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Diep-Nguyen, Arthur Francis

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Coxeter groups and the polynomial Hurwitz action

A dissertation submitted in partial satisfaction

of the requirements for the degree

Doctor of Philosophy

in

Mathematics

by

Arthur Diep-Nguyen

Committee in charge:

Professor Jon McCammond, Chair

Professor Stephen Bigelow

Professor Darren Long

Professor Fedya Manin, University of Toronto

September 2026

The dissertation of Arthur Diep-Nguyen is approved.

Professor Stephen Bigelow

Professor Darren Long

Professor Fedya Manin, University of Toronto

Professor Jon McCammond, Chair

September 2026

Coxeter groups and the polynomial Hurwitz action

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by

Arthur Diep-Nguyen

Dedicated to Kimchi, Bamao, and Momo, for lifting my spirits.

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Curriculum Vitæ

Arthur Diep-Nguyen

Education

- 2026 Ph.D. in Mathematics, University of California, Santa Barbara
- 2018 B.S. in Mathematics, Boston College

Abstract

Coxeter groups and the polynomial Hurwitz action

by

Arthur Diep-Nguyen

Artin groups are generalizations of Emil Artin's braid groups, and dual Artin groups are a variation defined using the Hurwitz action on reduced reflection factorizations of a fixed Coxeter element of the corresponding Coxeter group. There is a surjective group homomorphism from the Artin group to the corresponding dual Artin group. It is conjectured that the standard and dual Artin groups are always isomorphic. When the two groups are isomorphic, the properties of the dual Artin group can be used to tackle the word problem and $K(\pi, 1)$ conjecture for the standard Artin group.

This dissertation investigates the matrix representations and change of basis induced by the Hurwitz action on reduced reflection factorizations of Coxeter elements. When attention is restricted to powers of half-twists, the resulting matrix representations exhibit connections to Fibonacci and Chebyshev polynomials, and can be viewed geometrically as acting via rotations on certain planes in the standard geometric representation of the Coxeter system. Generalizing to powers of braids that twist a sequence of adjacent strands, the matrix representations exhibit connections to a generalization of the Fibonacci polynomials called polynomial wave sequences. This is a first step towards understanding all dual Artin groups in a uniform way.

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Chapter 1

Introduction

This dissertation focuses on the Hurwitz action on reduced reflection factorizations of a fixed Coxeter element of a Coxeter system. This is a braid group action that determines a surjective group homomorphism from the Artin group to the corresponding dual Artin group, and it is conjectured to always be an isomorphism. When the two groups are isomorphic, the properties of the dual Artin group can be used to tackle the word problem and $K(\pi, 1)$ conjecture for the standard Artin group. The goal of this dissertation is to investigate the Hurwitz action by viewing it through the lens of matrix representations. We first establish that the Hurwitz action on reduced reflection factorizations of a fixed Coxeter element induces a change of basis on the standard geometric representation of the Coxeter system, then explore the matrix representations that arise when applying the Hurwitz action via certain types of braids.

1.1 Coxeter groups, Artin groups, and the Hurwitz action on factorizations

This dissertation investigates the Hurwitz action on reduced reflection factorizations of Coxeter elements, which we have not yet discussed so far in this introductory chapter. This section explains some of the motivations for studying the Hurwitz action, focusing on the connections to Artin theory and the construction of dual Artin groups, then states the definition of the Hurwitz action itself. While this dissertation does not address Artin theory directly, we see the Hurwitz action as a worthy object of study in light of its importance in constructing dual Artin groups.

Coxeter groups are well-understood and well-behaved groups. A Coxeter group W may be defined via straightforward presentation in terms of a finite generating set S of involutions.

$$W := \left\langle S \left| \begin{array}{l} \forall s \in S, \quad s^2 = 1 \\ \forall s, t \in S, \quad (st)^{m(s,t)} = 1 \end{array} \right. \right\rangle = \left\langle S \left| \begin{array}{l} \forall s \in S, \quad s^2 = 1 \\ \forall s, t \in S, \quad \underbrace{sts \cdots}_{m(s,t)} = \underbrace{tst \cdots}_{m(s,t)} \end{array} \right. \right\rangle$$

The generators and relations may be encoded by a graph Γ called the Coxeter diagram, whereby we may write $W = \text{Cox}(\Gamma)$. Much is known about general Coxeter groups; below is a sampling of facts we know about all Coxeter groups.

- Every Coxeter group has a faithful linear representation.
- Every Coxeter group is automatic (in fact, biautomatic).
- Every Coxeter group has a solvable word problem.
- Every Coxeter group has a solvable conjugacy problem.

- The Coxeter groups that act geometrically on spheres are completely classified by their diagrams Γ .
- The Coxeter groups that act geometrically on Euclidean spaces are completely classified by their diagrams Γ .

Artin groups are related to Coxeter groups, in that they may be defined by a similar presentation in terms of a finite generating set S . Here, the generators are no longer involutions; they are required to have infinite order.

$$A := \langle S \mid \forall s, t \in S, \underbrace{sts \cdots}_{m(s,t)} = \underbrace{tst \cdots}_{m(s,t)} \rangle$$

The generators S and relation lengths $m(s, t)$ may again be encoded by Γ , whereby we may write $A = \text{Art}(\Gamma)$. Notice that this gives a natural surjection $\text{Art}(\Gamma) \twoheadrightarrow \text{Cox}(\Gamma)$.

Unlike their Coxeter cousins, Artin groups remain very mysterious; many basic questions remain open for general Artin groups. Below, we present a small sampling of such open conjectures.

Conjecture 1.1.1 (Four basic conjectures). *Here are four basic conjectures about Artin groups.*

- (A) *Every Artin group is torsion-free.*
- (B) *Every non-spherical irreducible Artin group has trivial center.*
- (C) *Every Artin group has a solvable word problem.*
- (D) *Artin groups satisfy the $K(\pi, 1)$ conjecture.*

Although these conjectures remain unanswered for vast swathes of Artin groups that exist, there are some classes of Artin groups for which these conjectures are answered. A powerful technique for tackling these conjectures is constructing the so-called *dual*

Artin group $A^* = \text{Art}^*(\Gamma, w)$. It is not known in general that A is always isomorphic to its dual A^* , but for every dual Artin group A^* that has been successfully analyzed, the isomorphism $A \cong A^*$ holds. This isomorphism is useful because the dual Artin groups that have been successfully analyzed all have nice properties that allow many conjectures to be answered. For example, every spherical Artin group A is isomorphic to its dual A^* , and A^* has a Garside structure that allows the above four conjectures to be answered affirmatively. More recently, it was shown that Euclidean Artin groups are also isomorphic to their duals, and that each dual embed nicely into a Garside supergroup. The resulting structure allowed the four conjectures to be answered affirmatively for Euclidean Artin groups. In the grand endeavor to understand Artin groups in general, dual Artin groups have so far proven to be fruitful and compelling.

How does one construct a dual Artin group? The dual Artin group $\text{Art}^*(\Gamma, w)$ is constructed in terms of generators and relations that arise from reduced reflection factorizations of a Coxeter element w from the corresponding Coxeter group $\text{Cox}(\Gamma)$. To obtain these reduced reflection factorizations, one may use the Hurwitz action, a braid group action that is transitive on these factorizations. We first define the braid group, itself an example of an Artin group, then define the Hurwitz action for an arbitrary group G .

Definition 1.1.2 (Braid group). The *braid group* on d strands is the group Braid_d with the following presentation.

$$\text{Braid}_d = \left\langle \sigma_1, \dots, \sigma_{d-1} \mid \begin{array}{l} \sigma_i \sigma_j \sigma_i = \sigma_j \sigma_i \sigma_j \text{ for } |i - j| = 1 \\ \sigma_i \sigma_j = \sigma_j \sigma_i \text{ for } |i - j| > 1 \end{array} \right\rangle$$

We call the above presentation the *standard presentation* for Braid_d , and each σ_i may be called the i -th *standard generator*.

Definition 1.1.3 (Hurwitz action). Let G be a group. The *Hurwitz action* of Braid_d on

G^d is defined by

$$\begin{aligned}\sigma_i \cdot (g_1, \dots, g_i, g_{i+1}, \dots, g_d) &= (g_1, \dots, g_i g_{i+1} g_i^{-1}, g_i, \dots, g_d), \\ \sigma_i^{-1} \cdot (g_1, \dots, g_i, g_{i+1}, \dots, g_d) &= (g_1, \dots, g_{i+1}, g_{i+1}^{-1} g_i g_{i+1}, \dots, g_d).\end{aligned}$$

We call the action by σ_i (respectively, σ_i^{-1}) the *i-th standard positive (resp. negative) Hurwitz move*.

The connection to Artin theory is an impetus for studying the Hurwitz action itself. This dissertation's investigation of the Hurwitz action on $\text{Red}_T(w)$ does not focus on Artin theory, but is motivated by the connection.

1.2 The major recurring concepts

This dissertation revolves around several recurring concepts, which we present together in this section.

We let $W = \text{Cox}(\Gamma)$ be a Coxeter group with generating set $S = \{s_1, s_2, \dots, s_d\}$. (Our chosen labeling of the generators s_i implies an ordering of S , so we may write S as the tuple $S = (s_1, s_2, \dots, s_d)$.)

The main recurring objects of this dissertation are

- the Coxeter element $c = s_1 s_2 \cdots s_d \in W$ that encodes the ordering $S = (s_1, s_2, \dots, s_d)$ of generators,
- the Wall forms of c and c^{-1} ,
- an upper unitriangular matrix C related to c and its Wall form, and
- the Hurwitz action on the set $\text{Red}_T(c)$ of reduced reflection factorizations of c .

We begin by introducing Coxeter elements in general.

Definition 1.2.1 (Coxeter element). A *Coxeter element* of a Coxeter group W is a product (in any order) of all elements of S , each generator occurring exactly once. Fixing an ordering $S = (s_1, \dots, s_d)$ of the generating set, the Coxeter elements are

$$\left\{ w = \prod_{i=1}^d s_{\tau(i)} \mid \tau \in \text{Sym}_d \right\}.$$

Each Coxeter element encodes a particular ordering of the generators. We let $c = s_1 s_2 \cdots s_d$ denote the Coxeter element that encodes the ordering $S = (s_1, s_2, \dots, s_d)$ implied by the labeling of the generators s_i . Sometimes, distinct orderings of the generators yield the same Coxeter element (e.g. when adjacent generators commute), sometimes conjugate Coxeter elements (e.g. every finite Coxeter group has exactly one conjugacy class of Coxeter elements), and sometimes completely distinct non-conjugate Coxeter elements.

The Coxeter group W has a faithful linear representation $\rho : W \rightarrow \text{GL}(V)$ to some d -dimensional real quadratic space (V, Q) , defined in terms of an ordered basis $(e_1, e_2, \dots, e_d)$ of unit vectors that corresponds with an ordering of the generators of S . (The image $\rho(W)$ lives in the generalized orthogonal group $\text{O}(V, Q)$ defined in terms of the quadratic form Q on V .) Fixing an ordering $S = (s_1, s_2, \dots, s_d)$ encoded by the Coxeter element $c = s_1 s_2 \cdots s_d$ leads to a representation ρ where each s_i acts as the reflection (with respect to the quadratic form of V) that sends the basis vector e_i to its negative. We call this representation ρ the *standard geometric representation* of W .

Each $\rho(s_i)$ is a reflection, so the isomorphic image $\rho(W)$ is generated by reflections. Clearly, each element of $\rho(W)$ can be written as a product of reflections; simply write in terms of the generating reflections $\rho(s_i)$, which we call *simple reflections*. (Identifying $W \cong \rho(W)$, we may say each element of W has a reflection factorization.) Instead of looking at factorizations in terms of the alphabet, we may examine factorizations with

respect to the set T of *all* reflections in W . We consider a reflection factorization of some $w \in W$ to be *reduced* if it has the shortest possible length among reflection factorizations of w . We let $\text{Red}_T(w)$ denote the set of all such reduced reflection factorizations of w .

Via the standard geometric representation $\rho : W \rightarrow \rho(W)$, we can represent each element w by a matrix $[\rho(w)]_{\mathcal{B}}$ with respect to some basis $\mathcal{B}$. (For simplicity, we may omit the subscript $\mathcal{B}$ when the basis is understood, and we may write $[w]$ instead of $[\rho(w)]$.) In the standard ordered basis $(e_1, e_2, \dots, e_d)$,

Recall that the Coxeter group W is defined by a generating set S of involutions, alongside relations $(s, t)^{m(s,t)} = 1$. With generators $S = (s_1, s_2, \dots, s_d)$, we write $m_{ij} := m(s_i, s_j)$. The relation lengths m_{ij} may be encoded by a so-called Gram matrix.

Definition 1.2.2 (Gram matrix). The *Gram matrix* B of the Coxeter group W is given by $B_{ij} = -\cos(\pi/m_{ij})$.

One of our main recurrent objects is an upper unitriangular matrix C that may be defined in terms of the Gram matrix B . This special matrix occurs frequently throughout this dissertation.

Definition 1.2.3. We define the upper unitriangular matrix C by

$$C_{ij} = \begin{cases} 1, & i = j \\ 0, & i > j \\ 2B_{ij}, & i < j \end{cases}$$

As it turns out, there is some relationship between the Coxeter element c and the upper unitriangular matrix C .

Theorem 1.2.4 (Matrix representation of c). *The matrix representation $[c]$ of the Coxeter element c with respect to the standard ordered basis $(e_1, e_2, \dots, e_d)$ is*

$$[c] = -C^{-1}C^{\top}.$$

We now introduce the notion of a Wall form. For each element $f \in O(V)$ of the orthogonal group, we may define its moved space $\text{Mov}(f)$, as well as a bilinear form χ_f on $\text{Mov}(f)$.

Definition 1.2.5 (Moved space). The *moved space* of f is $\text{Mov}(f) = \text{im}(\text{id} - f)$.

Definition 1.2.6 (Wall form). The *Wall form* of a linear isometry $f \in O(V)$ is the bilinear form χ_f on $\text{Mov}(f)$ defined as

$$\chi_f(u, v) = \beta(w, v),$$

where β is the polar form

$$\beta(u, v) = Q(u + v) - Q(u) - Q(v)$$

of the quadratic form Q associated with the quadratic space V , and where $w \in V$ is any vector satisfying $u = w - f(w)$.

The following theorem states that the Wall form χ_f (along with the moved space $\text{Mov}(f)$) is enough to completely recover the original isometry f .

Theorem 1.2.7 (Wall's parametrization). *The map $f \mapsto (\text{Mov}(f), \chi_f)$ is a one-to-one correspondence between the orthogonal group $O(V)$ and the set of pairs (W, χ) such that W is a subspace of V and χ is a non-degenerate bilinear form on W satisfying $\chi(u, u) = Q(u)$ for each $u \in W$.*

As it turns out, the Wall form of the Coxeter element c and its inverse c^{-1} may be written in terms of the upper unitriangular matrix C .

Theorem 1.2.8 (Matrix representations of χ_c and $\chi_{c^{-1}}$). *The matrix representations $[\chi_c]$ and $[\chi_{c^{-1}}]$ of the Wall forms χ_c and $\chi_{c^{-1}}$ with respect to the standard basis $(e_1, e_2, \dots, e_d)$ are*

$$[\chi_c] = C^\top \quad \text{and} \quad [\chi_{c^{-1}}] = C.$$

In light of the above theorem, we call $C^\top$ the *(standard) Wall form matrix* for the Wall form χ_c , since it is the matrix representation with respect to the *standard* basis. Likewise, we call C the (standard) Wall form matrix for $\chi_{c^{-1}}$. (Any matrix representation of χ_f may simply be called a Wall form matrix for χ_f .)

Remark 1.2.9 (Binary choices). The fact that $[\chi_c] = C^\top$ and $[\chi_{c^{-1}}] = C$ depends on multiple binary choices regarding definitions, notations, and conventions. For example, we have chosen for the matrix representations of linear maps to be written in terms of matrices acting on the left of column vectors, rather than acting on the right of row vectors, as this choice conforms with many authors, including relevant results in the literature that show $[c] = -C^{-1}C^\top$ [How82] [HK16]. Similarly, we have chosen to define Wall forms in line with the relevant literature [Tay92] [MP22].

Recall from Section 1.1 that the Hurwitz action is a braid group action on d -tuples of elements from a group G . Notice that value of the product of the d -tuple is invariant under the Hurwitz action. Fixing a Coxeter element w , we obtain the d -tuple $(s_{\tau(1)}, s_{\tau(2)}, \dots, s_{\tau(d)})$ encoding the reduced reflection factorization $w = s_{\tau(1)}s_{\tau(2)} \cdots s_{\tau(d)}$. Applying the Hurwitz action to this tuple apparently yields another reduced reflection factorization of w . Furthermore, the Hurwitz action is transitive on the set $\text{Red}_T(w)$ of all such reduced reflection factorizations.

Theorem 1.2.10 (Transitivity of the Hurwitz action). *Let w be a Coxeter element in W . Then the Hurwitz action on the set $\text{Red}_T(w)$ of reduced reflection factorizations of w is transitive.*

We may apply the Hurwitz action to get any reduced reflection factorization of the Coxeter element $c = s_1s_2 \cdots s_d$.

1.3 New results

This dissertation investigates the Hurwitz action from the perspective of matrix representations with respect to the standard geometric representation of the associated Coxeter group. By computing change-of-basis matrices and matrix representations induced by the Hurwitz action via particular braids, we obtain polynomial sequences that exhibit connections to combinatorics, graph theory, and numerical linear algebra. This section aims to summarize and contextualize the results of this dissertation.

Recall the standard geometric representation $\rho : W \rightarrow \mathrm{GL}(V)$. Each Coxeter group W comes equipped with a *root system* Φ , whose elements are unit vectors in V . There is a correspondence between the set of reflections T and pairs $\pm u$ of roots. Namely, for each root $u \in \Phi$, there is a reflection $r_u \in T$ that sends u to its negative $-u$. (Clearly, $r_u = r_{-u}$.)

The correspondence between reflections and roots allows us to take a reduced reflection factorization of the Coxeter element c and obtain a d -tuple of roots, which turns out to always form a basis of unit vectors in V . A preliminary result in this dissertation is that the Hurwitz action on reduced reflection factorizations of c induces a braid group action on an appropriate choice $\mathfrak{B}$ of ordered bases of roots corresponding with the reflection factorizations. First, we define how a standard braid generator σ_i (also known as a positive half-twist) acts on ordered bases of V . (We may analogously define the action via a negative half-twist σ_i^{-1} .) See Figure 1.1 for an example of a braid diagram of a half-twist.

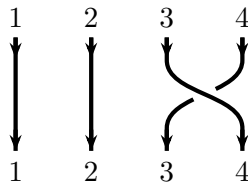


Figure 1.1: Positive half-twist $\sigma_3 \in \mathrm{Braid}_4$

Definition 1.3.1 (Hurwitz move by σ_i). Let $\mathcal{B} = (u_1, u_2, \dots, u_d)$ be some ordered basis of V . We define $\sigma_i \cdot \mathcal{B} = (v_1, v_2, \dots, v_d)$ to be the d -tuple whose j -th element is

$$v_j = \begin{cases} u_j, & j \notin \{i, i+1\} \\ xu_i - u_{i+1}, & j = i \\ u_i, & j = i+1 \end{cases}$$

where $x := \chi_{c^{-1}}(u_i, u_{i+1})$. (Notice that when $\mathcal{B}$ is the standard basis $(e_1, e_2, \dots, e_d)$, the coefficient x becomes $x = C_{i,i+1}$, where C is the upper unitriangular matrix from earlier.)

We call applying a half-twist $\sigma_i^\pm$ to an ordered basis $\mathcal{B}$ a *Hurwitz move* by $\sigma_i^\pm$ on $\mathcal{B}$. Notice that Hurwitz moves are defined in terms of the Wall form $\chi_{c^{-1}}$.

Suppose each u_i is the root corresponding to a reflection $r_i \in T$, and suppose $c = r_1 r_2 \cdots r_d$ is a reduced reflection factorization of the Coxeter element c . The Hurwitz action via σ_i gives another reduced reflection factorization of c , say $c = t_1 t_2 \cdots t_d$. It turns out that each vector v_j is a root in Φ , and that it is sent to its negative by the reflection t_j .

The foundation of this dissertation is the fact that the above definition for $\sigma_i \cdot \mathcal{B}$ extends to a braid group action on all ordered bases of roots obtainable by applying sequences of Hurwitz moves to the standard basis $\mathcal{B}_0 = (e_1, e_2, \dots, e_d)$.

Proposition 1.3.2 (Hurwitz action on ordered bases of roots). *Let $\mathcal{B}_0 = (e_1, e_2, \dots, e_d)$ be the standard basis of simple roots. Let $\mathfrak{B}$ denote the collection of ordered bases of roots obtainable by sequences of Hurwitz moves applied to $\mathcal{B}_0$. Then for each braid $\beta \in \text{Braid}_d$, the j -th root in the ordered basis $\beta \cdot \mathcal{B}_0$ is sent to its negative by the j -th reflection t_j in the reduced reflection factorization $c = t_1 t_2 \cdots t_d$ obtained by the Hurwitz action on the usual factorization $c = s_1 s_2 \cdots s_d$ via the braid β . Thus, the Hurwitz action on reduced reflection factorizations of the Coxeter element c induces a transitive braid group action on $\mathfrak{B}$, which we call the induced Hurwitz action on $\mathfrak{B}$.*

Say we have two bases $\mathcal{B}, \mathcal{B}' \in \mathfrak{B}$ related by $\mathcal{B}' = \beta \cdot \mathcal{B}$ for some $\beta \in \text{Braid}_d$. One may compute the change-of-basis matrix ${}_{\mathcal{B}}M_{\mathcal{B}'}$, called the *Hurwitz change-of-basis matrix* via β on $\mathcal{B}$. For example, writing $\mathcal{B} = (u_1, u_2, \dots, u_d)$ and letting $\beta = \sigma_i$, the change-of-basis matrix is the block diagonal matrix

$${}_{\mathcal{B}}M_{\sigma_i \mathcal{B}} = \begin{pmatrix} I_{i-1} & 0 & 0 \\ 0 & \mathbf{M} & 0 \\ 0 & 0 & I_{d-i-1} \end{pmatrix}$$

where the block $\mathbf{M}$ is the 2×2 matrix

$$\mathbf{M} = \begin{pmatrix} x & 1 \\ -1 & 0 \end{pmatrix}$$

with $x = \chi_{c^{-1}}(u_i, u_{i+1})$.

Using these change-of-basis matrices, we may compute matrix representations of elements of W (viewed as linear isometries in $O(V)$ via the standard geometric representation $\rho : W \rightarrow O(V)$), as well as the matrix representations of the Wall forms of these isometries. This dissertation obtains its results by investigating these various matrices,

Chapter 7 investigates the induced Hurwitz action via powers of half-twists σ_i . We start by explicitly computing $\sigma_i^n \cdot \mathcal{B}$, and obtain the following result.

Proposition 1.3.3 (Hurwitz action via σ_i^n). *The Hurwitz change-of-basis matrix ${}_{\mathcal{B}}M_{\sigma_i^n \mathcal{B}}$ satisfies*

$${}_{\mathcal{B}}M_{\sigma_i^n \mathcal{B}} = {}_{\mathcal{B}}M_{\sigma_i \mathcal{B}}^n$$

and may be written as the block diagonal matrix

$$\begin{pmatrix} I_{i-1} & 0 & 0 \\ 0 & \mathbf{M}^n & 0 \\ 0 & 0 & I_{d-i-1} \end{pmatrix}$$

with $\mathbf{M}$ as before.

A new result in this dissertation shows that the Hurwitz action via powers of half-twists are related to the Fibonacci polynomials $F_n(x)$ and Chebyshev polynomials of the second kind $U_n(x)$. We obtain a polynomial sequence $H_n(x)$ as follows.

Proposition 1.3.4 ($\mathbf{M}^n$ as a matrix of polynomials). *For $n \in \mathbb{Z}$, we have*

$$\mathbf{M}^n = \begin{pmatrix} H_n(x) & H_{n-1}(x) \\ -H_{n-1}(x) & -H_{n-2}(x) \end{pmatrix}$$

where the polynomials $H_n(x)$ are defined by the recurrence relation

$$H_{n+1}(x) = xH_n(x) - H_{n-1}(x)$$

with initial conditions $H_0(x) = 1$ and $H_1(x) = x$.

The polynomial sequence $H_n(x)$ satisfies

$$H_n(x) = i^{-n} F_{n+1}(ix) = U_n(x/2)$$

where i denotes the imaginary unit. Section 7.2 discusses the relationship between these three polynomial sequences and states some basic combinatorial data, such as the ordinary generating functions and closed formulas.

Another new finding is that the Hurwitz action via powers of half-twists is related to plane rotations. Let $U(\theta; j, j+1)$ denote the $d \times d$ matrix that acts by rotating the $(j, j+1)$ -coordinate plane in $\mathbb{R}^d$ by angle θ . Below is one of several new results about the relationship.

Proposition 1.3.5 (Non-free relation leads to $\hat{\theta}$ -rotation). *Suppose the relation length $m_{j,j+1} \geq 2$ between the simple reflections s_j and s_{j+1} is a positive integer. (In particular, suppose $m_{j,j+1} \neq \infty$, i.e. the relation between s_j and s_{j+1} is not free.) Define the angle $\theta = \pi - \frac{\pi}{m_{j,j+1}}$. Let $\mathcal{B}_0 = (e_1, e_2, \dots, e_d)$ be the standard basis. Treating the Hurwitz change-of-basis matrix ${}_{\mathcal{B}_0}M_{\sigma_j^n \mathcal{B}_0}$ as representing some linear automorphism $T : V \rightarrow V$,*

there exists a basis $\mathcal{B}$ where $[T]_{\mathcal{B}} = U(\theta^n; j, j+1)$.

Chapter 8 focuses on the Hurwitz action via powers of rotation braids $\sigma_j \sigma_{j-1} \cdots \sigma_{i+1} \sigma_i$ formed by twisting contiguous sequences of adjacent strands. When dealing with a Coxeter system of rank d , the most relevant braid is the braid formed by twisting all d strands, namely $\delta = \sigma_{d-1} \cdots \sigma_2 \sigma_1$. We call δ the *(positive) all-strand rotation*. (In the literature, δ is known as a Garside element of the dual braid monoid.) See Figure 1.2 for an example of a braid diagram the all-strand rotation δ .

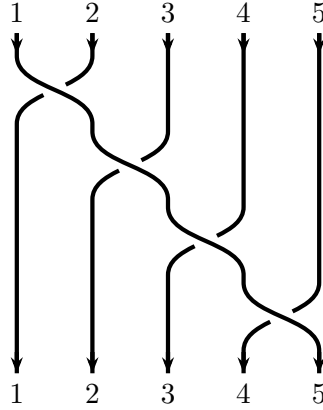


Figure 1.2: Positive all-strand rotation $\delta = \sigma_4 \sigma_3 \sigma_2 \sigma_1 \in \text{Braid}_5$

We begin with basic results, like computing the change-of-matrices induced by the Hurwitz action via δ . Let $\mathcal{B}_0 = (e_1, e_2, \dots, e_d)$ denote the standard basis of simple roots, and $\mathcal{B} = (u_1, u_2, \dots, u_d) \in \mathfrak{B}$ be some basis, say $\mathcal{B} = \beta \cdot \mathcal{B}_0$ for some braid $\beta \in \text{Braid}_d$. Let $\mathbf{C}$ denote the matrix representation of the Wall form χ_c with respect to the basis $\mathcal{B}$, i.e. the (i, j) -th entry of $\mathbf{C}$ is $\mathbf{C}_{ij} = \chi_c(u_i, u_j)$. The Hurwitz action via δ on $\mathcal{B}$ is given by the following proposition.

Proposition 1.3.6 (Change of basis given by all-strand rotation). *The Hurwitz change-of-basis matrix ${}_{\mathcal{B}}M_{\delta\mathcal{B}}$ given by the positive all-strand rotation δ may be written as the block matrix*

$${}_B M_{\delta B} = \begin{pmatrix} \mathbf{x}^\top & 1 \\ -I_{d-1} & \mathbf{0} \end{pmatrix}$$

where $\mathbf{x}^\top$ is the row vector

$$\mathbf{x}^\top = (x_1, x_2, \dots, x_{d-1})$$

whose entries are given by

$$x_k = \chi_{c^{-1}}(u_1, u_{k+1}) = \mathbf{C}_{1,k+1}.$$

Similarly, the Hurwitz change-of-basis matrix ${}_B M_{\delta^{-1}B}$ given by the negative all-strand rotation δ^{-1} may be written as the block matrix

$${}_B M_{\delta^{-1}B} = \begin{pmatrix} \mathbf{0} & -I_{d-1} \\ 1 & \mathbf{y}^\top \end{pmatrix}$$

where $\mathbf{y}^\top$ is the row vector

$$\mathbf{y}^\top = (y_1, y_2, \dots, y_{d-1})$$

whose entries are given by

$$y_k = \chi_{c^{-1}}(u_k, u_d) = \mathbf{C}_{kd}.$$

Upon further investigation, we discover periodic behavior in the change-of-basis matrices of the form ${}_{\delta^{n-1}B} M_{\delta^n B}$. Let $\mathfrak{C}$ denote the infinite block matrix where the block entries of the main diagonal are all $\mathbf{C}$, the block entries of the superdiagonal are all $\mathbf{C}^\top$, and all other entries are zero, as in Figure 1.3 below. The following proposition is a new result about the periodic behavior of the matrices ${}_{\delta^{n-1}B} M_{\delta^n B}$ with respect to the power n , with period d .

Proposition 1.3.7 (Change-of-basis matrices from $\delta^{n-1}B$ to $\delta^n B$). *For each $n \in \mathbb{Z}$, the Hurwitz change-of-basis matrix ${}_{\delta^{n-1}B} M_{\delta^n B}$ may be written as*

$$\mathfrak{C} = \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\ \dots & \mathbf{C} & \mathbf{C}^\top & \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots \\ \dots & \mathbf{0} & \mathbf{C} & \mathbf{C}^\top & \mathbf{0} & \mathbf{0} & \dots \\ \dots & \mathbf{0} & \mathbf{0} & \mathbf{C} & \mathbf{C}^\top & \mathbf{0} & \dots \\ \dots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{C} & \mathbf{C}^\top & \dots \\ \dots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{C} & \dots \\ \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Figure 1.3: The blocks forming the infinite matrix $\mathfrak{C}$

$$\delta^{n-1}\mathcal{B}M_{\delta^n\mathcal{B}} = \begin{pmatrix} x_1 & x_2 & \cdots & x_{d-1} & 1 \\ -1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -1 & 0 \end{pmatrix} = -A(x_1, x_2, \dots, x_{d-1}, 1)$$

where the row vector $\mathbf{x}^\top = (x_1, x_2, \dots, x_{d-1})$ equals the slice of the n -th row of the infinite matrix $\mathfrak{C}$, slicing inclusively from the $(n+1)$ -th column to the $(n+d-1)$ -th column, using 1-based indexing. Furthermore, we observe the following recurrence.

$$\delta^{n-1}\mathcal{B}M_{\delta^n\mathcal{B}} = \delta^{n+d-1}\mathcal{B}M_{\delta^{n+d}\mathcal{B}}$$

Each of these change-of-basis matrices takes the form of a *companion matrix*. We obtained a pleasant new formula relating the Hurwitz change-of-basis matrix ${}_{\mathcal{B}}M_{\delta^d\mathcal{B}}$ with matrix representations of the Coxeter element c , and with the upper unitriangular matrix C .

Corollary 1.3.7.1 (Relationship between ${}_{\mathcal{B}}M_{\delta^d\mathcal{B}}$ and c). *Let $\mathcal{B}_0$ denote the standard basis of simple roots. Let $[c]$ denote the matrix representation of the usual Coxeter element $c = s_1 s_2 \cdots s_d$ with respect to $\mathcal{B}_0$. Then*

$${}_{\mathcal{B}_0}M_{\delta^d\mathcal{B}_0} = (-1)^d [c] = (-1)^{d+1} C^{-1} C^\top.$$

One may analyze the associated *companion digraphs* and obtain formulas for the Hurwitz change-of-basis matrix ${}_B M_{\delta^n B}$ using known formulas for products of companion matrices. One may also apply a graph-theoretic technique called the *transfer-matrix method* to obtain explicit closed formulas for entries of our Hurwitz change-of-basis matrices. As an example, viewing the projective general linear group $\mathrm{PGL}(2, \mathbb{Z})$ as a rank-3 Coxeter group, we applied the transfer-matrix method to show new connections with Fibonacci numbers. Let G_{ij} denote the ordinary generating function for the (i, j) -th entry of the powers of the Coxeter element $[c]$. Tables 1.1 and 1.2 summarize some of the new data obtained by applying the transfer-matrix method.

n	(1, 1)	(1, 2)	(1, 3)	(2, 1)	(2, 2)	(2, 3)	(3, 1)	(3, 2)	(3, 3)
0	1	0	0	0	1	0	0	0	1
1	0	3	-2	1	3	-2	0	2	-1
2	3	5	-4	3	8	-6	2	4	-3
3	5	16	-12	8	21	-16	4	12	-9
4	16	39	-30	21	55	-42	12	30	-23
5	39	105	-80	55	144	-110	30	80	-61
6	105	272	-208	144	377	-288	80	208	-159
7	272	715	-546	377	987	-754	208	546	-417
8	715	1869	-1428	987	2584	-1974	546	1428	-1091
9	1869	4896	-3740	2584	6765	-5168	1428	3740	-2857
10	4896	12815	-9790	6765	17711	-13530	3740	9790	-7479

Table 1.1: The first several terms of each entry $([c]^n)_{ij}$ for $\mathrm{PGL}(2, \mathbb{Z})$

In the same way that half-twists are associated with univariate polynomials with connections to the Fibonacci and Chebyshev polynomials, all-strand rotation braids are associated with multivariate polynomials that are related to a generalization of the Fibonacci polynomials, called *polynomial wave sequences*. The above results about $\mathrm{PGL}(2, \mathbb{Z})$ can be seen as a special case of wave sequences.

1.4 Organization of the dissertation

This section provides a roadmap of the dissertation, to which the reader may refer when navigating the document. We summarize the content of each chapter so that the reader may identify their destinations of interest, and suggest some routes that the reader may take. See Figure 1.4 below for a dependency diagram of the chapters.

Part I lays out the relevant background for investigating the Hurwitz action, and consists of Chapters 2 through 5, which are mostly independent; Chapter 2 discusses the linear algebra of quadratic spaces, Chapters 3 and 4 discuss Coxeter theory, and Chapter 5 discusses the Hurwitz action and its connections to Artin theory. Readers may skip the chapters with whose content they are already familiar, and may proceed directly to Part II, which contains the new results of this dissertation. Those readers who want further reading on these background topics may seek the references provided near the relevant definitions and theorems. Besides discussing background material, the chapters of Part I also establish and explain the notation and conventions used throughout this dissertation, as certain objects in this dissertation are described using multiple different conventions throughout the literature. We invite even those readers familiar with the background material to return to Part I when they would like to review the notation and conventions.

Chapter 2 reviews the linear algebra of quadratic spaces. In particular, this chapter discusses Wall forms, which are (not necessarily symmetric) bilinear forms that correspond bijectively with and encode information from linear endomorphisms of a quadratic space. This chapter also clarifies conventions and notation for linear actions and matrix representations, since multiple different conventions exist in the literature.

Next, Chapter 3 provides a brief overview of Coxeter theory, while Chapter 4 explains some of the geometry associated with Coxeter groups. The first of these two chapters

reminds the reader of many relevant objects, such as Coxeter systems, Coxeter matrices, Coxeter groups, Coxeter diagrams, and Coxeter elements. The second of these chapters discusses a geometric realization called the standard geometric representation, a faithful linear representation of a Coxeter group, whereby we may view Coxeter groups as linear groups acting by (products of) reflections represented as matrices. We also briefly discuss root systems, which arise from the standard geometric representation.

Finally, Chapter 5 reviews braid groups, introduces the Hurwitz action, and discusses the relevance to Artin groups. The Hurwitz action is introduced as a braid group action on tuples of elements from an arbitrary group. As it turns out, the Hurwitz action is transitive on reduced reflection factorizations of a Coxeter element. The reflections obtained from these factorizations may be used to define the dual Artin group, providing an avenue for studying the standard Artin group.

In Part II, we begin our investigation of the Hurwitz action, consisting of Chapters 6 through 8. Chapter 6 lays out some preliminary results about the Hurwitz action as a change of basis. Chapters 7 and 8 rely on the foundation laid by Chapter 6, but are relatively independent from each other; Chapter 7 focuses on the Hurwitz action via half-twists, while Chapter 8 focuses on the Hurwitz action via rotation braids that twist a sequence of adjacent strands about each other. The results from 8 are a generalization of those in Chapter 7.

We start with Chapter 6, which lays out various preliminaries. We discuss matrix representations of Coxeter elements and their Wall forms. Fixing a Coxeter element, we may apply the Hurwitz action to obtain all reduced reflection factorizations. We show that these reduced reflection factorizations map faithfully onto ordered bases of roots in the standard geometric representation, whereby we can view the Hurwitz action as inducing a change of basis. The remaining chapters are dedicated to investigating this induced Hurwitz change-of-basis when applying certain types of braids, exploring

unexpected connections to various combinatorial objects.

Chapter 7 restricts our attention to the Hurwitz action via powers of half-twists. This chapter explicitly computes the change-of-basis matrices that arise from these powers of half-twists. These change-of-basis matrices and matrix representations exhibit connections to Fibonacci and Chebyshev polynomials; we state some relevant data, such as ordinary generating functions and closed form expressions. We may also view these Hurwitz change-of-basis matrices as acting via rotations on certain planes in the standard geometric representation of the Coxeter system.

Chapter 8 focuses on the Hurwitz action via powers of rotation braids that twist a sequence of adjacent strands, and generalizes some of the results of the previous chapter. When we twist all strands of the braid group about each other, the resulting Hurwitz change-of-basis matrix ends up being a companion matrix. With some elementary linear algebra, we reveal that the Coxeter element may be written as a product of these companion matrices, and that an appropriate Hurwitz change-of-basis matrix may be written in terms of the Coxeter element. We also apply the transfer-matrix method, a graph-theoretic and combinatorial technique, to investigate the integer and polynomial sequences arising from these Hurwitz change-of-basis matrices. Analogous to the way that half-twists relate to Fibonacci polynomials, all-strand rotation braids relate to a generalization of the Fibonacci polynomials, which we call polynomial wave sequences.

We provide Figure 1.4 below as a roadmap for navigating the dissertation. Solid arrows indicate that the chapter at the end of the arrow depends on the chapter at the start of the arrow. Dashed arrows indicate that the chapter at the start of the arrow provides useful context for the chapter at the end of the arrow.

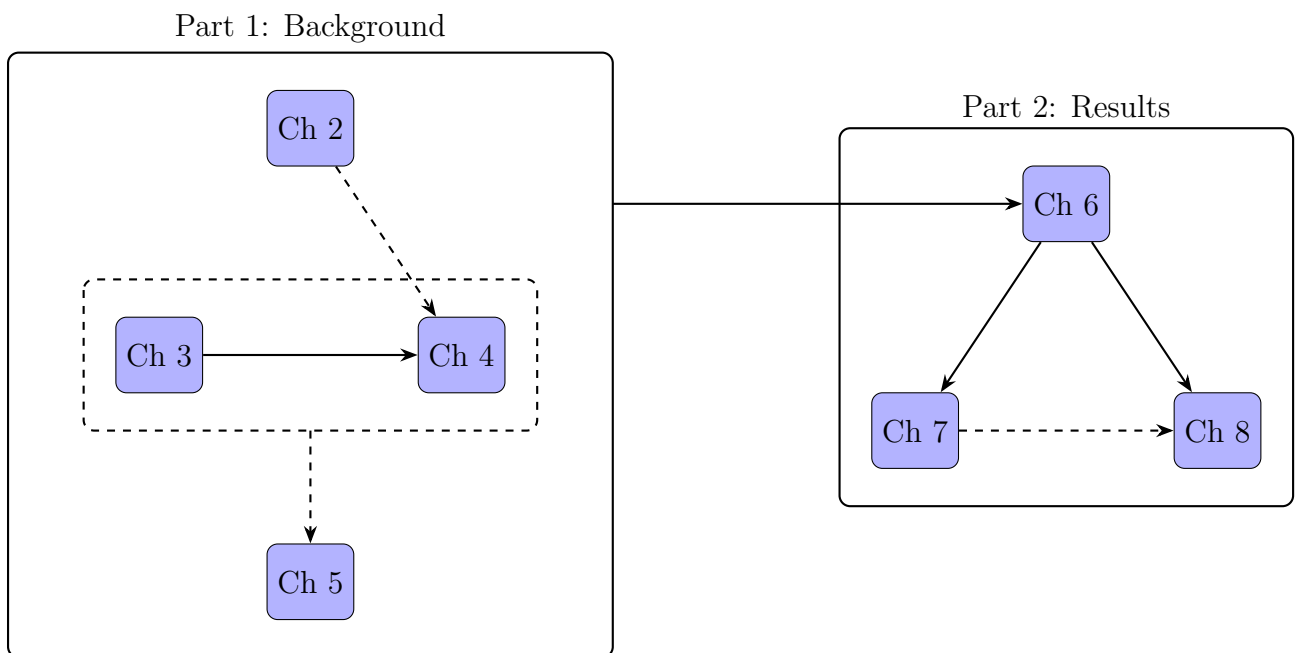


Figure 1.4: Roadmap of the dissertation

(i, j)	$G_{ij}(\lambda)$	$a_n = ([c]^n)_{ij}$
$(1, 1)$	$\frac{(\lambda - 1)^2}{(\lambda + 1)(\lambda^2 - 3\lambda + 1)}$	$a_n = f_{n+1}^2 - f_n^2 = f_{n+2}f_{n-1}$
$(1, 2)$	$-\frac{\lambda(\lambda - 3)}{(\lambda + 1)(\lambda^2 - 3\lambda + 1)}$	$a_n = f_{n+2}^2 - f_{n+1}^2 = f_{n+3}f_n$
$(1, 3)$	$-\frac{2\lambda}{(\lambda + 1)(\lambda^2 - 3\lambda + 1)}$	$a_n = -2f_nf_{n+1}$
$(2, 1)$	$\frac{\lambda}{\lambda^2 - 3\lambda + 1}$	$a_n = f_{2n}$
$(2, 2)$	$\frac{1}{\lambda^2 - 3\lambda + 1}$	$a_n = f_{2(n+1)}$
$(2, 3)$	$-\frac{2\lambda}{\lambda^2 - 3\lambda + 1}$	$a_n = -2f_{2n}$
$(3, 1)$	$\frac{2\lambda^2}{(\lambda + 1)(\lambda^2 - 3\lambda + 1)}$	$a_n = 2f_{n-1}f_n$
$(3, 2)$	$\frac{2\lambda}{(\lambda + 1)(\lambda^2 - 3\lambda + 1)}$	$a_n = 2f_nf_{n+1}$
$(3, 3)$	$-\frac{3\lambda^2 + 3\lambda - 1}{(\lambda + 1)(\lambda^2 - 3\lambda + 1)}$	$a_n = -(f_{n-1}^2 + f_{n-2}^2 + 2f_{n-1}f_n + 3f_{n-2}f_{n-1})$

Table 1.2: Data for sequences $([c]^n)_{ij}$ arising from $\text{PGL}(2, \mathbb{Z})$

Part I

Background

The chapters in this first part of the dissertation reviews the background material necessary for the new results in the second part. Chapter 2 discusses the linear algebra of quadratic spaces, including a brief explanation of Wall forms. Chapter 3 provides an introduction to Coxeter theory, with guiding examples to illustrate relevant objects like Coxeter matrices and Coxeter diagrams. Chapter 4 explains the standard geometric representation of a Coxeter group, whereby the Coxeter group acts linearly and faithfully on a real quadratic space. Chapter 5 states the definition of the Hurwitz action, and explains its relevance in constructing dual Artin groups.

Chapter 2

Quadratic spaces and reflections

This chapter discusses the linear algebra necessary for investigating the change of basis induced by the Hurwitz action. Section 2.1 clarifies many of the conventions and notation that are used throughout the dissertation, particularly regarding linear actions and matrix representations. In Section 2.2, we discuss the linear algebra of quadratic spaces. Section 2.3 focuses on reflections as linear maps in quadratic spaces, since a major point of discussion throughout the dissertation is reflection factorizations of Coxeter elements. Finally, Section 2.4 discusses Wall forms of linear endomorphisms; a Wall form (along with its moved space) encodes enough information to fully recover the original linear endomorphism.

2.1 Basics and conventions

In this section, we first review some elementary definitions and establish our notation. We also introduce various conventions, such as our choice to treat linear actions as left actions rather than right actions.

This dissertation mostly discusses real vector spaces, but we write the following gen-

eral definitions for arbitrary vector spaces.

Definition 2.1.1 (Vector space of matrices). Let $\mathbb{F}$ be a field. We denote by $\mathcal{M}_{n \times m}(\mathbb{F})$ the vector space of $n \times m$ matrices over $\mathbb{F}$. When $\mathbb{F}$ is understood, we may simply write $\mathcal{M}_{n \times m}$.

Certain matrices occur frequently throughout this dissertation, so we state their definitions in this section. One such example is the set of standard basis matrices, i.e. the standard basis elements from $\mathcal{M}_{n \times m}$.

Definition 2.1.2 (Standard basis matrix). Let $E_{ij} \in \mathcal{M}_{n \times m}$ denote the matrix whose (i, j) -th entry is 1, and whose other entries are all 0. We call E_{ij} the (i, j) -th *standard basis matrix* in $\mathcal{M}_{n \times m}$.

Remark 2.1.3 (Warning about subscripts). In Definition 2.1.2, we use the subscript to index the standard basis matrices, i.e. E_{ij} denotes the (i, j) -th standard basis matrix. However, throughout this dissertation, we also use subscripts to index the entries of a matrix, i.e. for some matrix A , we let A_{ij} denote the (i, j) -th entry of A , the entry from the i -th row and j -th column of A . We try to avoid conflicts of these two different uses of subscripts, and hope that the intention of subscripts is clear from context.

Remark 2.1.4 (Multiplying by E_{ij}). Let $M \in \mathcal{M}_{n \times m}$, and let $E_{ij} \in \mathcal{M}_{k \times m}$. Then $E_{ij}M$ is the $k \times m$ matrix whose i -th row equals the j -th row of M , and whose other entries are all zero. Similarly, letting $E_{ij} \in \mathcal{M}_{m \times k}$, we find that ME_{ij} is the $n \times k$ matrix whose j -th column equals the i -th column of M , and whose other entries are all zero.

The following lemmas are useful for computations involving E_{ij} , and are easy to verify. We use these lemmas later in this dissertation, such as the proof of Lemma 8.2.3.

Another useful matrix that occurs later in this dissertation is the reversal matrix. The definition below follows notation from [HJ12, Section 0.9.5].

Definition 2.1.5 (Reversal matrix). The $n \times n$ *reversal matrix* (or *exchange matrix*), denoted R_n , is the matrix where each entry on the antidiagonal is 1, and all other entries are zero.

$$R_n = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

Remark 2.1.6 (Conjugating by the reversal matrix). Notice that the reversal matrix is an involution, and $R_n = R_n^{-1} = R_n^\top$. Let $M \in \mathcal{M}_{n \times n}$. Left-multiplying by R_n flips the matrix M vertically, and right-multiplying by R_n flips the matrix M horizontally.

$$\begin{aligned} R_n \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} &= \begin{pmatrix} a_{n1} & a_{n2} & \cdots & a_{nn} \\ \vdots & \vdots & \ddots & \vdots \\ a_{21} & a_{22} & \cdots & a_{2n} \\ a_{11} & a_{12} & \cdots & a_{1n} \end{pmatrix} \\ \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} R_n &= \begin{pmatrix} a_{1n} & \cdots & a_{12} & a_{11} \\ a_{2n} & \cdots & a_{22} & a_{21} \\ \vdots & \ddots & \vdots & \vdots \\ a_{nn} & \cdots & a_{n2} & a_{n1} \end{pmatrix} \end{aligned}$$

Conjugating by R_n results in a π -rotation of the entries of M . We write $M^R := R_n M R_n$ to denote conjugating of M by the reversal matrix R_n .

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}^R = \begin{pmatrix} a_{nn} & \cdots & a_{n2} & a_{n1} \\ \vdots & \ddots & \vdots & \vdots \\ a_{2n} & \cdots & a_{22} & a_{21} \\ a_{1n} & \cdots & a_{12} & a_{11} \end{pmatrix}$$

Notice that the following matrix operations are all involutions, and that they pairwise commute.

- π -rotation R

- Transpose $\top$
- Matrix inverse
- Complex conjugation

The operation $\mathbf{RT}$ is equivalent to reflecting across the antidiagonal.

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}^{\mathbf{RT}} = \begin{pmatrix} a_{nn} & \cdots & a_{2n} & a_{1n} \\ \vdots & \ddots & \vdots & \vdots \\ a_{n2} & \cdots & a_{22} & a_{12} \\ a_{n1} & \cdots & a_{21} & a_{11} \end{pmatrix}$$

The reversal matrix is an example of a permutation matrix, corresponding to the permutation $\pi : [n] \rightarrow [n]$ defined by $\pi(k) = n - k + 1$. Permutation matrices arise in several places throughout this dissertation, so we state the definition below.

Definition 2.1.7 (Permutation matrix). A *permutation matrix* is a square matrix where each column and each row has exactly one entry equal to one, and where all the other entries equal zero. Given a permutation $\pi \in \text{Sym}_n$, the permutation matrix P_π associated with π is defined as the $n \times n$ matrix such that left-multiplication by P_π permutes rows by the left-action of π . Equivalently, right-multiplication by P_π permutes columns by the right-action of π .

Example 2.1.8 (Permutation matrix). Consider the permutation $\pi = (1, 3, 5) \in \text{Sym}_5$. Following Definition 2.1.7, the corresponding permutation matrix is

$$P_\pi = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

It is easy to confirm that left-multiplication permutes the rows by π , so that the i -th row of $M \in \mathcal{M}_{5 \times 5}$ equals the $\pi(i)$ -th row of $P_\pi M$.

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} \end{pmatrix} = \begin{pmatrix} a_{51} & a_{52} & a_{53} & a_{54} & a_{55} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \end{pmatrix}$$

Right-multiplying by the same matrix P_π permutes the columns by π^{-1} , so that the i -th column of M equals the $\pi^{-1}(i)$ -th column of MP_π .

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} a_{13} & a_{12} & a_{15} & a_{14} & a_{11} \\ a_{23} & a_{22} & a_{25} & a_{24} & a_{21} \\ a_{33} & a_{32} & a_{35} & a_{34} & a_{31} \\ a_{43} & a_{42} & a_{45} & a_{44} & a_{41} \\ a_{53} & a_{52} & a_{55} & a_{54} & a_{51} \end{pmatrix}$$

We now turn our attention to matrix representations of vectors, linear transformations, and bilinear forms. Let $\mathbb{F}$ be a field, and let V and W be finite-dimensional vector spaces over $\mathbb{F}$. Let $\mathcal{B} = (v_1, \dots, v_n)$ be an ordered basis for V and let $\mathcal{B}' = (w_1, \dots, w_m)$ be an ordered basis for some W . For the definitions below, we adapt the notation used in [HJ12, Section 0.10].

Definition 2.1.9 (Representations of vectors). Say $v \in V$ with

$$v = \sum_{i=1}^n \alpha_i v_i.$$

The $\mathcal{B}$ -coordinate representation of v is the column vector

$$_{\mathcal{B}}[v] = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix},$$

whereby we may identify V with $\mathbb{F}^n$. We may want to use the row vector representation, denoted by

$$[v]_{\mathcal{B}}^{\top} = [\alpha_1 \quad \alpha_2 \quad \cdots \quad \alpha_n].$$

(Notice that the basis is indicated as a left subscript for column vectors, and as a right subscript for row vectors.) When the basis $\mathcal{B}$ is understood, we may simplify notation by writing $[v]$ or $\mathbf{v}$ instead of $_{\mathcal{B}}[v]$.

Definition 2.1.10 (Representations of linear transformations). Let $T : V \rightarrow W$ be a linear transformation. The $\mathcal{B}$ - $\mathcal{B}'$ *basis representation* of T is the $m \times n$ matrix whose i -th column is the column vector $_{\mathcal{B}'}[T(v_i)]$, the $\mathcal{B}'$ -coordinate representation of $T(v_i)$. We denote the $\mathcal{B}$ - $\mathcal{B}'$ basis representation of T by

$$_{\mathcal{B}'}[T]_{\mathcal{B}} = \begin{bmatrix} _{\mathcal{B}'}[T(v_1)] & _{\mathcal{B}'}[T(v_2)] & \cdots & _{\mathcal{B}'}[T(v_n)] \end{bmatrix}.$$

In the special case where $W = V$ and $\mathcal{B}' = \mathcal{B}$, we call $_{\mathcal{B}}[T]_{\mathcal{B}}$ the $\mathcal{B}$ *basis representation* of T , and may write $[T]_{\mathcal{B}}$ for brevity.

Remark 2.1.11 (Warning about notation). In the name “ $\mathcal{B}$ - $\mathcal{B}'$ basis representation,” the input basis $\mathcal{B}$ for the space V is on the left, and the output basis $\mathcal{B}'$ for the space W is on the right. This ordering corresponds to how we write $T : V \rightarrow W$, where the domain V is on the left and the codomain W is on the right. However, in the notation $_{\mathcal{B}'}[T]_{\mathcal{B}}$, the input basis $\mathcal{B}$ is on the right, and the output basis $\mathcal{B}'$ is on the left. We treat the matrix $_{\mathcal{B}'}[T]_{\mathcal{B}}$ as acting on the left of column vectors, so vectors that appear on the right of $_{\mathcal{B}'}[T]_{\mathcal{B}}$ come from V and are written with respect to $\mathcal{B}$, hence the notation.

Remark 2.1.12 (Left vs right actions). A linear transformation $T : V \rightarrow W$ between finite-dimensional vector spaces may be written as a matrix A with respect to fixed bases of V and W . However, there are two possible conventions for writing the matrix. For $x \in V$, let $\mathbf{x}$ denote the column vector representing x with respect to some fixed basis of V . We may represent T as a matrix A that

1. acts on the left of column vectors, i.e. $T(x) = A\mathbf{x}$, or
2. acts on the right of row vectors, i.e. $T(x) = \mathbf{x}^\top A^\top$.

Following [HJ12], Definition 2.1.10 and the corresponding notation ${}_{\mathcal{B}'}[T]_{\mathcal{B}}$ assumes the first convention, where input vectors are written with respect to $\mathcal{B}$, and output vectors are written with respect to $\mathcal{B}'$. In other words, if $T(x) = y$, then we have

$${}_{\mathcal{B}'}[T]_{\mathcal{B}}[x] = {}_{\mathcal{B}'}[y].$$

Later in this dissertation, a key result from the literature (see Theorem 6.1.1) is stated using the first convention, following [How82, Theorem 2.1] and [HK16, Proposition 2.5]. This convention also agrees with our conventions for the standard geometric representation $\rho : W \rightarrow GL(V)$ discussed in Section 4.1, which follows [Hum90] by using functional notation, as discussed in Remark 3.1.4. Hence, throughout this dissertation, unless otherwise stated, we represent linear transformations as matrices acting on the left of column vectors. Similarly, when a bilinear form B is represented by some matrix M with respect to some basis, we mean $B(u, v) = \mathbf{u}^\top M \mathbf{v}$.

Remark 2.1.13 (Notation for composition of linear maps). Consider linear maps $T : V \rightarrow W$ and $S : W \rightarrow Z$ with bases $\mathcal{B}$, $\mathcal{B}'$, and $\mathcal{B}''$ for V , W , and Z , respectively. The matrix representation of the composition $S \circ T$ with respect to these bases is

$${}_{\mathcal{B}''}[S \circ T]_{\mathcal{B}} = {}_{\mathcal{B}''}[S]_{\mathcal{B}'} {}_{\mathcal{B}'}[T]_{\mathcal{B}}.$$

The above notation has a redundant subscript $\mathcal{B}'$ between S and T , which we may omit for visual clarity. For example, for $z = (S \circ T)(v)$, we may write

$${}_{\mathcal{B}''}[z] = {}_{\mathcal{B}''}[S \circ T]_{\mathcal{B}}[v] = {}_{\mathcal{B}''}[S]_{\mathcal{B}'}[T]_{\mathcal{B}}[v].$$

Having defined matrix representations for vectors and linear transformations, we state

the definition of matrix representations for bilinear forms. Let $\mathcal{B} = (v_1, \dots, v_n)$ and $\mathcal{B}' = (w_1, \dots, w_n)$ be two bases of V .

Definition 2.1.14 (Representations of bilinear forms). Let $B : V \times V \rightarrow \mathbb{F}$ be a bilinear form. The $\mathcal{B}$ - $\mathcal{B}'$ *basis representation* of B is the $n \times n$ matrix ${}_{\mathcal{B}'}[B]_{\mathcal{B}}$ whose (i, j) -th entry equals $B(w_i, v_j)$, whereby for each $x, y \in V$, we have

$$B(x, y) = [x]_{\mathcal{B}'}^{\top} [B]_{\mathcal{B}} [y].$$

In the special case where $\mathcal{B}' = \mathcal{B}$, we call ${}_{\mathcal{B}}[B]_{\mathcal{B}}$ the $\mathcal{B}$ *basis representation* of B , and may write $[B]_{\mathcal{B}}$ for brevity.

Definition 2.1.15 (Change of basis). Let $I : V \rightarrow V$ denote the identity map. The $\mathcal{B}$ - $\mathcal{B}'$ *change-of-basis matrix* is ${}_{\mathcal{B}'}[I]_{\mathcal{B}}$. Notice that $({}_{\mathcal{B}'}[I]_{\mathcal{B}})^{-1} = {}_{\mathcal{B}}[I]_{\mathcal{B}'}$.

Given a linear transformation $T : V \rightarrow V$, we may use change-of-basis matrices to compute its representation in a new basis.

$${}_{\mathcal{B}'}[T]_{\mathcal{B}'} = {}_{\mathcal{B}'}[I]_{\mathcal{B}} [T]_{\mathcal{B}} [I]_{\mathcal{B}'} = ({}_{\mathcal{B}}[I]_{\mathcal{B}'})^{-1} ({}_{\mathcal{B}}[T]_{\mathcal{B}}) ({}_{\mathcal{B}}[I]_{\mathcal{B}'})$$

Similarly, given a bilinear form $B : V \times V \rightarrow \mathbb{F}$, we may compute its matrix representation in a different basis.

$${}_{\mathcal{B}'}[B]_{\mathcal{B}'} = ({}_{\mathcal{B}}[I]_{\mathcal{B}'})^{\top} ({}_{\mathcal{B}}[B]_{\mathcal{B}}) ({}_{\mathcal{B}}[I]_{\mathcal{B}'})$$

2.2 Quadratic spaces

We now review the linear algebra of quadratic spaces. The quadratic spaces discussed in this dissertation are real vector spaces, but we provide the general definitions for vector spaces over arbitrary fields.

Definition 2.2.1 (Quadratic spaces). Let V be a finite-dimensional vector space over a field $\mathbb{F}$. A *quadratic form* on V is a map $Q : V \rightarrow \mathbb{F}$ satisfying the following:

- For each $a \in \mathbb{F}$ and $v \in V$, we have $Q(av) = a^2Q(v)$.
- The map $\beta(u, v) = Q(u + v) - Q(u) - Q(v)$ is bilinear.

The pair (V, Q) is called a *quadratic space*, and the symmetric bilinear form β is called the *polar form* of Q .

Definition 2.2.2 (Definite forms). Let (V, Q) be a quadratic space over $\mathbb{R}$. The quadratic form Q is called *positive definite* (respectively, *negative definite*) if, for each nonzero $v \in V$, we have $Q(v) > 0$ (resp. $Q(v) < 0$). Similarly, Q is called *positive semidefinite* (resp. *negative semidefinite*) if, for each nonzero $v \in V$, we have $Q(v) \geq 0$ (resp. $Q(v) \leq 0$). We say that Q is *definite* (resp. *semidefinite*) if it is positive or negative definite (resp. positive or negative semidefinite); otherwise, we say that Q is *indefinite*.

Up to this point, we let $\mathbb{F}$ be an arbitrary field. For the rest of this chapter, we impose a few minor restrictions. Let $\mathbb{F}$ be a field with characteristic greater than 2, and let (V, Q) be a *non-degenerate* quadratic form, i.e. the polar form β is non-degenerate, i.e. if for each $v \in V$ we have $\beta(u, v) = 0$, then $u = 0$.

Definition 2.2.3 (Singular vectors). A nonzero vector $u \in V$ is called *isotropic* if $\beta(u, u) = 0$, and is called *singular* if $Q(u) = 0$. Since $\mathbb{F}$ has characteristic greater than 2, these two definitions coincide.

Definition 2.2.4 (Orthogonal group). The *orthogonal group* of a quadratic space (V, Q) is

$$O(V, Q) := \{f \in \text{GL}(V) \mid \forall v \in V, Q(f(v)) = Q(v)\}.$$

The elements of $O(V, Q)$ are called *(linear) isometries*. We may write $O(V)$ for brevity.

Definition 2.2.5 (Special orthogonal group). The *special orthogonal group* of a quadratic space (V, Q) is

$$\mathrm{SO}(V, Q) := \{f \in \mathrm{O}(V, Q) \mid \det(f) = 1\}.$$

The elements of $\mathrm{SO}(V, Q)$ are called *rotations*, and $\mathrm{SO}(V, Q)$ may be called the *rotation group* of (V, Q) . When Q is understood, we may write $\mathrm{SO}(V)$ for brevity.

Definition 2.2.6 (Orthogonal subspace). Given a linear subspace $W \subseteq V$, the corresponding *orthogonal subspace* is

$$W^\perp := \{v \in V \mid \forall w \in W, \beta(v, w) = 0\}.$$

Definition 2.2.7 (Fixed and moved space). The *fixed space* of $f \in \mathrm{O}(V)$ is $\mathrm{Fix}(f) := \ker(\mathrm{id} - f)$, and the *moved space* of f is $\mathrm{Mov}(f) = \mathrm{im}(\mathrm{id} - f)$.

Definition 2.2.8 (Bilinear forms). Let V be a vector space over a field $\mathbb{F}$. A bilinear form is a bilinear map $B : V \times V \rightarrow \mathbb{F}$. We say that B is *symmetric* when for each $u, v \in V$, we have $B(u, v) = B(v, u)$.

Remark 2.2.9 (Quadratic forms and symmetric bilinear forms). Quadratic forms Q over V correspond with symmetric bilinear forms B over V via $B = \frac{1}{2}\beta$, whereby $Q(v) = B(v, v)$. When referring to a quadratic space (V, Q) , we may equivalently refer to the space by referring to its bilinear form, writing (V, B) . A symmetric bilinear form B is called *definite*, *semidefinite*, etc. according to its corresponding quadratic form Q .

2.3 Reflections

This section reviews the linear algebra of reflections in quadratic spaces.

Definition 2.3.1 (Reflections). Let (V, Q) be a quadratic space. A *reflection* is a non-trivial isometry $r \in O(V)$ that fixes pointwise some codimension-1 subspace (a *hyperplane*).

Remark 2.3.2 (Basic facts about reflections). Every reflection in $O(V)$ can be written in the form

$$r_v(u) = u - \frac{\beta(u, v)}{Q(v)}v$$

for some non-singular v , and r_v is called the reflection with respect to v . Notice the following facts about reflections.

- The reflection r_v is an involution that fixes the hyperplane $\langle v \rangle^\perp$, sends v to $-v$, and has determinant -1 .
- For each nonzero scalar multiple w of v , we have $r_v = r_w$.
- The set of reflections is closed under conjugation by isometries. In particular, for each $f \in O(V)$, we have $fr_vf^{-1} = r_{f(v)}$.

Definition 2.3.3 (Reflection groups). A group $G \leq O(V)$ generated by reflections is called a *reflection group*.

Definition 2.3.4 (Reflection length). If $f \in O(V)$ can be written as a product of reflections, we say its *reflection length* is the minimal length k of a factorization $f = r_1r_2 \cdots r_k$ as a product of reflections.

The following theorem [MP22, Theorem 2.7] provides a formula for the reflection length of any isometry.

Theorem 2.3.5 (Reflection length). *Suppose $\mathbb{F} \neq \mathbb{F}_2$, let (V, Q) be a non-degenerate quadratic space over $\mathbb{F}$, and let $f \in O(V, Q)$ be an isometry different from the identity. Then f may be written as a product of reflections, where the reflection length is*

- $\dim(\text{Mov}(f))$ if $\text{Mov}(f)$ is not totally singular, and
- $\dim(\text{Mov}(f)) + 2$ otherwise.

In particular, f can be written as a product of at most $\dim(V)$ reflections.

For a thorough discussion of reflection factorizations in quadratic spaces over fields with characteristic greater than 2, see [MP22].

2.4 Wall forms

Introduced in [Wal59], the Wall form of a linear isometry f is a (not necessarily symmetric) bilinear form on the moved space $\text{Mov}(f)$ that encodes enough information to recover f . Later in this dissertation, we focus on reflection factorizations of Coxeter elements, and find that the Wall form of a Coxeter element is a useful way to encode information about the Coxeter element itself. We discuss the Wall form using the notation of [Tay92, Chapter 11], which differs from the original notation in [Wal59] and [Wal63].

As mentioned in Definition 2.2.7, each linear isometry $f \in O(V)$ comes with a fixed space $\text{Fix}(f)$ of vectors in V fixed by f , as well as a moved space $\text{Mov}(f)$ consisting of “movement vectors” $f(u) - u$ for $u \in V$. The Wall form is a bilinear form χ_f on $\text{Mov}(f)$. In Theorem 2.4.3 below, we see that the Wall form (along with the moved space) encodes enough information to completely recover the original isometry f .

Definition 2.4.1 (Wall form). The *Wall form* of a linear isometry $f \in O(V, Q)$ is the bilinear form χ_f on $\text{Mov}(f)$ defined as

$$\chi_f(u, v) = \beta(w, v),$$

where β is the polar form of the quadratic form Q , and where $w \in V$ is any vector satisfying $u = w - f(w)$.

Since the Wall form is not necessarily symmetric, there is a notion of left and right orthogonal complement with respect to the form.

Definition 2.4.2 (Orthogonal complements). Let χ be a non-degenerate bilinear form on a finite-dimensional vector space W . The *left* and *right orthogonal complement* of a subspace $U \subseteq W$ with respect to χ are

$$\begin{aligned} U^\triangleleft &:= \{v \in W \mid \forall u \in U, \chi(v, u) = 0\} \\ U^\triangleright &:= \{v \in W \mid \forall u \in U, \chi(u, v) = 0\} \end{aligned}$$

respectively.

The following theorem shows that the Wall form χ_f is sufficient to recover the linear isometry f . For the proof of this theorem, see [Tay92, Theorem 11.32] or [MP22, Theorem 1.6].

Theorem 2.4.3 (Wall's parametrization). *The map $f \mapsto (\text{Mov}(f), \chi_f)$ is a one-to-one correspondence between the orthogonal group $O(V)$ and the set of pairs (W, χ) such that W is a subspace of V and χ is a non-degenerate bilinear form on W satisfying $\chi(u, u) = Q(u)$ for each $u \in W$.*

The following lemma from [Tay92, Lemma 11.34] lists several basic properties of the Wall form. One may also find this lemma in [MP22, Lemma 1.7].

Lemma 2.4.4. *For each $f \in O(V)$ and $u, v \in \text{Mov}(f)$, the following properties hold.*

- (i) $\chi_f(u, v) + \chi_f(v, u) = \beta(u, v)$,
- (ii) $\chi_f(f(u), v) = -\chi_f(v, u)$,
- (iii) $\text{Mov}(f) = \text{Mov}(f^{-1})$ and $\chi_{f^{-1}}(u, v) = \chi_f(v, u)$,

(iv) for each $g \in O(V)$, we have $\text{Mov}(gfg^{-1}) = g(\text{Mov}(f))$ and $\chi_{gfg^{-1}}(g(u), g(v)) = \chi_f(u, v)$.

The following lemma explains what happens to a linear isometry when left- or right-multiplied by a reflection, and is used in the proof of Theorem 2.3.5. For a proof of this lemma, see [MP22, Lemma 2.6].

Lemma 2.4.5. *Let $f \in O(V)$ be an isometry, and let $v \in V$ be a non-singular vector.*

- (a) *If $v \in \text{Mov}(f)$, then $\text{Mov}(r_v f) = \langle v \rangle^\perp$, where the right orthogonal complement is taken inside $\text{Mov}(f)$ with respect to the Wall form χ_f . In particular, $\dim(\text{Mov}(r_v f)) = \dim(\text{Mov}(f)) - 1$.*
- (b) *If $v \notin \text{Mov}(f)$, then $\text{Mov}(r_v f) = \text{Mov}(f) \oplus \langle v \rangle$. In particular, $\dim(\text{Mov}(r_v f)) = \dim(\text{Mov}(f)) + 1$.*

Chapter 3

Introduction to Coxeter theory

Coxeter groups are important objects in geometric group theory, with a storied history tracing back to when H.S.M Coxeter himself introduced finite and affine Coxeter groups in [Cox34] as generalizations of reflection groups, and classified the finite Coxeter groups in [Cox35]. Bourbaki's thorough treatment of Coxeter groups in 1968 [Bou68] established a faithful linear representation for general Coxeter groups and illustrated the deep connections between Coxeter groups and semisimple Lie theory. (For an English translation of the original French text by Bourbaki, see [Bou08].) Coxeter groups are well understood.

Coxeter groups serve as the algebraic foundation of this dissertation, so this chapter reviews the Coxeter theory needed to investigate the Hurwitz action on reflection factorizations of Coxeter elements. Section 3.1 discusses symmetric groups as a motivating example of Coxeter groups. Section 3.2 states the abstract definition of Coxeter groups, and discusses various objects associated with Coxeter groups, such as Coxeter matrices and Coxeter diagrams. Section 3.3 briefly discusses Coxeter elements, a central object of study in this dissertation. For further reading on Coxeter groups, see [Hum90].

3.1 Symmetric groups

In this section, we examine the symmetric groups as a motivating example. We first establish some conventions for discussing permutations and symmetric groups, then present the usual group presentation for the symmetric group. This group presentation can be generalized to obtain the formal definition of a Coxeter group.

Let A be a set, and let $n \in \mathbb{N}$. Following [Sta11, Example 1.1.1], we write $[n] := \{1, 2, \dots, n\}$. Below, we state the definitions of permutation and symmetric group.

Definition 3.1.1 (Permutation). A *permutation* of a set A is a bijection from A to itself.

Definition 3.1.2 (Symmetric group). The symmetric group Sym_A , called the *symmetric group on A* , is the group of permutations of A under composition. We use the notation Sym_n to denote the symmetric group on $[n]$, and call this group the *symmetric group on n elements*, or the *symmetric group of degree n* .

Recall that each $s \in \text{Sym}_n$ may be written as a product of cycles. We follow the cycle notation used in [DF03, Section 1.3].

Definition 3.1.3 (Cycle notation). The cycle $(a_1, a_2, \dots, a_m)$ is the permutation that, for each $i \in [m]$, sends a_i to a_{i+1} , and sends a_m to a_1 . A cycle of length m is called an *m -cycle*.

Remark 3.1.4 (Functional vs algebraist notation). Let $\sigma, \tau \in \text{Sym}_A$. As noted in [McC06, Section 4], there are two natural conventions to denote composition of permutations. In *functional notation*, $\sigma\tau$ denotes the permutation that sends x to $\sigma(\tau(x))$, whereby we read permutations *from right to left*. For example, the product of cycles $(1, 2)(1, 3)$ equals $(1, 3, 2)$. However, in *algebraist notation*, permutations are read *from left to right*, whereby the product $(1, 2)(1, 3)$ equals $(1, 2, 3)$. Following [DF03, Section 1.3], we use functional notation for permutations throughout this dissertation. We also use functional

notation to describe other objects of interest throughout this dissertation. For example, for the standard geometric representation $\rho : W \rightarrow \text{GL}(V)$ discussed in Section 4.1 we follow [Hum90] by seeing elements of the Coxeter group W as functions, whereby $(w_1 w_2 w_3)(v) = w_1(w_2(w_3(v)))$. Although this convention may be somewhat unintuitive since it reads from right to left (whereas English is read from left to right), this convention keeps our notation consistent with some relevant results from the literature. For example, this dissertation uses results from [How82] and [HK16], both of which treat elements of Coxeter groups as functions, reading multiplication of these elements as function composition from right to left.

Transpositions are permutations that can be written as 2-cycles. We may use transpositions to generate the symmetric group.

Definition 3.1.5 (Transpositions). A *transposition* is a permutation that swaps two elements and leaves the other elements fixed. In cycle notation, a transposition is a 2-cycle. An *adjacent transposition* in Sym_n is a transposition of the form $(i, i + 1)$.

Remark 3.1.6 (Generating a symmetric group by adjacent transpositions). Let $s_i \in \text{Sym}_{d+1}$ denote the transposition $(i, i + 1)$. The symmetric group Sym_{d+1} has the following group presentation in terms of adjacent transpositions.

$$\text{Sym}_{d+1} = \left\langle s_1, s_2, \dots, s_d \mid \forall i, j \in [d], \begin{array}{l} s_i^2 = \text{id}, \\ s_i s_j = s_j s_i \text{ if } |i - j| > 1, \\ s_i s_j s_i = s_j s_i s_j \text{ if } |i - j| = 1 \end{array} \right\rangle$$

Example 3.1.7 (Symmetric group on 4 elements). The symmetric group on 4 elements has the following presentation in terms of relations.

$$\begin{aligned} \text{Sym}_4 &= \left\langle s_1, s_2, s_3 \mid \begin{array}{l} s_1^2 = s_2^2 = s_3^2 = \text{id}, \quad s_1 s_2 s_1 = s_2 s_1 s_2, \\ s_2 s_3 s_2 = s_3 s_2 s_3, \quad s_1 s_3 = s_3 s_1 \end{array} \right\rangle \\ &= \langle s_1, s_2, s_3 \mid s_1^2 = s_2^2 = s_3^2 = (s_1 s_2)^3 = (s_2 s_3)^3 = (s_1 s_3)^2 = \text{id} \rangle \end{aligned}$$

We may also write the presentation in terms of relators.

$$\text{Sym}_4 = \langle s_1, s_2, s_3 \mid s_1^2, s_2^2, s_3^2, (s_1 s_2)^3, (s_2 s_3)^3, (s_1 s_3)^2 \rangle$$

The following example illustrates our choice of conventions for computing and visualizing products of adjacent transpositions.

Example 3.1.8 (Visualizing products of adjacent transpositions). We may visualize a product of adjacent transpositions as a set of strands, as in Figure 3.1, without distinguishing between over- and under-crossings of strands, with the domain at the top of the diagram and the image at the bottom. This figure represents the permutation

$$(2, 3)(3, 4)(1, 2)(2, 3)(3, 4) = (1, 3, 2, 4).$$

We emphasize that the transpositions at the *top* of the diagram are the transpositions that are applied *first*, i.e. the transpositions on the *right* side of the product (in functional notation). Figure 3.2 reorients the diagram so that the composition of transpositions can be read in the same order as the product in functional notation; the domain is on the right, and the image on the left.

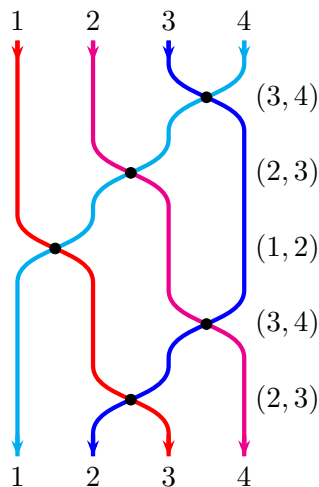


Figure 3.1: Strands representing a product of adjacent transpositions

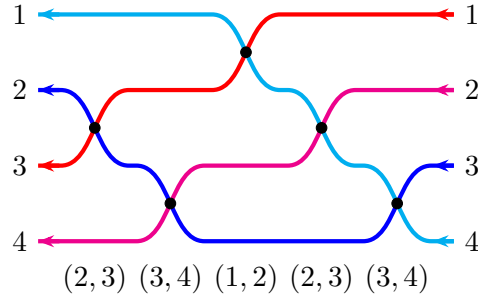


Figure 3.2: Reoriented permutation diagram

3.2 Coxeter groups

This section arrives at the formal definition of Coxeter groups by generalizing the presentation of symmetric groups. We also present some structures associated with Coxeter groups, such as the Coxeter matrix and the Coxeter diagram. Finally, we introduce Coxeter elements, which play a crucial role throughout this dissertation.

As discussed in Section 3.1, symmetric groups are groups of permutations, and can be generated by the set of adjacent transpositions. Recall from Remark 3.1.6 that symmetric groups may be defined in terms of an abstract group presentation; the generators are all involutions, and the defining relations are of the form $(s_i s_j)^2 = \text{id}$ or $(s_i s_j)^3 = \text{id}$. This definition may be generalized as follows: taking a finite collection of generating involutions $S = (s_1, s_2, \dots, s_d)$, we impose relations of the form $(s_i s_j)^{m_{ij}}$ where $m_{ij} \in \{2, 3, \dots, \infty\}$. This generalization is precisely the formal definition of a Coxeter group.

Definition 3.2.1 (Coxeter systems and Coxeter groups). Let S be a finite set. A *Coxeter system* is a pair (W, S) where W is a group that can be written with a presentation

$$W = \langle S \mid \forall s, t \in S, (st)^{m(s,t)} = \text{id} \rangle$$

satisfying the following:

- For each $s \in S$, we have $m(s, s) = 1$, i.e. each s is an involution, and

- For each pair $s, t \in S$, we have $m(s, t) = m(t, s) \in \{2, 3, \dots, \infty\}$.
- $m(s, t) = \infty$ means that there is no relation between s and t .

The group W is called a *Coxeter group*, and S is called the *(standard) generating set* of (W, S) . The cardinality of S is called the *rank* of (W, S) . We may write m_{st} and $m(s, t)$ interchangeably.

Viewed as a Coxeter group, the symmetric group Sym_{d+1} of degree $d+1$ is a Coxeter group of rank d ; it is generated by d involutions (the adjacent transpositions) whose pairwise relations satisfy the constraints above.

Remark 3.2.2 (Notation for Coxeter systems). We use the following notation and conventions.

- When the Coxeter system (W, S) is understood, we may refer to only the Coxeter group W for brevity.
- We may also write (W, S) to emphasize the generating set S when discussing the group W .
- Let $S = \{s_1, s_2, \dots, s_d\}$. We may use the indices instead of the elements themselves, i.e. we write $m_{ij} = m(s_i, s_j)$.
- If we need to fix the order of the elements of S , then we may write S as a tuple, i.e. $S = (s_1, s_2, \dots, s_d)$.

Let (W, S) be a Coxeter system, and let $I \subseteq S$. Let W_I be the subgroup of W generated by I . Consider the Coxeter system $(\overline{W_I}, I)$ that inherits the values m_{st} from (W, S) . Then W_I is canonically isomorphic to the Coxeter group $\overline{W_I}$, and is called a *parabolic subgroup* of (W, S) . A more complete discussion of parabolic subgroups can be found in [Hum90, Section 5.5] and [BB06, Section 2.4].

Theorem 3.2.3 (Parabolic subgroups). *For each $I \subseteq S$, the parabolic subgroup (W_I, I) inheriting the values m_{st} from (W, S) forms a Coxeter system.*

Furthermore, the parabolic subgroups of (W, S) form a sublattice of the subgroup lattice of W , and the map $I \mapsto W_I$ induces a lattice isomorphism from the subsets of S to the parabolic subgroups of (W, S) .

The Coxeter system (W, S) may be encoded in several equivalent forms, including the Coxeter matrix and the Coxeter diagram. We state the definitions below.

Definition 3.2.4 (Coxeter matrix). Let S be a finite set. A *Coxeter matrix* is a square matrix M indexed by S , satisfying the following:

- For each $s \in S$, we have $M_{ss} = 1$.
- For each $s, t \in S$, we have $M_{st} = M_{ts} \in \{2, 3, \dots, \infty\}$.

A Coxeter system (W, S) corresponds with the Coxeter matrix given by $M_{st} = m_{st}$.

Definition 3.2.5 (Coxeter diagram). Let S be a finite set. A *Coxeter diagram* (or *Coxeter graph*) is an edge-labeled simple graph Γ whose vertex set is S , and where each edge $\{s, t\}$ is either unlabeled or adorned with a label from the set $\{4, 5, \dots, \infty\}$.

A Coxeter system (W, S) corresponds with the Coxeter diagram whose vertex set is S , and whose edge set E is described as follows:

- For each $s \neq t$, we have $\{s, t\} \notin E$ if and only if $m_{st} = 2$.
- For each $s \neq t$, we have an unlabeled edge $\{s, t\} \in E$ if and only if $m_{st} = 3$.
- For each $s \neq t$, we have a labeled edge $\{s, t\} \in E$ adorned with the label m_{st} if and only if $m_{st} \geq 4$.

A Coxeter diagram encodes all the relation lengths m_{st} necessary to recover the Coxeter group itself. We may write $\text{Cox}(\Gamma)$ to denote the Coxeter group corresponding to the Coxeter diagram Γ . We say $\text{Cox}(\Gamma)$ is *irreducible* if Γ is connected.

Remark 3.2.6 (Subdiagrams and parabolic subgroups). Let Γ be a Coxeter diagram on the vertex set S , and let Γ_I be the subgraph induced by a subset $I \subseteq S$. Then Γ_I is a Coxeter diagram, and $W_I = \text{Cox}(\Gamma_I)$ is the parabolic subgroup (of $W = \text{Cox}(\Gamma)$) generated by I . Indeed, this relationship between subdiagrams of Γ and parabolic subgroups of $\text{Cox}(\Gamma)$ is simply a rewording of the lattice isomorphism from Theorem 3.2.3.

In Coxeter and Artin theory, there is an alternative way to write the Coxeter diagram. The “classical” way to encode a Coxeter system as a graph is the Coxeter diagram defined in Definition 3.2.5, and are written to emphasize the connection to Dynkin diagrams in the classification of semisimple Lie groups, as in [Bou08]. The “modern” way is what we call the *presentation diagram*, and is used in the study of right-angled Artin groups, for example. Both approaches have their advantages, but for consistency, any Coxeter graph in this dissertation can be assumed to be the classical Coxeter diagram unless otherwise specified. For completeness, we state the definition of presentation diagram below.

Definition 3.2.7 (Presentation diagram). Let S be a finite set. A *presentation diagram* is an edge-labeled simple graph Γ whose vertex set is S , and where each edge $\{s, t\}$ is either unlabeled or adorned with a label from the set $\{3, 4, \dots\}$.

A Coxeter system (W, S) corresponds with the presentation diagram whose vertex set is S , and whose edge set E is described as follows:

- For each $s \neq t$, we have $\{s, t\} \notin E$ if and only if $m_{st} = \infty$.
- For each $s \neq t$, we have an unlabeled edge $\{s, t\} \in E$ if and only if $m_{st} = 2$.
- For each $s \neq t$, we have a labeled edge $\{s, t\} \in E$ adorned with the label m_{st} if and only if $3 \leq m_{st} < \infty$.

Example 3.2.8 (Coxeter matrix and diagram for Sym_4). The symmetric group Sym_4 from Example 3.1.7, is a Coxeter group. Looking at the values m_{ij} for Sym_4 , we see that the corresponding Coxeter matrix is

$$\begin{pmatrix} 1 & 3 & 2 \\ 3 & 1 & 3 \\ 2 & 3 & 1 \end{pmatrix}.$$

The Coxeter diagram and presentation diagram for Sym_4 are shown in Figure 3.3.



Figure 3.3: Coxeter diagram Γ and presentation diagram Γ' of Sym_4

Let us present a few more examples of Coxeter groups. This chapter returns repeatedly to these examples.

Example 3.2.9 (Sym_{d+1} as a Coxeter group). We have already seen from Remark 3.1.6 that symmetric groups are Coxeter groups whose standard generating set consists of the adjacent transpositions. The Coxeter matrix M of the symmetric group Sym_{d+1} is given by

$$M_{ij} = \begin{cases} 1, & i = j \\ 2, & |i - j| > 1 \\ 3, & |i - j| = 1 \end{cases}$$

and the Coxeter diagram is the path graph on d vertices with no edge labels.

Remark 3.2.10 (Diagrams of finite Coxeter groups). In the classification of irreducible finite Coxeter groups, the path graph on n vertices is called the diagram of type A_n , so we may write $\text{Sym}_{d+1} = \text{Cox}(A_d)$. We briefly discuss finite Coxeter groups (and other special cases) in Section 4.3. See Figure 4.1 for the diagrams of all irreducible finite Coxeter groups.

Example 3.2.11 ($\mathrm{PGL}(2, \mathbb{Z})$ as a Coxeter group). Let $S = (s_1, s_2, s_3)$, and consider the Coxeter system (W, S) given by $m_{12} = 3$, $m_{13} = 2$, and $m_{23} = \infty$, i.e. by the Coxeter matrix

$$\begin{pmatrix} 1 & 3 & 2 \\ 3 & 1 & \infty \\ 2 & \infty & 1 \end{pmatrix}.$$

The corresponding Coxeter diagram Γ is shown in Figure 3.4. As it turns out, W is isomorphic to the projective general linear group $\mathrm{PGL}(2, \mathbb{Z})$. Denoting the canonical map $\mathrm{GL}(2, \mathbb{Z}) \rightarrow \mathrm{PGL}(2, \mathbb{Z})$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

the assignment

$$s_1 \mapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad s_2 \mapsto \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}, \quad s_3 \mapsto \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

induces an isomorphism $W \rightarrow \mathrm{PGL}(2, \mathbb{Z})$. For a more detailed discussion, see [Hum90, Chapter 5, Example 2].

$$\Gamma = \begin{array}{c} \bullet \text{---} \bullet \text{---} \bullet \\ s_1 \quad s_2 \quad s_3 \end{array}$$

∞

Figure 3.4: Coxeter diagram Γ of $\mathrm{PGL}(2, \mathbb{Z})$

3.3 Coxeter elements

To finish this chapter, we briefly turn our attention to Coxeter elements, which are distinguished elements that are related to basic polynomial invariants of the Coxeter group. We provide only a definition below; see [Hum90, Sections 3.16-3.20] for a discussion of Coxeter elements for a finite Coxeter group, and [Hum90, Section 8.4] for arbitrary

Coxeter groups.

Definition 3.3.1 (Coxeter element). Let (W, S) be a Coxeter system. A *Coxeter element* w of (W, S) is a product (in any order) of all elements of S , each element occurring exactly once. Fixing an ordering $S = (s_1, \dots, s_d)$, the Coxeter elements are

$$\left\{ w = \prod_{i=1}^d s_{\tau(i)} \mid \tau \in \text{Sym}_d \right\}.$$

Coxeter elements play an important role in Artin theory. In Chapter 4, we view Coxeter groups as reflection groups, whereby Coxeter elements in a rank- d Coxeter group may be written as a product of d reflections. As stated later in Chapter 5, the Hurwitz action is transitive on the reduced reflection factorizations of a fixed Coxeter element, and the reflections that arise among these reflection factorizations are used to construct the corresponding dual Artin group.

In Part II of this dissertation, Coxeter elements play the role of encoding orderings of the generating set S of the Coxeter system, which is necessary when we investigate matrix representations with respect to bases of roots that correspond with the aforementioned reflection factorizations. Indeed, the particular Coxeter element $c = s_1 s_2 \cdots s_d$ that encodes the fixed ordering $S = (s_1, s_2, \dots, s_d)$ appears in the literature; for example, see Theorem 6.1.1, which is a restatement of [How82, Theorem 2.1] and [HK16, Proposition 2.5].

Chapter 4

Geometry of Coxeter groups

In Chapter 3, we viewed Coxeter groups as abstract groups defined in terms of presentations of a certain form. Coxeter groups can also be viewed from a geometric perspective. Section 4.1 describes the standard geometric representation, whereby Coxeter groups act linearly and faithfully by reflections on real quadratic spaces. Section 4.2 discusses root systems and how they are related to reflections. Finally, Section 4.3 lists a few special cases of Coxeter groups, so that we may refer to them in later chapters.

4.1 The standard geometric representation

This section discusses a faithful linear representation ρ of the Coxeter group W . [BB06, Chapter 4] calls ρ the *standard geometric representation* of W , so we also use this name. Applying the standard geometric representation, we find that any Coxeter group (W, S) has a natural realization as a reflection group, acting by reflections on a real quadratic space.

First, we can associate with each Coxeter system (W, S) a real quadratic space, defined in terms of a symmetric bilinear form.

Definition 4.1.1 (Standard symmetric bilinear form for a Coxeter system). Fix an ordering of S , say $S = (s_1, \dots, s_d)$. We obtain the pair (V, B) , where V is a real d -dimensional vector space with a distinguished ordered basis $\mathcal{B} = (e_1, \dots, e_d)$, and where $B : V \times V \rightarrow \mathbb{R}$ is a symmetric bilinear form defined by

$$B(e_i, e_j) = -\cos\left(\frac{\pi}{m_{ij}}\right).$$

We call B the *standard* symmetric bilinear form induced by (W, S) . We may refer to (V, B) interchangeably with the corresponding quadratic space (V, Q) . For a reason discussed later in Section 4.2, we call the elements of $\mathcal{B}$ the *simple roots* of (W, S) .

Remark 4.1.2 (Convention for $m_{ij} = \infty$). When we have the free relation $m_{ij} = \infty$, we use the convention

$$-\cos\left(\frac{\pi}{m_{ij}}\right) = -\cos(0) = -1.$$

Remark 4.1.3 (Geometry on V). The bilinear form B gives us notions of length and angle. For $v \in V$, we may write

$$\|v\| := B(v, v) = Q(v)$$

to denote the “length” of v . Similarly, for $u, v \in V$, we obtain a notion of “angle” between u and v by setting

$$\cos(\theta) := \frac{B(u, v)}{\|u\|\|v\|}.$$

Next, let us introduce the idea of a Gram matrix. In [HJ12, Section 7.2], Gram matrices are defined for inner product spaces. In this dissertation, we expand this definition to apply to vector spaces endowed with a (possibly non-positive-definite) symmetric bilinear form.

Definition 4.1.4 (Gram matrix). Let $\mathbb{F}$ be either $\mathbb{C}$ or $\mathbb{R}$, and let V be a vector space over $\mathbb{F}$ endowed with a symmetric bilinear form $B : V \times V \rightarrow \mathbb{F}$. Let $v_1, \dots, v_m \in V$. The *Gram matrix* with respect to $v_1, \dots, v_m$ is the $m \times m$ matrix G defined by $G_{ij} = B(v_i, v_j)$.

Remark 4.1.5 (Gram matrices and symmetric bilinear forms). When $\mathcal{B}(v_1, \dots, v_m)$ is an ordered basis for V , we say G is *the* Gram matrix for V (with respect to the basis $\mathcal{B}$ and the form B). As a minor abuse of notation, we may write B to denote both the bilinear form B and its matrix representation with respect to $\mathcal{B}$, which is precisely the Gram matrix defined above.

Definition 4.1.6 (Standard Gram matrix of a Coxeter system). The *(standard) Gram matrix* of (W, S) is the matrix corresponding to the standard symmetric bilinear form B . In other words, the Gram matrix is defined as

$$B_{ij} = -\cos\left(\frac{\pi}{m_{ij}}\right)$$

Below, we distinguish a matrix J defined by $J = 2B$, where B is the Gram matrix. This matrix J is called a Schläfli matrix by some authors [Joh18, Section 11.5], but is also called a *Cartan matrix* by other authors [How82].

Definition 4.1.7 (Schläfli matrix). Let B be the standard Gram matrix for a Coxeter system (W, S) . The *Schläfli matrix* for (W, S) is defined as $J = 2B$.

Let's look at the Gram and Schläfli matrices of some familiar Coxeter groups.

Example 4.1.8 (Gram matrix of Sym_{d+1}). We have already seen the Coxeter matrix and diagram of Sym_{d+1} in Example 3.2.9. The Gram matrix B of Sym_{d+1} is given by

$$B_{ij} = \begin{cases} 1, & i = j \\ 0, & |i - j| > 1 \\ -\frac{1}{2}, & |i - j| = 1. \end{cases}$$

We may prefer to use the Schläfli matrix of Sym_{d+1} , since all the entries are integers. For example, the Schläfli matrix of Sym_4 is

$$\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

Example 4.1.9 (Gram matrix of $\text{PGL}(2, \mathbb{Z})$). We have already seen the Coxeter matrix and diagram of $\text{PGL}(2, \mathbb{Z})$ in Example 3.2.11. The Gram matrix of $\text{PGL}(2, \mathbb{Z})$ is

$$\begin{pmatrix} 1 & -\frac{1}{2} & 0 \\ -\frac{1}{2} & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix}.$$

Remark 4.1.10 (Coxeter groups and quadratic spaces). Let (V, Q) be the real quadratic space associated with (W, S) as in Definition 4.1.1, and let B and J be the Gram and Schläfli matrices, respectively. Writing elements $v \in V$ as column vectors $[v]$ with respect to the ordered basis $\mathcal{B} = (e_1, \dots, e_d)$, we have

$$Q(v) = [v]^\top B[v].$$

Notice that the polar form β of Q is given by

$$\beta(v, w) = [v]^\top J[w].$$

Via this real quadratic space (V, Q) associated with (W, S) , we may see W as a reflection group. It suffices to find an isomorphism between W and a subgroup of $O(V)$ generated by reflections, so it seems sensible to start by mapping the standard generators of W to reflections in $O(V)$.

Definition 4.1.11 (Simple reflections). For each standard generator $s_i \in S$, define a reflection $r_i \in O(V)$ by

$$r_i(v) = v - 2B(v, e_i)e_i.$$

We call r_i the *simple reflection* corresponding to s_i .

Notice that each simple reflection preserves the form B (and thus the form Q). Each r_i really is a reflection in $O(V)$, and each element of the subgroup of $O(V)$ generated by the simple reflections also preserves the form B .

As discussed in [Bou08, Section 5.4], [Hum90, Section 5.3], and [BB06, Section 4.2], the assignment $s_i \mapsto r_i$ extends to a unique homomorphism ρ that allows us to view W as a reflection group. Following the terminology of [BB06], we call ρ the *standard geometric representation* of (W, S) . We emphasize that this ρ is the *standard* geometric representation because there are other valid geometric representations arising from different choices of bilinear form B , as discussed in [BB06, Section 4.2]. For this reason, we also emphasize that the bilinear form defined in Definition 4.1.1 is the *standard* one.

Theorem 4.1.12 (Standard geometric representation). *There is a unique homomorphism $\rho : W \rightarrow \mathrm{GL}(V)$ extending the assignment $s_i \mapsto r_i$. This homomorphism is injective, and the group $\rho(W)$ preserves the form B on V . Furthermore, for each pair $s_i, s_j \in S$, the order of $\rho(s_i s_j)$ is m_{ij} .*

The standard geometric representation allows us to identify W with $\rho(W)$. Moving forward, we may abuse notation via the identification $W \cong \rho(W)$.

- We may refer to W and $\rho(W)$ interchangeably, and we may call W a reflection group.
- We may refer to s_i and r_i interchangeably, and we may call s_i a simple reflection.
- We may describe $w \in W$ as a linear isometry in $\rho(W)$.

Following [Hum90], we use functional notation when discussing W and $\rho(W)$, i.e. $(w_1 w_2)(v) = w_1(w_2(v))$.

Using the standard geometric representation, let us examine some familiar Coxeter groups, seeing them now as reflection groups.

Example 4.1.13 (Standard geometric representation of Sym_{d+1}). Recall that the symmetric group Sym_{d+1} is generated by the adjacent transpositions

$$s_i = (i, i + 1).$$

By the standard geometric representation ρ , we may view Sym_{d+1} as a reflection group acting on the real d -dimensional vector space V endowed with the symmetric bilinear form B . We may identify each generator s_i with the corresponding simple reflection $r_i = \rho(s_i)$ given by

$$r_i(v) = v - 2B(v, e_i)e_i.$$

In the ordered basis $\mathcal{B} = (e_1, \dots, e_d)$ of simple roots, the matrix representation of B is the Gram matrix (also denoted B) discussed in Example 4.1.8. We may compute the matrix representation of each simple reflection. Letting $\mathbf{v}$ denote the column vector representing $v \in V$ with respect to the standard basis of simple roots, we obtain

$$\begin{aligned} [r_i]\mathbf{v} &= I\mathbf{v} - 2(\mathbf{e}_i^\top B\mathbf{v})\mathbf{e}_i \\ &= I\mathbf{v} - 2\mathbf{e}_i(\mathbf{e}_i^\top B\mathbf{v}) \\ &= I\mathbf{v} - 2E_{ii}B\mathbf{v} \\ [r_i] &= I - E_{ii}J \end{aligned}$$

where E_{ij} denotes the standard basis matrix from Definition 2.1.2, and where J denotes the Schläfli matrix.

Using Sym_4 as a concrete example, we have

$$\begin{aligned}
 [r_1] &= I - E_{11}J \\
 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \\
 &= \begin{pmatrix} -1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
 [r_2] &= I - E_{22}J = \begin{pmatrix} 1 & 0 & 0 \\ 1 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \\
 [r_3] &= I - E_{33}J = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & -1 \end{pmatrix}
 \end{aligned}$$

We know that each r_i sends e_i to $-e_i$. We may also compute the planes $H_i = \text{Fix}(r_i)$.

$$\begin{aligned}
 H_1 &= \{(a, b, c)^\top \in \mathbb{R}^3 \mid 2a = b\}, \\
 H_2 &= \{(a, b, c)^\top \in \mathbb{R}^3 \mid 2b = a + c\}, \\
 H_3 &= \{(a, b, c)^\top \in \mathbb{R}^3 \mid 2c = b\}.
 \end{aligned}$$

Notice how the computation of $[r_i] = I - E_{ii}J$ did not depend on anything particular to Sym_{d+1} ; the formula holds true for any (W, S) . More generally, we have the following fact for $r_u \in O(V)$.

Proposition 4.1.14 (Matrix representation of reflections). *Let $u \in V$ be a unit vector (with respect to the quadratic form Q), and let $\mathbf{u}$ denote the column vector of u with respect to the standard basis of simple roots. The matrix representation of r_u with respect to the standard basis is*

$$[r_u] = I - \mathbf{u}\mathbf{u}^\top J.$$

Proof. Recall that

$$r_u(v) = v - \frac{2B(u, v)}{B(u, u)}u.$$

As matrices and column/row vectors, we have

$$\begin{aligned} [r_u]\mathbf{v} &= I\mathbf{v} - 2(\mathbf{u}^\top B\mathbf{v})\mathbf{u} \\ &= I\mathbf{v} - 2\mathbf{u}(\mathbf{u}^\top B\mathbf{v}) \\ &= I\mathbf{v} - 2(\mathbf{u}\mathbf{u}^\top)B\mathbf{v} \\ &= I\mathbf{v} - (\mathbf{u}\mathbf{u}^\top)(2B)\mathbf{v} \\ &= I\mathbf{v} - \mathbf{u}\mathbf{u}^\top J\mathbf{v} \\ &= (I - \mathbf{u}\mathbf{u}^\top J)\mathbf{v}. \end{aligned}$$

□

Let us continue with some more examples of the standard geometric representation for some familiar Coxeter groups.

Example 4.1.15 (Standard geometric representation of $\mathrm{PGL}(2, \mathbb{Z})$). Recall from Example 3.2.11 that the standard generators of $\mathrm{PGL}(2, \mathbb{Z})$ are

$$s_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad s_2 = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}, \quad s_3 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Let B be the Gram matrix of $\mathrm{PGL}(2, \mathbb{Z})$, seen in Example 4.1.9. In the basis of simple roots, we obtain the following matrix representations of the simple reflections.

$$\begin{aligned}
[r_1] &= I - E_{11}J = \begin{pmatrix} -1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
[r_2] &= I - E_{22}J = \begin{pmatrix} 1 & 0 & 0 \\ 1 & -1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \\
[r_3] &= I - E_{33}J = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & -1 \end{pmatrix}.
\end{aligned}$$

We may also compute the fixed planes H_i .

$$\begin{aligned}
H_1 &= \{(a, b, c)^\top \in \mathbb{R}^3 \mid 2a = b\}, \\
H_2 &= \{(a, b, c)^\top \in \mathbb{R}^3 \mid 2b = a + 2c\}, \\
H_3 &= \{(a, b, c)^\top \in \mathbb{R}^3 \mid b = c\}.
\end{aligned}$$

4.2 The root system

This section briefly discusses the notion of a root system for a Coxeter group. Much can be said about root systems, but we present only the basic ideas needed for this dissertation. The most important idea is the correspondence between roots and reflections; in Chapter 6, we use this correspondence to reinterpret the Hurwitz action on reduced reflection factorizations of a Coxeter element as a change of basis between ordered bases of roots.

As before, let (W, S) be a Coxeter system. Recall from Definition 4.1.11 that for each generator $s_i \in S$, we call $r_i = \rho(s_i)$ a *simple reflection*. (We may abuse notation by referring to s_i itself as a simple reflection. Indeed, we may identify each element $w \in W$ with its image $\rho(w)$ via the standard geometric representation ρ .) The simple reflection r_i is the reflection that sends the standard basis vector e_i to its negative $-e_i$. We call

these e_i 's *simple roots*.

Definition 4.2.1 (Simple roots). The *simple roots* are precisely the standard basis vectors $\{e_i\}$. We may denote the set of simple roots by Δ .

The W -orbit of the simple roots is called the root system.

Definition 4.2.2 (Root system). The *root system* Φ of a Coxeter system (W, S) is the collection of vectors

$$\Phi = \{w(e_i) \mid e_i \in \Delta, w \in W\}.$$

The elements of Φ are called *roots* of W .

Notice that all roots are unit vectors with respect to the standard symmetric bilinear form B from Definition 4.1.1; each root is in the W -orbit of a simple root, each simple root is a unit vector, and W preserves the form B .

Since the simple roots form a basis of V , we may write each root u uniquely as a $\mathbb{R}$ -linear combination of simple roots.

$$u = \sum_{i=1}^d c_i e_i$$

Writing u in this way gives us a notion of positive and negative roots.

Definition 4.2.3 (Positive and negative roots). A root $u \in \Phi$ is called *positive* (respectively, *negative*) if all $c_i \geq 0$ (resp. all $c_i \leq 0$). We let Φ^+ and Φ^- denote the sets of positive and negative roots, respectively.

It may not be immediately obvious, but the positive and negative roots form a partition of the entire root system, i.e. $\Phi = \Phi^+ \sqcup \Phi^-$. See [Hum90, Section 5.4] or [BB06, Section 4.4] for a proof of this fact.

What can we say about the set of *all* reflections in a Coxeter group? Let T denote the set of conjugates of simple reflections in some Coxeter group W .

$$T := \{ws_iw^{-1} \mid s_i \in S, w \in W\}$$

Clearly, each element of T is an involution. In fact, (the image via the standard geometric representation of) each element of T is an actual reflection in the sense of Definition 2.3.1. In particular, the element ws_iw^{-1} acts as the reflection r_u sending the root $u = w(e_i)$ to its negative $-u$. Notice that as a reflection, $r_u = \rho(ws_iw^{-1})$ depends only on u , not on the choice of w and s_i . It is clear now that T is the set of *all* reflections r_u where u is a root in Φ .

$$T = \{r_u \mid u \in \Phi\}$$

We have the following correspondence between positive roots and reflections, as discussed in [Hum90, Section 5.7] and [BB06, Proposition 4.4.5].

Theorem 4.2.4 (Correspondence between roots and reflections). *The map $\tau : \Phi^+ \rightarrow T$ given by $\tau : u \mapsto r_u$ is a bijective correspondence between positive roots and reflections.*

Remark 4.2.5 (Lax vectors). In the language of [CF97], we can think of $u \in \Phi$ as a *strict vector*, and the corresponding pair $\pm u$ as a *lax vector*. Since positive and negative roots $\pm u$ are naturally paired with the same reflection $r_u = r_{-u}$, we may also think of the map τ as a bijection $\pm u \mapsto r_u$ between lax roots and reflections of (W, S) .

4.3 Special cases of Coxeter groups

This section briefly discusses important special cases of Coxeter groups, namely spherical, Euclidean, hyperbolic, and crystallographic Coxeter groups.

The standard symmetric bilinear form B (or its Gram matrix) can be used to identify certain classes of Coxeter groups. For example, a Coxeter group is finite if and only if B is positive-definite; for a proof of this fact, see [Hum90, Section 6.4]. (Finite Coxeter groups are also called *spherical*, because they are subgroups of isometry groups of spheres.) The finite Coxeter groups are precisely the finite reflection groups (the finite groups generated by reflections in real Euclidean spaces); for a proof of this fact, see [Hum90, Section 6.4]. The Coxeter diagrams for all irreducible finite Coxeter groups are presented below in Figure 4.1. In this figure, each symbol has a subscript, which equals the number of vertices in the corresponding diagram (equivalently, the rank of the corresponding Coxeter group). For example, the diagram of type A_n has n vertices, indicating the group $\text{Cox}(A_n)$ has rank n . For a thorough discussion of finite Coxeter groups, see Chapters 1 and 2 of [Hum90].

Another important class of Coxeter groups is the class of *affine* (or *Euclidean*) Coxeter groups. An *affine reflection group* is an (infinite) group generated by affine reflections in (real) Euclidean space; these groups are discussed in detail in [Hum90, Chapter 4]. The affine reflection groups are precisely the Coxeter groups where the Gram matrix B is positive semidefinite (and *not* positive definite), as discussed in [Hum90, Section 6.5]. The Coxeter diagrams for all irreducible affine Coxeter groups are presented below in Figure 4.2; compare with Figure 4.1, which shows the diagrams for all irreducible finite Coxeter groups.

There are a few non-equivalent definitions of *hyperbolic* Coxeter groups. The definition used by [Hum90, Section 6.8] is as follows: Given a Coxeter system (W, S) of rank d , let C denote the cone

$$C := \{v \in V \mid \forall i \in [d], B(v, e_i) > 0\} = \left\{ \sum_{i=1}^d c_i e_i \mid c_i > 0 \right\}$$

where the basis vectors $\{e_i\}$ are the simple roots. An (irreducible) Coxeter group W of

rank d is called hyperbolic if its Gram matrix B has signature $(d - 1, 1)$ (i.e. has $d - 1$ positive eigenvalues, 1 negative eigenvalue, and 0 zero eigenvalues), and if for each $v \in C$ we have $B(v, v) < 0$. (This definition is not equivalent to hyperbolicity in the sense of Gromov [Gro87]; for a characterization of hyperbolic Coxeter groups in this sense, see Moussong's dissertation [Mou88].) There is a way to view a hyperbolic Coxeter group of rank d as a group of reflections in $(d - 1)$ -dimensional hyperbolic space, hence the name. All triangle groups (equivalently, Coxeter groups of rank 3) that are irreducible (i.e. the Coxeter diagram Γ is connected) and not of positive type (i.e. not spherical) are hyperbolic. For the complete lists of all compact and noncompact hyperbolic Coxeter groups of rank $d \geq 4$, see [Hum90, Section 6.9].

Yet another nice class of Coxeter groups defined by their geometric properties is the class of *crystallographic* Coxeter groups. A Coxeter group W is called crystallographic (relative to its standard geometric representation $\rho : W \rightarrow \mathrm{GL}(V)$) if W stabilizes a lattice L in V . The following proposition, which may be found in [Hum90, Section 6.6], characterizes the crystallographic Coxeter groups.

Proposition 4.3.1 (Crystallographic Coxeter groups). *A Coxeter group W is crystallographic (relative to ρ) if and only if the following conditions are met:*

- *For each pair $s \neq t$ in S , we have $m(s, t) \in \{2, 3, 4, 6, \infty\}$, and*
- *For each circuit in the Coxeter diagram Γ , the number of edges labeled 4 (respectively, 6) is even.*

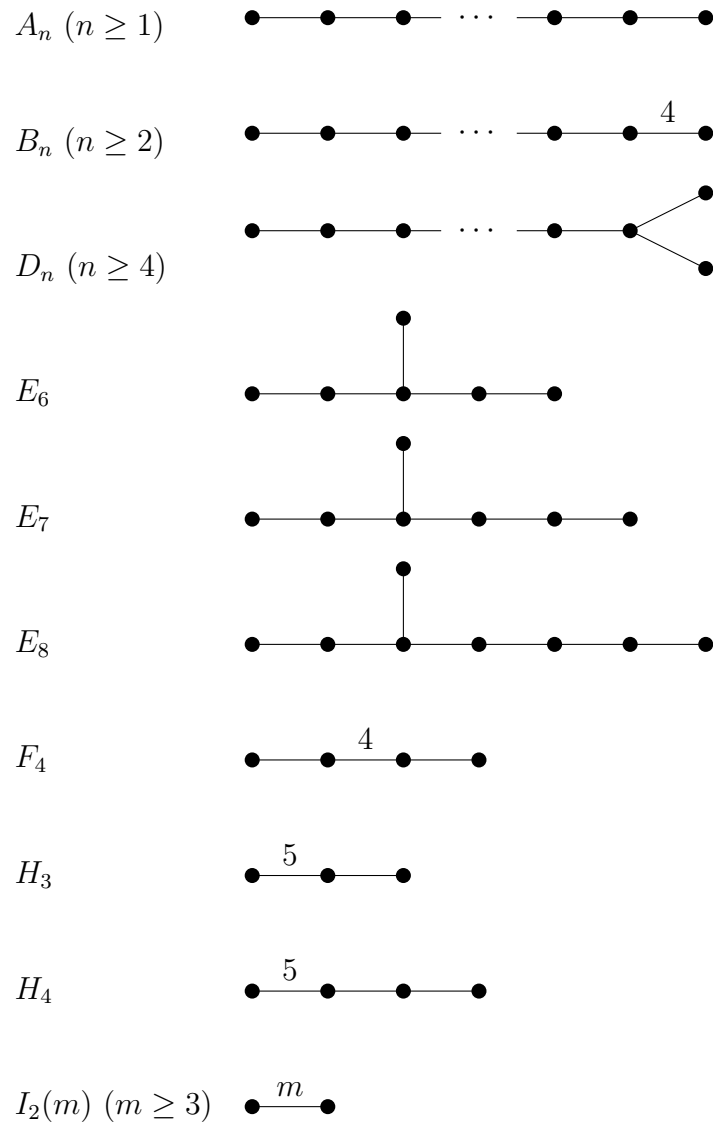


Figure 4.1: Diagrams for the irreducible finite Coxeter groups

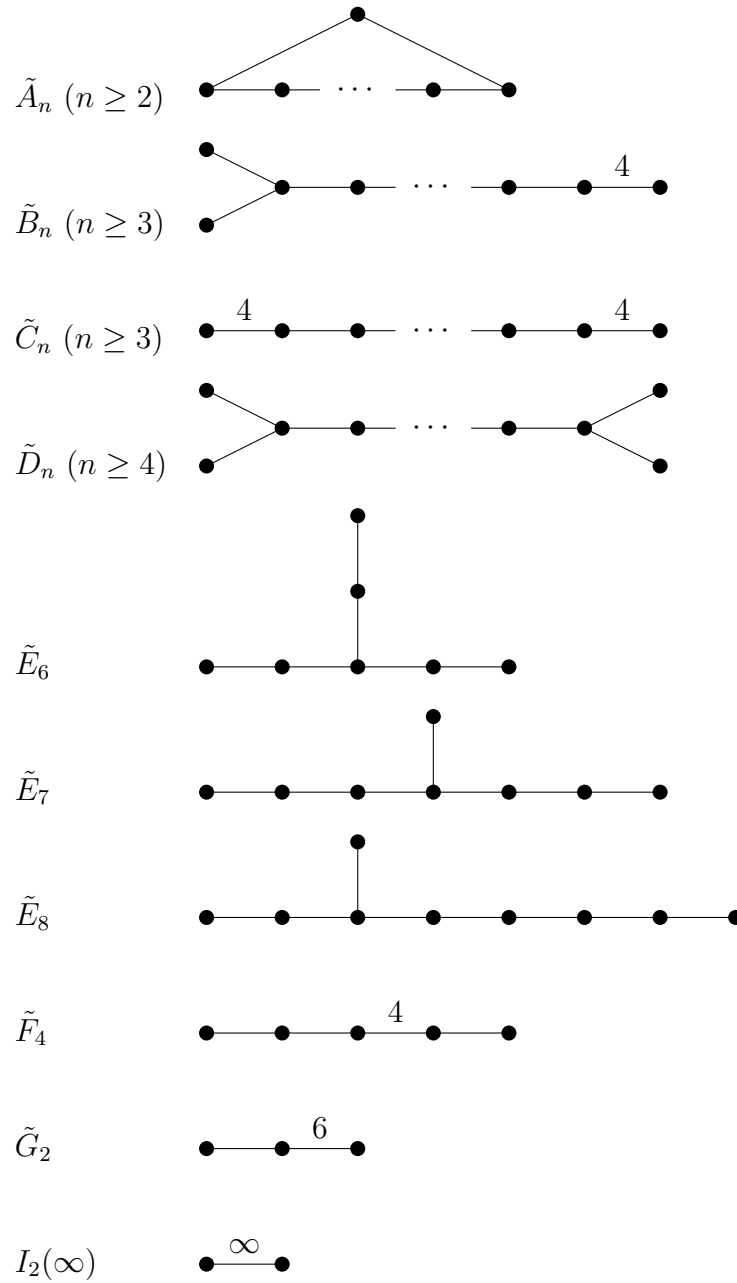


Figure 4.2: Diagrams for the irreducible affine Coxeter groups

Chapter 5

The Hurwitz action on reflection factorizations

The Hurwitz action on reduced reflection factorizations of Coxeter elements is used in the literature to construct dual Artin groups, and is the main object of study in this dissertation. Since the Hurwitz action is a braid group action, we begin this chapter with some preliminaries on the braid group. Next, we define the Hurwitz action, then explain the relevance to Artin theory.

5.1 Braid groups

The braid group is an important object in topology, whose main relevance to this dissertation is the Hurwitz action. In 1925, Emil Artin introduced braid groups [Art25], described as groups whose elements are n -stranded braids with concatenation as the group operation. In [Art47], Artin provided an explicit presentation for braid groups, as well as a topological definition where braid groups can be viewed as fundamental groups of certain configuration spaces. In this section, we discuss braid groups to the extent

that is necessary for this dissertation. For further reading, see the surveys [Mag74] and [BB05].

Braid groups may be defined topologically, but for the purpose of this dissertation, we focus more on the abstract definition via group presentation, provided below.

Definition 5.1.1 (Braid group). The *braid group* on d strands is the group Braid_d with the following presentation.

$$\text{Braid}_d = \left\langle \sigma_1, \dots, \sigma_{d-1} \mid \begin{array}{l} \sigma_i \sigma_j \sigma_i = \sigma_j \sigma_i \sigma_j \text{ for } |i - j| = 1 \\ \sigma_i \sigma_j = \sigma_j \sigma_i \text{ for } |i - j| > 1 \end{array} \right\rangle$$

We call the above presentation the *standard presentation* for Braid_d , and each σ_i may be called the i -th *standard generator*.

Elements of braid groups can be thought of as (isotopy classes of) geometric braids of strands in space. Intuitively, consider d parallel strands. The identity element in Braid_d can be represented by the braid where no strands cross. The standard generator σ_i can be thought of as the braid that crosses the i -th strand *over* the $(i + 1)$ -th strand, so we call σ_i the *i -th standard positive half-twist*. (Similarly, we may call σ_i^{-1} the *i -th standard negative half-twist*.) Multiplying braids can be thought of as juxtaposition, concatenation, isotopy, and rescaling. We may conveniently depict geometric braids via braid diagrams; Figures 5.1 and 5.2 are examples of braid diagrams for the braid $\sigma_2 \sigma_3^{-1} \sigma_1 \sigma_2^{-1} \sigma_3 \in \text{Braid}_4$. Two braid diagrams correspond to the same element of the braid group when one can be jiggled around to the other without moving the endpoints.

Remark 5.1.2 (Over- vs under-crossing). The choice to define σ_i as the i -th strand crossing *over* the $(i + 1)$ -th strand was arbitrary; we could have instead defined it as the *under*-crossing, as in [BB05, Figure 1]. We follow [Bir75] in defining σ_i as the over-crossing.

Example 5.1.3 (Visualizing an element of Braid_4). Consider the element $\sigma_2 \sigma_3^{-1} \sigma_1 \sigma_2^{-1} \sigma_3 \in \text{Braid}_4$. Using *functional notation* as in Remark 3.1.4, we first apply σ_3 , then σ_2^{-1} , and so

on. The braid diagram is given in Figure 5.1; compare to Figure 3.1. Figure 5.2 reorients the diagram so that the composition of half-twists can be read in the same order as the product in functional notation; compare to Figure 3.2.

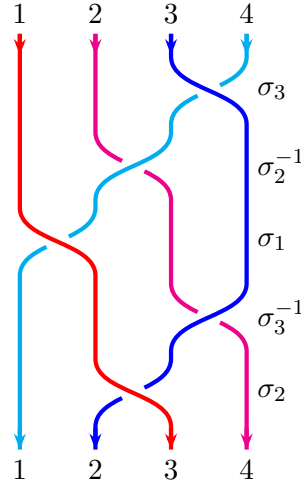


Figure 5.1: Vertical braid diagram for $\sigma_2 \sigma_3^{-1} \sigma_1 \sigma_2^{-1} \sigma_3 \in \text{Braid}_4$

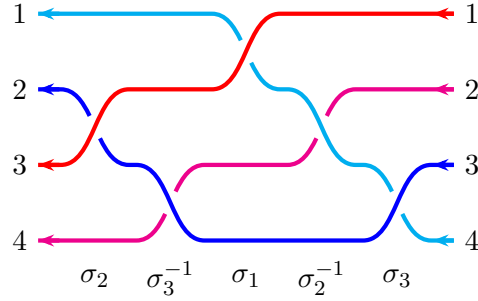


Figure 5.2: Horizontal braid diagram for $\sigma_2 \sigma_3^{-1} \sigma_1 \sigma_2^{-1} \sigma_3 \in \text{Braid}_4$

5.2 The Hurwitz action

The Hurwitz action is a braid group action that can be used to construct dual Artin groups, an important object in Artin theory. This section first defines the Hurwitz action in general terms, then states a crucial fact about the Hurwitz action: it is transitive on

reduced reflection factorizations of Coxeter elements. We finish up the section by briefly discussing Artin groups, and how the Hurwitz action is relevant to Artin theory.

Let us first define the Hurwitz action. We follow the conventions of [Bau+14] and [MS17, Definition 5.9] for the definition below.

Definition 5.2.1 (Hurwitz action). Let G be a group. The *Hurwitz action* of Braid_d on G^d is defined by

$$\begin{aligned}\sigma_i \cdot (g_1, \dots, g_i, g_{i+1}, \dots, g_d) &= (g_1, \dots, g_i g_{i+1} g_i^{-1}, g_i, \dots, g_d), \\ \sigma_i^{-1} \cdot (g_1, \dots, g_i, g_{i+1}, \dots, g_d) &= (g_1, \dots, g_{i+1}, g_{i+1}^{-1} g_i g_{i+1}, \dots, g_d).\end{aligned}$$

We call the action by σ_i (respectively, σ_i^{-1}) the *i -th standard positive (resp. negative) Hurwitz move*.

Notice that value of the product of the d -tuple is invariant under the Hurwitz action. Given a d -tuple $(g_1, g_2, \dots, g_d) \in G$, let

$$g = g_1 g_2 \cdots g_d$$

denote the product. Fixing g , we can obtain factorizations of g into d factors by applying the Hurwitz action to $(g_1, g_2, \dots, g_d)$. However, the Hurwitz action need not be transitive on such factorizations.

We now turn our attention back to Coxeter groups. As in Section 3.2, let (W, S) be a Coxeter system with a fixed ordering $S = (s_1, s_2, \dots, s_d)$ of the generators, and let $c = s_1 s_2 \cdots s_d$ denote the Coxeter element of (W, S) that encodes this ordering. Let $\rho : W \rightarrow \text{GL}(V)$ be the standard geometric representation, as in Theorem 4.1.12. As in Section 4.2, let

$$T := \{wsw^{-1} \mid s \in S, w \in W\}$$

denote the set of all reflections in W .

Each element $w \in W$ certainly admits a factorization into reflections; indeed, simply write w in terms of the generating set S , the simple reflections. We may also consider factorizations with respect to the alphabet T of all reflections. The element w may admit many reflection factorizations, some of different length. A reflection factorization of w that minimizes the length among all such reflection factorizations is called a *reduced* reflection factorization of w .

Definition 5.2.2 (Reduced reflection factorizations). For $w \in W$, a *reduced reflection factorization* is a shortest length factorization (not necessarily unique) of w into reflections. We denote by $\text{Red}_T(w)$ the set of reduced reflection factorizations of w .

Fixing w , we may consider the Hurwitz action on $\text{Red}_T(w)$. For $w = r_{u_1} r_{u_2} \cdots r_{u_n}$ with $r_{u_i} \in T$, the Hurwitz action on the tuple $(r_{u_1}, r_{u_2}, \dots, r_{u_n})$ is

$$\sigma_i \cdot (r_{u_1}, \dots, r_{u_i}, r_{u_{i+1}}, \dots, r_{u_n}) = (r_{u_1}, \dots, r_{u_i} r_{u_{i+1}} r_{u_i}^{-1}, r_{u_i}, \dots, r_{u_n}).$$

Clearly, this is still a reflection factorization of w .

The following theorem by Igusa and Schiffler establishes that the Hurwitz action is transitive on reflection factorizations of Coxeter elements in Coxeter groups [IS10, Theorem 1.4]. See [Bau+14, Theorem 1.3] for a short proof of the theorem.

Theorem 5.2.3 (Transitivity of the Hurwitz action). *Let w be a Coxeter element in W . Then the Hurwitz action on $\text{Red}_T(w)$ is transitive.*

5.3 Artin groups

To appreciate the importance of the transitivity of the Hurwitz action on reduced reflection factorizations of Coxeter elements, we now discuss Artin groups, which are closely

related to Coxeter groups. While Coxeter groups are very well understood, Artin groups remain mysterious, and many basic questions remain open for general Artin groups. In the various cases where these questions have been answered, so-called dual Artin groups have played a crucial role. The Hurwitz action is used to construct these dual Artin groups.

Jacques Tits introduced Artin groups in 1966 [Tit66], defining them in terms of an abstract presentation, presented below. Compare this presentation to the presentation that defines Coxeter groups in Definition 3.2.1.

Definition 5.3.1 (Artin groups). An *Artin group* is a group A with a generating set S and a presentation

$$A := \langle S \mid \forall s, t \in S, \underbrace{sts \cdots}_{m(s,t)} = \underbrace{tst \cdots}_{m(s,t)} \rangle$$

such that

- each generator $s \in S$ satisfies $m(s, s) = \infty$, and
- each pair $s, t \in S$ of distinct generators has exactly one relation $\underbrace{sts \cdots}_{m(s,t)} = \underbrace{tst \cdots}_{m(s,t)}$ where $m(s, t) \in \{2, 3, \dots, \infty\}$.

(Here, $m(s, t) = \infty$ indicates no relation between s and t .)

In the same way that a Coxeter group is entirely determined by the relation lengths $m(s, t)$ between generators $s, t \in S$, an Artin group is entirely determined by the relation lengths $m(s, t)$. Recall from Section 3.2 that this data is completely encoded by the Coxeter matrix M , as well as by the Coxeter diagram Γ . Thus, from the Coxeter diagram Γ arises both the Coxeter group $\text{Cox}(\Gamma)$ and the Artin group $\text{Art}(\Gamma)$, giving a natural correspondence between Coxeter groups and Artin groups. Much of the language used

for Coxeter groups is applied to the corresponding Artin groups, e.g. if $\text{Cox}(\Gamma)$ is spherical/Euclidean/hyperbolic/irreducible/etc., we say $\text{Art}(\Gamma)$ is spherical/Euclidean/hyperbolic/irreducible/etc.

Example 5.3.2 (Sym_d and Braid_d). Recall from Example 3.2.8 that the Coxeter matrix for the symmetric group Sym_d is

$$\begin{pmatrix} 1 & 3 & 2 \\ 3 & 1 & 3 \\ 2 & 3 & 1 \end{pmatrix}.$$

The Coxeter diagram Γ for Sym_4 is given in Figure 3.3. We may write $\text{Sym}_4 = \text{Cox}(\Gamma)$. One might notice that the braid group Braid_4 has the same relation lengths as Sym_4 , so $\text{Braid}_4 = \text{Art}(\Gamma)$. More generally, if Γ is the Coxeter diagram for Sym_d , then $\text{Braid}_d = \text{Art}(\Gamma)$.

Even though Coxeter groups and Artin groups are defined by related abstract presentations, there is currently a large gap between what is known about Coxeter groups and what is known about Artin groups. The theory of general Coxeter groups is well developed. As discussed in Section 4.1, every Coxeter group has a faithful linear representation. Coxeter groups are known to be automatic [DS91] [BH93], which implies that the word problem for a Coxeter group is solvable in quadratic time. (In fact, Coxeter groups are biautomatic [OP25], so they have a solvable conjugacy problem.) Coxeter groups are also CAT(0) groups [Mou88]. Many of the theorems of geometric group theory apply to Coxeter groups, and they have nice algorithmic properties.

In contrast, little is known about Artin groups. In their survey [GP12], Godelle and Paris emphasize the following four conjectures that remain open for Artin groups in general. There are many other open conjectures about Artin groups, such as the conjugacy problem, automaticity, linearity, etc.

Conjecture 5.3.3 (Four basic conjectures). *[GP12] highlights the following conjectures.*

- (A) *Every Artin group is torsion-free.*
- (B) *Every non-spherical irreducible Artin group has trivial center.*
- (C) *Every Artin group has a solvable word problem.*
- (D) *Artin groups satisfy the $K(\pi, 1)$ conjecture.*

These conjectures are answered for some classes of Artin groups. McCammond's survey on Artin groups [McC17a] lists the classes of Artin groups for which these conjectures are answered, and examines the groups for which these conjectures remain open. As discussed in Charney's survey on right-angled Artin groups (or RAAGs) [Cha07], RAAGs are fundamental groups of CAT(0) cube complexes, which implies that the four conjectures (and others) hold for RAAGs. The papers [Del72] and [BS72] show that spherical Artin groups satisfy all four conjectures. Further study on spherical Artin groups show that a spherical Artin group is isomorphic to its *dual Artin group* [Bes03] [BW02], which has a Garside structure that gives it nice algorithmic properties.

Before discussing more Artin groups for which these conjectures have been answered, we first define dual Artin groups below. Let (W, S) be a Coxeter system as in Definition 3.2.1, with some Coxeter diagram Γ . Let T be the full set of reflections

$$T := \{ws w^{-1} \mid s \in S, w \in W\}$$

as discussed in Section 4.2. Consider the (left) Cayley graph of W with respect to the generating set T , and for elements $g, h \in W$, let $[g, h]^W$ denote the interval of group elements between g and h , given a graded poset structure based on the distance between vertices in the corresponding Cayley graph. Let w be a Coxeter element of W , and let $R \subseteq T$ be the subset of reflections that appear as edge labels in the interval $[1, w]^W$.

Definition 5.3.4 (Dual Artin group). The *dual Artin group* with respect to w , denoted $\text{Art}^*(\Gamma, w)$, is the group generated by R subject only to the relations that appear in

the interval $[1, w]^W$. We refer to the defining presentation of $\text{Art}^*(\Gamma, w)$ as the *dual presentation*.

The maximal chains in the interval $[1, w]^W$ are precisely the reduced reflection factorizations of the Coxeter element w . We know by Theorem 5.2.3 that the Hurwitz action is transitive on such reflection factorizations, allowing us to construct $[1, w]^W$ (and thus $\text{Art}^*(\Gamma, w)$) using the Hurwitz action.

The injection $S \hookrightarrow R$ extends to a group homomorphism $\text{Art}(\Gamma) \rightarrow \text{Art}^*(\Gamma, w)$, but it is not known in general if the standard Artin group $\text{Art}(\Gamma)$ is isomorphic to its dual $\text{Art}^*(\Gamma, w)$. The importance of the dual Artin group comes from Garside structures. We do not define Garside structures as it is not needed in this dissertation; for further reading on Garside structures, see [Deh+15] and [McC05]. Garside structures allow us to answer the aforementioned conjectures, using the following two facts.

Proposition 5.3.5 (Garside structures). *If $W = \text{Cox}(\Gamma)$ is a Coxeter group generated by its full set of reflections T with Coxeter element w , and if the interval $[1, w]^W$ is a lattice, then the dual Artin group $\text{Art}^*(\Gamma, w)$ is a Garside group.*

Theorem 5.3.6 (Consequences). *If G is a Garside group, then its elements have normal forms, and G has a finite-dimensional classifying space whose dimension equals the length of the longest chain in the interval defining G as a Garside group. Consequently, G has a decidable word problem and is torsion-free.*

Dual Artin groups and Garside structures have proven to be a very successful tool for answering questions about Artin groups. Every dual Artin group that has been successfully analyzed has been shown to be isomorphic to its standard Artin group, as in the aforementioned case of spherical Artin groups [Bes03] [BW02], so it is conjectured that all Artin groups are isomorphic to their duals. In [MS17], McCammond and Sulway show that the Euclidean Artin groups are isomorphic to their duals, and that

each dual Artin group injects into a Garside supergroup, whereby every Euclidean Artin group is a torsion-free group, has trivial center, has a solvable word problem, and has a finite-dimensional classifying space; see [McC17b] for notes on the theory leading up to the results in [MS17]. In [PS21], Paolini and Salvetti use the Garside structure from [MS17] to show that Euclidean Artin groups satisfy the $K(\pi, 1)$ conjecture; see [PS25] for notes on the proof. More recently, in [DPS24], Delucchi, Paolini, and Salvetti extend these methods to prove the $K(\pi, 1)$ conjecture for all rank-3 Artin groups. In [Res24], Resteghini finds sufficient conditions on the Hurwitz action for which the standard and dual Artin group are isomorphic.

Part II

Hurwitz action on Wall form matrices

In this second part of the dissertation, we focus our attention on the Hurwitz action. Chapter 6 starts with the matrix representations of Coxeter elements and their Wall forms, then shows how the Hurwitz action on reduced reflection factorizations of Coxeter elements induces a change-of-basis on the real quadratic space from the standard geometric representation of the Coxeter group. The change of basis comes from the correspondence between reflections and roots; each reduced reflection factorization of a Coxeter element corresponds to an ordered tuple of linearly independent roots (up to a choice of sign for each root). Fixing a choice of signs on the roots, we explicitly state the change-of-basis matrices and show that the induced Hurwitz action on these ordered bases of roots is well-defined.

The remaining chapters explore the matrix representations that arise when applying the Hurwitz action via certain classes of braids. Chapter 7 focuses on powers of half-twists of two adjacent strands. After explicitly computing the Hurwitz change-of-basis matrices for powers of half-twists, we state new results that show connections between these braids and Fibonacci/Chebyshev polynomials. We also show how the Hurwitz action via these braids can be interpreted as plane rotations, whose angle and type (Euclidean, hyperbolic, and otherwise) depend on the Coxeter system and on which pair of strands are twisted.

Chapter 8 generalizes the previous results to rotation braids that twist multiple adjacent strands together. We begin by explicitly computing the Hurwitz change-of-basis matrices. The resulting patterns can be simply written in terms of (the matrix representation of) the Wall form of the chosen Coxeter element. We explore connections to companion matrices, and find a simple formula for these change-of-basis matrices in terms of (the matrix representation of) the Coxeter element itself. We use a combinatorial technique called the transfer-matrix method to compute closed formulas of the entries of these change-of-basis matrices. In a similar way to how we found that half-twists are related to the univariate Fibonacci and Chebyshev polynomials, we show that rotation

braids of adjacent strands are related to a multivariate generalization of the Fibonacci polynomials.

Chapter 6

Wall form matrices of Coxeter elements

Up to this point, we have seen Coxeter elements, Wall forms, and the Hurwitz action in isolation. This chapter investigates matrix representations of the Wall form of Coxeter elements and illustrates how the Hurwitz action on reduced reflection factorizations of a Coxeter element induces a change of basis among these matrix representations.

Throughout this chapter, as in Chapters 3 and 4, let (W, S) denote a Coxeter system of rank d , with standard generating set $S = (s_1, s_2, \dots, s_d)$. Let c denote the Coxeter element induced by the given ordering of the generators, namely $c = s_1 s_2 \cdots s_d$. Identify elements w of W with their linear actions $\rho(w)$ via the standard geometric representation $\rho : W \rightarrow \mathrm{GL}(V)$. In particular, identify standard generators s_i with the corresponding reflection $r_i = \rho(s_i)$, and the Coxeter element c with its linear action $\rho(c)$. Under this identification, we may omit ρ from the notation, writing the Wall form of the linear map $\rho(w)$ as χ_w instead of $\chi_{\rho(w)}$. Let B refer to both the standard symmetric bilinear form and the Gram matrix of (W, S) , as in Definitions 4.1.1 and 4.1.4. Let J refer to the Schläfli matrix $J = 2B$, as in Definition 4.1.7.

This chapter first establishes connections between matrix representations of the Coxeter elements c and c^{-1} and of the Wall forms χ_c and $\chi_{c^{-1}}$. Next, we introduce the induced Hurwitz action on ordered bases of roots of the Coxeter system (W, S) , and investigates the matrix representations of $\chi_{c^{-1}}$ under the Hurwitz action. Finally, we establish some basic properties of the Hurwitz action on these Wall form matrices, which we use in future chapters.

6.1 Matrix representations of Coxeter elements

This section discusses matrix representations of the Coxeter element $c = s_1 s_2 \cdots s_d$. First, we discuss the matrix representation with respect to the standard basis $(e_1, e_2, \dots, e_d)$ of simple roots. When no basis is specified, we understand $[c]$ to denote the matrix representation of the Coxeter element c with respect to the standard basis. After computing this matrix representation explicitly, we generalize these results to arbitrary Coxeter elements with respect to two natural bases, which we call the word basis and action basis.

As it turns out, the matrix representations of both the Coxeter element c and its Wall form χ_c may be written in terms of the following $d \times d$ upper unitriangular matrix C , defined by

$$C_{ij} = \begin{cases} 1, & i = j \\ 0, & i > j \\ 2B(e_i, e_j), & i < j \end{cases}$$

where B is the standard bilinear form from Definition 4.1.1. Notice that the Schläfli matrix from Definition 4.1.7 satisfies $J = C + C^\top$, i.e. C and $C^\top$ are the upper and lower unitriangular parts of J .

The following theorem from [How82, Theorem 2.1] and [HK16, Proposition 2.5] establishes the relationship between the Coxeter element $c = s_1 s_2 \cdots s_d$ and the matrix

C .

Theorem 6.1.1 (Matrix representation of c). *The matrix of c with respect to the standard basis of simple roots is*

$$[c] = -C^{-1}C^{\top}.$$

To gain familiarity with the relationship between c and C , let us examine a few concrete examples.

Example 6.1.2 (Coxeter element for a rank-3 Coxeter system). Let (W, S) be a rank-3 Coxeter system with generating set $S = (s_1, s_2, s_3)$ and some Gram matrix B defined by $B_{ij} = -\cos(\pi/m_{ij})$. Write the Schläfli matrix as

$$J = 2B = \begin{pmatrix} 2 & a_1 & b_1 \\ a_1 & 2 & a_2 \\ b_1 & a_2 & 2 \end{pmatrix}.$$

(This particular labeling of the entries of J comes from the labelings we use for higher-rank examples in Chapters 7 and 8.) We know C is the upper unitriangular matrix satisfying $J = C + C^{\top}$.

$$C = \begin{pmatrix} 1 & a_1 & b_1 \\ 0 & 1 & a_2 \\ 0 & 0 & 1 \end{pmatrix}.$$

We may also compute C^{-1} and $-C^{-1}C^{\top}$.

$$C^{-1} = \begin{pmatrix} 1 & -a_1 & a_1a_2 - b_1 \\ 0 & 1 & -a_2 \\ 0 & 0 & 1 \end{pmatrix},$$

$$-C^{-1}C^{\top} = \begin{pmatrix} -a_1b_1a_2 + a_1^2 + b_1^2 - 1 & -a_1a_2^2 + b_1a_2 + a_1 & -a_1a_2 + b_1 \\ b_1a_2 - a_1 & a_2^2 - 1 & a_2 \\ -b_1 & -a_2 & -1 \end{pmatrix}.$$

We know from Theorem 6.1.1 that the matrix representation of $c = s_1 s_2 s_3$ (with respect to the standard basis of simple roots) is $[c] = -C^{-1}C^\top$. We may confirm this fact by computing the matrix of each simple reflection r_i and showing that $[r_1 r_2 r_3] = -C^{-1}C^\top$.

$$\begin{aligned}
[r_1] &= I - E_{1,1}J = \begin{pmatrix} -1 & -a_1 & -b_1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
[r_2] &= I - E_{2,2}J = \begin{pmatrix} 1 & 0 & 0 \\ -a_1 & -1 & -a_2 \\ 0 & 0 & 1 \end{pmatrix} \\
[r_3] &= I - E_{3,3}J = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -b_1 & -a_2 & -1 \end{pmatrix} \\
[r_1 r_2 r_3] &= \begin{pmatrix} -a_1 b_1 a_2 + a_1^2 + b_1^2 - 1 & -a_1 a_2^2 + b_1 a_2 + a_1 & -a_1 a_2 + b_1 \\ b_1 a_2 - a_1 & a_2^2 - 1 & a_2 \\ -b_1 & -a_2 & -1 \end{pmatrix} \\
&= -C^{-1}C^\top
\end{aligned}$$

Example 6.1.3 (Coxeter element for Sym_4). In Example 4.1.13, we saw the matrix representations $[r_1], [r_2], [r_3]$ of the simple reflections of the symmetric group $\text{Sym}_4 = \text{Cox}(A_3)$ in the standard basis $\mathcal{B} = (e_1, e_2, e_3)$ of simple roots. The usual Coxeter element $c = s_1 s_2 s_3$ has the matrix representation

$$[c] = [r_1][r_2][r_3] = \begin{pmatrix} -1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 1 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}.$$

The matrix C is

$$C = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}.$$

We may confirm that $[c] = -C^{-1}C^\top$.

$$-C^{-1}C^{\top} = - \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix} = [c].$$

Of course, we obtain the same result when plugging in $a_1 = a_2 = -2\cos(\pi/3) = -1$ and $b_1 = -2\cos(\pi/2) = 0$ into Example 6.1.2.

Example 6.1.4 ($\mathrm{PGL}(2, \mathbb{Z})$). Recall from Examples 3.2.11 and 4.1.9 that the projective general linear group $\mathrm{PGL}(2, \mathbb{Z})$ is a rank-3 Coxeter group whose Gram matrix is

$$\begin{pmatrix} 1 & -\frac{1}{2} & 0 \\ -\frac{1}{2} & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix},$$

from which we obtain the upper unitriangular matrix

$$C = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}$$

In Example 4.1.15, we obtained the matrix representations

$$[r_1] = \begin{pmatrix} -1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad [r_2] = \begin{pmatrix} 1 & 0 & 0 \\ 1 & -1 & 2 \\ 0 & 0 & 1 \end{pmatrix}, \quad [r_3] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & -1 \end{pmatrix}$$

for the simple reflections. On the one hand, we may compute the matrix representation $[c]$ of the Coxeter element $c = s_1 s_2 s_3$ as the product of the simple reflections.

$$[c] = [r_1][r_2][r_3] = \begin{pmatrix} -1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 1 & -1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 3 & -2 \\ 1 & 3 & -2 \\ 0 & 2 & -1 \end{pmatrix}.$$

On the other hand, we may compute $[c]$ as the product $-C^{-1}C^{\top}$, which gives the same result.

$$-C^{-1}C^{\top} = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -2 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 3 & -2 \\ 1 & 3 & -2 \\ 0 & 2 & -1 \end{pmatrix}.$$

When calculating matrix representations in different bases of various Coxeter elements, I observed matrix representations for arbitrary Coxeter elements of (W, S) that look similar to the matrix representation in Theorem 6.1.1. In Corollary 6.1.7.1, I show that we may obtain these matrix representations simply by reindexing the rows and columns of the upper unitriangular matrix C . Before discussing this corollary, we introduce two natural ordered bases induced by the Coxeter element.

Definition 6.1.5 (Word vs action order). Let $\tau \in \text{Sym}_d$ be some permutation. The Coxeter element $c_\tau = s_{\tau(1)}s_{\tau(2)} \cdots s_{\tau(d)}$ induces two natural bases on V , namely the *word basis* $\mathcal{B}_{\text{word}(\tau)} = (e_{\tau(1)}, e_{\tau(2)}, \dots, e_{\tau(d)})$ from reading left-to-right, and the *action basis* $\mathcal{B}_{\text{act}(\tau)} = (e_{\tau(d)}, \dots, e_{\tau(2)}, e_{\tau(1)})$ from reading right-to-left. (We may also use the subscripts $\text{word}(c_\tau)$ and $\text{act}(c_\tau)$ to emphasize the Coxeter element c_τ . When τ and c_τ are sufficiently clear from context, we may denote the word and action bases by $\mathcal{B}_{\text{word}}$ and $\mathcal{B}_{\text{act}}$ instead.) We call the order $(\tau(1), \dots, \tau(d))$ the *word order* induced by c_τ , because the i -th term corresponds with the i -th letter in c_τ . We call the reverse order $(\tau(d), \dots, \tau(1))$ the *action order* induced by c_τ , because the i -th term corresponds to the i -th simple reflection that acts on the left when viewing c_τ as an linear action via the standard geometric representation.

Let us see some examples of certain Coxeter elements written with respect to these two bases.

Example 6.1.6 (Coxeter element for a rank-3 Coxeter system). As in Example 6.1.2, let (W, S) be a rank-3 Coxeter system with generating set $S = (s_1, s_2, s_3)$, whose Gram matrix B is given by $B_{ij} = \cos(\pi/m_{ij})$. (Equivalently, the standard bilinear form B is defined by $B(e_i, e_j) = \cos(\pi/m_{ij})$.) Consider the Coxeter element $c = s_1s_2s_3$. The word basis is precisely the standard basis (e_1, e_2, e_3) ; in Example 6.1.2, we discussed the matrix representation $[c]_{\text{word}} = -C^{-1}C^\top$ with respect to this basis. To change to the

action basis (e_3, e_2, e_1) , we need only reverse the order of rows and columns, which can be accomplished by conjugating by the reversal matrix R_3 . (As in Remark 2.1.6, we use the superscript R to denote conjugating by the reversal matrix.) Thus, we obtain

$$[c]_{\text{act}} = -C^{-R}C^{R\top} = -(C^{-1}C^\top)^R.$$

Below, we show the relevant matrices.

$$\begin{aligned} C^R &= \begin{pmatrix} 1 & 0 & 0 \\ a_2 & 1 & 0 \\ b_1 & a_1 & 0 \end{pmatrix} \\ C^{-R} &= \begin{pmatrix} 1 & 0 & 0 \\ -a_2 & 1 & 0 \\ a_1a_2 - b_1 & -a_1 & 0 \end{pmatrix} \\ -(C^{-1}C^\top)^R &= \begin{pmatrix} -1 & -a_2 & -b_1 \\ a_2 & a_2^2 - 1 & b_1a_2 - a_1 \\ -a_1a_2 + b_1 & -a_1a_2^2 + b_1a_2 + a_1 & -a_1b_1a_2 + a_1^2 + b_1^2 - 1 \end{pmatrix} \end{aligned}$$

Example 6.1.7 (A different Coxeter element). In the same rank-3 Coxeter system (W, S) as Example 6.1.6, consider another Coxeter element, say $c_\tau = s_2s_1s_3$ induced by the transposition $\tau = (1, 2)$. The change-of-basis matrices from the standard basis $\mathcal{B}_{\text{std}} = (e_1, e_2, e_3)$ to the word basis $\mathcal{B}_{\text{word}} = (e_2, e_1, e_3)$ and the action basis $\mathcal{B}_{\text{act}} = (e_3, e_1, e_2)$ are the permutation matrices given below.

$${}_{\text{word}}[I]_{\text{std}} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad {}_{\text{act}}[I]_{\text{std}} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Knowing $[r_i]_{\text{std}}$ from Example 6.1.2, we may easily compute $[r_i]_{\text{word}}$ and $[r_i]_{\text{act}}$.

$$\begin{aligned}
[r_1]_{\text{word}} &= {}_{\text{word}}[I]_{\text{std}}[r_1]_{\text{std}}[I]_{\text{word}} = \begin{pmatrix} 1 & 0 & 0 \\ -a_1 & -1 & -b_1 \\ 0 & 0 & 1 \end{pmatrix} \\
[r_2]_{\text{word}} &= {}_{\text{word}}[I]_{\text{std}}[r_2]_{\text{std}}[I]_{\text{word}} = \begin{pmatrix} -1 & -a_1 & -a_2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
[r_3]_{\text{word}} &= {}_{\text{word}}[I]_{\text{std}}[r_3]_{\text{std}}[I]_{\text{word}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -a_2 & -b_1 & -1 \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
[r_1]_{\text{act}} &= {}_{\text{act}}[I]_{\text{std}}[r_1]_{\text{std}}[I]_{\text{act}} = \begin{pmatrix} 1 & 0 & 0 \\ -b_1 & -1 & -a_1 \\ 0 & 0 & 1 \end{pmatrix} \\
[r_2]_{\text{act}} &= {}_{\text{act}}[I]_{\text{std}}[r_2]_{\text{std}}[I]_{\text{act}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -a_2 & -a_1 & -1 \end{pmatrix} \\
[r_3]_{\text{act}} &= {}_{\text{act}}[I]_{\text{std}}[r_3]_{\text{std}}[I]_{\text{act}} = \begin{pmatrix} -1 & -b_1 & -a_2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}
\end{aligned}$$

Now we may compute the matrix representations of $c_\tau = r_2 r_1 r_3$.

$$\begin{aligned}
[r_2 r_1 r_3]_{\text{word}} &= \begin{pmatrix} -a_1 a_2 b_1 + a_1^2 + a_2^2 - 1 & -a_1 b_1^2 + a_2 b_1 + a_1 & -a_1 b_1 + a_2 \\ a_2 b_1 - a_1 & b_1^2 - 1 & b_1 \\ -a_2 & -b_1 & -1 \end{pmatrix} \\
[r_2 r_1 r_3]_{\text{act}} &= \begin{pmatrix} -1 & -b_1 & -a_2 \\ b_1 & b_1^2 - 1 & a_2 b_1 - a_1 \\ -a_1 b_1 + a_2 & -a_1 b_1^2 + a_2 b_1 + a_1 & -a_1 a_2 b_1 + a_1^2 + a_2^2 - 1 \end{pmatrix} \\
&= [r_2 r_1 r_3]_{\text{word}}^{\text{R}}
\end{aligned}$$

In the same way that the Coxeter element c has the matrix representations $[c]_{\text{word}(\text{id})} = -C^{-1}C^\top$ and $[c]_{\text{act}(\text{id})} = -(C^{-1}C^\top)^{\text{R}}$, we may write the Coxeter element c_τ in terms of a certain upper unitriangular matrix C_τ .

$$C_\tau = \begin{pmatrix} 1 & a_1 & a_2 \\ 0 & 1 & b_1 \\ 0 & 0 & 1 \end{pmatrix}, \quad C_\tau^{-1} = \begin{pmatrix} 1 & -a_1 & a_1 b_1 - a_2 \\ 0 & 1 & -b_1 \\ 0 & 0 & 1 \end{pmatrix}$$

By straightforward matrix multiplication, we obtain the following matrix representations for c_τ .

$$[c_\tau]_{\text{word}(\tau)} = -C_\tau^{-1} C_\tau^\top, \quad [c_\tau]_{\text{act}(\tau)} = -(C_\tau^{-1} C_\tau^\top)^R.$$

The following corollary shows that, as in the above examples, reordering the basis vectors via the permutation τ leads to upper and lower unitriangular matrix representations of the Coxeter element c_τ .

Corollary 6.1.7.1 (Matrix representations of an arbitrary Coxeter element). *Let $\tau \in \text{Sym}_d$ be a permutation, and let $c_\tau = s_{\tau(1)} s_{\tau(2)} \cdots s_{\tau(d)}$ be a Coxeter element of (W, S) . Then the matrix representation of c_τ with respect to the word basis of c_τ is*

$$[c_\tau]_{\text{word}} = -C_\tau^{-1} C_\tau^\top$$

where C_τ is the $d \times d$ upper unitriangular matrix defined by

$$(C_\tau)_{ij} = \begin{cases} 1, & \tau(i) = \tau(j) \\ 0, & \tau(i) > \tau(j) \\ 2B(e_{\tau(i)}, e_{\tau(j)}), & \tau(i) < \tau(j) \end{cases}$$

Additionally, the matrix representation of c_τ with respect to the action basis of c_τ is

$$[c_\tau]_{\text{act}} = -(C_\tau^{-1} C_\tau^\top)^R$$

where the superscript R denotes conjugating by the reversal matrix, as in Remark 2.1.6.

Proof. Consider the reordered generating set $S_\tau = (s_{\tau(1)}, s_{\tau(2)}, \dots, s_{\tau(d)})$. By Theorem 6.1.1 applied to the Coxeter element c_τ and the Coxeter system (W, S_τ) , we conclude

$$[c_\tau]_{\text{word}} = -C_\tau^{-1} C_\tau^\top.$$

The change-of-basis matrix ${}_{\text{word}}[I]_{\text{act}}$ is precisely the reversal matrix R_d , which reverses the order of the basis vectors. Thus, we conclude

$$[c_\tau]_{\text{act}} = -(C_\tau^{-1} C_\tau^\top)^{\text{R}}.$$

□

Remark 6.1.8. In the above proof, all we are doing is permuting basis elements by τ . Indeed, we simply have

$$C_\tau = {}_{\text{word}(\tau)}[I]_{\text{std}} C[I]_{\text{word}(\tau)}.$$

Now let

$$J_\tau = {}_{\text{word}(\tau)}[I]_{\text{std}} J[I]_{\text{word}(\tau)}$$

where J is the Schläfli matrix. Since $J = C + C^\top$, we obtain $J_\tau = C_\tau + C_\tau^\top$. Notice that $J_\tau = [\beta]_{\text{word}}$ where β is the polar form of the quadratic form Q associated with the standard geometric representation of (W, S) .

Notice that Corollary 6.1.7.1 gives us an *LU decomposition* (or *LU factorization*) of the matrix representation of the Coxeter element c_τ with respect to its action basis. Namely, we have

$$c_\tau = -C_\tau^{-\text{R}} C_\tau^{\text{R}\top}$$

where $C_\tau^{-\text{R}}$ is lower (uni)triangular and $C_\tau^{\text{R}\top}$ is upper (uni)triangular. For more about LU decompositions, see [HJ12, Section 3.5].

6.2 Matrix representations of Wall forms of Coxeter elements

Whereas the previous section discusses matrix representations of Coxeter elements, this section investigates matrix representations of the corresponding Wall forms. In particular, we discuss the matrix representation of the Wall forms χ_c and $\chi_{c^{-1}}$ with respect to the standard basis $(e_1, e_2, \dots, e_d)$ of simple roots. (When no basis is specified, we understand $[\chi_c]$ to denote the matrix representation of the Wall form of the Coxeter element $c = s_1 s_2 \cdots s_d$ with respect to the standard basis.) We also generalize our results to Wall forms of arbitrary Coxeter elements with respect to the word and action bases.

We saw in Theorem 6.1.1 that the matrix representation $[c]$ may be written in terms of the upper unitriangular matrix C introduced in Section 6.1. I show in the following theorem that the Wall form χ_c may also be written in terms of C .

Theorem 6.2.1 (Matrix representation of χ_c). *The matrix representation of the Wall form χ_c with respect to the standard basis $(e_1, e_2, \dots, e_d)$ of simple roots (i.e. the word basis of c) is*

$$[\chi_c] = C^\top.$$

Proof. Recall $\rho(c) = r_1 r_2 \cdots r_d$. (To simplify notation, we identify c and $\rho(c)$.) Clearly, we have $\text{Mov}(r_d) = \text{span}\{e_d\}$. By Lemma 2.4.5, since $e_{d-1} \notin \text{span}\{e_d\}$, we have $\text{Mov}(r_{d-1} r_d) = \text{span}\{e_{d-1}, e_d\}$. Continuing this process, we find that $\text{Mov}(c) = V$. In other words, c is fixed-point free (besides the origin), and in particular, the map $\text{id} - c$ is invertible, so for each $u \in V$, there exists $w = (\text{id} - c)^{-1}(u)$.

Recall from Remark 2.2.9 that the quadratic form $Q(u) = B(u, u)$ has the polar form $\beta(u, v) = 2B(u, v)$. In the standard geometric representation, the bilinear form B is

given by the Gram matrix B , so the polar form of Q has the matrix $[\beta] = J$ with respect to the standard basis of simple roots, where $J = 2B$ is the Schläfli matrix.

By the definition of Wall form, we may write χ_c in terms of the polar form β , namely

$$\chi_c(u, v) = \beta(w, v)$$

where $w = (\text{id} - c)^{-1}(u)$. Applying Theorem 6.1.1, we find that the linear map $(\text{id} - c)^{-1}$ has the matrix representation

$$[(\text{id} - c)^{-1}] = (I + C^{-1}C^\top)^{-1}$$

with respect to the standard basis. Since $[\beta] = J$, we find that the Wall form χ_c has the matrix representation

$$[\chi_c] = (I + C^{-1}C^\top)^{-\top} J$$

with respect to the standard basis. To prove the desired result $[\chi_c] = C^\top$, it suffices to show that $(I + C^{-1}C^\top)^\top C^\top = J$, which we can easily compute.

$$(I + C^{-1}C^\top)^\top C^\top = (I + CC^{-\top})C^\top = C^\top + C = J.$$

□

At this point, we have seen that the lower unitriangular matrix $C^\top$ is the matrix representation of the Wall form χ_c of the Coxeter element $c = s_1 s_2 \cdots s_d$, with respect to the standard basis of simple roots. By Lemma 2.4.4, we see that the upper unitriangular matrix C is the matrix representation of the Wall form $\chi_{c^{-1}}$ of the Coxeter element $c^{-1} = s_d \cdots s_2 s_1$, again with respect to the standard basis. As it turns out, for any Coxeter element of (W, S) , the Wall form is represented by an upper (respectively, lower) unitriangular matrix in the action basis (resp. word basis) induced by the Coxeter element. Corollary 6.2.1.1 below states the matrix representations of the Wall forms of

both c_τ and c_τ^{-1} with respect to the action and word bases.

Corollary 6.2.1.1 (Matrix representations of Wall forms). *Let $\tau \in \text{Sym}_d$ be some permutation, and let $c_\tau = s_{\tau(1)}s_{\tau(2)} \cdots s_{\tau(d)}$ be some Coxeter element of (W, S) . Let χ denote the Wall form of c_τ , and let χ' denote the Wall form of c_τ^{-1} . Then we obtain the following matrix representations of χ and χ' .*

$$(i) \quad [\chi]_{\text{word}(c_\tau)} = [\chi]_{\text{act}(c_\tau^{-1})} = C_\tau^\top$$

$$(ii) \quad [\chi]_{\text{act}(c_\tau)} = [\chi]_{\text{word}(c_\tau^{-1})} = C_\tau^{R\top}$$

$$(iii) \quad [\chi']_{\text{word}(c_\tau^{-1})} = [\chi']_{\text{act}(c_\tau)} = C_\tau^R$$

$$(iv) \quad [\chi']_{\text{act}(c_\tau^{-1})} = [\chi']_{\text{word}(c_\tau)} = C_\tau$$

Here, the matrix C_τ is the upper unitriangular matrix from Corollary 6.1.7.1, and the superscript R denotes conjugating by the reversal matrix, as in Remark 2.1.6.

Proof. The relevant bases are

$$\begin{aligned} \mathcal{B}_{\text{word}(c_\tau)} &= (e_{\tau(1)}, e_{\tau(2)}, \dots, e_{\tau(d)}) = \mathcal{B}_{\text{act}(c_\tau^{-1})} \\ \mathcal{B}_{\text{word}(c_\tau^{-1})} &= (e_{\tau(d)}, \dots, e_{\tau(2)}, e_{\tau(1)}) = \mathcal{B}_{\text{act}(c_\tau)} \end{aligned}$$

so it is clear that $[\chi]_{\text{word}(c_\tau)} = [\chi]_{\text{act}(c_\tau^{-1})}$, and so on.

As in the proof of Theorem 6.2.1, the linear map $\text{id} - c_\tau$ is invertible, so the Wall form χ in terms of the polar form β is

$$\chi(u, v) = \beta(w, v)$$

where $w = (\text{id} - c_\tau)^{-1}(u)$. Applying the same reasoning as in the proof of Theorem 6.2.1, we find that the matrix representation of χ with respect to the word basis of c_τ is

$$[\chi]_{\text{word}(c_\tau)} = (I + C_\tau^{-1}C_\tau^\top)^{-\top} J_\tau = C_\tau^\top,$$

as desired. To obtain $[\chi]_{\text{act}(c_\tau)}$, we achieve the desired change of basis by conjugating by the reversal matrix, which reverses the order of the basis vectors. At this point, we have obtained representations (i) and (ii).

By Property (iii) from Lemma 2.4.4, we know that $[\chi'] = [\chi]^\top$ (with respect to the same basis), so we obtain representations (iii) and (iv) by taking the transpose of representations (ii) and (i), respectively. $\square$

6.3 Hurwitz action as a change of basis

This section describes the Hurwitz action on reduced reflection factorizations of Coxeter elements in terms of matrices. By associating reduced reflection factorizations of the Coxeter element $c = s_1 s_2 \cdots s_d$ with ordered bases of roots, we may view the Hurwitz action as inducing a change of basis, which we may compute explicitly with matrices.

As before, we identify W with its linear representation $\rho(W)$, thereby viewing the Coxeter element $c = s_1 s_2 \cdots s_d$ as a product of simple reflections $r_1 r_2 \cdots r_d$. The proof of Theorem 6.2.1, showed that $\text{Mov}(c) = V$, so by Theorem 2.3.5, the reflection length of c is d .

Recall from Section 4.2 that there is a bijective correspondence between reflections $r_u \in T$ and “lax” roots $\pm u \in \Phi$. (As mentioned previously, the distinction between *strict* vectors v and *lax* vectors $\pm v$ comes from [CF97].) Thus, each reduced reflection factorization $c = r_{u_1} r_{u_2} \cdots r_{u_d}$ encodes a d -tuple of lax roots $(\pm u_1, \pm u_2, \dots, \pm u_d)$. Arbitrarily choosing a sign for each u_i yields a basis of V , which we denote as $(u_1, u_2, \dots, u_d)$ without loss of generality.

Lemma 6.3.1 (Reflection factorizations of c determine a basis). *Let $c = r_{u_1} r_{u_2} \cdots r_{u_d}$ be a reduced reflection factorization encoding a tuple $(u_1, u_2, \dots, u_d)$ of roots, with signs chosen arbitrarily from the lax roots. Then $(u_1, u_2, \dots, u_d)$ is a basis for V .*

Proof. From the definition of moved space, it is easy to see that for each $f_1, f_2 \in \mathcal{O}(V)$, we have $\text{Mov}(f_1 f_2) \subseteq \text{Mov}(f_1) + \text{Mov}(f_2)$. Thus, we have

$$V = \text{Mov}(c) \subseteq \text{Mov}(r_{u_1}) + \text{Mov}(r_{u_2}) + \cdots + \text{Mov}(r_{u_d}) \subseteq V,$$

so we attain equality

$$V = \text{Mov}(r_{u_1}) + \text{Mov}(r_{u_2}) + \cdots + \text{Mov}(r_{u_d}).$$

Since $\text{Mov}(r_u) = \langle u \rangle$, we have

$$V = \text{span}\{u_1, u_2, \dots, u_d\}.$$

Since $\dim(V) = d$, we conclude that $(u_1, u_2, \dots, u_d)$ is a basis for V . $\square$

Recall from Theorem 5.2.3 that the Hurwitz action is transitive on reduced reflection factorizations of a Coxeter element. Hence, for two reflection factorizations

$$c = r_{u_1} r_{u_2} \cdots r_{u_d} = r_{v_1} r_{v_2} \cdots r_{v_d},$$

there exists a braid $\sigma \in \text{Braid}_d$ sending one ordered lax basis $(\pm u_1, \pm u_2, \dots, \pm u_d)$ to the other lax basis $(\pm v_1, \pm v_2, \dots, \pm v_d)$. With respect to the correspondence between reflections and lax roots, one choice of signs for $(\pm u_1, \dots, \pm u_d)$ is equivalent to any another. However, we want to choose a choice of signs so that the Hurwitz action on reduced reflection factorizations of c lifts to an action on appropriate d -tuples of roots. We now define one such valid choice, whereby the Hurwitz action induces a change of basis on V .

Definition 6.3.2 (Change of basis given by a half-twist). Let $\mathcal{B} = (u_1, \dots, u_d)$ be a basis for V , consisting of (strict) roots (with signs chosen arbitrarily from the lax roots) corresponding to a minimal length reflection factorization of c . Let $x = \chi_{c^{-1}}(u_i, u_{i+1})$. (Notice that $x = \beta(u_i, u_{i+1}) = 2B(u_i, u_{i+1})$ where β is the polar form of the quadratic

form Q from the standard geometric representation, and B is the standard bilinear form for (W, S) .) For each positive half-twist $\sigma_i \in \text{Braid}_d$, we define ${}_B M_{\sigma_i \cdot \mathcal{B}}$ to be the block diagonal matrix

$${}_B M_{\sigma_i \cdot \mathcal{B}} = \begin{pmatrix} I_{i-1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{M} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & I_{d-i-1} \end{pmatrix}$$

where $\mathbf{M}$ is the 2×2 matrix

$$\mathbf{M} = \begin{pmatrix} x & 1 \\ -1 & 0 \end{pmatrix}.$$

For each negative half-twist σ_i^{-1} , define ${}_B M_{\sigma_i^{-1} \cdot \mathcal{B}} = {}_B M_{\sigma_i \cdot \mathcal{B}}^{-1}$, which we may write as

$${}_B M_{\sigma_i^{-1} \cdot \mathcal{B}} = \begin{pmatrix} I_{i-1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{M}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & I_{d-i-1} \end{pmatrix}$$

with

$$\mathbf{M}^{-1} = \begin{pmatrix} 0 & -1 \\ 1 & x \end{pmatrix}.$$

When σ_i and $\mathcal{B}$ are understood, we may suppress the subscripts to write M instead of ${}_B M_{\sigma_i \cdot \mathcal{B}}$.

The positive half-twist σ_i gives us a change-of-basis matrix ${}_B M_{\sigma_i \cdot \mathcal{B}} = {}_B [I]_{\sigma_i \cdot \mathcal{B}}$ where the basis $\sigma_i \cdot \mathcal{B} = (v_1, v_2, \dots, v_d)$ is given by

$$v_j = \begin{cases} u_j, & j \notin \{i, i+1\} \\ xu_i - u_{i+1}, & j = i \\ u_i, & j = i+1 \end{cases}$$

where $x = \chi_{c^{-1}}(u_i, u_{i+1})$. Similarly, the negative half-twist σ_i^{-1} gives us a change-of-basis matrix ${}_B M_{\sigma_i^{-1} \cdot \mathcal{B}} = {}_B [I]_{\sigma_i^{-1} \cdot \mathcal{B}}$ where the basis $\sigma_i^{-1} \cdot \mathcal{B} = (w_1, w_2, \dots, w_d)$ is given by

$$w_j = \begin{cases} u_j, & j \notin \{i, i+1\} \\ u_{i+1}, & j = i \\ xu_{i+1} - u_i, & j = i+1 \end{cases}$$

One may easily confirm that if $\mathcal{B}' = \sigma_i \cdot \mathcal{B}$, then $\sigma_i^{-1} \cdot \mathcal{B}' = \mathcal{B}$, as expected.

Remark 6.3.3 (Other valid induced Hurwitz actions). Later in this section, Proposition 6.3.8 states that induced Hurwitz action on d -tuples of roots arising from the change of basis given in Definition 6.3.2 is well-defined; the proof confirms that the braid relations hold. However, there are other valid choices on the signs (choosing strict roots from lax roots) that also lead to a well-defined Hurwitz action on d -tuples of roots. For example, we could have defined the change-of-basis matrix ${}_{\mathcal{B}}M_{\sigma_i\mathcal{B}}$ as

$${}_{\mathcal{B}}M_{\sigma_i\mathcal{B}} = \begin{pmatrix} I_{i-1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\mathbf{M} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & I_{d-i-1} \end{pmatrix}$$

where $\mathbf{M}$ is the 2×2 matrix given in Definition 6.3.2. This alternative choice for the change-of-basis matrix also leads to a valid induced Hurwitz action on d -tuples of roots; one may verify that the braid relations hold.

Example 6.3.4 (Positive half-twist acting on the standard basis). Let $\mathcal{B}_0 = (e_1, e_2, \dots, e_d)$ be the standard basis of simple roots. Then the new basis $\sigma_i \cdot \mathcal{B}_0 = (u_1, u_2, \dots, u_d)$ induced by the i -th positive half-twist is given by

$$u_j = \begin{cases} e_j, & j \notin \{i, i+1\} \\ xe_i - e_{i+1}, & j = i \\ e_i, & j = i+1 \end{cases}$$

where $x = \chi_{c^{-1}}(e_i, e_{i+1}) = C_{i,i+1}$. The corresponding $\sigma_i\mathcal{B}_0\text{-}\mathcal{B}_0$ change-of-basis matrix is the block diagonal matrix

$$M = {}_{\mathcal{B}_0}[I]_{\sigma_i\mathcal{B}_0} = \begin{pmatrix} I_{i-1} & 0 & 0 \\ 0 & \mathbf{M} & 0 \\ 0 & 0 & I_{d-i-1} \end{pmatrix}$$

where $\mathbf{M}$ is the 2×2 matrix

$$\mathbf{M} = \begin{pmatrix} x & 1 \\ -1 & 0 \end{pmatrix}.$$

Notice that the basis $\sigma_i \mathcal{B}_0$ consists of roots, corresponding to the reduced reflection factorization given by the Hurwitz action $\sigma_i \cdot (r_1, r_2, \dots, r_d)$. Indeed, writing $(r'_1, r'_2, \dots, r'_d) = \sigma_i \cdot (r_1, r_2, \dots, r_d)$, the j -th reflection in r'_j is

$$r'_j = \begin{cases} r_j, & j \notin \{i, i+1\} \\ r_i r_{i+1} r_i^{-1}, & j = i \\ r_i, & j = i+1 \end{cases}$$

Clearly, for $j \neq i$, the basis vector u_j is a simple root corresponding to the reflection r'_j , so $r'_j = r_{u_j}$. We may confirm that $u_i = x e_i - e_{i+1}$ is a root corresponding to r'_i (i.e. $r'_i = r_{u_i}$) by showing that $Q(u_i) = 1$ and $r'_i(u_i) = -u_i$. First, we compute $Q(u_i)$ as follows.

$$\begin{aligned} Q(u_i) &= Q(x e_i - e_{i+1}) \\ &= x^2 B(e_i, e_i) - 2x B(e_i, e_{i+1}) + B(e_{i+1}, e_{i+1}) \\ &= (2B_{i,i+1})^2 - 4B_{i,i+1}^2 + 1 \\ &= 1 \end{aligned}$$

Next, we compute $r'_i(u_i)$.

$$\begin{aligned}
r'_i(u_i) &= (r_i r_{i+1} r_i^{-1})(x e_i - e_{i+1}) \\
&= (r_i r_{i+1} r_i)(x e_i - e_{i+1}) \\
&= (r_i r_{i+1})(-x e_i - (e_{i+1} - 2B(e_i, e_{i+1})e_i)) \\
&= (r_i r_{i+1})(-x e_i - (e_{i+1} - x e_i)) \\
&= (r_i r_{i+1})(-e_{i+1}) \\
&= r_i(e_{i+1}) \\
&= e_{i+1} - 2B(e_i, e_{i+1})e_i \\
&= e_{i+1} - x e_i \\
&= -u_i
\end{aligned}$$

Thus, the basis $\sigma_i \cdot \mathcal{B}_0$ is an ordered basis of roots corresponding to the reflections in the reduced reflection factorization given by the Hurwitz action via σ_i on $c = r_1 r_2 \cdots r_d$.

Starting with the standard basis $\mathcal{B}_0 = (e_1, \dots, e_d)$ of simple roots, we may consider the set $\mathfrak{B}$ of (ordered) bases obtainable by applying sequences of positive and negative half-twists. As it turns out, definition of $\sigma_i \cdot \mathcal{B}$ from Definition 6.3.2 extends to an induced braid action $\text{Braid}_d \times \mathfrak{B} \rightarrow \mathfrak{B}$, a Hurwitz action on bases associated with reduced reflection factorizations of the usual Coxeter element $c = s_1 s_2 \cdots s_d$. We refer to a change of basis given by this action as a *Hurwitz change of basis*, and each basis in $\mathfrak{B}$ as a *Hurwitz basis*. Before discussing this Hurwitz action on bases in Proposition 6.3.8 below, let us illustrate this action with a familiar example.

Example 6.3.5 (Hurwitz action for a rank-3 Coxeter system). As in Example 6.1.2, let (W, S) be a rank-3 Coxeter system with generating set $S = (s_1, s_2, s_3)$ and Schläfli matrix

$$J = \begin{pmatrix} 2 & a_1 & b_1 \\ a_1 & 2 & a_2 \\ b_1 & a_2 & 2 \end{pmatrix},$$

whereby the Wall form matrix $C = [\chi_{c^{-1}}]_{\mathcal{B}}$ of the Coxeter element $c^{-1} = s_3 s_2 s_1$ with respect to the standard basis $\mathcal{B}_0 = (e_1, e_2, e_3)$ of simple roots is

$$C = \begin{pmatrix} 1 & a_1 & b_1 \\ 0 & 1 & a_2 \\ 0 & 0 & 1 \end{pmatrix}.$$

Let us compute the bases $\sigma_1\sigma_2\sigma_1\mathcal{B}_0$ and $\sigma_2\sigma_1\sigma_2\mathcal{B}_0$. (Since the Hurwitz action is a braid action, these bases should be the same.) First, denote the relevant bases as follows.

$$\begin{aligned} \sigma_1\mathcal{B}_0 &= (u_1, u_2, u_3), & \sigma_2\mathcal{B}_0 &= (u'_1, u'_2, u'_3) \\ \sigma_2\sigma_1\mathcal{B}_0 &= (v_1, v_2, v_3), & \sigma_1\sigma_2\mathcal{B}_0 &= (v'_1, v'_2, v'_3) \\ \sigma_1\sigma_2\sigma_1\mathcal{B}_0 &= (w_1, w_2, w_3), & \sigma_2\sigma_1\sigma_2\mathcal{B}_0 &= (w'_1, w'_2, w'_3). \end{aligned}$$

Below are the corresponding change-of-basis matrices.

$$\begin{aligned} \mathcal{B}_0 M_{\sigma_1\mathcal{B}_0} &= \begin{pmatrix} x & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & \mathcal{B}_0 M_{\sigma_2\mathcal{B}_0} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & x' & 1 \\ 0 & -1 & 0 \end{pmatrix}, \\ \sigma_1\mathcal{B}_0 M_{\sigma_2\sigma_1\mathcal{B}_0} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & y & 1 \\ 0 & -1 & 0 \end{pmatrix}, & \sigma_2\mathcal{B}_0 M_{\sigma_1\sigma_2\mathcal{B}_0} &= \begin{pmatrix} y' & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\ \sigma_2\sigma_1\mathcal{B}_0 M_{\sigma_1\sigma_2\sigma_1\mathcal{B}_0} &= \begin{pmatrix} z & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & \sigma_1\sigma_2\mathcal{B}_0 M_{\sigma_2\sigma_1\sigma_2\mathcal{B}_0} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & z' & 1 \\ 0 & -1 & 0 \end{pmatrix}. \end{aligned}$$

We may compute the entries using the Wall form $\chi_{c^{-1}}$.

$$\begin{aligned} x &= \chi_{c^{-1}}(e_1, e_2), & x' &= \chi_{c^{-1}}(e_2, e_3), \\ y &= \chi_{c^{-1}}(u_2, u_3), & y' &= \chi_{c^{-1}}(u'_1, u'_2), \\ z &= \chi_{c^{-1}}(v_1, v_2), & z' &= \chi_{c^{-1}}(v'_2, v'_3). \end{aligned}$$

We know $x = C_{12} = a_1$ and $x' = C_{23} = a_2$. We also know that $\sigma_1\mathcal{B}_0$ and $\sigma_2\mathcal{B}_0$ are given by

$$\begin{aligned}\sigma_1\mathcal{B}_0 &= (u_1, u_2, u_3) = (a_1e_1 - e_2, e_1, e_3) \\ \sigma_2\mathcal{B}_0 &= (u'_1, u'_2, u'_3) = (e_1, a_2e_2 - e_3, e_2)\end{aligned}$$

so we may compute y and y' as follows.

$$\begin{aligned}y &= \chi_{c^{-1}}(u_2, u_3) = \chi_{c^{-1}}(e_1, e_3) = b_1 \\ y' &= \chi_{c^{-1}}(u'_1, u'_2) = \chi_{c^{-1}}(e_1, a_2e_2 - e_3) = a_1a_2 - b_1\end{aligned}$$

The bases $\sigma_2\sigma_1\mathcal{B}_0$ and $\sigma_1\sigma_2\mathcal{B}_0$ are

$$\begin{aligned}\sigma_2\sigma_1\mathcal{B}_0 &= (v_1, v_2, v_3) = (u_1, yu_2 - u_3, u_2) = (a_1e_1 - e_2, b_1e_1 - e_3, e_1) \\ \sigma_1\sigma_2\mathcal{B}_0 &= (v'_1, v'_2, v'_3) = (y'u'_1 - u'_2, u'_1, u'_3) = ((a_1a_2 - b_1)e_1 - a_2e_2 + e_3, e_1, e_2)\end{aligned}$$

so we may compute z and z' as follows.

$$\begin{aligned}z &= \chi_{c^{-1}}(v_1, v_2) = \chi_{c^{-1}}(a_1e_1 - e_2, b_1e_1 - e_3) = a_1b_1 - a_1b_1 - 0 + a_2 = a_2 \\ z' &= \chi_{c^{-1}}(v'_2, v'_3) = \chi_{c^{-1}}(e_1, e_2) = a_1\end{aligned}$$

Thus, the bases $\sigma_1\sigma_2\sigma_1\mathcal{B}_0$ and $\sigma_2\sigma_1\sigma_2\mathcal{B}_0$ are

$$\begin{aligned}\sigma_1\sigma_2\sigma_1\mathcal{B}_0 &= (w_1, w_2, w_3) \\ &= (zv_1 - v_2, v_1, v_3) \\ &= (a_1(a_1e_1 - e_2) - (b_1e_1 - e_3), a_1e_1 - e_2, e_1) \\ &= ((a_1^2 - b_1)e_1 - a_1e_2 + e_3, a_1e_1 - e_2, e_1) \\ &= (v'_1, z'v'_2 - v'_3, v'_2) \\ &= (w'_1, w'_2, w'_3) \\ &= \sigma_2\sigma_1\sigma_2\mathcal{B}_0,\end{aligned}$$

as expected. The change-of-basis matrix ${}_{\mathcal{B}_0}M_{\sigma_1\sigma_2\sigma_1\mathcal{B}_0} = {}_{\mathcal{B}_0}M_{\sigma_2\sigma_1\sigma_2\mathcal{B}_0}$ is given below.

$$\begin{aligned}
\begin{pmatrix} a_1 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & b_1 & 1 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} a_2 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} &= \begin{pmatrix} a_1 a_2 - b_1 & a_1 & 1 \\ -a_2 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 \\ 0 & a_2 & 1 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} a_1 a_2 - b_1 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & a_1 & 1 \\ 0 & -1 & 0 \end{pmatrix}
\end{aligned}$$

Later in this section, Proposition 6.3.8 shows that the change of basis described by Definition 6.3.2 indeed extends to a well-defined Hurwitz action; in particular, two distinct sequences of half-twists representing the same braid always have the same change-of-basis matrix. Before stating this proposition, we introduce some useful lemmas about the change of basis in terms of sequences of half-twists.

Recall the matrix representation C of the Wall form $\chi_{c^{-1}}$ with respect to the standard basis of simple roots, defined in Section 6.1. This matrix representation is upper unitriangular. Equivalently, the following two facts about the Wall form $\chi_{c^{-1}}$ hold:

- (i) For each $i \in [d]$, we have $\chi_{c^{-1}}(e_i, e_i) = 1$, and
- (ii) For each pair $i > j$, we have $\chi_{c^{-1}}(e_i, e_j) = 0$.

The following lemma shows that the matrix representation of $\chi_{c^{-1}}$ remains upper unitriangular after applying the Hurwitz change of basis given by any sequence of half-twists.

Lemma 6.3.6. *Consider some sequence $\beta = \sigma_{i_n}^{\epsilon_n} \cdots \sigma_{i_2}^{\epsilon_2} \sigma_{i_1}^{\epsilon_1}$ of (positive and/or negative) half-twists. Let $\mathcal{B} = (u_1, u_2, \dots, u_d)$ be the basis obtained by the change of basis given by applying the sequence β to the standard basis $\mathcal{B}_0 = (e_1, e_2, \dots, e_d)$ of simple roots, written $\mathcal{B} = \beta \cdot \mathcal{B}_0$. Then*

- (i) *for each $i \in [d]$, we have $\chi_{c^{-1}}(u_i, u_i) = 1$, and*
- (ii) *for each pair $i > j$, we have $\chi_{c^{-1}}(u_i, u_j) = 0$.*

Proof. We may proceed by induction on the length n of the sequence $\beta = \sigma_{i_n}^{\epsilon_n} \cdots \sigma_{i_2}^{\epsilon_2} \sigma_{i_1}^{\epsilon_1}$.

To simplify notation, we denote $\chi = \chi_{c^{-1}}$.

For the base case $n = 1$, write $\sigma_i = \sigma_{i_1}$ and $\epsilon = \epsilon_1$ for simplicity. We must consider two cases, namely $\beta = \sigma_i^\pm$. For the positive case $\epsilon = +1$, the basis $\mathcal{B} = \sigma_i \cdot \mathcal{B}_0 = (u_1, u_2, \dots, u_d)$ is given by

$$u_j = \begin{cases} e_j, & j \notin \{i, i+1\} \\ \chi(e_i, e_{i+1})e_i - e_{i+1}, & j = i \\ e_i, & j = i+1 \end{cases}$$

as in Definition 6.3.2 and Example 6.3.4. For each $j \in [d]$, we have

$$\chi(u_j, u_j) = \begin{cases} \chi(e_j, e_j) = 1, & j \notin \{i, i+1\} \\ \chi(e_i, e_i) = 1, & j = i+1 \\ \chi(\chi(e_i, e_{i+1})e_i - e_{i+1}, \chi(e_i, e_{i+1})e_i - e_{i+1}), & j = i. \end{cases}$$

In the third case, where $j = i$, we compute the following.

$$\begin{aligned} \chi(u_i, u_i) &= \chi(\chi(e_i, e_{i+1})e_i - e_{i+1}, \chi(e_i, e_{i+1})e_i - e_{i+1}) \\ &= \chi(e_i, e_{i+1})^2 \chi(e_i, e_i) + \chi(e_i, e_{i+1})(\chi(e_i, -e_{i+1}) + \chi(-e_{i+1}, e_i)) + \chi(-e_{i+1}, -e_{i+1}) \\ &= \chi(e_i, e_{i+1})^2 - \chi(e_i, e_{i+1})^2 + 1 \\ &= 1. \end{aligned}$$

Thus, statement (i) holds.

Continuing with the base case, now suppose $j > k$. We have four subcases.

- (a) $j, k \notin \{i, i+1\}$,
- (b) $j = i > k$,
- (c) $j = i+1$ with $k < i$, and
- (d) $j = i+1$ with $k = i$.

For subcase (a), it is clear that $\chi(u_j, u_k) = \chi(e_j, e_k) = 0$. Similarly, for subcase (c), we immediately obtain $\chi(u_j, u_k) = \chi(e_i, e_k) = 0$. For subcase (b), we compute

$$\begin{aligned}\chi(u_j, u_k) &= \chi(\chi(e_i, e_{i+1})e_i - e_{i+1}, e_k) \\ &= \chi(e_i, e_{i+1})\chi(e_i, e_k) - \chi(e_{i+1}, e_k) \\ &= 0 - 0 = 0.\end{aligned}$$

Finally, for subcase (d), we compute

$$\begin{aligned}\chi(u_j, u_k) &= \chi(e_i, \chi(e_i, e_{i+1})e_i - e_{i+1}) \\ &= \chi(e_i, e_{i+1})\chi(e_i, e_i) - \chi(e_i, e_{i+1}) \\ &= \chi(e_i, e_{i+1}) - \chi(e_i, e_{i+1}) = 0.\end{aligned}$$

Thus, statement (ii) holds.

For the negative case $\epsilon = -1$, the basis $\mathcal{B}' = \sigma_i^{-1} \cdot \mathcal{B} = (u_1, u_2, \dots, u_d)$ is given by

$$u_j = \begin{cases} e_j, & j \notin \{i, i+1\} \\ e_{i+1}, & j = i \\ \chi(e_i, e_{i+1})e_{i+1} - e_i, & j = i+1 \end{cases}$$

as in Definition 6.3.2. The proof that statements (i) and (ii) hold are similar to the positive case, so we omit the proof and consider the base case done.

For the inductive step, suppose that the proposition holds for products of half-twists up to length n , written $\beta = \sigma_{i_n}^{\epsilon_n} \cdots \sigma_{i_2}^{\epsilon_2} \sigma_{i_1}^{\epsilon_1}$. It suffices to show that it holds for the braids $\sigma_{i_{n+1}}^{\epsilon_{n+1}} \beta$ with $\epsilon_{n+1} = \pm 1$. For simplicity of notation, write $\sigma_i = \sigma_{i_{n+1}}$ and $\epsilon_{n+1} = \epsilon$. As with the base case, we show that the proposition holds for the positive case $\epsilon = +1$, and omit the proof of the negative case $\epsilon = -1$, since the proofs are substantially similar.

Write $\mathcal{B} = \beta \cdot \mathcal{B}_0 = (u_1, \dots, u_d)$ and $\mathcal{B}' = \sigma_i \cdot \mathcal{B} = (v_1, \dots, v_d)$. For each $j \in [d]$, we have

$$\chi(v_j, v_j) = \begin{cases} \chi(u_j, u_j), & j \notin \{i, i+1\} \\ \chi(u_i, u_i), & j = i+1 \\ \chi(\chi(u_i, u_{i+1})u_i - u_{i+1}, \chi(u_i, u_{i+1})u_i - u_{i+1}), & j = i \end{cases}$$

By the induction hypothesis, we know $\chi(u_j, u_j) = 1$, so for $j \neq i$, it is immediately clear that $\chi(v_j, v_j) = 1$. As in the analogous part of the base case, one may easily compute that $\chi(v_i, v_i) = 1$, whereby statement (i) holds.

Now suppose $j > k$. As in the base case, we have four subcases.

- (a) $j, k \notin \{i, i+1\}$,
- (b) $j = i > k$,
- (c) $j = i+1$ with $k < i$, and
- (d) $j = i+1$ with $k = i$.

For subcase (a), it is clear that $\chi(v_j, v_k) = \chi(v_j, v_k) = 0$. Similarly, for subcase (c), we immediately obtain $\chi(v_j, v_k) = \chi(v_i, v_k) = 0$. As in the base case, the computations for subcases (b) and (d) are straightforward enough that we omit them. Thus, statement (ii) holds. $\square$

For the Wall form matrix $C = [\chi_{c^{-1}}]_{\mathcal{B}_0}$ with respect to the standard basis $\mathcal{B}_0 = (e_1, e_2, \dots, e_d)$ of simple roots, let $\beta \cdot C$ denote the matrix representation of the Wall form $\chi_{c^{-1}}$ after applying the Hurwitz action via the braid $\beta \in \text{Braid}_d$ (treating β as a sequence of half-twists). The following corollary shows that $\beta \cdot C$ is always upper unitriangular.

Corollary 6.3.6.1 ($\beta \cdot C$ is upper unitriangular). *Let $\beta \in \text{Braid}_d$ be a sequence of half-twists, and let $\mathcal{B}_0$ denote the standard basis of simple roots. Then the Wall form matrix $\beta \cdot C = [\chi_{c^{-1}}]_{\beta \cdot \mathcal{B}_0}$ is upper unitriangular.*

Proof. Write $\beta \cdot \mathcal{B}_0 = (u_1, u_2, \dots, u_d)$. For each (i, j) , the (i, j) -th entry of $\beta \cdot C$ is

$$(\beta \cdot C)_{ij} = \chi_{c^{-1}}(u_i, u_j).$$

By Lemma 6.3.6, we immediately conclude that $\beta \cdot C$ is upper unitriangular. $\square$

Another useful fact is that the Hurwitz action via a half-twist σ_i on a basis $\mathcal{B}$ preserves the Wall form $\chi_{c^{-1}}$ applied to the i -th and $(i+1)$ -th basis elements.

Lemma 6.3.7. *Consider the positive half-twist $\sigma_i \in \text{Braid}_d$. Let $\mathcal{B} = (u_1, u_2, \dots, u_d)$ be any basis obtainable by the change of basis given by a sequence of half-twists applied to the standard basis $\mathcal{B}_0 = (e_1, e_2, \dots, e_d)$ of simple roots, and let $\mathcal{B}' = (v_1, v_2, \dots, v_d) = \sigma_i \cdot \mathcal{B}$. Then*

$$\chi_{c^{-1}}(v_i, v_{i+1}) = \chi_{c^{-1}}(u_i, u_{i+1}).$$

Proof. For simplicity of notation, write $\chi = \chi_{c^{-1}}$. As discussed in Section 6.3, we know that

$$v_i = \chi(u_i, u_{i+1})u_i - u_{i+1} \quad \text{and} \quad v_{i+1} = u_i.$$

From Lemma 6.3.6, we know $\chi(u_i, u_i) = 1$ and $\chi(u_{i+1}, u_i) = 0$. We compute $\chi(v_i, v_{i+1})$ as follows.

$$\begin{aligned} \chi(v_i, v_{i+1}) &= \chi(\chi(u_i, u_{i+1})u_i - u_{i+1}, u_i) \\ &= \chi(u_i, u_{i+1})\chi(u_i, u_i) - \chi(u_{i+1}, u_i) \\ &= \chi(u_i, u_{i+1}). \end{aligned}$$

$\square$

The following proposition states that by extending the definition of $\sigma_i \cdot \mathcal{B}$ from Definition 6.3.2, the Hurwitz action on reflection factorizations of the Coxeter element c indeed

induces a Hurwitz action on the collection $\mathfrak{B}$ of ordered bases obtainable by applying sequences of half-twists, as described earlier in this section.

Proposition 6.3.8 (Hurwitz action as a change of basis). *Let $\mathfrak{B}$ denote the collection of ordered bases of roots obtainable by sequences of Hurwitz moves applied to the standard basis of simple roots $(e_1, e_2, \dots, e_d)$. Then the definition of $\sigma_i \cdot \mathcal{B}$ from Definition 6.3.2 extends to a braid action $\text{Braid}_d \times \mathfrak{B} \rightarrow \mathfrak{B}$. Furthermore, for each braid $\beta \in \text{Braid}_d$, the j -th vector in the ordered basis $\beta \cdot (e_1, e_2, \dots, e_d)$ is a root from the root system Φ of the Coxeter system (W, S) , and is sent to its negative by the j -th reflection t_j in the reduced reflection factorization $c = t_1 t_2 \cdots t_d$ obtained by the Hurwitz action on the usual factorization $c = s_1 s_2 \cdots s_d$ via the braid β . Thus, the Hurwitz action on reduced reflection factorizations of the Coxeter element c induces a transitive braid group action on $\mathfrak{B}$, which we call the induced Hurwitz action on $\mathfrak{B}$.*

Proof. To show that the induced Hurwitz action is well-defined, it suffices to show that the braid relations hold. Let $\mathcal{B} = (u_1, u_2, \dots, u_d) \in \mathfrak{B}$. To simplify notation, let $\chi = \chi_{c^{-1}}$. For integers $i, j \in [d]$ satisfying $|i - j| > 2$, we may easily show that the commuting relation $\sigma_i \sigma_j = \sigma_j \sigma_i$ holds. Without loss of generality, let $i < j$. Applying Definition 6.3.2, the basis $\sigma_j \mathcal{B} = (v_1, v_2, \dots, v_d)$ is given by

$$v_k = \begin{cases} u_k, & k \notin \{j, j+1\} \\ x_j u_j - u_{j+1}, & k = j \\ u_j, & k = j+1 \end{cases}$$

where $x_j = \chi(u_j, u_{j+1})$. Since $|i - j| > 2$, we have $v_i = u_i$ and $v_{i+1} = u_{i+1}$. Letting $x_i = \chi(v_i, v_{i+1}) = \chi(u_i, u_{i+1})$, the basis $\sigma_i \sigma_j \mathcal{B} = (w_1, w_2, \dots, w_d)$ is

$$w_k = \begin{cases} v_k, & k \notin \{i, i+1\} \\ x_i v_i - v_{i+1}, & k = i \\ v_{i+1}, & k = i+1 \end{cases}$$

which can be rewritten as

$$w_k = \begin{cases} u_k, & k \notin \{i, i+1, j, j+1\} \\ x_i u_i - u_{i+1}, & k = i \\ u_i, & k = i+1 \\ x_j u_j - u_{j+1}, & k = j \\ u_j, & k = j+1 \end{cases}$$

By a very similar computation, we may confirm that $\sigma_j \sigma_i \mathcal{B} = \sigma_i \sigma_j \mathcal{B}$.

Now for the braid relation $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$, consider the bases

$$\begin{aligned} \sigma_i \mathcal{B} &= (v_1, v_2, \dots, v_d), & \sigma_{i+1} \mathcal{B} &= (v'_1, v'_2, \dots, v'_d) \\ \sigma_{i+1} \sigma_i \mathcal{B} &= (w_1, w_2, \dots, w_d), & \sigma_i \sigma_{i+1} \mathcal{B} &= (w'_1, w'_2, \dots, w'_d) \\ \sigma_i \sigma_{i+1} \sigma_i \mathcal{B} &= (z_1, z_2, \dots, z_d), & \sigma_{i+1} \sigma_i \sigma_{i+1} \mathcal{B} &= (z'_1, z'_2, \dots, z'_d). \end{aligned}$$

We may compute these bases in succession by repeatedly applying Definition 6.3.2. We easily see that $\sigma_i \mathcal{B}$ is given by

$$v_j = \begin{cases} u_j, & j \notin \{i, i+1\} \\ x u_i - u_{i+1}, & j = i \\ u_i, & j = i+1 \end{cases}$$

where $x = \chi(u_i, u_{i+1})$. Next, we may write the elements of $\sigma_{i+1} \sigma_i \mathcal{B}$ as

$$w_j = \begin{cases} v_j, & j \notin \{i+1, i+2\} \\ y v_{i+1} - v_{i+2}, & j = i+1 \\ v_{i+1}, & j = i+2 \end{cases}$$

where $y = \chi(v_{i+1}, v_{i+2}) = \chi(u_i, u_{i+2})$. Knowing the v_j 's, we can rewrite the w_j 's as follows.

$$w_j = \begin{cases} u_j, & j \notin \{i, i+1, i+2\} \\ x u_i - u_{i+1}, & j = i \\ y u_i - u_{i+2}, & j = i+1 \\ u_i, & j = i+2 \end{cases}$$

Continuing with $\sigma_i \sigma_{i+1} \sigma_i \mathcal{B}$, we have

$$z_j = \begin{cases} w_j, & j \notin \{i, i+1\} \\ zw_i - w_{i+1}, & j = i \\ w_i, & j = i+1 \end{cases}$$

where

$$z = \chi(w_i, w_{i+1}) = \chi(xu_i - u_{i+1}, yu_i - u_{i+2}).$$

We know from Lemma 6.3.6 that $\chi(u_i, u_i) = 1$ and $\chi(u_{i+1}, u_i) = 0$, so we obtain

$$\begin{aligned} z &= xy\chi(u_i, u_i) - x\chi(u_i, u_{i+2}) - y\chi(u_{i+1}, u_i) + \chi(u_{i+1}, u_{i+2}) \\ &= xy - xy - 0 + \chi(u_{i+1}, u_{i+2}) \\ &= \chi(u_{i+1}, u_{i+2}). \end{aligned}$$

Knowing the w_j 's, we can rewrite the z_j 's as follows.

$$z_j = \begin{cases} u_j, & j \notin \{i, i+1, i+2\} \\ (xz - y)u_i - u_{i+1} + u_{i+2}, & j = i \\ xu_i - u_{i+1}, & j = i+1 \\ u_i, & j = i+2 \end{cases}$$

By a similar procedure, we may compute the remaining three bases. The basis $\sigma_{i+1}\mathcal{B}$ is given by

$$v'_j = \begin{cases} u_j, & j \notin \{i+1, i+2\} \\ x'u_{i+1} - u_{i+2}, & j = i+1 \\ u_{i+1}, & j = i+2 \end{cases}$$

where $x' = \chi(u_{i+1}, u_{i+2}) = z$. Next, the basis $\sigma_i\sigma_{i+1}\mathcal{B}$ is given by

$$w'_j = \begin{cases} u_j, & j \notin \{i, i+1, i+2\} \\ y'u_i - x'u_{i+1} + u_{i+2}, & j = i \\ u_i, & j = i+1 \\ u_{i+1}, & j = i+2 \end{cases}$$

where

$$\begin{aligned}
y' &= \chi(v'_i, v'_{i+1}) \\
&= \chi(u_i, x'u_{i+1} - u_{i+2}) \\
&= x'\chi(u_i, u_{i+1}) - \chi(u_i, u_{i+2}) \\
&= xz - y.
\end{aligned}$$

Finally, the basis $\sigma_{i+1}\sigma_i\sigma_{i+1}\mathcal{B}$ is given by

$$z'_j = \begin{cases} u_j, & j \notin \{i, i+1, i+2\} \\ y'u_i - x'u_{i+1} + u_{i+2}, & j = i \\ z'u_i - u_{i+1}, & j = i+1 \\ u_i, & j = i+2 \end{cases}$$

where $z = \chi(w'_{i+1}, w'_{i+2}) = \chi(u_i, u_{i+1}) = x$. One can see now that the braid relation holds, so the braid action is well-defined, as desired.

Let $\mathcal{B}_0 = (e_1, e_2, \dots, e_d)$ denote the standard basis of simple roots. We want to show that for each braid $\beta \in \text{Braid}_d$ and for each $j \in [d]$, the j -th vector u_j in the ordered basis $\beta \cdot \mathcal{B}$ is a root in Φ corresponding to the j -th reflection t_j from the reduced reflection factorization $c = t_1 t_2 \cdots t_d$ given by the Hurwitz action via β on the usual reflection factorization $c = s_1 s_2 \cdots s_d$. We may prove this fact by induction on the word length of β from the alphabet of half-twists (the standard generators of Braid_d).

Example 6.3.4 already proves the base case. For the inductive step, suppose the statement holds for braids of length n . Let β be a braid with word length n . Write the induced basis $\mathcal{B} = \beta \cdot \mathcal{B}_0$ as $\mathcal{B} = (u_1, u_2, \dots, u_d)$, and write reflection factorization given by the Hurwitz action via β as $t_1 t_2 \cdots t_d$. By the induction hypothesis, each u_i is a root satisfying $t_i(u_i) = -u_i$. Now consider some half-twist σ_i^ϵ with $\epsilon = \pm 1$. We consider only the positive case $\epsilon = +1$, since the proof of the negative case is substantially similar. The Hurwitz action via σ_i on the reflection factorization $t_1 t_2 \cdots t_d$ gives a new reflection factorization $t'_1 t'_2 \cdots t'_d$ where each t'_j is given as follows.

$$t'_j = \begin{cases} t_j, & j \notin \{i, i+1\} \\ t_i t_{i+1} t_i^{-1}, & j = i \\ t_i, & j = i+1 \end{cases}$$

The Hurwitz action via σ_i on the basis $\mathcal{B}$ gives a new basis $\mathcal{B}' = \sigma_i \cdot \mathcal{B}$, written $\mathcal{B}' = (v_1, v_2, \dots, v_d)$, where the basis vectors v_j are

$$v_j = \begin{cases} u_j, & j \notin \{i, i+1\} \\ xu_i - u_{i+1}, & j = i \\ u_i, & j = i+1 \end{cases}$$

where $x = \chi(u_i, u_{i+1})$. Clearly, for $j \notin \{i, i+1\}$, the basis vector $v_j = u_j$ is a root satisfying $t'_j(v_j) = t_j(u_j) = -u_j$. Similarly, we easily find $t'_{i+1}(v_{i+1}) = t_i(u_i) = -u_i$. It remains to show that $t'_i(v_i) = -v_i$. First, we have

$$\begin{aligned} t'_i(v_i) &= (t_i t_{i+1} t_i^{-1})(xu_i - u_{i+1}) \\ &= (t_i t_{i+1})(-xu_i - t_i(u_{i+1})) \end{aligned}$$

Since u_{i+1} is the root corresponding to the reflection t_i , we know we can rewrite t_i as follows.

$$t_i(u) = u - 2B(u, u_i)u_i$$

Continuing to compute $t'_i(v_i)$, we find

$$\begin{aligned} t'_i(v_i) &= (t_i t_{i+1})(-xu_i - (u_{i+1} - 2B(u_i, u_{i+1})u_i)) \\ &= (t_i t'_{i+1})(-xu_i - (u_{i+1} - xu_i)) \\ &= (t_i t_{i+1})(-u_{i+1}) \\ &= t_i(u_{i+1}) \\ &= u_{i+1} - 2B(u_i, u_{i+1})u_i \\ &= u_{i+1} - xu_i \\ &= -v_i \end{aligned}$$

as desired. $\square$

Remark 6.3.9 (Notation for the Hurwitz action as a change of basis). Let $\beta \in \text{Braid}_d$, and let $\mathcal{B}, \mathcal{B}' \in \mathfrak{B}$ such that $\beta \cdot \mathcal{B} = \mathcal{B}'$. Let $x \in V$ be a vector, let $T : V \rightarrow V$ be a linear transformation, and let $B : V \times V \rightarrow \mathbb{R}$ be a bilinear form. We denote their representations given by the Hurwitz action via the arbitrary braid β as

$$\begin{aligned}\beta \cdot {}_{\mathcal{B}}[x] &:= {}_{\mathcal{B}'}[x] = {}_{\mathcal{B}'}[I]_{\mathcal{B}}[x] = ({}_{\mathcal{B}}M_{\beta \cdot \mathcal{B}})^{-1}({}_{\mathcal{B}}[x]) \\ \beta \cdot [T]_{\mathcal{B}} &:= [T]_{\mathcal{B}'} = {}_{\mathcal{B}'}[I]_{\mathcal{B}}[T]_{\mathcal{B}}[I]_{\mathcal{B}'} = ({}_{\mathcal{B}}M_{\beta \cdot \mathcal{B}})^{-1}([T]_{\mathcal{B}})({}_{\mathcal{B}}M_{\beta \cdot \mathcal{B}}) \\ \beta \cdot [B]_{\mathcal{B}} &:= [B]_{\mathcal{B}'} = ({}_{\mathcal{B}}[I]_{\mathcal{B}'})^{\top}([B]_{\mathcal{B}})({}_{\mathcal{B}}[I]_{\mathcal{B}'}) = ({}_{\mathcal{B}}M_{\beta \cdot \mathcal{B}})^{\top}([B]_{\mathcal{B}})({}_{\mathcal{B}}M_{\beta \cdot \mathcal{B}})\end{aligned}$$

When the bases are understood, we may omit the subscripts and superscripts denoting the bases, instead simply writing M to denote the $\mathcal{B}$ - $\mathcal{B}'$ change-of-basis matrix ${}_{\mathcal{B}}M_{\sigma \cdot \mathcal{B}}$ induced by the Hurwitz action on the basis $\mathcal{B}$.

These Hurwitz change-of-basis matrices are used frequently in both this chapter and later chapters to compute matrix representations of linear maps and bilinear forms with respect to various bases induced by the Hurwitz action.

As stated in Theorem 5.2.3, the Hurwitz action is transitive on reduced reflection factorizations of a fixed Coxeter element. As a consequence, the Hurwitz action is also transitive on the matrix representations of the Wall form χ_c (and $\chi_{c^{-1}}$, respectively) in the bases given by the reflection factorizations of the Coxeter element $c = s_1 s_2 \cdots s_d$. We refer to these matrix representations as the *Wall form matrices* of c (and c^{-1} , resp.). Since $C^{\top} = [\chi_c]_{\mathcal{B}_0}$ is the matrix representation of χ_c in the standard basis $\mathcal{B}_0 = (e_1, e_2, \dots, e_d)$ of simple roots, we call $C^{\top}$ the *standard* Wall form matrix of c . Similarly, C is the standard Wall form matrix of c^{-1} .

Remark 6.3.10 (The effects of many binary choices). In results throughout this chapter and in later chapters, we focus on the Wall form matrix C of the Coxeter element c^{-1} .

Notice that many binary choices have been baked into the definition of C ; we list a few below.

- We choose to use functional notation (reading right-to-left) instead of algebraic notation (reading left-to-right); see Remark 3.1.4. (This convention applies to how we read permutations, braids, elements of Coxeter groups viewed from the perspective of the standard geometric representation, etc.)
- We choose to represent linear maps by matrices acting on the left of column vectors, instead of on the right of row vectors.
- We choose to define $c = s_1 s_2 \cdots s_d$ instead of $c = s_d \cdots s_2 s_1$.
- We choose to define the Hurwitz action via σ_i as

$$\sigma_i \cdot (g_1, \dots, g_i, g_{i+1}, \dots, g_d) = (g_1, \dots, g_i g_{i+1} g_i^{-1}, g_i, \dots, g_d)$$

instead of

$$\sigma_i \cdot (g_1, \dots, g_i, g_{i+1}, \dots, g_d) = (g_1, \dots, g_{i+1}, g_{i+1}^{-1} g_i g_{i+1}, \dots, g_d).$$

Different choices of conventions would alter the aesthetics of certain statements and formulas. For example, I find myself dissatisfied that, using the current conventions and definitions, we have the identities $[\chi_c] = C^\top$ and $[\chi_{c^{-1}}] = C$; I would find $[\chi_c] = C$ and $[\chi_{c^{-1}}] = C^\top$ more aesthetically pleasing. However, whenever we face a binary choice on how to define something, we choose to always use the conventions found in the relevant literature. For example, our definition of the standard geometric representation is written using the same conventions as [Hum90], [BB06], [How82], etc.; these authors also all use functional notation and represent linear maps by matrices acting on the left of column vectors. Using these conventions means that the formula

$$[c] = -C^{-1}C^\top$$

may be stated exactly as written in [How82] and [HK16], where the upper unitriangular matrix C can be found. Our definition of the Wall form χ_f of a linear map f uses the conventions of [Tay92] and [MP22]. Our definition of the Hurwitz action on n -tuples follows the conventions of [Bau+14] and [MS17]. The unfortunate side effect of using all these conventions together is that some statements and formulas look a tad strange, like the aforementioned identities $[\chi_c] = C^\top$ and $[\chi_{c^{-1}}] = C$.

Chapter 7

Hurwitz action via powers of half-twists

The previous chapter establishes that the Hurwitz action on reduced reflection factorizations of the Coxeter element $c = s_1 s_2 \cdots s_d$ induces a Hurwitz action on ordered bases of roots, whereby braids acts as a change of basis. This chapter investigates the Hurwitz action via powers of half-twists, especially on matrix representations of the Wall form $\chi_{c^{-1}}$. We first explicitly compute the relevant change-of-basis matrices, then explore the connections with Fibonacci/Chebyshev polynomials and rotation matrices.

7.1 Matrix representations given by powers of half-twists

This section discusses how matrix representations change when applying the change of basis induced by the Hurwitz action via powers of a single half-twist. We explicitly compute the change-of-basis matrix, show the effect on the matrix representations of the Wall form $\chi_{c^{-1}}$, and write the change-of-basis matrix in terms of a certain polynomial

sequence.

Recall from Section 6.3 the change-of-basis matrix $M = {}_{\mathcal{B}}M_{\sigma_i \cdot \mathcal{B}}$ given by the Hurwitz action via a single positive half-twist σ_i for some basis $\mathcal{B} = (u_1, u_2, \dots, u_d) \in \mathfrak{B}$ of roots.

$$M = \begin{pmatrix} I_{i-1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{M} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & I_{d-i-1} \end{pmatrix}, \quad \mathbf{M} = \begin{pmatrix} x & 1 \\ -1 & 0 \end{pmatrix}, \quad x = \chi_{c^{-1}}(u_i, u_{i+1}).$$

When $\mathcal{B}$ is the standard basis $\mathcal{B}_0 = (e_1, e_2, \dots, e_d)$ of simple roots, the change-of-basis matrix M depends on only $x = \chi_{c^{-1}}(e_i, e_{i+1}) = C_{i,i+1}$ which remains fixed by the Hurwitz action via σ_i . Thus, we may easily compute the change-of-basis matrix for the Hurwitz action via powers of σ_i .

Proposition 7.1.1 (Change-of-basis matrix for σ_i^n). *Let $\mathcal{B}_0 = (e_1, e_2, \dots, e_d)$ be the standard basis of simple roots. For each $n \in \mathbb{Z}$, let $\mathcal{B}_n = \sigma_i^n \cdot \mathcal{B}_0$. Then the $\mathcal{B}_n$ - $\mathcal{B}_{n-1}$ change-of-basis matrix ${}_{\mathcal{B}_{n-1}}M_{\mathcal{B}_n}$ equals ${}_{\mathcal{B}_0}M_{\mathcal{B}_1}$. In particular, the $\mathcal{B}_n$ - $\mathcal{B}_0$ change-of-basis matrix is*

$${}_{\mathcal{B}_0}M_{\mathcal{B}_n} = M^n = \begin{pmatrix} I_{i-1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{M}^n & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & I_{d-i-1} \end{pmatrix}.$$

where $M = {}_{\mathcal{B}_0}M_{\mathcal{B}_1}$ and $\mathbf{M}$ are the matrices defined in Example 6.3.4. In other words, we have

$${}_{\mathcal{B}_0}M_{\mathcal{B}_n} = ({}_{\mathcal{B}_0}M_{\mathcal{B}_1})^n.$$

The above proposition is a special case of a more general statement, Proposition 7.1.2 given below.

Proposition 7.1.2 (Change-of-basis matrix for σ_i^n). *Let $\mathcal{B}_0$ be any basis in $\mathfrak{B}$. For each $n \in \mathbb{Z}$, let $\mathcal{B}_n = \sigma_i^n \cdot \mathcal{B}_0$. Then the $\mathcal{B}_n$ - $\mathcal{B}_{n-1}$ change-of-basis matrix ${}_{\mathcal{B}_{n-1}}M_{\mathcal{B}_n}$ equals ${}_{\mathcal{B}_0}M_{\mathcal{B}_1}$. In particular, the $\mathcal{B}_n$ - $\mathcal{B}_0$ change-of-basis matrix is*

$${}_{\mathcal{B}_0}M_{\mathcal{B}_n} = M^n = \begin{pmatrix} I_{i-1} & 0 & 0 \\ 0 & \mathbf{M}^n & 0 \\ 0 & 0 & I_{d-i-1} \end{pmatrix}.$$

where $M = {}_{\mathcal{B}_0}M_{\mathcal{B}_1}$ and $\mathbf{M}$ are the matrices defined in Definition 6.3.2. In other words, we have

$${}_{\mathcal{B}_0}M_{\mathcal{B}_n} = ({}_{\mathcal{B}_0}M_{\mathcal{B}_1})^n.$$

Proof. We prove the statement by induction in the case where $n \geq 0$; we omit the proof for $n < 0$ because it is substantially similar. For each $n \in \mathbb{N}$, write $\mathcal{B}_n = (u_1^{(n)}, u_2^{(n)}, \dots, u_d^{(n)})$. The base cases $n \in \{0, 1\}$ are trivially true. For the inductive step, suppose the proposition statement holds up to some n , i.e. for each $1 \leq k \leq n$, we have ${}_{\mathcal{B}_{k-1}}M_{\mathcal{B}_k} = {}_{\mathcal{B}_0}M_{\mathcal{B}_1}$. In particular, we have

$$\chi_{c^{-1}}(u_i^{(k-1)}, u_{i+1}^{(k-1)}) = \chi_{c^{-1}}(u_i^{(0)}, u_{i+1}^{(0)}).$$

By Definition 6.3.2, we know that

$${}_{\mathcal{B}_n}M_{\mathcal{B}_{n+1}} = \begin{pmatrix} I_{i-1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & x & 1 & \mathbf{0} \\ \mathbf{0} & -1 & 0 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & I_{d-i-1} \end{pmatrix}$$

where $x = \chi_{c^{-1}}(u_i^{(n)}, u_{i+1}^{(n)})$. By Lemma 6.3.7, we know that

$$\chi_{c^{-1}}(u_i^{(n)}, u_{i+1}^{(n)}) = \chi_{c^{-1}}(u_i^{(n-1)}, u_{i+1}^{(n-1)}) = \chi_{c^{-1}}(u_i^{(0)}, u_{i+1}^{(0)}),$$

so ${}_{\mathcal{B}_n}M_{\mathcal{B}_{n+1}} = {}_{\mathcal{B}_0}M_{\mathcal{B}_1}$, as desired. $\square$

For the rest of the chapter, let $\mathcal{B}_0 = (e_1, e_2, \dots, e_d)$ be the standard basis of simple roots, and let $\mathcal{B}_n = \sigma_i^n \cdot \mathcal{B}_0$, unless otherwise specified. Recall from Section 6.1 and Section 6.2 that the Wall form matrix $C = [\chi_{c^{-1}}]_{\mathcal{B}_0}$ is a $d \times d$ upper unitriangular matrix. We may write C as a block matrix, as follows.

$$C = \begin{pmatrix} \mathbf{A} & \mathbf{B} & \mathbf{C} \\ \mathbf{0} & \mathbf{D} & \mathbf{E} \\ \mathbf{0} & \mathbf{0} & \mathbf{F} \end{pmatrix}. \quad (7.1)$$

Here, we mean that

- $\mathbf{A}$ is the $(i-1) \times (i-1)$ upper unitriangular block sliced from C , given by the 1st through $(i-1)$ -th rows and columns,
- $\mathbf{B}$ is the $(i-1) \times 2$ block sliced from C given by the 1st through $(i-1)$ -th rows at the i -th and $(i+1)$ -th columns,
- $\mathbf{C}$ is the $(i-1) \times (d-i-1)$ block sliced from C , given by the 1st through $(i-1)$ -th rows at the $(i+2)$ -th through d -th columns,
- $\mathbf{D}$ is the 2×2 upper unitriangular block sliced from C , given by the i -th and $(i+1)$ -th rows and columns,
- $\mathbf{E}$ is the $2 \times (d-i-1)$ block sliced from C , given by the i -th and $(i+1)$ -th rows at the $(i+2)$ -th through d -th columns of C , and
- $\mathbf{F}$ is the $(d-i-1) \times (d-i-1)$ upper unitriangular block sliced from C , given by the $(i+2)$ -th through d -th rows and columns.

The matrix representations of the Wall form matrix C when we apply the Hurwitz action via σ_i^n follows a predictable pattern. Before stating the general formula (Corollary 7.1.3.1 below), let us consider an illustrative example.

Example 7.1.3 (Computing $\sigma_4^n \cdot C$). Let (W, S) be a rank-9 Coxeter system with Schläfli matrix J . As defined in Section 6.1, the Wall form matrix $C = [\chi_{c^{-1}}]_{\mathcal{B}_0}$ is

$$C = \begin{pmatrix} 1 & J_{12} & J_{13} & J_{14} & J_{15} & J_{16} & J_{17} & J_{18} & J_{19} \\ 0 & 1 & J_{23} & J_{24} & J_{25} & J_{26} & J_{27} & J_{28} & J_{29} \\ 0 & 0 & 1 & J_{34} & J_{35} & J_{36} & J_{37} & J_{38} & J_{39} \\ 0 & 0 & 0 & 1 & J_{45} & J_{46} & J_{47} & J_{48} & J_{49} \\ 0 & 0 & 0 & 0 & 1 & J_{56} & J_{57} & J_{58} & J_{59} \\ 0 & 0 & 0 & 0 & 0 & 1 & J_{67} & J_{68} & J_{69} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & J_{78} & J_{79} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & J_{89} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

By Theorem 6.2.1, we know that C is the matrix representation (with respect to the standard basis $\mathcal{B}_0 = (e_1, \dots, e_9)$ of simple roots) of the Wall form $\chi_{c^{-1}}$, where $\rho : W \rightarrow \mathrm{GL}(V)$ is the standard geometric representation of W from Theorem 4.1.12, and $c = s_1 s_2 \cdots s_9$ is the Coxeter element encoding the fixed ordering $S = (s_1, s_2, \dots, s_9)$ as in Section 3.2. The Hurwitz action by powers of the positive half-twist $\sigma_4 \in \mathrm{Braid}_9$ induces a change of basis. Writing $\mathcal{B}_n = \sigma_4^n \cdot \mathcal{B}_0$, we want to compute $\sigma_4^n \cdot C := [\chi_{c^{-1}}]_{\mathcal{B}_n}$. Writing C as a block matrix as in Equation 7.1, the blocks constituting C are

$$\begin{aligned} \mathbf{A} &= \begin{pmatrix} 1 & J_{12} & J_{13} \\ 0 & 1 & J_{23} \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} J_{14} & J_{15} \\ J_{24} & J_{25} \\ J_{34} & J_{35} \end{pmatrix}, \quad \mathbf{C} = \begin{pmatrix} J_{16} & J_{17} & J_{18} & J_{19} \\ J_{26} & J_{27} & J_{28} & J_{29} \\ J_{36} & J_{37} & J_{38} & J_{39} \end{pmatrix}, \\ \mathbf{D} &= \begin{pmatrix} 1 & J_{45} \\ 0 & 1 \end{pmatrix}, \quad \mathbf{E} = \begin{pmatrix} J_{46} & J_{47} & J_{48} & J_{49} \\ J_{56} & J_{57} & J_{58} & J_{59} \end{pmatrix}, \quad \mathbf{F} = \begin{pmatrix} 1 & J_{78} & J_{79} \\ 0 & 1 & J_{89} \\ 0 & 0 & 1 \end{pmatrix}. \end{aligned}$$

Following Definition 6.3.2, the change-of-basis matrix ${}_{\mathcal{B}_0}M_{\mathcal{B}_1}$ induced by the Hurwitz action via σ_4 is the block matrix

$${}_{\mathcal{B}_0}M_{\mathcal{B}_1} = \begin{pmatrix} I_3 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{M} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & I_4 \end{pmatrix}$$

where the 2×2 block $\mathbf{M}$ is

$$\mathbf{M} = \begin{pmatrix} \chi_{c^{-1}}(e_4, e_5) & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} J_{45} & 1 \\ -1 & 0 \end{pmatrix}$$

Following Definition 2.1.14, we may compute the matrix representation $\sigma_4 \cdot C = [\chi_{c^{-1}}]_{\mathcal{B}_1}$ of the Wall form $\chi_{c^{-1}}$.

$$\begin{aligned} \sigma_4 \cdot C &= (\mathcal{B}_0 M_{\mathcal{B}_1})^\top C (\mathcal{B}_0 M_{\mathcal{B}_1}) \\ &= \begin{pmatrix} I_3 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{M}^\top & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & I_4 \end{pmatrix} \begin{pmatrix} \mathbf{A} & \mathbf{B} & \mathbf{C} \\ \mathbf{0} & \mathbf{D} & \mathbf{E} \\ \mathbf{0} & \mathbf{0} & \mathbf{F} \end{pmatrix} \begin{pmatrix} I_3 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{M} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & I_4 \end{pmatrix} \\ &= \begin{pmatrix} \mathbf{A} & \mathbf{BM} & \mathbf{C} \\ \mathbf{0} & \mathbf{M}^\top \mathbf{D} \mathbf{M} & \mathbf{M}^\top \mathbf{E} \\ \mathbf{0} & \mathbf{0} & \mathbf{F} \end{pmatrix} \end{aligned}$$

The blocks $\mathbf{A}$, $\mathbf{C}$, and $\mathbf{F}$ have remained unchanged. The middle 2×2 block has also remained unchanged.

$$\mathbf{M}^\top \mathbf{D} \mathbf{M} = \begin{pmatrix} J_{45} & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & J_{45} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} J_{45} & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & J_{45} \\ 0 & 1 \end{pmatrix} = \mathbf{D}.$$

The only blocks that have changed are the ones above and to the right of $\mathbf{D}$.

$$\begin{aligned} \mathbf{BM} &= \begin{pmatrix} J_{14} & J_{15} \\ J_{24} & J_{25} \\ J_{34} & J_{35} \end{pmatrix} \begin{pmatrix} J_{45} & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} J_{14}J_{45} - J_{15} & J_{14} \\ J_{24}J_{45} - J_{25} & J_{24} \\ J_{34}J_{45} - J_{35} & J_{34} \end{pmatrix} \\ \mathbf{M}^\top \mathbf{E} &= \begin{pmatrix} J_{45} & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} J_{46} & J_{47} & J_{48} & J_{49} \\ J_{56} & J_{57} & J_{58} & J_{59} \end{pmatrix} \\ &= \begin{pmatrix} J_{45}J_{46} - J_{56} & J_{45}J_{47} - J_{57} & J_{45}J_{48} - J_{58} & J_{45}J_{49} - J_{59} \\ J_{46} & J_{47} & J_{48} & J_{49} \end{pmatrix} \end{aligned}$$

At this point, we have

$$\sigma_4 \cdot C = \begin{pmatrix} \mathbf{A} & \mathbf{BM} & \mathbf{C} \\ \mathbf{0} & \mathbf{D} & \mathbf{M}^\top \mathbf{E} \\ \mathbf{0} & \mathbf{0} & \mathbf{F} \end{pmatrix}. \quad (7.2)$$

To compute the matrix representation $\sigma_4^2 \cdot C = [\chi_{c^{-1}}]_{\mathcal{B}_2}$, we may apply to $\sigma_4 \cdot C$ the change-of-basis matrix $\mathcal{B}_1 M_{\mathcal{B}_2}$. Following Definition 6.3.2, we know that $\mathcal{B}_1 M_{\mathcal{B}_2}$ depends only on the $(4, 5)$ -th entry of $\sigma_4 \cdot C$. By Equation 7.2, we know that the $(4, 5)$ -th entry

of $\sigma_4 \cdot C$ equals that of C , so the change-of-basis matrix ${}_{\mathcal{B}_1}M_{\mathcal{B}_2}$ equals ${}_{\mathcal{B}_0}M_{\mathcal{B}_1}$, as already shown in Proposition 7.1.1. We may now easily compute $\sigma_4^2 \cdot C$.

$$\begin{aligned} \sigma_4^2 \cdot C &= {}_{\mathcal{B}_1}M_{\mathcal{B}_2}^\top (\sigma_4 \cdot C) {}_{\mathcal{B}_1}M_{\mathcal{B}_2} \\ &= \begin{pmatrix} I_3 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{M}^\top & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & I_4 \end{pmatrix} \begin{pmatrix} \mathbf{A} & \mathbf{B}\mathbf{M} & \mathbf{C} \\ \mathbf{0} & \mathbf{D} & \mathbf{M}^\top \mathbf{E} \\ \mathbf{0} & \mathbf{0} & \mathbf{F} \end{pmatrix} \begin{pmatrix} I_3 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{M} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & I_4 \end{pmatrix} \\ &= \begin{pmatrix} \mathbf{A} & \mathbf{B}\mathbf{M}^2 & \mathbf{C} \\ \mathbf{0} & \mathbf{D} & (\mathbf{M}^\top)^2 \mathbf{E} \\ \mathbf{0} & \mathbf{0} & \mathbf{F} \end{pmatrix}. \end{aligned}$$

A predictable pattern is beginning to emerge.

Corollary 7.1.3.1 (Representations $\sigma_i^n \cdot C$). *Write the Wall form matrix $C = [\chi_{c^{-1}}]_{\mathcal{B}_0}$ as a block matrix*

$$\begin{pmatrix} \mathbf{A} & \mathbf{B} & \mathbf{C} \\ \mathbf{0} & \mathbf{D} & \mathbf{E} \\ \mathbf{0} & \mathbf{0} & \mathbf{F} \end{pmatrix}$$

as in Equation 7.1. Then $\sigma_i^n \cdot C := [\chi_{c^{-1}}]_{\mathcal{B}_n}$ equals

$$\begin{pmatrix} \mathbf{A} & \mathbf{B}\mathbf{M}^n & \mathbf{C} \\ \mathbf{0} & \mathbf{D} & (\mathbf{M}^\top)^n \mathbf{E} \\ \mathbf{0} & \mathbf{0} & \mathbf{F} \end{pmatrix}$$

where $\mathbf{M}$ is the 2×2 matrix

$$\mathbf{M} = \begin{pmatrix} \chi_{c^{-1}}(e_i, e_{i+1}) & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} C_{i,i+1} & 1 \\ -1 & 0 \end{pmatrix},$$

as in Definition 6.3.2.

Proof. By Proposition 7.1.1, we know $\sigma_i^n \cdot C = [\chi_{c^{-1}}]_{\mathcal{B}_n} = (\mathbf{M}^n)^\top C \mathbf{M}^n$. We have

$$\begin{aligned}
(M^n)^\top C M^n &= \begin{pmatrix} I_{i-1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & (\mathbf{M}^n)^\top & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & I_{d-i-1} \end{pmatrix} \begin{pmatrix} \mathbf{A} & \mathbf{B} & \mathbf{C} \\ \mathbf{0} & \mathbf{D} & \mathbf{E} \\ \mathbf{0} & \mathbf{0} & \mathbf{F} \end{pmatrix} \begin{pmatrix} I_{i-1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{M}^n & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & I_{d-i-1} \end{pmatrix} \\
&= \begin{pmatrix} \mathbf{A} & \mathbf{B}\mathbf{M}^n & \mathbf{C} \\ \mathbf{0} & (\mathbf{M}^\top)^n \mathbf{D} \mathbf{M}^n & (\mathbf{M}^\top)^n \mathbf{E} \\ \mathbf{0} & \mathbf{0} & \mathbf{F} \end{pmatrix}.
\end{aligned}$$

As in Example 7.1.3, we have $\mathbf{M}^\top \mathbf{D} \mathbf{M} = \mathbf{D}$. Therefore, we have

$$(M^n)^\top C M^n = \begin{pmatrix} \mathbf{A} & \mathbf{B}\mathbf{M}^n & \mathbf{C} \\ \mathbf{0} & \mathbf{D} & (\mathbf{M}^\top)^n \mathbf{E} \\ \mathbf{0} & \mathbf{0} & \mathbf{F} \end{pmatrix},$$

as desired. $\square$

Evidently, $\mathbf{M}$ determines the Hurwitz action of the positive half-twist σ_i on the Wall form matrix C . Below, we investigate $\mathbf{M}$.

Let the off-diagonal entries of the Schläfli matrix J be indeterminate variables, whereby we may specify the particular Coxeter system (W, S) by plugging in values into these variables. (Recall that $J_{ij} = 2B_{ij} = -2\cos(\pi/m_{ij})$, so when we are viewing J as the Schläfli matrix for a Coxeter system (W, S) , the values that we are allowed to plug into the variable at J_{ij} must lie in the interval $[-2, 0]$. We need not care about this detail for now; later in this chapter, Section 7.3.4 addresses this restriction.) Since C is the unique upper unitriangular matrix satisfying $J = C + C^\top$, we know C inherits these indeterminate variables from J . In other words, we may think of $x = C_{i,i+1}$ as an indeterminate, whereby the entries of $\mathbf{M}^n$ are polynomials in x . We obtain a predictable sequence of polynomials.

Proposition 7.1.4 ($\mathbf{M}^n$ as a matrix of polynomials). *For $n \in \mathbb{Z}$, we have*

$$\mathbf{M}^n = \begin{pmatrix} H_n(x) & H_{n-1}(x) \\ -H_{n-1}(x) & -H_{n-2}(x) \end{pmatrix}$$

where the polynomials $H_n(x)$ are defined by the recurrence relation

$$H_{n+1}(x) = xH_n(x) - H_{n-1}(x)$$

with initial conditions $H_0(x) = 1$ and $H_1(x) = x$.

Proof. Using the recurrence relation, we obtain $H_{-1}(x) = 0$ and $H_{-2}(x) = -1$, whereby it is clear that

$$\mathbf{M}^1 = \begin{pmatrix} H_1(x) & H_0(x) \\ -H_0(x) & -H_{-1}(x) \end{pmatrix}, \quad \mathbf{M}^0 = \begin{pmatrix} H_0(x) & H_{-1}(x) \\ -H_{-1}(x) & -H_{-2}(x) \end{pmatrix}.$$

To show that the proposition holds for $n \in \mathbb{N}$, simply use induction. To show that the proposition holds for negative integers, notice that for $n \in \mathbb{N}$, we may show $H_{-n} = -H_{n-2}$ by induction. Indeed, recall from Remark 2.1.6 that $\mathbf{M}^R$ denotes conjugating $\mathbf{M}$ by the reversal matrix. One may easily confirm that $\mathbf{M}^{-1} = \mathbf{M}^R$, and thus $\mathbf{M}^{-n} = (\mathbf{M}^n)^R$. Thus, for $n \in \mathbb{N}$, we have

$$\begin{aligned} \mathbf{M}^{-n} &= \begin{pmatrix} H_n(x) & H_{n-1}(x) \\ -H_{n-1}(x) & -H_{n-2}(x) \end{pmatrix}^R \\ &= \begin{pmatrix} -H_{n-2} & -H_{n-1} \\ H_{n-1} & H_n \end{pmatrix} \\ &= \begin{pmatrix} H_{-n}(x) & H_{-(n-1)}(x) \\ -H_{-(n-1)}(x) & -H_{-(n-2)}(x) \end{pmatrix} \end{aligned}$$

as desired. □

We may easily prove the following identity, which resembles Cassini's identity

$$f_{n-1}f_{n+1} - f_n^2 = (-1)^n$$

for the Fibonacci numbers, as well as the Turán inequalities

$$\begin{aligned} T_n(x)^2 - T_{n-1}(x)T_{n+1}(x) &= 1 - x^2 > 0, & x \in (-1, 1) \\ U_n(x)^2 - U_{n-1}(x)U_{n+1}(x) &= 1 > 0, & x \in \mathbb{R} \end{aligned}$$

for the Chebyshev polynomials T_n and U_n of the first and second kind, respectively. (Throughout this chapter, we state a few well-known identities for the Fibonacci numbers, which can be found in many texts. One can find Cassini's identity in [Knu97, Section 1.2.8].)

Proposition 7.1.5 (Cassini-like identity). *For $n \in \mathbb{Z}$, we have*

$$H_n(x)^2 - H_{n-1}(x)H_{n+1}(x) = 1.$$

Proof. We can prove this identity for non-negative integers n using induction. The base case is

$$H_0^2 - H_{-1}H_1 = 1^2 - 0 \cdot 1 = 1.$$

For the inductive step, we have

$$\begin{aligned} H_n^2 - H_{n-1}H_{n+1} &= H_n^2 - H_{n-1}(xH_n - H_{n-1}) \\ &= H_n^2 - xH_nH_{n-1} + H_{n-1}^2 \\ &= H_n(H_n - xH_{n-1}) + H_{n-1}^2 \\ &= H_{n-1}^2 - H_nH_{n-2} = 1 \end{aligned}$$

Using the fact that $H_{-n} = -H_{n-2}$, we can extend this fact to all $n \in \mathbb{Z}$. □

7.2 Relationship with Fibonacci and Chebyshev polynomials

The previous section shows that the change-of-basis matrices M^n induced by the Hurwitz action on the standard basis $\mathcal{B}_0$ of simple roots via the n -th power of the positive half-twist σ_i may be written in terms of a polynomial sequence $H_n(x)$. This section focuses on the connections between $H_n(x)$ and other important polynomial sequences, namely the Fibonacci polynomials and the Chebyshev polynomials of the second kind.

The Fibonacci polynomials are a sequence of polynomials generalizing the Fibonacci numbers.

Definition 7.2.1 (Fibonacci polynomials). The *Fibonacci polynomials* $F_n(x)$ are defined by the recurrence relation

$$F_{n+1}(x) = xF_n(x) + F_{n-1}(x)$$

with initial conditions $F_0(x) = 0$ and $F_1(x) = 1$.

The Fibonacci polynomials $F_n(x)$ are so named because the defining recurrence relation and initial conditions resemble those of the Fibonacci numbers f_n , and because we can recover the Fibonacci numbers via $F_n(1) = f_n$.

Remark 7.2.2 (Indexing for the Fibonacci polynomials). The Fibonacci numbers f_n are typically indexed with the initial conditions $f_0 = 0$ and $f_1 = 1$ [Sta11], so it is completely reasonable to index the Fibonacci polynomials in the same way. Following [WP69] and [HB73], Definition 7.2.1 uses this standard indexing scheme for the Fibonacci polynomials. An alternative indexing scheme would be to set $F_0(x) = 1$ and $F_1(x) = x$, which pleasingly gives us $\deg(F_n) = n$ for all non-negative integers n . To avoid confusion, we let $F_n(x)$ denote the Fibonacci polynomials with the standard indexing scheme, and let

$F_n^*(x)$ denote the index-shifted Fibonacci polynomials with initial conditions $F_0^*(x) = 1$ and $F_1^*(x) = x$, whereby $F_n^*(x) = F_{n+1}(x)$. This style of notation for index-shifted sequences can be found in [Olv+10].

The Chebyshev polynomials of the second kind are a sequence of polynomials related to the Fibonacci polynomials, also defined by 2nd order recurrence relation. For more about Chebyshev polynomials, see [Riv20].

Definition 7.2.3 (Chebyshev polynomials of the second kind). The *Chebyshev polynomials of the second kind* $U_n(x)$ are defined by the recurrence relation

$$U_{n+1}(x) = 2xU_n(x) - U_{n-1}(x)$$

with initial conditions $U_0(x) = 1$ and $U_1(x) = 2x$.

The polynomial sequence H_n that I discovered as arising from the Hurwitz action via powers of a half-twist is related to F_n and U_n . The relationship between the three polynomial sequences becomes immediately apparent upon examining the first several terms, presented in Table 7.1. One may also see this relationship by comparing the generating matrices, as in Table 7.2. In the following discussion, we refer to the index-shifted Fibonacci polynomials $F_n^*(x) = F_{n+1}(x)$, so that the three polynomial sequences have degrees that agree with their indices.

$$\deg(F_n^*) = \deg(U_n) = \deg(H_n) = n.$$

From each sequence, we may derive the others.

Proposition 7.2.4 (Relationship between the polynomial sequences). *For $n \in \mathbb{Z}$, we have*

$$(i) \quad F_n^*(x) = i^n H_n(-ix) = i^n U_n(-ix/2)$$

$$(ii) \ U_n(x) = H_n(2x) = i^{-n} F_n^*(2ix)$$

$$(iii) \ H_n(x) = i^{-n} F_n^*(ix) = U_n(x/2)$$

Proof. Use induction to prove one of the three equalities. The other two equalities follow by substitution of variables. $\square$

Remark 7.2.5 (In the context of Coxeter systems). In this chapter, we've found it convenient to temporarily forget that the Wall form matrix C defined in Section 6.1 comes from the Schläfli matrix of a Coxeter system, instead treating C as an arbitrary upper unitriangular matrix where the off-diagonal entries are indeterminate variables. Returning to the context of Coxeter systems, recall that the entries of C are parameters that, when taking on appropriate values, determine a Coxeter system (W, S) . Namely, the entry $C_{i,j}$ determines the length m_{ij} of the relation between s_i and s_j . In terms of $m_{i,i+1}$, the parameter $x = C_{i,i+1}$ is

$$x = C_{i,i+1} = J_{i,i+1} = 2B_{i,i+1} = 2B(e_i, e_{i+1}) = -2 \cos(\pi/m_{i,i+1}).$$

Plugging this into Equation (iii) from Proposition 7.2.4, we have

$$H_n(x) = U_n(B_{i,i+1}).$$

Thus, the change-of-basis matrix M^n induced by the Hurwitz action $\sigma_i^n \cdot \mathcal{B}_0$ can be defined by plugging in $B_{i,i+1}$ into the n -th Chebyshev polynomial of the second kind.

Since our polynomial sequence $H_n(x)$ can be written in terms of the Fibonacci and Chebyshev polynomials, we can derive many identities of $H_n(x)$ from the known identities of $F_n(x)$ and $U_n(x)$. Indeed, [FMM18] provides over a hundred identities for what they call generalized Fibonacci polynomials, of which $F_n(x)$, $U_n(x)$, and $H_n(x)$ are all examples. The Cassini-like identity from Proposition 7.1.5 can be seen as a special case

of [FMM18, Proposition 2(1)]. Applying [FMM18, Proposition 2(2)], we obtain the following identity for $H_n(x)$, which resembles the Catalan identity

$$f_n^2 - f_{n-r}f_{n+r} = (-1)^{n-r}f_r^2$$

for the Fibonacci numbers.

Proposition 7.2.6 (Catalan-like identity). *For $n, r \in \mathbb{Z}$, we have*

$$H_n(x)^2 - H_{n-r}(x)H_{n+r}(x) = (-1)^{n-r}H_r(x)^2.$$

One may compute the ordinary generating functions for the three polynomial sequences, presented below.

Proposition 7.2.7 (OGFs). *The corresponding ordinary generating functions are*

$$\begin{aligned} f^*(t, x) &:= \sum_{n=0}^{\infty} F_n^*(x)t^n = \frac{1}{1 - xt - t^2}, \\ u(t, x) &:= \sum_{n=0}^{\infty} U_n^*(x)t^n = \frac{1}{1 - 2xt + t^2}, \\ h(t, x) &:= \sum_{n=0}^{\infty} H_n(x)t^n = \frac{1}{1 - xt + t^2}. \end{aligned}$$

The ordinary generating functions $f(t, x)$ and $u(t, x)$ for the Fibonacci polynomials $F_n(x)$ and Chebyshev polynomials of the second kind $U_n(x)$, respectively, are well known. The formula

$$u(t, x) = \frac{1}{1 - 2xt + t^2}$$

may be found in [Olv+10, Equation 18.12.10] [Riv20, Section 1.5.3], and the formula

$$f(t, x) = \frac{t}{1 - xt - t^2}$$

may be found in [WP69] [Kos01, Chapter 37] [Kos19, Section 31.5]. Knowing the closed form for $f(t, x)$, we may obtain the formula

$$f^*(t, x) = \frac{1}{1 - xt - t^2},$$

via a well-known rule for computing ordinary generating functions of index-shifted sequences [Wil05, Rule 1]. To compute the ordinary generating function $h(t, x)$ for the polynomial sequence $H_n(x)$, we may use Equation (iii) from Proposition 7.2.4; since $H_n(x) = U_n(x/2)$, we conclude

$$h(t, x) = u(t, x/2) = \frac{1}{1 - xt + t^2}.$$

For each of the three polynomial sequences, we could have computed the ordinary generating functions by using the standard method for computing OGFs of sequences in one variable defined by a linear recurrence relation [Sta11, Theorem 4.1.1] [Wil05, Section 1.2.1]. (As an aside, Exercise 1.5.13 and Equation 1.107 from [Riv20] can be used to compute the ordinary generating function of $U_n(x/2) = H_n(x)$.) To illustrate the OGF computation, the proof below uses the standard methods to compute the formula for $h(t, x)$ presented in Proposition 7.2.7.

Proof. Recall the linear recurrence relation $H_{n+2}(x) = xH_{n+1}(x) - H_n(x)$ and initial conditions $H(0) = 1$ and $H_1(x) = x$ from Proposition 7.1.4. Multiplying both sides of the recurrence by t^{n+2} and summing over all $n \in \mathbb{N}$, we obtain

$$\sum_{n=0}^{\infty} H_{n+2}(x)t^{n+2} = x \sum_{n=0}^{\infty} H_{n+1}(x)t^{n+2} - \sum_{n=0}^{\infty} H_n(x)t^{n+2}$$

which we can rewrite as

$$\sum_{n=2}^{\infty} H_n(x)t^n = xt \sum_{n=1}^{\infty} H_n(x)t^n - t^2 h(t, x).$$

Applying the definition $h(t, x) := \sum_{n=0}^{\infty} H_n(x)t^n$, we obtain

$$h(t, x) - H_1(x)t - H_0(x) = xt[h(t, x) - H_0(x)] - t^2h(t, x)$$

which gives us

$$h(t, x) - xt - 1 = xth(t, x) - xt - t^2h(t, x).$$

Isolating $h(t, x)$, we finally obtain the desired result.

$$h(t, x) = \frac{1}{1 - xt + t^2}$$

□

Recall Binet's explicit closed formula for the n -th Fibonacci number f_n , stated in numerous combinatorics texts [Kos01, Theorem 5.6] [Sta11, Example 4.1.2].

Theorem 7.2.8 (Binet's formula). *Let φ denote the golden ratio, and let ψ denote its conjugate.*

$$\varphi = \frac{1 + \sqrt{5}}{2} \quad \text{and} \quad \psi = \frac{1 - \sqrt{5}}{2}$$

Then

$$f_n = \frac{\varphi^n - \psi^n}{\varphi - \psi} = \frac{\varphi^n - \psi^n}{\sqrt{5}}.$$

Akin to Binet's formula for the Fibonacci numbers, we may compute closed forms for the three polynomial sequences, presented below.

Proposition 7.2.9 (Binet-like formulas). *The closed forms for these polynomial sequences are*

$$\begin{aligned} F_n^*(x) &= \frac{\alpha(x)^{n+1} - \beta(x)^{n+1}}{\alpha(x) - \beta(x)}, \\ H_n(x) &= \frac{\gamma(x)^{n+1} - \delta(x)^{n+1}}{\gamma(x) - \delta(x)}, \\ U_n(x) &= \frac{\gamma(2x)^{n+1} - \delta(2x)^{n+1}}{\gamma(2x) - \delta(2x)} \end{aligned}$$

where

$$\begin{aligned} \alpha(x) &= \frac{x + \sqrt{x^2 + 4}}{2}, & \gamma(x) &= \frac{x + \sqrt{x^2 - 4}}{2}, \\ \beta(x) &= \frac{x - \sqrt{x^2 + 4}}{2}, & \delta(x) &= \frac{x - \sqrt{x^2 - 4}}{2}. \end{aligned}$$

The Binet-like formula

$$F_n(x) = \frac{\alpha(x)^n - \beta(x)^n}{\alpha(x) - \beta(x)} = \frac{(x + \sqrt{x^2 + 4})^n - (x - \sqrt{x^2 + 4})^n}{2^n \sqrt{x^2 + 4}}$$

for the Fibonacci polynomials is well known, and may be found in [Kos01, Chapter 38] [Kos19, Chapter 31]. It is immediately clear that by shifting the index, we obtain the desired formula for $F_n^*(x)$. The Binet-like formula

$$U_n(x) = \frac{\gamma(2x)^{n+1} - \delta(2x)^{n+1}}{\gamma(2x) - \delta(2x)} = \frac{(x + \sqrt{x^2 - 1})^{n+1} - (x - \sqrt{x^2 - 1})^{n+1}}{2\sqrt{x^2 - 1}}$$

for the Chebyshev polynomials of the second kind is also well-known, and may be found in [Riv20, Exercise 1.2.20]. (Many Binet-like formulas, including the formulas for $F_n(x)$ and $U_n(x)$ stated above, are compiled by Rigoberto Flórez and collaborators [FHM18, Section 2.1] [FMM18, Section 2] [FS22, Section 2] [Col+25, Section 2.1].) By applying the identity $H_n(x) = U_n(x/2)$ from Proposition 7.2.4, we obtain the desired Binet-like formula

$$H_n(x) = \frac{\gamma(x)^{n+1} - \delta(x)^{n+1}}{\gamma(x) - \delta(x)} = \frac{(x + \sqrt{x^2 - 4})^{n+1} - (x - \sqrt{x^2 - 4})^{n+1}}{2^n \sqrt{x^2 - 4}}$$

for our polynomial sequence $H_n(x)$.

We can easily compute the above three Binet-like formulas using any of the usual methods for computing the original Binet formula. As an illustration, we provide a proof below for the computation of the Binet-like formula for $H_n(x)$ presented in Proposition 7.2.9.

Proof. Recall the ordinary generating function

$$h(t, x) := \sum_{n=0}^{\infty} H_n(x) t^n = \frac{1}{1 - xt + t^2}$$

from Proposition 7.2.7. Let $g(t, x)$ be the (-1) -shifted OGF for $H_n(x)$.

$$g(t, x) := \sum_{n=0}^{\infty} H_{n-1}(x) t^n$$

By [Wil05, Rule 1], we know

$$g(t, x) = \frac{t}{1 - xt + t^2}.$$

The partial fraction decomposition of $g(t, x)$ is

$$\begin{aligned} g(t, x) &= \frac{1}{\sqrt{x^2 - 4}} \left(\frac{1}{1 - \gamma(x)t} - \frac{1}{1 - \delta(x)t} \right) \\ &= \frac{1}{\sqrt{x^2 - 4}} \left(\sum_{n=0}^{\infty} \gamma(x)^n t^n - \sum_{n=0}^{\infty} \delta(x)^n t^n \right) \\ &= \sum_{n=0}^{\infty} \frac{\gamma(x)^n - \delta(x)^n}{\gamma(x) - \delta(x)} t^n. \end{aligned}$$

Therefore, after shifting the indices back, we get the desired formula

$$H_n(x) = \frac{\gamma(x)^{n+1} - \delta(x)^{n+1}}{\gamma(x) - \delta(x)}.$$

□

7.3 Relationship with rotation matrices

The change-of-basis matrix $M = {}_{\sigma_j \mathcal{B}} M_{\mathcal{B}}$ discussed throughout this chapter may be reinterpreted as representing a linear map with respect to some basis. This section shows that M may be thought of as a rotation matrix, where the particular notion of “rotation” depends on the $(j, j+1)$ -th entry of the Wall form matrix $C = [\chi_{c^{-1}}]_{\mathcal{B}}$.

In Section 7.1, the Wall form matrix $C = [\chi_{c^{-1}}]_{\mathcal{B}_0}$ is written as an upper unitriangular matrix whose entries are indeterminate variables. Set $x = C_{j,j+1}$. Let $\mathcal{B}_1 = \sigma_j \cdot \mathcal{B}_0$, let M denote the $\mathcal{B}_1$ - $\mathcal{B}_0$ change-of-basis matrix induced by the Hurwitz action via the positive half-twist σ_j , and let $\mathbf{M}$ denote the 2×2 submatrix of M sliced from the j -th and $(j+1)$ -th rows and columns, as in Example 6.3.4 and Proposition 7.1.1. For the aesthetics of upcoming computations, let $\hat{x} = \frac{x}{2}$, whereby $\mathbf{M}$ may be written as

$$\mathbf{M} = \begin{pmatrix} 2\hat{x} & 1 \\ -1 & 0 \end{pmatrix} \quad (7.3)$$

By a quick calculation, we see that $\mathbf{M}$ has eigenvalues

$$\lambda = \hat{x} \pm \sqrt{\hat{x}^2 - 1}. \quad (7.4)$$

As mentioned in Remark 4.1.3, the standard symmetric bilinear form B from Definition 4.1.1 gives us the following notion of angle θ between vectors $u, v \in V$.

$$\cos(\theta) := \frac{B(u, v)}{\|u\| \|v\|}.$$

For example, the angle θ_{jk} between the standard basis vectors e_j and e_k is

$$\theta_{jk} = \pi - \frac{\pi}{m_{jk}}.$$

Recall as in Remark 7.2.5 that $\hat{x} = B(e_j, e_{j+1}) = \cos(\theta_{j,j+1})$. (For brevity, we denote $\hat{\theta} = \theta_{j,j+1}$.) We must have $\hat{x} \in [-1, 0]$. More specifically, we have

$$\begin{cases} \hat{x} = 0, & m_{j,j+1} = 2 \\ \hat{x} \in (-1, 0), & 3 \leq m_{j,j+1} < \infty \\ \hat{x} = -1, & m_{j,j+1} = \infty \end{cases} \quad (7.5)$$

The eigenvalues λ from Equation 7.4 take the form

$$\lambda = \cos(\hat{\theta}) \pm i \sin(\hat{\theta}) = e^{\pm i\hat{\theta}}.$$

Below, we investigate what happens to M in the three cases from Equation 7.5, and how the change-of-basis matrix induced by the Hurwitz action via powers of a positive half-twist can be seen as a rotation matrix.

7.3.1 First case $m_{j,j+1} = 2$

Let us first consider the very straightforward case where $m_{j,j+1} = 2$, i.e. s_j and s_{j+1} commute. Referring back to Equation 7.4, this case is where $\hat{x} = \cos(\pi/2) = 0$. Plugging this value of $\hat{x}$ into Equation 7.3, we obtain

$$\mathbf{M} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The eigenvalues of $\mathbf{M}$ are simply $\lambda = \pm i$.

Up to this point, we have thought of the change-of-basis matrix M (induced by the Hurwitz action via σ_j , as in Example 6.3.4 and Proposition 7.1.1) as the $\mathcal{B}_1$ - $\mathcal{B}_0$ change-of-basis matrix ${}_{\mathcal{B}_0}[I]_{\mathcal{B}_1}$. We may alternatively think of M as representing a linear automorphism on $V \cong \mathbb{R}^d$. In other words, letting $\mathcal{B} = (u_1, \dots, u_d)$ be an arbitrary basis of V , there is some linear automorphism $T : V \rightarrow V$ for which $M = [T]_{\mathcal{B}}$.

In Proposition 7.3.2 below, we show that there exists a choice of basis $\mathcal{B}'$ whereby $[T]_{\mathcal{B}'}$ is a plane rotation. A plane rotation of a coordinate plane is sometimes called a Givens rotation [HJ12, Example 2.1.11] [GVL13, Section 5.1.8].

Definition 7.3.1 (Givens rotation). The *Givens rotation* (or *plane rotation*) of the (i, j) -coordinate plane in $\mathbb{R}^n$ by angle θ , denoted by $U(\theta; i, j)$, is defined as the result of replacing the (i, i) - and (j, j) -entries of the identity matrix I_n with $\cos(\theta)$, replacing the (i, j) -entry with $-\sin(\theta)$, and replacing the (j, i) -entry with $\sin(\theta)$.

$$U(\theta; i, j) = \begin{pmatrix} 1 & & & & & & & & \\ & \ddots & & & & & & & \\ & & 1 & & & & & & \\ & & & \cos(\theta) & & & & & \\ & & & & 1 & & & & \\ & & & & & \ddots & & & \\ & & & & & & 1 & & \\ & & & \sin(\theta) & & & & \cos(\theta) & \\ & & & & & & & & 1 & \\ & & & & & & & & & \ddots & \\ & & & & & & & & & & 1 \end{pmatrix}$$

As discussed in [HJ12, Example 2.1.11], the Givens rotation $U(\theta; i, j)$ rotates by angle θ the (i, j) -coordinate plane in $\mathbb{R}^n$ while fixing pointwise the orthogonal complement. If some isometry $T \in O(V)$ is represented by a Givens rotation in some basis, then T is certainly a rotation in the sense of Definition 2.2.5, at least with respect to the standard Euclidean norm. (Of course, the converse is not true in general; for example, consider double rotations in $\mathbb{R}^4$.)

The following proposition states that when there is a commuting relation $s_j s_{j+1} = s_{j+1} s_j$, then there exists a basis where the linear map given by the Hurwitz action via the positive half-twist σ_j can be thought of as a $\pi/2$ -rotation of the $(j, j+1)$ -coordinate plane.

Proposition 7.3.2 (Commuting relation leads to a $\frac{\pi}{2}$ -rotation). *Suppose $m_{j,j+1} = 2$, i.e. the standard generators s_j and s_{j+1} commute. Then there exists a basis $\mathcal{B}'$ where $[T]_{\mathcal{B}'} = U(\frac{\pi}{2}; j, j+1)$.*

Proof. Writing $\mathcal{B} = (u_1, u_2, \dots, u_d)$, define $\mathcal{B}' = (v_1, v_2, \dots, v_d)$ by

$$v_i = \begin{cases} u_i, & i \neq j+1 \\ -u_i, & i = j+1 \end{cases}$$

Clearly, $\mathcal{B}'$ is indeed a basis for V . The change-of-basis matrix $A = {}_{\mathcal{B}}[I]_{\mathcal{B}'}$ is the result of taking the $d \times d$ identity matrix and flipping the sign of the $(j+1, j+1)$ -entry. We have

$$\begin{aligned} [T]_{\mathcal{B}'} &= {}_{\mathcal{B}'}[I]_{\mathcal{B}}[T]_{\mathcal{B}}[I]_{\mathcal{B}'} = A^{-1}MA \\ &= \begin{pmatrix} I_{j-1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & 0 & -1 & \mathbf{0} \\ \mathbf{0} & 1 & 0 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & I_{d-j-1} \end{pmatrix} \\ &= \begin{pmatrix} I_{j-1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \cos(\pi/2) & -\sin(\pi/2) & \mathbf{0} \\ \mathbf{0} & \sin(\pi/2) & \cos(\pi/2) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & I_{d-j-1} \end{pmatrix} \\ &= U\left(\frac{\pi}{2}; j, j+1\right). \end{aligned}$$

□

This proposition shows that when $m_{j,j+1} = 2$, the change of basis induced by the Hurwitz action via powers of σ_j can be thought of as rotating a plane by integer multiples of $\frac{\pi}{2}$. In particular, the rotation $T \in \text{SO}(V)$ given by $[T]_{\mathcal{B}} = M$ can be thought of as the plane rotation that (setwise) preserves the cylinder

$$\left\{ \sum_{i=1}^d x_i v_i \in V \mid x_j^2 + x_{j+1}^2 = 1 \right\}$$

by rotating the (v_j, v_{j+1}) -plane by angle $\frac{\pi}{2}$, as in Figure 7.1. When viewed in the original

basis $\mathcal{B}$, the rotation T preserves (setwise) the cylinder

$$\left\{ \sum_{i=1}^d x_i u_i \in V \mid x_j^2 + x_{j+1}^2 = 1 \right\}$$

as in Figure 7.2.

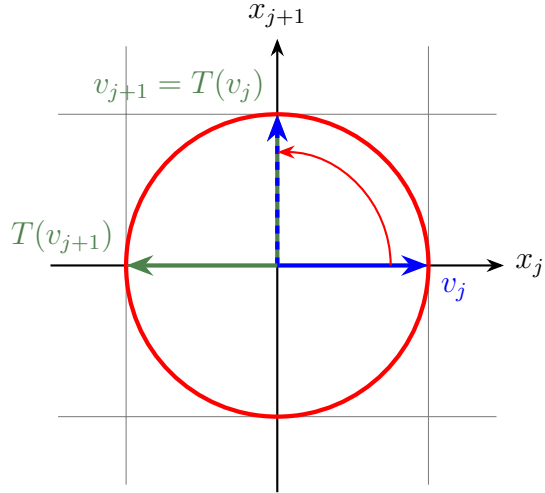


Figure 7.1: Rotation T of the (v_j, v_{j+1}) -plane in basis $\mathcal{B}'$ when $m_{j,j+1} = 2$

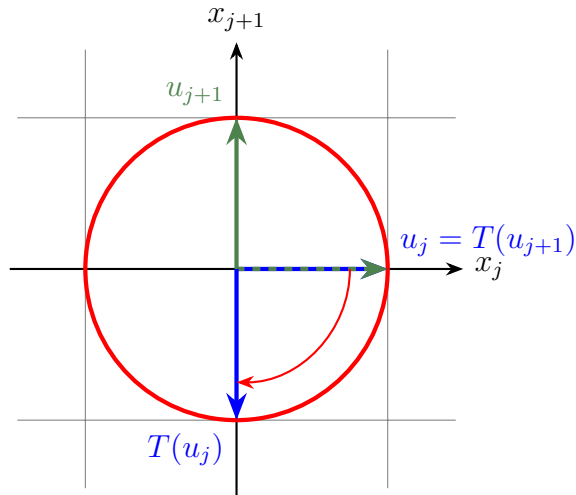


Figure 7.2: Rotation T of the (u_j, u_{j+1}) -plane in basis $\mathcal{B}$ when $m_{j,j+1} = 2$

7.3.2 Second case $m_{j,j+1} \in [3, \infty)$

Moving on to the second case from Equation 7.5, suppose $m_{j,j+1} \in [3, \infty]$, whereby $\hat{\theta} = \pi - \frac{\pi}{m_{j,j+1}}$ and $\hat{x} = \cos(\hat{\theta})$. As discussed previously, the eigenvalues of the matrix

$$\mathbf{M} = \begin{pmatrix} 2\hat{x} & 1 \\ -1 & 0 \end{pmatrix}$$

from Equation 7.3 are the pair of complex conjugates $\lambda = e^{\pm i\hat{\theta}}$. Following the same conventions from Section 7.3.1, we may think of the $\mathcal{B}_0$ - $\mathcal{B}_1$ change-of-basis $M = {}_{\mathcal{B}_0}[I]_{\mathcal{B}_1}$ as instead representing some linear automorphism. Namely, given an arbitrary basis $\mathcal{B} = (u_1, u_2, \dots, u_d)$, there exists a linear automorphism $T : V \rightarrow V$ such that $M = [T]_{\mathcal{B}}$.

As it turns out, we may extend Proposition 7.3.2 to work for all integers $m_{j,j+1} \geq 2$. In other words, for all integers $m_{j,j+1} \geq 2$, the matrix M (originally defined as the change-of-basis matrix induced by the Hurwitz action via σ_j on the standard basis $\mathcal{B}_0$ of simple roots) can be seen as a Givens rotation by angle $\hat{\theta}$. The proof follows the same pattern as that of Proposition 7.3.2.

Proposition 7.3.3 (Non-free relation leads to $\hat{\theta}$ -rotation). *Suppose $m_{j,j+1} \geq 2$ is a positive integer. (In particular, suppose $m_{j,j+1} \neq \infty$, i.e. the relation between the standard generators s_j and s_{j+1} is not free.) There exists a basis $\mathcal{B}'$ where $[T]_{\mathcal{B}'} = U(\hat{\theta}; j, j+1)$.*

Proof. Writing $\mathcal{B} = (u_1, u_2, \dots, u_d)$, define $\mathcal{B}' = (v_1, v_2, \dots, v_d)$ by

$$v_i = \begin{cases} u_i, & i \neq j+1 \\ \frac{\hat{x}}{\sqrt{1-\hat{x}^2}}u_j - \frac{1}{\sqrt{1-\hat{x}^2}}u_{j+1}, & i = j+1 \end{cases}$$

We may also write v_{j+1} as

$$v_{j+1} = \cot(\hat{\theta})u_j - \csc(\hat{\theta})u_{j+1}.$$

Clearly, $\mathcal{B}'$ is a basis for V . The change-of-basis matrix $A = {}_{\mathcal{B}}[I]_{\mathcal{B}'}$ may be written as a block diagonal matrix

$$A = \begin{pmatrix} I_{j-1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & I_{d-j-1} \end{pmatrix}$$

where the 2×2 block $\mathbf{A}$ sliced from the j -th and $(j+1)$ -th rows and columns of A is

$$\mathbf{A} = \begin{pmatrix} 1 & \frac{\hat{x}}{\sqrt{1-\hat{x}^2}} \\ 0 & -\frac{1}{\sqrt{1-\hat{x}^2}} \end{pmatrix}.$$

As in the proof of Proposition 7.3.2, we have

$$\begin{aligned} [T]_{\mathcal{B}'} &= A^{-1}MA \\ &= \begin{pmatrix} I_{j-1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{A}^{-1}\mathbf{M}\mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & I_{d-j-1} \end{pmatrix}. \end{aligned}$$

We may easily compute $\mathbf{A}^{-1}\mathbf{M}\mathbf{A}$.

$$\begin{aligned} \mathbf{A}^{-1}\mathbf{M}\mathbf{A} &= \begin{pmatrix} 1 & \frac{\hat{x}}{\sqrt{1-\hat{x}^2}} \\ 0 & -\frac{1}{\sqrt{1-\hat{x}^2}} \end{pmatrix}^{-1} \begin{pmatrix} 2\hat{x} & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & \frac{\hat{x}}{\sqrt{1-\hat{x}^2}} \\ 0 & -\frac{1}{\sqrt{1-\hat{x}^2}} \end{pmatrix} \\ &= \begin{pmatrix} 1 & \cos(\hat{\theta}) \\ 0 & -\sin(\hat{\theta}) \end{pmatrix} \begin{pmatrix} 2\cos(\hat{\theta}) & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & \cot(\hat{\theta}) \\ 0 & -\csc(\hat{\theta}) \end{pmatrix} \\ &= \begin{pmatrix} \cos(\hat{\theta}) & -\sin(\hat{\theta}) \\ \sin(\hat{\theta}) & \cos(\hat{\theta}) \end{pmatrix}. \end{aligned}$$

Therefore, $[T]_{\mathcal{B}'} = U(\hat{\theta}; j, j+1)$. □

This proposition shows that when s_j and s_{j+1} have a non-free relation, the change of basis induced by the Hurwitz action via powers of σ_j can be thought of as rotating a plane by integer multiples of $\hat{\theta}$. In particular, the rotation $T \in \mathrm{SO}(V)$ given by $[T]_{\mathcal{B}} = M$ can be thought of as the plane rotation that (setwise) preserves the cylinder

$$\left\{ \sum_{i=1}^d x_i v_i \in V \mid x_j^2 + x_{j+1}^2 = 1 \right\}$$

by rotating the (v_j, v_{j+1}) -plane by angle $\hat{\theta} = \pi - \pi/m_{j,j+1}$. For example, when $m_{j,j+1} = 3$, we obtain a rotation of the (v_j, v_{j+1}) -plane by angle $2\pi/3$. When viewed in the original basis $\mathcal{B}$, the rotation T preserves (setwise) the elliptic cylinder

$$\left\{ \sum_{i=1}^d x_i u_i \in V \mid (x_j + \cos(\hat{\theta})x_{j+1})^2 + (-\sin(\hat{\theta})x_{j+1})^2 = 1 \right\}.$$

Figures 7.3 and 7.4 show the rotation T when $m_{j,j+1} = 3$ in the bases $\mathcal{B}'$ and $\mathcal{B}$, respectively.

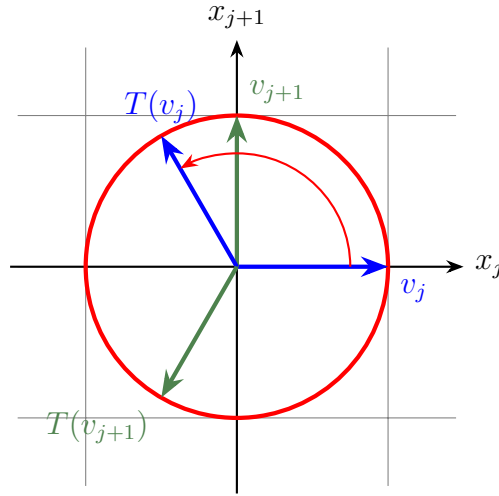


Figure 7.3: Rotation T of the (v_j, v_{j+1}) -plane in basis $\mathcal{B}'$ when $m_{j,j+1} = 3$

7.3.3 Third case $m_{j,j+1} = \infty$

Now let us consider the third case from Equation 7.5, where $m_{j,j+1} = \infty$. Treating $\hat{\theta}$ as the “angle” between the simple roots e_j and e_{j+1} given by the standard bilinear form B , we have $\hat{\theta} = \pi$ and $\hat{x} = \cos(\hat{\theta}) = -1$. Examining the change-of-basis matrix M and the 2×2 submatrix $\mathbf{M}$, we see that

$$\mathbf{M} = \begin{pmatrix} -2 & 1 \\ -1 & 0 \end{pmatrix}$$

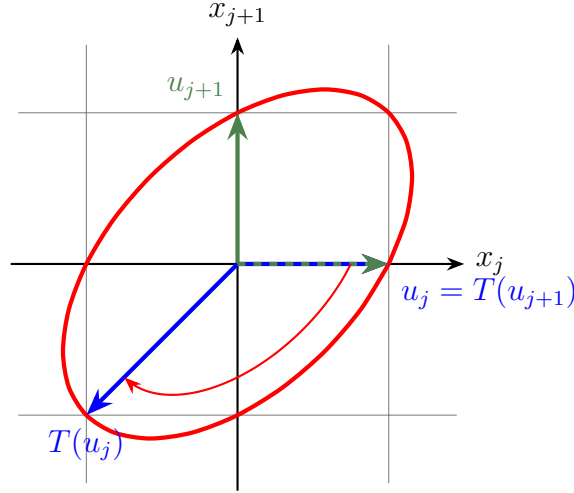


Figure 7.4: Rotation T of the (u_j, u_{j+1}) -plane in basis $\mathcal{B}$ when $m_{j,j+1} = 3$

has the eigenvalue $\lambda = -1$ with algebraic multiplicity 2. Thus, we have $\det(M) = 1$, so like the previous two cases, M is a rotation matrix in the sense of Definition 2.2.5. However, unlike the previous two cases, we are unable to realize M as a Givens rotation $U(\hat{\theta}; j, j+1)$.

Proposition 7.3.4 (Free relation does not lead to a π -rotation). *Suppose $m_{j,j+1} = \infty$, i.e. the relation between the standard generators s_j and s_{j+1} is free. Then M is not similar to $U(\pi; j, j+1)$.*

Proof. Notice that $U(\pi; j, j+1)$ is a diagonal matrix. However, $\mathbf{M}$ (and thus M) is defective. Indeed, the eigenvalue $\lambda = -1$ has algebraic multiplicity 2, but its eigenspace is spanned by the column vector $(1, 1)^\top$. $\square$

Is there some way to think of M as representing a rotation in $\text{SO}(V)$ that rotates by angle $\hat{\theta} = \pi$ in some sense? Perhaps there is. As before, we may think of the change-of-basis matrix M as representing a linear automorphism $T : V \rightarrow V$ with respect to some basis $\mathcal{B} = (u_1, u_2, \dots, u_d)$. Additionally, consider M as parameterized by $m_{j,j+1} \in \{2, 3, 4, \dots, \infty\}$, and consider an ellipse fixed (setwise) by M , as in Figures 7.2 and 7.4.

As $m_{j,j+1}$ increases, the ellipse grows longer and thinner, and points on one long side are sent by M to the other long side as depicted in Figure 7.5. Taking the limit $m_{j,j+1} \rightarrow \infty$, the ellipse approaches two lines $x_{j+1} = x_j \pm 1$ as in Figure 7.6. We can see the motion induced by M as rotating all the way around the “ellipse” (past the point at infinity), as in Figure 7.7.

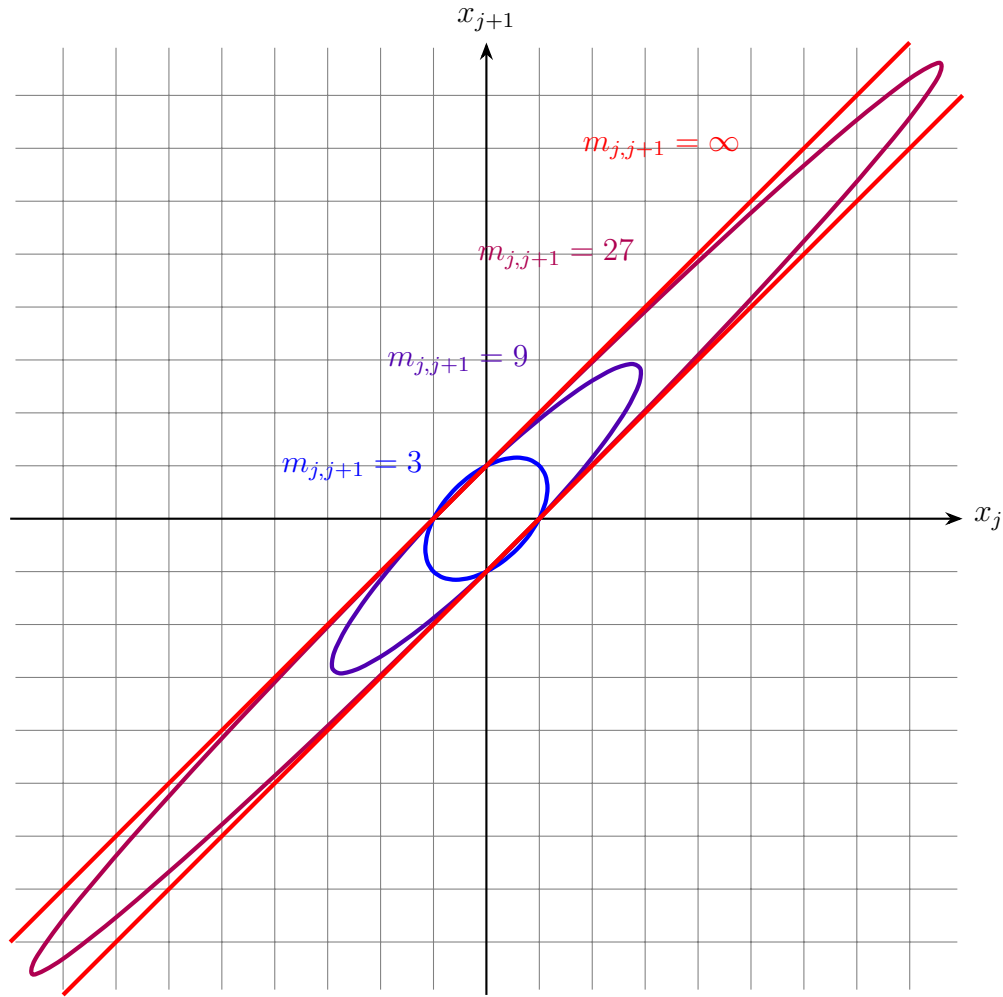
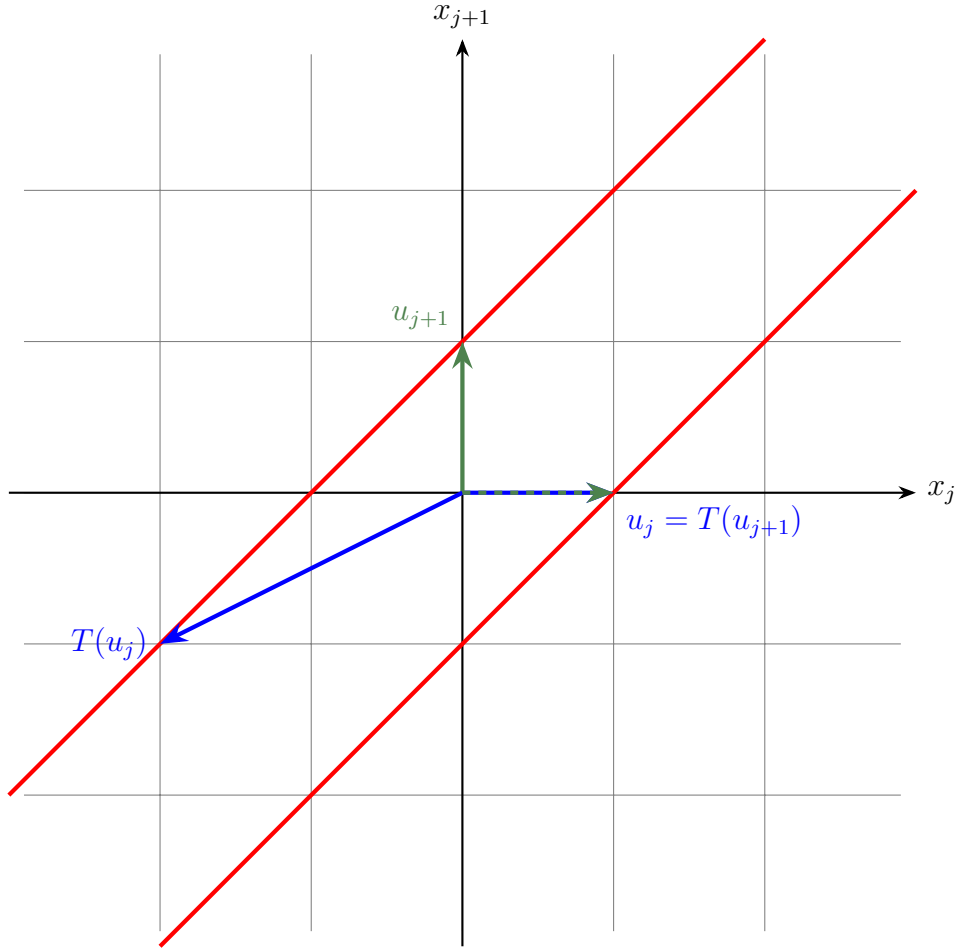
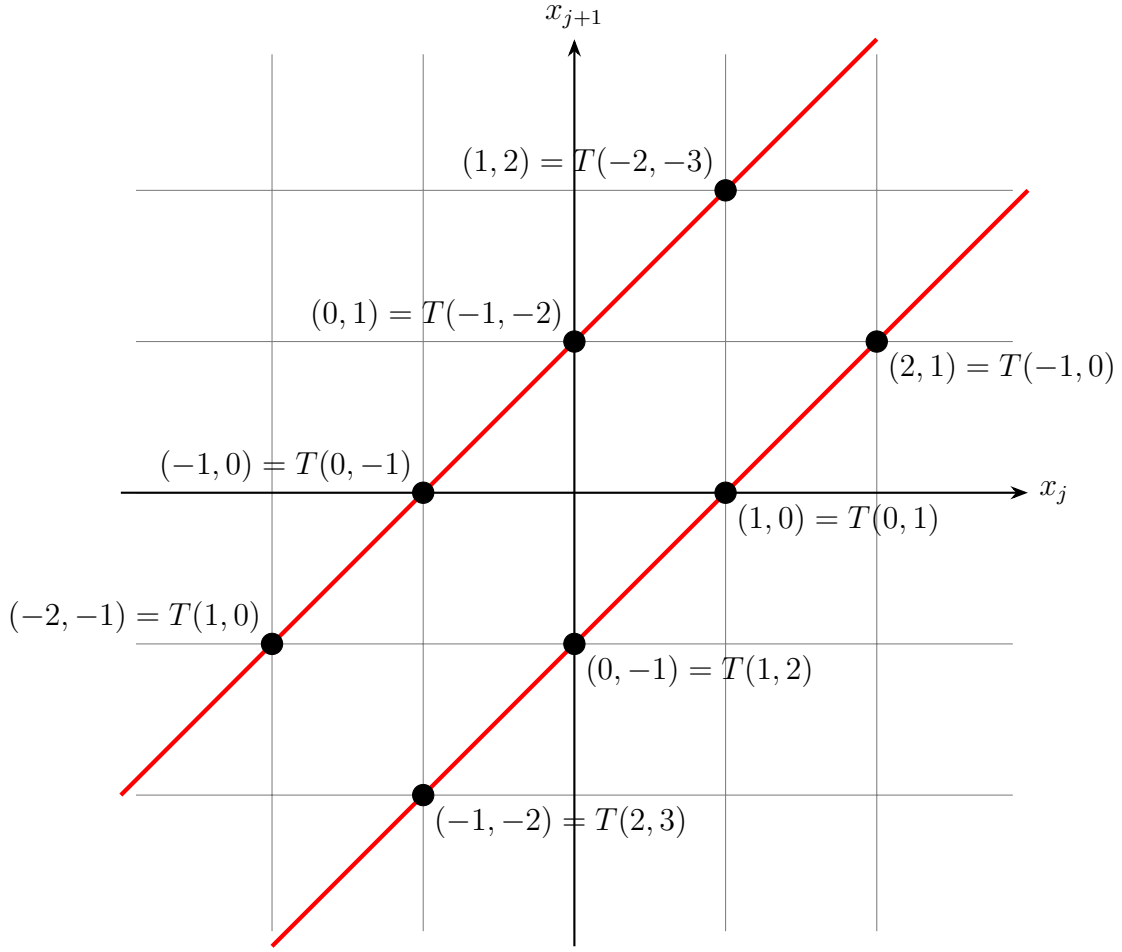


Figure 7.5: Setwise-fixed ellipses as $m_{j,j+1} \rightarrow \infty$ (i.e. $\hat{\theta} \rightarrow \pi$)

Figure 7.6: Two lines fixed setwise by M when $m_{j,j+1} = \infty$

7.3.4 Additional considerations

Up to this point in this chapter, we have used the definition $\hat{x} = \frac{x}{2}$ where x denotes the (j, j) -entry of the change-of-basis matrix $M = {}_{\mathcal{B}_0}[I]_{\sigma_j \cdot \mathcal{B}_0}$ induced by the Hurwitz action via σ_j on the standard basis $\mathcal{B}_0 = (e_1, e_2, \dots, e_d)$ of simple roots, as in Definition 6.3.2. (Equivalently, x denotes the $(j, j+1)$ -entry of $C = [\chi_{c^{-1}}]_{\mathcal{B}_0}$.) We were interested in this particular matrix M because the change-of-basis matrix induced by the Hurwitz action via σ_j^n on $\mathcal{B}_0$ is simply M^n , as shown in Proposition 7.1.1. By these definitions, we have $\hat{x} = B(e_j, e_{j+1}) = \cos(\pi - \frac{\pi}{m_{j,j+1}})$. In particular $\hat{x} \in [-1, 0]$, as in Equation 7.5.

Figure 7.7: Motion induced by M when $m_{j,j+1} = \infty$

We may also consider the Hurwitz action via σ_j^n on some other basis $\mathcal{B} = (u_1, u_2, \dots, u_d) \in \mathfrak{B}$, as in Proposition 7.1.2. In this case, the (j, j) -th entry of the change-of-basis matrix $M = {}_{\mathcal{B}}M_{\sigma_j \mathcal{B}}$ is $x = 2\hat{x}$ where $\hat{x} = B(u_j, u_{j+1})$. Here, we may find cases where $\hat{x} \notin [-1, 0]$. Indeed, consider the following example.

Example 7.3.5 (Applying σ_1^n to $\sigma_2 \cdot \mathcal{B}_0$). Suppose we have a rank-3 Coxeter system (W, S) . The basis $\mathcal{B} = \sigma_2 \cdot \mathcal{B}_0 \in \mathfrak{B}$ is

$$\mathcal{B} = (e_1, \chi_{c^{-1}}(e_2, e_3)e_2 - e_3, e_2) = (e_1, C_{23}e_2 - e_3, e_2).$$

Now consider applying the Hurwitz action via σ_1 . The change-of-basis matrix is

$${}_{\mathcal{B}}M_{\sigma_1\mathcal{B}} = \begin{pmatrix} \chi_{c^{-1}}(e_1, C_{23}e_2 - e_3) & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Computing $\chi_{c^{-1}}(e_1, C_{23}e_2 - e_3)$, we find

$$\begin{aligned} \chi_{c^{-1}}(e_1, C_{23}e_2 - e_3) &= C_{12}C_{23} - C_{13} \\ &= 2(B_{12}B_{23} - B_{13}). \end{aligned}$$

Now we have

$${}_{\mathcal{B}_u}M_{\sigma_1\mathcal{B}_u} = \begin{pmatrix} 2\hat{x} & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

where $\hat{x} = B_{12}B_{23} - B_{13}$. Since each $B_{ij} \in [-1, 0]$ for $i \neq j$, we must have $\hat{x} \geq 0$. Depending on our original Gram matrix B , or equivalently, the relation lengths m_{ij} , we may have $\hat{x} \notin [-1, 0]$. For example, if we have a commuting relation $m_{12} = 2$, a non-free relation $m_{13} < \infty$, and any relation m_{23} , then we have

$$\hat{x} = 0 \cdot B_{23} - B_{13} = \cos(\pi/m_{13}) \in [0, 1].$$

If instead we have non-commuting relations $m_{12}, m_{23} > 2$ and a free relation $m_{13} = \infty$, then $0 < B_{12}B_{23} < 1$ and $B_{13} = -1$, in which case $\hat{x} > 1$.

Evidently, we may encounter a change-of-basis matrix $M = {}_{\mathcal{B}}M_{\sigma_j\mathcal{B}}$ where $\hat{x} = \frac{1}{2}M_{jj} \notin [-1, 0]$. As it turns out, we may still view M as some form of rotation.

Understanding M when $\hat{x} \in (0, 1]$

Suppose we have a Coxeter system (W, S) and basis $\mathcal{B} = (u_1, u_2, \dots, u_d) \in \mathfrak{B}$ such that the (j, j) -th entry $M_{jj} = 2\hat{x}$ of the change-of-basis matrix $M = {}_{\mathcal{B}}M_{\sigma_j\mathcal{B}}$ satisfies $\hat{x} \in (0, 1]$, as shown in Example 7.3.5. As we have done throughout this section, we may view M not

as a change-of-basis matrix, but as representing a linear automorphism T with respect to some basis, say $\mathcal{B}$.

In Sections 7.3.1, 7.3.2, and 7.3.2, the value $\hat{x}$ was defined concretely as the cosine of a relevant angle, namely $\hat{x} = \cos(\hat{\theta})$ where $\hat{\theta} = \pi - \pi/m_{j,j+1}$. In our current case, we are not starting with an angle $\hat{\theta}$, but rather with $\hat{x}$ itself. Since $0 < \hat{x} \leq 1$, we may instead *define* the angle $\hat{\theta}$ as $0 \leq \hat{\theta} < \frac{\pi}{2}$ satisfying $\hat{x} = \cos(\hat{\theta})$.

Applying the same analysis as in those previous sections, when $0 < \hat{x} < 1$, we find that M is similar to a Givens rotation $U(\hat{\theta}; j, j+1)$. When $\hat{x} = 0$ (so $\hat{\theta} = \frac{\pi}{2}$), we are in the case already discussed in Section 7.3.1. When $\hat{x} = 1$ (so $\hat{\theta} = 0$), the 2×2 submatrix $\mathbf{M}$ sliced from the j -th and $(j+1)$ -th rows and columns

$$\mathbf{M} = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}$$

has the eigenvalue $\lambda = 1$ with multiplicity 2, so M is a rotation in the sense of Definition 2.2.5. However, in this case, M is *not* similar to the Givens rotation $U(0; j, j+1)$ because M is defective. Applying the same analysis as in Section 7.3.2, we may consider the limiting behavior of M as $\hat{\theta} \rightarrow 0$, we find that as the angle $\hat{\theta}$ decreases to zero, the corresponding setwise-fixed ellipses in the (u_j, u_{j+1}) -plane elongate along their main axis (the $x_{j+1} = -x_j$ line), approaching two setwise-fixed lines $x_{j+1} = -x_j \pm 1$, as depicted in Figure 7.8. Compare with Figure 7.5, which depicts when $m_{j,j+1} \rightarrow \infty$ (i.e. $\hat{\theta} \rightarrow \pi$).

Understanding M when $|\hat{x}| > 1$

Now suppose we have a Coxeter system (W, S) and basis $\mathcal{B} = (u_1, u_2, \dots, u_d) \in \mathfrak{B}$ such that the (j, j) -th entry $M_{jj} = 2\hat{x}$ of the change-of-basis matrix $M = {}_{\mathcal{B}}M_{\sigma_j \cdot \mathcal{B}}$ satisfies $|\hat{x}| > 1$, as shown in Example 7.3.5. Again, we view M not as a change-of-basis matrix, but as representing a linear automorphism T with respect to some basis, say $\mathcal{B}$.

When $|\hat{x}| > 1$, the eigenvalues $\lambda = \hat{x} \pm \sqrt{\hat{x}^2 - 1}$ of $\mathbf{M}$ are now distinct real numbers.

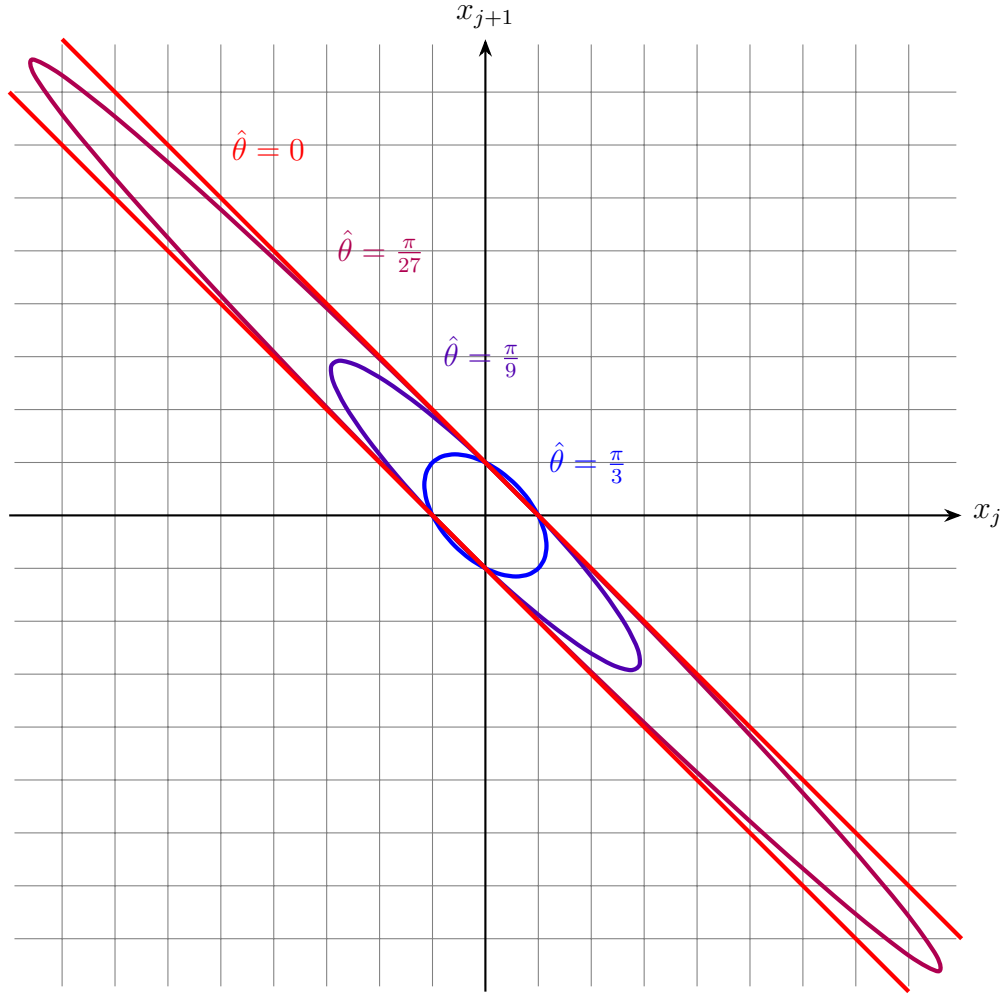


Figure 7.8: Setwise-fixed ellipses as $\hat{x} \rightarrow 1$ (i.e. $\hat{\theta} \rightarrow 0$)

Since

$$(\hat{x} + \sqrt{\hat{x}^2 - 1})(\hat{x} - \sqrt{\hat{x}^2 - 1}) = 1,$$

we see that $\det(M) = 1$. With an appropriate quadratic form, we may see M as a rotation in the sense of Definition 2.2.5.

Whereas $\hat{x}$ was defined in Sections 7.3.1, 7.3.2, and 7.3.2 as the cosine of the angle $\hat{\theta} = \pi - \pi/m_{j,j+1}$, we now define $\hat{\theta}$ in terms of $\hat{x}$. In particular, let $\hat{\theta}$ be the *hyperbolic angle* satisfying $\hat{x} = \cosh(\hat{\theta})$, whereby the eigenvalues λ of $\mathbf{M}$ satisfy

$$\lambda = \cosh(\hat{\theta}) \pm \sinh(\hat{\theta}) = e^{\pm \hat{\theta}}.$$

Given an appropriate basis $\mathcal{B}' = (v_1, v_2, \dots, v_d)$, we may view M as representing a *hyperbolic rotation* T of the (v_j, v_{j+1}) -plane. As noted by Émile Borel in [Bor14], a hyperbolic rotation of angle θ fixes (setwise) the unit hyperbola and preserves the area of hyperbolic sectors as they are rotated, in the same way that a usual (circular) rotation fixes (setwise) the unit circle and preserves the area of (circular) sectors as they are rotated.

Definition 7.3.6 (Hyperbolic rotation). The *hyperbolic rotation* by angle θ of the plane $\mathbb{R}^2$ is the map

$$(x, y) \mapsto (\cosh(\theta)x + \sinh(\theta)y, \sinh(\theta)x + \cosh(\theta)y).$$

Proposition 7.3.7 (M as a hyperbolic rotation). *Suppose $|\hat{x}| > 1$. There exists a basis $\mathcal{B}' = (v_1, v_2, \dots, v_d)$ where $[T]_{\mathcal{B}'}$ is a hyperbolic rotation of the (v_j, v_{j+1}) -plane by angle $-\frac{\pi}{2} < \hat{\theta} < \frac{\pi}{2}$ satisfying $\hat{x} = \cosh(\hat{\theta})$.*

Proof. Writing $\mathcal{B} = (u_1, u_2, \dots, u_d)$, define $\mathcal{B}' = (v_1, v_2, \dots, v_d)$ by

$$v_i = \begin{cases} u_i, & i \neq j+1 \\ \frac{\hat{x}}{\sqrt{\hat{x}^2-1}}u_j - \frac{1}{\sqrt{\hat{x}^2-1}}u_{j+1} & i = j+1 \end{cases}$$

With $\hat{x} = \cosh(\hat{\theta})$, we may also write v_{j+1} as

$$v_{j+1} = \coth(\hat{\theta})u_j - \operatorname{csch}(\hat{\theta})u_{j+1}.$$

Clearly, $\mathcal{B}'$ is a basis for V . The change-of-basis matrix $A = {}_{\mathcal{B}}[I]_{\mathcal{B}'}$ may be written as a block diagonal matrix

$$A = \begin{pmatrix} I_{j-1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & I_{d-j-1} \end{pmatrix}$$

where the 2×2 block $\mathbf{A}$ sliced from the j -th and $(j+1)$ -th rows and columns of A is

$$\mathbf{A} = \begin{pmatrix} 1 & \frac{\hat{x}}{\sqrt{\hat{x}^2-1}} \\ 0 & -\frac{1}{\sqrt{\hat{x}^2-1}} \end{pmatrix}.$$

As in the proofs of Propositions 7.3.2 and 7.3.3, we have

$$\begin{aligned} [T]_{\mathcal{B}'} &= A^{-1}MA \\ &= \begin{pmatrix} I_{j-1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{A}^{-1}\mathbf{M}\mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & I_{d-j-1} \end{pmatrix}. \end{aligned}$$

We may easily compute $\mathbf{A}^{-1}\mathbf{M}\mathbf{A}$.

$$\begin{aligned} \mathbf{A}^{-1}\mathbf{M}\mathbf{A} &= \begin{pmatrix} 1 & \frac{\hat{x}}{\sqrt{\hat{x}^2-1}} \\ 0 & -\frac{1}{\sqrt{\hat{x}^2-1}} \end{pmatrix}^{-1} \begin{pmatrix} 2\hat{x} & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & \frac{\hat{x}}{\sqrt{\hat{x}^2-1}} \\ 0 & -\frac{1}{\sqrt{\hat{x}^2-1}} \end{pmatrix} \\ &= \begin{pmatrix} 1 & \cosh(\hat{\theta}) \\ 0 & -\sinh(\hat{\theta}) \end{pmatrix} \begin{pmatrix} 2\cosh(\hat{\theta}) & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & \coth(\hat{\theta}) \\ 0 & -\operatorname{csch}(\hat{\theta}) \end{pmatrix} \\ &= \begin{pmatrix} \cosh(\hat{\theta}) & \sinh(\hat{\theta}) \\ \sinh(\hat{\theta}) & \cosh(\hat{\theta}) \end{pmatrix}. \end{aligned}$$

Therefore, T acts on the (v_j, v_{j+1}) -plane as a hyperbolic rotation by angle $\hat{\theta}$. □

Evidently, we can think of the linear isometry T given by $[T]_{\mathcal{B}}$ as the hyperbolic rotation that (setwise) preserves the hyperbolic cylinder

$$\left\{ \sum_{i=1}^d x_i v_i \in V \mid x_j^2 - x_{j+1}^2 = 1 \right\}$$

by “rotating” the (v_j, v_{j+1}) -plane by hyperbolic angle $\hat{\theta}$. When viewed in the original basis $\mathcal{B}$, the hyperbolic rotation T preserves (setwise) the hyperbolic cylinder

$$\left\{ \sum_{i=1}^d x_i u_i \in V \mid (x_j + \cosh(\hat{\theta})x_{j+1})^2 - (-\sinh(\hat{\theta})x_{j+1})^2 = 1 \right\}.$$

n	F_n^*
0	1
1	x
2	$x^2 + 1$
3	$x^3 + 2x$
4	$x^4 + 3x^2 + 1$
5	$x^5 + 4x^3 + 3x$
6	$x^6 + 5x^4 + 6x^2 + 1$
7	$x^7 + 6x^5 + 10x^3 + 4x$
8	$x^8 + 7x^6 + 15x^4 + 10x^2 + 1$
9	$x^9 + 8x^7 + 21x^5 + 20x^3 + 5x$
10	$x^{10} + 9x^8 + 28x^6 + 35x^4 + 15x^2 + 1$
n	U_n
0	1
1	$2x$
2	$4x^2 - 1$
3	$8x^3 - 4x$
4	$16x^4 - 12x^2 + 1$
5	$32x^5 - 32x^3 + 6x$
6	$64x^6 - 80x^4 + 24x^2 - 1$
7	$128x^7 - 192x^5 + 80x^3 - 8x$
8	$256x^8 - 448x^6 + 240x^4 - 40x^2 + 1$
9	$512x^9 - 1024x^7 + 672x^5 - 160x^3 + 10x$
10	$1024x^{10} - 2304x^8 + 1792x^6 - 560x^4 + 60x^2 - 1$
n	H_n
0	1
1	x
2	$x^2 - 1$
3	$x^3 - 2x$
4	$x^4 - 3x^2 + 1$
5	$x^5 - 4x^3 + 3x$
6	$x^6 - 5x^4 + 6x^2 - 1$
7	$x^7 - 6x^5 + 10x^3 - 4x$
8	$x^8 - 7x^6 + 15x^4 - 10x^2 + 1$
9	$x^9 - 8x^7 + 21x^5 - 20x^3 + 5x$
10	$x^{10} - 9x^8 + 28x^6 - 35x^4 + 15x^2 - 1$

Table 7.1: The first several terms of the three polynomial sequences

Sequence	Generating matrix	Comments
f_n	$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$	Called the Lucas matrix in [Kin51]; called the Q -matrix in [BH63] [Hog68] [Gou81] [Kos01, Chapter 32]
$F_n(x)$	$\begin{pmatrix} x & 1 \\ 1 & 0 \end{pmatrix}$	Called the Q -matrix in [Kos19, Chapter 32]
$U_n(x)$	$\begin{pmatrix} 2x & 1 \\ -1 & 0 \end{pmatrix}$	
$H_n(x)$	$\begin{pmatrix} x & 1 \\ -1 & 0 \end{pmatrix}$	M from Definition 6.3.2

Table 7.2: Generating matrices of relevant sequences

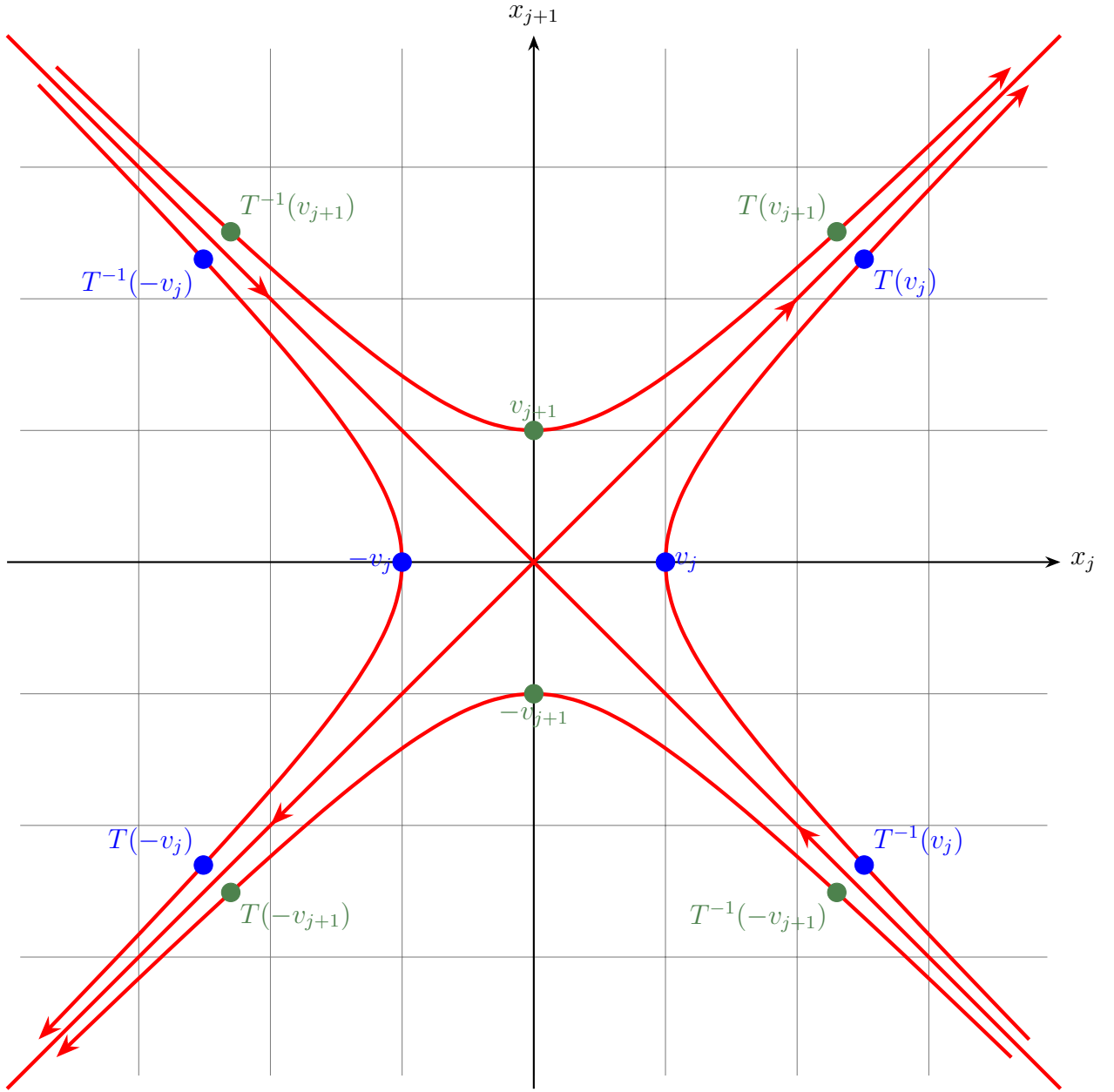


Figure 7.9: Hyperbolic rotation T of the (v_j, v_{j+1}) -plane in basis $\mathcal{B}'$ by angle $\frac{\pi}{2}$

Chapter 8

Hurwitz action via powers of rotation braids

This chapter turns its attention to the change of bases and matrix representations given by the Hurwitz action via a certain type of braid called rotation braids. Using combinatorial and graph-theoretic techniques such as the transfer-matrix method, we find that both the Hurwitz change-of-basis matrix and the Coxeter element can be written as a product of companion matrices. Furthermore, analogous to the connection between half-twists and Fibonacci polynomials discussed in Section 7.2, this chapter investigates the connection between rotation braids and a generalization of Fibonacci polynomials, which we call polynomial wave sequences; see Section 8.4.

As in previous chapters, we let (W, S) be a rank- d Coxeter system with generating set $S = (s_1, s_2, \dots, s_d)$ and Gram matrix B given by $B_{ij} = -\cos(\pi/m_{ij})$. We focus attention on the Coxeter element $c = s_1 s_2 \cdots s_d$ and its inverse c^{-1} , as well as the Wall forms χ_c and $\chi_{c^{-1}}$. We let C denote the upper unitriangular matrix introduced in Section 6.1; recall that C is the matrix representation of the Wall form $\chi_{c^{-1}}$ with respect to the standard ordered basis $(e_1, e_2, \dots, e_d)$ of simple roots. Since we refer frequently to this standard

basis, throughout this chapter, we let $\mathcal{B}_0$ denote this standard basis unless otherwise specified.

8.1 Rotating adjacent strands

This section introduces rotation braids and explicitly computes the Hurwitz change-of-basis matrices given by rotation braids. We also compute matrix representations of Coxeter elements and Wall forms with respect to these bases. We compare these findings to the analogous results coming from half-twists. In particular, whereas the Hurwitz change-of-basis matrices from powers of half-twists are powers of the change-of-basis matrix of the original half-twist, the Hurwitz change-of-basis matrices from arbitrary powers of rotation braids are *not* powers of the change-of-basis matrix of the original rotation braid.

In [MM09], Margalit and McCammond discuss a presentation for the braid group called the *rotation presentation*, generated by braids consisting of rotating strands around each other, called *rotation braids*. In this chapter, we need not define rotation braids in full generality. Instead, we discuss rotation braids of a certain form, namely those formed by rotating adjacent strands.

Definition 8.1.1 (Rotating adjacent strands). For integers $1 \leq i < j \leq d$, a *positive rotation* of the strands $\{i, i+1, \dots, j\}$ is a positive power of the braid $\sigma_{j-1}\sigma_{j-2} \cdots \sigma_{i+1}\sigma_i \in \text{Braid}_d$. A *negative rotation* of the i -th through j -th strands is the inverse of a positive rotation.

Example 8.1.2 (Half-twists are rotations of adjacent strands). The half-twists described in Section 5.1 are a special case of rotating by adjacent strands: powers of σ_i are rotations of the strands i and $i+1$.

Remark 8.1.3 (Functional notation and binary choices). As discussed in Section 5.1, we write braids using “functional notation.” (We first mentioned functional notation in Remark 3.1.4.) Notice that Definition 8.1.1 describes rotation braids $\sigma_{j-1}\sigma_{j-2}\cdots\sigma_{i+1}\sigma_i \in \text{Braid}_d$ where the half-twists with the *lowest* indices are applied first. This ordering of indices is in the opposite order as the usual Coxeter element $c = s_1s_2\cdots s_d \in W$, but in the same order as its inverse c^{-1} whose Wall form matrix $C = [\chi_{c^{-1}}]$ appears frequently in Chapters 6, 7, and 8. This choice of ordering is yet another binary choice to add to the list of binary choices discussed in Remark 6.3.10.

In the notion of rotation described by [MM09], we can see the rotation braid

$$(\sigma_{j-1}\sigma_{j-2}\cdots\sigma_{i+1}\sigma_i)^k$$

as rotating the strands $\{i, i+1, \dots, j\}$ around each other by an angle of $\frac{2\pi}{k(j-i)}$ radians. See Figure 8.1 and Figure 8.2 for examples of rotation braid diagrams. Figure 8.3 is a redrawing of Figure 8.1 that emphasizes the geometric rotation of strands in space.

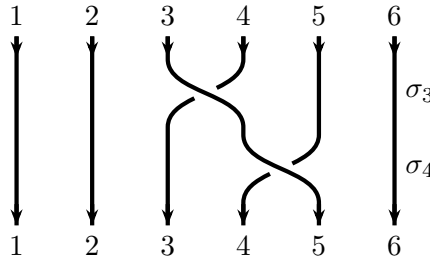
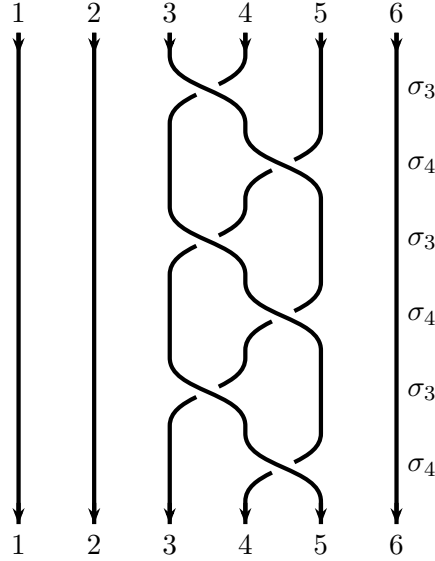


Figure 8.1: Positive rotation braid $\sigma_4\sigma_3 \in \text{Braid}_6$

Let $\mathcal{B} = (u_1, u_2, \dots, u_d) \in \mathfrak{B}$ be some basis obtainable by applying the Hurwitz action via some braid $\beta \in \text{Braid}_d$ to the standard basis $\mathcal{B}_0 = (e_1, e_2, \dots, e_d)$ of simple roots, i.e. $\mathcal{B} = \beta \cdot \mathcal{B}_0$. For $1 \leq i < j \leq d$, let $\mathcal{B}'$ denote the basis obtained by applying the Hurwitz action via the positive rotation braid $\delta = \sigma_{j-1}\sigma_{j-2}\cdots\sigma_{i+1}\sigma_i$ to $\mathcal{B}$, i.e. $\mathcal{B}' = \delta \cdot \mathcal{B}$. I discovered that the resulting Hurwitz change-of-basis matrix is a block diagonal matrix. Compare with the change-of-basis matrix given by a half-twist, stated in Definition 6.3.2.

Figure 8.2: Pure positive rotation braid $(\sigma_4\sigma_3)^3 \in \text{PBraid}_6$

Proposition 8.1.4 (Change of basis given by rotating adjacent strands). *The Hurwitz change-of-basis matrix $M = {}_{\mathcal{B}}[I]_{\mathcal{B}'}$ may be written as the block diagonal matrix*

$$M = \begin{pmatrix} I_{i-1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{M} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & I_{d-j} \end{pmatrix}$$

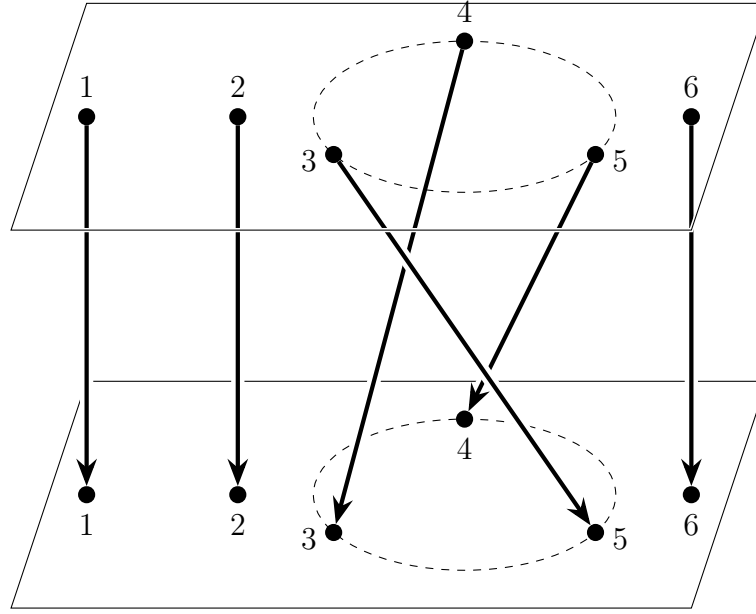
where the block $\mathbf{M}$ is the $(j-i+1) \times (j-i+1)$ matrix

$$\mathbf{M} = \begin{pmatrix} x_i & x_{i+1} & \cdots & x_{j-1} & 1 \\ -1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -1 & 0 \end{pmatrix}$$

and where for each integer $k \in [i, j-1]$, the entry x_k is

$$x_k = \chi_{c^{-1}}(u_i, u_{k+1}) = (\beta \cdot C)_{i,k+1}.$$

More explicitly, the block $\mathbf{M}$ is itself a block matrix

Figure 8.3: $\frac{2\pi}{3}$ -rotation of the strands 3, 4, and 5 by $\sigma_4\sigma_3 \in \text{Braid}_6$

$$\mathbf{M} = \begin{pmatrix} \mathbf{x}^\top & 1 \\ -I_{j-i} & \mathbf{0} \end{pmatrix}$$

where $\mathbf{x}^\top$ is the row vector whose k -th entry is x_k . To emphasize the bases $\mathcal{B}$ and $\mathcal{B}'$, we may write $M = {}_{\mathcal{B}}M_{\mathcal{B}'}$.

Proof. We proceed by induction on the number $n = j - i + 1$ of rotated strands. In the base case $n = 2$, the rotation braid δ is simply the positive half-twist σ_i ; the change-of-basis matrix given in Definition 6.3.2 agrees with the proposition statement. For the inductive step, suppose the proposition holds up to $n = j - i + 1$ strands. Write the basis

$$\mathcal{B}' = \sigma_{j-1} \cdots \sigma_{i+1} \sigma_i \cdot \mathcal{B}$$

as $\mathcal{B}' = (v_1, v_2, \dots, v_d)$. Let $\mathcal{B}'' = \sigma_j \cdot \mathcal{B}'$, and write $\mathcal{B}'' = (w_1, w_2, \dots, w_d)$. The Hurwitz change-of-basis matrix ${}_{\mathcal{B}}M_{\mathcal{B}''}$ may be written as the product ${}_{\mathcal{B}}M_{\mathcal{B}''} = {}_{\mathcal{B}}M_{\mathcal{B}'} M_{\mathcal{B}''}$ of block diagonal matrices. The matrix ${}_{\mathcal{B}}M_{\mathcal{B}'}$ is given in the proposition statement, and the matrix ${}_{\mathcal{B}'}M_{\mathcal{B}''}$ is

$${}_{\mathcal{B}'} M_{\mathcal{B}''} = \begin{pmatrix} I_{j-1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{M}' & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & I_{d-j-1} \end{pmatrix}$$

where the block $\mathbf{M}'$ is the 2×2 matrix

$$\mathbf{M}' = \begin{pmatrix} x_j & 1 \\ -1 & 0 \end{pmatrix}$$

with entry $x_j = \chi_{c^{-1}}(v_j, v_{j+1})$. One may confirm $v_j = u_i$ and $v_{j+1} = u_{j+1}$, so $x_j = \chi_{c^{-1}}(u_i, u_{j+1})$. By simply multiplying the two change-of-basis matrices, we obtain the product

$${}_{\mathcal{B}} M_{\mathcal{B}''} = \begin{pmatrix} I_{i-1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{M}'' & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & I_{d-j-1} \end{pmatrix}$$

where the block $\mathbf{M}''$ is the $(j-i+2) \times (j-i+2)$ matrix

$$\mathbf{M}'' = \begin{pmatrix} x_i & x_{i+1} & \cdots & x_j & 1 \\ -1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -1 & 0 \end{pmatrix},$$

as desired. □

Remark 8.1.5 (Positive vs negative rotations). Proposition 8.1.4 states the Hurwitz change-of-basis matrix $M = {}_{\mathcal{B}}[I]_{\delta\mathcal{B}}$ given by applying the positive rotation $\delta = \sigma_{j-1} \cdots \sigma_{i+1} \sigma_i$. Consider the basis $\delta^{-1} \cdot \mathcal{B}$ obtained by applying the negative rotation $\delta^{-1} = \sigma_i^{-1} \sigma_{i+1}^{-1} \cdots \sigma_j^{-1}$ to the initial basis $\mathcal{B}$. The Hurwitz change-of-basis matrix ${}_{\mathcal{B}}[I]_{\delta^{-1}\mathcal{B}}$ and may be written as the following block diagonal matrix.

$${}_{\mathcal{B}}[I]_{\delta^{-1}\mathcal{B}} = \begin{pmatrix} I_{i-1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{M}' & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & I_{d-j} \end{pmatrix}$$

Here, the block $\mathbf{M}'$ is

$$\mathbf{M}' = \begin{pmatrix} 0 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -1 & 0 \\ 0 & 0 & \cdots & 0 & -1 \\ 1 & x'_i & \cdots & x'_{j-i} & x'_j \end{pmatrix}$$

where for each integer $k \in [i, j-1]$, the entry x'_k is

$$x'_k = \chi_{c^{-1}}(u_k, u_j).$$

Notice that unlike with half-twists in Definition 6.3.2, where the change-of-basis matrix of a negative half-twist is the inverse of the change-of-basis matrix of the corresponding positive half-twist, the change-of-basis matrix of a negative rotation braid is not generally the inverse of the change-of-basis matrix of the corresponding positive rotation braid. We compute M^{-1} and $\mathbf{M}^{-1}$ below.

$$M^{-1} = \begin{pmatrix} I_{i-1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{M}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & I_{d-j} \end{pmatrix}, \quad \mathbf{M}^{-1} = \begin{pmatrix} 0 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -1 & 0 \\ 0 & 0 & \cdots & 0 & -1 \\ 1 & x_i & \cdots & x_{j-1} & x_j \end{pmatrix} \neq \mathbf{M}'.$$

Since $\mathbf{M}' \neq \mathbf{M}^{-1}$, we conclude ${}_B[I]_{\delta^{-1}B} \neq M^{-1}$. This result is sensible because $M^{-1} = {}_{\delta B}[I]_B$, which we should not generally expect to equal ${}_B[I]_{\delta^{-1}B}$.

The matrix $\mathbf{M}$ from Proposition 8.1.4 takes the form of a companion matrix, which we define below.

Definition 8.1.6 (Companion matrix). An $n \times n$ *companion matrix* is a matrix A of the form

$$A = \begin{pmatrix} -u_1 & -u_2 & -u_3 & \cdots & -u_{n-1} & -u_n \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix}$$

where $u_1, u_2, \dots, u_n$ may be regarded as indeterminates. In other words, A is the block matrix

$$A = \begin{pmatrix} -\mathbf{u}^\top & -u_n \\ I_{n-1} & \mathbf{0} \end{pmatrix}$$

where

- $\mathbf{u}^\top$ is the row vector $(u_1, u_2, \dots, u_{n-1})$,
- I_{n-1} is the $(n-1) \times (n-1)$ identity matrix, and
- $\mathbf{0}$ is the all-zero column vector of length $n-1$.

To indicate the entries u_j , we may write $A(u_1, u_2, \dots, u_n)$.

Remark 8.1.7 (Different conventions for companion matrices). There are multiple conventions for writing companion matrices. Definition 8.1.6 follows the convention used in [CL96], [KV04], and [LD11]. Alternatively, we could have defined $A = A(u_1, u_2, \dots, u_n)$ as

$$A = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & -u_1 \\ 1 & 0 & 0 & \cdots & 0 & -u_2 \\ 0 & 1 & 0 & \cdots & 0 & -u_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & -u_{n-1} \\ 0 & 0 & 0 & \cdots & 1 & -u_n \end{pmatrix}$$

as in [HJ12] and [LN02]. Yet another definition would be

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ u_1 & u_2 & u_3 & \cdots & u_{n-1} & u_n \end{pmatrix}$$

as in [LT85]. These various definitions differ only by a few binary choices.

- Ordering of coefficients u_i
- Transpose $\top$
- Conjugation by reversal matrix, as in Remark 2.1.6

These binary choices do not affect relevant matrix properties, e.g. companion matrices defined by all the definitions have the same characteristic polynomial (assuming appropriate orderings of coefficients), so we need not worry about the different conventions; we simply use the convention stated in Definition 8.1.6 (unless otherwise specified) because the Hurwitz change-of-basis matrices given by positive rotations of adjacent strands take this form.

Remark 8.1.8 (Companion matrices and monic polynomials). Using the definition of $A(u_0, u_1, \dots, u_{n-1})$ from Definition 8.1.6, there is a correspondence between monic polynomials

$$p(t) = t^n + \sum_{i=0}^{n-1} u_{n-i} t^i$$

and companion matrices $A(u_0, u_1, \dots, u_{n-1})$. (When using the definition from [HJ12], the companion matrix $A(u_0, u_1, \dots, u_{n-1})$ corresponds with

$$p(t) = t^n + \sum_{i=0}^{n-1} u_i t^i,$$

which is admittedly more pleasing.) Indeed, each monic polynomial is both the minimal polynomial and characteristic polynomial of its companion matrix. For a discussion of companion matrices, see [HJ12, Section 3.3].

The matrix $\mathbf{M}$ from Proposition 8.1.4 may be written

$$\mathbf{M} = -A(x_i, x_{i+1}, \dots, x_{j-1}, 1).$$

In Section 8.2, we exploit this companion matrix form of $\mathbf{M}$ to explicitly compute powers of Hurwitz change-of-basis matrices given by rotating all strands together. For now, let us see walk through an example of the Hurwitz action via rotating adjacent strands.

Example 8.1.9 (Rotating 4 strands in a rank-6 Coxeter system). Let (W, S) denote a rank-6 Coxeter system with generating set $S = (s_1, s_2, s_3, s_4, s_5, s_6)$ and Gram matrix B given by $B_{ij} = -\cos(\pi/m_{ij})$. As usual, let c denote the Coxeter element $c = s_1 s_2 \cdots s_6$. In the standard basis $\mathcal{B}_0 = (e_1, e_2, \dots, e_d)$ of simple roots, the Wall form $\chi_{c^{-1}}$ is represented by an upper unitriangular matrix C , which we may write as

$$C = \begin{pmatrix} 1 & a_1 & b_1 & c_1 & d_1 & e_1 \\ 0 & 1 & a_2 & b_2 & c_2 & d_2 \\ 0 & 0 & 1 & a_3 & b_3 & c_3 \\ 0 & 0 & 0 & 1 & a_4 & b_4 \\ 0 & 0 & 0 & 0 & 1 & a_5 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Consider the rotation braid $\delta = \sigma_4 \sigma_3 \sigma_2$, which rotates strands 2, 3, 4, and 5 by a $\pi/2$ rotation, as depicted in Figure 8.4. The Hurwitz change-of-basis matrix ${}_{\mathcal{B}_0} M_{\delta \mathcal{B}_0}$ is

$${}_{\mathcal{B}_0} M_{\delta \mathcal{B}_0} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & a_2 & b_2 & c_2 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

The matrix representation $\delta \cdot C = [\chi_{c^{-1}}]_{\delta \mathcal{B}_0}$ of the Wall form $\chi_{c^{-1}}$ with respect to the

basis $\delta \cdot \mathcal{B}_0$ is

$$\begin{aligned} \beta \cdot C &= ({}_{\mathcal{B}_0} M_{\delta \mathcal{B}_0})^\top C_{\mathcal{B}_0} M_{\delta \mathcal{B}_0} \\ &= \begin{pmatrix} 1 & a_1 a_2 - b_1 & a_1 b_2 - c_1 & a_1 c_2 - d_1 & a_1 & e_1 \\ 0 & 1 & a_3 & b_3 & a_2 & a_2 d_2 - c_3 \\ 0 & 0 & 1 & a_4 & b_2 & b_2 d_2 - b_4 \\ 0 & 0 & 0 & 1 & c_2 & c_2 d_2 - a_5 \\ 0 & 0 & 0 & 0 & 1 & d_2 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \end{aligned}$$

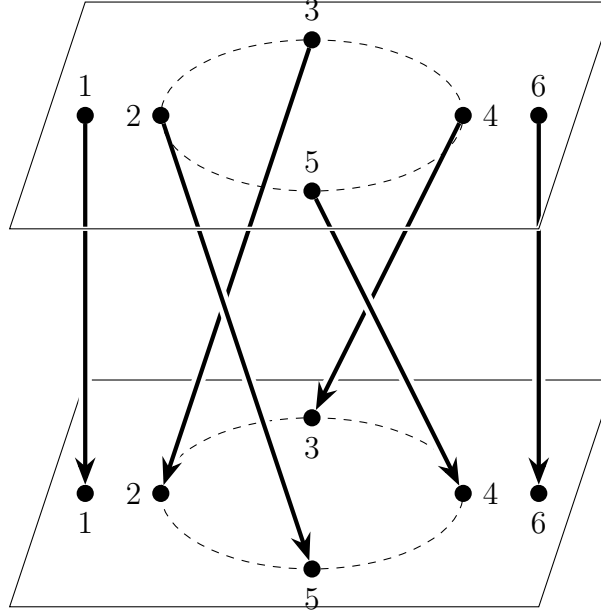


Figure 8.4: $\frac{\pi}{2}$ -rotation of strands 2 through 5 by $\sigma_4 \sigma_3 \sigma_2 \in \text{Braid}_6$

Example 8.1.10 (Powers of rotation braids). Continuing with the same setup as Example 8.1.9, what happens if we apply a power of the rotation braid δ ? The change-of-basis matrix ${}_{\mathcal{B}} M_{\delta^2 \mathcal{B}}$ may be computed as the product ${}_{\mathcal{B}} M_{\delta^2 \mathcal{B}} = {}_{\mathcal{B}} M_{\delta \mathcal{B}} M_{\delta^2 \mathcal{B}}$. Writing $\delta \cdot \mathcal{B} = (u_1, u_2, \dots, u_d)$, we know $\chi_{c^{-1}}(u_k, u_\ell) = (\beta \cdot C)_{k\ell}$, which we use to compute the change-of-basis matrix ${}_{\delta \mathcal{B}} M_{\delta^2 \mathcal{B}}$.

$${}_{\delta\mathcal{B}}M_{\delta^2\mathcal{B}} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & a_3 & b_3 & a_2 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

By matrix multiplication, we obtain the change-of-basis matrix

$${}_{\mathcal{B}}M_{\delta^2\mathcal{B}} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & a_2a_3 - b_2 & a_2b_3 - c_2 & a_2^2 - 1 & a_2 & 0 \\ 0 & -a_3 & -b_3 & -a_2 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Notice that ${}_{\mathcal{B}}M_{\delta^2\mathcal{B}} \neq ({}_{\mathcal{B}}M_{\delta\mathcal{B}})^2$. Unlike with half-twists, where ${}_{\mathcal{B}}M_{\sigma_i^n\mathcal{B}} = ({}_{\mathcal{B}}M_{\sigma_i\mathcal{B}})^n$ as in Proposition 7.1.2, the Hurwitz change-of-basis matrix for the n -th power of a rotation braid δ does not generally equal the n -th power of the change-of-basis matrix for δ . Indeed, as n grows, the change-of-basis matrix ${}_{\mathcal{B}}M_{\delta^n\mathcal{B}}$ becomes increasingly complicated.

When computing Hurwitz change-of-basis matrices ${}_{\mathcal{B}}M_{\delta^n\mathcal{B}}$ given by powers of the rotation braid δ , I found a pattern in the product of intermediate change-of-basis matrices, which we illustrate in the following example.

Example 8.1.11 (Patterns in the change-of-basis matrices). Although the Hurwitz change-of-basis matrix ${}_{\mathcal{B}}M_{\delta^n\mathcal{B}}$ is complicated, it can be written as a product

$${}_{\mathcal{B}}M_{\delta^n\mathcal{B}} = {}_{\mathcal{B}}M_{\delta\mathcal{B}} {}_{\delta\mathcal{B}}M_{\delta^2\mathcal{B}} \cdots {}_{\delta^{n-1}\mathcal{B}}M_{\delta^n\mathcal{B}}$$

where the factors ${}_{\delta^k\mathcal{B}}M_{\delta^{k+1}\mathcal{B}}$ are (negatives of) companion matrices, as in Proposition 8.1.4. Continuing with the same setup from Examples 8.1.9 and 8.1.10, we have already seen the following change-of-basis matrices.

$${}_{\mathcal{B}}M_{\delta\mathcal{B}} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & a_2 & b_2 & c_2 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad {}_{\delta\mathcal{B}}M_{\delta^2\mathcal{B}} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & a_3 & b_3 & a_2 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

By the same process as before, we may compute the change-of-basis matrices for higher powers of δ .

$$\begin{aligned} {}_{\delta^2\mathcal{B}}M_{\delta^3\mathcal{B}} &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & a_4 & b_2 & a_3 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, & {}_{\delta^3\mathcal{B}}M_{\delta^4\mathcal{B}} &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & c_2 & b_3 & a_4 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \\ {}_{\delta^4\mathcal{B}}M_{\delta^5\mathcal{B}} &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & a_2 & b_2 & c_2 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, & {}_{\delta^5\mathcal{B}}M_{\delta^6\mathcal{B}} &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & a_3 & b_3 & a_2 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

First, notice that all the coefficients are simply entries of the Wall form matrix $C = [\chi_{c^{-1}}]_{\mathcal{B}}$. Next, notice that ${}_{\mathcal{B}}M_{\delta\mathcal{B}} = {}_{\delta^4\mathcal{B}}M_{\delta^5\mathcal{B}}$ and ${}_{\delta\mathcal{B}}M_{\delta^2\mathcal{B}} = {}_{\delta^5\mathcal{B}}M_{\delta^6\mathcal{B}}$. To understand what is happening here, we use a special device: let $\mathbf{C}$ denote the 4×4 submatrix sliced from the 2nd through 5th rows and columns of C .

$$\mathbf{C} = \begin{pmatrix} 1 & a_2 & b_2 & c_2 \\ 0 & 1 & a_3 & b_3 \\ 0 & 0 & 1 & a_4 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Consider the infinite block matrix $\mathfrak{C}$ where the block entries of the main diagonal are all $\mathbf{C}$, the block entries of the superdiagonal are all $\mathbf{C}^\top$, and all other entries are zero. Figure 8.5 illustrates $\mathfrak{C}$ in block form, while Figure 8.6 depicts $\mathfrak{C}$ in terms of its entries.

Using 1-based indexing to indicate the rows and columns, the submatrix sliced from the 1st through 4th rows and columns of $\mathfrak{C}$ is precisely $\mathbf{C}$. It appears that for each n , the coefficients of the change-of-basis matrix ${}_{\delta^{n-1}\mathcal{B}}M_{\delta^n\mathcal{B}}$ come from the n -th row of $\mathfrak{C}$. If this relationship indeed holds, then we obtain ${}_{\delta^{n-1}\mathcal{B}}M_{\delta^n\mathcal{B}} = {}_{\delta^{n+3}\mathcal{B}}M_{\delta^{n+4}\mathcal{B}}$ due to the periodicity along the diagonals of $\mathfrak{C}$.

$$\mathfrak{C} = \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\ \dots & \mathbf{C} & \mathbf{C}^\top & \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots \\ \dots & \mathbf{0} & \mathbf{C} & \mathbf{C}^\top & \mathbf{0} & \mathbf{0} & \dots \\ \dots & \mathbf{0} & \mathbf{0} & \mathbf{C} & \mathbf{C}^\top & \mathbf{0} & \dots \\ \dots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{C} & \mathbf{C}^\top & \dots \\ \dots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{C} & \dots \\ \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Figure 8.5: The blocks forming the infinite matrix $\mathfrak{C}$

$$\mathfrak{C} = \begin{pmatrix} \dots & -3 & -2 & -1 & 0 & 1 & 2 & 3 & 4 & \dots \\ \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\ \dots & 1 & a_2 & b_2 & c_2 & 1 & 0 & 0 & 0 & \dots \\ \dots & 0 & 1 & a_3 & b_3 & a_2 & 1 & 0 & 0 & \dots \\ \dots & 0 & 0 & 1 & a_4 & b_2 & a_3 & 1 & 0 & \dots \\ \dots & 0 & 0 & 0 & 1 & c_2 & b_3 & a_4 & 1 & \dots \\ \dots & 0 & 0 & 0 & 0 & 1 & a_2 & b_2 & c_2 & \dots \\ \dots & 0 & 0 & 0 & 0 & 0 & 1 & a_3 & b_3 & \dots \\ \dots & 0 & 0 & 1 & 0 & 0 & 0 & 1 & a_4 & \dots \\ \dots & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & \dots \\ \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{matrix} \vdots \\ -3 \\ -2 \\ -1 \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ \vdots \end{matrix}$$

Figure 8.6: The entries forming the infinite matrix $\mathfrak{C}$

I also discovered that similar periodic behavior appears in the matrix representations of the Wall form $\chi_{c^{-1}}$. We illustrate this behavior with an example below.

Example 8.1.12 (Patterns in matrix representations of Wall forms). Continuing with the same setup from Example 8.1.11, consider the submatrix sliced from the 2nd through 5th rows and columns of the Wall form matrix $\delta^n \cdot C = [\chi_{c^{-1}}]_{\delta^n \mathcal{B}}$. For low powers $n \in \{1, 2, 3, 4\}$, these submatrices are

$$\begin{pmatrix} 1 & a_3 & b_3 & a_2 \\ 0 & 1 & a_4 & b_2 \\ 0 & 0 & 1 & c_2 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & a_4 & b_2 & a_3 \\ 0 & 1 & c_2 & b_3 \\ 0 & 0 & 1 & a_2 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & c_2 & b_3 & a_4 \\ 0 & 1 & a_2 & b_2 \\ 0 & 0 & 1 & a_3 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & a_2 & b_2 & c_2 \\ 0 & 1 & a_3 & b_3 \\ 0 & 0 & 1 & a_4 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

respectively. Notice that the submatrix sliced from $\delta^4 \cdot C$ equals the corresponding submatrix sliced from the Wall form matrix $C = [\chi_{c^{-1}}]_{\mathcal{B}}$ itself, and both equal the 4×4 matrix $\mathbf{C}$ from Example 8.1.11. Recall the infinite matrix $\mathfrak{C}$ depicted in Figures 8.5 and 8.6. It appears that for each n , the submatrix sliced from the 2nd through 5th rows and columns of $\delta^n \cdot C$ equals the submatrix sliced from the $(n+1)$ -th through $(n+4)$ -th rows and columns of the infinite matrix $\mathfrak{C}$. (The above statement would be neater with 0-based indexing, but we continue with 1-based indexing unless otherwise specified, since it used throughout this dissertation.) If this relationship indeed holds, then the submatrix sliced from $\delta^n \cdot C$ equals the same submatrix sliced from $\delta^{n+4} \cdot C$.

I discovered that the apparent periodicity observed in Examples 8.1.11 and 8.1.12 indeed hold, as stated in Proposition 8.1.13 below. In preparation for the proposition statement, we clarify some notation. Let $\mathcal{B} \in \mathfrak{B}$ be some basis obtainable by applying the Hurwitz action via some braid $\beta \in \text{Braid}_d$ to the standard basis $\mathcal{B}_0 = (e_1, e_2, \dots, e_d)$ of simple roots, so $\mathcal{B} = \beta \cdot \mathcal{B}_0$. As before, let $\delta \in \text{Braid}_d$ denote the rotation braid $\delta = \sigma_{j-1} \cdots \sigma_{i+1} \sigma_i$. Let $\mathbf{C}$ denote the submatrix sliced from the i -th through j -th rows and columns of the Wall form matrix $[\chi_{c^{-1}}]_{\mathcal{B}}$. Let $\mathfrak{C}$ denote infinite block matrix where the block entries of the main diagonal are all $\mathbf{C}$, the block entries of the superdiagonal are all $\mathbf{C}^\top$, and all other entries are zero, as in Example 8.1.11.

Proposition 8.1.13 (Periodicity). *For each $n \in \mathbb{Z}$, the following two facts hold.*

1. *The Hurwitz change-of-basis matrix $M = {}_{\delta^{n-1}\mathcal{B}}M_{\delta^n\mathcal{B}}$ may be written as the block diagonal matrix*

$$M = \begin{pmatrix} I_{i-1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{M} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & I_{d-j} \end{pmatrix}$$

where the block $\mathbf{M}$ is the $(j-i+1) \times (j-i+1)$ matrix

$$\mathbf{M} = \begin{pmatrix} x_i^{(n)} & x_{i+1}^{(n)} & \cdots & x_{j-1}^{(n)} & 1 \\ -1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -1 & 0 \end{pmatrix} = -A(x_i^{(n)}, x_{i+1}^{(n)}, \dots, x_{j-1}^{(n)}, 1)$$

where the row vector $\mathbf{x}^\top = (x_i^{(n)}, x_{i+1}^{(n)}, \dots, x_{j-1}^{(n)})$ equals the slice of the n -th row of the infinite matrix $\mathfrak{C}$, slicing from the $(n+1)$ -th column to the $(n+j-i)$ -th column, inclusive, using 1-based indexing.

2. *The $(j-i+1) \times (j-i+1)$ submatrix sliced from the i -th through j -th rows and columns of the Wall form matrix $[\chi_{c-1}]_{\delta^n\mathcal{B}}$ is precisely the submatrix sliced from the $(n+1)$ -th through $(n+j-i+2)$ -th rows and columns (with 1-based indexing) of the infinite matrix $\mathfrak{C}$.*

Furthermore, we observe the following two recurrence relations, both with period $j-i+1$.

- (i) *The Hurwitz change-of-basis matrices satisfy the recurrence*

$${}_{\delta^{n-1}\mathcal{B}}M_{\delta^n\mathcal{B}} = {}_{\delta^{n+j-i}\mathcal{B}}M_{\delta^{n+j-i+1}\mathcal{B}}.$$

- (ii) *The submatrix sliced from the Wall form matrix $[\chi_{c-1}]_{\delta^n\mathcal{B}}$ equals the same submatrix sliced from $[\chi_{c-1}]_{\delta^{n+j-i+1}\mathcal{B}}$.*

To prove this proposition, we use the following key observations.

Lemma 8.1.14. *Let $\mathbf{x}$ be a column vector of length $k - 1$, let $\mathbf{0}$ denote the zero column vector of length $k - 1$, let A be a $(k - 1) \times (k - 1)$ matrix, and let I denote the $(k - 1) \times (k - 1)$ identity matrix. Then we obtain the following equality of $k \times k$ block matrices.*

$$\begin{pmatrix} \mathbf{x} & -I \\ 1 & \mathbf{0}^\top \end{pmatrix} \begin{pmatrix} 1 & \mathbf{x}^\top \\ \mathbf{0} & A \end{pmatrix} \begin{pmatrix} \mathbf{x}^\top & 1 \\ -I & \mathbf{0} \end{pmatrix} = \begin{pmatrix} A & \mathbf{x} \\ \mathbf{0}^\top & 1 \end{pmatrix}$$

Proof. Use matrix multiplication. □

Lemma 8.1.15. *Let $\mathbf{C}$ be a $k \times k$ upper unitriangular matrix, and let $\mathfrak{C}$ be the infinite block matrix where the block entries of the main diagonal are all $\mathbf{C}$, the block entries of the superdiagonal are all $\mathbf{C}^\top$, and all other entries are zero, as depicted in Figure 8.5. For each $n \in \mathbb{Z}$, let $\mathbf{A}_n$ denote the $k \times k$ upper unitriangular matrix sliced from the n -th through $(n + k + 1)$ -th rows and columns of $\mathfrak{C}$. Write $\mathbf{A}_n$ as a block matrix*

$$\mathbf{A}_n = \begin{pmatrix} 1 & \mathbf{x}_n^\top \\ \mathbf{0} & A_n \end{pmatrix}$$

where $\mathbf{x}_n^\top$ is a row vector of length $k - 1$, $\mathbf{0}$ is the all-zero column vector of length $k - 1$, and the block A_n is a $(k - 1) \times (k - 1)$ upper unitriangular matrix. Then for each $n \in \mathbb{Z}$, we have

$$\mathbf{A}_{n+1} = \begin{pmatrix} A_n & \mathbf{x}_n \\ \mathbf{0}^\top & 1 \end{pmatrix}.$$

Proof. Simply slide a $k \times k$ “window” along the main (block) diagonal of the infinite matrix $\mathfrak{C}$, and the pattern becomes self-evident. □

We now prove Proposition 8.1.13.

Proof. For positive integers $n \in \mathbb{N}$, we proceed by induction on n . (We may also use induction for non-positive integers; we omit the proof for that case.) Proposition 8.1.4

immediately gives us the base case $n = 1$ for Statement 1. For the base case for Statement 2, write the Wall form matrix $[\chi_{c^{-1}}]_{\mathcal{B}}$ as the $d \times d$ block matrix

$$[\chi_{c^{-1}}]_{\mathcal{B}} = \begin{pmatrix} * & * & * \\ * & \mathbf{C} & * \\ * & * & * \end{pmatrix}$$

where the block $\mathbf{C}$ is the $(j-i+1) \times (j-i+1)$ upper unitriangular matrix sliced from the i -th through j -th rows and columns of $[\chi_{c^{-1}}]_{\mathcal{B}}$, and where each $*$ denotes its respective block. By Proposition 8.1.4, the Wall form matrix $[\chi_{c^{-1}}]_{\delta\mathcal{B}}$ may be computed as

$$\begin{aligned} [\chi_{c^{-1}}]_{\delta\mathcal{B}} &= ({}_{\mathcal{B}}M_{\delta\mathcal{B}})^{\top} [\chi_{c^{-1}}]_{\mathcal{B}} M_{\delta\mathcal{B}} \\ &= \begin{pmatrix} I_{i-1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{M}^{\top} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & I_{d-j} \end{pmatrix} \begin{pmatrix} * & * & * \\ * & \mathbf{C} & * \\ * & * & * \end{pmatrix} \begin{pmatrix} I_{i-1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{M} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & I_{d-j} \end{pmatrix} \\ &= \begin{pmatrix} * & * & * \\ * & \mathbf{M}^{\top} \mathbf{C} \mathbf{M} & * \\ * & * & * \end{pmatrix}. \end{aligned}$$

We care about only the $(j-i+1) \times (j-i+1)$ submatrix $\mathbf{M}^{\top} \mathbf{C} \mathbf{M}$ sliced from the i -th through j -th rows and columns of $[\chi_{c^{-1}}]_{\delta\mathcal{B}}$. Writing the basis $\mathcal{B}$ as $(u_1, u_2, \dots, u_d)$, we may write the upper unitriangular matrix $\mathbf{C}$ as a block matrix

$$\mathbf{C} = \begin{pmatrix} 1 & \mathbf{x}^{\top} \\ \mathbf{0} & A \end{pmatrix}$$

where the entries of the row vector $\mathbf{x}^{\top} = (x_i, x_{i+1}, \dots, x_{j-1})$ are given by $x_k = \chi_{c^{-1}}(u_i, u_{k+1})$.

Notice that by Proposition 8.1.4, the matrix $\mathbf{M}$ is a companion-like matrix that may be written in block form as

$$\mathbf{M} = \begin{pmatrix} \mathbf{x}^{\top} & 1 \\ -I_{j-i} & \mathbf{0} \end{pmatrix}.$$

Applying Lemma 8.1.14 to $\mathbf{M}^{\top} \mathbf{C} \mathbf{M}$, we find

$$\mathbf{M}^\top \mathbf{C} \mathbf{M} = \begin{pmatrix} A & \mathbf{x} \\ \mathbf{0}^\top & 1 \end{pmatrix}.$$

Applying Lemma 8.1.15, we find that $\mathbf{M}^\top \mathbf{C} \mathbf{M}$ is precisely the submatrix sliced from the 2nd through $(j - i + 3)$ -th rows and columns of the infinite matrix $\mathfrak{C}$ as desired, thereby proving the base case for Statement 2.

For the inductive step, suppose that for some fixed $n \in \mathbb{N}$, both Statements 1 and 2 hold. First, we want to show that Statement 1 holds for $n + 1$. For each $\ell \in \mathbb{Z}$, write the Hurwitz basis $\delta^\ell \cdot \mathcal{B}$ as $(u_1^{(\ell)}, u_2^{(\ell)}, \dots, u_d^{(\ell)})$. By Proposition 8.1.4, the Hurwitz change-of-basis matrix ${}_{\delta^n \mathcal{B}} M_{\delta^{n+1} \mathcal{B}}$ may be written as the block diagonal matrix

$${}_{\delta^n \mathcal{B}} M_{\delta^{n+1} \mathcal{B}} = \begin{pmatrix} I_{i-1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_{n+1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & I_{d-j} \end{pmatrix}$$

where the block $\mathbf{M}_{n+1}$ is the $(j - i + 1) \times (j - i + 1)$ matrix

$$\mathbf{M}_{n+1} = \begin{pmatrix} x_i^{(n+1)} & x_{i+1}^{(n+1)} & \cdots & x_{j-1}^{(n+1)} & 1 \\ -1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -1 & 0 \end{pmatrix}$$

and where for each integer $k \in [i, j - 1]$, the entry $x_k^{(n+1)}$ is

$$x_k^{(n+1)} = \chi_{c^{-1}}(u_i^{(n)}, u_{k+1}^{(n)}) = ([\chi_{c^{-1}}]_{\delta^n \mathcal{B}})_{i, k+1}.$$

By the induction hypothesis for Statement 2, we find that $x_k^{(n+1)}$ is the $(n + 1, n + 1 + k)$ -th entry of the infinite matrix $\mathfrak{C}$, so the row vector $\mathbf{x}_{n+1}^\top = (x_i^{(n+1)}, x_{i+1}^{(n+1)}, \dots, x_{j-1}^{(n+1)})$ is precisely the slice of the $(n + 1)$ -th row of $\mathfrak{C}$ from columns $n + 2$ to $n + j - i + 1$ as desired, thereby finishing the inductive step for Statement 1.

Next, we want to show that Statement 2 holds for $n + 1$. By the induction hypothesis for Statement 2, the Wall form matrix $[\chi_{c^{-1}}]_{\delta^n \mathcal{B}}$ may be written as a block matrix

$$[\chi_{c^{-1}}]_{\delta^n \mathcal{B}} = \begin{pmatrix} * & * & * \\ * & \mathbf{C}_n & * \\ * & * & * \end{pmatrix}$$

where the block $\mathbf{C}_n$ (sliced from the i -th through j -th rows and columns of $[\chi_{c^{-1}}]_{\delta^n \mathcal{B}}$ is precisely the $(j - i + 1) \times (j - i + 1)$ submatrix sliced from the $(n + 1)$ -th through $(n + j - i + 2)$ -th rows and columns of the infinite matrix $\mathfrak{C}$. By Proposition 8.1.4, the Wall form matrix $[\chi_{c^{-1}}]_{\delta^{n+1} \mathcal{B}}$ may be computed as

$$\begin{aligned} [\chi_{c^{-1}}]_{\delta^{n+1} \mathcal{B}} &= (\delta^n \mathcal{B} M_{\delta^{n+1} \mathcal{B}})^\top [\chi_{c^{-1}}]_{\delta^n \mathcal{B}} M_{\delta^{n+1} \mathcal{B}} \\ &= \begin{pmatrix} I_{i-1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_{n+1}^\top & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & I_{d-j} \end{pmatrix} \begin{pmatrix} * & * & * \\ * & \mathbf{C}_n & * \\ * & * & * \end{pmatrix} \begin{pmatrix} I_{i-1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_{n+1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & I_{d-j} \end{pmatrix} \\ &= \begin{pmatrix} * & * & * \\ * & \mathbf{M}_{n+1}^\top \mathbf{C}_n \mathbf{M}_{n+1} & * \\ * & * & * \end{pmatrix}. \end{aligned}$$

We may write the upper unitriangular matrix $\mathbf{C}_n$ as a block matrix

$$\mathbf{C}_n = \begin{pmatrix} 1 & \mathbf{x}_n^\top \\ \mathbf{0} & A_n \end{pmatrix}$$

where the entries of the row vector $\mathbf{x}_n^\top = (x_i^{(n)}, x_{i+1}^{(n)}, \dots, x_{j-1}^{(n)})$ are given by $x_k^{(n)} = \chi_{c^{-1}}(u_i^{(n-1)}, u_{k+1}^{(n-1)})$. By Proposition 8.1.4 applied to $\delta^n \mathcal{B} M_{\delta^{n+1} \mathcal{B}}$, we may write the matrix $\mathbf{M}_{n+1}$ as the block matrix

$$\mathbf{M}_{n+1} = \begin{pmatrix} \mathbf{x}_n^\top & 1 \\ -I_{j-i} & \mathbf{0} \end{pmatrix}.$$

Applying Lemma 8.1.14 in the same way as it was used for the base case of Statement 2, we find that

$$\mathbf{M}_{n+1}^\top \mathbf{C}_n \mathbf{M}_{n+1} = \begin{pmatrix} A_n & \mathbf{x}_n \\ \mathbf{0}^\top & 1 \end{pmatrix}.$$

By the induction hypothesis for Statement 2, we know $\mathbf{C}_n$ is precisely the submatrix

sliced from the $(n+1)$ -th through $(n+j-i+2)$ -th rows and columns of $\mathfrak{C}$. Applying Lemma 8.1.15, the matrix $\mathbf{M}_{n+1}^\top \mathbf{C}_n \mathbf{M}_{n+1}$ equals the submatrix sliced from the $(n+2)$ -th through $(n+j-i+3)$ -th rows and columns of the infinite matrix $\mathfrak{C}$ as desired, thereby finishing the inductive step for Statement 2.

The recurrence relations in Statements (i) and (ii) follow immediately from the periodic construction of $\mathfrak{C}$. $\square$

We may view Statement 2 of Proposition 8.1.13 as a statement about the Wall form $\chi_{c^{-1}}$ evaluated on basis vectors coming from Hurwitz bases. Taking some Hurwitz basis $\mathcal{B} \in \mathfrak{B}$, write $\mathcal{B} = (u_1^{(0)}, u_2^{(0)}, \dots, u_d^{(0)})$. Write $\delta^n \cdot \mathcal{B} = (u_1^{(n)}, u_2^{(n)}, \dots, u_d^{(n)})$ to denote the basis given by the Hurwitz action via δ^n on the basis $\mathcal{B}$. Statement 2 of Proposition 8.1.13 tells us how to evaluate the Wall form $\chi_{c^{-1}}$ on basis vectors $u_k^{(n)}$. As an illustration, letting $\mathcal{B}_0$ denote the standard basis of simple roots, the matrix $\delta \cdot C = [\chi_{c^{-1}}]_{\delta \mathcal{B}_0}$ from Example 8.1.12 gives us the following evaluations of $\chi_{c^{-1}}$.

$$\begin{aligned} \chi_{c^{-1}}(u_2^{(1)}, u_3^{(1)}) &= a_3, & \chi_{c^{-1}}(u_2^{(1)}, u_4^{(1)}) &= b_3, & \chi_{c^{-1}}(u_2^{(1)}, u_5^{(1)}) &= a_2, \\ \chi_{c^{-1}}(u_3^{(1)}, u_4^{(1)}) &= a_4, & \chi_{c^{-1}}(u_3^{(1)}, u_5^{(1)}) &= b_2, & \chi_{c^{-1}}(u_4^{(1)}, u_5^{(1)}) &= c_2. \end{aligned}$$

Using the periodic construction of the infinite matrix $\mathfrak{C}$ from Figure 8.6, we conclude that the Wall form $\chi_{c^{-1}}$ from Example 8.1.12 satisfies $\chi_{c^{-1}}(u_k^{(5)}, u_\ell^{(5)}) = \chi_{c^{-1}}(u_k^{(1)}, u_\ell^{(1)})$ for $2 \leq k < \ell \leq 5$. The following corollary simply rephrases Statement 2 of Proposition 8.1.13 in this language.

Corollary 8.1.15.1. *For integers $i \leq k < \ell \leq j$ and $n \in \mathbb{Z}$, the Wall form $\chi_{c^{-1}}(u_k^{(n)}, u_\ell^{(n)})$ equals the entry from the $(k-1+1+n)$ -th row and $(\ell-i+1+n)$ -th column of the infinite matrix $\mathfrak{C}$ from Proposition 8.1.13. In particular, $\chi_{c^{-1}}$ satisfies the recurrence*

$$\chi_{c^{-1}}(u_k^{(n)}, u_\ell^{(n)}) = \chi_{c^{-1}}(u_k^{(n+j-i+1)}, u_\ell^{(n+j-i+1)}).$$

Evidently, the Hurwitz change of bases obtained by rotating adjacent strands preserves some geometric data with respect to the basis vectors. For $k < \ell$, we know

$$\chi_{c^{-1}}(u_k^{(n)}, u_\ell^{(n)}) = 2B(u_k^{(n)}, u_\ell^{(n)})$$

where B is the standard bilinear form of (W, S) . As in Remark 4.1.3, we may let θ satisfying $B(u, v) = \cos(\theta)$ denote the “angle” between unit vectors $u, v \in V$. From this perspective, Corollary 8.1.15.1 essentially says that the Hurwitz change of basis from applying the $(j - i + 1)$ -th power of the rotation braid $\delta = \sigma_{j-1} \cdots \sigma_{i+1} \sigma_i$ preserves the pairwise angles between the basis vectors indexed inclusively from i to j .

8.2 Rotating with all strands

Whereas Section 8.1 investigates the general case of rotating adjacent strands, this section focuses on the special case where we rotate all d strands. In this special case, we observe periodicity in the Hurwitz change-of-basis matrices for powers of all-strand rotations. Furthermore, we find that the change-of-basis matrix can be decomposed as a product of companion matrices, and use the transfer-matrix method to explicitly compute the entries of this product. We also find that the matrix representation of the usual Coxeter element $c = s_1 s_2 \cdots s_d$ can be written as a product of these companion matrices.

Consider the positive rotation braid given by rotating all d strands, namely the braid $\delta = \sigma_{d-1} \cdots \sigma_2 \sigma_1$. We call this braid the *positive all-strand rotation*, viewed as rotating all d strands about each other by a $2\pi/d$ rotation. (Those familiar with Garside structures may recognize the all-strand rotation as a Garside element of the dual braid monoid. For further reading on Garside structures, see [Deh+15]; for further reading on the dual braid monoid, see [Bes03].) For the rest of the chapter, let δ denote the positive all-strand rotation braid, unless otherwise specified. For each power $\ell \in \mathbb{Z}$, the braid δ^ℓ rotates the

d strands by a $2\pi\ell/n$ rotation. When n is an integer multiple of d , the braid δ^n is a pure braid; the example $\delta^4 \in \text{PBraid}_4$ is depicted in Figure 8.7.

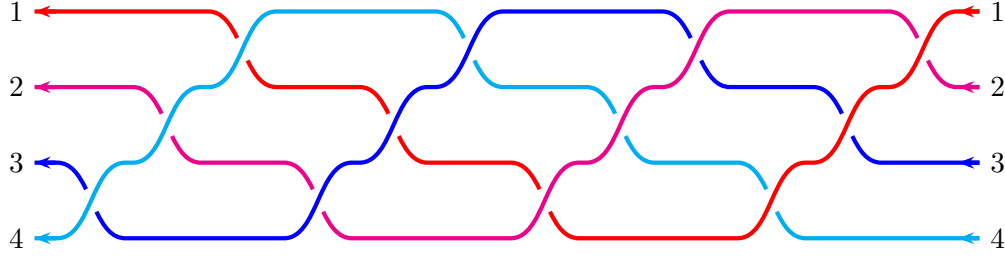


Figure 8.7: Pure all-strand rotation $\delta^4 = (\sigma_3\sigma_2\sigma_1)^4 \in \text{PBraid}_4$

Let $\mathcal{B} = (u_1, u_2, \dots, u_d) \in \mathfrak{B}$ be some basis obtainable by applying the Hurwitz action via some braid $\beta \in \text{Braid}_d$ to the standard basis of simple roots. Let $\mathbf{C} = [\chi_{c^{-1}}]_{\mathcal{B}}$ denote the matrix representation of the Wall form $\chi_{c^{-1}}$ with respect to this Hurwitz basis $\mathcal{B}$. In other words, let $\mathbf{C} = \beta \cdot C$ where C is the upper unitriangular matrix representing the $\chi_{c^{-1}}$ with respect to the standard basis. We obtain the corresponding Hurwitz change-of-basis matrices as a special case of Proposition 8.1.4.

Proposition 8.2.1 (Change of basis given by all-strand rotation). *The Hurwitz change-of-basis matrix $M = {}_{\mathcal{B}}[I]_{\delta\mathcal{B}}$ given by the positive all-strand rotation may be written as the block matrix*

$$M = \begin{pmatrix} \mathbf{x}^\top & 1 \\ -I_{d-1} & \mathbf{0} \end{pmatrix}$$

where $\mathbf{x}^\top$ is the row vector

$$\mathbf{x}^\top = (x_1, x_2, \dots, x_{d-1})$$

whose entries are given by

$$x_k = \chi_{c^{-1}}(u_1, u_{k+1}) = \mathbf{C}_{1,k+1}.$$

Similarly, the Hurwitz change-of-basis matrix $M' = {}_{\mathcal{B}}[I]_{\delta^{-1}\mathcal{B}}$ given by the negative all-strand rotation may be written as the block matrix

$$M' = \begin{pmatrix} \mathbf{0} & -I_{d-1} \\ 1 & \mathbf{y}^\top \end{pmatrix}$$

where $\mathbf{y}^\top$ is the row vector

$$\mathbf{y}^\top = (y_1, y_2, \dots, y_{d-1})$$

whose entries are given by

$$y_k = \chi_{c^{-1}}(u_k, u_d) = \mathbf{C}_{kd}.$$

Proof. Simply set $i = 1$ and $j = d$ in Proposition 8.1.4. □

Recall from Definition 8.1.6 that $A(u_1, u_2, \dots, u_n)$ denotes the companion matrix with coefficients $u_1, u_2, \dots, u_n$. Notice that the change-of-basis matrix M in Proposition 8.2.1 can be written as the negative of a companion matrix, namely

$$M = -A(x_1, x_2, \dots, x_{d-1}, 1).$$

As a special case of Statement 1 of Proposition 8.1.13, the following proposition shows that the coefficients of the change-of-basis matrix ${}_{\delta^{n-1}\mathcal{B}}M_{\delta^n\mathcal{B}}$ obey a predictable periodic pattern. As before, let $\mathcal{B} = (u_1, u_2, \dots, u_d) \in \mathfrak{B}$ be some basis obtainable by applying the Hurwitz action via some braid $\beta \in \text{Braid}_d$ to the standard basis of simple roots, and let $\mathbf{C} = \beta \cdot C$. Let $\mathfrak{C}$ denote the infinite block matrix where the block entries of the main diagonal are all $\mathbf{C}$, the block entries of the superdiagonal are all $\mathbf{C}^\top$, and all other entries are zero, as in Figure 8.5.

Proposition 8.2.2 (Change-of-basis matrices from $\delta^{n-1}\mathcal{B}$ to $\delta^n\mathcal{B}$). *For each $n \in \mathbb{Z}$, the Hurwitz change-of-basis matrix ${}_{\delta^{n-1}\mathcal{B}}M_{\delta^n\mathcal{B}}$ may be written as*

$$\delta^{n-1}\mathcal{B}M_{\delta^n\mathcal{B}} = \begin{pmatrix} x_1 & x_2 & \cdots & x_{d-1} & 1 \\ -1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -1 & 0 \end{pmatrix} = -A(x_1, x_2, \dots, x_{d-1}, 1)$$

where the row vector $\mathbf{x}^\top = (x_1, x_2, \dots, x_{d-1})$ equals the slice of the n -th row of the infinite matrix $\mathfrak{C}$, slicing inclusively from the $(n+1)$ -th column to the $(n+d-1)$ -th column, using 1-based indexing. Furthermore, we observe the following recurrence.

$$\delta^{n-1}\mathcal{B}M_{\delta^n\mathcal{B}} = \delta^{n+d-1}\mathcal{B}M_{\delta^{n+d}\mathcal{B}} \quad (8.1)$$

Proof. Simply set $i = 1$ and $j = d$ in Statement 1 of Proposition 8.1.13. $\square$

I discovered a pleasant formula for the change-of-basis matrix $\delta^{n-1}\mathcal{B}M_{\delta^n\mathcal{B}}$. First, let π denote the cyclic permutation $(1, 2, \dots, d) \in \text{Sym}_d$. Recall from Definition 2.1.7 that P_π denotes the permutation matrix associated with π .

$$P_\pi = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix} = \begin{pmatrix} \mathbf{0}^\top & 1 \\ I_{d-1} & 0 \end{pmatrix}$$

Also recall from Definition 2.1.2 that E_{ij} denotes the (i, j) -th standard basis matrix.

Corollary 8.2.2.1 (Formula for $\delta^{n-1}\mathcal{B}M_{\delta^n\mathcal{B}}$). *Let $\mathbf{J} = \mathbf{C} + \mathbf{C}^\top$. Write $n = qd + r$, as in Euclidean division. Using 1-based indexing for the rows and columns of matrices, the change-of-basis matrix $\delta^{n-1}\mathcal{B}M_{\delta^n\mathcal{B}}$ may be written as*

$$\delta^{n-1}\mathcal{B}M_{\delta^n\mathcal{B}} = \begin{cases} E_{1r}\mathbf{J}P_\pi^r - P_\pi, & r \neq 0 \\ E_{1d}\mathbf{J} - P_\pi, & r = 0 \end{cases}$$

Equivalently, we may write

$$\delta^n \mathcal{B} M_{\delta^{n+1} \mathcal{B}} = E_{1,r+1} \mathbf{J} P_\pi^{r+1} - P_\pi. \quad (8.2)$$

Proof. By Equation 8.1, we know $\delta^n \mathcal{B} M_{\delta^{n+1} \mathcal{B}} = \delta^r \mathcal{B} M_{\delta^{r+1} \mathcal{B}}$. The product $\mathbf{J} P_\pi^{r+1}$ is the result of permuting the columns of $\mathbf{J}$ by the permutation π^{r+1} . By Remark 2.1.4, the product $E_{1,r+1} \mathbf{J} P_\pi^r$ is the matrix whose 1st row equals the $(r+1)$ -th row of $\mathbf{J} P_\pi^{r+1}$. One may easily confirm that the matrix $E_{1,r+1} \mathbf{J} P_\pi^{r+1} - P_\pi$ equals $-A(x_1, x_2, \dots, x_{d-1}, 1)$ from Proposition 8.2.2, where the row vector $\mathbf{x}^\top = (x_1, x_2, \dots, x_{d-1})$ equals the slice of the $(r+1)$ -th row of infinite matrix $\mathfrak{C}$, slicing inclusively from the $(r+2)$ -th column to the $(r+d)$ -th column. $\square$

Let $A(n)$ denote the companion matrix satisfying

$$\delta^{n-1} \mathcal{B} M_{\delta^n \mathcal{B}} = -A(n),$$

i.e. the companion matrix from Proposition 8.2.2. By Corollary 8.2.2.1, we obtain the formula

$$A(n+1) = P_\pi - E_{1,r+1} \mathbf{J} P_\pi^{r+1}. \quad (8.3)$$

The following corollary of Proposition 8.2.2 shows that we may write the Hurwitz change-of-basis matrix ${}_{\mathcal{B}} M_{\delta^n \mathcal{B}}$ as a product of (negatives of) companion matrices whose coefficients obey the aforementioned periodic pattern. Later in this section, we use this pattern to compute the entries of ${}_{\mathcal{B}} M_{\delta^n \mathcal{B}}$.

Corollary 8.2.2.2 (Product of companion matrices). *For each $n \in \mathbb{Z}$, the Hurwitz change-of-basis matrix ${}_{\mathcal{B}} M_{\delta^n \mathcal{B}}$ may be written as the product*

$${}_{\mathcal{B}} M_{\delta^n \mathcal{B}} = \prod_{i=1}^n (-A(i)) = (-1)^n \prod_{i=1}^n A(i). \quad (8.4)$$

Furthermore, writing $n = qd + r$ as in Euclidean division, we obtain

$${}_B M_{\delta^n B} = ({}_B M_{\delta^d B})^q {}_B M_{\delta^r B} = (-1)^n \left(\prod_{i=1}^d A(i) \right)^q \prod_{k=1}^r A(k). \quad (8.5)$$

Proof. We obtain Equation 8.4 by applying Proposition 8.2.2 to the product

$${}_B M_{\delta^n B} = \prod_{i=1}^n {}_{\delta^{i-1} B} M_{\delta^i B}.$$

Furthermore, notice that Equation 8.1 implies

$${}_B M_{\delta^{qd} B} = ({}_B M_{\delta^d B})^q \quad \text{and} \quad {}_{\delta^{qd} B} M_{\delta^{qd+r} B} = {}_B M_{\delta^r B},$$

so we obtain Equation 8.5 as follows.

$${}_B M_{\delta^n B} = {}_B M_{\delta^{qd} B} {}_B M_{\delta^{qd+r} B} = ({}_B M_{\delta^d B})^q {}_B M_{\delta^r B}$$

□

The above propositions and corollaries are written in terms of an arbitrary Hurwitz basis $B \in \mathfrak{B}$. When we apply these propositions to the standard basis of simple roots B_0 , something beautiful happens. In Theorem 8.2.4 below, we find that (the matrix representation in the standard basis of) the usual Coxeter element $c = s_1 s_2 \cdots s_d$ can be written as a product of companion matrices. Furthermore, in Corollary 8.2.4.1, we find that the Hurwitz change-of-basis matrix ${}_{B_0} M_{\delta^d B_0}$ has a simple formula in terms of c .

Before discussing these results, we establish some preliminary setup. First, the block $\mathbf{C}$ from the infinite matrix $\mathfrak{C}$ equals the upper unitriangular matrix C that represents the Wall form $\chi_{c^{-1}}$ with respect to the standard basis. Anything defined in terms of $\mathfrak{C}$, such as the companion matrices $A(i)$, draw their entries from C and $C^\top$. The matrix $\mathbf{J} = \mathbf{C} + \mathbf{C}^\top$ becomes the Schläfli matrix $J = C + C^\top$. The previous formulas from Equation 8.2 and Equation 8.3 may be written with J in place of $\mathbf{J}$. Consider the permutation matrix P_π given by the cyclic permutation $\pi = (1, 2, \dots, d) \in \text{Sym}_d$, as before. The following

lemma states that simple reflections may be written in terms of the companion matrices found in Proposition 8.2.2.

Lemma 8.2.3 (Matrix representation of simple reflections). *The i -th simple reflection $r_i = \rho(s_i)$ has matrix representation*

$$[r_i] = P_\pi^{i-1} A(i) P_\pi^{-i}$$

with respect to the standard basis of simple roots. Equivalently, we may write

$$A(i) = P_\pi^{1-i} [r_i] P_\pi^i.$$

Proof. By Proposition 4.1.14, we know $[r_i] = I - E_{ii}J$. Recall from Remark 2.1.4 for each $d \times d$ matrix M , the product $E_{ij}M$ is the matrix whose i -th row equals the j -th row of M , and whose other entries are all zero. Using the fact that left-multiplying by P_π permutes rows by the cyclic permutation $\pi = (1, 2, \dots, d)$, we know that $E_{ii} = P_\pi^{i-1} E_{1i}$. Thus, we find that

$$\begin{aligned} [r_i] &= I - E_{ii}J \\ &= P_\pi^0 - P_\pi^{i-1} E_{1i}J \\ &= P_\pi^{i-1} (P_\pi - E_{1i}J P_\pi^i) P_\pi^{-i}. \end{aligned}$$

Applying Equation 8.3, we conclude that

$$[r_i] = P_\pi^{i-1} A(i) P_\pi^{-i}.$$

□

As a consequence of this lemma, we obtain the telescoping product

$$\prod_{k=i}^j A(k) = \prod_{k=i}^j P_\pi^{1-k} [r_k] P_\pi^k = P_\pi^{1-i} \left(\prod_{k=i}^j [r_k] \right) P_\pi^j$$

for integers $1 \leq i \leq j \leq d$. In particular, we obtain

$$\prod_{k=1}^n A(k) = \left(\prod_{k=1}^n [r_k] \right) P_\pi^n \quad (8.6)$$

for $1 \leq n \leq d$. We obtain the following beautiful formula for the matrix representation $[c]$ of the Coxeter element $c = s_1 s_2 \cdots s_d$.

Theorem 8.2.4 (The Coxeter element as a product of companion matrices). *Let $[c]$ denote the matrix representation of the usual Coxeter element $c = s_1 s_2 \cdots s_d$ with respect to the standard basis of simple roots. Then*

$$[c] = A(1)A(2) \cdots A(d).$$

Proof. Set $n = d$ in Equation 8.6. Since $\pi^d = \text{id}$, we know $P_\pi^d = I$, so we obtain

$$[c] = [r_1][r_2] \cdots [r_d] = A(1)A(2) \cdots A(d)$$

as desired. □

We may write the Hurwitz change-of-basis matrix ${}_{\mathcal{B}_0} M_{\delta^d \mathcal{B}_0}$ in terms of $[c]$. As usual, C denotes the upper unitriangular matrix that represents the Wall form χ_{c^1} with respect to the standard basis, as defined in Chapter 6.

Corollary 8.2.4.1 (Relationship between ${}_{\mathcal{B}} M_{\delta^d \mathcal{B}}$ and c). *Let $[c]$ denote the matrix representation of the usual Coxeter element $c = s_1 s_2 \cdots s_d$ with respect to the standard basis $\mathcal{B}_0$ of simple roots. Then*

$${}_{\mathcal{B}_0} M_{\delta^d \mathcal{B}_0} = (-1)^d [c] = (-1)^{d+1} C^{-1} C^\top.$$

Proof. Applying Corollary 8.2.2.2 and Theorem 8.2.4, we obtain

$${}_{\mathcal{B}_0} M_{\delta^d \mathcal{B}_0} = (-1)^d [c],$$

as desired. Applying Theorem 6.1.1, we may rewrite this as

$$_{\mathcal{B}_0} M_{\delta^d \mathcal{B}_0} = (-1)^{d+1} C^{-1} C^\top.$$

□

So far in this section, we have focused mostly on Hurwitz change-of-basis matrices given by powers of the all-strand rotation δ , especially the periodic behavior with respect to the integer power n for change-of-basis matrices of the form $_{\delta^{n-1}\mathcal{B}} M_{\delta^n \mathcal{B}}$. The observed periodicity applies not only to Hurwitz change-of-basis matrices, but also to matrix representations of the Wall form $\chi_{c^{-1}}$. Returning to the general setup used for Proposition 8.2.1, let $\mathbf{C}$ denote the matrix representation of $\chi_{c^{-1}}$ with respect to some Hurwitz basis $\mathcal{B} \in \mathfrak{B}$, and let $\mathfrak{C}$ denote the infinite matrix depicted in Figure 8.5. As a special case of Statement 2 of Proposition 8.1.13, the matrix representations of the Wall form $\chi_{c^{-1}}$ exhibit the following periodic behavior, with period d .

Proposition 8.2.5 (Periodicity with all-strand rotations). *For each $n \in \mathbb{Z}$, the Wall form matrix $\delta^n \cdot \mathbf{C} = [\chi_{c^{-1}}]_{\delta^n \mathcal{B}}$ equals the $d \times d$ submatrix sliced from the $(n+1)$ -th through $(n+d+1)$ -th rows and columns of $\mathfrak{C}$. (Here, we use 1-based indexing, as in Example 8.1.12.) Furthermore, $\delta^n \cdot \mathbf{C} = \delta^{n+d} \cdot \mathbf{C}$. In particular, $\delta^d \cdot \mathbf{C} = \mathbf{C}$.*

Proof. Simply set $i = 1$ and $j = d$ in Statement 2 of Proposition 8.1.13. □

Since each Wall form matrix $\delta^n \cdot \mathbf{C}$ is a matrix representation of the Wall form $\chi_{c^{-1}}$, Proposition 8.2.5 may be viewed as a statement about evaluating the $\chi_{c^{-1}}$ on the basis vectors from Hurwitz bases obtained by powers of the all-strand rotation δ . In particular, while the Hurwitz bases $\delta^d \cdot \mathcal{B}$ and $\mathcal{B}$ may differ, the Hurwitz action by δ^d preserves the Wall form $\chi_{c^{-1}}$ when evaluated at pairs of basis vectors. More explicitly, writing $\mathcal{B} = (u_1, u_2, \dots, u_d)$ and $\delta^d \cdot \mathcal{B} = (v_1, v_2, \dots, v_d)$, we find that $\chi_{c^{-1}}(u_i, u_j) = \chi_{c^{-1}}(v_i, v_j)$ for each pair (i, j) . Due to this periodicity in the Wall form matrices $\delta^n \cdot \mathbf{C}$, when discussing

representations of $\chi_{c^{-1}}$ throughout the rest of this chapter, we need only consider the powers of δ up to the d -th power.

For each Hurwitz basis $\mathcal{B} = (u_1, u_2, \dots, u_d) \in \mathfrak{B}$ and each pair of distinct indices $i < j$, the Wall form $\chi_{c^{-1}}$ may be easily evaluated as

$$\chi_{c^{-1}}(u_i, u_j) = 2B(u_i, u_j)$$

where B is the standard bilinear form for the Coxeter system (W, S) . As in Remark 4.1.3, we may let θ satisfying $B(u, v) = \cos(\theta)$ denote the “angle” between unit vectors $u, v \in V$. From this perspective, Proposition 8.2.5 essentially says that the Hurwitz action via δ^d preserves the pairwise angles between basis vectors.

8.3 Applying the transfer-matrix method

In Section 8.2, we saw several formulas for the Hurwitz change-of-basis matrix ${}_B M_{\delta^n \mathcal{B}}$, where $\delta = \sigma_{d-1} \cdots \sigma_2 \sigma_1 \in \text{Braid}_d$ denotes the positive all-strand rotation. In particular, we discovered that ${}_B M_{\delta^n \mathcal{B}}$ can be written as a product of n companion matrices. Calculating such a product becomes computationally expensive as n gets large, so we want explicit closed formulas for the entries of this product. In Chapter 7, we were able to find closed formulas for the entries of the change-of-basis matrices given by powers of half-twists, and related these formulas to Fibonacci and Chebyshev polynomials. This section discusses an approach to finding formulas for the entries of ${}_B M_{\delta^n \mathcal{B}}$ using the transfer-matrix method, a technique at the intersection of graph theory, combinatorics, and linear algebra.

As in Section 8.2, let $\mathcal{B} \in \mathfrak{B}$ denote an arbitrary Hurwitz basis, let $\mathbf{C}$ denote the matrix representation of the Wall form $\chi_{c^{-1}}$ with respect to this basis $\mathcal{B}$, and let $\mathfrak{C}$ denote the infinite matrix constructed from $\mathbf{C}$ as in Figure 8.5. Let $A(i)$ denote the companion

matrix whose coefficients come from the i -th row of $\mathfrak{C}$, i.e. the companion matrix from Proposition 8.2.2. From Corollary 8.2.2.2, we may write the Hurwitz change-of-basis matrix ${}_B M_{\delta^n B}$ as the product

$${}_B M_{\delta^n B} = (-1)^n \left(\prod_{i=1}^d A(i) \right)^q \prod_{k=1}^r A(k)$$

of companion matrices. Naturally, we are particularly interested in computing such products.

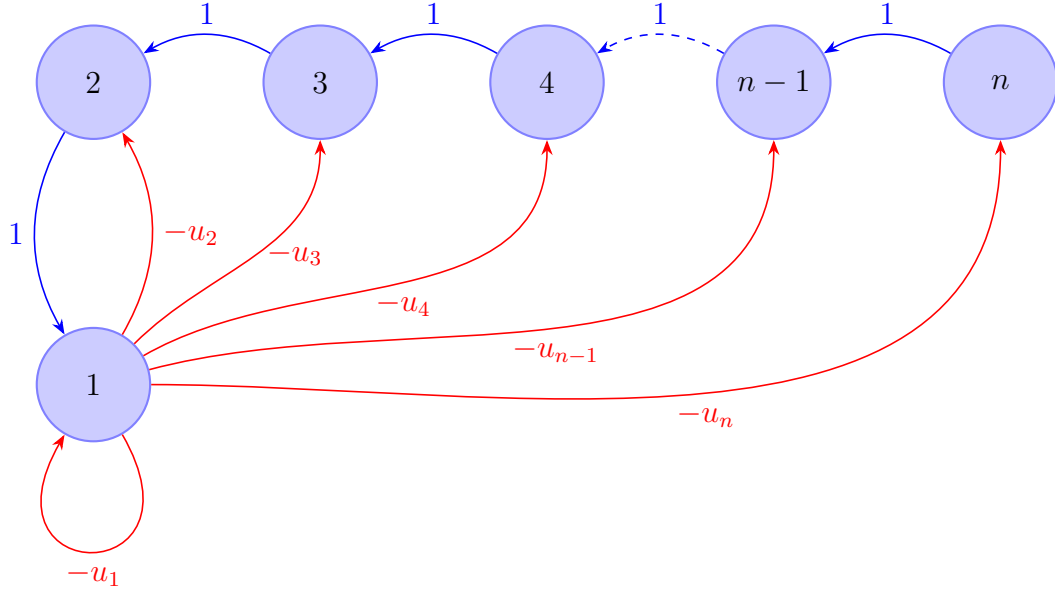
[CL96] introduces a combinatorial technique for computing powers A^n of a companion matrix A by viewing the (i, j) -th entry of A^n in terms of (i, j) -walks in the so-called *companion digraph* associated with A , then applying the transfer-matrix method to the companion digraph. [LD11] uses a different combinatorial approach to compute products of companion matrices, still employing the companion digraph as an auxiliary tool.

Definition 8.3.1 (Companion digraph). Given a companion matrix $A = A(u_1, \dots, u_n)$, the *companion digraph* is the weighted directed graph Γ_A on n vertices for which A is its adjacency matrix. Concretely, let $[n]$ be the vertex set of Γ_A . The edge set contains a directed edge $e = (i, j)$ with weight A_{ij} precisely when A_{ij} is nonzero.

Figure 8.8 depicts the companion digraph Γ_A for the companion matrix $A = A(u_1, \dots, u_n)$. In this figure, each directed edge of the form $(k, k-1)$ has the label 1, and is colored blue; each directed edge of the form $(1, k)$ has the label $-u_k$, and is colored red.

Unaware of the results from [CL96] and [LD11], my initial attempts at computing the entries of ${}_B M_{\delta^n B}$ sought to apply the transfer-matrix method to the companion digraphs associated with the product $A(1)A(2) \cdots A(n)$ of companion matrices.

Say we have an arbitrary square matrix A , which we can treat as a weighted adjacency matrix for some weighted directed graph Γ . To compute the (i, j) -th entry of A^n , consider the (i, j) -walks of length n in Γ . For some such walk γ , we may obtain the weight $w(\gamma)$

Figure 8.8: Companion digraph Γ_A for $A(u_1, \dots, u_n)$

as the product of edge weights along γ . In other words, if γ traverses the edge sequence $(e_1, e_2, \dots, e_n)$ and $w(e_k)$ denotes the weight of edge e_k , then the weight of γ is

$$w(\gamma) = \prod_{k=1}^n w(e_k).$$

The (i, j) -th entry of A^n is then the sum of walk weights, indexed over all (i, j) -walks of length n in Γ .

$$(A^n)_{ij} = \sum_{(i,j)\text{-walks } \gamma} w(\gamma)$$

The transfer-matrix method involves analyzing the ordinary generating function

$$F_{ij}(\lambda) = \sum_{n=0}^{\infty} (A^n)_{ij} \lambda^n$$

to find a closed form expression of $(A^n)_{ij}$. The typical process involves writing $F_{ij}(\lambda)$ as a rational generating function, then computing the power series expansion of the partial fraction decomposition. For a thorough discussion of the transfer-matrix method, see

[Sta11, Section 4.7]. Another helpful resource is [BC08, Section 3.1].

My first idea was to adapt the transfer-matrix method to work on products of companion matrices, not just powers of matrices. Suppose we have a sequence $A(1), A(2), A(3), \dots$ of companion matrices of the same size. They all share the same companion digraph Γ , except for the edge weights, which depend on the coefficients of the particular companion matrix $A(k)$. We can treat each $A(k)$ as a weighted adjacency matrix for Γ . To compute the (i, j) -th entry of the product $A(1)A(2) \cdots A(n)$, consider the (i, j) -walks of length n in Γ . For some such walk γ , we may obtain the weight $w(\gamma)$ of the walk as follows: the weight $w(\gamma)$ is the product of edge weights in the walk γ , where the weight on the k -th edge comes from the companion matrix $A(k)$. More explicitly, say the k -th edge e_k in the walk starts at vertex a and ends at vertex b . The weight $w(e_k)$ of the edge e_k is the (a, b) -entry of the companion matrix $A(k)$, and the weight $w(\gamma)$ is the product

$$w(\gamma) = \prod_{k=1}^n w(e_k).$$

The (i, j) -th entry of the product $A(1)A(2) \cdots A(n)$ is then the sum of such weights, indexing over all (i, j) -walks in Γ .

$$(A(1)A(2) \cdots A(n))_{ij} = \sum_{(i, j)\text{-walks } \gamma} w(\gamma)$$

We may compute the (i, j) -th entry of the product $A(1)A(2) \cdots A(n)$ by analyzing the generating function

$$F_{ij}(\lambda) = \sum_{n=0}^{\infty} \left(\prod_{k=0}^n A(k) \right)_{ij} \lambda^n$$

to find a closed form expression for $(A(1)A(2) \cdots A(n))_{ij}$.

I found that applying the transfer-matrix method to the product $A(1)A(2) \cdots A(n)$ of companion matrices was difficult; the transfer-matrix method works well for powers

of matrices, but is more difficult to apply to products of matrices, even if those matrices share a high relative number of explicitly zero entries. Later, I learned of a formula for the product of companion matrices [LD11, Theorem 4.4], as well as a formula for the (i, j) -th entry of such a product [LD11, Equation 25]. These formulas are well suited for computing products of companion matrices.

My own breakthrough came when I decided to apply the transfer-matrix method to powers of the particular change-of-basis matrix ${}_B M_{\delta^d B}$. Indeed, we know from Corollary 8.2.2.2 that for powers $n = dq + r$, the change-of-basis matrix ${}_B M_{\delta^n B}$ can be written as the product

$${}_B M_{\delta^n B} = ({}_B M_{\delta^d B})^q {}_B M_{\delta^r B} = (-1)^n \left(\prod_{i=1}^d A(i) \right)^q \prod_{k=1}^r A(k).$$

To compute ${}_B M_{\delta^n B}$, we need only compute powers of

$${}_B M_{\delta^d B} = \prod_{i=1}^d A(i)$$

and multiply on the right by some product

$$\prod_{i=1}^r A(i)$$

whose length r is at most $d - 1$.

A key observation that I found immensely helpful for computing entries of $({}_B M_{\delta^d B})^n$ is the formula I discovered for writing $({}_B M_{\delta^d B})^n$ in terms of the matrix representation of the Coxeter element $c = s_1 s_2 \cdots s_d$, stated in Corollary 8.2.4.1. Namely, letting $\mathcal{B}_0$ denote the standard basis of simple roots, we may write

$${}_{\mathcal{B}_0} M_{\delta^d \mathcal{B}_0} = (-1)^{d+1} C^{-1} C^\top$$

where C is the upper unitriangular matrix defined in Section 6.1. For the rest of this

section, we let $\mathcal{B}$ denote the standard basis of simple roots, omitting the subscript for simplicity of notation. Below, we illustrate how to compute entries of $({}_B M_{\delta^d \mathcal{B}})^n$ using the transfer-matrix method.

We apply the transfer-matrix method not to companion digraphs, but to another directed graph defined in terms of the upper unitriangular matrix C representing the Wall form $\chi_{c^{-1}}$ in the standard basis. Let $\mathcal{A}$ denote the block matrix

$$\mathcal{A} = \begin{pmatrix} \mathbf{0} & C^{-1} \\ C^\top & \mathbf{0} \end{pmatrix}.$$

Notice that

$$\mathcal{A}^2 = \begin{pmatrix} C^{-1}C^\top & \mathbf{0} \\ \mathbf{0} & C^\top C^{-1} \end{pmatrix}.$$

Clearly, we obtain

$$\mathcal{A}^{2n} = \begin{pmatrix} (C^{-1}C^\top)^n & \mathbf{0} \\ \mathbf{0} & (C^\top C^{-1})^n \end{pmatrix}.$$

We observe that for indices $1 \leq i, j \leq d$, the (i, j) -th entry of $(C^{-1}C^\top)^n$ is

$$((C^{-1}C^\top)^n)_{ij} = (\mathcal{A}^{2n})_{ij}.$$

More importantly, applying Corollary 8.2.4.1, we conclude that the (i, j) -th entry of $({}_B M_{\delta^d \mathcal{B}})^n$ is

$$({}_B M_{\delta^d \mathcal{B}})^n_{ij} = (-1)^{n(d+1)} (\mathcal{A}^{2n})_{ij}.$$

Viewing $\mathcal{A}$ as the weighted adjacency matrix of a directed graph, we may apply the transfer-matrix method to compute the entries of $\mathcal{A}^n$. Since $({}_B M_{\delta^d \mathcal{B}})^n$ may be written in terms of $\mathcal{A}^{2n}$, we may also apply the transfer matrix method to $\mathcal{A}^{2n}$ instead.

Remark 8.3.2 (Symplectic matrices). The weighted adjacency matrix $\mathcal{A}$ defined above is an example of a *symplectic matrix*. A symplectic vector space is a vector space over

a field $\mathbb{F}$ equipped with a *symplectic (bilinear) form*, i.e. a bilinear form that is skew-symmetric and non-degenerate. Any transformation that preserves this form is called a *symplectic transformation*, and a matrix is called *symplectic* if it represents a symplectic transformation. For further reading on symplectic matrices, see [Vai87, Chapter 2] and [Rad24, Chapter 1].

The directed graph Γ whose weighted adjacency matrix is $\mathcal{A}$ has the vertex set $V(\Gamma) = [2d] := \{1, 2, \dots, 2d\}$. For indices $1 \leq i \leq j \leq d$, the directed edge $(i, d + j)$ of Γ is the (i, j) -th entry of C^{-1} . Similarly, for $1 \leq j \leq i \leq d$, the directed edge $(i + d, j)$ is the (i, j) -th entry of $C^\top$. As an illustration, consider the following rank-3 example.

Example 8.3.3 (Digraph Γ for rank-3 Coxeter system). Let (W, S) denote a rank-3 Coxeter system. As in Example 6.1.2, we write the upper unitriangular matrix C as

$$C = \begin{pmatrix} 1 & a_1 & b_1 \\ 0 & 1 & a_2 \\ 0 & 0 & 1 \end{pmatrix}.$$

Write the inverse C^{-1} as

$$C^{-1} = \begin{pmatrix} 1 & A_1 & B_1 \\ 0 & 1 & A_2 \\ 0 & 0 & 1 \end{pmatrix}.$$

The values of the entries of C^{-1} are

$$A_1 = -a_1, \quad A_2 = -a_2, \quad B_1 = a_1 a_2 - b_1,$$

but we suppress these values for now. The weighted adjacency matrix $\mathcal{A}$ is

$$\mathcal{A} = \begin{pmatrix} 0 & 0 & 0 & 1 & A_1 & B_1 \\ 0 & 0 & 0 & 0 & 1 & A_2 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ a_1 & 1 & 0 & 0 & 0 & 0 \\ b_1 & a_2 & 1 & 0 & 0 & 0 \end{pmatrix}. \tag{8.7}$$

The associated digraph Γ is illustrated in Figure 8.9. For simplicity, we may depict Γ as an undirected graph, remembering that the weights on a given undirect edge depends on the direction of traversal along the edge, as in Figure 8.10.

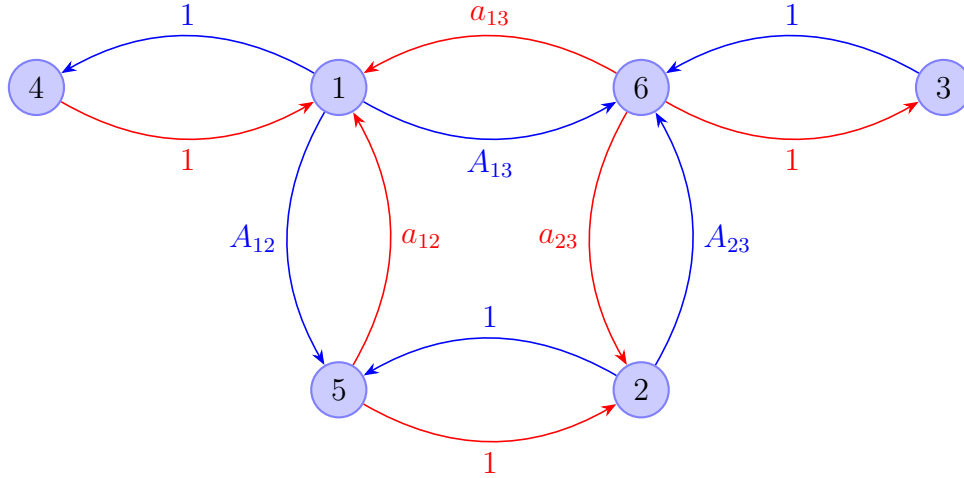


Figure 8.9: Digraph Γ associated with $\mathcal{A}$

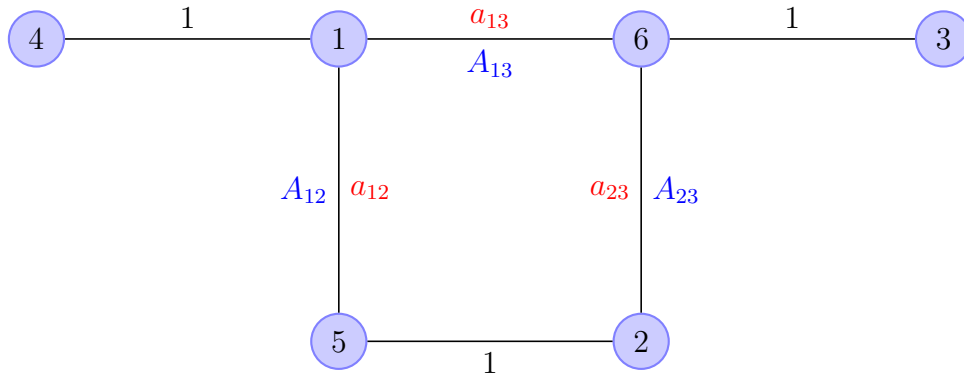


Figure 8.10: Simplified depiction of the graph Γ

We may define the ordinary generating function

$$F_{ij}(\lambda) = \sum_{n=0}^{\infty} (\mathcal{A}^n)_{ij} \lambda^n$$

in terms of the (i, j) -th entry $(\mathcal{A}^n)_{ij}$ of $\mathcal{A}$. As before, $(\mathcal{A}^n)_{ij}$ can be thought of as the sum of weights of (i, j) -walks in Γ of length n . For any matrix M , let $(M : j, i)$ denote the

matrix obtained by removing the j -th row and i -th column of M . The transfer-matrix method uses the following theorem [Sta11, Theorem 4.7.2] to write the OGF $F_{ij}(\lambda)$ as a rational function of λ .

Theorem 8.3.4 (Rational generating function for transfer-matrix method). *Let $F_{ij}(\lambda)$ denote the ordinary generating function*

$$F_{ij}(\lambda) = \sum_{n=0}^{\infty} (A^n)_{ij} \lambda^n$$

for some square matrix A . Then

$$F_{ij}(\lambda) = \frac{(-1)^{i+j} \det(I - \lambda A : j, i)}{\det(I - \lambda A)}.$$

In particular, $F_{ij}(\lambda)$ is a rational function of λ whose degree is strictly less than the multiplicity of 0 as an eigenvalue of A .

As an illustration, let us apply the transfer-matrix method to some specific examples, starting with the symmetric group Sym_4 . Of course, applying the transfer-matrix method to Sym_4 is overkill because Sym_4 is finite, whereby powers of $C^{-1}C^\top$ must be periodic. In particular, since the Coxeter element

$$c = s_1 s_2 s_3 = (1, 2)(2, 3)(3, 4) = (1, 2, 3, 4)$$

has order 4, so does the matrix $C^{-1}C^\top$. Still, applying the transfer-matrix method to this simple example serves as a useful illustration before we move on to trickier examples.

Example 8.3.5 (Transfer-matrix method for Sym_4). Recall from Example 6.1.3 the symmetric group Sym_4 is associated with the upper unitriangular matrix

$$C = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix},$$

from which we obtain the weighted adjacency matrix

$$\mathcal{A} = \begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

To find the entries of $(C^{-1}C^\top)^n$, we may apply the transfer-matrix method to $\mathcal{A}$ or $\mathcal{A}^2$.

Let us compute the $(1, 1)$ -th entry of $(C^{-1}C^\top)^n$ by applying the transfer-matrix method to $\mathcal{A}^2$. Consider the ordinary generating function

$$F_{11}(\lambda) = \sum_{n=0}^{\infty} (\mathcal{A}^{2n})_{11} \lambda^n.$$

By Theorem 8.3.4, we may write $F_{11}(\lambda)$ as the rational generating function

$$F_{11}(\lambda) = \frac{\det(I - \lambda \mathcal{A}^2 : 1, 1)}{\det(I - \lambda \mathcal{A}^2)}.$$

Using Sagemath, I first calculated the determinants

$$\begin{aligned} \det(I - \lambda \mathcal{A}^2 : 1, 1) &= -\lambda^5 + 2\lambda^4 - 3\lambda^3 + 3\lambda^2 - 2\lambda + 1 \\ &= -(\lambda^2 - \lambda + 1)(\lambda^2 + 1)(\lambda - 1) \\ \det(I - \lambda \mathcal{A}^2) &= \lambda^6 - 2\lambda^5 + 3\lambda^4 - 4\lambda^3 + 3\lambda^2 - 2\lambda + 1 \\ &= (\lambda^2 + 1)^2(\lambda - 1)^2 \end{aligned}$$

Now it is clear that

$$F_{11}(\lambda) = \frac{-\lambda^2 + \lambda - 1}{(\lambda - i)(\lambda + i)(\lambda - 1)},$$

where i is the imaginary unit. The partial fraction decomposition of $F_{11}(\lambda)$ is

$$\begin{aligned}
F_{11}(\lambda) &= \frac{1}{2(1-\lambda)} + \frac{i}{(2-2i)(\lambda+i)} - \frac{i}{(2+2i)(\lambda-i)} \\
&= \frac{1}{2(1-\lambda)} + \frac{1}{(2-2i)(1-i\lambda)} + \frac{1}{(2+2i)(1+i\lambda)},
\end{aligned}$$

which has the power series expansion

$$\begin{aligned}
F_{11}(\lambda) &= \frac{1}{2} \sum_{n=0}^{\infty} \lambda^n + \frac{1}{(2-2i)} \sum_{n=0}^{\infty} (i\lambda)^n + \frac{1}{(2+2i)} \sum_{n=0}^{\infty} (-i\lambda)^n \\
&= \sum_{n=0}^{\infty} \left(\frac{1}{2} + \frac{(1+(-1)^n)i^n + (1+(-1)^{n+1})i^{n+1}}{4} \right) \lambda^n.
\end{aligned}$$

Thus, we conclude

$$\begin{aligned}
((C^{-1}C^\top)^n)_{11} &= (\mathcal{A}^{2n})_{11} \\
&= \frac{1}{2} + \frac{(1+(-1)^n)i^n + (1+(-1)^{n+1})i^{n+1}}{4} \\
&= \begin{cases} 1, & n \equiv 0, 3 \pmod{4} \\ 0, & n \equiv 1, 2 \pmod{4} \end{cases}
\end{aligned}$$

This result is exactly what we find when we compute $(C^{-1}C^\top)^n$ directly.

$$\begin{aligned}
(C^{-1}C^\top)^{4k} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & (C^{-1}C^\top)^{4k+1} &= \begin{pmatrix} 0 & 0 & 1 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{pmatrix}, \\
(C^{-1}C^\top)^{4k+2} &= \begin{pmatrix} 0 & -1 & 1 \\ 0 & -1 & 0 \\ 1 & -1 & 0 \end{pmatrix}, & (C^{-1}C^\top)^{4k+3} &= \begin{pmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 1 & 0 & 0 \end{pmatrix}.
\end{aligned}$$

Applying Corollaries 8.2.2.2 and 8.2.4.1, we conclude that the $(1,1)$ -th entry of the change-of-basis matrix ${}_B M_{\delta^{4n}\mathcal{B}}$ is

$$({}_{\mathcal{B}}M_{\delta^{4n}\mathcal{B}})_{11} = (-(C^{-1}C^{\top})^n)_{11} = \begin{cases} 1, & n \equiv 0 \pmod{4} \\ -1, & n \equiv 3 \pmod{4} \\ 0, & \text{otherwise} \end{cases}$$

We can do the same for all pairs of indices $1 \leq i, j \leq 3$, though we omit those computations here.

We reiterate that using the transfer-matrix method for such a small group like Sym_4 is excessive, but hopefully the above example has provided some familiarity with the transfer-matrix method. We now apply the transfer-matrix method to a familiar Coxeter group of infinite order.

Example 8.3.6 (Transfer-matrix method for $\text{PGL}(2, \mathbb{Z})$). Examples 3.2.11, 4.1.9, 4.1.15, and 6.1.4 discuss the projective general linear group $\text{PGL}(2, \mathbb{Z})$ as a rank-3 Coxeter group. In particular, the upper unitriangular matrix C representing the Wall form $\chi_{c^{-1}}$ with respect to the standard basis of simple roots is

$$C = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix},$$

and the matrix representation of the Coxeter element c with respect to the standard basis is

$$[c] = -C^{-1}C^{\top} = \begin{pmatrix} 0 & 3 & -2 \\ 1 & 3 & -2 \\ 0 & 2 & -1 \end{pmatrix}.$$

Although we can compute the entries of $(C^{-1}C^{\top})^n$ by repeated matrix multiplication, we would like a closed form expression for these entries. From the matrix C , we obtain the weighted adjacency matrix

$$\mathcal{A} = \begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -2 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

Let us compute the $(1, 1)$ -th entry of $(C^{-1}C^\top)^n$ by applying the transfer-matrix method to $\mathcal{A}^2$. Consider the ordinary generating function

$$F_{11}(\lambda) = \sum_{n=0}^{\infty} (\mathcal{A}^{2n})_{11} \lambda^n,$$

which by Theorem 8.3.4 may be written as the rational generating function

$$F_{11}(\lambda) = \frac{\det(I - \lambda \mathcal{A}^2 : 1, 1)}{\det(I - \lambda \mathcal{A}^2)}.$$

Using SageMath, I computed the following determinants.

$$\begin{aligned} \det(I - \lambda \mathcal{A}^2 : 1, 1) &= -\lambda^5 - 4\lambda^4 - 3\lambda^3 + 3\lambda^2 + 4\lambda + 1 \\ &= -(\lambda^2 + 3\lambda + 1)(\lambda + 1)^2(\lambda - 1) \\ \det(I - \lambda \mathcal{A}^2) &= \lambda^6 + 4\lambda^5 - 10\lambda^3 + 4\lambda + 1 \\ &= (\lambda^2 + 3\lambda + 1)^2(\lambda - 1)^2 \end{aligned}$$

We obtain

$$F_{11}(\lambda) = -\frac{(\lambda + 1)^2}{(\lambda^2 + 3\lambda + 1)(\lambda - 1)} = -\frac{(\lambda + 1)^2}{(\lambda - (\varphi - 2))(\lambda - (-\varphi^{-1} - 2))(\lambda - 1)}$$

where φ denotes the golden ratio. I used SageMath to compute the following partial fraction decomposition.

$$\begin{aligned}
F_{11}(\lambda) &= \frac{\sqrt{5}-1}{10(\lambda-(\varphi-2))} - \frac{\sqrt{5}+1}{10(\lambda+(\varphi^{-1}+2))} - \frac{4}{5(\lambda-1)} \\
&= \frac{1-\sqrt{5}}{10((\varphi-2)-\lambda)} - \frac{\sqrt{5}+1}{10((\varphi^{-1}+2)+\lambda)} + \frac{4}{5(1-\lambda)}
\end{aligned}$$

We may now compute the power series expansion.

$$F_{11}(\lambda) = \frac{1-\sqrt{5}}{10(\varphi-2)} \sum_{n=0}^{\infty} \left(\frac{\lambda}{\varphi-2} \right)^n - \frac{\sqrt{5}+1}{10(\varphi^{-1}+2)} \sum_{n=0}^{\infty} \left(\frac{-\lambda}{\varphi^{-1}+2} \right)^n + \frac{4}{5} \sum_{n=0}^{\infty} \lambda^n$$

We finally obtain the following closed form expression.

$$\begin{aligned}
((C^{-1}C^{\top})^n)_{11} &= (\mathcal{A}^{2n})_{11} = \frac{1-\sqrt{5}}{10(\varphi-2)^{n+1}} + \frac{(-1)^{n+1}(\sqrt{5}+1)}{10(\varphi^{-1}+2)^{n+1}} + \frac{4}{5} \\
&= \frac{(1-\sqrt{5})(\varphi^{-1}+2)^{n+1} + (-1)^{n+1}(\sqrt{5}+1)(\varphi-2)^{n+1}}{10 \cdot (-2)^{n+1}} + \frac{4}{5} \\
&= \frac{(-1)^{n+1}(1-\sqrt{5})(\varphi^{-1}+2)^{n+1} + (\sqrt{5}+1)(\varphi-2)^{n+1}}{10 \cdot 2^{n+1}} + \frac{4}{5}
\end{aligned}$$

Evaluating the above formula for some values of n , one may check that it agrees with the values of $((C^{-1}C^{\top})^n)_{11}$ obtained via matrix multiplication. For the other pairs of indices $1 \leq i, j \leq 3$, one may similarly use the transfer-matrix method to compute closed form expressions for $((C^{-1}C^{\top})^n)_{ij}$. For brevity, we omit those calculations.

Having seen the golden ratio φ appear in the generating function $F_{11}(\lambda)$ for $\text{PGL}(2, \mathbb{Z})$, one might expect a connection with the Fibonacci numbers. Upon inspecting the integer sequences that arise from the Coxeter element $[c] = -C^{-1}C^{\top}$ for the projective general linear group $\text{PGL}(2, \mathbb{Z})$ as in Example 8.3.6, I discovered a pleasant connection between $[c]$ and the Fibonacci numbers f_n . (As in Section 7.2, we index the Fibonacci numbers using initial values $f_0 = 0$ and $f_1 = 1$.) By examination of the first several terms of $([c]^n)_{ij}$, I conjectured the following:

- $([c]^n)_{11}$ and $([c]^n)_{12}$ are given by the sequence [OEI26, A226205] up to indexing offset,
- $([c]^n)_{13}$, $([c]^n)_{31}$, and $([c]^n)_{32}$ are given by the sequence [OEI26, A079472] up to sign and indexing offset,
- $([c]^n)_{21}$ and $([c]^n)_{22}$ are given by the sequence [OEI26, A001906] up to indexing offset,
- $([c]^n)_{23}$ is given by the sequence [OEI26, A025169] up to sign and indexing offset, and
- $([c]^n)_{33}$ is given by the sequence [OEI26, A061647] up to sign and indexing offset.

Let $G_{ij}(\lambda)$ denote the generating function

$$G_{ij}(\lambda) = \sum_{n=0}^{\infty} ([c]^n)_{ij} \lambda^n.$$

I confirmed that the above conjectures hold by applying Theorem 8.3.4 to each $G_{ij}(\lambda)$, then comparing the rational generating function I obtained to the ones found on OEIS. Below, Table 8.1 provides the first several terms of each of the sequences $([c]^n)_{ij}$, while Table 8.2 presents some additional data. The second column of Table 8.2 states each rational generating function $G_{ij}(\lambda)$, while the third column gives a formula for each $([c]^n)_{ij}$ in terms of the Fibonacci numbers. These Fibonacci formulas can also be found in the OEIS pages for the aforementioned sequences. The generating functions $G_{ij}(\lambda)$ are organized as a matrix in Equations 8.8 through 8.10 below.

$$\begin{pmatrix} G_{11}(\lambda) & G_{12}(\lambda) & G_{13}(\lambda) \\ G_{21}(\lambda) & G_{22}(\lambda) & G_{23}(\lambda) \\ G_{31}(\lambda) & G_{32}(\lambda) & G_{33}(\lambda) \end{pmatrix} \quad (8.8)$$

$$= \frac{1}{(\lambda+1)(\lambda^2-3\lambda+1)} \begin{pmatrix} \lambda^2-2\lambda+1 & -\lambda^2+3\lambda & -2\lambda \\ \lambda^2+\lambda & \lambda+1 & -2\lambda^2-2\lambda \\ 2\lambda^2 & 2\lambda & -3\lambda^2-3\lambda+1 \end{pmatrix} \quad (8.9)$$

$$= \frac{1}{(\lambda+1)(\lambda^2-3\lambda+1)} \left(I + \lambda \begin{pmatrix} -2 & 3 & -2 \\ 1 & 1 & -2 \\ 0 & 2 & -3 \end{pmatrix} + \lambda^2 \begin{pmatrix} 1 & -1 & 0 \\ 1 & 0 & -2 \\ 2 & 0 & -3 \end{pmatrix} \right) \quad (8.10)$$

n	(1, 1)	(1, 2)	(1, 3)	(2, 1)	(2, 2)	(2, 3)	(3, 1)	(3, 2)	(3, 3)
0	1	0	0	0	1	0	0	0	1
1	0	3	-2	1	3	-2	0	2	-1
2	3	5	-4	3	8	-6	2	4	-3
3	5	16	-12	8	21	-16	4	12	-9
4	16	39	-30	21	55	-42	12	30	-23
5	39	105	-80	55	144	-110	30	80	-61
6	105	272	-208	144	377	-288	80	208	-159
7	272	715	-546	377	987	-754	208	546	-417
8	715	1869	-1428	987	2584	-1974	546	1428	-1091
9	1869	4896	-3740	2584	6765	-5168	1428	3740	-2857
10	4896	12815	-9790	6765	17711	-13530	3740	9790	-7479

Table 8.1: The first several terms of each entry $([c]^n)_{ij}$ for $\mathrm{PGL}(2, \mathbb{Z})$

We may calculate explicit closed formulas for each $([c]^n)_{ij}$ by finding the power series expansion of $G_{ij}(\lambda)$, but we omit this step for now; as seen in Example 8.3.6, the closed formulas are very long expressions. Some of the relevant OEIS pages provide these closed formulas [OEI26, A226205, A079472].

At this point, we have seen the transfer-matrix method applied to two different rank-3 Coxeter systems. We may apply the transfer-matrix method to an arbitrary rank-3 Coxeter system, i.e. with the upper unitriangular matrix

$$C = \begin{pmatrix} 1 & a_1 & b_1 \\ 0 & 1 & a_2 \\ 0 & 0 & 1 \end{pmatrix}$$

where the entries a_1 , a_2 , and b_2 are left as indeterminate variables. By plugging in values into these three variables, we may obtain results analogous to Examples 8.3.5 and 8.3.6 for any rank-3 Coxeter system we wish. Applying Theorem 8.3.4, we may easily obtain each rational generating function $F_{ij}(\lambda)$. If we are able to compute the power series expansion of $F_{ij}(\lambda)$, then we obtain a closed form expression for $((C^{-1}C^\top)^n)_{ij}$ in terms of the variables a_1 , a_2 , and b_2 . In other words, we obtain a polynomial sequence

$$((C^{-1}C^\top)^n)_{ij} = p_n(a_1, a_2, b_2).$$

However, obtaining this polynomial sequence is easier said than done; while we may obtain the rational generating function $F_{ij}(\lambda)$ without much difficulty, computing the power series expansion is quite a task. The task becomes even more difficult for Coxeter systems of higher rank. The following rank-3 example demonstrates how to compute the rational generating functions $F_{ij}(\lambda)$ and its partial fraction decomposition, but omits the calculation of the full power series expansion.

Example 8.3.7 (Transfer-matrix method for rank-3 Coxeter system). Recall the setup from Example 8.3.3. To compute entries of powers of $C^{-1}C^\top$, we may apply the transfer-matrix method to the weighted adjacency matrix $\mathcal{A}$ from Equation 8.7, or to its square $\mathcal{A}^2$. Consider the ordinary generating function

$$F_{11}(\lambda) = \sum_{n=0}^{\infty} (\mathcal{A}^{2n})_{11} \lambda^n.$$

By Theorem 8.3.4, we obtain the rational generating function

$$F_{11}(\lambda) = \frac{\det(I - \lambda \mathcal{A}^2 : 1, 1)}{\det(I - \lambda \mathcal{A}^2)}.$$

Using SageMath, I computed the various determinants.

$$\begin{aligned}
\det(I - \lambda \mathcal{A}^2 : 1, 1) &= \lambda^4 - (a_2 A_2 + 2)\lambda^2 + 1 \\
&= \lambda^4 + (a_2^2 - 2)\lambda^2 + 1 \\
\det(I - \lambda \mathcal{A}^2) &= -\lambda^6 + (a_1 a_2 A_1 A_2 - A_1 A_2 b_1 - a_1 a_2 B_1 + a_1 A_1 + a_2 A_2 + b_1 B_1 + 3)\lambda^4 \\
&\quad - (a_1 A_1 + a_2 A_2 + b_2 B_2 + 3)\lambda^2 + 1 \\
&= -\lambda^6 + (a_1 a_2 b_1 - a_1^2 - a_2^2 - b_1^2 + 3)\lambda^4 \\
&\quad - (a_1 a_2 b_1 - a_1^2 - a_2^2 - b_1^2 + 3)\lambda^2 + 1
\end{aligned}$$

Using SageMath, I computed the following partial fraction decomposition.

$$\begin{aligned}
F_{11}(\lambda) &= -\frac{a_2^2}{2(a_1 a_2 b_1 - a_1^2 - a_2^2 - b_1^2)(\lambda + 1)} \\
&\quad + \frac{a_2^2}{2(a_1 a_2 b_1 - a_1^2 - a_2^2 - b_1^2)(\lambda - 1)} \\
&\quad + \frac{-(a_1 a_2 b_1 - a_1^2 - b_1^2)\lambda^2 + (a_1 a_2 b_1 - a_1^2 - b_1^2)}{(a_1 a_2 b_1 - a_1^2 - a_2^2 - b_1^2)(\lambda^4 - (a_1 a_2 b_1 - a_1^2 - a_2^2 - b_1^2 + 2)\lambda^2 + 1)}
\end{aligned}$$

Finding the full power series expansion is tricky, so we stop here for now.

8.4 Wave sequences

Recall from Section 7.2 that the Hurwitz change-of-basis matrix for powers of a half-twist σ_i is a block diagonal matrix whose non-trivial 2×2 block generates a sequence of univariate polynomials that are closely related to Fibonacci polynomials. One may see the Hurwitz change-of-basis for powers of the all-strand rotation braid δ as generating a sequence of multivariate polynomials. This section focuses on these multivariate polynomials, and their relationship to a generalization of the Fibonacci polynomials, which we call *polynomial wave sequences*.

Recall that the Fibonacci numbers f_n may be computed using powers of the 2×2 upper antitriangular matrix whose upper antitriangular entries are all 1, as mentioned in

Table 7.2.

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n \begin{pmatrix} f_1 \\ f_0 \end{pmatrix} = \begin{pmatrix} f_{n+1} \\ f_n \end{pmatrix}, \quad \begin{pmatrix} f_1 \\ f_0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

We have already seen the Fibonacci polynomials $F_n(x)$ as one way of generalizing the Fibonacci numbers. In a similar fashion, we may generate $F_n(x)$ using powers of the 2×2 upper triangular matrix shown in Table 7.2, presented again below.

$$\begin{pmatrix} x & 1 \\ 1 & 0 \end{pmatrix}^n \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} F_{n+1}(x) \\ F_n(x) \end{pmatrix}$$

Another way of generalizing the Fibonacci numbers is to define a sequence $w_n^{(d)}$ using powers of the $d \times d$ upper antitriangular matrix whose upper antitriangular entries are all 1.

$$\begin{pmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 1 & 1 & \cdots & 1 & 0 \\ 1 & 1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 & 0 \\ 1 & 1 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}^n \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} w_{(n+1)(d-1)}^{(d)} \\ w_{(n+1)(d-1)-1}^{(d)} \\ w_{(n+1)(d-1)-2}^{(d)} \\ \vdots \\ w_{n(d-1)+1}^{(d)} \\ w_{n(d-1)}^{(d)} \end{pmatrix}.$$

Such sequences are called *d-wave sequences* (or just *wave sequences*) by [Lam99] and [OEI26]. (They are also called *cascade numbers* by [Knu98, Section 5.4.3].) The name comes from a wave-like diagram that can be used to generate the sequence; see Figure 8.11 for the diagram corresponding to the 4-wave sequence. See Table 8.3 to see the first several terms of wave sequences for several values of d . Observe that the Fibonacci sequence is the 2-wave sequence.

Remark 8.4.1 (Indexing wave sequences). We use different indexing n for the wave sequence $w_n^{(d)}$ than [OEI26] to emphasize that wave sequences are a generalization of the Fibonacci sequence. The standard indexing for the Fibonacci numbers $f_n = w_n^{(2)}$ begins

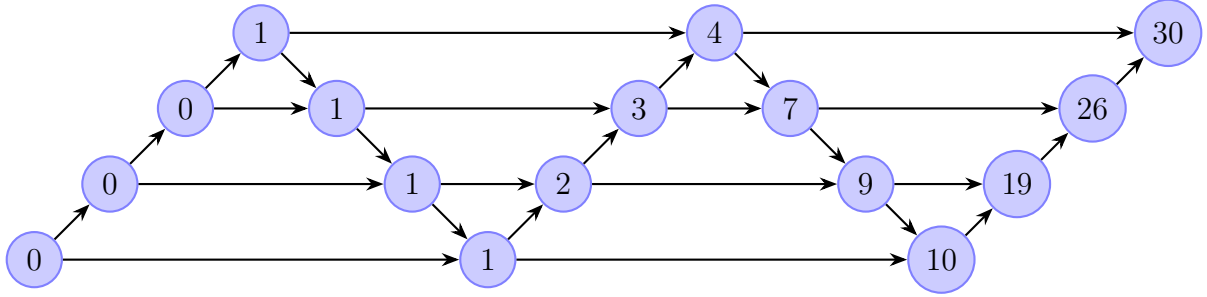


Figure 8.11: Wave diagram for the 4-wave sequence

with the initial values $f_0 = 0$ and $f_1 = 1$. By analogy, we choose the initial values

$$w_n^{(d)} = \begin{cases} 0, & 0 \leq n \leq d-2 \\ 1, & n = d-1 \end{cases}$$

for the wave sequence $w_n^{(d)}$.

As discussed in [Lam99], we may obtain the defining recurrence relation for the wave sequence $w_n^{(d)}$ by analyzing the characteristic equation of its generating matrix. Table 8.4 presents the recurrence relations for wave sequences for several values of d . As an illustration for how to compute the recurrence relation, Example 8.4.2 walks through the computation for the 4-wave sequence.

Example 8.4.2 (Recurrence relation for $w_n^{(4)}$). For simplicity of notation, we write $w_n = w_n^{(4)}$ to denote the 4-wave sequence. The generating matrix

$$M = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

for w_n has the characteristic polynomial

$$\chi(\lambda) = \lambda^4 - 2\lambda^3 - 3\lambda^2 + \lambda + 1.$$

By the Cayley-Hamilton Theorem, we obtain

$$M^4 = 2M^3 + 3M^2 - M - I.$$

Thus, for each k , we have

$$\begin{aligned} \begin{pmatrix} w_{3k+3} \\ w_{3k+2} \\ w_{3k+1} \\ w_{3k} \end{pmatrix} &= M^k \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \\ &= (2M^3 + 3M^2 - M - I)M^{k-4} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \\ &= (2M^3 + 3M^2 - M - I) \begin{pmatrix} w_{3(k-4)+3} \\ w_{3(k-4)+2} \\ w_{3(k-4)+1} \\ w_{3(k-4)} \end{pmatrix} \\ &= 2 \begin{pmatrix} w_{3(k-1)+3} \\ w_{3(k-1)+2} \\ w_{3(k-1)+1} \\ w_{3(k-1)} \end{pmatrix} + 3 \begin{pmatrix} w_{3(k-2)+3} \\ w_{3(k-2)+2} \\ w_{3(k-2)+1} \\ w_{3(k-2)} \end{pmatrix} - \begin{pmatrix} w_{3(k-3)+3} \\ w_{3(k-3)+2} \\ w_{3(k-3)+1} \\ w_{3(k-3)} \end{pmatrix} - \begin{pmatrix} w_{3(k-4)+3} \\ w_{3(k-4)+2} \\ w_{3(k-4)+1} \\ w_{3(k-4)} \end{pmatrix} \end{aligned}$$

We obtain the recurrence relation

$$w_n = 2w_{n-3} + 3w_{n-6} - w_{n-9} - w_{n-12}.$$

Remark 8.4.3 (Polygonal diagonals). A remarkable property of wave sequences is their relationship to diagonals of regular polygons. It is well-known that the quotient of consecutive Fibonacci numbers $f_n = w_n^{(2)}$ tends to the golden ratio φ .

$$\lim_{n \rightarrow \infty} \frac{f_{n+1}}{f_n} = \varphi = \frac{1 + \sqrt{5}}{2}$$

The golden ratio is the ratio of a regular pentagon's diagonal to its side, and for any interval of length $a + b$ where $\frac{a+b}{a} = \frac{a}{b}$, the unique solution is $\frac{a}{b} = \varphi$. We may obtain similarly nice ratios of diagonals of n -gons for odd n , and as it turns out, quotients

of consecutive wave numbers tend to these ratios. For example, let ρ and σ denote the lengths of the short and long diagonals (respectively) of the regular unit heptagon. These heptagonal diagonals satisfy the following amazing facts.

- The diagonals satisfy the ratios

$$1 : \rho : \sigma = \rho : (1 + \sigma) : (\rho + \sigma) = \sigma : (\rho + \sigma) : (1 + \rho + \sigma).$$

- These diagonals are the unique real solutions to

$$1 : x : y = x : (1 + y) : (x + y) = y : (x + y) : (1 + x + y).$$

- The 3-wave sequence $w_n^{(3)}$ satisfies the following limit of ratios.

$$w_{3n+2}^{(3)} : w_{3n+1}^{(3)} : w_{3n}^{(3)} \xrightarrow{n \rightarrow \infty} \sigma : \rho : 1$$

More generally, consider a regular unit $(2k+1)$ -gon, which has $k-1$ distinct diagonal lengths, say $\alpha_{k-1} > \alpha_{k-2} > \cdots > \alpha_2 > \alpha_1 > 1$. These diagonals have nice ratios among each other, and the wave sequence $w_n^{(k)}$ satisfies the following limit of ratios.

$$w_{kn+(k-1)}^{(k)} : w_{kn+(k-2)}^{(k)} : \cdots : w_{kn+2}^{(k)} : w_{kn+1}^{(k)} : w_{kn}^{(k)} \xrightarrow{n \rightarrow \infty} \alpha_{k-1} : \alpha_{k-2} : \cdots : \alpha_2 : \alpha_1 : 1$$

For further reading on polygonal diagonals, see [Ste97], which discusses the heptagonal case in detail, and also touches upon the enneagonal (nonagonal) case. Notice that in the equations where powers of one polygonal diagonal are written as linear combinations of the polygonal diagonals (and the constant 1), the corresponding wave numbers arise as coefficients, such as the 3-wave numbers appearing in [Ste97, Equation 7]. The relationship between wave sequences and polygonal diagonals is also mentioned in [Knu98, Section 5.4.3].

In Section 7.1, we found a univariate polynomial sequence $H_n(x)$, arising as entries of the 2×2 upper unitriangular matrix $\mathbf{M}$ from the Hurwitz action via half-twists. In a similar fashion, we may obtain a multivariate polynomial sequence from the Hurwitz action via all-strand rotations. As before, consider a rank- d Coxeter system (W, S) , let $\mathcal{B}$ denote the standard basis of simple roots, and let C denote the matrix representation of the Wall form $\chi_{c^{-1}}$ of the usual Coxeter element $c = s_1 s_2 \cdots s_d$ with respect to $\mathcal{B}$. Treat the above-diagonal entries of C as indeterminate variables, so plugging in values determines the Coxeter system (W, S) . Recall from Section 8.2 that the Hurwitz change-of-basis matrix ${}_B M_{\delta^{nd} \mathcal{B}}$ may be written as a power of $[c] = -C^{-1}C^\top$.

$${}_B M_{\delta^{nd} \mathcal{B}} = {}_B M_{\delta^d \mathcal{B}}^n = (-1)^{n(d+1)} (C^{-1}C^\top)^n.$$

We may define $H_n^{(d)}$ as the sequence of multivariate polynomials given by

$$M_n \cdots M_2 M_1 \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} H_{(n+1)(d-1)}^{(d)} \\ H_{(n+1)(d-1)-1}^{(d)} \\ H_{(n+1)(d-1)-2}^{(d)} \\ \vdots \\ H_{n(d-1)+1}^{(d)} \\ H_{n(d-1)}^{(d)} \end{pmatrix}$$

where each U_k is a $d \times d$ upper anti-unitriangular matrix given by

$$U_k = \begin{cases} RC^\top, & k \equiv 1 \pmod{2} \\ C^{-1}R, & k \equiv 0 \pmod{2} \end{cases}$$

(Here, R denotes the $d \times d$ reversal matrix from Definition 2.1.5.) Notice that $C^{-1}C^\top = (C^{-1}R)(RC^\top)$, so the Hurwitz change-of-basis matrix ${}_B M_{\delta^{nd} \mathcal{B}}$ may be written as the product

$${}_B M_{\delta^{nd} \mathcal{B}} = (-1)^{n(d+1)} ((C^{-1}R)(RC^\top))^n = (-1)^{n(d+1)} U_{2n} \cdots U_2 U_1.$$

In the same way that Fibonacci numbers f_n generalize to Fibonacci polynomials $F_n(x)$ by substituting entries of the 2×2 upper antitriangular generating matrix with indeterminate variables, we may generalize the wave numbers $w_n^{(d)}$ by substituting entries of the $d \times d$ upper antitriangular generating matrix with indeterminate variables. The new generating matrix determines a sequence of multivariate polynomials $W_n^{(d)}$; we present some examples of these generating matrices in Table 8.5. (At the time of writing, I am unaware if the resulting sequences $W_n^{(d)}$ are discussed in the literature or have an established name; for now, we call them *polynomial wave sequences*.) Notice that, in the same way that the Fibonacci polynomials yield the Fibonacci numbers via $F_n(1) = f_n$, setting all entries of the wave polynomial $W_n^{(d)}$ to 1 yields the wave number $w_n^{(d)}$. Furthermore, in the same way that $H_n(x)$ is related to $F_n(x)$ (as discussed in Section 7.2), the multivariate polynomial sequences $H_n^{(d)}$ arising from ${}_B M_{\delta^{nd} B}$ (or equivalently, from $((C^{-1}R)(RC^\top))^n$) are related to the polynomial wave sequences. We summarize the relationship in Figure 8.12.

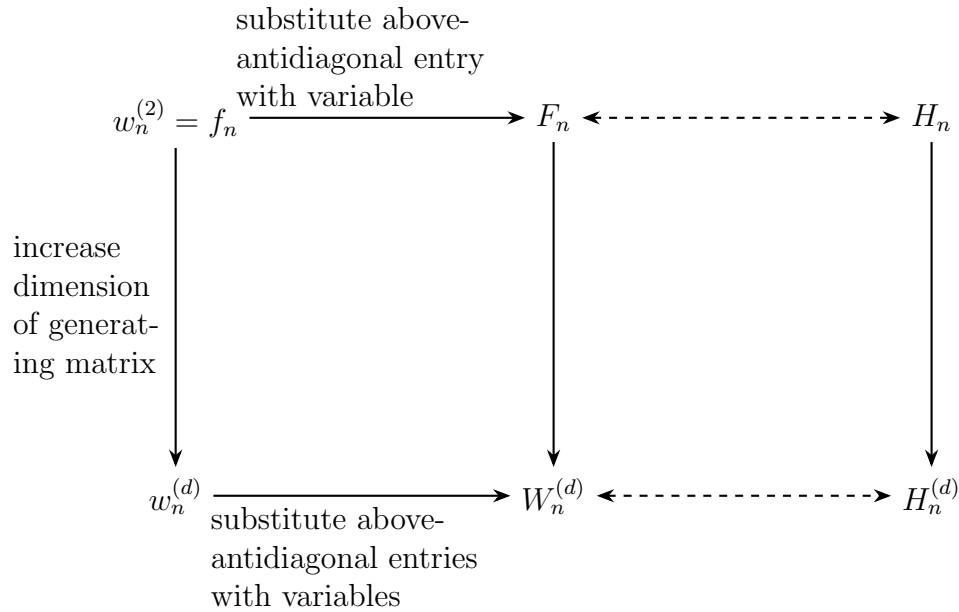


Figure 8.12: Diagram of various generalizations of the Fibonacci numbers

As an illustrative example, let us see examine the rank-3 versions of both the original polynomial wave sequence and the related Hurwitz polynomial wave sequence.

Example 8.4.4 (Rank-3 polynomial wave sequences). The wave sequence $w_n^{(3)}$ is generated by a 3×3 upper antitriangular matrix. For simplicity of notation, we suppress the superscript (3), writing $w_n = w_n^{(3)}$.

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}^n \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} w_{2n+2} \\ w_{2n+1} \\ w_{2n} \end{pmatrix}$$

We can replace the above-antidiagonal entries of the generating matrix with indeterminate variables to obtain the polynomial wave sequence $W_n^{(3)}$. As with the wave sequence w_n , we suppress the superscript (3), writing $W_n = W_n^{(3)}$.

$$\begin{pmatrix} b_1 & a_2 & 1 \\ a_1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}^n \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} W_{2n+2} \\ W_{2n+1} \\ W_{2n} \end{pmatrix}$$

As in Example 8.3.3, write

$$C = \begin{pmatrix} 1 & a_1 & b_1 \\ 0 & 1 & a_2 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad C^{-1} = \begin{pmatrix} 1 & A_1 & B_1 \\ 0 & 1 & A_2 \\ 0 & 0 & 1 \end{pmatrix}$$

where

$$A_1 = -a_1, \quad A_2 = -a_2, \quad B_1 = a_1 a_2 - b_1.$$

Writing

$$U_k = \begin{cases} KC^\top, & k \equiv 1 \pmod{2} \\ C^{-1}K, & k \equiv 0 \pmod{2} \end{cases}$$

we may now define the Hurwitz polynomial wave sequence $H_n^{(3)}$. Below, we suppress the superscript (3), writing $H_n = H_n^{(3)}$.

$$\begin{aligned}
U_n \cdots U_3 U_2 U_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} &= \underbrace{\begin{pmatrix} * & * & 1 \\ * & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \cdots \begin{pmatrix} b_1 & a_2 & 1 \\ a_1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} B_1 & A_1 & 1 \\ A_2 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} b_1 & a_2 & 1 \\ a_1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}}_{n \text{ matrices}} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} H_{2n+2} \\ H_{2n+1} \\ H_{2n} \end{pmatrix}.
\end{aligned}$$

We present the first several terms of both sequences in Table 8.6. Following the technique used in Example 8.4.2, we may compute the defining recurrence relation for the polynomial wave sequence $W_n^{(3)}$. By the Cayley-Hamilton Theorem, the generating matrix

$$M = \begin{pmatrix} b_1 & a_2 & 1 \\ a_1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

is annihilated by its characteristic polynomial

$$\chi(x) = x^3 + (-b_1 - 1)x^2 + (-a_1 a_2 + b_1 - 1)x + 1,$$

whereby we obtain the recurrence relation

$$W_n^{(3)} = (b_1 + 1)W_{n-2}^{(3)} + (a_1 a_2 - b_1 + 1)W_{n-4}^{(3)} - W_{n-6}^{(3)}.$$

Similarly, for $n = 2k$, using the characteristic polynomial

$$\chi(x) = x^3 + (-a_1 a_2 b_1 + a_1^2 + a_2^2 + b_1^2 - 3)x^2 + (a_1 a_2 b_1 - a_1^2 - a_2^2 - b_1^2 + 3)x - 1$$

for $C^{-1}C^\top$, we obtain the recurrence relation

$$H_n^{(3)} = (a_1 a_2 b_1 - a_1^2 - a_2^2 - b_1^2 + 3)H_{n-2}^{(3)} - (a_1 a_2 b_1 - a_1^2 - a_2^2 - b_1^2 + 3)H_{n-4}^{(3)} + H_{n-6}^{(3)}.$$

8.5 N -bonacci sequences

The Hurwitz change-of-basis matrices given by powers of the all-strand rotation braid δ are related not only to wave sequences, but also to another generalization of the Fibonacci numbers called the N -bonacci numbers. This section lays out some of the basic connections.

The N -bonacci numbers are another generalization of the Fibonacci numbers. The names *tribonacci* and *tetranacci* were coined by [Fei63], whence the term *N -bonacci* is derived. We state the definition of the N -bonacci numbers below, and Table 8.7 presents the first several terms of some N -bonacci sequences for a few low values of N . Clearly, the Fibonacci numbers f_n are precisely the 2-bonacci numbers $f_n^{(2)}$.

Definition 8.5.1 (N -bonacci numbers). Fix an integer $N \geq 2$. The N -bonacci numbers $f_n^{(N)}$ are defined by the recurrence relation

$$f_n^{(N)} = \sum_{k=1}^N f_{n-k}^{(N)}$$

with the initial conditions

$$f_n^{(N)} = \begin{cases} 0, & 0 \leq n \leq N-2 \\ 1, & n = N-1 \end{cases}$$

Remark 8.5.2 (Choice of initial values). In the literature, there are several sequences that share the name *tribonacci*, *tetranacci*, and so on, all with the same recurrence relation, but with different initial values, as seen on [OEI26]. For example, *tribonacci* may refer to the sequence A000073, agreeing with Definition 8.5.1, but another popular choice of initial values

$$f_0^{(3)} = f_1^{(3)} = f_2^{(3)} = 1$$

gives us the sequence A000213. The same could be said of tetranacci, pentanacci, hexanacci, and so on. In this section, any N -bonacci sequence refers to those sequences fitting Definition 8.5.1.

Notice that the N -bonacci numbers are generated by an $N \times N$ companion matrix. Writing $f_n = f_n^{(N)}$ for simplicity of notation, we have

$$\begin{pmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix}^n \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} f_{n+N-1} \\ f_{n+N-2} \\ f_{n+N-3} \\ \vdots \\ f_{n+1} \\ f_n \end{pmatrix}.$$

One might generalize the N -bonacci numbers by considering the sequence of N -variable polynomials $F_n^{(N)}$ generated by powers of an arbitrary companion matrix, whose coefficients we may treat as indeterminate variables.

$$\begin{pmatrix} a_1 & a_2 & a_3 & \cdots & a_{N-1} & a_N \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix}^n \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} F_{n+N-1}^{(N)} \\ F_{n+N-2}^{(N)} \\ F_{n+N-3}^{(N)} \\ \vdots \\ F_{n+1}^{(N)} \\ F_n^{(N)} \end{pmatrix}$$

Notice that when $N = 2$ and $a_N = 1$, we recover the Fibonacci polynomials, i.e. $F_n^{(2)}(x, 1) = F_n(x)$. Generalizing to higher N , we may consider the $(N - 1)$ -variable polynomials

$$\mathcal{F}_n^{(N)}(a_1, a_2, \dots, a_{N-1}) = F_n^{(N)}(a_1, a_2, \dots, a_{N-1}, 1)$$

obtained by always setting the coefficient $a_N = 1$ in the companion matrix used to generate the sequence. Perhaps a reasonable name for $\mathcal{F}_n^{(N)}$ would be the N -bonacci polynomials, since this sequence generalizes both the N -bonacci numbers and the Fibonacci

polynomials.

As discussed in Section 8.2, the Hurwitz change-of-basis matrix ${}_B M_{\delta^n B}$ is a product of companion matrices ${}_{\delta^{k-1} B} M_{\delta^k B}$. These companion matrices exhibit periodicity with respect to the power k of the all-strand rotation braid δ . We can view the Hurwitz change-of-basis matrix ${}_B M_{\delta^n B}$ as a distant relative of the generating matrix for the N -bonacci polynomials; instead of obtaining multivariate polynomials by powering up a single companion matrix, we obtain multivariate polynomials by taking a (periodic) product of companion matrices.

(i, j)	$G_{ij}(\lambda)$	$a_n = ([c]^n)_{ij}$	OEIS
$(1, 1)$	$\frac{(\lambda - 1)^2}{(\lambda + 1)(\lambda^2 - 3\lambda + 1)}$	$a_n = f_{n+1}^2 - f_n^2 = f_{n+2}f_{n-1}$	A226205
$(1, 2)$	$-\frac{\lambda(\lambda - 3)}{(\lambda + 1)(\lambda^2 - 3\lambda + 1)}$	$a_n = f_{n+2}^2 - f_{n+1}^2 = f_{n+3}f_n$	A226205
$(1, 3)$	$-\frac{2\lambda}{(\lambda + 1)(\lambda^2 - 3\lambda + 1)}$	$a_n = -2f_nf_{n+1}$	A079472
$(2, 1)$	$\frac{\lambda}{\lambda^2 - 3\lambda + 1}$	$a_n = f_{2n}$	A001906
$(2, 2)$	$\frac{1}{\lambda^2 - 3\lambda + 1}$	$a_n = f_{2(n+1)}$	A001906
$(2, 3)$	$-\frac{2\lambda}{\lambda^2 - 3\lambda + 1}$	$a_n = -2f_{2n}$	A025169
$(3, 1)$	$\frac{2\lambda^2}{(\lambda + 1)(\lambda^2 - 3\lambda + 1)}$	$a_n = 2f_{n-1}f_n$	A079472
$(3, 2)$	$\frac{2\lambda}{(\lambda + 1)(\lambda^2 - 3\lambda + 1)}$	$a_n = 2f_nf_{n+1}$	A079472
$(3, 3)$	$-\frac{3\lambda^2 + 3\lambda - 1}{(\lambda + 1)(\lambda^2 - 3\lambda + 1)}$	$a_n = -(f_{n-1}^2 + f_{n-2}^2 + 2f_{n-1}f_n + 3f_{n-2}f_{n-1})$	A061647

Table 8.2: Data for sequences $([c]^n)_{ij}$ arising from $\text{PGL}(2, \mathbb{Z})$

n	$w_n^{(2)}$ (A000045)	$w_n^{(3)}$ (A187070)	$w_n^{(4)}$ (A038197)	$w_n^{(5)}$ (A038201)
0	0	0	0	0
1	1	0	0	0
2	1	1	0	0
3	2	1	1	0
4	3	1	1	1
5	5	2	1	1
6	8	3	1	1
7	13	5	2	1
8	21	6	3	1
9	34	11	4	2
10	55	14	7	3
11	89	25	9	4
12	144	31	10	5
13	233	56	19	9
14	377	70	26	12
15	610	126	30	14
16	987	157	56	15
17	1597	283	75	29
18	2584	353	85	41
19	4181	636	160	50
20	6765	793	216	55

Table 8.3: First several terms of some wave sequences

d	Recurrence relation
2	$w_n^{(2)} = w_{n-1}^{(2)} + w_{n-2}^{(2)}$
3	$w_n^{(3)} = 2w_{n-2}^{(3)} + w_{n-4}^{(3)} - w_{n-6}^{(3)}$
4	$w_n^{(4)} = 2w_{n-3}^{(4)} + 3w_{n-6}^{(4)} - w_{n-9}^{(4)} - w_{n-12}^{(4)}$
5	$w_n^{(5)} = 3w_{n-4}^{(5)} + 3w_{n-8}^{(5)} - 4w_{n-12}^{(5)} - w_{n-16}^{(5)} + w_{n-20}^{(5)}$

Table 8.4: Recurrence relations for the wave sequences

d	Generating matrix
2	$\begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix}$
3	$\begin{pmatrix} b_1 & a_2 & 1 \\ a_1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$
4	$\begin{pmatrix} c_1 & b_2 & a_3 & 1 \\ b_1 & a_2 & 1 & 0 \\ a_1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$
5	$\begin{pmatrix} d_1 & c_2 & b_3 & a_4 & 1 \\ c_1 & b_2 & a_3 & 1 & 0 \\ b_1 & a_2 & 1 & 0 & 0 \\ a_1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$

Table 8.5: Generating matrices for polynomial wave sequences

n	$W_n^{(3)}$	$H_n^{(3)}$
0	0	0
1	0	0
2	1	1
3	a_1	a_1
4	b_1	b_1
5	$a_1b_1 + a_1$	$A_2b_1 + a_1$
6	$a_1a_2 + b_1^2 + 1$	$a_1A_1 + b_1B_1 + 1$
7	$a_1^2a_2 + a_1b_1^2 + a_1b_1 + 2a_1$	$a_1^2A_1 + a_1b_1B_1 + A_2b_1 + 2a_1$
8	$2a_1a_2b_1 + b_1^3 + a_1a_2 + 2b_1$	$a_1A_1b_1 + a_2A_2b_1 + b_1^2B_1 + a_1a_2 + 2b_1$
n	$W_n^{(3)}$	$H_n^{(3)}$ simplified
0	0	0
1	0	0
2	1	1
3	a_1	a_1
4	b_1	b_1
5	$a_1b_1 + a_1$	$-a_2b_1 + a_1$
6	$a_1a_2 + b_1^2 + 1$	$a_1a_2b_1 - a_1^2 - b_1^2 + 1$
7	$a_1^2a_2 + a_1b_1^2 + a_1b_1 + 2a_1$	$a_1^2a_2b_1 - a_1^3 - a_1b_1^2 - a_2b_1 + 2a_1$
8	$2a_1a_2b_1 + b_1^3 + a_1a_2 + 2b_1$	$a_1a_2b_1^2 - a_1^2b_1 - a_2^2b_1 - b_1^3 + a_1a_2 + 2b_1$

Table 8.6: First several terms of $W_n^{(3)}$ and $H_n^{(3)}$

n	$f_n^{(2)}$ (A000045)	$f_n^{(3)}$ (A000073)	$f_n^{(4)}$ (A000078)	$f_n^{(5)}$ (A001591)	$f_n^{(6)}$ (A001592)
0	0	0	0	0	0
1	1	0	0	0	0
2	1	1	0	0	0
3	2	1	1	0	0
4	3	2	1	1	0
5	5	4	2	1	1
6	8	7	4	2	1
7	13	13	8	4	2
8	21	24	15	8	4
9	34	44	29	16	8
10	55	81	56	31	16
11	89	149	108	61	32
12	144	274	208	120	63
13	233	504	401	236	125
14	377	927	773	464	248
15	610	1705	1490	912	492
16	987	3136	2872	1793	976
17	1597	5768	5536	3525	1936
18	2584	10609	10671	6930	3840
19	4181	19513	20569	13624	7617
20	6765	35890	39648	26784	15109

Table 8.7: First several terms of some N -bonacci sequences

Chapter 9

Future work

This brief chapter lists several directions for future investigation.

9.1 Investigating polynomial sequences

There are a few avenues for continuing to investigate the polynomial sequences discussed in Chapter 8.

In Section 8.3, we applied the transfer-matrix method to Sym_4 and $\text{PGL}(2, \mathbb{Z})$, obtaining closed formulas for integer sequences arising from entries of powers of the Coxeter element $[c] = -C^{-1}C^\top$. Applying the transfer-matrix method to the Wall form matrix C of an arbitrary rank-3 Coxeter system, we obtained the generating functions for multivariate polynomial sequences, again arising from entries of powers of $[c]$. Plugging in values for the indeterminate variables allows us to obtain the generating functions for any rank-3 Coxeter system we wish. The next step would be to compute the closed formulas for these multivariate polynomial sequences, which may be done by fully computing the power series expansions of the generating functions. The step afterwards would be to replicate these computations for higher-rank Coxeter systems.

We can also investigate the polynomial wave sequences and N -bonacci polynomial sequences discussed in Sections 8.4 and 8.5, respectively. Those sections discussed only basic data, namely the defining recurrence relation and the generating matrix. We would like to obtain additional data.

- Identities
- Rational generating functions
- Closed formulas
- Relationships with well-known sequences
- Combinatorial interpretation

9.2 Investigating more types of braids

In Chapter 7, we investigated the Hurwitz action via powers of half-twists; in Chapter 8, we investigated the Hurwitz action via powers of rotation braids of adjacent strands, focusing especially on the all-strand rotation δ . Ultimately, we want to understand the change-of-basis matrices and Wall form matrices obtained by applying the Hurwitz action via *arbitrary* braids. We may already compute these matrices using the techniques discussed in Chapter 6, but it is difficult to discern patterns in the resulting matrices. One future direction is to identify new techniques for extracting patterns from the Wall form matrices obtained via arbitrary sequences of Hurwitz moves. Another is to apply our existing techniques to other The following subsections discuss some types of braids that could be investigated.

9.2.1 Hurwitz action via powers of arbitrary rotation braids

In our investigation of the Hurwitz action, we used the standard presentation for the braid group, writing arbitrary braids as sequences of half-twists. It might be fruitful to consider other presentations for the braid group. One such presentation is the *rotation presentation* from [MM09]. We already investigated a special case of rotation braids, namely the rotation of adjacent strands. We could immediately apply techniques used in this dissertation to powers of generators in the rotation presentation.

One difficulty would be the number of generators to investigate. There are only $d - 1$ standard generators (positive half-twists) in the standard presentation of Braid_d , and they may be analyzed uniformly (so only one generic case needs to be analyzed) as in Chapter 7. However, there are $2^d - d - 1$ generating rotations in the rotation presentation of Braid_d , and it is not clear that they can be analyzed uniformly. One might start by analyzing rotations that exhibit nice properties, developing techniques that could help with analyzing arbitrary rotation braids.

9.2.2 Swing and twist braids

We are interested in the Hurwitz action because of its role in constructing dual Artin groups; see Definition 5.3.4. The generators in the presentation of the dual Artin group $\text{Art}^*(\Gamma, w)$ are precisely the reflections that appear in reduced reflection factorizations of the Coxeter element w ; we may obtain all such reflections by applying the Hurwitz action, since it is transitive on the set $\text{Red}_T(w)$ of such reflection factorizations. It turns out that one may obtain all generators for the defining presentation of $\text{Art}^*(\Gamma, w)$ by applying the *pure* Hurwitz action, i.e. the Hurwitz action via only pure braids.

Indeed, write $w = r_1 r_2 \cdots r_d$, and suppose r is some generator in the dual presentation, i.e. there exists some braid $\beta \in \text{Braid}_d$ such that r appears in the d -tuple $\beta \cdot (r_1, r_2, \dots, r_d)$,

say at the k -th position. Let $\pi \in \text{Sym}_d$ be the image of β via the natural surjection $\text{Braid}_d \twoheadrightarrow \text{Sym}_d$. Let $\gamma \in \text{Braid}_d$ be a braid that surjects onto π^{-1} , subject to the condition that the k -th strand always crosses *over* other strands, never under. Then the braid $\gamma\beta$ is a pure braid where r appears in the d -tuple $\gamma\beta \cdot (r_1, r_2, \dots, r_d)$.

Example 9.2.1 (Pure Hurwitz action). As an illustration of how to obtain a pure Hurwitz action that gives the desired generator of $\text{Art}^*(\Gamma, w)$, consider the braid $\beta = \sigma_4\sigma_3^{-1}\sigma_4\sigma_1\sigma_2^{-1}\sigma_3^{-1}\sigma_2$, and suppose r occurs in the 4th position of $\beta \cdot (r_1, r_2, r_3, r_4, r_5)$. Letting $\gamma = \sigma_3^{-1}\sigma_2\sigma_1\sigma_3\sigma_4\sigma_3^{-1}$, we find that $\gamma\beta \in \text{PBraid}_5$, and r occurs in the 3rd position of $\gamma\beta \cdot (r_1, r_2, r_3, r_4, r_5)$. See Figure 9.1 for the braid diagram.

One future direction for investigation is to focus on the change-of-basis matrices and Wall form matrices that arise from the pure Hurwitz action, since the pure Hurwitz action is sufficient for finding all generators in the dual presentation. In the same way that we investigated half-twists of adjacent strands (the generators of the standard presentation of the braid group), we may investigate the generators of nice presentations of the pure braid group. Two such presentations are the *twist presentation* and *swing presentation* from [MM09]. We could immediately apply the techniques used in this dissertation to powers of generators of these two presentations.

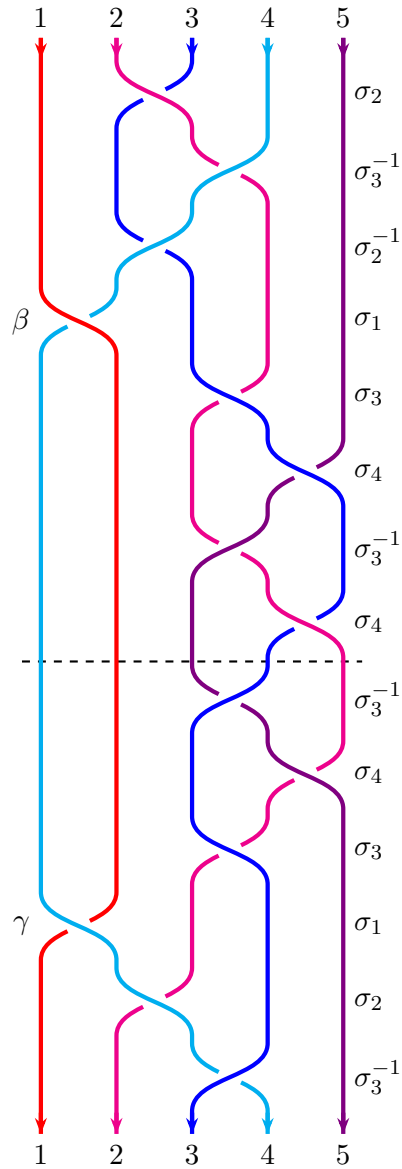


Figure 9.1: Pure braid $\gamma\beta$ obtained from $\beta = \sigma_4\sigma_3^{-1}\sigma_4\sigma_1\sigma_2^{-1}\sigma_3^{-1}\sigma_2$

Appendix A

Code for computing the Hurwitz action on Wall forms

Below is some of my SageMath code. Listings A.2 through A.5 state the functions that I wrote for computing the Hurwitz action as a change-of-basis. Listing A.1 defines the symbolic variables, multivariate polynomial ring, and matrices used in the docstrings of the aforementioned functions. These functions (and other functions that I wrote for the experimental mathematics that went into writing this dissertation) can be found in the file `wallform.sage` at my Github repository [DN26].

```
1 var("a1 a2 a3 a4 a5 b1 b2 b3 b4 c1 c2 c3 d1 d2 e1")
2 S.<a1,a2,a3,a4,a5,b1,b2,b3,b4,c1,c2,c3,d1,d2,e1> = PolynomialRing(ZZ)
3
4 # Examples of Wall form matrices
5 C3 = matrix([[1,a1,b1],[0,1,a2],[0,0,1]])
6 C4 = matrix([[1,a1,b1,c1],[0,1,a2,b2],[0,0,1,a3],[0,0,0,1]])
7 C5 = matrix([[1,a1,b1,c1,d1],
8              [0,1,a2,b2,c2],
9              [0,0,1,a3,b3],
10             [0,0,0,1,a4],
```

Appendix A. Code for computing the Hurwitz action on Wall forms

```
11         [0,0,0,0,1]])
12 C6 = matrix([[1,a1,b1,c1,d1,e1],
13             [0,1,a2,b2,c2,d2],
14             [0,0,1,a3,b3,c3],
15             [0,0,0,1,a4,b4],
16             [0,0,0,0,1,a5],
17             [0,0,0,0,0,1]])
```

Listing A.1: Variables and matrices used in the docstring examples

```
1 def hurwitz_mov_mtx(C,j):
2     """Returns the change-of-basis matrix for a Hurwitz move.
3
4     Consider the Hurwitz action of the 'j'-th positive half-twist
5     braid (i.e. the 'j'-th standard generator of the braid group)
6     on a Wall form matrix 'C'.
7
8     This function returns the corresponding change-of-basis matrix.
9     In particular, if ' $j * C = M^T * C * M$ ', then this function returns
10    ' $M$ '.
11
12    Parameters
13    -----
14    C : Matrix
15        The nxn Wall form matrix.
16    j : Integer
17        Positive integer from the interval [1,n-1], representing the
18        'j'-th positive half-twist.
19
20    Returns
21    -----
22    Matrix
```

```

23
24 Examples
25 -----
26
27 sage: hurwitz_mov_mtx(C3,2)
28 [ 1  0  0]
29 [ 0 a2  1]
30 [ 0 -1  0]
31
32 sage: hurwitz_mov_mtx(C5,3)
33 [ 1  0  0  0  0]
34 [ 0  1  0  0  0]
35 [ 0  0 a3  1  0]
36 [ 0  0 -1  0  0]
37 [ 0  0  0  0  1]
38
39 """
40
41 i = abs(j)
42 s = sgn(j)
43 n = len(C[0])
44
45 B = (copy(identity_matrix(n))).change_ring(C.base_ring())
46 p = C[i-1,i]
47 B[i-1,i-1] = p
48 B[i,i-1] = -1
49 B[i-1,i] = 1
50 B[i,i] = 0
51
52 # If s == -1, then the entries of B^s become FractionFieldElement
53 # when they are supposed to be MPolynomial_libsingular
54 res = (B^s).change_ring(C.base_ring())
55
56 return res

```

Listing A.2: Change-of-basis matrix for a Hurwitz move via σ_j

```

1 def hurwitz_mov_res(C,j):
2     """Returns the result of a Hurwitz move.
3
4     Consider the Hurwitz action of the 'j'-th positive half-twist
5     braid (i.e. the 'j'-th standard generator of the braid group)
6     on a Wall form matrix 'C'.
7
8     This function returns the matrix of the Wall form after the
9     change of basis.
10
11     Parameters
12     -----
13     C : Matrix
14         The nxn Wall form matrix.
15     j : Integer
16         Positive integer from the interval [1,n-1], representing the
17         'j'-th positive half-twist.
18
19     Returns
20     -----
21     Matrix
22
23     Examples
24     -----
25
26     sage: hurwitz_mov_res(C3,2)
27     [      1 a1*a2 - b1      a1]
28     [      0          1      a2]
29     [      0          0      1]
30     """
31     B = hurwitz_mov_mtx(C,j)

```

```
32     return transpose(B)*C*B
```

Listing A.3: Wall form matrix after a Hurwitz move via σ_j

```
1 def hurwitz_act_res(C,l):
2     """Returns the result of a sequence of Hurwitz moves.
3
4     Consider the Hurwitz action (given by a sequence of Hurwitz
5     moves) on a Wall form matrix 'C'.
6     The Hurwitz action causes a change of basis.
7
8     This function returns the matrix of the Wall form after the
9     change of basis.
10
11     This is a left-action, so we first apply the right-most braid in
12     the sequence, eventually ending with the left-most braid.
13     '[ln, ..., l3, l2, l1] * C = [ln, ..., l3, l2] * (l1 * C)''
14
15     Parameters
16     -----
17     C : Matrix
18         The nxn Wall form matrix.
19     l : list[Integer]
20         List of integers from the interval [1,n-1], representing a
21         sequence of positive half-twists.
22
23     Returns
24     -----
25     Matrix
26
27     Examples
28     -----
```

```

29
30 sage: hurwitz_act_res(C3,2)
31 [      1 a1*a2 - b1      a1]
32 [      0      1      a2]
33 [      0      0      1]
34 sage: hurwitz_act_res(C4,[])
35 [ 1 a1 b1 c1]
36 [ 0  1 a2 b2]
37 [ 0  0  1 a3]
38 [ 0  0  0  1]
39 sage: hurwitz_act_res(C3,[2])
40 [      1 a1*a2 - b1      a1]
41 [      0      1      a2]
42 [      0      0      1]
43 sage: hurwitz_act_res(C3,[1,2])
44 [      1      a1*a2 - b1 a1^2*a2 - a1*b1 - a2]
45 [      0      1      a1]
46 [      0      0      1]
47 sage: hurwitz_act_res(C3,[2,1])
48 [ 1 a2 a1]
49 [ 0  1 b1]
50 [ 0  0  1]
51 """
52 match l:
53     case Integer():
54         return hurwitz_mov_res(C,l)
55     case []:
56         return C
57     case [num]:
58         return hurwitz_mov_res(C,num)
59     case list():

```

```

60         return hurwitz_act_res(hurwitz_mov_res(C,l[-1]),l[:-1])
61     case _:
62         raise TypeError("Only integers or lists of integers are
63                          allowed")

```

Listing A.4: Wall form matrix after Hurwitz action

```

1 def hurwitz_act_mtx(C,l):
2     """Returns the c.o.b. matrix for a sequence of Hurwitz moves.
3
4     Consider the Hurwitz action (given by a sequence of Hurwitz
5     moves) on a Wall form matrix 'C'.
6
7     This function returns the corresponding change-of-basis matrix.
8     In particular, if ' $l * C = M^T * C * M$ ', then this function returns
9     ' $M$ '.
10
11     Parameters
12     -----
13     C : Matrix
14         The nxn Wall form matrix.
15     l : list[Integer]
16         List of integers from the interval [1,n-1], representing a
17         sequence of positive half-twists.
18
19     Returns
20     -----
21     Matrix
22
23     Examples
24     -----
25

```

```

26     sage: hurwitz_act_mtx(C3,2)
27     [ 1  0  0]
28     [ 0 a2  1]
29     [ 0 -1  0]
30     sage: hurwitz_act_mtx(C3,[2])
31     [ 1  0  0]
32     [ 0 a2  1]
33     [ 0 -1  0]
34     sage: hurwitz_act_mtx(C4,[ ])
35     [1 0 0 0]
36     [0 1 0 0]
37     [0 0 1 0]
38     [0 0 0 1]
39     sage: hurwitz_act_mtx(C3,[1,2])
40     [a1*a2 - b1          1          0]
41     [          -a2          0          1]
42     [          1          0          0]
43     sage: hurwitz_act_mtx(C3,[2,1])
44     [a1 b1  1]
45     [-1  0  0]
46     [ 0 -1  0]
47     """
48     match l:
49         case Integer():
50             return hurwitz_mov_mtx(C,l)
51         case []:
52             return identity_matrix(C.nrows()).change_ring(
53                 C.base_ring())
54         case [num]:
55             return hurwitz_mov_mtx(C,num)
56         case list():

```

```
57         return (hurwitz_mov_mtx(C,l[-1]) *
58                 hurwitz_act_mtx(hurwitz_mov_res(C,l[-1]),l[:-1]))
59     case _:
60         raise TypeError("Only integers or lists of integers are
61                           allowed")
```

Listing A.5: Change-of-basis matrix after Hurwitz action

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