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A Parametrix Construction for Low Regularity Wave Equations and Spectral Rigidity for Two Dimensional Periodic Schroedinger Operators

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A Parametrix Construction for Low Regularity Wave Equations and Spectral Rigidity for Two Dimensional Periodic Schrödinger Operators

A dissertation submitted in partial satisfaction
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by

Alden Marie Waters

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Abstract of the Dissertation

A Parametrix Construction for Low Regularity Wave Equations and Spectral Rigidity for Two Dimensional Periodic Schrödinger Operators

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Alden Marie Waters

Doctor of Philosophy in Mathematics

University of California, Los Angeles, 2012

Professor James Ralston IV, Chair

This dissertation consists of two parts. In the first half, we construct a frame of complex Gaussians for the space of $L^2(\mathbb{R}^n)$ functions. When propagated along bicharacteristics for the wave equation, the frame can be used to build a parametrix with suitable error terms. When the coefficients of the wave equation have more regularity, propagated frame functions become Gaussian beams.

In the latter half, we consider two dimensional real-valued analytic potentials for the Schrödinger equation which are periodic over a lattice $\mathbb{L}$. Under certain assumptions on the form of the potential and the lattice $\mathbb{L}$, we can show there is a large class of analytic potentials which are Floquet rigid and dense in the set of $C^\infty(\mathbb{R}^2/\mathbb{L})$ potentials. The result extends the work of Eskin et. al, in "On isospectral periodic potentials in $\mathbb{R}^n$, II."

The dissertation of Alden Marie Waters is approved.

Richard Korf

John Garnett

Christoph Thiele

James Ralston IV, Committee Chair

University of California, Los Angeles

2012

To my parents, Candice and Raymond Waters

and my grandfather, Bror Charles Seaburg who taught me about numbers as a little girl

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Acknowledgments

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Vita

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CHAPTER 1

Introduction

This dissertation consists of two parts which are reproductions of my previous work. The first part is a reproduction, with minor changes, of my paper [Wat11] and the second part is a reproduction of [Wat12]. The significance of both halves is outlined below.

1.1 Gaussian Frames for Parametrices

I have completed an investigation into the feasibility of a certain parametrix construction for the wave equation. In my paper [Wat11], which is the basis for the first half of my dissertation, I consider the Cauchy problem for the wave equation

$$\Box u(t,x) = \frac{\partial^2}{\partial t^2} u(t,x) - \sum_{j,k=1}^{n} a_{jk}(x,t) \frac{\partial^2}{\partial x_i \partial x_k} u(t,x) = 0, \tag{1.1.1}$$

with $u(t,x) : [-T,T] \times \mathbb{R}^n \to \mathbb{R}$. The coefficients $(a_{jk}(x,t))_{j,k=1}^n$ are assumed to form a uniformly positive definite and bounded matrix. Following Hart Smith's paper [Smi98], I assume only that the coefficients $a_{jk}(x,t)$ are $C^{1,1}$. I consider a family of complex Gaussians indexed by $\gamma \in \Gamma$,

$$\phi_\gamma(x) = C_{\gamma,n} \exp(i\xi_\gamma \cdot (x - x_\gamma) - |\xi_\gamma|\,|x - x_\gamma|^2), \tag{1.1.2}$$

along with a natural choice of a family (x_γ, ξ_γ) in $\mathbb{R}^{2n}$, and I prove that they satisfy a frame condition for $H^m(\mathbb{R}^n)$ functions as in [SHB04]:

Theorem 1. *[Wat11] There exist constants C_1 and C_2 with $0 < C_1 < C_2 < \infty$ such that,* *for all m with $0 \le m < \infty$, the inequalities*

$$C_1||f(x)||^2_{H^m(\mathbb{R}^n)} \le \sum_{\gamma\in\Gamma} |(1+|\xi_\gamma|^2)^{\frac{m}{2}} \langle \phi_\gamma(x), f(x)\rangle|^2 \le C_2||f(x)||^2_{H^m(\mathbb{R}^n)}$$

hold for all $f \in H^m(\mathbb{R}^n)$.

If we use the pairs (x_γ, ξ_γ) in (1.1.2) as initial data for bicharacteristics determined by the wave equation, then along the solution curves, or null-bicharacteristics, we may propagate the functions in (1.1.2) to obtain $\phi_\gamma(t,x)$ in a simple way to approximately solve $\Box u(t,x) = 0$. The frame condition ensures that, when P and P^* are given by

$$P : f(x) \mapsto \{\langle f(x), \phi_\gamma(x)\rangle\}_{\gamma\in\Gamma}, \qquad P^* : \{\langle f(x), \phi_\gamma(x)\rangle\}_{\gamma\in\Gamma} \mapsto \sum_{\gamma\in\Gamma} \langle f(x), \phi_\gamma(x)\rangle \phi_\gamma(x),$$

the operator $\Pi = P^*P$ is bounded and invertible on its range. The operators P and P^* allow us to think of the action of hyperbolic operators in terms of weighted $l^2(\Gamma)$ sequences, which are sometimes easier to understand and manipulate. Let the propagation operator $E(t)$ be defined as follows:

$$E(t)\Pi f(x) = \sum_{\gamma\in\Gamma} \langle f(x), \phi_\gamma(x)\rangle \phi_\gamma(t,x).$$

The simplicity of the construction of $E(t)$ is the main reason that a frame of Gaussian functions proves valuable. Using the idea of transferring $H^m(\mathbb{R}^n)$ norms to to weighted $l^2(\Gamma)$ sequences, I prove

Theorem 2. *$E(t)$ is an operator of order zero, and $\Box E(t)$ is an operator of order one.*

The main theorem in my paper, Theorem 3.2 in [Wat11] may be stated as follows:

Theorem 3. *If $-1 \le m \le 2$, with $f \in H^{m+1}(\mathbb{R}^n)$ and $h \in H^m(\mathbb{R}^n)$, and if $F \in L^1([-T,T]; H^m(\mathbb{R}^n))$, then there exists a $G \in L^1([-T,T]; H^m(\mathbb{R}^n))$ such that*

$$u(t,x) = \mathcal{C}(t,0)f(x) + \mathcal{S}(t,0)h(x) + \int_0^t \mathcal{S}(t,s)G(s,x)\,ds$$

and

$$||G||_{L^1([-T,T];H^m(\mathbb{R}^n))} \le C(T)\left(||f||_{H^{m+1}(\mathbb{R}^n)} + ||h||_{H^m(\mathbb{R}^n)} + ||F||_{L^1([-T,T];H^m(\mathbb{R}^n))}\right)$$

solves the Cauchy problem

$$\Box u(t,x) = (\partial_t^2 - A(t,x,\partial_x))u(t,x) = F(t,x)$$
$$u(t,x)|_{t=0} = f(x)$$
$$\partial_t u(t,x)|_{t=0} = h(x)$$

in the weak sense. If f and h are both identically zero and F is also zero for all $t \in [-T,T]$, then G and u will vanish as well. Here $\mathcal{C}(t,0)f(x)$ and $\mathcal{S}(t,0)h(x)$ are defined in terms of linear combinations of the propagated functions and the sets of inner products $\{\langle f(x), \phi_\gamma(x)\rangle\}_{\gamma\in\Gamma}$ and $\{\langle h(x), \phi_\gamma(x)\rangle\}_{\gamma\in\Gamma}$.

1.2 Spectral Analysis for Periodic Two Dimensional Schrödinger Potentials:

In my work on inverse problems, I consider the Schrodinger operator, P, such that

$$P : u(x) \mapsto (-\Delta + q(x))u(x) \tag{1.2.1}$$

where $q(x) : \mathbb{R}^n \to \mathbb{R}$ is periodic over a vector lattice $\mathbb{L} \subset \mathbb{R}^n$, i.e.,

$$q(x+d) = q(x), \qquad \forall d \in \mathbb{L}.$$

I assume that the lattice satisfies the condition

$$|d| = |d'| \Rightarrow d = \pm d', \qquad \forall d, d' \in \mathbb{L} \tag{1.2.2}$$

and study the set of $\lambda \in \mathbb{R}$ for which the problem

$$(-\Delta + q)u(x) = \lambda u(x), \quad u(x+d) = \exp(2\pi i k \cdot d)u(x), \quad \forall d \in \mathbb{L} \tag{1.2.3}$$

has a solution. If, for some k in $\mathbb{R}^n$, there exists a function u not identically zero solving (6.0.1), we say that $\lambda \in \mathrm{Spec}_k(-\Delta + q)$, and we refer to $\bigcup_k \mathrm{Spec}_k(-\Delta + q)$ as the Floquet spectrum. In the case $k = 0$, we refer to $\mathrm{Spec}_0(-\Delta + q)$ simply as the spectrum, denoted by $\mathrm{Spec}(-\Delta + q)$.

I have begun a study of the Floquet spectrum in 2 dimensions which will naturally extend to dimensions 3 and higher. We say that two potentials, q and $\tilde{q}$, are Floquet isospectral if

$$\mathrm{Spec}_k(-\Delta + q) = \mathrm{Spec}_k(-\Delta + \tilde{q}), \quad \forall k \in \mathbb{R}^n,$$

and we say that they are isospectral if equality need only hold for the spectrum. We say that a certain potential q is Floquet rigid if there are only a finite number of potentials which are Floquet isospectral to q; similarly, if only a finite number of potentials are isospectral to q, we say that q is spectrally rigid. Under the assumptions that the the lattice satisfies the condition (1.2.2) and the potentials are analytic, the potentials are Floquet isospectral whenever they are isospectral, which simplifies the analysis as in [ERT84a] and [ERT84b].

In the second half of my dissertation, I will show that, under certain hypotheses, there is a larger set of smooth analytic periodic $L^2(\mathbb{R}^2/\mathbb{L})$ potentials than those considered in [ERT84b] which are Floquet rigid

CHAPTER 2

Background for the Parametrix Construction

In [Smi98], Hart Smith constructed a parametrix solution for the wave equation using a frame that is now called curvelets. We construct, in this dissertation section, a new frame out of Gaussian functions. When a Gaussian function is propagated along the ray, it becomes a Gaussian beam, which looks like a Gaussian distribution on planes perpendicular to a ray in space-time. The existence of such solutions has been known to the pure mathematics community since the 1960s. Recently, there has been a revival of interest in Gaussian beams, given their robustness in approximating solutions to PDEs.

Nicolay Tanushev numerically simulated mountain waves with a high degree of accuracy using superpositions of high frequency Gaussian beams in [Tan08]. Gaussian beams are concentrated along a single ray, and thus it is desirable to use many of them to represent a solution because a global solution is rarely concentrated along a single curve [Ral82]. Tanushev's dissertation showed that Gaussian beams have several major advantages over other techniques used to numerically approximate the solution to a mountain wave. Motivated by these numerical calculations, we will show that a frame consisting entirely of complex Gaussians can be used to build an accurate parametrix to the wave equation.

The idea of using complex Gaussians to build an accurate parametrix is not new. Daniel Tataru, in [Tat00], constructed a parametrix to the wave equation with low regularity coefficients using a modified FBI transform. While the solution in his paper is elegant, numerical calculations with such a construction would be difficult, if not impossible. Representing initial data in terms of a frame of Gaussians may lead to more viable and accurate numerical solutions, as done with frames of curvelets in [AHS08].

For the first half of this dissertation we will consider the wave equation,

$$\partial_t^2 u(t,x) - A(t,x,\partial_x)u(t,x) = \partial_t^2 u(t,x) - \sum_{1\le i,j\le n} a_{ij}(t,x)\partial_{x_i}\partial_{x_j}u(t,x) = 0,$$

and we let

$$A(x,t) = \{a_{ij}(x,t)\}_{1\le i,j\le n}.$$

We assume that the matrix A is uniformly positive definite and bounded – that is, there exists a constant $C > 0$ with

$$\frac{|\xi|^2}{C} \le \sum_{1\le i,j\le n} a_{ij}(t,x)\xi_i\xi_j \le C|\xi|^2$$

for all (t,x,ξ) in $[-T,T]\times\mathbb{R}^n\times\mathbb{R}^n$. Here, T is fixed and finite. Furthermore, we assume the entries of the matrix, denoted $a_{ij}(t,x)$ with $(t,x)\in[-T,T]\times\mathbb{R}^n$, are in $C^{1,1}$. Coefficients which are $C^{1,1}$ are of interest because they are minimally regular; they satisfy a Lipschitz condition in x and t,

$$|a_{ij}(t,x) - a_{ij}(t',x')| \le C(|t-t'| + |x-x'|),$$

and their first derivatives in x satisfy a Lipschitz condition

$$|\nabla_x a_{ij}(t,x) - \nabla_x a_{ij}(t,x')| \le C|x-x'|.$$

This dissertation section is divided into three major parts. The focus of Chapter 3 is the introduction of a frame of Gaussian functions, which will represent elements of the Hilbert space $L^2(\mathbb{R}^n)$. Theorem 4 is the main topic of Chapter 3, which shows not only that the essential $L^2(\mathbb{R}^n)$ estimate for a frame holds, but also that we also have stronger estimates for weighted sequences of frame functions in terms of Sobolev norms. The proof of Theorem 4 consists of two technical lemmas which introduce notation that will be used in Chapter 4.

Building on the framework of Chapter 3, Chapter 4 details the construction of what are usually called Gaussian beams (when the higher regularity cases are considered) and shows how they are propagated in space-time. The main theorem in Chapter 4 is Theorem

6, which shows that the propagated frame operators are bounded in the appropriate little-ℓ sequence spaces. These sequence spaces correspond to the natural Sobolev norms of the functions which are used as initial data. This chapter contains the necessary estimates for the construction of a parametrix for the wave equation with $C^{1,1}$ coefficients. Finally, Chapter 5 follows the work of [Smi98] very closely and contains the actual parametrix construction.

CHAPTER 3

Construction of the Frame

Let the set of functions $\{\phi_\gamma(x)\}_{\gamma\in\Gamma}$ be defined as follows:

$$\phi_\gamma(x) := \left(\frac{|\xi_\gamma|\Delta x_\gamma}{2\pi}\right)^{\frac{n}{2}} \exp\left(i\xi_\gamma \cdot (x - x_\gamma) - |\xi_\gamma||x - x_\gamma|^2\right),$$

where γ is the index $\gamma = (i, k, \alpha)$ with $i \in I_k$, and where I_k is a finite subset of integers which depends on $k \in \mathbb{N}$ and $\alpha \in \mathbb{Z}^n$. In the first two lemmas we will pick $\xi_\gamma = 2^k\omega_{i,k}$ a vector in $\mathbb{R}^n$ with $\frac{1}{2} \leq |\omega_{i,k}| < 1$, and $x_\gamma = \Delta x_\gamma \alpha$ another vector in $\mathbb{R}^n$ with Δx_γ a scale factor depending on k. We will show that these vectors can be chosen so that the set of functions $\{\phi_\gamma(x)\}_{\gamma\in\Gamma}$ form a frame for $L^2(\mathbb{R}^n)$. Not only will our chosen set of $\{\phi_\gamma(x)\}_{\gamma\in\Gamma}$ form a frame in $L^2(\mathbb{R}^n)$, but we also will show that weighted sequences of frame functions are comparable to the m^{th} Sobolev norm (provided it exists) of any $f(x)$. In particular, we have

Theorem 4. *For any finite $m \geq 0$ and $f(x) \in H^m(\mathbb{R}^n)$ there exist constants C_1 and C_2, independent of γ and with $0 < C_1 \leq C_2$, such that the following holds:*

$$0 < C_1 \,||f(x)||^2_{\dot{H}^m(\mathbb{R}^n)} \leq \sum_\gamma |2^{km}c(\gamma)|^2 \leq C_2 \,||f(x)||^2_{\dot{H}^m(\mathbb{R}^n)}, \tag{3.0.1}$$

with

$$c(\gamma) = \int_{\mathbb{R}^n} \overline{\phi_\gamma(x)} f(x)\, dx.$$

For this dissertation we will use the convention that the Fourier Transform for a function $h(u) \in L^2(\mathbb{R}^n)$ is defined as

$$\hat{h}(\eta) := \int_{\mathbb{R}^n} e^{-i\eta\cdot u} h(u)\, du.$$

We will also need to introduce the following functions:

$$\psi_\gamma(w) := \left(\frac{|\xi_\gamma|}{2\pi}\right)^{\frac{n}{2}} \exp\left(i\xi_\gamma \cdot w - |\xi_\gamma||w|^2\right).$$

The only difference between $\psi_\gamma(x-u)$ and $\phi_\gamma(x)$ is that the discrete variable x_γ is now a continuous one, u, and there is no factor of $(\Delta x_\gamma)^{\frac{n}{2}}$. Here we note that

$$|\hat{\psi}_\gamma(\xi)|^2 = 2^{-n} \exp\left(-\frac{|\xi - \xi_\gamma|^2}{2|\xi_\gamma|}\right).$$

In Lemma 1 we construct an approximate partition of unity from the sum of the squares of the Fourier transforms of the $\psi_\gamma(w)$.

Lemma 1. *One can chose $\omega_{i,k}$, i in I_k, $k \in \mathbb{N}$ with $\frac{1}{2} \le |\omega_{i,k}| < 1$ so that the inequalities*

$$0 < C_1'|\xi|^{2m} \le \sum_{(i,k)} 2^{2km} \exp\left(-\frac{|\xi - 2^k\omega_{i,k}|^2}{2|2^k\omega_{i,k}|}\right) \le C_2'|\xi|^{2m} \tag{3.0.2}$$

hold for all $\xi \in \mathbb{R}^n/\{0\}$, $0 \le m < \infty$, finite. Here C_1' and C_2' are constants independent of ξ.

For clarity, we will save the proof of Lemma 1 and Lemma 2 below until the end of the proof of Theorem 4. For Lemma 2, we will pick Δx_γ so that we can approximate the center term in the inequality (3.0.1) by an expression which no longer involves α, effectively turning the summation over α into an integral.

Lemma 2. *For fixed $k \in \mathbb{N}$, let Δx_γ equal $C_\epsilon 2^{-\frac{k}{2}-\epsilon k}$ with $\epsilon > 0$ and C_ϵ a small constant independent of k and dependent on ϵ. Then, for every $\epsilon > 0$, there exists a choice of C_ϵ such that the following holds:*

$$\Bigg|\sum_\gamma \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} 2^{2km} f(x)\phi_\gamma(x-\alpha\Delta x_\gamma)\overline{f(x')\phi_\gamma(x'-\alpha\Delta x_\gamma)}\,dx\,dx' -$$
$$\sum_{(i,k)} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} 2^{2km}\psi_\gamma(x-u)f(x)\overline{\psi_\gamma(x'-u)f(x')}\,du\,dx\,dx'\Bigg| \le \frac{\pi^n e^{-1}}{2}\,||f||^2_{L^2(\mathbb{R}^n)}.$$

Proof of Theorem 4. If we let

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} 2^{2km}\psi_\gamma(x-u)\overline{\psi_\gamma(x'-u)}f(x)f(x')\,du\,dx\,dx',$$

then the kernel of this expression can be rewritten as

$$\int_{\mathbb{R}^n} 2^{2km}\psi_\gamma(x-u)\overline{\psi_\gamma(x'-u)}\,du = (2\pi)^n \int_{\mathbb{R}^n} e^{i\xi\cdot(x-x')}2^{2km}|\widehat{\psi_\gamma(\xi)}|^2\,d\xi,$$

since the Fourier Transform is an isometry on $L^2(\mathbb{R}^n)$. As remarked earlier,

$$|\widehat{\psi_\gamma(\xi)}|^2 = 2^{-n}\exp\left(-\frac{|\xi-\xi_\gamma|^2}{2|\xi_\gamma|}\right),$$

so that, by Lemma 1 and Fubini's Theorem,

$$\begin{aligned}\pi^n C_1' \left|\left||\xi|^m \widehat{f(\xi)}\right|\right|^2_{L^2(\mathbb{R}^n)} &\leq \sum_{(i,k)}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} 2^{2km}\psi_\gamma(x-u)\overline{\psi_\gamma(x'-u)}f(x)f(x')\,du\,dx\,dx' \\ &\leq \pi^n C_2' \left|\left||\xi|^m \widehat{f(\xi)}\right|\right|^2_{L^2(\mathbb{R}^n)},\end{aligned}$$

which is equivalent to

$$\begin{aligned}\pi^n C_1' \,||f(x)||^2_{\dot{H}^m(\mathbb{R}^n)} &\leq \sum_{(i,k)}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} 2^{2km}\psi_\gamma(x-u)\overline{\psi_\gamma(x'-u)}f(x)f(x')\,du\,dx\,dx' \\ &\leq \pi^n C_2' \,||f(x)||^2_{\dot{H}^m(\mathbb{R}^n)}.\end{aligned} \tag{3.0.3}$$

From Lemma 2, we also have

$$\begin{aligned}&\left|\sum_\gamma \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} 2^{2km} f(x)\phi_\gamma(x-\alpha\Delta x_\gamma)\overline{f(x')\phi_\gamma(x'-\alpha\Delta x_\gamma)}\,dx\,dx' - \right. \\ &\left.\sum_{(i,k)}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} 2^{2km}\psi_\gamma(x-u)f(x)\overline{\psi_\gamma(x'-u)f(x')}\,du\,dx\,dx'\right| \leq \frac{\pi^n e^{-1}}{2}\,||f||^2_{L^2(\mathbb{R}^n)},\end{aligned} \tag{3.0.4}$$

but, since $\frac{e^{-1}}{2} < C_1' = e^{-1}$ and $C_2' > C_1'$,we can combine inequalities (3.0.3) and (3.0.4) to conclude

$$\begin{aligned}C_1\,||f||^2_{\dot{H}^m(\mathbb{R}^n)} &\leq \left|\sum_\gamma \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} 2^{2km} f(x)\phi_\gamma(x-x_\gamma)\overline{f(x')\phi_\gamma(x'-x_\gamma)}\,dx\,dx'\right| \\ &\leq C_2\,||f||^2_{\dot{H}^m(\mathbb{R}^n)},\end{aligned}$$

which is the result (3.0.1). □

Proof of Lemma 1. Since $\xi \in \mathbb{R}^n/\{0\}$, we begin by considering $\mathbb{R}^n$ as an infinite union of dyadic annuli, each of which we will cover with real Gaussians which are centered at our choice of $2^k\omega_{i,k}$. In every annulus $2^{k-1} \leq |\xi| < 2^k$, for all $k \in \mathbb{N}$, we choose the vectors $2^k\omega_{i,k}$ such that, for all $i \neq j$, we have $|2^k\omega_{i,k} - 2^k\omega_{j,k}| > 2^{\frac{k}{2}}$, while allowing the number of $2^k\omega_{i,k}$ in each annulus to be as large as possible. The index set I_k is finite, as the volume of every annulus is finite.

Fixing ξ for the rest of this proof, ξ must lie in an annulus $2^{k-1} \leq |\xi| < 2^k$ for some fixed k in $\mathbb{N}$. As a result of our choice of vectors, for all $\xi \in \mathbb{R}^n$ there exists at least one point $2^k\omega_{i,k}$ for which the inequality $|\xi - 2^k\omega_{i,k}| < 2^{\frac{k}{2}}$ holds. This condition gives a lower bound:

$$|\xi|^{2m}e^{-1} \leq \sum_{(i,k)} 2^{2km} \exp\left(-\frac{|\xi - 2^k\omega_{i,k}|^2}{2|2^k\omega_{i,k}|}\right).$$

To show the sum is bounded above, we will consider sets of indices $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$, and $\mathcal{E}$, whose union contains all the indices (i,k) in γ and show that the contribution to the sum from each of these sets is bounded by a constant multiple of $|\xi|^{2m}$. The cases $k = 0, 1$ are easy, so we consider $k \geq 2$.

First let $\mathcal{A}$ consist of those indices (i,q) for which $|\xi - 2^q\omega_{i,q}| < 2^{\frac{k}{2}}$. Clearly, q can only be equal to $k-1, k$, or $k+1$. Fixing q for the moment and setting $r = 2^{\frac{q}{2}}$, if we consider a ball B of radius $\frac{r}{2}$ centered at each $2^q\omega_{i,q}$, then, for all pairs $(i,q),(j,q) \in \mathcal{A}$ with $i \neq j$, we have $B(2^q\omega_{i,q}, \frac{r}{2}) \cap B(2^q\omega_{j,q}, \frac{r}{2}) = \emptyset$. But by the triangle inequality, all balls of radius $\frac{r}{2}$ with centers that have indices in $\mathcal{A}$ are contained in a ball of radius $\frac{3r}{2}$ around ξ. Therefore, the total number of balls N is bounded, as

$$Vol\left(B\left(\xi, \frac{3r}{2}\right)\right) \geq NVol\left(B\left(2^q\omega_{i,q}, \frac{r}{2}\right)\right),$$

which implies $N \leq 3^n$. Since there are only three possible values q can take, the total contribution for the set of indices $\mathcal{A}$ to the sum is bounded by 3^{n+1}.

For the second set, let $\mathcal{B}$ consist of those indices (i,q) for which the inequality $2^{\frac{k}{2}} \leq |\xi - 2^q\omega_{i,q}| < 2^k$ holds and $|k-q| \leq 1$. We can write $\mathcal{B}$ as a collection of subsets $\mathcal{B}_j$ such that

$$\mathcal{B} = \bigcup_{j=2}^{2^{\frac{k}{2}}} \mathcal{B}_j,$$

were $\mathcal{B}_j$ denotes the set of indices for which $(j-1)2^{\frac{k}{2}} \leq |\xi - 2^q\omega_{i,q}| < j2^{\frac{k}{2}}$. As before, we consider balls of radius $\frac{r}{2} = 2^{\frac{q}{2}-1}$ centered at each $2^q\omega_{i,q}$ such that for all pairs $(i,q),(j,q) \in \mathcal{B}_j$ with $i \neq j$, $B(2^q\omega_{i,q}, \frac{r}{2}) \cap B(2^q\omega_{j,q}, \frac{r}{2}) = \emptyset$. By the triangle inequality, all balls with centers that have indices in $\mathcal{B}_j$ are contained in an annulus centered about ξ with inner radius $(j-1)r - \frac{r}{2}$ and outer radius $jr + \frac{r}{2}$. The total number of indices for fixed q in each set $\mathcal{B}_j$ is bounded, as

$$Vol\left(B\left(\xi, jr + \frac{r}{2}\right)\right) - Vol\left(B\left(\xi, (j-1)r - \frac{r}{2}\right)\right) \geq NVol\left(B\left(2^q\omega_{i,q}, \frac{r}{2}\right)\right),$$

which implies

$$N \leq 2^n\left(\left(j + \frac{1}{2}\right)^n - \left(j - \frac{3}{2}\right)^n\right).$$

Since q can take only three possible values, multiplying this last bound by 3 gives a bound on the total number of indices in each set B_j. Because of the restriction on the size of $|\xi - 2^q\omega_{i,q}|$ and the fact that $|q-k| \leq 1$, the inequality

$$\frac{(j-1)^2}{2} \leq \frac{|\xi - 2^q\omega_{i,q}|^2}{2|2^q\omega_{i,q}|} \leq 4j^2$$

holds for each tuple in $\mathcal{B}_j$. Summing over all of the sets $\mathcal{B}_j$,

$$\begin{aligned}
&\sum_j \sum_{(i,q)\in\mathcal{B}_j} 2^{2km} \exp\left(\frac{-|\xi - 2^q\omega_{i,q}|^2}{2|2^q\omega_{i,q}|}\right) \\
&< \sum_{j=2}^{2^{\frac{k}{2}}} 2^{2km} 3(2^n)\left(\left(j + \frac{1}{2}\right)^n - \left(j - \frac{3}{2}\right)^n\right) \exp\left(-\frac{(j-1)^2}{2}\right) \\
&\leq \sum_{j=2}^{\infty} 2^{2km} 3(2^n)(j+1)^n \exp\left(-\frac{(j-1)^2}{2}\right).
\end{aligned}$$

The sum

$$\sum_{j=2}^{\infty} 3(2^n)(j+1)^n \exp\left(-\frac{(j-1)^2}{2}\right)$$

is finite; furthermore, it is uniformly bounded regardless of the choice of k and hence of ξ. Therefore since $2^{k-1} \leq |\xi| < 2^k$, the total contribution from the set $\mathcal{B}$ is bounded by a constant times $|\xi|^{2m}$.

Next, let $\mathcal{C}$ be the set of indices (i,q) for which $|\xi - 2^q\omega_{i,q}| > 2^k$ holds and also $|k-q| \leq 1$. As before, for each fixed q, we take balls of radius $\frac{r}{2} = 2^{\frac{q}{2}-1}$ centered at each $2^q\omega_{i,q}$ so that, for all pairs $(i,q),(j,q) \in \mathcal{C}$ with $i \neq j$, we have $B(2^q\omega_{i,q}, \frac{r}{2}) \cap B(2^q\omega_{j,q}, \frac{r}{2}) = \emptyset$. All balls with centers that have indices in $\mathcal{C}$ are contained in an annulus centered about the origin with inner radius $r^2 - \frac{r}{2}$ and outer radius $r^2 + \frac{r}{2}$. Since we have removed a number of the vectors because their indices are in $\mathcal{B}$, the total number of indices, N, for fixed q is over-estimated as follows:

$$Vol\left(B\left(0, r^2 + \frac{r}{2}\right)\right) - Vol\left(B\left(0, r^2 - \frac{r}{2}\right)\right) \geq N Vol\left(B\left(2^q\omega_{i,q}, \frac{r}{2}\right)\right),$$

which implies

$$N \leq 2^n \left(\left(r + \frac{1}{2}\right)^n - \left(r - \frac{1}{2}\right)^n\right).$$

Since $|\xi - 2^q\omega_{i,q}| > 2^k$ for all $(i,q) \in \mathcal{C}$, the inequality

$$2^{k-2} < \frac{2^{2k}}{2^{q+1}} \leq \frac{|\xi - 2^q\omega_{i,q}|^2}{2|2^q\omega_{i,q}|}$$

holds for each point $2^q\omega_{i,q}$ with indices in $\mathcal{C}$. Then the contribution from the set $\mathcal{C}$ is bounded in terms of a sum over k as

$$\begin{aligned}
&\sum_{(i,q)\in\mathcal{C}} 2^{2km} \exp\left(\frac{-|\xi - 2^q\omega_{i,q}|^2}{2|2^q\omega_{i,q}|}\right) \\
&< \sum_{q=k-1}^{k+1} 2^{2km} 2^n \left(\left(2^{\frac{q}{2}} + \frac{1}{2}\right)^n - \left(2^{\frac{q}{2}} - \frac{1}{2}\right)^n\right) \exp\left(-2^{k-2}\right).
\end{aligned}$$

But, since

$$\begin{aligned}
&\sum_{q=k-1}^{k+1} 2^n \left(\left(2^{\frac{q}{2}} + \frac{1}{2}\right)^n - \left(2^{\frac{q}{2}} - \frac{1}{2}\right)^n\right) \exp\left(-2^{k-2}\right) \\
&< 3(2^n) \left(\left(2^{\frac{k+1}{2}} + \frac{1}{2}\right)^n - \left(2^{\frac{k+1}{2}} - \frac{1}{2}\right)^n\right) \exp\left(-2^{k-2}\right),
\end{aligned} \tag{3.0.5}$$

and since $2^{kn}\exp\left(-2^{k-2}\right) \to 0$ for all finite n in $\mathbb{R}$ as $k \to \infty$, (3.0.5) is bounded is independently of ξ. So we can conclude that, since $2^{k-1} \leq |\xi| < 2^k$, the total contribution from the set $\mathcal{C}$ is bounded by a constant times $|\xi|^{2m}$ as well.

Now, let $\mathcal{D}$ be the set of indices (i, q) for which $q < k - 1$. To find the number of vectors in $\mathcal{D}$ for fixed q, we again take balls of radius $\frac{r}{2} = 2^{\frac{q}{2}-1}$ centered at each $2^q\omega_{i,q}$ so that, for all pairs $(i, q), (j, q) \in \mathcal{D}$ with $i \neq j$, $B(2^q\omega_{i,q}, \frac{r}{2}) \cap B(2^q\omega_{j,q}, \frac{r}{2}) = \emptyset$. By the triangle inequality, all balls with centers that have indices in $\mathcal{D}$ are contained in an annulus centered about the origin with inner radius $r^2 - \frac{r}{2}$ and outer radius $r^2 + \frac{r}{2}$. The total number of indices N is bounded, as

$$Vol\left(B\left(0, r^2 + \frac{r}{2}\right)\right) - Vol\left(B\left(0, r^2 - \frac{r}{2}\right)\right) \geq NVol\left(B\left(2^q\omega_{i,q}, \frac{r}{2}\right)\right),$$

which gives

$$N \leq 2^n\left(\left(r + \frac{1}{2}\right)^n - \left(r - \frac{1}{2}\right)^n\right).$$

We can conclude there are at most $2^n\left(\left(2^{\frac{q}{2}} + \frac{1}{2}\right)^n - \left(2^{\frac{q}{2}} - \frac{1}{2}\right)^n\right)$ vectors for fixed q. Since, for these indices, $q < k - 1$, the inequality

$$2^{q-1} \leq \frac{(2^{k-1} - 2^q)^2}{2^{q+1}} \leq \frac{|\xi - 2^q\omega_{i,q}|^2}{2|2^q\omega_{i,q}|}$$

holds for each (i, q) in $\mathcal{D}$. The total contribution from the set $\mathcal{D}$ is also bounded by a constant times $|\xi|^{2m}$:

$$\begin{aligned}
&\sum_{(i,q)\in\mathcal{D}} 2^{2km}\exp\left(\frac{-|\xi - 2^q\omega_{i,q}|^2}{2|2^q\omega_{i,q}|}\right) \\
&< \sum_{q=1}^{k-2} 2^{2km}2^n\left(\left(2^{\frac{q}{2}} + \frac{1}{2}\right)^n - \left(2^{\frac{q}{2}} - \frac{1}{2}\right)^n\right)\exp\left(-2^{q-1}\right) \\
&< \sum_{q=1}^{\infty} 2^{2km}2^n(2^{\frac{q}{2}} + 1)^n\exp\left(-2^{q-1}\right),
\end{aligned}$$

since the sum

$$\sum_{q=1}^{\infty} 2^n(2^{\frac{q}{2}} + 1)^n\exp\left(-2^{q-1}\right)$$

is uniformly bounded with respect to k.

The final set $\mathcal{E}$ contributing to the sum consists of the indices (i,q) for which $q > k+1$. Again, as above, the total number of vectors N for fixed q is at most

$$2^n \left(\left(2^{\frac{q}{2}} + \frac{1}{2} \right)^n - \left(2^{\frac{q}{2}} - \frac{1}{2} \right)^n \right).$$

To find the exponential contribution for each $q > k+1$, note that, for each $(i,q) \in \mathcal{E}$,

$$2^{q-5} \leq \frac{(2^{q-1} - 2^{q-2})^2}{2^{q+1}} \leq \frac{|\xi - 2^q \omega_{i,q}|^2}{2|2^q \omega_{i,q}|},$$

and hence

$$\sum_{(i,q)\in\mathcal{E}} 2^{2km} \exp\left(\frac{-|\xi - 2^q \omega_{i,q}|^2}{2|2^q \omega_{i,q}|} \right) < \sum_{q=k+2}^{\infty} 2^{2km} 2^n \left(2^{\frac{q}{2}} + 1 \right)^n \exp\left(-2^{q-5} \right).$$

The sum

$$\sum_{q=k+2}^{\infty} 2^n \left(2^{\frac{q}{2}} + 1 \right)^n \exp\left(-2^{q-5} \right)$$

is convergent and bounded independently of k and q. Therefore the total contribution from the set $\mathcal{E}$ is bounded by a constant times $|\xi|^{2m}$ as well. This completes the proof of the Lemma. The construction of the approximate partition of unity is similar in idea to the construction of almost orthogonal frames in Meyer's book, [MC97]. The ξ_γ which are further away from the variable ξ contribute less to the the partition than those which are close. □

Proof of Lemma 2. For convenience we let:

$$g_\gamma(u,x,x')(\Delta_\gamma x)^n = 2^{2km} \phi_\gamma(x-u)\overline{\phi_\gamma(x'-u)},$$

which implies that the operator

$$\sum_\gamma \left(2^{2km} \phi_\gamma(x - \alpha\Delta x_\gamma)\overline{\phi_\gamma(x' - \alpha\Delta x_\gamma)} \right)$$

is equal to

$$\sum_{(i,k)} \sum_{\alpha\in\mathbb{Z}^n} (g_\gamma(\alpha\Delta x_\gamma, x, x'))(\Delta x_\gamma)^n.$$

We will rewrite the sum over α above using the Poisson summation formula. Recall:

Theorem 5. *[Hor03] (Poisson Summation Formula) Let a be constant, $h(u) \in \mathcal{S}(\mathbb{R}^n)$, and $\alpha, \beta \in \mathbb{Z}^n$. The following holds:*

$$a^n \sum_{\alpha \in \mathbb{Z}^n} h(a\alpha) = \sum_{\beta \in \mathbb{Z}^n} \hat{h}\left(\frac{2\pi\beta}{a}\right).$$

Since, by definition,

$$\hat{g}_\gamma(\eta, x, x') = \int_{\mathbb{R}^n} e^{-i\eta\cdot u} g_\gamma(u, x, x')\, du,$$

we start by computing

$$\begin{aligned}
&g_\gamma(u, x, x')\\
&\quad = 2^{2km}\left(\frac{|\xi_\gamma|}{2\pi}\right)^n \exp\left(i(u-x)\cdot\xi_\gamma - |\xi_\gamma|(u-x)^2 - i(u-x')\cdot\xi_\gamma - |\xi_\gamma|(u-x')^2\right)\\
&\quad = 2^{2km}\left(\frac{|\xi_\gamma|}{2\pi}\right)^n \exp\left(i(x'-x)\cdot\xi_\gamma\right)\exp\left(|\xi_\gamma|\left(-2u^2 + 2u(x+x') - x^2 - x'^2\right)\right)\\
&\quad = 2^{2km}\left(\frac{|\xi_\gamma|}{2\pi}\right)^n \exp\left(i(x-x')\cdot\xi_\gamma\right)\exp\left(-2|\xi_\gamma|\left(u - \left(\frac{x+x'}{2}\right)\right)^2\right)\exp\left(-\frac{|\xi_\gamma|(x-x')^2}{2}\right),
\end{aligned}$$

which, by a standard result on the Fourier transform of a Gaussian (see Appendix A), gives

$$\begin{aligned}
&\hat{g}_\gamma(\eta, x, x') \qquad (3.0.6)\\
&= 2^{2km}\left(\frac{\pi}{2|\xi_\gamma|}\right)^{\frac{n}{2}}\left(\frac{|\xi_\gamma|}{2\pi}\right)^n \exp\left(i(x-x')\cdot\xi_\gamma + i\eta\cdot\left(\frac{x+x'}{2}\right)\right)\exp\left(-\frac{\eta^2}{8|\xi_\gamma|} - \frac{|\xi_\gamma|(x-x')^2}{2}\right).
\end{aligned}$$

Now we notice that

$$\hat{g}_\gamma(0, x, x') = \int_{\mathbb{R}^n} g_\gamma(u, x, x')\, du,$$

so, by applying the Poisson summation formula, we obtain

$$\sum_{\alpha\in\mathbb{Z}^n} (g_\gamma(\alpha\Delta x_\gamma, x, x'))(\Delta x_\gamma)^n = \int_{\mathbb{R}^n} g_\gamma(u, x, x')\, du + \sum_{\beta\in\mathbb{Z}^n \beta\neq 0} \hat{g}_\gamma\left(\frac{2\pi\beta}{\Delta x_\gamma}, x, x'\right),$$

where $\hat{g}_\gamma(\eta, x, x')$ is given explicitly by (3.0.6). From this we can conclude the left hand side of (3.0.4) is

$$\left|\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\sum_{(k,i)}\sum_{\beta\in\mathbb{Z}^n, \beta\neq 0} \hat{g}_\gamma\left(\frac{2\pi\beta}{\Delta x_\gamma}, x, x'\right) f(x)\overline{f(x')}\, dx\, dx'\right|.$$

By symmetry of the integrands in x and x', if we use Schur's lemma, the inequality in (3.0.4) follows from the estimate

$$\sup_{x\in\mathbb{R}^n}\int_{\mathbb{R}^n}\left|\sum_{(i,k)}\sum_{\beta\in\mathbb{Z}^n,\beta\neq 0}\hat{g}_\gamma\left(\frac{2\pi\beta}{\Delta x_\gamma},x,x'\right)\right|dx'<\sqrt{\frac{\pi^n e^{-1}}{2}}. \tag{3.0.7}$$

If we examine the integrand in the left-hand side of (3.0.7) we find, from equality (3.0.6), that

$$\left|\sum_{\beta\in\mathbb{Z}^n,\beta\neq 0}\hat{g}_\gamma\left(\frac{2\pi\beta}{\Delta x_\gamma},x,x'\right)\right|\leq\sum_{\beta\in\mathbb{Z}^n,\beta\neq 0}\left|\hat{g}_\gamma\left(\frac{2\pi\beta}{\Delta x_\gamma},x,x'\right)\right|$$
$$=\sum_{\beta\in\mathbb{Z}^n,\beta\neq 0}2^{2km}\left(\frac{|\xi_\gamma|}{8\pi}\right)^{\frac{n}{2}}\exp\left(-\frac{(2\pi\beta)^2}{8|\xi_\gamma|(\Delta x_\gamma)^2}-\frac{|\xi_\gamma|(x-x')^2}{2}\right).$$

Integrating both sides of the above inequality with respect to x' gives

$$\sup_{x\in\mathbb{R}^n}\int_{\mathbb{R}^n}\left|\sum_{\beta\in\mathbb{Z}^n,\beta\neq 0}\hat{g}_\gamma\left(\frac{2\pi\beta}{\Delta x_\gamma},x,x'\right)\right|dx'\leq\sum_{\beta\in\mathbb{Z}^n,\beta\neq 0}2^{2km}2^{-n}\exp\left(-\frac{(2\pi\beta)^2}{8|\xi_\gamma|(\Delta x_\gamma)^2}\right).$$

Let $\beta=(\beta_1,\beta_2,...,\beta_n)$. Then since, with this notation, $\beta_i\in\mathbb{Z}$ is indexed independently of $\beta_j\in\mathbb{Z}$ for all $i\neq j$, we have

$$\sum_{\beta\in\mathbb{Z}^n}\left(\prod_{i=1}^n\exp\left(-\frac{(2\pi\beta_i)^2}{8|\xi_\gamma|(\Delta x_\gamma)^2}\right)\right)=\prod_{i=1}^n\left(\sum_{\beta_i\in\mathbb{Z}}\exp\left(-\frac{(2\pi\beta_i)^2}{8|\xi_\gamma|(\Delta x_\gamma)^2}\right)\right).$$

As $\beta\neq 0$, at least one of the $\beta_i's$ must also be nonzero. Without loss of generality, take $\beta_n\neq 0$, and then

$$\sum_{\beta\in\mathbb{Z}^n,\beta\neq 0}\exp\left(-\frac{(2\pi\beta)^2}{8|\xi_\gamma|(\Delta x_\gamma)^2}\right) \tag{3.0.8}$$
$$\leq n\left(\prod_{i=1}^{n-1}\left(\sum_{\beta_i\in\mathbb{Z}}\exp\left(-\frac{(2\pi\beta_i)^2}{8|\xi_\gamma|(\Delta x_\gamma)^2}\right)\right)\right)\left(\sum_{\beta_n\in\mathbb{Z},\beta_n\neq 0}\exp\left(-\frac{(2\pi\beta_n)^2}{8|\xi_\gamma|(\Delta x_\gamma)^2}\right)\right).$$

To put a bound on this last expression, we now need to pick Δx_γ. Let

$$a(k)=\frac{(2\pi)^2}{8|\xi_\gamma|(\Delta x_\gamma)^2}.$$

Then, since Δx_γ is of the form $C_\epsilon 2^{-\frac{k}{2}-\epsilon k}$, we have that

$$a(k)=\frac{\pi^2 2^{\epsilon k}}{2C_\epsilon}.$$

Now if we pick $C_\epsilon < 4$, then, for all k, $a(k) > 1$ provided $\epsilon \geq 0$. Therefore, for any such choice of C_ϵ, we have

$$\sum_{\beta_i \in \mathbb{Z}} \exp\left(-\frac{(\beta_i)^2}{8|\xi_\gamma|(\Delta x_\gamma)^2}\right) < 2\sum_{\beta_i \in \mathbb{N}} \exp\left(-a(k)\beta_i\right) = \frac{2}{1-e^{-a(k)}} < \frac{2}{1-e^{-1}},$$

which ensures the first product in (3.0.8) is uniformly bounded independently of k:

$$\left(\prod_{i=1}^{n-1}\left(\sum_{\beta_i \in \mathbb{Z}} \exp\left(-\frac{(2\pi\beta_i)^2}{8|\xi_\gamma|(\Delta x_\gamma)^2}\right)\right)\right) < \left(\frac{2}{1-e^{-a(k)}}\right)^{n-1} < \left(\frac{2e}{e-1}\right)^{n-1}.$$

From Lemma 1, the number of the $2^k\omega_{i,k}$ can be over-estimated by by $2^n(2^{\frac{k}{2}}+1)^n$ for any fixed k, so, combining estimates,

$$\begin{aligned}&\sum_{(i,k)} \sum_{\beta \in \mathbb{Z}^n, \beta \neq 0} 2^{2km} 2^{-n} \exp\left(-\frac{(2\pi\beta)^2}{8|\xi_\gamma|(\Delta x_\gamma)^2}\right) \\ &< \sum_{k=1}^{\infty} 2^{2km} n(2^{\frac{k}{2}}+1)^n \left(\frac{2e}{e-1}\right)^{n-1} \exp\left(-a(k)\right).\end{aligned}$$

Since $\epsilon > 0$, $\exp(-a(k))$ dominates any power of 2^k. Thus as long as C_ϵ is chosen sufficiently small we can make this sum less than $\sqrt{\frac{\pi^n e^{-1}}{2}}$, which concludes the proof of inequality (3.0.7).

□

CHAPTER 4

Operator Norm Estimates

From Theorem 4 in Chapter 3, the operator $P_1^m(f(y)) = \{c(\gamma)\}_{\gamma\in\Gamma}$, where

$$c(\gamma) = \int_{\mathbb{R}^n} 2^{km}\overline{\phi_\gamma(x)}f(x)\,dx,$$

is a one-to-one bounded mapping of $H^m(\mathbb{R}^n)$ into the space of sequences which are convergent in $l^2(\Gamma)$ when weighted with 2^{km}. Let $P_2^m = P_1^{m*}$ be defined as follows:

$$P_2^m : l^2(\Gamma) \to L^2(\mathbb{R}^n), \qquad P_2^m(\{c(\gamma)\}) = \sum_\gamma 2^{km}c(\gamma)\phi_\gamma(y).$$

Now recall that $\Box$ is an operator of order m if $\Box$ maps $H^r(\mathbb{R}^n) \to H^{r-m}(\mathbb{R}^n)$. In Chapter 3, we showed that $\Pi^m = P_2^m \circ P_1^m$ is an operator of order $2m$. Let I denote the identity operator. As there exist constants C_1' and C_2' such that $C_1'I \le \Pi^0 \le C_2'I$, in $L^2(\mathbb{R}^n)$ norm sense, P_1^0 is bounded and invertible on its range. The construction of P_1^0 and P_2^0 allows us to translate the characterization of functions and operators in $H^m(\mathbb{R}^n)$ to the framework of weighted sequences in $l^2(\Gamma)$. Armed with the frame operators, we will show that, when the frame functions are propagated along bicharacteristics for the wave equation, their Sobolev norm is preserved. This will help us also show that the the action of the operator $\Box(x,t,\partial_x,\partial_t) = \partial_t^2 - A(x,t,\partial_x)$ on the parametrix is order 1. The estimates established in this chapter will ultimately be useful in building the parametrix in Chapter 5.

First we recall that $\Box(x,t,\partial_x,\partial_t) = \partial_t^2 - A(x,t,\partial_x)$ has principal symbol $p(x,t,\xi,\tau) = \tau^2 - \sum_{i,j} a_{i,j}(x,t)\xi_i\xi_j$. The bicharacteristics associated to p are

$$\frac{dt}{ds} = p_\tau, \qquad \frac{dx_j}{ds} = p_{\xi_j}, \qquad \frac{d\xi_j}{ds} = -p_{x_j}, \qquad \frac{d\tau}{ds} = -p_t. \tag{4.0.1}$$

Setting $q = \left(\sum_{i,j} a_{i,j}\xi_i\xi_j\right)^{\frac{1}{2}}$, we find that $p = (\tau - q)(\tau + q)$. There are two choices for null bicharacteristics. Here we assume that $\tau = q$, so that the set of bicharacteristic equations (4.0.1) become

$$\begin{aligned} \frac{dt}{ds} &= p_\tau = 2\tau = 2q, \qquad \frac{dx}{dt} = \frac{p_\xi}{2q} = q_\xi, \\ \frac{d\xi}{dt} &= \frac{-p_x}{2q} = -q_x, \qquad \frac{d\tau}{dt} = 1. \end{aligned} \tag{4.0.2}$$

Define

$$(x_\gamma(t, t', x_\gamma, \xi_\gamma), \xi_\gamma(t, t', x_\gamma, \xi_\gamma))$$

as the solution to the system (4.0.2) at time t with initial conditions

$$(x_\gamma(t', t', x_\gamma, \xi_\gamma), \xi_\gamma(t', t', x_\gamma, \xi_\gamma)) = (x_\gamma, \xi_\gamma),$$

where (x_γ, ξ_γ) are given in Lemmas 1 and 2 of Chapter 3. We let $\mathcal{U}(t, t')$ denote the the evolution operator associated to this transformation. Often we will abbreviate

$$\mathcal{U}(t, 0)(x_\gamma, \xi_\gamma) = (x_\gamma(t, 0, x_\gamma, \xi_\gamma), \xi_\gamma(t, 0, x_\gamma, \xi_\gamma))$$

as

$$(x_\gamma(t), \xi_\gamma(t)),$$

and

$$\mathcal{U}(0, t)(x_\gamma, \xi_\gamma) = (x_\gamma(0, t, x_\gamma, \xi_\gamma), \xi_\gamma(0, t, x_\gamma, \xi_\gamma))$$

as

$$(x_\gamma(-t), \xi_\gamma(-t)).$$

Let

$$\phi_\gamma(t, x) = \left(\frac{|\xi_\gamma(t)|\Delta x_\gamma}{2\pi}\right)^{\frac{n}{2}} \exp\left(i\xi(t) \cdot (x - x_\gamma(t)) - |\xi_\gamma(t)||x - x_\gamma(t)|^2\right).$$

Then define $E(t)$ to be the propagation operator acting on $f(x) \in L^2(\mathbb{R}^n)$ as follows:

$$\begin{aligned} \Pi^0 E(t) \Pi^0 f &= P_2^0 B_E(t) P_1^0 f \\ &= \sum_{\gamma,\gamma'} b_E(\gamma, \gamma', t) c(\gamma') \phi_\gamma(x), \end{aligned}$$

where

$$b_E(\gamma,\gamma',t) = \int_{\mathbb{R}^n} \overline{\phi_\gamma(x)}\phi_{\gamma'}(t,x)\,dx$$

denotes the entries of the matrix $B_E(t)$. As a result, $\Box E(t)$ is defined by the following equation:

$$\begin{aligned}\Pi^0\Box E(t)\Pi^0 f &= P_2^0 B_\Box(t) P_1^0 f \\ &= \sum_{\gamma,\gamma'} b_\Box(\gamma,\gamma',t)c(\gamma')\phi_\gamma(x),\end{aligned}$$

where

$$b_\Box(\gamma,\gamma',t) = \int_{\mathbb{R}^n} \overline{\phi_\gamma(x)}\Box\phi_{\gamma'}(t,x)\,dx$$

denotes the entries of the matrix $B_\Box(t)$. The central theorem of this Chapter is then:

Theorem 6. *$E(t)$ is a bounded operator of order 0, and $\Box E(t)$ is a bounded operator of order 1.*

From Chapter 3, Π^0 is bounded and invertible, and also, by Theorem 4, we know the relationship of the frame to the Sobolev norm of $f(x)$. Therefore, to prove Theorem 3 by Schur's lemma, it suffices to show

$$\sum_\gamma |b_E(\gamma,\gamma',t)| \le C, \qquad \sum_{\gamma'} |b_E(\gamma,\gamma',t)| \le C, \tag{4.0.3}$$

and

$$\sum_\gamma |b_\Box(\gamma,\gamma',t)| \le C2^{k'}, \qquad \sum_{\gamma'} |b_\Box(\gamma,\gamma',t)| \le C2^{k}, \tag{4.0.4}$$

where C denotes a constant independent of γ and γ'. We will also show that this constant is uniform for all $t \in [-T,T]$.

We start by examining $\Box\phi_\gamma(t,x)$:

Lemma 3.

$$\Box\phi_\gamma(t,x) = \left(\frac{|\xi_\gamma(t)|\Delta x_\gamma}{2\pi}\right)^{\frac{n}{2}} \times \left(p(x,t,\psi_x,\psi_t)e^{i\psi} + \mathcal{O}(|\xi_\gamma(t)|)e^{i\psi}\right),$$

where

$$\psi(t,x,x_\gamma(t),\xi_\gamma(t)) = \xi_\gamma(t)\cdot(x-x_\gamma(t)) + i|\xi_\gamma(t)||x-x_\gamma(t)|^2$$

and

$$p(t,x,\psi_x,\psi_t) = \mathcal{O}(|\xi_\gamma(t)|^2|x-x_\gamma(t)|^2).$$

Proof. As $p(t,x,\psi_x,\psi_t)$ is positive and homogeneous of degree two in $|\xi_\gamma(t)|$, the desired conclusion will follow if, on null-bicharacteristics $(t,x_\gamma(t),\xi_\gamma(t))$, we can show that

$$\nabla_x p(t,x,\psi_x(t,x_\gamma(t),\xi_\gamma(t)),\psi_t(t,x_\gamma(t),\xi_\gamma(t))) = 0.$$

Computing $\nabla_x p(t,x,\psi_x,\psi_t)$,

$$\frac{\partial}{\partial x_j} p(t,x,\psi_x,\psi_t) = p_{x_j} + p_{\xi_l}\psi_{x_l x_j} + p_\tau \psi_{\tau x_j}. \tag{4.0.5}$$

Dividing (4.0.5) by $2q$ and substituting the equations in (4.0.2) into the right hand side of (4.0.5), we obtain

$$-\frac{d\xi_j}{dt} + \frac{dx_l}{dt}\psi_{x_l x_j} + \psi_{t x_j} \tag{4.0.6}$$

As $\psi_{x_j}(t,x_\gamma(t),\xi_\gamma(t)) = \xi_j(t)$, differentiating $\xi_j(t)$ with respect to t we have

$$\frac{d\xi_j}{dt} = \frac{dx_l}{dt}\psi_{x_l x_j} + \psi_{t x_j}. \tag{4.0.7}$$

Substituting (4.0.7) into (4.0.6) implies (4.0.6) is 0, which happens if and only if (4.0.5) vanishes on null bicharacteristics. □

With Lemma 3 in mind, we consider the entries of the matrices $B_E(t)$ and $B_\square(t)$. First we set

$$\beta^0_{\gamma,\gamma'} = \left(\frac{|\xi_\gamma||\xi_{\gamma'}(t)|\Delta x_\gamma \Delta x_{\gamma'}}{(2\pi)^2}\right)^{\frac{n}{2}},$$

and then

$$b_E(\gamma,\gamma',t)$$
$$= \beta^0_{\gamma,\gamma'} \int_{\mathbb{R}^n} \exp\left(i(x-x_{\gamma'}(t))\cdot\xi_{\gamma'}(t) - i(x-x_\gamma)\cdot\xi_\gamma - |\xi_{\gamma'}(t)||x-x_{\gamma'}(t)|^2 - |\xi_\gamma|||x-x_\gamma|^2\right)\,dx$$

and, to leading order,

$$
\begin{aligned}
&b_{\square}(\gamma,\gamma',t) \qquad (4.0.8)\\
&= \beta^0_{\gamma,\gamma'} \int_{\mathbb{R}^n} \exp\left(i(x-x_{\gamma'}(t))\cdot\xi_{\gamma'}(t) - i(x-x_\gamma)\cdot\xi_\gamma - |\xi_{\gamma'}(t)||x-x_{\gamma'}(t)|^2 - |\xi_\gamma||x-x_\gamma|^2\right)\\
&\times |x-x_{\gamma'}(t)|^2|\xi_{\gamma'}(t)|^2\,dx.
\end{aligned}
$$

The first inner product, $b_E(\gamma,\gamma',t)$, is evaluated via

$$
\begin{aligned}
&b_E(\gamma,\gamma',t)\\
&= \beta^0_{\gamma,\gamma'} \int_{\mathbb{R}^n} \exp\left(i(x-x_{\gamma'}(t))\cdot\xi_{\gamma'}(t) - i(x-x_\gamma)\cdot\xi_\gamma - |\xi_{\gamma'}(t)||x-x_{\gamma'}(t)|^2 - |\xi_\gamma|||x-x_\gamma|^2\right)\,dx\\
&= \beta^0_{\gamma,\gamma'} \exp\left(ix_\gamma\cdot\xi_\gamma - ix_{\gamma'}(t)\cdot\xi_{\gamma'}(t)\right)\\
&= \int_{\mathbb{R}^n} \exp\left(ix\cdot(\xi_{\gamma'}(t)-\xi_\gamma)\right)\exp\left(-\left(|\xi_\gamma|+|\xi_{\gamma'}(t)|\right)\left|x-\frac{|\xi_\gamma|x_\gamma+|\xi_{\gamma'}(t)|x_{\gamma'}(t)}{|\xi_\gamma|+|\xi_{\gamma'}|}\right|^2\right)\\
&\times \exp\left(-\frac{|\xi_{\gamma'}(t)||\xi_\gamma|}{|\xi_\gamma|+|\xi_{\gamma'}(t)|}|x_{\gamma'}(t)-x_\gamma|^2\right)\,dx.
\end{aligned}
$$

Making the change of variable

$$
y = x - \frac{|\xi_\gamma|x_\gamma + |\xi_{\gamma'}(t)|x_{\gamma'}(t)}{|\xi_\gamma|+|\xi_{\gamma'}(t)|}, \qquad (4.0.9)
$$

we see that $b_E(\gamma,\gamma',t)$ takes the form of the Fourier transform of a Gaussian integral which we can evaluate (see Appendix B), obtaining

$$
\begin{aligned}
&b_E(\gamma,\gamma',t)\\
&= \beta_{\gamma,\gamma'} \exp\left(i\left(x_\gamma\cdot\xi_\gamma - x_{\gamma'}(t)\cdot\xi_{\gamma'}(t) + \frac{|\xi_\gamma|x_\gamma+|\xi_{\gamma'}(t)|x_{\gamma'}(t)}{|\xi_\gamma|+|\xi_{\gamma'}(t)|}\cdot(\xi_{\gamma'}(t)-\xi_\gamma)\right)\right)\\
&\times \exp\left(-\frac{|\xi_{\gamma'}(t)-\xi_\gamma|^2}{4(|\xi_\gamma|+|\xi_{\gamma'}(t)|)}\right)\exp\left(-\frac{|\xi_{\gamma'}(t)||\xi_\gamma|}{|\xi_\gamma|+|\xi_{\gamma'}(t)|}\,|x_{\gamma'}(t)-x_\gamma|^2\right),
\end{aligned}
$$

Therefore

$$
\beta_{\gamma,\gamma'} = \left(\frac{|\xi_\gamma||\xi_{\gamma'}(t)|\Delta x_\gamma \Delta x_{\gamma'}}{4\pi(|\xi_{\gamma'}(t)|+|\xi_\gamma|)}\right)^{\frac{n}{2}}, \qquad (4.0.10)
$$

so that

$$|b_E(\gamma,\gamma',t)| \tag{4.0.11}$$
$$\le \beta_{\gamma,\gamma'}\exp\left(-\frac{|\xi_{\gamma'}(t)-\xi_\gamma|^2}{4(|\xi_\gamma|+|\xi_{\gamma'}(t)|)}\right)\exp\left(-\frac{|\xi_{\gamma'}(t)||\xi_\gamma|}{|\xi_\gamma|+|\xi_{\gamma'}(t)|}|x_{\gamma'}(t)-x_\gamma|^2\right).$$

For the integral $b_\square(\gamma,\gamma',t)$ we make the same substitution (4.0.9) into (4.0.8). Then we set

$$\eta=\xi_{\gamma'}(t)-\xi_\gamma,\qquad c=|\xi_\gamma|+|\xi_{\gamma'}(t)|,$$

and

$$b=\frac{|\xi_\gamma|(x_\gamma-x_{\gamma'}(t))}{|\xi_\gamma|+|\xi_{\gamma'}(t)|},$$

so that we can apply the estimates in Appendix A. These give that $|b_\square(\gamma,\gamma',t)|$ is equal to

$$\beta_{\gamma,\gamma'}|\xi_{\gamma'}|^2\exp\left(-\frac{\eta^2}{4c}\right)\left|-\frac{\eta^2}{4c^2}+\frac{ib\eta}{c}+b^2+\frac{1}{2c}\right|.$$

Applying Cauchy-Schwartz and back substituting values for η, c, and b, we have that, for C a constant independent of γ,γ',

$$|b_\square(\gamma,\gamma',t)|\le C\beta_{\gamma,\gamma'}\left(\frac{|\xi_{\gamma'}(t)|^2|\xi_\gamma-\xi_{\gamma'}(t)|^2}{(|\xi_{\gamma'}(t)|+|\xi_\gamma|)^2}+\frac{|\xi_{\gamma'}(t)|^2|\xi_\gamma|^2|x_\gamma-x_{\gamma'}(t)|^2}{(|\xi_\gamma|+|\xi_{\gamma'}(t)|)^2}\right)$$
$$\times\exp\left(-\frac{|\xi_{\gamma'}(t)-\xi_\gamma|^2}{4(|\xi_\gamma|+|\xi_{\gamma'}(t)|)}\right)\exp\left(-\frac{|\xi_{\gamma'}(t)||\xi_\gamma|}{|\xi_\gamma|+|\xi_{\gamma'}(t)|}|x_{\gamma'}(t)-x_\gamma|^2\right)$$
$$\le C\beta_{\gamma,\gamma'}\left(|\xi_\gamma-\xi_{\gamma'}(t)|^2+|\xi_{\gamma'}(t)||\xi_\gamma||x_\gamma-x_{\gamma'}(t)|^2\right)$$
$$\times\exp\left(-\frac{|\xi_{\gamma'}(t)-\xi_\gamma|^2}{4(|\xi_\gamma|+|\xi_{\gamma'}(t)|)}\right)\exp\left(-\frac{|\xi_{\gamma'}(t)||\xi_\gamma|}{|\xi_\gamma|+|\xi_{\gamma'}(t)|}|x_{\gamma'}(t)-x_\gamma|^2\right).$$

The next two Lemmas characterize properties of the evolution operator $U(t,t')$ acting on the lattice, and they will assist us in obtaining the bounds (4.0.3) and (4.0.4).

Lemma 4. *We let*

$$A(x,t)=\{a_{ij}(x,t)\}_{1\le i,j\le n}$$

be a real symmetric $n\times n$ matrix with entries $a_{ij}(x,t)$ in $C^{1,1}$, as in the introduction. For the rest of this lemma, $C>0$ denotes a constant which is independent of the essential variables.

Furthermore, as before, $A(t,x)$ is bounded and positive definite and, we let

$$q(x,t,\xi) = \left(\sum_{i,j} a_{ij}(x,t)\xi_i\xi_j \right)^{\frac{1}{2}}.$$

If we consider the system

$$\frac{dx}{dt} = q_\xi, \qquad \frac{d\xi}{dt} = -q_x, \tag{4.0.12}$$

with initial conditions $|x(0)| < R$ and $\frac{1}{a} < |\xi(0)| < a$ for some finite $a, R > 0$, then solutions to the system (9.0.13) satisfy the following two conditions:

1. *$|x(t) - x(0)| < C\sqrt{n}|T|$, and*

2. *For all finite $T > 0$, there exists a constant $C(T,a)$ such that*

$$\frac{1}{C(T,a)} < |\omega(t)| < C(T,a)$$

   whenever $|t| < T$.

Proof. We prove condition (1) first and then condition (2).

1. Computing q_{ξ_i},

$$\frac{\partial}{\partial \xi_i} \left(\sum_{ij} a_{ij}(x,t)\xi_i\xi_j \right)^{\frac{1}{2}} = \frac{\sum_j a_{ij}(x,t)\xi_j}{\left(\sum_{i,j} a_{ij}(x,t)\xi_i\xi_j \right)^{\frac{1}{2}}} < C, \tag{4.0.13}$$

   since the expression in the middle of (4.0.13) is homogeneous of degree 0 and the numerator and denominator are both bounded above and below on $|\xi| = 1$. From (9.0.13) we then have

$$\left| \frac{dx_i}{dt} \right| < C,$$

   which implies

$$\left| \frac{dx}{dt} \right| < C\sqrt{n}.$$

   Integrating this inequality gives (1).

2. Differentiating q with respect to x, we have

$$\left|\frac{d\xi}{dt}\right| = \left|\frac{\sum_{ij}(a_{ij})_x(x,t)\xi_i\xi_j}{2\left(\sum_{i,j} a_{i,j}(x,t)\xi_i\xi_j\right)^{\frac{1}{2}}}\right| < C|\xi| \tag{4.0.14}$$

for some C independent of ξ, x, and t, since the expression in the middle of (4.0.14) is homogeneous of degree 1 and the numerator and denominator are both bounded above and below on $|\xi| = 1$. Using Gronwall's inequality gives

$$\frac{1}{C(T,a)} < |\xi(0)|\exp(-Ct) < |\xi(t)| < |\xi(0)|\exp(Ct) < C(T,a) \tag{4.0.15}$$

This results in the desired conclusion for finite T, that is, if $\frac{1}{a} < |\xi(0)| < a$ then, for any $t \in [-T, T]$, there exists $C(T,a)$ such that condition (4.0.15) holds.

□

Recall that $U(t,t')$ is the evolution operator associated to (9.0.13), and

$$U(t,t')(x_\gamma, \xi_\gamma) = (x_\gamma(t,t',x_\gamma,\xi_\gamma), \xi_\gamma(t,t',x_\gamma,\xi_\gamma)).$$

By homogeneity, if c is a constant, then the above equation scales as follows:

$$(x_\gamma(t,t',x_\gamma,\xi_\gamma), c\xi_\gamma(t,t',x_\gamma,\xi_\gamma)) = (x_\gamma(t,t',x_\gamma,c\xi_\gamma), \xi_\gamma(t,t',x_\gamma,c\xi_\gamma)). \tag{4.0.16}$$

Since $\xi_\gamma = 2^k\omega_{i,k}$ with $\frac{1}{2} \leq |\omega_{i,k}| < 1$, the relationship (4.0.16) with $c = 2^{-k}$ gives that the pair $(x_\gamma, \omega_\gamma)$ lies in a compact subset of $\mathbb{R}^n \times (\mathbb{R}^n/\{0\})$ whenever $|x_\gamma| < R$ for R a constant independent of γ. We note that Lemma 4 then applies to $(x_\gamma, \omega_\gamma)$, and so we have a bound on the size of $x_\gamma(t)$ and $\xi_\gamma = 2^k\omega_{i,k}(t)$ in terms of the initial data.

Because our frame is similar to an almost orthogonal frame in type, it makes sense that the pairs of initial data which are close together in frequency contribute the most to the absolute value of the inner products in the sums in (4.0.3) and (4.0.4). However, we have an extra variable α since we have a non-compactly supported set of frame functions. Therefore we will use the term ”close in frequency” to mean that the pairs (x_γ, ξ_γ) and $(x_{\gamma'}, \xi_{\gamma'})$ from Chapter 3 satisfy not only the condition $|x_\gamma|, |x_{\gamma'}| < R$ but also that $|k - k'| \leq k_0$, where k_0 is a finite constant independent of γ, γ'. In Lemma 5, we will show that close pairs of lattice variables have an extra property beyond that of Lemma 4, which makes it possible to compute the bounds on (4.0.3) and (4.0.4).

First we see, by equation (4.0.16), that for all such close pairs with $c = 2^{-k'}$ (where here without loss of generality we have taken $k' \leq k$) the corresponding scaled pairs $(x_\gamma, 2^{k-k'}\omega_\gamma)$ and $(x_{\gamma'}, \omega_{\gamma'})$ lie in the same compact subset $[-R, R]^n \times [\frac{1}{2}, 2^{k_0}]^n$ of $\mathbb{R}^n \times (\mathbb{R}^n/\{0\})$. Thus we can conclude from Lemma 4 that the transformation $\mathcal{U}(t, 0)$ is invertible and Lipschitz with uniform Lipschitz constant when acting on $(x_\gamma, 2^{k-k'}\omega_\gamma)$ and $(x_{\gamma'}, \omega_{\gamma'})$. In other words, for all close pairs and for all $t \in [-T, T]$ with T fixed and finite, there exist nonzero constants D_1 and D_2, independent of γ, γ', with

$$D_1 d^2((x_\gamma(t), 2^{k-k'}\omega_\gamma(t)); (x_{\gamma'}(t), \omega_{\gamma'}(t))) \leq d^2((x_\gamma, 2^{k-k'}\omega_\gamma); (x_{\gamma'}, \omega_{\gamma'})) \tag{4.0.17}$$
$$\leq \; D_2 d^2((x_\gamma(t), 2^{k-k'}\omega_\gamma(t)); (x_{\gamma'}(t), \omega_{\gamma'}(t))),$$

where d denotes the usual Euclidean distance. We will abbreviate this type of equivalence relationship, where the left hand side is bounded above and below by multiples of the right hand side, by $\sim$, so that inequality (4.0.17) can be rewritten as

$$d^2((x_\gamma, 2^{k-k'}\omega_\gamma); (x_{\gamma'}, \omega_{\gamma'})) \sim d^2((x_\gamma(t), 2^{k-k'}\omega_\gamma(t)); (x_{\gamma'}(t), \omega_{\gamma'}(t))).$$

The inequality (4.0.17) allows us to obtain another similar relationship which is crucial in the computations to obtain bounds on the action of the matrices $B_E(t)$ and $B_\Box(t)$.

Lemma 5. *For pairs (x_γ, ξ_γ) and $(x_{\gamma'}, \xi_{\gamma'})$ such that $|x_\gamma|, |x_{\gamma'}| < R$ where $0 < R < \infty$ and $|k - k'| \leq k_0$, with R and k_0 independent of γ and γ', the following holds:*

$$d^2(\mathcal{U}(t, 0)(x_{\gamma'}, \omega_{\gamma'}); (x_\gamma, 2^{k-k'}\omega_\gamma)) \sim d^2(\mathcal{U}(0, t)(x_\gamma, 2^{k-k'}\omega_\gamma); (x_{\gamma'}, \omega_{\gamma'})).$$

Proof. Since $\mathcal{U}(t, 0) \circ \mathcal{U}(0, t) = I$, the right hand side of the relationship,

$$d^2(\mathcal{U}(0, t)(x_\gamma, 2^{k-k'}\omega_\gamma); (x_{\gamma'}, \omega_{\gamma'})),$$

can be expressed as

$$d^2(\mathcal{U}(0,t)(x_\gamma, 2^{k-k'}\omega_\gamma); \mathcal{U}(0,t)\mathcal{U}(t,0)(x_{\gamma'}, \omega_{\gamma'}))$$

and from estimate (4.0.17) we obtain the desired conclusion. □

With these Lemmas, we can now calculate a bound on

$$\sum_{\gamma'} |b_E(\gamma, \gamma', t)| \tag{4.0.18}$$

for fixed γ. We break this sum into three pieces: in region 1, $\gamma' : k' < k - k_0$, in region 2, $\gamma' : |k' - k| \leq k_0$, and in region 3, $\gamma' : k' > k + k_0$ where, for all $t \in [-T, T]$ with $T < \infty$, $k_0 = \max\{2\log_2 C(T,a), 1\}$. For the rest of this argument, let $D > 0$ denote a constant which is independent of k' and k and which is uniform for all $t \in [-T, T]$.

We will apply Lemma 4 in each region to subsets of the initial data $(x_{\gamma'}, \omega_{\gamma'})$ as outlined earlier. Here we must cut off the $x_{\gamma'}$'s so that $|x_{\gamma'}| < R$, for some large positive R. This corresponds to having the initial data with support living in a ball of radius R.

Again, because of the similarity of the frame to an almost orthogonal frame, in regions 1 and 3 from Lemma 4 the exponential term from the bound on each inner product will dominate the sum, but in region 2 the argument is more subtle. In each case, formula (4.0.16) and Lemma 4 imply that:

$$\begin{aligned} \beta_{\gamma,\gamma'} &= \left(\frac{|\xi_\gamma||\xi_{\gamma'}(t)|\Delta x_\gamma \Delta x_{\gamma'}}{4\pi(|\xi_\gamma| + |\xi_{\gamma'}|)} \right)^{\frac{n}{2}} \\ &= \left(\frac{C_\epsilon^2 2^{\frac{k}{2} - \epsilon k} 2^{\frac{k'}{2} - \epsilon k'} |\omega_\gamma||\omega_{\gamma'}(t)|}{4\pi(2^k|\omega_\gamma| + 2^{k'}|\omega_{\gamma'}(t)|)} \right)^{\frac{n}{2}} \leq \left(\frac{C_\epsilon^2 2^{\frac{k}{2} - \epsilon k} 2^{\frac{k'}{2} - \epsilon k'} C(T,a)}{4\pi(2^{k-1} + \frac{2^{k'}}{C(T,a)})} \right)^{\frac{n}{2}}. \end{aligned} \tag{4.0.19}$$

The right hand side of (4.0.11) contains a product of two exponentials, with arguments

$$-\frac{|2^k\omega_\gamma - 2^{k'}\omega_{\gamma'}(t)|^2}{4(2^k|\omega_\gamma| + 2^{k'}|\omega_{\gamma'}(t)|)} \tag{4.0.20}$$

and

$$-\frac{|\xi_\gamma||\xi_{\gamma'}(t)|}{|\xi_\gamma| + |\xi_{\gamma'}(t)|}|x_\gamma - x_{\gamma'}(t)|^2. \tag{4.0.21}$$

In region 1, Lemma 4 implies a lower bound on (4.0.20)

$$\frac{|2^k\omega_\gamma - 2^{k'}\omega_{\gamma'}(t)|^2}{4(2^k|\omega_\gamma| + 2^{k'}|\omega_{\gamma'}(t)|)} > \frac{\left(2^{k-1} - 2^{k'}C(T,a)\right)^2}{4(2^k + 2^{k'}C(T,a))}$$
$$> \frac{2^{k-5}\left(1 - 2^{k'-k+1}C(T,a)\right)^2}{(1 + 2^{k'-k}C(T,a))} > \frac{2^{k-5}\left(1 - \frac{2}{C(T,a)}\right)^2}{\left(1 + \frac{1}{C(T,a)}\right)} = 2^k D.$$

For (4.0.21), we only know that

$$|x_\gamma - x_{\gamma'}(t)|^2 \geq 0,$$

which gives

$$\exp\left(-|x_\gamma - x_{\gamma'}(t)|^2\right) \leq 1.$$

Since, by assumption, $|x_{\gamma'}| < R$ and $x_{\gamma'} = \Delta x_{\gamma'}\alpha' = C_\epsilon 2^{-\frac{k'}{2}-\epsilon k}\alpha'$ for fixed k', by scaling we have $|\alpha'| < RC_\epsilon^{-1}2^{\frac{k'}{2}+\epsilon k'}$. Bounding the number of points in both $\mathbb{Z}^n$ and in this ball by $D(2^{\frac{k'}{2}+\epsilon k'})^n$, we obtain a bound on the number of $x_{\gamma'}$ for fixed (i,k'). While the position of the $x_{\gamma'}$ may change, their total number does not change when they are propagated. From Lemma 1, there are $\mathcal{O}(2^{\frac{k'n}{2}})$ vectors $\omega_{i,k'}$ in each annulus indexed by k'. Applying (4.0.19) in region 1, we find, since $k' < k - k_0$,

$$\beta_{\gamma,\gamma'} \leq \left(\frac{C_\epsilon^2 2^{\frac{k}{2}-\epsilon k}2^{\frac{k'}{2}-\epsilon k'}C(T,a)}{4\pi(2^{k-1} + \frac{2^{k'}}{C(T,a)})}\right)^{\frac{n}{2}} \leq D\left(2^{-\frac{k}{2}-\epsilon k}2^{\frac{k'}{2}-\epsilon k'}\right)^{\frac{n}{2}}.$$

Combining estimates gives

$$\sum_{\gamma':k'<k-k_0} |b_E(\gamma,\gamma',t)| = \mathcal{O}\left(\sum_{\substack{(i,k')\\ k'<k-k_0}} 2^{-\frac{nk}{4}-\frac{\epsilon nk}{2}}2^{\frac{3nk'}{4}+\frac{\epsilon nk'}{2}}\exp\left(-D2^k\right)\right) \tag{4.0.22}$$
$$=\mathcal{O}\left(\sum_{k'<k-k_0} 2^{nk'}\exp\left(-D2^k\right)\right) = \mathcal{O}\left(2^{nk}\exp\left(-D2^k\right)\right).$$

But since $2^{nk}\exp\left(-D2^k\right) \to 0$ as $k \to \infty$, the contribution from (4.0.22) is bounded independently of γ, γ'.

Similarly, in region 3, an application of Lemma 4 to the first part of the exponential contribution (4.0.20) gives

$$\frac{|2^k\omega_\gamma(0) - 2^{k'}\omega_{\gamma'}(t)|^2}{4(2^k|\omega_\gamma(0)| + 2^{k'}|\omega_{\gamma'}(t)|)} > \frac{\left(\frac{2^{k'}}{C(T,a)} - 2^k\right)^2}{4(2^k + 2^{k'}C(T,a))}$$
$$> \frac{2^{k'-3}}{C(T,a)^3} - \frac{2^{k-2}}{C(T,a)^2} > D2^{k'}.$$

Again, the same estimates as in region 1 for the number of the $x_{\gamma'}(t)$ and their exponential contribution hold, and the number of vectors $\omega_{i,k'}$ in each annulus for fixed k' is still $\mathcal{O}(2^{\frac{nk'}{2}})$. For the size of $\beta_{\gamma,\gamma'}$ from (4.0.19) and the fact $k' > k + k_0$, we have

$$\beta_{\gamma,\gamma'} \leq \left(\frac{C_\epsilon^2 2^{\frac{k}{2}-\epsilon k}2^{\frac{k'}{2}-\epsilon k'}C(T,a)}{4\pi(2^{k-1} + \frac{2^{k'}}{C(T,a)})}\right)^{\frac{n}{2}} < D\left(2^{\frac{k}{2}-\epsilon k}2^{-\frac{k'}{2}-\epsilon k'}\right)^{\frac{n}{2}}.$$

Thus

$$\sum_{\gamma':k'>k+k_0} |b_E(\gamma,\gamma',t)| = \mathcal{O}\left(\sum_{\substack{(i,k')\\ k'>k+k_0}} 2^{\frac{nk}{4}-\frac{\epsilon nk}{2}}2^{\frac{nk'}{4}+\frac{\epsilon nk'}{2}}\exp\left(-D2^{k'}\right)\right) \tag{4.0.23}$$
$$= \mathcal{O}\left(\sum_{k'>k+k_0} 2^{\frac{3nk'}{4}+\frac{kn}{4}+\frac{\epsilon n(k'-k)}{2}}\exp\left(-D2^{k'}\right)\right).$$

By hypothesis, $k' > k + k_0$, so the exponential term dominates the sum here as well, and so the contribution from (4.0.23) is bounded independently of γ, γ'.

If we try to simply apply Lemma 4 in region 2, as we did in regions 1 and 3, we get a constant bound on the exponential contributions (4.0.20) and (4.0.21) which is not enough to dominate the contributions to the sum from the number of x_γ and ξ_γ. Therefore the application of Lemma 5 to the exponential term is essential in region 2 since the treatment of the exponential contribution to the summation is more delicate there. The key is that the additional Lemma 5 allows us to sum over unpropagated variables which, from the construction, are fixed in space.

In region 2, by homogeneity and the fact $|k - k'| \leq k_0$, the entire exponential term can be re-written as follows:

$$\frac{|\xi_{\gamma'}(t)-\xi_\gamma|^2}{4(|\xi_\gamma|+|\xi_{\gamma'}(t)|)}+\frac{|\xi_{\gamma'}(t)||\xi_\gamma|}{|\xi_\gamma|+|\xi_{\gamma'}(t)|}\,|x_{\gamma'}(t)-x_\gamma|^2 \tag{4.0.24}$$
$$\sim 2^{k'}d^2(\mathcal{U}(t,0)(x_\gamma,2^{k-k'}\omega_\gamma);(x_{\gamma'},\omega_{\gamma'})).$$

Applying Lemma (5) to (4.0.24), we obtain

$$\frac{|\xi_{\gamma'}(t)-\xi_\gamma|^2}{4(|\xi_\gamma|+|\xi_{\gamma'}(t)|)}+\frac{|\xi_{\gamma'}(t)||\xi_\gamma|}{|\xi_\gamma|+|\xi_{\gamma'}(t)|}\,|x_{\gamma'}(t)-x_\gamma|^2 \tag{4.0.25}$$
$$\sim 2^{k'}d^2((x_\gamma,2^{k-k'}\omega_\gamma);\mathcal{U}(0,t)(x_{\gamma'},\omega_{\gamma'})),$$

where the constants in this equivalence relation may depend on k_0 but are uniform in T.

Since $x_{\gamma'}=\Delta x_{\gamma'}\alpha'$, we can factor out the scaling $\Delta x_{\gamma'}=C_\epsilon 2^{-\frac{k'}{2}-\epsilon k'}$ from part of the right hand side of (4.0.25), giving

$$2^{k'}\left|x_{\gamma'}-x_\gamma(-t)\right|^2=C_\epsilon^2 2^{-2\epsilon k'}\left|\alpha'-\epsilon_0\right|^2,$$

where we have set $\epsilon_0=(\Delta x_{\gamma'})^{-1}x_\gamma(-t)$. Substituting $\lambda=C_\epsilon^{-2}2^{2\epsilon k'}$, an application of the integral estimates in Appendix A gives

$$\sum_{\alpha'\in\mathbb{Z}^n,|\alpha'|<R}\exp\left(-2^{k'}\left|x_{\gamma'}-x_\gamma(-t)\right|^2\right)<\sum_{\alpha'\in\mathbb{Z}^n}\exp\left(-C_\epsilon^2 2^{-2\epsilon k'}\left|\alpha'-\epsilon_0\right|^2\right) \tag{4.0.26}$$
$$=\mathcal{O}\left(2^{\epsilon k' n}\right).$$

But, looking at inequality (4.0.19), the k' dependence in this last bound is exactly canceled by the size of $\beta_{\gamma,\gamma'}$ in region 2. Since $|k-k'|\le k_0$, the other part of the exponential contribution may be tackled with an argument similar to that of Lemma 1 applied to $\xi=2^k\omega_\gamma(-t)$. This implies that the sum

$$\sum_{\substack{(i,k')\\|k'-k|\le k_0}}\exp\left(2^{-k'}\left|2^{k'}\omega_{\gamma'}-2^k\omega_\gamma(-t)\right|^2\right) \tag{4.0.27}$$

is bounded independently of γ,γ'. From these bounds and from the equivalence relation (4.0.25), we can conclude

$$\sum_{\gamma':|k'-k|\le k_0}|b_E(\gamma,\gamma',t)|=\mathcal{O}\left(1\right), \tag{4.0.28}$$

and thus

$$\sum_{\gamma'} |b_E(\gamma, \gamma', t)| = \mathcal{O}\left(1\right).$$

If we reverse the roles of γ and γ', we can run a similar argument to the one above to bound

$$\sum_{\gamma} |b_E(\gamma, \gamma', t)|.$$

The bounds in each of the regions $|k - k'| \leq k_0$, $k > k' + k_0$ and $k < k' - k_0$ will follow almost identically. The main difference in the argument will be that, in the region where $|k - k'| \leq k_0$, we do not need to apply Lemma 5 since the γ variables are not propagated. In this way, we obtain the desired bound (4.0.3).

For the estimate (4.0.4), we examine

$$\sum_{\gamma'} |b_\Box(\gamma, \gamma', t)|. \tag{4.0.29}$$

The only difference between the bound on $|b_E(\gamma, \gamma', t)|$ and the bound on $|b_\Box(\gamma, \gamma', t)|$ is the factor of $|\xi_\gamma - \xi_{\gamma'}(t)|^2 + |\xi_{\gamma'}(t)||\xi_\gamma||x_\gamma - x_{\gamma'}(t)|^2$. In region 1, application of Lemma 4 gives

$$\begin{aligned}
&|\xi_\gamma - \xi_{\gamma'}(t)|^2 + |\xi_{\gamma'}(t)||\xi_\gamma||x_\gamma - x_{\gamma'}(t)|^2 \\
&\leq (|\xi_\gamma| + |\xi_{\gamma'}(t)|)^2 + |\xi_{\gamma'}(t)||\xi_\gamma|(|x_\gamma - x_{\gamma'}| + |x_{\gamma'}(t) - x_{\gamma'}|)^2 \\
&\leq \left|2^k + 2^{k'} C(T, a)\right|^2 + 2^k 2^{k'} C(T, a)(R + T)^2 \leq D 2^{2k}
\end{aligned}$$

The rest of the estimates on $\beta_{\gamma,\gamma'}$ and the exponential contribution stay the same. Therefore, by (4.0.22),

$$\begin{aligned}
\sum_{\gamma' : k' < k - k_0} |b_\Box(\gamma, \gamma', t)| &= \mathcal{O}\left(\sum_{\substack{(i,k') \\ k' < k - k_0}} 2^{2k} 2^{-\frac{nk}{4} - \frac{\epsilon nk}{2}} 2^{\frac{3nk'}{4} + \frac{\epsilon nk'}{2}} \exp\left(-D 2^k\right) \right) \\
&= \mathcal{O}\left(\sum_{k' < k - k_0} 2^{2k} 2^{nk'} \exp\left(-D 2^k\right) \right) = \mathcal{O}\left(2^{(n+2)k} \exp\left(-D 2^k\right)\right)
\end{aligned}$$

and, as $k \to \infty$, we see that $2^{(n+2)k} \exp(-D 2^k) \to 0$, so the sum in question is also uniformly bounded independently of γ, γ'

Similarly, in region 3, by Lemma 4, the extra factor is bounded by

$$\begin{aligned}&|\xi_\gamma - \xi_{\gamma'}(t)|^2 + |\xi_{\gamma'}(t)||\xi_\gamma||x_\gamma - x_{\gamma'}(t)|^2 \\ &\leq (|\xi_\gamma| + |\xi_{\gamma'}(t)|)^2 + |\xi_{\gamma'}(t)||\xi_\gamma|(|x_\gamma - x_{\gamma'}| + |x_{\gamma'}(t) - x_{\gamma'}|)^2 \\ &\leq \left|2^k + 2^{k'}C(T,a)\right|^2 + 2^k 2^{k'} C(T,a)(R+T)^2 \leq D 2^{2k'}.\end{aligned}$$

Again, the rest of the estimates stay the same, so that, analogously to (4.0.23),

$$\begin{aligned}\sum_{\gamma':k'>k+k_0} |b_\Box(\gamma,\gamma',t)| &= \mathcal{O}\left(\sum_{\substack{(i,k')\\ k'>k+k_0}} 2^{2k'} 2^{\frac{nk}{4}-\frac{\epsilon nk}{2}} 2^{\frac{nk'}{4}+\frac{\epsilon nk'}{2}} \exp\left(-D2^{k'}\right)\right) \qquad (4.0.30)\\ &= \mathcal{O}\left(\sum_{k'>k+k_0} 2^{2k'} 2^{\frac{3nk'}{4}+\frac{kn}{4}+\frac{\epsilon n(k'-k)}{2}} \exp\left(-D2^{k'}\right)\right),\end{aligned}$$

and, as before, this sum also converges independently of γ, γ' since $k' > k + k_0$.

The only region where the extra factor in question makes a difference is in region 2. As in the treatment of the sum of $|b_E(\gamma,\gamma')|$ over γ', Lemma 5 is again crucial. By homogeneity and Lemma 5, the extra factor in the bounds for $|b_\Box(\gamma,\gamma',t)|$ can be rewritten as

$$|\xi_\gamma - \xi_{\gamma'}(t)|^2 + |\xi_{\gamma'}(t)||\xi_\gamma||x_\gamma - x_{\gamma'}(t)|^2 \sim 2^{2k'} d^2(\mathcal{U}'(0,t)(x_\gamma, 2^{k-k'}\omega_\gamma); (x_{\gamma'}, \omega_{\gamma'})),$$

and the exponential factor in the bounds still follows the equivalence relation (4.0.25). With these relationships in mind, we split the sum

$$\begin{aligned}&\sum_{\gamma':|k-k'|\leq k_0} \beta_{\gamma,\gamma'}(|\xi_\gamma - \xi_{\gamma'}(t)|^2 + |\xi_{\gamma'}(t)||\xi_\gamma||x_\gamma - x_{\gamma'}(t)|^2) \qquad (4.0.31)\\ &\times \exp\left(-\frac{|2^k\omega_\gamma - 2^{k'}\omega_{\gamma'}(t)|^2}{4(2^k|\omega_\gamma| + 2^{k'}|\omega_{\gamma'}(t)|)} - \frac{|\xi_\gamma||\xi_{\gamma'}(t)|}{|\xi_\gamma| + |\xi_{\gamma'}(t)|}|x_\gamma - x_{\gamma'}(t)|^2\right)\end{aligned}$$

into two pieces which together are equivalent under the $\sim$ relationship to (4.0.31). These sums are

$$\begin{aligned}&\sum_{\gamma':|k-k'|\leq k_0} \beta_{\gamma,\gamma'} 2^{2k'} |x_{\gamma'} - x_\gamma(-t)|^2 \exp\left(-2^{k'}|x_{\gamma'} - x_\gamma(-t)|^2\right) \qquad (4.0.32)\\ &\times \exp\left(2^{-k'}\left|2^{k'}\omega_{\gamma'} - 2^k\omega_\gamma(-t)\right|^2\right)\end{aligned}$$

and

$$\sum_{\gamma':|k-k'|\leq k_0} \beta_{\gamma,\gamma'} \left|2^{k'}\omega_{\gamma'} - 2^k\omega_\gamma(-t)\right|^2 \exp\left(-2^{k'}\left|x_{\gamma'} - x_\gamma(-t)\right|^2\right) \tag{4.0.33}$$
$$\times \exp\left(2^{-k'}\left|2^{k'}\omega_{\gamma'} - 2^k\omega_\gamma(-t)\right|^2\right).$$

To handle the sum (4.0.32), since again $x_{\gamma'} = \Delta x_{\gamma'}\alpha'$, we can factor out the scaling $\Delta x_{\gamma'} = C_\epsilon 2^{-\frac{k'}{2}-\epsilon k'}$ from part of the right hand side of (4.0.25) and obtain

$$2^{k'}\left|x_{\gamma'} - x_\gamma(-t)\right|^2 = C_\epsilon^2 2^{-2\epsilon k'}\left|\alpha' - \epsilon_0\right|^2.$$

We can also factor the scaling from the new multiplying factor in (4.0.32) which gives

$$2^{2k'}\left|x_{\gamma'} - x_\gamma(-t)\right|^2 = C_\epsilon^2 2^{-2\epsilon k'+k'}\left|\alpha' - \epsilon_0\right|^2.$$

In both cases we have set $\epsilon_0 = (\Delta x_{\gamma'})^{-1}x_\gamma(-t)$. Substituing $\lambda = C_\epsilon^{-2}2^{2\epsilon k'}$, an application of the second integral estimate in Appendix B gives

$$\sum_{\alpha'\in\mathbb{Z}^n,|\alpha'|<R} \left(2^{2k'}\left|x_{\gamma'} - x_\gamma(-t)\right|^2\right)\exp\left(-2^{k'}\left|x_{\gamma'} - x_\gamma(-t)\right|^2\right) \tag{4.0.34}$$
$$< \sum_{\alpha'\in\mathbb{Z}^n}\left(C_\epsilon^2 2^{-2\epsilon k'+k'}\left|\alpha' - \epsilon_0\right|^2\right)\exp\left(-C_\epsilon^2 2^{-2\epsilon k'}\left|\alpha' - \epsilon_0\right|^2\right)$$
$$= \mathcal{O}\left(2^{k'}(2^{2\epsilon k'})^{\frac{n}{2}}\right).$$

Using the previous estimates (4.0.19) and (4.0.27), and the fact $|k-k'| \leq k_0$, the sum (4.0.32) is $\mathcal{O}(2^k)$.

For the second sum (4.0.33), estimate (4.0.26) still applies for the sum over α' so we are reduced to examining

$$\sum_{\substack{(i,k')\\|k'-k|\leq k_0}} \left|2^{k'}\omega_{\gamma'} - 2^k\omega_\gamma(-t)\right|^2 \exp\left(2^{-k'}\left|2^{k'}\omega_{\gamma'} - 2^k\omega_\gamma(-t)\right|^2\right). \tag{4.0.35}$$

Now if we consider the same sets defined in Lemma 1, with $\xi = 2^k\omega_\gamma(-t)$, for the first set $\mathcal{A}$ we get

$$\left|2^{k'}\omega_{\gamma'} - 2^k\omega_\gamma(-t)\right|^2 \leq 2^k.$$

This implies, from previous bounds on the number of $\omega_{\gamma'}$ in $\mathcal{A}$, that

$$\sum_{\mathcal{A}} \left|2^{k'}\omega_{\gamma'} - 2^k\omega_\gamma(-t)\right|^2 \exp\left(2^{-k'}\left|2^{k'}\omega_{\gamma'} - 2^k\omega_\gamma(-t)\right|^2\right) \leq 3^{n+1}2^k$$
$$\leq D2^k.$$

In each of the sets $\mathcal{B}_j$,

$$\left|2^{k'}\omega_{\gamma'} - 2^k\omega_\gamma(-t)\right|^2 \leq j^2 2^k,$$

and similarly, from an argument in Lemma 1, we can deduce that

$$\sum_{\mathcal{B}} \left|2^{k'}\omega_{\gamma'} - 2^k\omega_\gamma(-t)\right|^2 \exp\left(2^{-k'}\left|2^{k'}\omega_{\gamma'} - 2^k\omega_\gamma(-t)\right|^2\right)$$
$$\leq \sum_{j=1}^{\infty} 2^n 2^k (j+1)^{n+2} \exp\left(-\frac{(j-1)^2}{2}\right) \leq D2^k.$$

Now it is easy to see that there is only a small (or 0) contribution coming from the sets $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{E}$ since $|k-k'| \leq k_0$, and this contribution is uniformly bounded independently of γ, γ'. From here it follows that the second sum (4.0.33) is $\mathcal{O}(2^k)$. Combining the estimates above gives

$$\sum_{\gamma' : |k'-k| \leq k_0} |b_\square(\gamma, \gamma', t)| \leq D2^k.$$

Since the contribution from regions 1 and 3 was uniformly bounded independently of γ, γ', we find

$$\sum_{\gamma'} |b_\square(\gamma, \gamma', t)| \leq D2^k.$$

By symmetry, we can use similar estimates to obtain the second bound in 4.0.4. Again, the main difference will be that there is no need to apply Lemma 5 in region 2. The combination of these estimates concludes the theorem.

CHAPTER 5

Construction of the Parametrix

With the frame of functions established, we turn our attention to constructing an appropriate parametrix for the Cauchy problem

$$\begin{aligned}
&\Box u(t,x) = (\partial_t^2 - A(t,x,\partial_x))u(t,x) = 0\\
&u(t,x)|_{t=0} = f(x)\\
&\partial_t u(t,x)|_{t=0} = h(x),
\end{aligned}$$

where $f(x)$ and $h(x)$ are functions in $L^2(\mathbb{R}^n)$. We will construct operators $\mathcal{C}(t,t')$ and $\mathcal{S}(t,t')$ out of families of functions which are related to the frame functions. This chapter will follow the work of [Smi98] very closely.

As earlier $\mathcal{U}(t,t')$ denotes the evolution operator associated to $\mathcal{H}^- = \tau - q$. Additionally we denote the evolution operator associated to the Hamiltonian $\mathcal{H}^+ = \tau + q$ as $\mathcal{V}(t,t')$. We set

$$\mathcal{U}(t,t')(x_\gamma(0),\xi_\gamma(0)) = (x_\gamma^+(t,t'),\xi_\gamma^+(t,t'))$$

and

$$\mathcal{V}(t,t')(x_\gamma(0),\xi_\gamma(0)) = (x_\gamma^-(t,t'),\xi_\gamma^-(t,t')).$$

Accordingly,

$$\begin{aligned}
&\phi_\gamma^\pm(t,t',x)\\
&= \left(\frac{|\xi_\gamma^\pm(t,t')|\Delta x_\gamma}{2\pi}\right)^{\frac{n}{2}} \exp\left(i\xi_\gamma^\pm(t,t')\cdot(x - x_\gamma^\pm(t,t')) - |\xi_\gamma^\pm(t,t')||x - x_\gamma^\pm(t,t')|^2\right),
\end{aligned}$$

and we let

$$\Omega_\gamma^\pm(t,t',x) = \frac{\phi_\gamma^\pm(t,t',x)}{q(t',x_\gamma,\xi_\gamma)}$$

From these definitions, we construct the following operators $\mathcal{C}(t,t')$ such that

$$\begin{aligned}\Pi^0\mathcal{C}(t,t')\Pi^0 f &= P_2^0 B_{\mathcal{C}}(t,t')P_1^0 f\\ &= \sum_{\gamma,\gamma'} b_{\mathcal{C}}(\gamma,\gamma',t)c(\gamma')\phi_\gamma(x)\end{aligned}$$

and $\mathcal{S}(t,t')$ such that

$$\begin{aligned}\Pi^0\mathcal{S}(t,t')\Pi^0 f &= P_2^0 B_{\mathcal{S}}(t,t')P_1^0 f\\ &= \sum_{\gamma,\gamma'} b_{\mathcal{S}}(\gamma,\gamma',t)c(\gamma')\phi_\gamma(x).\end{aligned}$$

Here

$$b_{\mathcal{C}}(\gamma,\gamma',t) = \frac{1}{2}\int_{\mathbb{R}^n} \overline{\phi_\gamma(x)}\left(\phi^+_{\gamma'}(t,t',x) + \phi^-_{\gamma'}(t,t',x)\right)\,dx$$

and

$$b_{\mathcal{S}}(\gamma,\gamma',t) = \frac{1}{2}\int_{\mathbb{R}^n} \overline{\phi_\gamma(x)}\left(\Omega^+_{\gamma'}(t,t',x) - \Omega^-_{\gamma'}(t,t',x)\right)\,dx$$

denote the entries of the matrices $B_{\mathcal{C}}(t,t')$ and $B_{\mathcal{S}}(t,t')$ respectively.

Theorem 7. *$\Box\mathcal{C}(t,t')$ and $\Box\mathcal{S}(t,t')$ are bounded operators of order one and zero respectively, with operator norms which are uniformly bounded on intervals where $t-t'$ is finite. Furthermore,*

$$\mathcal{C}(t',t)\sim I \qquad \partial_t\mathcal{C}(t',t') = 0$$

and

$$\mathcal{S}(t',t') = 0 \qquad \partial_t\mathcal{S}(t',t')\sim I$$

to leading order.

Proof. The first statement is an immediate extension of Theorem 6 in Chapter 4. The first set of operator estimates follow directly from the definition of Π^0 and the calculations in Chapter 4. For the second set of estimates, the result (where $q = q(t', x_\gamma, \xi_\gamma)$)

$$((\partial_t\Omega^+(t',t',y) - \partial_t\Omega^-(t',t',y)) = \left(\left(1-\frac{q_t}{q^2}\right) - \left(-1+\frac{q_t}{q^2}\right)\right)\phi_\gamma(y)$$

is easy, as on null bicharacteristics $\tau = \pm q$ so by homogeneity

$$\frac{q_t}{q^2} = \mathcal{O}\left(\frac{1}{2^{k'}}\right).$$

From the proceeding arguments, if we define $u(t,x)$ as

$$u(t,x) = \mathcal{S}(t,t')h(x) + \mathcal{C}(t,t')f(x),$$

then $u(t,x)$ is the desired parametrix solution to the Cauchy problem. $\square$

Theorem 8. *If $-1 \leq m \leq 2$, if $f \in H^{m+1}(\mathbb{R}^n)$, if $h \in H^m(\mathbb{R}^n)$, and if $F \in L^1([-T,T]; H^m(\mathbb{R}^n))$, then there exists a $G \in L^1([-T,T]; H^m(\mathbb{R}^n))$ such that*

$$u(t,x) = \mathcal{C}(t,0)f(x) + \mathcal{S}(t,0)h(x) + \int_0^t (\mathcal{S}(t,s)G(s,x)\,ds$$

and

$$||G||_{L^1([-T,T];H^m(\mathbb{R}^n))} \leq C(T)\left(||f||_{H^{m+1}(\mathbb{R}^n)} + ||h||_{H^m(\mathbb{R}^n)} + ||F||_{L^1([-T,T];H^m(\mathbb{R}^n))}\right)$$

solves the Cauchy problem

$$\begin{aligned}
&\Box u(t,x) = (\partial_t^2 - A(t,x,\partial_x))u(t,x) = F(t,x) \\
&u(t,x)|_{t=0} = f(x) \\
&\partial_t u(t,x)|_{t=0} = h(x)
\end{aligned}$$

in the weak sense. If f and h are both identically zero and F is also zero for all $t \in [-T,T]$, then G and u will vanish as well.

Proof. As per [Smi98], we will show the existence of such a G using Volterra iteration. Assuming $G \in L^1([-T,T],;H^m(\mathbb{R}^n))$, we let

$$v(t,x) = \int_0^t \mathcal{S}(t,s)G(s,x)\,ds.$$

Because $\mathcal{S}(t,t')$ and $\partial_t\mathcal{S}(t,t')$ are both strongly continuous operators and $\mathcal{S}(t,t) = 0$, we have $v(t,x)$ is in $C([-T,T]; H^{m+1}(\mathbb{R}^n)) \cap C^1([-T,T]; H^m(\mathbb{R}^n))$, and also

$$\partial_t v(t,x) = \int_0^t \partial_t \mathcal{S}(t,s) G(s,x)\, ds,$$

so it follows that

$$v(0,x) = 0 \qquad \partial_t v(t,x)|_{t=0} = 0.$$

Furthermore, differentiating in the sense of distributions, we obtain

$$\partial_t^2 v(t,x) = G(t,x) + \int_0^t \partial_t^2 \mathcal{S}(t,s) G(s,x)\, ds.$$

We can conclude $u(t,x)$ of the form in Theorem 8 is a weak solution to the Cauchy problem if the following Volterra equation

$$G(t,x) + \int_0^t \Box S(t,s) G(s,x)\, ds = F(t,x) - \Box\left(\mathcal{C}(t,0) f(x) + \mathcal{S}(t,0) h(x)\right) \tag{5.0.1}$$

holds. Equation (5.0.1) can be solved by iteration since the operator norm of $S(t,s)$ is uniformly bounded on finite intervals of time by Theorem 6. Setting

$$G(t,x) = F(t,x) + \sum_{n=1}^{\infty} G_n(t,x) \tag{5.0.2}$$

with

$$G_n(t,x) = \int_0^t \int_0^{s_1} \ldots \int_0^{s_{n-1}} S(t,s_1) S(s_1,s_2) \ldots$$

$$S(s_{n-1},s_n) F(s_n,x)\, ds_n \ldots ds_1$$

we see that $G(t,x)$ is a solution to the equation

$$G(t,x) + \int_0^t S(t,s) G(s,x)\, ds = F(t,x).$$

As the series in (5.0.2) converges in $L^1([-T,T],; H^m(\mathbb{R}^n))$ with norm bounded by $\exp(TC(T)\,||F||)$, this finishes the Theorem. $\Box$

CHAPTER 6

Background for Schrödinger Operators

The focus of the second half of this dissertation is the class of Schrödinger operators

$$P : u(x) \mapsto (-\Delta + q(x))u(x),$$

where

$$\Delta = \sum_{j=1}^{2} \frac{\partial^2}{\partial x_j^2},$$

and

$$q(x) : \mathbb{R}^2 \to \mathbb{R}$$

is a real-valued periodic potential over a lattice, $\mathbb{L} \subset \mathbb{R}^2$. In other words we have

$$q(x+d) = q(x) \qquad \forall d \in \mathbb{L}.$$

We will study the question of spectral rigidity for the operator P and derive results which could extend to $\mathbb{R}^n$ for $n \geq 3$. We consider the set of λ in $\mathbb{R}$ for which the self-adjoint eigenvalue problem

$$Pu(x) = \lambda u(x) \qquad u(x+d) = \exp(2\pi i k \cdot d)\, u(x) \tag{6.0.1}$$

has a solution for k in $\mathbb{R}^2$ and d in $\mathbb{L}$. When there is a nonzero solution to (6.0.1) we say that λ is in $\mathrm{Spec}_k(-\Delta + q)$. We refer to

$$\bigcup_k \mathrm{Spec}_k(-\Delta + q)$$

as the Floquet spectrum. However, when $k = 0$, we simply say 'spectrum' which we denote by $\mathrm{Spec}(-\Delta + q)$. In two dimensions, two potentials q and $\tilde{q}$ are Floquet isospectral if

$$\mathrm{Spec}_k(-\Delta + q) = \mathrm{Spec}_k(-\Delta + \tilde{q}) \qquad \forall k \in \mathbb{R}^2$$

and isospectral if Spec$(-\Delta + q)$ =Spec$(-\Delta + \tilde{q})$. Following the convention in [ERT84b], we consider a potential to be Floquet (spectrally) rigid if there are only a finite number of potentials modulo translations which are Floquet isospectral (resp. isospectral) to it.

In [ERT84a], Eskin et al. showed that under the assumptions

1. q is real analytic

2. $\mathbb{L}$ has the property $|d| = |d'| \Rightarrow d = \pm d'$ for all d, d' in $\mathbb{L}$

then Spec$(-\Delta + q)$ determines Spec$_k(-\Delta + q)$ for all k in $\mathbb{R}^n$.

In the sequel to [ERT84a], [ERT84b], Eskin, et al., show that there is a set of analytic potentials satisfying the conditions (1) and (2) which are dense in $C^\infty(\mathbb{R}^2/\mathbb{L})$ such that if $q(x)$ is in this set, then $q(x)$ is Floquet rigid. Furthermore, there is a smaller, but still dense set of analytic potentials in $C^\infty(\mathbb{R}^2/\mathbb{L})$ such that if $q(x)$ is in this set and $\tilde{q}(x)$ is Floquet isospectral to $q(x)$ then, $\tilde{q}(x) = q(\pm x + a)$ where a is an arbitrary constant. Under the assumptions (1) and (2), if a potential in $\mathbb{R}^2$ is spectrally rigid (resp. unique) then it is Floquet rigid (resp unique), so their results are also true with the words "Floquet rigid" (resp. unique) replacing "isospectrally rigid" (resp unique). The main result of this dissertation section is to show that there is a more general class of potentials which satisfy the conditions for Floquet rigidity than in [ERT84b].

CHAPTER 7

The Isospectral Manifold in $\mathbb{R}^1$

In $\mathbb{R}^1$ the structure of the isospectral sets of periodic potentials has been well studied and contains many results which are useful in higher dimensions. In $\mathbb{R}^1$ the Schrödinger operator becomes Hill's operator.

$$-\frac{d^2}{ds^2} + q(s)$$

where $q(s)$ has period 1 and is real-valued. We start by assuming that q is at least three times differentiable, so that we can use many of the standard results which may be found in Magnus and Winkler, [MW79]. For the rest of this dissertation, we will also assume that $q(x)$ has mean zero. We look at the set of λ where there is a solution to

$$\begin{aligned} -\frac{d^2\phi(s)}{ds^2} + q(s)\phi(s) &= \lambda\phi(s) \qquad (7.0.1) \\ \phi(s+1) &= (-1)^m\phi(s). \end{aligned}$$

The scalars λ are known as the periodic and anti-periodic eigenvalues. Through curious use of notation, the scalar, $\lambda_m^{\pm}$, denotes the eigenvalue corresponding to the eigenfunction $\phi_m^{\pm}(s+1) = (-1)^m\phi_m^{\pm}(s)$ so that

$$\lambda_0 < \lambda_1^- \leq \lambda_1^+ < \lambda_2^- \leq \lambda_2^+ ... \qquad (7.0.2)$$

Hence the periodic spectrum consists of $\{\lambda_m^{\pm},\, m \text{ even}\}$ and the antiperiodic spectrum is $\{\lambda_m^{\pm},\, m \text{ odd}\}$.

If we change the problem (7.0.1) so that $\phi(s)$ obeys the boundary condition

$$\phi(0) = \phi(1) = 0,$$

then the associated spectrum is called the Dirichlet spectrum. The Dirichlet spectrum are denoted $\mu_m(q)$ and they interlace the periodic and anti-periodic spectra. We will often use the fact

$$|\lambda_m^+ - \lambda_n^+| = \mathcal{O}(|m^2 - n^2|), \tag{7.0.3}$$

and find it worthwhile to mention it here. Although $\lambda_m^+ < \lambda_{m+1}^-$, it is possible to have $\lambda_m^- = \lambda_m^+$. The spectrum of

$$-\frac{d^2}{ds^2} + q(s)$$

as an operator in $L^2(\mathbb{R})$ is

$$\bigcup_{m=0}^{\infty} [\lambda_m^+, \lambda_{m+1}^-]$$

Each of the intervals $[\lambda_m^+, \lambda_{m+1}^-]$ in the union above is called a "band", or interval of stability. The complement of the set of bands is union of the intervals $(\lambda_m^-, \lambda_m^+)$ which are called "gaps" or intervals of stability. In each gap, the operator $-\frac{d^2}{ds^2} + q(s)$ does not have a bounded eigenfunction. A gap is referred to as open whenever $\lambda_m^- < \lambda_m^+$ and closed if $\lambda_m^- = \lambda_m^+$. The length of a gap is denoted as γ_m.

In [GT84] Garnett and Trubowitz gave a compete characterization of the gaps for q in $L^2_{\mathbb{R}}[0,1]$.

Theorem 9. *[GT84] Let γ_n, $n \geq 1$, be any sequence of nonnegative numbers satisfying*

$$\sum_{n \geq 1} \gamma_n^2 < \infty$$

Then there is a way of placing the sequence of open tiles of lengths γ_n, $n \geq 1$ in order on the positive axis $(0, \infty)$ so that the complement is the set of bands for a function q in $L^2_{\mathbb{R}}[0,1]$. In other words, the map

$$q \to \gamma(q) = \{\gamma_n(q)\}_{n \geq 1}, \tag{7.0.4}$$

from $L^2_{\mathbb{R}}[0,1]$ to $(l^2)^+$, is onto.

Furthermore if we multiply the gap lengths γ_m by ϵ where ϵ is in $[0,1]$ then the map (7.0.4) is still onto. The fundamental result in $\mathbb{R}^1$ is that the set of analytic periodic potentials $M(\epsilon)$ with the same periodic and anti-periodic spectra is equivalent to a torus with dimension equal to I [MT76]. Here I is the number of m for which $\lambda_m^- < \lambda_m^+$. The coordinates $\alpha_m(q)$, on this manifold with m referring to the m^{th} gap on $q(s)$, are related to the Dirichlet spectra and the gap lengths. They are defined as follows

$$\sin^2 \alpha_m(q) = \frac{\mu_m(q) - \lambda_m^-}{\lambda_m^+ - \lambda_m^-} \qquad -\frac{\pi}{2} < \alpha_m \leq \frac{\pi}{2} \tag{7.0.5}$$

where $\mu_m(q)$ is the Dirichlet eigenvalue for q such that $\lambda_m^- \leq \mu_m(q) \leq \lambda_m^+$. These coordinates are further discussed in Chapter 4.

Finally we will need the fact that all the gap lengths are exponentially decreasing if and only if $q(s)$ is real analytic. Whenever q has only a finite number of open gaps, then q must be real analytic, [Tru77]. The analyticity of $q(s)$ with finitely many gaps is crucial in many of the proofs of the theorems in this dissertation.

CHAPTER 8

Review of Necessary Results in $\mathbb{R}^n$

We outline some necessary results and definitions from [ERT84a] and [ERT84b] which will be used in the rest of this dissertation. Let $\mathbb{L}$ be an n-dimensional lattice generated by n vectors $v_1, v_2, ..., v_n$. We can then consider it's dual $\mathbb{L}^*$ where

$$\mathbb{L}^* = \{\delta \in \mathbb{R}^n : \delta \cdot v \in \mathbb{Z}, \forall v \in \mathbb{L}\},$$

to be generated by some basis $\delta_1, \delta_2, ..., \delta_n$. A function is periodic over the lattice $\mathbb{L}$ if $q(x + d) = q(x)$ for all d in $\mathbb{L}$. For any arbitrary lattice $\mathbb{L}$ satisfying condition (2) and basis fixed as above, let $\mathbb{S}^*$ be the set of fundamental directions for $\mathbb{L}$, that is

$$\mathbb{S}^* = \{\delta \in \mathbb{L}^* : \delta \cdot d = 1 \text{ for some } d \in \mathbb{L}\}.$$

It is clear that whenever δ is in $\mathbb{S}^*$ then $-\delta$ is also in this set, so we reduce the set to $\mathbb{S}$ by only picking δ in $\mathbb{S}^*$. Therefore any element of $\mathbb{L}^*/\{0\}$ has a unique representation as $m\delta$ with δ in $\mathbb{S}$ and m in $\mathbb{Z}$.

If q is a function which is periodic over $\mathbb{L}$, then it has the following Fourier series representation

$$q(x) = \sum_{\delta \in \mathbb{L}^*} a_\delta \exp\left(2\pi i \delta \cdot x\right)$$

with

$$a_\delta = \frac{1}{Vol(\Gamma)} \int_\Gamma q(x) \exp\left(-2\pi i \delta \cdot x\right) \, dx$$

where Γ the fundamental domain of the lattice $\mathbb{L}$ as given by

$$\Gamma = \{s_1 v_1 + ... + s_n v_n : 0 \leq s_i \leq 1\}.$$

If we write

$$|\delta|^2 q_\delta(s) = \sum_{n\in\mathbb{Z}} a_{n\delta} \exp(2\pi ins)$$

then we have that

$$q(x) = \sum_{\delta\in\mathbb{S}} \sum_{n\in\mathbb{Z}} a_{n\delta} \exp\left(2\pi in\delta\cdot x\right) = \sum_{\delta\in\mathbb{S}} |\delta|^2 q_\delta(\delta\cdot x)$$

where each $q_\delta(s)$ is a periodic potential on $\mathbb{R}^1$. These one-dimensional potentials $q_\delta(s)'s$ are called directional potentials. The assumption that $q(x)$ has mean zero is equivalent to setting $a_0 = 0$ for all the directional potentials.

Theorem 2 in ([ERT84a], [ERT84b]) states that

Theorem 10. *Spec$(-\Delta + q)$ determines*

$$\mathrm{Spec}_k\left(-\frac{d^2}{ds^2} + q_\delta(s)\right) \qquad \forall\delta\in\mathbb{S}, k\in\mathbb{R}^n$$

The theorems in $\mathbb{R}^1$ we mentioned will help reduce the study of periodic potentials in $\mathbb{R}^n$ to the study of $\mathbb{R}^1$ potentials, about which much more is known.

CHAPTER 9

Potentials in $\mathbb{R}^2$

Following [ERT84b], for the rest of this dissertation we assume that the elements of the lattice $\mathbb{L}$ satisfy condition (2) as stated in the introduction, and we consider analytic periodic potentials $q(x)$ such that $q(x+d) = q(x)$ for all d in $\mathbb{L}$. We also only consider potentials with a finite number of directional potentials. For this Chapter, we make the additional assumptions that the number of gaps in each direction δ_j is finite, and that there are at least 3 directions. This setup differs from [ERT84b] where two of the directional potentials were fixed translates of the one gap potentials and the other directions were viewed as perturbations of the zero potential.

Under these assumptions we can simplify the form of $q(x)$ as follows

$$q(x) = \sum_{j=1}^{S} |\delta_j|^2 q_j(\delta_j \cdot x). \tag{9.0.1}$$

Each one dimensional directional potential $q_j(\delta_j \cdot x)$ corresponds to a one dimensional operator with corresponding eigenvalue and eigenfunction pair $(\lambda, \phi(s))$ satisfying

$$-\frac{d^2}{ds^2}\phi(s) + q_j(s)\phi(s) = \lambda\phi(s). \tag{9.0.2}$$

In order to simplify the computations needed in this dissertation we make the following assumptions (*)

1. $\delta_3 = \delta_1 + \delta_2$

2. q_1, q_2 and q_3 have the same number of open gaps

We will discuss how, given sufficient time and energy, using spectral invariants and the standard perturbation techniques that one could remove the assumptions (*). The invariants are derived from the trace theorems. If we let the fundamental solution of the heat equation

$$\frac{\partial u}{\partial t} = \Delta u - qu \qquad u(0,x) = f(x) \tag{9.0.3}$$

on $\mathbb{R}^n$ be $G(x,y,t)$ then

$$\sum_{\lambda \in \mathrm{Spec}_k} \exp(-\lambda t) = \sum_{d \in \mathbb{L}} \exp(-2\pi i k \cdot d) \int_{\Gamma} G(x+d, x, t)\, dx \tag{9.0.4}$$

Therefore if one knows $\mathrm{Spec}_k(-\Delta + q)$ for all k, then one knows

$$\int_{\Gamma} G(x+d, x, t)\, dx \qquad \forall t > 0, d \in \mathbb{L} \tag{9.0.5}$$

In [ERT84a] and [ERT84b], they derive Theorem 10 from the asymptotics of

$$\int_{\Gamma} G(x+Nd+e, x, t)\, dx \qquad \forall t > 0, d \in \mathbb{L} \tag{9.0.6}$$

as $N \to \infty$.

Theorem 10 has the consequence that the set of real-analytic $\tilde{q}(x)$ isospectral to $q(x)$ can be identified with a subset of a real analytic manifold

$$M = T_1 \times T_2 \times \ldots \times T_S.$$

Here each torus T_j has dimension equal to the number of open gaps associated to each directional potential $q(\delta_j \cdot x)$; we call this set I_j. This manifold M has dimension $\sum_j |I_j| = N$. Again, the coordinates on the manifold $\alpha_{j,m}(q)$ are given for each j by (7.0.5).

In our case, we would like our set of potentials which we will call $M(\epsilon)$ to have open gap lengths which are parametrized as follows. Let E_0 denote the set

$$\{(j,m) : (j,m) = (1,1), (2,1)\},$$

and E_1 denote the set

$$\{(j,m) : j \leq 2, m > 1\}.$$

Now we let ϵ be the vector with four components $(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4)$ so we can parametrize the new gap lengths so they depend on $\vec{\epsilon}$ and γ as follows

$$\gamma_{j,m}(\epsilon, \gamma) = \epsilon_j \gamma_{j,m} \text{ for } (j,m) \in E_0$$

$$\gamma_{j,m}(\epsilon, \gamma) = \epsilon_4 \gamma_{j,m} \text{ for } (j,m) \in E_1$$

$$\gamma_{3,m}(\epsilon, \gamma) = \epsilon_3 \gamma_{3,m} \text{ for } m \in I_3$$

$$\gamma_{j,m}(\epsilon, \gamma) = \epsilon_4 \gamma_{j,m} \text{ for } j > 3, m \in I_j$$

and are associated with the potential $q(\epsilon, x, \alpha)$. Here, suppressing the "q", we have $\alpha = \{\alpha_{j,m}\}$ is the rescaled vector of coordinates, where for each directional potential, the coordinates are given by (7.0.5). Notice that we have also written our gap lengths in terms of finitely many parameters and this does not destroy the fact the mapping (7.0.4) is onto and in this case analytic.

The following spectral invariants are derived from higher order terms in the asymptotics of 9.0.6 in [ERT84b] which we will use in our computations:

Theorem 11. *The periodic and anti-periodic spectra for the one dimensional potentials $q_\delta(x)$ which form $q(x)$ and the invariants*

$$\begin{aligned} &\Phi_{\delta_j,m}(\epsilon, \alpha) \\ &= \Phi_{j,m}(\epsilon, \alpha) = \int_\Gamma |h(\epsilon, x, \alpha)|^2 (\phi^{\pm}_{j,m}(\epsilon, \delta_j \cdot x, \alpha))^2 \, dx \end{aligned} \tag{9.0.7}$$

when $\lambda^+_{j,m} > \lambda^-_{j,m}$ and

$$\begin{aligned} &\Phi_{\delta_j,m}(\epsilon, \alpha) = \Phi_{j,m}(\epsilon, \alpha) \\ &= \int_\Gamma |h(\epsilon, x, \alpha)|^2 \left((\phi^+_{j,m}(\epsilon, \delta_j \cdot x, \alpha))^2 + (\phi^-_{j,m}(\epsilon, \delta_j \cdot x, \alpha))^2 \right) \, dx \end{aligned} \tag{9.0.8}$$

when $\lambda^+_{j,m} = \lambda^-_{j,m}$ maybe recovered from the spectra of $q(x)$. Here $\alpha = \{\alpha_{j,m}\}$ is the collection of coordinates associated to each gap length and we have set

$$h(\epsilon, x, \alpha) = \sum_{\substack{e \in \mathbb{S} \\ e \cdot d_j \neq 0}} \frac{e}{e \cdot d_j} q_e(\epsilon, e \cdot x, \alpha)$$

with $\delta_j \cdot d_j = 0$, and d_j of minimal length.

Setting $\Phi^{+}_{\delta_j,m}(\epsilon,\alpha) = \Phi_{j,m}(\epsilon,\alpha)$, then the number of invariants with $\lambda^{+}_{m} > \lambda^{-}_{m}$ has dimension equal to the manifold $M(\epsilon)$. We would like to show that the Jacobian determinant of the invariants with respect to the coordinates α is nonzero so that we may apply the implicit function theorem.

We will primarily be calculating the spectral invariants for potentials at a specific parameter $\epsilon = \epsilon_0$. We let ϵ_0 be the vector with $(\epsilon_1, \epsilon_2, 0, 0)$ where ϵ_1 and ϵ_2 are in $(0,1)$. When $\epsilon = \epsilon_0$ the potential $q(\epsilon_0, x, \alpha)$ has

$$\gamma_{j,m}(\epsilon_0,\gamma) = \epsilon_j \gamma_{j,m} \text{ for } (j,m) \in E_0$$
$$\gamma_{j,m}(\epsilon_0,\gamma) = 0 \text{ for } (j,m) \in E_0^c$$

for gap lengths. The potential $q(\epsilon_0, x, \alpha)$ is therefore the sum of 2 potentials with only one gap, one in each direction δ_j, $j = 1, 2$. The rest of the directional potentials are zero. While the limit $q(\epsilon_0, x, \alpha)$ coincides with the form of the potential as calculated in [ERT84b], one specific difference remains- the first two directional have finitely many gaps, they are not just translates of the $\wp$ function. We will Taylor expand the Jacobian determinant with respect to ϵ_3 around $\epsilon \neq \epsilon_0$ and use these computations to show that the Jacobian determinant for certain fixed α is not identically zero.

For the rest of this dissertation, we let $\wp(s+\frac{i\tau}{2},\tau)$ denote a general normalized Weierstrass $\wp$ function. Whenever the parameter τ is real and greater than zero, then $\wp(s+\frac{i\tau}{2},\tau)$ is real-valued with periods 1 and τ [SS03]. The real-valued $\wp$-function is always even about $\frac{1}{2}$, and by a theorem of Hochstadt [Hoc65], all one gap potentials are translates of the $\wp$-function. The directional potential, in the limit, $q_j(\epsilon_0, s, \alpha) = \wp(s + \frac{i\tau_j}{2} + \nu_j, \tau_j)$ has eigenfunctions which satisfy the following equation:

$$-\frac{d^2}{ds^2}\phi(\epsilon_0,s,\alpha) + q_j(s)\phi(\epsilon_0,s,\alpha) = \lambda\phi(\epsilon_0,s,\alpha).$$

where $q_j(\epsilon_0, 0, \alpha) = \wp(\frac{i\tau_j}{2} + \nu_j, \tau_j)$ has bands given by

$$[-\wp\left(\frac{1}{2}\right), -\wp\left(\frac{i\tau_j+1}{2}\right)] \cup [-\wp\left(\frac{i\tau_j}{2}\right), +\infty). \tag{9.0.9}$$

Aligning the classical elliptic function theory with spectral theory [Cai06] we have that,

$$-\wp\left(\frac{1}{2}\right) = \lambda_0 \qquad -\wp\left(\frac{i\tau_j+1}{2}\right) = \lambda_1^- \qquad -\wp\left(\frac{i\tau_j}{2}\right) = \lambda_1^+. \tag{9.0.10}$$

We will need the parameters τ_j later in the computation of the Fourier coefficients of the $\wp$ function and the perturbation calculations for the eigenfunctions. From equation (9.0.9) we know that they are related to the ϵ_j as follows

$$\wp\left(\frac{i\tau_j+1}{2}\right) - \wp\left(\frac{i\tau_j}{2}\right) = \epsilon_j\gamma_{j,1} \tag{9.0.11}$$

for $j = 1, 2$. Therefore if we pick ϵ_j, we pick τ_j and vice versa.

Since any potential $q(x, \epsilon, \alpha)$ is always Floquet isospectral to $q(\pm x + a, \epsilon, \alpha)$ where a is arbitrary, we cannot hope to remove the sign or translation degeneracy. We know that when $\epsilon = \epsilon_0$ that $\delta_1 \cdot a = \nu_1$ and $\delta_2 \cdot a = \nu_2$, so for simplicity we fix a so when $\epsilon = \epsilon_0$ then $a = 0$. As a result we have that

$$q_j(s, \alpha, \epsilon_0) = \wp_j(s + \frac{i\tau_j}{2}, \tau_j) = \sum_{n\in\mathbb{N}} a_n^j \cos(2\pi ns)$$

for $j = 1, 2$, where the coefficients a_n^j are given by Appendix C. We consider our manifold $M(\epsilon)$ of potentials which have translation fixed as above.

In order to prove that $M(\epsilon)$ actually is an analytic manifold with coordinates $\alpha = \{\alpha_{j,m}(q)\}$ we must first remind the reader of a few definitions involved in the selection of the coordinates $\{\alpha_{j,m}\}$ defined by (7.0.5) as they are related to the Dirichlet spectra $\mu_{j,m}(q)$ of the operator. We define the discriminant $\Delta(\lambda)$ as follows

$$\Delta^2(\lambda) - 4 = 4(\lambda_0 - \lambda)\prod_{n=1}^{\infty}\frac{(\lambda_n^+ - \lambda)(\lambda_n^- - \lambda)}{n^4\pi^4}. \tag{9.0.12}$$

Let $\mu_m(s, q_j) = \mu_{j,m}(\epsilon, s, \alpha)$ be the the solution to the system (where here we are suppressing the j)

$$\frac{d\mu_m(\epsilon, s, \alpha)}{ds} = m^2\pi^2\frac{\sqrt{\Delta^2(\mu_m) - 4}}{\prod_{\substack{n\in I,\\ n\neq m}}(\mu_n(\epsilon, s, \alpha) - \mu_m(\epsilon, s, \alpha))/n^2\pi^2} \tag{9.0.13}$$

with $\mu_{j,m}(\epsilon, 0, \alpha) = \mu_m(0, q_j), k \in I$, where the choice of signs is initially by the sign of numerator, and changes whenever $\mu_{j,m}(\epsilon, s, \alpha)$ hits $\lambda^{\pm}_{j,m}$. The proof of analyticity of μ by examining (9.0.13) remains almost exactly the same as in [ERT84b] and is omitted here. Since there are a finite number of coordinates, it is easy to see that analyticity in each coordinate is preserved, and hence $M(\epsilon)$ is still an analytic manifold

By McKean-Van Moerbeke [MT76], the initial value the sum of the initial values, $\mu_{j,m}(\epsilon, 0, \alpha)$, is related to each directional potential $q_j(\epsilon, 0, \alpha)$ in the following way

$$q_j(\epsilon, 0, \alpha) = \lambda_0 + \sum_{m \in I_j} (\lambda^+_{j,m} + \lambda^-_{j,m} - 2\mu_{j,m}(\epsilon, 0, \alpha))$$

and this relationship remains true when the parameter s is varied

$$q_j(\epsilon, s, \alpha) = \lambda_0 + \sum_{m \in I_j} (\lambda^+_{j,m} + \lambda^-_{j,m} - 2\mu_{j,m}(\epsilon, s, \alpha)). \tag{9.0.14}$$

Using a combination of formulas on pp. 325 and 329, in [Tru77], the eigenfunctions for each directional potential corresponding to $\lambda^+_{j,m}$ for all j can be written as

$$(\phi^+_m(\epsilon, s, \alpha))^2 = \prod_{n \in I_j} \left(\frac{\lambda^+_m - \mu_n(\epsilon, s, \alpha)}{\lambda^+_m - \dot{\lambda}_n} \right) \tag{9.0.15}$$

where $\dot{\lambda}_m$ is the zero of $\frac{\partial \Delta}{\partial \lambda}$ lying between λ^-_m and λ^+_m. It is important to note here that the formula in [ERT84b] is a misprint. We will also need the derivatives of the eigenfunctions which from equation (9.0.15) are

$$2\phi^+_m(\epsilon, s, \alpha) \frac{d\phi^+_m(\epsilon, s, \alpha)}{ds} = \sum_{n \in I_j} \frac{-1}{\lambda^+_n - \dot{\lambda}_k} \left(\frac{d\mu_n(\epsilon, s, \alpha)}{ds} \right) \prod_{k \neq n} \frac{\lambda^+_m - \mu_k(\epsilon, s, \alpha)}{\lambda^+_m - \dot{\lambda}_k} \tag{9.0.16}$$

with the derivative for $\phi^-(\epsilon_0, s, \alpha)$ computed similarly. Let us start by considering the eigenfunctions for those directional potentials with $j > 3$. Because we are looking for the root between $\lambda^+_{j,m}$ and $\lambda^-_{j,m}$ when $\epsilon = \epsilon_0$, we make the substitution $\lambda = \lambda^-_{j,m} + \epsilon_4 \gamma_{j,m} \tilde{\lambda}$ into (9.0.12) to find that

$$\Delta^2(\tilde{\lambda}) - 4 = \epsilon_4^2 \tilde{\lambda}(1 - \tilde{\lambda}) f(\epsilon_4 \tilde{\lambda}, \epsilon_4)$$

where $f(z, \epsilon_4)$ is analytic and $f(0, 0) = \gamma^2_m \neq 0$. Therefore for ϵ_4 sufficiently small, $\dot{\lambda}_m$ corresponds to the root of

$$0 = (1 - 2\tilde{\lambda}) f(\epsilon_4 \tilde{\lambda}, \epsilon_4) + \epsilon_4 \tilde{\lambda}(1 - \tilde{\lambda}) \frac{\partial f}{\partial z}(\epsilon_4 \tilde{\lambda}, \epsilon_4)$$

near $\tilde{\lambda} = \frac{1}{2}$. As a result, the following estimate holds

$$\frac{\lambda_m^+(\epsilon) - \lambda_m^-(\epsilon)}{\lambda_m^+(\epsilon) - \dot{\lambda}_m(\epsilon)} = 2 + \mathcal{O}(\epsilon_4). \tag{9.0.17}$$

giving that

$$\frac{\lambda_m^+(\epsilon_0) - \mu_m(\epsilon_0, \alpha, s)}{\lambda_m^+(\epsilon_0) - \dot{\lambda}_m(\epsilon_0)} = 2\cos^2(\tilde{\alpha}_m(s, \alpha)). \tag{9.0.18}$$

The variable $\tilde{\alpha}_m(s, \alpha)$ denotes the solution to the system (9.0.13) where $\epsilon = \epsilon_0$ with initial condition α under the change of variables (7.0.5). The same estimates above are true for the eigenfunctions $\phi_{3,m}^+(\epsilon_0, s, \alpha)$, m in I_3 when expanded with respect to ϵ_3. We can conclude for all $j \geq 3$

$$\phi_{j,m}^+(\epsilon_0, s, \alpha) = \sqrt{2}\cos\tilde{\alpha}_{j,m}(s, \alpha) \tag{9.0.19}$$

where we know we have picked the right sign by verifying the derivative (9.0.16) in the limit.

Now we consider the case when $j \leq 2$. When $\epsilon = \epsilon_0$, we have for all $n > 1$ that $\lambda_{j,n}^+ = \lambda_n^- = \mu_n = \dot{\lambda}_n$ so that terms in the product (9.0.15) where $n \neq m$ and $n > 1$ become

$$\frac{\lambda_{j,m}^+(\epsilon_0) - \mu_{j,n}(\epsilon_0, s, \alpha)}{\lambda_{j,m}^+(\epsilon_0) - \dot{\lambda}_{j,n}(\epsilon_0)} = 1, \tag{9.0.20}$$

and for $n = 1$ we have

$$\frac{\lambda_{j,m}^+(\epsilon_0) - \mu_{j,1}(\epsilon_0)}{\lambda_{j,m}^+(\epsilon_0) - \dot{\lambda}_{j,1}(\epsilon_0)} = \frac{\lambda_{j,m}^+(\epsilon_j) - \lambda_{j,1}^-(\epsilon_j) - \epsilon_j\gamma_{j,1}\sin^2(\tilde{\alpha}_{j,1}(s, \alpha))}{\lambda_{j,m}^+ - \dot{\lambda}_{j,1}(\epsilon_j)}. \tag{9.0.21}$$

Combining equations (9.0.18) (which is still true for $j \leq 2$) and (9.0.20), we see that for $\epsilon = \epsilon_0$, and (j, m) in E_1,

$$(\phi_{j,m}^+(\epsilon_0, \alpha, s))^2 = 2\cos^2(\tilde{\alpha}_m(s, \alpha))\left(\frac{\lambda_{j,m}^+(\epsilon_j) - \lambda_{j,1}^-(\epsilon_j) - \epsilon_j\gamma_{j,1}\sin^2(\tilde{\alpha}_{j,1}(s, \alpha))}{\lambda_{j,m}^+(\epsilon_j) - \dot{\lambda}_{j,1}(\epsilon_j)}\right). \tag{9.0.22}$$

Comparing with the derivative computed in (9.0.16) we know that the correct choice of sign is

$$(\phi_{j,m}^+(\epsilon_0, \alpha, s)) = \sqrt{2}\cos(\tilde{\alpha}_m(s, \alpha))\sqrt{\frac{\lambda_{j,m}^+(\epsilon_j) - \lambda_{j,1}^-(\epsilon_j) - \epsilon_j\gamma_{j,1}\sin^2(\tilde{\alpha}_{j,1}(s, \alpha))}{\lambda_{j,m}^+(\epsilon_j) - \dot{\lambda}_{j,1}(\epsilon_j)}}. \tag{9.0.23}$$

The introduction of this setup provides the necessary background to introduce the following theorem:

Theorem 12. *For all but an analytic set of* (ϵ_3, ϵ_4) *in* $[0,1]^2$*, there is an open set of potentials satisfying the hypotheses (1),(2) and (*) in* $M(\epsilon)$ *which are isospectral to only a finite number of other analytic potentials.*

In order to find the Jacobian corresponding to the invariants as given by equation (9.0.7), we must first figure out what it means to calculate their derivatives with respect to $\{\alpha_{j,m}\}$ with (j,m) in E_0^c. We start with the following lemma

Lemma 6. *For* (j,m) *in* E_0^c*, we have*

$$\frac{\partial \tilde{\alpha}_{j,m}(s,\alpha)}{\partial \alpha_{j,m}} = 1, \qquad \text{and} \qquad \frac{\partial \tilde{\alpha}_{j,m}(s,\alpha)}{\partial \alpha_{r,k}} = 0 \qquad \text{when } (r,k) \neq (j,k)$$

Proof. Examining (9.0.13) under the change of variables given by (7.0.5) for (j,m) in E_1 and $\epsilon = \epsilon_0$

$$\frac{d\tilde{\alpha}_{j,m}(s,\alpha)}{ds} = \frac{\sqrt{(\lambda_{j,m}^+ - \lambda_0)(\lambda_{j,m}^+ - \lambda_{j,1}^+)(\lambda_{j,m}^+ - \lambda_{j,1}^-)}}{\lambda_{j,m}^+ - \lambda_{j,1}^- - \epsilon_j \gamma_{j,1} \sin^2 \tilde{\alpha}_{j,1}(s,\alpha)} \tag{9.0.24}$$

Therefore $\tilde{\alpha}_{j,m}(s,\alpha)$ depends only on $\alpha_{j,1}$ and the initial data for $\tilde{\alpha}_{j,m}(0,\alpha) = \alpha_{j,m}$ so the result follows.

The case whenever $j \geq 3$ and $\epsilon = \epsilon_0$, is much easier to compute. We have for all such corresponding m

$$\frac{d\tilde{\alpha}_{j,m}(s)}{ds} = m\pi \tag{9.0.25}$$

so again the result follows by the same reasoning above. □

For the computations done in Appendix D, we need to know that when $\epsilon_j = 0$, (9.0.23) agrees with the limit one would expect. In other words for (j,m) in E_1, we have

$$\phi_{j,m}^+(\epsilon_0, \alpha, s) = \sqrt{2}\cos(\pi m s + \alpha_{j,m}) + \mathcal{O}(\epsilon_j) \tag{9.0.26}$$

which is easily verifiable by Lemma 6, and the estimates (9.0.17) and (9.0.20). We have computed the eigenfunctions in (9.0.23) to illustrate that they are expressed in terms of elliptic functions, and therefore the invariants will not be explicitly computable.

We can now prove the main Lemma. If we consider a potential $q(\epsilon, x, \alpha)$ in $M(\epsilon)$ then it is associated to a fixed set of coordinates α. Let $\det(J)(\epsilon, \alpha)$ be the Jacobian determinant of the invariants $\Phi_{j,m}(\epsilon, \alpha)$ with respect to the coordinates $\{\alpha_{j,m}\}$ with j, m in E_0^c, and $\det(J)(\epsilon, \alpha)$ is an $(N-2) \times (N-2)$ determinant.

The proof of Theorem 12 will be based on the following Lemma:

Lemma 7. *There is a choice of* ϵ_1, ϵ_2 *in* $[0, 1]$ *such that on a dense open set of* α*,*

$$\det(J)(\epsilon, \alpha) \neq 0 \tag{9.0.27}$$

Proof. We will proceed by showing that for all $k = 1$ to $n-1$

$$\frac{\partial^k \det(J)}{\partial \epsilon_3^k}(\epsilon_0, \alpha) = 0$$

while

$$\frac{\partial^n \det(J)}{\partial \epsilon_3^n}(\epsilon_0, \alpha) \neq 0$$

where $n = |I_1| + |I_2| - 2 = |E_1|$. The desired result will follows since we notice that if for some n

$$\frac{\partial^n \det(J)}{\partial \epsilon_3^n}(\epsilon_0, \alpha) \neq 0 \qquad \text{and} \qquad \det(J)(\epsilon, \alpha) \equiv 0$$

then this is a contradiction since all of the derivatives of $\det(J)(\epsilon, \alpha)$ evaluated at any ϵ should be identically zero as well, since $\det(J)(\epsilon, \alpha)$ is an analytic function of ϵ.

Now we proceed to calculate the derivatives of $\det(J)(\epsilon, \alpha)$. Let the columns $v_i(\epsilon, \alpha)$ of $\det(J)(\epsilon, \alpha)$ be indexed by i where i ranges from 1 to $N-2$. Each i corresponds to a pair of indices (j, m) such that

$$v_i(\epsilon, \alpha) = \nabla_\alpha \Phi_{j,m}(\epsilon, \alpha)$$

where we are considering the pairs (j, m) ordered first by the j and then by the m. The perturbation calculations to find the derivatives of the invariants are located in Appendices. In order to examine the Jacobian further, we need the following key observations:

1. $\frac{\partial q_j}{\partial \alpha_{l,k}}(\epsilon_0, \delta_j \cdot x, \alpha) = 0 \qquad \forall (l,k) \in E_o^c, \text{ and } \forall j$

2. $\frac{\partial (\phi_{j,m}^+)^2}{\partial \alpha_{l,k}}(\epsilon_0, \delta_j \cdot x, \alpha) = 0 \qquad \forall (l,k), (j,m) \in E_0^c \text{ unless } (j,k) = (l,m)$

3. $\frac{\partial q_j}{\partial \epsilon_3}(\epsilon_0, \delta_j \cdot x, \alpha) = \frac{\partial (\phi_{j,m}^+)^2}{\partial \epsilon_3}(\epsilon_0, \delta_j \cdot x, \alpha) = 0 \qquad \forall j \neq 3$

The first two observations follow from Lemma 6 and formulae (9.0.14) and (9.0.15), respectively. The last observation follows from the parametrization of the open gaps since only $q_3(\epsilon, \delta_3 \cdot x, \alpha)$ and $\phi_{3,m}(\epsilon, \delta_3 \cdot x, \alpha)$ for m in I_3 depend on ϵ_3.

Going back to equation (9.0.7), each invariant has the form as follows

$$\Phi_{j,m}(\epsilon, \alpha) = \int_\Gamma \left| \sum_{\substack{l \in N \\ l \neq j}} \frac{\delta_l}{\delta_l \cdot d_j} q_l(\epsilon, \delta_l \cdot x, \alpha) \right|^2 (\phi_{j,m}^+(\epsilon, \delta_j \cdot x, \alpha))^2 \, dx \tag{9.0.28}$$

Now we let D denote a generic constant independent of the coordinates. When $\epsilon = \epsilon_0$ the form of the invariants (9.0.28) for $j \geq 3$ coincides with that of [ERT84b]. Since δ_1 and δ_2 form a basis for $\mathbb{S}$, we know that there exists a nonzero pair of integers (p_l, r_l) such that for any third vector $\delta_l \neq \delta_1, \delta_2$ we have $\delta_l = p_l \delta_1 + r_l \delta_2$. Therefore when $j \geq 3$

$$\Phi_{j,m}(\epsilon_0, \alpha) = D \int_0^1 \int_0^1 (\wp_2(t + \frac{i\tau_2}{2}, \tau_2))(\wp_1(s + \frac{i\tau_1}{2}, \tau_1)) \cos^2(\pi m(p_j s + r_j t) + \alpha_{j,m}) \, ds \, dt + D \tag{9.0.29}$$

Exactly as in [ERT84b], we have that when (j, m) is such that $j \geq 3$

$$\Phi_{j,m}(\epsilon_0, \alpha) = c_{1,2,j} a^1_{mp_j} a^2_{mr_j} \cos 2\alpha_{j,m} + D$$

The coefficients $c_{1,2,j} a^1_{mp_j} a^2_{mr_j}$ are independent of the coordinates and nonzero. They can be found in Appendix C. However for j in $\{1, 2\}$, we come across the degeneracy that

$$\frac{\partial \Phi_{j,m}}{\partial \alpha_{l,k}}(\epsilon_0, \alpha) = 0 \tag{9.0.30}$$

for all (l, k) in E_0^c. We know from our observations (1) and (2) that (9.0.30) holds except for possibly when $(l, k) = (j, m)$. In this case since again δ_1 and δ_2 form a basis for $\mathbb{S}$ we can write

$$\frac{\partial \Phi_{j,m}}{\partial \alpha_{j,m}}(\epsilon_0,\alpha) = \int\limits_{\Gamma} \left| \frac{\delta_l}{\delta_l \cdot d_j} q_l(\epsilon_0, \delta_l \cdot x, \alpha) \right|^2 \frac{\partial (\phi^+_{j,m})^2}{\partial \alpha_{j,m}}(\epsilon_0, \delta_j \cdot x, \alpha))\, dx \tag{9.0.31}$$

$$= D \int\limits_0^1 \wp_l^2(s + \frac{i\tau_l}{2}, \tau_l)\, ds \int\limits_0^1 \frac{\partial (\phi^+_{j,m})^2}{\partial \alpha_{j,m}}(\epsilon_0, t, \alpha)\, dt$$

where $l \neq j$ and l is in $\{1,2\}$. But since we consider our eigenfunctions as normalized for all (j,m), e.g. $||\phi^+_{j,m}(\epsilon, \delta_j \cdot x, \alpha)||_{L^2(\mathbb{R})} = 1$, the right hand side of (9.0.31) is just zero.

Therefore for all i from 1 to n we have

$$v_i(\epsilon_0, \alpha) = 0.$$

while for all i from $n+1$ to $(N-2)$ we see that

$$(v_i(\epsilon_0,\alpha))^t_l = \left\{ \begin{array}{cc} 0 & l = 1,..,i-1 \\ c_{1,2,j} a^1_{mp_j} a^2_{mr_j} \sin 2\alpha_{j,m} & l = i \\ 0 & l > i \end{array} \right\}. \tag{9.0.32}$$

Because the determinant is a multi-linear function of its rows, we may write

$$\det(J)(\epsilon_0,\alpha) = \det\left(v_1, v_2, ..., v_n, v_{n+1}, .., v_{N-2}\right)$$

It is now clear that for all $k = 1$ to $n-1$

$$\frac{\partial^k \det(J)}{\partial \epsilon_3^k}(\epsilon_0, \alpha) = 0$$

however for $k = n$ we have

$$\frac{\partial^n \det(J)}{\partial \epsilon_3^n}(\epsilon_0,\alpha) = C(n) \det\left(\frac{\partial v_1}{\partial \epsilon_3}, \frac{\partial v_2}{\partial \epsilon_3}, .., \frac{\partial v_n}{\partial \epsilon_3}, v_{n+1}, ..., v_{N-2} \right). \tag{9.0.33}$$

where $C(n)$ is a constant depending on n only.

From observations (1-3) we know for j in $\{1,2\}$

$$\frac{\partial^2 \Phi_{j,m}}{\partial \epsilon_3 \partial \alpha_{l,k}}(\epsilon_0, \alpha) = 0$$

except for possibly when $l = 3$ or $(l, k) = (j, m)$. We then note that corresponding rows with $1 \leq i \leq n$ in (9.0.34) take the form

$$\left(\frac{\partial v_i}{\partial \epsilon_3}\right)^t_l = \left\{ \begin{array}{cc} 0 & l = 1, .., i-1 \\ \frac{\partial^2 \Phi_{j,m}}{\partial \alpha_{j,m} \partial \epsilon_3}(\epsilon_0, \alpha) & l = i \\ 0 & r > l > i \\ \frac{\partial^2 \Phi_{j,m}}{\partial \alpha_{3,j} \partial \epsilon_3}(\epsilon_0, \alpha) & i = r...k \\ 0 & l > r \end{array} \right\} \tag{9.0.34}$$

Here the index r corresponds to $(3, 1)$ and $k - r = |I_3|$. We can conclude from (9.0.32) and (9.0.34) the determinant (9.0.33) is an upper triangular one. The determinant (9.0.33) looks like

$$\left| \begin{array}{cc} A & B \\ 0 & C \end{array} \right|$$

where A is an $n \times n$ block diagonal matrix, and C is an $(N - n - 2) \times (N - n - 2)$ block diagonal matrix. If the diagonal entries in the upper triangular determinant (9.0.34) are nonzero, then we will arrive at the desired result that

$$\frac{\partial^n \det(J)}{\partial^n \epsilon_3}(\epsilon_0, \alpha) \neq 0 \tag{9.0.35}$$

The collection of diagonal entries for (j, m) in E_1 corresponding the block A, for $1 \leq i \leq n$ are $\frac{\partial^2 \Phi_{j,m}}{\partial \epsilon_3 \partial \alpha_{j,m}}(\epsilon_0, \alpha)$. From Appendix D, we know that there is a choice of ϵ_1 and ϵ_2 so that these invariants are nonzero except on an analytic set of $\alpha_{j,m}$. Also from Appendix D and equation (D.0.22), whenever $i > n$ we have diagonal entries corresponding to (j, m) with $j \geq 3$, corresponding to the block C are

$$\frac{\partial \Phi_{j,m}}{\partial \alpha_{j,m}}(\epsilon_0, \alpha) = -2c_{1,2,j} a^1_{mp_j} a^2_{mr_j} \sin 2\alpha_{j,m} \tag{9.0.36}$$

These entries are only zero whenever $\alpha_{j,m} \equiv 0 \mod \pi/2$ for $j \geq 3$. The lemma is finished.

Remark : It should be possible to remove the assumption (*) by using the standard perturbation series to calculate $(\phi^+_{j,m}(\epsilon_j, s, \alpha))^2$ around $\epsilon_j = 0$. If δ_3 were generically of the from $p_3\delta_1 + r_3\delta_2$, then we conjecture that (D.0.8) is nonzero provided we expanded the eigenfunctions to order n with n satisfying the relation $m \pm l = np_3$ or $m \pm l = nr_3$ for some l in $\mathbb{N}$. The calculations required to do so are difficult. This conjecture is discussed further in Appendix D □

Proof of Theorem 12. This proof is very similar to the one in [ERT84b] and is again included for completeness. Let us start by assuming the matrix J is invertible on $M(\epsilon)$ except for on an analytic set, say U, of (ϵ_3, ϵ_4) Recall that on the manifold ϵ_j and the corresponding $\alpha_{j,1}$ for $j = 1, 2$ are fixed. Then given some $\tilde{\epsilon}$ with variable components (ϵ_3, ϵ_4) in $[0, 1]^2/U$, we let

$$F = \{\alpha : \frac{\partial \Phi}{\partial \alpha}(\tilde{\epsilon}, \alpha) = 0\}.$$

Since

$$\Phi(\tilde{\epsilon}, \alpha) : M(\tilde{\epsilon}) \to \mathbb{R}^{N-2},$$

the corollary follows if we can show that the set $\Phi^{-1}(\Phi(F)^c)$ is open and dense. We know the set is open since Φ^{-1} is open, and F is compact. If we assume that it is not dense, then the set contains contains an open set O which also contains a point α_0 which is not in F. Because the Jacobian is nonzero, Φ is a homeomorphism on a neighborhood of α_0, which implies $\Phi(F)$ contains an open set. The last statement contradicts Sard's theorem. Now we assume that $\Phi(\alpha_1)$ is not in $\Phi(F)$ and $\Phi^{-1}(\Phi(F))$ is infinite. Let α_2 be an accumulation point of $\Phi^{-1}(\Phi(\alpha_1))$. Because Φ is continuous, $\Phi(\alpha_2) = \Phi(\alpha_1)$ and $\frac{\partial \Phi}{\partial \alpha_2} \neq 0$. It follows that there is a neighborhood, N, of α_2 such that α is in N and $\Phi(\alpha) = \Phi(\alpha_2)$ implies $\alpha = \alpha_2$. This is a contradiction to our assumption so we know $\Phi^{-1}(\Phi(\alpha_1))$ is finite. Because Φ is a spectral invariant, then $\Phi^{-1}(\Phi(F)^c)$ is a subset of the manifold which satisfies the conditions of Theorem 12. □

This theorem has a nice corollary if we make the following observations:

1. Any two directions δ_1 and δ_2 form a basis for the lattice $\mathbb{L}$, so our choice of basis and translate is arbitrary.

2. The potentials on $M(\epsilon)$ satisfying the conditions of the theorem are dense in the set of all analytic potentials in the C^∞ topology.

3. The set of smooth periodic potentials which are a sum of only a finite number of directional potentials each with a finite number of gaps in each direction are dense in the set of finite gap periodic potentials in the $C^\infty(\mathbb{R}^2/\mathbb{L})$ topology.

4. The set of finite gap potentials is dense in the set of all $C^6(\mathbb{R}^2/\mathbb{L})$ potentials in the C^∞ topology.

Corollary 1. *The set of analytically rigid potentials is dense in the set of smooth potentials on $\mathbb{R}^2/\mathbb{L}$ in the $C^\infty(\mathbb{R}^2/\mathbb{L})$ topology*

APPENDIX A

Integration by Parts for Gaussians

It is well known that

$$\int_{\mathbb{R}^n} \exp(iy \cdot \eta) \exp\left(-cy^2\right)\, dy = \left(\frac{\pi}{c}\right)^{\frac{n}{2}} \exp\left(-\frac{\eta^2}{4c}\right). \tag{A.0.1}$$

We will use this fact to help us evaluate integrals of the form

$$\int_{\mathbb{R}^n} (y+b) \exp(iy \cdot \eta) \exp\left(-cy^2\right)\, dy \tag{A.0.2}$$

and

$$\int_{\mathbb{R}^n} |y+b|^2 \exp(iy \cdot \eta) \exp\left(-cy^2\right)\, dy. \tag{A.0.3}$$

Recall that, for c a constant, $\eta, y \in \mathbb{R}^n$, and α a multi-index with n components,

$$i^{|\alpha|}\eta^\alpha \int_{\mathbb{R}^n} \exp\left(-cy^2\right) \exp(i\eta \cdot y)\, dy = (-1)^{|\alpha|} \int_{\mathbb{R}^n} \partial_y^\alpha (\exp\left(-cy^2\right)) \exp(i\eta \cdot y)\, dy, \tag{A.0.4}$$

and also

$$\begin{aligned} \frac{\partial}{\partial y} \exp\left(-cy^2\right) &= -2cy \exp\left(-cy^2\right) \\ \frac{\partial^2}{\partial y^2} \exp\left(-cy^2\right) &= \left(-2c + 4c^2y^2\right) \exp\left(-cy^2\right). \end{aligned}$$

With these equalities in mind, (A.0.2) is equal to

$$\begin{aligned} &\frac{-1}{2c} \int_{\mathbb{R}^n} \partial_y (\exp\left(-cy^2\right)) \exp(i\eta \cdot y)\, dy + b \int_{\mathbb{R}^n} (\exp\left(-cy^2\right)) \exp(i\eta \cdot y)\, dy \\ &= \left(\frac{\pi}{c}\right)^{\frac{n}{2}} \exp\left(-\frac{\eta^2}{4c}\right) \left(\frac{i\eta}{2c} + b\right). \end{aligned}$$

We can also expand and re-write the integral in (A.0.3) so it is equal to

$$\frac{1}{4c^2}\int\limits_{\mathbb{R}^n} \partial_y^2(\exp\left(-cy^2\right))\exp\left(i\eta\cdot y\right)\,dy - \frac{b}{c}\int\limits_{\mathbb{R}^n} \partial_y(\exp\left(-cy^2\right))\exp\left(i\eta\cdot y\right)\,dy$$
$$+\left(b^2+\frac{1}{2c}\right)\int\limits_{\mathbb{R}^n}(\exp\left(-cy^2\right))\exp\left(i\eta\cdot y\right)\,dy$$

Using the integration by parts formula, (A.0.4) is just

$$\left(\frac{\pi}{c}\right)^{\frac{n}{2}}\exp\left(-\frac{\eta^2}{4c}\right)\left(-\frac{\eta^2}{4c^2}+\frac{ib\eta}{c}+b^2+\frac{1}{2c}\right). \tag{A.0.5}$$

APPENDIX B

Euler Summation for the Theta Function

An integer valued function $h(\alpha)$ may be estimated by the Euler summation formula

$$\sum_{a\leq\alpha\leq b} h(\alpha) = \int_a^b h(x)\,dx + \sum_{j=1}^{m} \frac{B_j}{j!} h^{(j-1)}(x)|_{x=a}^{x=b} + R_m. \tag{B.0.1}$$

where B_j is the j^{th} Bernoulli number and $h^{(j)}(x)$ denotes the j^{th} derivative of $h(x)$. The remainder R_m is defined as

$$R_m = (-1)^{m+1} \int_{\mathbb{R}} \frac{B_m(\{x\})}{m!} h^m(x)\,dx.$$

The notation $\{x\}$ denotes the fractional part of x, and $B_m(\{x\})$ denotes the m^{th} Bernoulli polynomial. Formula (B.0.1) is derived in *Concrete Mathematics*, [GKP94].

Fix $\epsilon_0 \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$, with $\lambda \geq 1$. We wish to use the Euler summation formula to estimate the sums

$$\sum_{\alpha\in\mathbb{Z}^n} \exp\left(-\frac{|\alpha-\epsilon_0|^2}{\lambda}\right) \tag{B.0.2}$$

and

$$\sum_{\alpha\in\mathbb{Z}^n} \frac{|\alpha-\epsilon_0|^2}{\lambda} \exp\left(-\frac{|\alpha-\epsilon_0|^2}{\lambda}\right) \tag{B.0.3}$$

in terms of the parameter λ. Since the variables $\alpha_1, \alpha_2, ...\alpha_n$ are indexed independently of each other, we may re-write (B.0.2) as

$$\sum_{\alpha_i\in\mathbb{Z}} \left(\prod_{i=1}^{n} \exp\left(-\frac{|\alpha_i-\epsilon_{0_i}|^2}{\lambda}\right)\right) = \prod_{i=1}^{n} \left(\sum_{\alpha_i\in\mathbb{Z}} \exp\left(-\frac{|\alpha_i-\epsilon_{0_i}|^2}{\lambda}\right)\right). \tag{B.0.4}$$

We apply the Euler summation formula with $m = 2$ to the sum in parentheses on the right hand side of (B.0.4), so that $h(x) = \exp\left(-\frac{|x-\epsilon_{0_i}|^2}{\lambda}\right)$. Letting $g(x) = \exp(-x^2)$, via the change of variables $x = \sqrt{\lambda}u + \epsilon_{0_i}$,

$$R_2 = \frac{-1}{2}\int_{\mathbb{R}} B_2(\{x\})h''(x)\,dx = \frac{-1}{2\sqrt{\lambda}}\int_{\mathbb{R}} B_2(\{\sqrt{\lambda}(u+\epsilon_{0_i})\})g''(u)\,du.$$

By properties of the Bernoulli numbers (again, cf [GKP94])

$$|B_2(\{\sqrt{\lambda}u + \epsilon_{0_i}\})| \leq B_2 = \frac{1}{6}.$$

Integrating by parts gives

$$\left|\int_{\mathbb{R}} g''(u)\,du\right| \leq \int_{\mathbb{R}} (4u^2+2)e^{-u^2}\,du = 4\sqrt{\pi}$$

and

$$|R_2| < \sqrt{\frac{\pi}{\lambda}}.$$

The second term on the right hand side in the Euler summation formula vanishes:

$$\sum_{j=1}^{m} \frac{B_j}{j!}h^{j-1}(x)|_{x=-\infty}^{x=\infty} = 0.$$

As a result,

$$\sum_{\alpha\in\mathbb{Z}^n} \exp\left(-\frac{|\alpha-\epsilon_0|^2}{\lambda}\right) \leq (2\pi\lambda)^{\frac{n}{2}}.$$

The second sum (B.0.3) can be re-written as

$$\sum_{i=1}^{n}\left(\sum_{\alpha_i\in\mathbb{Z}} \frac{|\alpha_i-\epsilon_{0_i}|^2}{\lambda}\exp\left(-\frac{|\alpha_i-\epsilon_{0_i}|^2}{\lambda}\right)\right)\sum_{\alpha'\in\mathbb{Z}^{n-1}}\exp\left(-\frac{|\alpha'-\epsilon_0'|^2}{\lambda}\right).$$

Here, $\alpha' = (\alpha_1, \alpha_2, \ldots\widehat{\alpha}_i, \ldots\alpha_n)$ and $\epsilon_0' = (\epsilon_{0_1}, \epsilon_{0_2}, \ldots\widehat{\epsilon}_{0_i}, \ldots\epsilon_{0_n})$. Applying the Euler summation formula to

$$\sum_{\alpha_i\in\mathbb{Z}} \frac{|\alpha_i-\epsilon_{0_i}|^2}{\lambda}\exp\left(-\frac{|\alpha_i-\epsilon_{0_i}|^2}{\lambda}\right) \tag{B.0.5}$$

gives that (B.0.5) is also $\mathcal{O}(\sqrt{\lambda})$. This follows since

$$\int_{-\infty}^{\infty} \frac{x^2}{\lambda}\exp\left(-\frac{x^2}{\lambda}\right)\,dx = \frac{\sqrt{\pi\lambda}}{2}. \tag{B.0.6}$$

The details are left to the reader. Therefore (B.0.3) is $\mathcal{O}((\lambda)^{\frac{n}{2}})$ as well.

APPENDIX C

Fourier Coefficients of the $\wp$ Function

As detailed in Chapter 8, the $\wp$-function depends on a parameter $\tau_j > 0$. The complex valued function $\wp(z, \tau)$ is given by

$$\wp(z,\tau) = \frac{1}{z^2} + \sum_{(m,n)\in\mathbb{Z}^2/0} \left(\frac{1}{(z-n-im\tau)^2} - \frac{1}{(n+im\tau)^2} \right)$$

which as before is real on the line $x + \frac{i\tau_j}{2}$ and setting,

$$a = e^{-2\pi\tau_j} \quad b = e^{2\pi i(x+\frac{i\tau_j}{2})}$$

gives

$$\frac{1}{(2\pi i)^2}\wp(x,\tau) = \frac{1}{12} + \sum_{n=-\infty}^{\infty} \frac{ab}{(1-a^m b)^2} - 2\sum_{n=1}^{\infty} \frac{na^n}{1-a^n}.$$

Because

$$\frac{a^m b}{(1-a^m b)^2} = \sum_{n=1}^{\infty} n(a^m b)^n \quad m \geq 0$$

and

$$\frac{a^m b}{(1-a^m b)^2} = \sum_{n=1}^{\infty} n(a^{-m} b^{-1})^n \quad m < 0$$

the representation

$$\frac{1}{(2\pi i)^2}\wp(x,\tau)$$
$$= \frac{1}{12} + \sum_{n=1}^{\infty} na^{\frac{n}{2}} e^{2\pi inx} + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} n(a^{n(m+\frac{1}{2})}e^{2\pi inx} + a^{n(m-\frac{1}{2})}e^{-2\pi ix}) - 2\sum_{n=1}^{\infty} \frac{na^n}{1-a^n}.$$

Changing the order of summation we get

$$\frac{-1}{4\pi^2}\wp(x,\tau) = \frac{1}{12} + \sum_{n=1}^{\infty} \frac{2na^{\frac{n}{2}}}{1-a^n}\cos(2\pi nx) - 2\sum_{n=1}^{\infty} \frac{na^n}{1-a^n}.$$

Therefore the Fourier coefficients for the $\wp$ functions in the first three directions are given by:

$$a_n^j = \frac{-8\pi^2 n \exp(-\pi n \tau_j)}{1 - \exp(-2\pi n \tau_j)} \qquad \text{for } n \geq 1 \tag{C.0.1}$$

$$a_0 = -\frac{\pi^2}{3} + 8\pi^2 \sum_{n=1}^{\infty} \frac{n \exp(-\pi n \tau_j)}{1 - \exp(-\pi n \tau_j)} \tag{C.0.2}$$

where $j = 1, 2$. The appropriate τ_j will depend on the choice of ϵ_j as given in Chapter 8.

APPENDIX D

Calculation of the Invariants

In order to prove Lemma 7 we need to show that there exist ϵ_1 and ϵ_2 in $[0,1]$ such that

$$\frac{\partial^2 \Phi_{1,m}}{\partial \epsilon_3 \partial \alpha_{1,m}}(\epsilon_0, \alpha) \qquad \text{and} \qquad \frac{\partial^2 \Phi_{2,n}}{\partial \epsilon_3 \partial \alpha_{2,n}}(\epsilon_0, \alpha) \tag{D.0.1}$$

are nonzero except perhaps on an analytic set of α.

We know by (9.0.7)

$$\Phi_{j,m}(\epsilon, \alpha) = \int_\Gamma \left| \sum_{j \neq k} \frac{\delta_k}{\delta_k \cdot d_j} q_k(\epsilon, \delta_k \cdot x, \alpha) \right|^2 (\phi^+_{j,m}(\epsilon, \delta_j \cdot x, \alpha))^2 \, dx \tag{D.0.2}$$

Each $q_j(\epsilon, \delta_j \cdot x, \alpha)$ is independent of ϵ_3 when $j \neq 3$. Furthermore since $q_k(\epsilon, \delta_k \cdot x, \alpha)$ and $(\phi^+_{k,m}(\epsilon, \delta_j \cdot x, \alpha))^2$ are independent of $\mu_{j,m}(\epsilon, \delta_j \cdot x, \alpha)$ for all $j \neq k$, so only the function $(\phi^+_{j,m}(\epsilon, \delta_j \cdot x, \alpha))^2$ depends on $\alpha_{j,m}$ in the above integral. As a result we can write

$$\frac{\partial^2 \Phi_{j,m}}{\partial \epsilon_3 \partial \alpha_{j,m}}(\epsilon, \alpha) = \int_\Gamma \frac{\partial}{\partial \epsilon_3} \left| \sum_{j \neq k} \frac{\delta_k}{\delta_k \cdot d_j} q_k(\epsilon, \delta_k \cdot x, \alpha) \right|^2 \frac{\partial}{\partial \alpha_{j,m}} (\phi^+_{j,m}(\epsilon, \delta_j \cdot x, \alpha))^2 \, dx \tag{D.0.3}$$

Whenever $\epsilon = \epsilon_0$, then $q_3(\epsilon_0, \delta_3 \cdot x, \alpha) = 0$ and the derivative $\partial_{\epsilon_3} q_3(\epsilon_0, \delta_3 \cdot x, \alpha)$ can be calculated using the Fredholm alternative as in [ERT84b]. Following Appendix I of [ERT84b], we may write

$$\frac{\partial q_3}{\partial \epsilon_3}(\epsilon_0, \delta_3 \cdot x, \alpha) = \sum_{n \in I_3} \gamma_{3,n} \cos(2\pi \delta_3 \cdot x + 2\alpha_{3,n}). \tag{D.0.4}$$

Also from the derivation of equation (9.0.19), we can conclude that

$$(\phi^+_{j,m}(\epsilon_0, s, \alpha))^2 = 2\cos^2(\pi m(\delta_j \cdot x) + \alpha_{j,m}) + \mathcal{O}(\epsilon_j) \tag{D.0.5}$$

where by Lemma 6 the order terms are bounded by $\epsilon_j C$ where C depends only on $\alpha_{j,m}$. Hence from analytic perturbation theory and the derivation of (9.0.19) we can use (D.0.5) to conclude that

$$\frac{\partial(\phi^+_{j,m})^2}{\partial\alpha_{j,m}}(\epsilon_0, \delta_j \cdot x, \alpha) = -2\sin(2\pi(\delta_j \cdot x)m + 2\alpha_{j,m}) + \mathcal{O}(\epsilon_j) \tag{D.0.6}$$

where the $\mathcal{O}(\epsilon_j)$ terms are bounded by $\epsilon_j C$ with C a constant depending only on the coordinate $\alpha_{j,m}$. Because any two directions δ_1 and δ_2 in $\mathbb{S}$ form a basis, we know that there exists a nonzero pair of integers (p_l, r_l) such that for any third vector $\delta_l \neq \delta_1, \delta_2$ we have $\delta_l = p_l\delta_1 + r_l\delta_2$. For easier computations we make the initial variable change $\delta_1 \cdot x = s$ and $\delta_2 \cdot x = t$, with the associated Jacobian, Vol(Γ), and rewrite the invariants. We also let D denote a generic constant which is independent of the coordinates, and we let

$$c_{l,k,j} = \frac{\delta_l \cdot \delta_j}{2(\delta_l \cdot d_j)(\delta_k \cdot d_j)}(\mathrm{Vol}(\Gamma)). \tag{D.0.7}$$

From statements (1-3) in Chapter 9, (D.0.4), (D.0.5) and (D.0.3), when $\epsilon = \epsilon_0$, we have

$$\begin{aligned} &(c_{3,l,j}\mathrm{Vol}(\Gamma))^{-1}\frac{\partial^2\Phi_{j,m}}{\partial\epsilon_3\partial\alpha_{j,m}}(\epsilon_0, \alpha) = \\ &4\int_0^1\int_0^1 \left(\sum_{n\in I_3}\gamma_{3,n}\cos(2\pi n(s+t) + 2\alpha_{3,n})\right)\wp_l(t + i\frac{\tau_l}{2}, \tau_l))\frac{\partial(\phi^+_{j,m})^2}{\partial\alpha_{j,m}}(\epsilon_0, s, \alpha))^2\,ds\,dt = \\ &2\sum_{n\in I_3}\gamma_{3,n}a^l_n\int_0^1 \cos(2\pi ns + 2\alpha_{3,n})\frac{\partial(\phi^+_{j,m})^2}{\partial\alpha_{j,m}}(\epsilon_0, s, \alpha)\,ds \end{aligned} \tag{D.0.8}$$

where $0 \le j, l \le 2, j \neq l$.

When $j = 1$, by the hypothesis (*) on the number of open gaps that q_3 has, the right hand side of (D.0.8) is just

$$2a^2_m\gamma_{3,m}\sin(2\alpha_{3,m} - 2\alpha_{1,m}) + \mathcal{O}(\epsilon_1) \tag{D.0.9}$$

Here the $\mathcal{O}(\epsilon_1)$ terms are bounded by $\epsilon_1 C$ where the constant depends only on $\alpha_{1,m}$ and $\alpha_{3,n}$ for all $n \in I_3$. We recall that $a^l_n \to 0$ as $\epsilon_l \to 0$ for all n in $\mathbb{N}$ and $l = 1, 2$ since a^l_n is related to ϵ_l by Equation (9.0.11) and (C.0.1) However, we can make the constant uniform in ϵ_2. If we let

$$\sup_{s\in[0,1]}\left|\frac{\partial(\phi^+_{1,m})^2}{\partial\alpha_{1,m}}(\epsilon_0,s,\alpha)\right| = M_m < \infty \tag{D.0.10}$$

then this follows from the rough estimate

$$\begin{aligned}&|\sum_{n\in I_3}\int_0^1\int_0^1\left(\sum_{n\in I_3}\gamma_{3,n}\cos(2\pi n(s+t)+2\alpha_{3,n})\right)p_2(t+i\frac{\tau_2}{2},\tau_2)) \\ &\times\left(\frac{\partial(\phi^+_{1,m})^2}{\partial\alpha_{1,m}}(\epsilon_0,s,\alpha))^2-\sin(2\pi ms+2\alpha_{1,m})\right)ds\,dt| \le \\ &\sum_{n\in I_3}\gamma_{3,n}a_n^2\cos(2\alpha_{3,n})\,(M_m+2)\le 2n\,(M_m+2)\end{aligned} \tag{D.0.11}$$

since the gap lengths $\gamma_{3,n}$ and the Fourier coefficients a_n^2 are exponentially decreasing. Now let β in $(0,1)$ be a small fixed parameter. We consider the set of α such that

$$|2\alpha_{3,m}-2\alpha_{1,m}-k\pi|\ge\beta \qquad \forall k\in\mathbb{Z},\ m\in I_1 \tag{D.0.12}$$

We let this set be denoted as A_1, and note that its complement is an analytic set. Therefore provided we chose ϵ_1 and ϵ_2 which satisfy the inequality

$$(M_m+2)\epsilon_1 < \frac{|a_m^2|\gamma_{3,m}}{2n}\sin(\beta) \tag{D.0.13}$$

for all m in I_1 and α in A_1 then (D.0.8) is nonzero for $j=1$ and all m in I_1. The tricky step is to prove that we can pick ϵ_1,ϵ_2 in $(0,1)$ such that D.0.13 holds for all m in I_1 but also so

$$\frac{\partial^2\Phi_{2,n}}{\partial\epsilon_3\partial\alpha_{2,n}}(\epsilon_0,\alpha)\ne 0 \tag{D.0.14}$$

for all n in I_2 except on an analytic set of α.

Because for small ϵ_1, $a^1_{n_1} > a^1_{n_2}$ whenever $n_2>n_1$ the right hand side of (D.0.8) is already written in ascending order in ϵ_1 for $j=2, l=1$. Let

$$b_{j,m,n}(\epsilon_0,\alpha)=\int_0^1\cos(2\pi ns+2\alpha_{3,n})\frac{\partial(\phi^+_{j,m})^2}{\partial\alpha_{j,m}}(\epsilon_0,s,\alpha)\,ds. \tag{D.0.15}$$

Since we do not know if $b_{2,m,n}(\epsilon_0,\alpha)\equiv 0$ in α for all $m\ne n$, we pick ϵ_1 as follows. Say $b_{2,m,1}(\epsilon_0,\alpha)$ is nonzero except on an analytic set of α, and then let the set where $b_{2,m,1}(\epsilon_0,\alpha)=$

0 be denoted as $A^c_{2,m,1}$. If we can prove that for $j = 2, l = 1$, (D.0.8) is nonzero for some α, then it will be nonzero on some open dense set of α's. The easiest α to select is the one when $b_{2,m,1}(\epsilon_0, \alpha)$ is at its maximum. Hence we then pick ϵ_1 such that

$$\max_{\alpha \in A_{2,m,1}} |\gamma_{3,1} a^1_1 b_{2,m,1}(\epsilon_0, \alpha)| \geq \left| \sum_{\substack{k \in I_3 \\ k \neq 1}} \gamma_{3,k} a^1_k b_{2,m,k}(\epsilon_0, \alpha) \right| \tag{D.0.16}$$

where the max is taken over the possible values of $b_{2,m,1}(\epsilon_0, \alpha)$ with α in $A_{2,m,1}$, and we consider the right hand side of (D.0.16) to be evaluated at this α as well. If $b_{2,m,1}(\epsilon_0, \alpha) \equiv 0$ in α, but $b_{2,m,2}(\epsilon_0, \alpha)$ is nonzero except on an analytic set of $\alpha_{2,m}$, and let the set where $b_{2,m,2}(\epsilon_0, \alpha) = 0$ be denoted as $A^c_{2,m,2}$ then pick ϵ_1 such that

$$\max_{\alpha \in A_{2,m,2}} |\gamma_{3,2} a^1_2 b_{2,m,2}(\epsilon_0, \alpha)| \geq \left| \sum_{\substack{k \in I_3 \\ k > 2}} \gamma_{3,k} a^1_k b_{2,m,k}(\epsilon_0, \alpha) \right| \tag{D.0.17}$$

where again the max is taken over the possible values of $b_{2,m,2}(\epsilon_0, \alpha)$ with α in $A_{2,m,2}$. We continue this process inductively. As before, let β be a small parameter in $(0, 1)$. We now also consider the set of α such that

$$|2\alpha_{3,m} - 2\alpha_{2,m} - k\pi| \geq \beta \qquad \forall k \in \mathbb{Z},\ m \in I_2 \tag{D.0.18}$$

and let this set be denoted by $A_{2,m,m}$. We know

$$b_{2,m,m}(\epsilon_0, \alpha) = \sin(2\alpha_{3,m} - 2\alpha_{2,m}) + \mathcal{O}(\epsilon_2) \tag{D.0.19}$$

where the $\mathcal{O}(\epsilon_2)$ terms are bounded by $\epsilon_2 C$ where is a constant depending only on $\alpha_{2,m}$ and $\alpha_{3,n}$ for all n in I_3. Hence our selection process terminates because $b_{2,m,m}(\epsilon_0, \alpha)$ is not zero for α in $A_{2,m,m}$ provided we chose ϵ_2 such that

$$\epsilon_2 |C| < \sin(\beta) \tag{D.0.20}$$

Hence we pick ϵ_1 in terms of ϵ_2 so that

$$\min_n \max_{\alpha \in A_{2,m,n}} \left(|\gamma_{3,n} a^1_n b_{2,m,n}(\epsilon_0, \alpha)| \right) \geq \left| \sum_{\substack{l \in I_3 \\ k > n}} \gamma_{3,k} a^1_k b_{2,m,k}(\epsilon_0, \alpha) \right| \tag{D.0.21}$$

for all m in I_2 where the $\min_n$ is taken over those indices n for which $b_{2,m,n}(\epsilon_0, \alpha)$ is not identically zero in α. This choice of ϵ_1 and ϵ_2 is not in contradiction to our choice of ϵ_1 small compared to ϵ_2 since the right hand side of the inequality (D.0.21) always has a higher order function of ϵ_1 than the left hand side. Furthermore $b_{2,m,n} = 0$ for all $m \neq n$ whenever $\epsilon_2 = 0$, so the right hand side is bounded. We conjecture using a computer and the standard perturbation series for $b_{j,m,n}(\epsilon_0, \alpha)$ that the assumption q_1, q_2 and q_3 have the same number of gaps could be removed. However, this is computationally difficult since it has been verified $b_{j,m,n}(\epsilon_0, \alpha)$ is $\mathcal{O}(\epsilon_j^{|m-n|})$ for all m up to some sufficiently large values of m and n.

For the case with $j \geq 3$, the invariants are computed almost exactly the same way as in [ERT84b] because the form of the invariants coincides for these indices. In this case we have that

$$\Phi_{j,m}(\epsilon_0, \alpha) = c_{1,2,j} a^1_{mp_j} a^2_{mr_j} \cos(2\alpha_{j,m}) + D \tag{D.0.22}$$

References

[AHS08] F. Andersson, M. V. de Hoop, H. F. Smith, and G. Uhlmann. "A multi-scale approach to hyperbolic evolution equations with limited smoothness." *Comm. Partial Differential Equations*, **33**(4-6):988–1017, 2008.

[Cai06] K. Cai. "Dispersion for Schrödinger operators with one-gap periodic potentials on $\mathbb{R}^1$." *Dyn. Partial Differ. Equ.*, **3**(1):71–92, 2006.

[ERT84a] G. Eskin, J. Ralston, and E. Trubowitz. "On isospectral periodic potentials in $\mathbf{R}^n$." *Comm. Pure Appl. Math.*, **37**(6):715–753, 1984.

[ERT84b] G. Eskin, J. Ralston, and E. Trubowitz. "On isospectral periodic potentials in $\mathbf{R}^n$. II." *Comm. Pure Appl. Math.*, **37**(5):647–676, 1984.

[GKP94] Ronald L. Graham, Donald E. Knuth, and Oren Patashnik. *Concrete mathematics*. Addison-Wesley Publishing Company, Reading, MA, second edition, 1994. A foundation for computer science.

[GT84] J. Garnett and E. Trubowitz. "Gaps and bands of one-dimensional periodic Schrödinger operators." *Comment. Math. Helv.*, **59**(2):258–312, 1984.

[Hoc65] Harry Hochstadt. "On the determination of a Hill's equation from its spectrum." *Arch. Rational Mech. Anal.*, **19**:353–362, 1965.

[Hor03] Lars Hörmander. *The analysis of linear partial differential operators. I.* Classics in Mathematics. Springer-Verlag, Berlin, 2003. Distribution theory and Fourier analysis, Reprint of the second (1990) edition [Springer, Berlin; MR1065993 (91m:35001a)].

[MC97] Y. Meyer and R. Coifman. *Wavelets*, volume 48 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1997. Calderón-Zygmund and multilinear operators, Translated from the 1990 and 1991 French originals by David Salinger.

[MT76] H. P. McKean and E. Trubowitz. "Hill's operator and hyperelliptic function theory in the presence of infinitely many branch points." *Comm. Pure Appl. Math.*, **29**(2):143–226, 1976.

[MW79] Wilhelm Magnus and Stanley Winkler. *Hill's equation*. Dover Publications Inc., New York, 1979. Corrected reprint of the 1966 edition.

[Ral82] James Ralston. "Gaussian beams and the propagation of singularities." In *Studies in partial differential equations*, volume 23 of *MAA Stud. Math.*, pp. 206–248. Math. Assoc. America, Washington, DC, 1982.

[SHB04] A. Shlivinski, E. Heyman, A. Boag, and C. Letrou. "A phase-space beam summation formula for ultrawide-band radiation." *IEEE Transactions on Atennas and Prop.*, **52**(8):2042–2056, 2004.

[Smi98] H. F. Smith. "A parametrix construction for wave equations with $C^{1,1}$ coefficients." *Ann. Inst. Fourier (Grenoble)*, **48**(3):797–835, 1998.

[SS03] Elias M. Stein and Rami Shakarchi. *Complex analysis.* Princeton Lectures in Analysis, II. Princeton University Press, Princeton, NJ, 2003.

[Tan08] N. Tanushev. "Superpositions and higher order Gaussian beams." *Commun. Math. Sci.*, **6**(2):449–475, 2008.

[Tat00] Daniel Tataru. "Global Strichartz estimates for variable coefficient second order hyperbolic operators." In *Séminaire: Équations aux Dérivées Partielles, 1999–2000*, Sémin. Équ. Dériv. Partielles, pp. Exp. No. XI, 17. École Polytech., Palaiseau, 2000.

[Tru77] E. Trubowitz. "The inverse problem for periodic potentials." *Comm. Pure Appl. Math.*, **30**(3):321–337, 1977.

[Wat11] A. Waters. "A parametrix construction for the wave equation with low regularity coefficients using a frame of Gaussians." *Commun. Math. Sci.*, **9**(1):225–254, 2011.

[Wat12] A. Waters. "Spectral rigidity for periodic Schrödinger operators in Dimension 2." *Archive Submission*, **http://arxiv.org/abs/1203.2901**, March 2012.