P}^{A^*}$ are related by

$$\sigma Z_t^A = \sigma Z_t^A + \int_0^t (A_s^* - A_s)ds.$$

If (1.13) does not hold on a positive measure, choose A_t^* that maximizes $Y_t A_t^* - h(A_t^*)$ for all $t \geq 0$. Then the drift of $\hat{V}$ (under $\mathbf{P}^{A^*}$) is non-negative and positive on a set of positive measure. Thus, there exists a time $t > 0$ such that

$$E^{A^*}[\hat{V}_t] > \hat{V}_0 = W_0(C, A).$$

Since the agent gets utility $E^{A^*}[\hat{V}_t]$ if he follows A^* until time t and then switches to A, the strategy A is suboptimal.

Suppose (1.13) holds for the strategy A. Then $\hat{V}_t$ is a $\mathbf{P}^{A^*}$-supermartingale for any alternative strategy A^*. Moreover, since the process is $\{W_t(C, A)\}$ is bounded from below, we can add

$$\hat{V}_\infty = r \int_0^\infty e^{-rs}(u(C_s) - h(A_s^*))ds$$

as the last element of the supermartingale $\hat{V}$. Therefore,

$$W_0(C, A) = \hat{V}_0 \geq E^{A^*}[\hat{V}_\infty] = W_0(C, A^*),$$

so the strategy A is at least as good as any alternative strategy A^*. □