**Title**
Conformal Bach Flow



**Author**
Chen, Jiaqi

**Publication Date**
2020

Peer reviewed|Thesis/dissertation

UNIVERSITY OF CALIFORNIA
SANTA CRUZ

# CONFORMAL BACH FLOW

A dissertation submitted in partial satisfaction of the
requirements for the degree of

DOCTOR OF PHILOSOPHY

in

MATHEMATICS

by

**Jiaqi Chen**

September 2020

The Dissertation of Jiaqi Chen
is approved:

Prof. Jie Qing, Chair

Prof. Longzhi Lin

Prof. Richard Montgomery

Quentin Williams
Vice Provost and Dean of Graduate Studies

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**Abstract**

Conformal Bach Flow

by

Jiaqi Chen

In this thesis, we introduce a new type of geometric flow of Riemannian metrics based on Bach tensor and the gradient of Weyl curvature functional and coupled with an elliptic equation which preserves a constant scalar curvature along with this flow. We named this flow by conformal Bach flow. In this thesis, we first establish the short-time existence of the conformal Bach flow and its regularity. After that, some evolution equations of curvature tensor along this flow are derived and we use them to obtain the $L^2$ estimates of the curvature tensors. After that, these estimates help us characterize the finite-time singularity. We also prove a compactness theorem for a sequence of solutions with uniformly bounded curvature norms. Finally, some singularity model is investigated.
**Keywords**: Riemannian manifold, Geometric Flow, Bach tensor

# Chapter 1

# Introduction

## 1.1 Weyl Curvature

Weyl curvature, named after Hermann Weyl, plays an important role in modern physics. It is the trace-free part of Riemann curvature tensor, hence, it won't carry any information about the volume change but rather only how the shape of the body is distorted by the tidal force [54].

One famous hypothesis came out by Penrose in [41]. He argues that the universe must have been in a low entropy state initially in order for there now to be a second law of thermodynamics. There is no generally accepted definition of gravitational entropy but Penrose argues that low gravitational entropy must mean small Weyl's $L^2$ norm.

In this thesis, we will investigate a new type of geometric flow defined by the gradient of the Weyl functional. In dimension 4, such gradient is call Bach tensor, which was introduced by Rudolf Bach [2] to study conformal geometry in early

1920's, and is defined by

$$B_{ij} = \frac{1}{n-3}\nabla^k\nabla^l W_{ikjl} + \frac{1}{n-2}R^{kl}W_{ikjl} \tag{1.1.1}$$

In general relativity, such flow is proposed in [5, Chapter 7] to investigate the Hořava-Lifshitz gravity, but the short time existence of such flow was not established at that moment. We will present later that some modification is necessary and the gradient property of such modified flow will be preserved.

## 1.2 Conformal Geometry

Weyl curvature is also highly related to the sphere theorem in Riemannian geometry, especially in dimension 4. This theorem states that under what conditions on the curvature can we conclude that a smooth, a closed Riemannian manifold is diffeomorphic or conformal to the standard sphere?

First result is from the work of Margerin, a sharp pointwise geometric characterisation of the smooth structure of $\mathbb{S}^4$. The powerful tool in this approach is the Ricci flow.

**Theorem 1.2.1.** *[40, Thm 1] Given a closed manifold with positive Yamabe constant and the following curvature pinching condition:*

$$|W|^2 + 2|E|^2 < \frac{1}{6}R^2$$

*then $M^4$ is diffeomorphic to standard $\mathbb{S}^4$ or $\mathbb{RP}^4$.*

**Remark 1.2.2.** *This theorem is sharp, both $(\mathbb{CP}^2, g_{FS})$ and $(\mathbb{S}^3\times\mathbb{S}^1, g_{prod})$ satisfies the equality.*

This theorem is improved by A.Chang, P.Yang and M.Gursky, a global curvature condition is proposed to obtain the same result.

**Theorem 1.2.3.** *[10, Thm A'] Given a closed manifold with positive Yamabe constant and the following curvature pinching condition:*

$$\int_M |W|^2 + 2|E|^2 d\mu < \frac{1}{6}\int_M R^2 d\mu$$

*then $M^4$ is diffeomorphic to standard $\mathbb{S}^4$ or $\mathbb{RP}^4$.*

Later on, Q.Jie, A.Chang, P.Yang improved this result by directly comparing the curvature quantity to $16\pi^2$.

**Theorem 1.2.4.** *[11, Thm A] There is an $\epsilon > 0$, for any Bach flat manifold $(M^4, g)$ with positive Yamabe type, if*

$$\int_M -\frac{1}{2}|E|^2 + \frac{1}{6}R^2 d\mu > (1-\epsilon)16\pi^2$$

*then $(M^4, [g]) \overset{conf}{\cong} (\mathbb{S}^4, g_{\mathbb{S}^4})$.*

In this theorem, Bach flat condition is assumed, the proof of the theorem builds upon some estimates in the work of Tian-Viaclovsky [53] on the compactness of Bach-flat metrics on 4-manifolds.

Will Bach flow helps us here? We will see that both conformal Bach flow and modified Bach flow we are going to introduced are gradient flows of $L^2$-norm of Weyl curvature in dimension 4. If we can prove such flow exists for a long time with small initial data, i.e. $L^2$ norm of Weyl is small, such flow might deform the manifold to the standard sphere and we have hope to improve the gap theorem in [11, Thm A] without any assumption on Bach tensor.

## 1.3 High Order Geometric Flow

In this section, we introduce some histories in geometric flows.

**Ricci Flow**

One of the notable geometric flow is Ricci flow introduced by R. Hamilton in his famous paper [25, Page 259].

Given a Riemannian manifold $(M^n, g_0)$, the solution to Ricci flow is a one-parameter family of metric $g(t)$ defined by

$$\begin{cases} \dfrac{\partial}{\partial t} g = -2Ric[g] \\ g(0) = g_0 \end{cases}$$

For an arbitrary smooth initial metric, the flow will always exists at least for a short time, but finite time singularities may occur which causes the flow to terminate. Ricci flow is used to prove Thurston's geometrization conjecture and the Poincaré conjecture in [42] [44] [43].

With a huge success in the Ricci flow, researchers start investigating some other higher-order geometric flows. These types of flows mostly come from the gradient of a certain energy functional. Such monotonicity is crucial in many scenarios.

**Calabi Flow**

E. Calabi introduces a high order geometric flow in [7][8], he shows that the Calabi energy is decreasing along with the Calabi flow. It is expected that the Calabi flow should converge to a constant scalar curvature metric.

In the case of Riemann surfaces, Chruściel [17, Proposition 5.1] shows that the flow always converges to a constant curvature metric. After that, X.X.Chen [13] proves the same theorem with a different approach.

**Gradient Flow of L2 Functional of Riemann Curvature**

From 2008, Jeffery Streets published a series of papers [51][50] [49][48] to discuss a geometric flow which deforms metric under the gradient of the following functional,

$$\mathcal{F} = \int_M |Riem|^2 d\mu$$

Since the equation is fourth-order, maximum principle techniques are not readily available, J. Streets used integral estimate to investigate properties such as long time stability of this flow. We will discuss some details about J. Streets' work later.

**Ambient Obstruction Flow**

Another progress in this field is related to a family of tensors called obstruction tensor $\mathcal{O}_{ij}$ introduced by Fefferman and Graham in [20]. In [3, Theorem C], Bahuaud and Helliwell studied the following flow

$$\begin{cases} \dfrac{\partial}{\partial t} g = \mathcal{O} + c_n(-1)^{\frac{n}{2}}(\Delta^{\frac{n}{2}-1}R)g_{ij} \\ g(0) = g_0 \end{cases}$$

Short time existence and uniqueness [4] were proved. Since that Bach tensor is the obstruction tensor in four-dimensional case, the modified Bach flow was defined by

$$\begin{cases} \dfrac{\partial}{\partial t} g = B + \dfrac{1}{2(n-1)(n-2)}\Delta R g_{ij} \\ g(0) = g_0 \end{cases}$$

## 1.4 Outlines

In this thesis, we first introduce some preliminary results in Chapter 2, and we also provide lots of the calculation details in Appendix. In Chapter 3, we prove the short time existence of the conformal Bach flow, and we also show the uniqueness and regularity in this chapter. In Chapter 4, we derive the integral estimates for Riemann curvature tensor under the conformal Bach flow, the volume estimate and finite time singularity are investigated. In Chapter 5, we present the Cheeger Gromov convergence theorem, and we prove the compactness of solutions to conformal Bach flow. In the end, we investigate a special singularity model obtained by re-scaling the metric near singularity.

# Chapter 2

# Preliminaries

In this chapter, we will introduce some background in conformal geometry, properties of Bach tensor and some motivation of conformal Bach flow.

Let's first recall that on an n-dimensional manifold $(M^n, g)$, where n is at least 4, Bach tensor is introduced by Rudolf Bach [2] to study conformal geometry in early 1920's, and is defined by

$$B_{ij} = \frac{1}{n-3}\nabla^k\nabla^l W_{ikjl} + \frac{1}{n-2}R^{kl}W_{ikjl} \tag{2.0.1}$$

where $W_{ikjl}$ is the Weyl tensor (A.9.3). We also introduce some equivalent forms of Bach tensor in (A.11.2) and (A.11.6).

## 2.1 Bach Tensor in Conformal Geometry

One of the motivations we are interested in the Bach tensor is that Bach tensor is highly related to the conformal geometry, it is a so-called obstruction tensor in dimension 4. Such family of tensors comes out naturally in the ambient space construction, this idea comes from Fefferman and Graham [20].

**Bundle Structure of Conformal Manifold**

Consider a conformal manifold $(M^n, [g])$, we can specify the class $[g]$ as a subbundle of a bundle of symmetric 2-tensors on $M^n$. The reason is that for a conformal class $[g]$, each pair $g_1$ and $g_2$ in $[g]$ is only differed by a smooth positive function. This induces a trivialization of such subbundle, which is $\mathbb{R}^+ \times M$, we denote it by $\mathcal{G}$. The bundle structure of $\mathcal{G}$ is given by the projection and dilation as follows, for any $p \in M$ and $s \in \mathbb{R}^+$, we have

$$\pi(g|_p) = p : \mathcal{G} \to M \qquad \text{and} \qquad \delta_s(g|_p) = s^2(g|_p)$$

This metric bundle induces a symmetric 2-tensor on $\mathcal{G}$ naturally in the following sense. Let $z = (\alpha, p) \in \mathcal{G}$, consider the vectors $X$ and $Y$ in the tangent space $T_z\mathcal{G}$, we can define the symmetric 2-tensor by:

$$g_0(X, Y) := g|_p(d\pi(X), d\pi(Y))$$

**Remark 2.1.1.** *This symmetric 2-tensor is homogeneous because* $\delta_s^* g_0 = s^2 g_0$.

Let $\mathcal{S} = \frac{d}{ds}\Big|_{s=1} \delta_s$ be the vector field generated by the dilation. For any given representatives $g \in [g]$, we can define a natural coordinate of $\mathcal{G}$:

$$\mathcal{G} = \{\alpha^2 g|_p : \alpha = \mathbb{R}^+, p \in M^n\} = \{(\alpha, p) : \alpha = \mathbb{R}^+, p \in M^n\}$$

With this definition, the projection and dilation will be:

$$\pi(\alpha, p) = p \qquad \text{and} \qquad \delta_s(\alpha, p) = (s\alpha, p)$$

which gives us the vector field generated by the dilation: $\mathcal{S} = \alpha d\alpha$. The symmetric 2-tensor can also be defined as $g_0(X, Y) = \alpha^2 g|_p(d\pi(X), d\pi(Y))$.

**Remark 2.1.2.** *this symmetric 2-tensor is not Riemannian, because the projection will send $\mathcal{S}$ to zero. We can think $\mathcal{S}$ is in the direction of a single fiber.*

**Ambient Space Construction**

Suppose that $(M^n, [g])$ is a conformal manifold and $\mathcal{G}$ is the metric bundle we defined before. We consider the space $\mathcal{G} \times \mathbb{R}$ which is identified by the map:

$$\begin{aligned} i : \mathcal{G} &\longrightarrow \mathcal{G} \times \mathbb{R} \\ z \in \mathcal{G} &\longmapsto (z, 0) \in \mathcal{G} \times \mathbb{R} \end{aligned} \tag{2.1.1}$$

for any $z \in \mathcal{G}$.

Given a coordinate $(x^1, x^2, \cdots, x^n)$ on $M^n$, a representative $g \in [g]$, we often use coordinate $(\alpha, x^1, x^2, \cdots, x^n, \rho)$ on $\mathcal{G} \times \mathbb{R}$. Fefferman and Graham defined the pre-ambient space which means the coordinate system is normal. This is the analog to the normal coordinate system on Riemannian manifold. They showed that such coordinate exists and it is unique. In fact, those two extra coordinates are defined by shooting geodesic rays, which is also named Fermi coordinates.

Once we have this coordinate system, we can define a so called ambient space.

**Definition 2.1.3.** *[20] A pre-ambient space $(\mathcal{G} \times R, \tilde{g})$ of a conformal manifold $(M^n, [g])$ is called an ambient space if:*

*(a) when $n$ is odd, $Ric[\tilde{g}] = 0$ to infinite order at $\rho = 0$*

*(b) when $n$ is even, $Ric[\tilde{g}] = O(\rho^{\frac{n}{2}-1})$*

If we calculate the Ricci curvature directly, we will have the following ordinary differential equation system.

$$\begin{cases} \rho\Big[2g''_{ij} - 2g^{kl}g'_{ik}g'_{jl} + Tr(g')g'_{ij}\Big] = Tr(g')g_{ij} - (2-n)g'_{ij} - 2R_{ij} \\ \frac{1}{2}g^{kl}\Big(\nabla_k g'_{il} - \nabla_i g'_{kl}\Big) = 0 \\ Tr(g'') = \frac{1}{2}g^{kl}g^{pq}g'_{kp}g'_{lq} \end{cases} \tag{2.1.2}$$

To solve this ODE system, we consider a formal power series expansion as follows:

$$g_{ij} = g^{(0)}_{ij} + \rho g^{(1)}_{ij} + \rho^2 g^{(2)}_{ij} + \rho^3 g^{(3)}_{ij} + \cdots \tag{2.1.3}$$

combine (2.1.2) and (2.1.3), and we collect terms with the same degree of $\rho$, then we will have:

$$\begin{cases} -Tr(g^{(1)})g^{(0)}_{ij} + (2-n)g^{(1)}_{ij} + 2R_{ij} = 0 \\ \nabla^k g^{(1)}_{ik} - \nabla_i Tr(g^{(1)}) = 0 \\ 2Tr(g^{(2)}) - \frac{1}{2}g^{(1)kl}g^{(1)}_{kl} = 0 \end{cases} \tag{2.1.4}$$

and

$$\begin{aligned} &(8-2n)g^{(2)}_{ij} - 2g^{(1)k}_{i}g^{(1)}_{jk} - 2Tr(g^{(2)})g^{(0)}_{ij} + g^{(1)kl}g^{(1)}_{kl}g^{(0)}_{ij} \\ &= -\nabla^k\nabla_i g^{(1)}_{jk} - \nabla^k\nabla_j g^{(1)}_{ik} + \Delta g^{(1)}_{ij} + \nabla_i\nabla_j Tr(g^{(1)}) \end{aligned} \tag{2.1.5}$$

take trace for the first equation in (2.1.4),

$$Tr(g^{(1)}) = \frac{R}{n-1} \tag{2.1.6}$$

where $R$ is the scalar curvature. With this result, we can solve for $g^{(1)}$. It turns out we have:

$$g^{(1)}_{ij} = \frac{2}{n-2}\Big(R_{ij} - \frac{R}{2(n-1)}g^{(0)}_{ij}\Big) = 2A_{ij} \tag{2.1.7}$$

which is the twice of Schouten tensor(A.9.2).

From the other two equations in (2.1.4), we have the following results which will help us solve for $g^{(2)}$.

$$\begin{cases} Tr(g^{(2)}) = \frac{1}{4} g^{(1)kl} g^{(1)}_{kl} = A^{kl} A_{kl} \\ \nabla^k g^{(1)}_{ik} = \frac{1}{n-1} \nabla_i R \end{cases} \tag{2.1.8}$$

Now we can derive $g^{(2)}_{ij}$ by plugging previous results (2.1.6), (2.1.7) and (2.1.8). $g^{(2)}$ satisfies the following equation,

$$\begin{aligned} (8-2n) g^{(2)}_{ij} = 2\Big(\Delta A_{ij} - \nabla^k \nabla_j A_{ik}\Big) + 8 A_{ik} A^k_j - 2 A^{kl} A_{kl} g^{(0)}_{ij} \\ - 2\nabla^k \nabla_i A_{jk} + \frac{1}{n-1} \nabla_i \nabla_j R \end{aligned}$$

By the Ricci identity (A.6.1), we have:

$$\begin{aligned} \nabla^k \nabla_i A_{jk} &= \nabla_i \nabla^k A_{jk} - R_{ikj}{}^p A_{pk} - R_{ikk}{}^p A_{jp} \\ &= \frac{1}{2(n-1)} \nabla_i \nabla_j R - R_{ikj}{}^p A_{pk} + R_{ik} A^k_j \\ &= \frac{1}{2(n-1)} \nabla_i \nabla_j R - R_{ikj}{}^p A_{pk} + (n-2) A_{ik} A^k_j + \frac{1}{2(n-1)} R A_{ij} \end{aligned} \tag{2.1.9}$$

By the definition of Weyl tensor (A.9.3), we have:

$$\begin{aligned} R_{ikj}{}^p A_{pk} &= A^{kl} W_{ikjl} + A^{kl} (A \owedge g)_{ikjl} \\ &= A^{kl} W_{ikjl} + \frac{1}{n-1} A_{ij} + 2 A_{kl} A^{kl} g_{ij} - 4 A_{ik} A^k_j \end{aligned} \tag{2.1.10}$$

With Schouten tensor, Bach tensor can be rewritten as

$$B_{ij} = \Delta A_{ij} - \nabla^k \nabla_j A_{ik} + A^{kl} W_{ikjl} \tag{2.1.11}$$

In the end, we have:

$$(4-n)g_{ij}^{(2)} = B_{ij} + (n-4)A_{ik}A_j^k \tag{2.1.12}$$

This result shows that, when $n = 4$, if the Bach tensor does not vanish, the expansion is inconsistent. When $n$ is even and greater than 6, such tensor is called obstruction tensor $\mathcal{O}_{ij}$.

## 2.2 Property of Bach Tensor

In this section, we will introduce some properties of Bach tensor. We also have some variant forms for Bach tensor in Sec A.11.

**Proposition 2.2.1.** *Bach tensor is trace free.*

*Proof.* Bach tensor inherits this trace free property from Weyl tensor, which is the trace free part in Riemann curvature decomposition. □

**Proposition 2.2.2.** *Bach tensor is divergence free when $n = 4$.*

*Proof.* It is well-known that when $n \geq 4$, the divergence of Bach tensor is

$$\nabla^j B_{ij} = \frac{n-4}{(n-2)^2} C_{ijk} R^{jk} \tag{2.2.1}$$

where $C_{ijk}$ is Cotton tensor defined by

$$C_{ijk} = (n-2)\Big(\nabla_i A_{jk} - \nabla_j A_{ik}\Big) \tag{2.2.2}$$

All details are in Proposition A.11.2. This result also shows that when $n \geq 5$, Bach tensor is no longer the gradient of $L^2$ norm of Weyl curvature because the gradient of any Riemann functional has to be divergence free. □

**Remark 2.2.3.** *We remark that $C_{ijk}$ is skew-symmetric in the first two indices and trace-free in any two indices:*

$$C_{ijk} = -C_{jik} \tag{2.2.3}$$

$$g^{ij}C_{ijk} = g^{ik}C_{ijk} = 0 \tag{2.2.4}$$

**Proposition 2.2.4.** *If the Riemannian manifold $(M^n, g)$ is locally conformally flat or Einstein, then it is Bach flat, i.e., $B_{ij} = 0$.*

*Proof.* First, we say that a manifold is locally conformally flat if and only if its Weyl tensor vanishes[16, Page 29 Prop 1.62], therefore, its Bach tensor also vanishes.

Conversely, if $(M^n, g)$, $n \geq 4$, is Einstein, we have

$$R_{ij} = \frac{R}{n} g_{ij} \tag{2.2.5}$$

With Einstein condition (2.2.5), Weyl tensor can be written as:

$$\begin{aligned} W_{ikjl} &= R_{ikjl} - \frac{1}{n-2}(R_{ij}g_{kl} + R_{kl}g_{ij} - R_{il}g_{kj} - R_{kj}g_{il}) \\ &\quad + \frac{R}{(n-1)(n-2)}(g_{ij}g_{kl} - g_{il}g_{kj}) \\ &= R_{ikjl} - \frac{R}{n(n-1)}(g_{ij}g_{kl} - g_{il}g_{kj}) \end{aligned} \tag{2.2.6}$$

Since Weyl tensor is also trace free, we have:

$$R_{kl}W_{ikjl} = \frac{R}{n} g_{kl} W_{ikjl} = 0 \tag{2.2.7}$$

With (2.2.6) and (2.2.7), Bach tensor (2.0.1) will be:

$$\begin{aligned} B_{ij} &= \frac{1}{n-3}\nabla_k\nabla_l\left(R_{ikjl} - \frac{R}{n(n-1)}(g_{ij}g_{kl} - g_{il}g_{kj})\right) \\ &= \frac{1}{n-3}\nabla_k\nabla_l R_{ikjl} \end{aligned} \tag{2.2.8}$$

Combines with the second Bianchi identity, we conclude that

$$\begin{aligned} \nabla_l R_{ikjl} &= -\nabla_i R_{kljl} - \nabla_k R_{lijl} \\ &= -\nabla_i R_{kj} + \nabla_k R_{ij} \\ &= -\nabla_i \frac{R}{n} g_{kj} + \nabla_k \frac{R}{n} g_{ij} = 0 \end{aligned} \tag{2.2.9}$$

□

**Proposition 2.2.5** (Theorem 1.2, [9])**.** *If $(M^n, g)$ is a Bach flat gradient shrinking soliton, then it is either locally conformally flat or Einstein.*

**Proposition 2.2.6.** *Let $(M^4, g)$ be a closed four dimensional manifold, Bach flat metrics are the critical points of the conformally invariant functional on the space of metrics.*

$$\mathcal{F}_W = \int_M |W_g|^2 dV_g$$

*Proof.* We have all details in Appendix C.4 . □

**Remark 2.2.7.** *Another point of view to see the divergence free property of Bach tensor is that all of the gradients of Riemannian functional are divergence free(Proposition 4.11 in Page 119 [6]). Therefore, Bach tensor is divergence free in dimension 4, since it is the gradient of $\mathcal{F}_W$. In higher dimension, it is not divergence free, but the gradient of Weyl functional $\mathcal{B}$ remains divergence free.*

## 2.3 Gauss Bonnet Chern Formula

Give a closed Riemannian manifold $(M^4, g)$, the Riemann curvature decomposition is given by:

$$|Rm|^2 = |W|^2 + 2|E|^2 + \frac{1}{6}R^2 \tag{2.3.1}$$

We also recall the elementary symmetric polynomials

$$\sigma_k(\lambda_1, \lambda_2, \cdots, \lambda_n) = \sum_{i_1 < i_2 < \cdots < i_k} \lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_k}$$

where $\lambda_i's$ are the eigenvalues of the contract tensor $g^{-1}A$. We note that in dimension 4, $\sigma_2(g^{-1}A) = \frac{1}{24}R^2 - \frac{1}{2}|E|^2$ . For now, we simply write $\sigma_2(A)$ instead of $\sigma_2(g^{-1}A)$.

On a closed four manifold $M^4$, the Gauss-Bonnet-Chern formula is:

$$32\pi^2\chi(M^4) = \int_M |W|^2 - 2|E|^2 + \frac{1}{6}R^2 d\mu \tag{2.3.2}$$

In terms of $\sigma_2$, we have:

$$32\pi^2\chi(M^4) = \int_M |W|^2 + 4\sigma_2(A_g) d\mu \tag{2.3.3}$$

**Remark 2.3.1.** *By the conformal invariance of Weyl curvature tensor and Gauss-Bonnet-Chern, $\sigma_2(A)$ is also conformal invariant.*

## 2.4 Yamabe Problem

For a closed manifold $(M^n, g)$, one basic fact is that on such manifold, the Yamabe constant $Y(M, [g])$ can be achieved by a metric $g_0 \in [g]$ which is called the Yamabe metric with constant scalar curvature $R[g_0] = s_0$. Given a closed manifold $(M^n, g)$, the Yamabe constant is defined by:

$$Y_{[g]} = \inf_{\hat{g}\in[g]} \frac{\int_M R_{\hat{g}} d\mu_{\hat{g}}}{Vol(M^n, \hat{g})^{\frac{n-2}{n}}} \tag{2.4.1}$$

By the work of Trudinger, Aubin[52] and R.Schoen[46], the infimum of Yamabe functional for any conformal class $[g]$ can be achieved by some metric, and this so-called Yamabe metric $\hat{g} \in [g]$ has a constant scalar curvature.

For a four manifold with positive Yamabe constant, follow the solution for the Yamabe problem[52][46], we may assume that $g$ is the Yamabe metric which achieves Yamabe constant, then we have:

$$\int_M \sigma_2(A_g) d\mu \leq \int_M \frac{1}{24} R_g^2 d\mu = \frac{1}{24} \frac{\left(\int_M R_g d\mu\right)^2}{\int_M d\mu} \leq \frac{1}{24} \frac{\left(\int_M R_{g^4_{\mathbb{S}}} \mu_{g^4_{\mathbb{S}}}\right)^2}{\int_M d\mu_{g^4_{\mathbb{S}}}} = 16\pi^2 \tag{2.4.2}$$

The equality holds if and only if $(M^4, g)$ is conformal equivalent to the standard four sphere $(\mathbb{S}^4, g^4_{\mathbb{S}})$ with $Vol(\mathbb{S}^4) = \frac{8\pi^2}{3}$ and $R_{g^4_{\mathbb{S}}} = 12$.

## 2.5 Sobolev Constant

In this subsection, we introduce the Sobolev inequality on manifold. And we will show that in dimension 4, $L^2$ Sobolev inequality can be controlled by Yamabe constant.

**Sobolev Inequality**

Given a closed Riemannian manifold $(M^n, g)$, we define the Sobolev constant to be the best constant such that the following inequality holds:

$$\|u\|_{L^{\frac{2n}{n-2}}} \leq C_s(g) \left( \|\nabla u\|_{L^2} + Vol_g^{-\frac{2}{n}} \|u\|_{L^2} \right) \tag{2.5.1}$$

for any function $u \in C^1(M)$.

**Remark 2.5.1.** *The Sobolev inequality* (2.5.1) *is scale invariant.*

**Remark 2.5.2.** *In four dimension, we have:*

$$\|u\|_{L^4}^2 \leq C_s(g)\bigg( \|\nabla u\|_{L^2}^2 + Vol_g^{-\frac{1}{2}} \|u\|_{L^2}^2 \bigg) \tag{2.5.2}$$

We also introduce a multiplicative Sobolev inequality[33] here.

**Lemma 2.5.3.** *Given $(M^n, g)$ be a Riemannian manifold, we have the following multiplicative Sobolev inequality for all function $u \in C_0^1(M)$*

$$\|u\|_\infty \leq C_S C(m, n, p) \|u\|_m^{1-\alpha} (\|\nabla u\|_p + \|u\|_p)^\alpha \tag{2.5.3}$$

*in which $n < p \leq \infty$, $0 \leq m \leq \infty$ , $0 < \alpha \leq 1$ and satisfied*

$$\frac{1}{\alpha} = \big(\frac{1}{n} - \frac{1}{p}\big)m + 1$$

*Proof.* We present the detail proof in Theorem. E.3.1 □

**Sobolev Constant and Yamabe Constant**

In four dimension, we have the following conformal change of scalar curvature. If $u$ is a smooth function defined on manifold $(M^4, g)$, let $\hat{g} = u^2 g$, the scalar curvature is

$$\hat{R}u^3 = (6\Delta + R)u$$

The Yamabe constant can be written as:

$$Y_{[g]} = \inf_{u \neq 0} \frac{\int_M 6|\nabla u|^2 + R_g u^2 d\mu_g}{\Big(\int_M u^4 d\mu_g\Big)^{\frac{1}{2}}}$$

If we assume that $Y_{[g]} > 0$, we have:

$$\|u\|_{L^4}^2 \le \frac{6}{Y_{[g]}} \|\nabla u\|_{L^2}^2 + \frac{\max_{g\in[g]} R_g}{Y_{[g]}} \|u\|_{L^2}^2$$

Then we have:

$$C_s(g) \le \frac{\max\{6, R_g V^{\frac{1}{2}}\}}{Y_{[g]}} \tag{2.5.4}$$

Therefore, in dimension 4, if Yamabe constant has a lower bound and $R_g V^{\frac{1}{2}}$ has an upper bound, Sobolev constant is bounded above.

## 2.6 Conformal Ricci Flow

Given a Riemannian manifold $(M^n, g)$, the solution to Ricci flow is a one-parameter family of metric $g(t)$ defined by

$$\begin{cases} \frac{\partial}{\partial t} g = -2Ric[g] \\ g(0) = g_0 \end{cases}$$

For an arbitrary smooth initial metric, the flow will always last for a short time, but finite time singularities may occur which causes the flow to terminate. Ricci flow is used to prove Thurston's geometrization conjecture and the Poincaré conjecture in [42][43][44].

In [21], A. Fischer introduced a variation of the classical Ricci flow equation that modifies the unit volume constraint of that equation to a scalar curvature constraint. The resulting equations are named the conformal Ricci flow equations because of the role that conformal geometry plays in constraining the scalar curvature. These new equations are given by

$$\begin{cases} \frac{\partial g}{\partial t} = -2\Big(Ric + \frac{1}{n}Rg\Big) - pg \\ R[g] = -1 \end{cases}$$

for evolving metric $g$ and a scalar function $p$. The conformal Ricci flow equations are analogous to the Navier Stokes equations of fluid mechanics. Because of this analogy, the time-dependent function $p$ is called a pressure function and it serves as a Lagrange multiplier to conformally deform the metric flow so as to maintain the scalar curvature constraint.

A. Fischer proved the short time existence to this flow with restriction to negative Yamabe type. After that, P. Lu, J. Qing and Y, Zheng proved the short time existence for all Yamabe type in [38].

## 2.7 Conformal Bach Flow

Analog to the conformal Ricci flow, we propose the conformal Bach flow as follows. Suppose that $(M^n, g_0)$ is an $n$-dimensional Riemannian manifold with constant scalar curvature $s_0$ and $n \geq 4$, the conformal Bach flow is a family of metrics $\{g(t)\}_{t\in[0,T]}$ which satisfy

$$\begin{cases} \dfrac{\partial}{\partial t}g = 2(n-2)(B_g + pg) \quad \text{on } M \times [0,T], \\ R_{g(t)} = s_0 \quad \text{on } M \times [0,T], \end{cases} \tag{2.7.1}$$

where $p = p(x,t)$ is a family of functions on $M$. This is a fourth order evolution equation. The pressure term $p$ is the “conformal change” which keeps the metric having constant scalar curvature. One key observation is that the pressure function $p$ term also takes care of the role played by the term $\frac{1}{2(n-1)(n-2)}(\Delta R_g)g$ in [3].

Like conformal Ricci flow system in [21] and [38], we will first derive an equivalent form of conformal Bach flow which is coupled weakly-parabolic and elliptic equations system.

**Proposition 2.7.1.** *Conformal Bach flow* (2.7.1) *is equivalent to the following equations:*

$$\begin{cases} \frac{\partial}{\partial t} g_{ij} = 2(n-2)\Big(B_{ij} + \frac{1}{2(n-1)(n-2)}(\Delta R)g_{ij} + pg_{ij}\Big) \\ -(n-1)\Delta p - s_0 p = (n-2)A^{ij}B_{ij} - \nabla^i\nabla^j B_{ij} \end{cases} \tag{2.7.2}$$

*Proof.* Recall the variation of scalar curvature (B.6.1) is :

$$\frac{\partial}{\partial t} R = -\Delta H + \nabla^i\nabla^j h_{ij} - R^{ij}h_{ij} \tag{2.7.3}$$

In this case, $h_{ij} = 2(n-2)\Big(B_{ij} + \frac{1}{2(n-1)(n-2)}(\Delta R)g_{ij} + pg_{ij}\Big)$ and $H = tr_g h = \frac{n}{n-1}(\Delta)R + 2n(n-2)p$, therefore, we have:

$$\begin{aligned} \frac{\partial}{\partial t} R = -\Delta^2 R - \frac{1}{n-1} R\Delta R \\ - 2(n-2)\bigg((n-1)\Delta p + s_0 p - \nabla^i\nabla^j B_{ij} + R^{ij}B_{ij})\bigg) \end{aligned} \tag{2.7.4}$$

Since scalar curvature maintains a constant, left hand side vanished, so do the first two terms on the right hand side. Combine with the definition of Schouten tensor(A.9.2), we have

$$-(n-1)\Delta p - s_0 p = R_{ij}B_{ij} - \nabla_i\nabla_j B_{ij} = (n-2)A_{ij}B_{ij} - \nabla_i\nabla_j B_{ij} \tag{2.7.5}$$

Therefore, (2.7.1) is equivalent to:

$$\begin{cases} \dfrac{\partial}{\partial t} g_{ij} = 2(n-2)\Big(B_{ij} + \dfrac{1}{2(n-1)(n-2)}(\Delta R)g_{ij} + pg_{ij}\Big) \\ -(n-1)\Delta p - s_0 p = (n-2)A^{ij}B_{ij} - \nabla^i\nabla^j B_{ij} \end{cases} \tag{2.7.6}$$

□

**Remark 2.7.2.** *If $n = 4$, we have a simpler version because of the divergence free property of Bach tensor* (2.2.2)*:*

$$\begin{cases} \dfrac{\partial}{\partial t} g_{ij} = 4\Big(B_{ij} + \dfrac{1}{12}(\Delta R)g_{ij} + pg_{ij}\Big) \\ -3\Delta p - s_0 p = 2A_{ij}B_{ij} \end{cases} \tag{2.7.7}$$

Next, we will introduce some nice properties of conformal Bach flow.

**Lemma 2.7.3.** *Given a four dimensional closed manifold $(M^4, g)$, under the conformal Bach flow, the $L^2$ norm of Weyl curvature is non-increasing and the $\sigma_2$ integral is non-decreasing.*

*Proof.* Let $(M^4, g(t))$ be a solution to the conformal Bach flow, we directly compute:

$$\frac{d}{dt}\int_M |W|^2 dv = \int_M \langle -4B_{ij}, \frac{\partial}{\partial t} g(t)_{ij}\rangle dv = -16\int_M |B|^2 dv$$

Combine with the Gauss-Bonnet-Chern formula (2.3.3), the result follows. □

**Lemma 2.7.4.** *Given a four dimensional close manifold $(M^4, g)$, under the conformal Bach flow, the volume of the manifold has a lower bound.*

*Proof.* By a simple calculation, we have:

$$Vol = \int_M d\mu = \frac{24}{R^2_{g(t)}}\int_M \frac{1}{24}R^2_{g(t)} d\mu \geq \frac{24}{R^2_{g(t)}}\int_M \sigma_2[g(t)]d\mu \geq \frac{24}{R^2_{g(t)}}\int_M \sigma_2[g(0)]d\mu$$

Since the $\sigma_2$ integral is non-decreasing under conformal Bach flow, the volume of manifold will have a uniform lower bound depends on the initial curvature quantity. $\square$

## 2.8 Conformal Gradient Flow

In this section, we will introduce another geometric flow which evolves metrics under gradient of $L^2$ norm of Weyl curvature. On a closed n dimensional manifold, we define:

$$\mathcal{F}_W = \int_M |W|^2 d\mu \tag{2.8.1}$$

First, we recall that gradient of $L^2$ norm of Weyl curvature is given by:

$$\begin{aligned} grad\mathcal{F}_W = & -\frac{4(n-3)}{n-2}\Delta R_{ij} + \frac{2(n-3)}{(n-1)(n-2)}\Delta R g_{ij} + \frac{2(n-3)}{n-1}\nabla_i\nabla_j R \\ & - \frac{4(n-4)}{n-2}R^{pq}R_{ipjq} + 4R_{ip}R_j^p - \frac{4}{(n-1)(n-2)}RR_{ij} \\ & - 2R_{pqri}R^{pqr}{}_j + \frac{1}{2}|W|^2 g_{ij} \end{aligned} \tag{2.8.2}$$

The proof in given in Sec C.4. In dimension 4, as we mentioned before, this is Bach tensor. For the rest of this thesis, we denote $\mathcal{B} = grad\mathcal{F}_W$. We first introduce two properties of this tensor.

**Proposition 2.8.1.** *$\mathcal{B}$ is trace free and divergence free.*

*Proof.* These two properties comes from direct calculations. Another point of view to see the divergence free is that we know for any Riemann functional, then its gradient is divergence free. See details in [6, Definition 4.7, Definition 4.10, Proposition 4.11, Page 118-119] $\square$

Similar to conformal Bach flow, we first define the conformal gradient flow as:

$$\begin{cases} \dfrac{\partial}{\partial t}g = 2(n-2)(-\dfrac{1}{4(n-3)}\mathcal{B}(g(t)) + pg) \quad \text{on } M\times[0,T], \\ R_{g(t)} = s_0 \quad \text{on } M\times[0,T], \end{cases} \tag{2.8.3}$$

Next, we prove the following equivalent form for conformal gradient flow by explicit write out the second equation in (2.8.3).

**Proposition 2.8.2.** *The conformal gradient flow is equivalent to the follow system of partial differential equations.*

$$\begin{cases} \dfrac{\partial}{\partial t}g = 2(n-2)(-\dfrac{1}{4(n-3)}\mathcal{B} + \dfrac{1}{2(n-1)(n-2)}\Delta Rg + pg) \quad \textit{on } M\times[0,T], \\ -(n-1)\Delta p - s_0 p = -\frac{1}{4(n-3)}Ric\cdot\mathcal{B} \quad \textit{on } M\times[0,T], \end{cases} \tag{2.8.4}$$

*Proof.* Recall the variation of scalar curvature (B.6.1) is :

$$\frac{\partial}{\partial t}R = -\Delta H + \nabla^i\nabla^j h_{ij} - R^{ij}h_{ij} \tag{2.8.5}$$

In this case,

$$h = -\frac{n-2}{2(n-3)}\mathcal{B} + \frac{1}{(n-1)}\Delta Rg + 2(n-2)pg$$

and

$$H = tr_g h = \frac{n}{n-1}\Delta R + 2n(n-2)p$$

Therefore, (2.8.3) is equivalent to:

$$\begin{cases} \dfrac{\partial}{\partial t}g = 2(n-2)(-\dfrac{1}{4(n-3)}\mathcal{B} + \dfrac{1}{2(n-1)(n-2)}\Delta Rg + pg) \quad \text{on } M\times[0,T], \\ -(n-1)\Delta p - s_0 p = -\frac{1}{4(n-3)}Ric\cdot\mathcal{B} \quad \text{on } M\times[0,T], \end{cases} \tag{2.8.6}$$

$\square$

**Remark 2.8.3.** *We notice that both Bach tensor and* $\mathcal{B}$ *have the same leading terms up to a constant depending on the dimension of manifold, therefore, in this thesis, we consider the conformal Bach flow, all of the results are valid for conformal gradient flow, some of them are even simpler since* $\mathcal{B}$ *is divergence free for any dimension greater or equal to 4.*

# Chapter 3

# Short Time Existence to Conformal Bach Flow

In this chapter, we will prove the short time existence and uniqueness to the conformal Bach flow. After that, a bootstrap argument will be applied for the regularity. The main proof relies on the classical DeTurck's trick, which helps us deal with non-parabolic issue from Bach tensor. Then we use inverse function theorem and contractive mapping theorem in functional analysis to derive our results.

Recall that our conformal Bach flow is defined by:

$$\begin{cases} \dfrac{\partial}{\partial t} g = 2(n-2)(B + \dfrac{1}{2(n-1)(n-2)} \Delta R g + pg) \\ -(n-1)p - s_0 p = (n-2) A \cdot B - \nabla^2 B \\ g(0) = g_0 \\ R[g(t)] = s_0 \end{cases} \tag{3.0.1}$$

for some initial data $(M^n, g_0)$ with constant scalar curvature $R[g_0] = s_0$.

## 3.1 Linearization

In this section, we will introduce the symbol of operators. Most of the elementary definitions can be found in [15], but we will only discuss the Bach tensor and conformal Bach flow in the text.

**Linearization of a Nonlinear Operator**

The linearization is defined in analogy with the derivative of a function. For a nonlinear operator defined on vector bundle $V$, if $u : [0,1] \to C^\infty(V)$ is a time dependent section with

$$\begin{cases} u(0) = u_0 \\ u'(0) = v \end{cases} \tag{3.1.1}$$

we define the linearization of $L$ at $u_0$ to be the linear map $D[L] : C^\infty(V) \to C^\infty(V)$ so that

$$D[L](v) = \frac{d}{dt} L(u(t))\Big|_{t=0}$$

We now regard the Bach tensor $B$ as a nonlinear partial differential operator on the metric $g$, and it defines a map

$$B_g : C^\infty(S_2^+ T^* M) \longrightarrow C^\infty(S_2 T^* M)$$

where $S_2^+ T^* M$ is the space of positive definite symmetric 2 tensors.

**Proposition 3.1.1.** *Let $g(t)$ be a one parameter family of Riemannian metric on $M^n$ such that $g(0) = g_0$ and $\frac{d}{dt} g\Big|_{t=0} = h$, the linearization of Bach tensor at $t = 0$*

*is given by*

$$
\begin{aligned}
[D(B_g)(h)]_{ij} = & \frac{1}{2(n-2)}(-\Delta^2 h_{ij} + \Delta\nabla^k\nabla_j h_{ik} + \Delta\nabla^k\nabla_i h_{jk}) \\
& - \frac{1}{2(n-1)(n-2)}\Delta\Big[\nabla^k\nabla^l h_{kl}g_{ij} + \nabla_i\nabla_j(Tr_g(h)) - \Delta Tr_g(h)g_{ij}\Big] \\
& - \frac{1}{2(n-1)}\nabla_i\nabla_j\nabla^k\nabla^l h_{kl}
\end{aligned}
$$

*Proof.* Bach tensor is defined in (2.0.1),

$$B_{ij} = \frac{1}{n-3}\nabla^k\nabla^l W_{ikjl} + A^{kl}W_{ikjl}$$

We only need to calculate the first term since it contains fourth order derivatives of metric. Based on our results in appendix (B.4.1) (B.5.1) and (B.6.1), the result follows. □

**Symbol of a Nonlinear Differential Operator**

Let $V$ and $W$ be vector bundles over $M^n$, let

$$L : C^\infty(V) \longrightarrow C^\infty(W)$$

be a linear differential operator of order $k$, written as

$$L(V) := \sum_{|\alpha|\leq k} L_\alpha \partial_\alpha V$$

where $L_\alpha \in Hom(V, W)$ is a bundle homomorphism, i.e., a linear map on each single fiber and $\alpha$ is a multi-index. If $\zeta \in C^\infty(T^*M)$, the we say the total symbol of differential operator $L$ in the direction $\zeta$ is

$$\sigma[L](\zeta) := \sum_{|\alpha|\leq k} L_\alpha(\prod_j \zeta^j)$$

and the principal symbol of $L$ in the direction $\zeta$ is the top degree terms in the total symbol, which is denoted by

$$\hat{\sigma}[L](\zeta) := \sum_{|\alpha|=k} L_\alpha(\prod_j \zeta^j)$$

**Proposition 3.1.2.** *The principal symbol in the direction $\zeta$ of the linear differential operator $D(B_g)$ is*

$$\hat{\sigma}[D(B_g)](\zeta) : S_2T^*M \to S_2T^*M$$

*which is defined by:*

$$\begin{aligned}\Big[\hat{\sigma}[D(B_g)](\zeta)(h)\Big]_{ij} = \frac{n-3}{2(n-2)}\Big(&\zeta_l\zeta_l\zeta_k\zeta_j\tilde{g}_{ik} + \zeta_l\zeta_l\zeta_k\zeta_i h_{jk} - \zeta_l\zeta_l\zeta_k\zeta_k h_{ij}\Big)\\ &+ \frac{n-3}{2(n-1)(n-2)}\Big(\zeta_l\zeta_l\zeta_k\zeta_k Tr_g(h)g_{ij} - \zeta_k\zeta_k\zeta_i\zeta_j Tr_g(h)\Big)\\ &- \frac{n-3}{2(n-1)(n-2)}\zeta_q\zeta_q\zeta_k\zeta_l\tilde{g}_{kl}g_{ij} - \frac{n-3}{2(n-1)}\zeta_i\zeta_j\zeta_k\zeta_l\tilde{g}_{kl}\end{aligned}$$

*Proof.* By the definition of symbols, we simply replace all of the covariant derivatives in (3.1.1) by vectors $\zeta$ with the same index. □

**Ellipticity of a Nonlinear Operator**

A linear partial differential operator $L$ is said to be elliptic if its principal symbol $\hat{\sigma}[L]\zeta$ is an isomorphism whenever $\zeta$ is non-zero. Similarly, a nonlinear differential operator $\mathcal{M}$ is said to be elliptic is its linearization $D\mathcal{M}$ is elliptic.

However, one key observation is that the principal symbol of Bach tensor has some degeneracies ,therefore, Bach tensor is not elliptical.

Another way to see this is that from the result of principal symbols in (3.1.2), we choose $\zeta = \delta_i^1\zeta_i$, and the principal symbol in this direction will be:

$$
[\hat{\sigma}[D(B_g)](\zeta)(h)]_{ij} = \begin{cases} -\dfrac{n-3}{2(n-2)}\zeta_1^4 & \text{if } i \neq j \text{ and } i, j \neq 1 \\ -\dfrac{n-3}{2(n-2)}\zeta_1^4\left(h_{ii} - \dfrac{1}{n-1}\displaystyle\sum_{k=2}^{n} h_{kk}\right) & \text{if } i = j \neq 1 \\ 0 & \text{if } i = 1 \text{ and } j \neq 1 \\ 0 & \text{if } j = 1 \text{ and } i \neq 1 \\ 0 & \text{if } i = j = 1 \end{cases}
$$

It is clear that there are lots of degeneracies.

## 3.2 DeTurck's Trick

**Introduction**

For Ricci flow, the local existence and uniqueness on compact manifolds were first established by Hamilton in [25]. After that, DeTurck provided an elegant proof in [18], the method is called DeTurck's Trick nowadays. Essentially, DeTurck's trick will eliminate the degeneracy form Ricci curvature tensor by adding an extra term.

**Ricci Flow**

We will do a quick review for the DeTurck Ricci flow. The motivation comes from the linearization of Ricci tensor. Recall this variation formula in (B.5.1):

$$
\frac{\partial}{\partial t}R_{ij} = -\frac{1}{2}\Big(\Delta h_{ij} + \nabla_i\nabla_j Tr_g(h) - \nabla^p\nabla_i h_{jp} - \nabla^p\nabla_j h_{ip}\Big) \tag{3.2.1}
$$

combine with Ricci identity (A.6.1), we have another form of this variation

$$
\frac{\partial}{\partial t}R_{ij} = -\frac{1}{2}\big(\Delta_L h_{ij} + \nabla_i\nabla_j Tr_g(h) + \nabla_i\nabla^p h_{pj} + \nabla_j\nabla^p h_{pi}\big) \tag{3.2.2}
$$

where $\Delta_L$ is called Lichnerowicz Laplacian[36], and defined by

$$\Delta_L h_{ij} = \Delta h_{ij} + 2R_{ipjq}h^{pq} - R_{iq}h_j^p - R_{jp}h_i^p \tag{3.2.3}$$

The degeneracy comes form the last three terms in (3.2.2). We may write it as

$$-2[D(Ric_g)(h)]_{ij} = \Delta_L h_{ij} - \nabla_i V_j - \nabla_j V_i \tag{3.2.4}$$

in which,

$$V_i = \frac{1}{2}g^{pq}\Big(\nabla_p h_{qi} + \nabla_q h_{pi} - \nabla_i h_{pq}\Big) \tag{3.2.5}$$

This is exact the same as the variation formula of Christoffel symbol(B.3.1).

**Ricci DeTurck Flow**

Based on this observature, DeTurck modified the Ricci flow by adding an extra term which cancels last three terms in (3.2.2).

$$\begin{cases} \frac{\partial g}{\partial t} = -2Ric + \mathcal{L}_V(g) \\ g(0) = g_0 \end{cases} \tag{3.2.6}$$

$V$ is the vector field defined as

$$V^k = g^{ij}\Big(\Gamma(g)_{ij}^k - \Gamma(\tilde{g})_{ij}^k\Big) \tag{3.2.7}$$

where $\Gamma(\tilde{g})_{ij}^k$ is the Christoffel symbol with respect to arbitrary fixed metric $\tilde{g}$. We need a difference of two Christoffel symbols to turn in into a tensor. And this so called Ricci-DeTurck flow is a parabolic partial differential equation system, the existence and uniqueness are well understand.

**DeTurck Conformal Bach Flow**

Now we will discuss the conformal Bach flow, which is a little bit different. Recall that the linearization of Bach tensor is

$$
\begin{aligned}
[D(B_g)(h)]_{ij} = & \frac{1}{2(n-2)}(-\Delta^2 h_{ij} + \Delta\nabla^k\nabla_j h_{ik} + \Delta\nabla^k\nabla_i h_{jk}) \\
& - \frac{1}{2(n-1)(n-2)}\Delta\Big[\nabla^k\nabla^l h_{kl} g_{ij} + \nabla_i\nabla_j(Tr_g(h)) - \Delta Tr_g(h) g_{ij}\Big] \\
& - \frac{1}{2(n-1)}\nabla_i\nabla_j\nabla^k\nabla^l h_{kl}
\end{aligned}
$$

Note that schematically the Bach tensor can be written as

$$
B_{ij} = \frac{1}{n-2}\Delta R_{ij} - \frac{1}{2(n-1)(n-2)}(\Delta R)g - \frac{1}{2(n-1)}\nabla_i\nabla_j R + \text{ lower order terms}
$$

We notice that the leading term of Bach tensor is $\Delta Ric$, hence we may modify the standard choice of the vector field in the DeTurck's trick for Ricci flow by adding $\Delta_g$ to eliminate the degeneracy in $\frac{1}{n-2}\Delta R_{ij}$ caused by the symmetry of diffeomophisms. We define the first vector field to be:

$$
W_1^k = -g^{ij}\Delta_g\left(\Gamma_{ij}^k(g) - \Gamma_{ij}^k(\tilde{g})\right) \tag{3.2.8}
$$

for some fixed metric $\tilde{g}$.

The term $-\frac{1}{2(n-1)(n-2)}(\Delta R)g$ has to be taken care of as in [3] which partly explains why we has that extra term in our conformal Bach flow rather than the original one

Lastly the term $-\frac{1}{2(n-1)}\nabla_i\nabla_j R$ is of Hessian-type and can be taken care off by modifying the choice of the vector field, we define the second vector field to be:

$$
W_2^k = \frac{n-2}{2(n-1)}(\nabla^k R) \tag{3.2.9}
$$

Following the choice of the vector fields and fixing a $\tilde{g}$ with enough smoothness as a background metric, we define vector field

$$W^k = -g^{ij}\Delta_g\left(\Gamma^k_{ij}(g) - \Gamma^k_{ij}(\tilde{g})\right) + \frac{n-2}{2(n-1)}(\nabla^k R). \tag{3.2.10}$$

Recall that the Lie derivative $(\mathcal{L}_W g)_{ij} = \nabla_i W_j + \nabla_j W_i$. We define the DeTurck modified conformal Bach flow as

$$\begin{cases} \dfrac{\partial}{\partial t} g = 2(n-2)(B + \dfrac{1}{2(n-1)(n-2)}\Delta R g + pg) + \mathcal{L}_W g \\ -(n-1)p - s_0 p = (n-2)A \cdot B - \nabla^2 B \end{cases} \tag{3.2.11}$$

We will consider its initial value problem $g(0) = g_0$ where $R_{g_0} = s_0$.

Suppose we have a solution $(g(t), p(t)), t \in [0,T]$, of (3.2.11) such that each $g(t)$ is a complete metric, we call $g(t)$ a complete solution. Using the vector field $W$ we define an one-parameter family of diffeomorphism $\varphi_t : M \to M$, $t \in [0,T]$, by solving ordinary differential equations for each $x \in M$

$$\frac{\partial}{\partial t}\varphi_t(x) = -W_{g(t)}(\varphi_t(x)) \quad \text{with} \quad \varphi_0(x) = x. \tag{3.2.12}$$

**Lemma 3.2.1.** *Let $(g(t), p(t))$ be a complete solution of* (3.2.11) *on manifold $M^n$ with initial metric $g_0$ whose scalar curvature $R_{g_0} = s_0$. Let $\varphi_t$ be the solution of* (3.2.12)*. We define $\hat{g}(t) = \varphi_t^* g(t)$ and $\hat{p}(x,t) = p(\varphi_t(x), t)$. Then $(\hat{g}(t), \hat{p}(t))$ is a solution of conformal Bach flow with initial condition $g(0) = g_0$.*

*Proof.* We denote every operator with respect to $\hat{g}$ with a hat over it. A direct computation gives us:

$$\begin{aligned}\frac{\partial}{\partial t}\hat{g}(t) &= \varphi_t^*(\frac{\partial}{\partial t}g(t)) + \frac{\partial}{\partial s}\Big|_{s=0}(\varphi_{t+s}^* g(t)) \\ &= 2(n-2)\varphi_t^*(B + \frac{1}{2(n-1)(n-2)}(\Delta R)g + pg) \\ &+ \varphi_t^*(\mathcal{L}_{W_g}g) - \mathcal{L}_{(\varphi_t^{-1})_* W_g}(\varphi_t^* g)g(t)) \\ &= 2(n-2)(\hat{B} + \frac{1}{2(n-1)(n-2)}(\hat{\Delta}\hat{R})\hat{g} + \hat{p}\hat{g})\end{aligned}$$

This verifies the first equation in (3.0.1). The second equation in (3.0.1) follows from the same calculation. □

**Remark 3.2.2.** *For the conformal gradient flow* (2.8.4)*, the choice of DeTurck's terms are exactly the same. We define the DeTurck modified conformal gradient flow as*

$$\begin{cases}\frac{\partial}{\partial t}g = 2(n-2)(\mathcal{B} + \frac{1}{2(n-1)(n-2)}\Delta R g + pg) + \mathcal{L}_W g \\ -(n-1)p - s_0 p = (n-2)A \cdot \mathcal{B}\end{cases} \tag{3.2.13}$$

## 3.3 Inverse Function Theorem

To prove the short time existence theorem for the conformal Bach flow, we first prove the existence to the DeTurck conformal Bach flow via the inverse function theorem we will introduce in this subsection. After that, the lemma(3.2.1) we proved before will give us the short time existence to the original conformal Bach flow.

**Theorem 3.3.1.** *Suppose that $X$ and $Y$ are Banach spaces and*

$$\mathcal{M} : X \to Y$$

*is at least a $C^1$ map. Let $D\mathcal{M}$ be its differential. Suppose that there exists a point $x_0 \in X$ and there are positive numbers $C$ and $\epsilon$ such that*

(a) $\|(D\mathcal{M}(x))^{-1}\| \leq C$, *for any* $x \in B_\epsilon(x_0)$,

(b) $\|D\mathcal{M}(x_1) - D\mathcal{M}(x_2)\| \leq \frac{1}{2C}$, *for any* $x_1, x_2 \in B_\epsilon(x_0)$,

*Then if $\mathcal{M}$ satisfies*

$$\|\mathcal{M}(x_0)\| \leq \frac{\epsilon}{2C}$$

*there is an $x \in B_\epsilon(x_0)$ such that $\mathcal{M}(x) = 0$.*

*Proof.* We are using the Newton's law to construct a convergent sequence both spaces. We start with $x_0$, let $x_1$ be defined as

$$x_1 = x_0 - [D\mathcal{M}(x_0)]^{-1} \cdot \mathcal{M}(x_0) \tag{3.3.1}$$

Inductively, we have:

$$x_{i+1} = x_i - [D\mathcal{M}(x_i)]^{-1} \cdot \mathcal{M}(x_i) \tag{3.3.2}$$

We want to show that $\{x_i\}_{i=1}^{\infty}$ converges to the solution of $\mathcal{M}(x) = 0$. The first observation is

$$\|x_1 - x_0\| = \left\|[D\mathcal{M}(x_0)]^{-1} \cdot \mathcal{M}(x_0)\right\| \leq \left\|[D\mathcal{M}(x_0)]^{-1}\right\| \cdot \|\mathcal{M}(x_0)\| = \frac{\epsilon}{2}$$

by our assumption. Next, fundamental theorem of calculus tells us

$$\begin{aligned}
\mathcal{M}(x_1) &= \mathcal{M}(x_0) + \int_0^1 [D\mathcal{M}(x_0 + t(x_1 - x_0))] \cdot (x_1 - x_0) dt \\
&= \mathcal{M}(x_0) - \int_0^1 [D\mathcal{M}(x_0 + t(x_1 - x_0))] \cdot [D\mathcal{M}(x_0)]^{-1} \cdot \mathcal{M}(x_0) dt \\
&= \mathcal{M}(x_0) \int_0^1 [D\mathcal{M}(x_0) - D\mathcal{M}(x_0 + t(x_1 - x_0))] \cdot [D\mathcal{M}(x_0)]^{-1} dt \\
&\leq \mathcal{M}(x_0) \int_0^1 \frac{1}{2C} \cdot C dt \\
&= \frac{1}{2}\mathcal{M}(x_0)
\end{aligned}$$

In summary, we have the following results:

(a) $\|x_1 - x_0\| \leq \frac{1}{2}\epsilon$

(b) $\mathcal{M}(x_1) \leq \frac{1}{2}\mathcal{M}(x_0)$

Therefore, $\{x_i\}_{i=1}^{\infty}$ converges to some point $x \in B_\epsilon(x_0)$ such that $\mathcal{M}(x) = 0$. □

## 3.4 Partial Differential Equation Theorems

In this section, we will introduce the functional spaces we are focusing on and provide some key theorems we are going to use in our proof of the main theorem.

**Parabolic Hölder Space**

In this subsection, we will state the convention and notation we adopt for parabolic Hölder space. There are many books that are good for references in linear and nonlinear systems of parabolic equations. We will mostly use the book [39], in particular Theorem 5.1.21 in [39] for existence and standard estimates. We adopt definitions of parabolic Hölder spaces from [[39], p. 175-177]. We use the same notations for parabolic Hölder spaces for functions and tensor fields when there is no confusion. To define the norms for tensor fields we may use the initial metric and local coordinate charts.

Let $u$ be a function defined on $\bar{\Omega} \subset \mathbb{R}^n$, we define the Banach spaces of Hölder continuous functions $C^\alpha(\bar{\Omega})$, $C^{k+\alpha}(\bar{\Omega})$ for $k \in \mathbb{N}$ and $\alpha \in (0,1)$ to be

$$C^\alpha(\bar{\Omega}) := \left\{u \in C(\bar{\Omega}) : \|u\|_{C^\alpha} < \infty\right\} \tag{3.4.1}$$

where the norm $\|\cdot\|_{C^\alpha}$ defined as follows:

$$\|\cdot\|_{C^\alpha} = \|u\|_\infty + [u]_{C^\alpha} \tag{3.4.2}$$

and $[f]_{C^\alpha}$ is called the semi-norm,

$$[u]_{C^\alpha} = \sup_{x,y\in\bar{\Omega}} \frac{\|u(x) - u(y)\|}{|x-y|^\alpha} \tag{3.4.3}$$

Consequently, we define the $C^{k+\alpha}(\bar{\Omega})$ for $k \in \mathbb{N}$ and $\alpha \in (0,1)$ to be

$$C^{k+\alpha}(\bar{\Omega}) = \left\{u \in C^k(\bar{\Omega}) : u^{(k)} \in C^\alpha(\bar{\Omega})\right\} \tag{3.4.4}$$

and the norm $\|\cdot\|_{C^{k+\alpha}}$ is defined as

$$\|u\|_{C^{k+\alpha}} = \sum_{i=0}^{k} \left\|u^{(i)}\right\|_\infty + [u^{(k)}]_{C^\alpha} \tag{3.4.5}$$

Now, for a non-negative integer $k \in \mathbb{N}$ and $l \in \{0,1\}$, we define the parabolic Hölder space $C^{k+\alpha,l}(\bar{\Omega} \times [0,T])$ to be

$$C^{k+\alpha,l}(\bar{\Omega} \times [a,b]) = \left\{u \in C^{k,l}(\bar{\Omega} \times [0,T]), \|u\|_{C^{k+\alpha,l}} < \infty\right\} \tag{3.4.6}$$

where

$$\begin{cases} C^{k,l}(\bar{\Omega}\times[a,b]) = u : \begin{Bmatrix} u(x,\cdot)\in C^l([a,b]) \text{ for all } x\in\bar{\Omega} \\ u(\cdot,t)\in C^k(\bar{\Omega}) \text{ for all } t\in[a,b] \end{Bmatrix} \\ \|u\|_{C^{k+\alpha,l}} = \sum_{|\alpha|=0}^{k} \left\|D^{|\alpha|}u\right\|_\infty + \sum_{\beta=0}^{b} \left\|\partial_t^\beta u\right\|_\infty + [D^{k,l}u]_{C^\alpha} \end{cases} \tag{3.4.7}$$

**Schauder Estimate for Elliptic and Parabolic PDE**

Before we proceeding to the main proof, we need some preparation of Hölder estimates.

**Lemma 3.4.1.** *Let $k\geq 2$ be a constant. There is a constant $C$ independent of $T$ such that for any function $h\in C^{k+\alpha,1}(M^n\times[0,T])$ and $t_1,t_2\in[0,T]$, we have the following estimate*

$$\|h(\cdot,t_1)-h(\cdot,t_2)\|_{C^{k-2+\alpha,0}} \leq C\cdot|t_1-t_2|\cdot\|h\|_{C^{k+\alpha,1}} \tag{3.4.8}$$

In the following lemma, we will introduce a classical Schauder estimate for parabolic partial differential equation. We denote an elliptic operator of order $2m$ by $L$ which defined in local coordinate system as follows,

$$L = \sum_{|\alpha|=0}^{2m} a_\alpha D^\alpha$$

where $\alpha=(\alpha_1,\alpha_2,\cdots,\alpha_n)$ is a multi-index and $D^\alpha = \partial_{x^1}^{\alpha_1}\partial_{x^2}^{\alpha_2}\cdots\partial_{x^n}^{\alpha_n}$ is the spatial derivative.

**Lemma 3.4.2.** *Let $L$ be a linear elliptic differential operator of order $2m$ with $C^{\alpha,0}$-coefficients on a closed manifold $M^n$ acting on tensors. Then for every $f\in$*

*$C^{\alpha,0}(M^n \times [0,T])$, there is a unique solution $u \in C^{2m+\alpha,1}(M^n \times [0,T])$ of*

$$\frac{\partial}{\partial t}u = Lu + f \tag{3.4.9}$$

*And there is a constant $C$ such that the solution $u$ satisfies the following estimate:*

$$\|u\|_{C^{2m+\alpha,1}} \le C\,\|f\|_{C^{\alpha,0}} \tag{3.4.10}$$

*Furthermore, if the function $f$ has a better regularity, i.e., $f \in C^{k+\alpha,0}(M^n \times [0,T])$, then the existence theorem holds and the estimate will be also improved:*

$$\|u\|_{C^{2m+k+\alpha,1}} \le C\,\|f\|_{C^{k+\alpha,0}} \tag{3.4.11}$$

**Lemma 3.4.3.** *Let $g(t)$, $t \in [0,T]$, be a family of $C^{2+\alpha,0}-$metric such that the elliptic operator $(n-1)\Delta_{g(t)} + s_0$ is invertible for all $t \in [0,T]$. Then the following elliptic equation*

$$[(n-1)\Delta_{g(t)} + s_0]p(t) = \gamma(t) \tag{3.4.12}$$

*has a unique solution $p \in C^{2+\alpha,0}(M^n \times [0,T])$ for each $\gamma(t) \in C^{\alpha,0}(M^n \times [0,T])$. And there is a constant $C$ such that we have the following estimate:*

$$\|p\|_{C^{2+\alpha,0}} \le C\,\|\gamma\|_{C^{\alpha,0}} \tag{3.4.13}$$

*Moreover, if $g(t)$, $t \in [0,T]$ is a family of $C^{2+k+\alpha,0}-$metric and $\gamma \in C^{k-2+\alpha,0}(M^n \times [0,T])$, then the unique solution $p$ also has a better regularity,*

$$\|p\|_{C^{2+k+\alpha,0}} \le C\,\|\gamma\|_{C^{k+\alpha,0}} \tag{3.4.14}$$

## 3.5 Short Time Existence

In this section, we are going to prove our main result, the short time existence. Based on (3.2.1), we focus on the DeTurck conformal Bach flow (3.2.11). This is a partial differential equation system coupled with a parabolic equation and a elliptic one. Therefore, we need to decouple the system first.

**Decoupling**

Let $g(t) \in C^{4+\alpha}(M)$ such that the operator $(n-1)\Delta_{g(t)} + s_0$ is invertible. We define an operator $\mathcal{P}$

$$\mathcal{P} : C^{4+\alpha}(M) \to C^{2+\alpha}(M) \tag{3.5.1}$$

such that the image $\mathcal{P}(g)$ is a solution to the following elliptic equation

$$[(n-1)\Delta_{g(t)} + s_0]p(t) = -(n-2)A_{g(t)} \cdot B_{g(t)} + \nabla^2_{g(t)} B_{g(t)} \tag{3.5.2}$$

And we have the following lemma.

**Lemma 3.5.1.** *Let $M^n$ be a closed n-dimensional manifold with metric $g(t) \in C^{4+\alpha}(M \times [0,T])$. Suppose that the elliptic operator $(n-1)\Delta_g + s_0$ is invertible. Then there are positive constants $\epsilon$ and $C$ such that the operator $\mathcal{P}$ we defined in* (3.5.1) *satisfies the following Lipschitz property*

$$\|\mathcal{P}(g_1) - \mathcal{P}(g_2)\|_{C^{2+\alpha}} \leq C \|g_1 - g_2\|_{C^{4+\alpha}} \tag{3.5.3}$$

*Proof.* We define $T = -(n-2)A \cdot B + \nabla^2 B$ to be an operator acting on space of Riemannian manifold. Let $g_1, g_2 \in C^{4+\alpha}(M \times [0,T])$, then we have:

$$[(n-1)\Delta_{g_i}]\mathcal{P}(g_i) = T(g_i) \tag{3.5.4}$$

for $i = 1, 2$. By a simple telescoping idea, we have:

$$[(n-1)\Delta_{g_1}](\mathcal{P}(g_1) - \mathcal{P}(g_2)) = (n-1)(\Delta_{g_2} - \Delta_{g_1})\mathcal{P}(g_2) + T(g_1) - T(g_2) \quad (3.5.5)$$

we see that the right hand side of this equation has the following estimate

$$\begin{cases} \|(\Delta_{g_2} - \Delta_{g_1})\mathcal{P}(g_2)\|_{C^{\alpha,0}} \leq C\,\|g_2 - g_1\|_{C^{2+\alpha,0}} \\ \|T(g_1) - T(g_2)\|_{C^{\alpha,0}} \leq C\,\|g_2 - g_1\|_{C^{4+\alpha,0}} \end{cases} \quad (3.5.6)$$

for some universal constant $C$. Follow by these two estimate and Lemma (3.4.3), we have:

$$\|\mathcal{P}(g_1) - \mathcal{P}(g_2)\|_{C^{2+\alpha}} \leq C\,\|g_1 - g_2\|_{C^{4+\alpha}}$$

$\square$

**Remark 3.5.2.** *The second estimate in* (3.5.6) *is a direct result from lemma* (2.2.2)*. Recall that the divergence of Bach tensor is given by*

$$\nabla^j B_{ij} = \frac{n-4}{(n-2)^2} C_{ijk} R^{jk}$$

*The Bach tensor can be viewed as a fourth order operator, but its divergence is actually of third order, this is why we have the estimate above.*

Once we define the operator $\mathcal{P}$, we now rewrite the DeTurck conformal Bach flow as

$$\begin{cases} \frac{\partial}{\partial t} g - 2(n-2)(B + \frac{1}{2(n-1)(n-2)}\Delta R g(t)) - \mathcal{L}_W g(t) - 2(n-2)\mathcal{P}[g(t)]g(t) = 0 \\ g(0) = g_0 \\ R[g(t)] = s_0 \end{cases} \quad (3.5.7)$$

For our convenience, we define another operator $\mathcal{F}$ to be:

$$\mathcal{F}[g(t)] = 2(n-2)(B + \frac{1}{2(n-1)(n-2)}\Delta R g(t)) + \mathcal{L}_W g(t) \tag{3.5.8}$$

and we consider the following decoupled DeTurck conformal Bach flow:

$$\begin{cases} \frac{\partial}{\partial t} g - \mathcal{F}[g(t)] - 2(n-2)\mathcal{P}[g(t)]g(t) = 0 \\ g(0) = g_0 \\ R[g(t)] = s_0 \end{cases} \tag{3.5.9}$$

Clearly, this equation is equivalent to (3.2.11) and we define the following operator:

$$\mathcal{M}[g(t)] = \frac{\partial}{\partial t} g - \mathcal{F}[g(t)] - 2(n-2)\mathcal{P}[g(t)]g(t) \tag{3.5.10}$$

Our target is to solve $\mathcal{M}[g(t)] = 0$ by using the inverse function theorem (3.3.1).

**Linearization of Operator $\mathcal{M}$**

We now compute the linearization of $\mathcal{M}[g(t)]$ by computing $\mathcal{F}[g(t)]$ and $\mathcal{P}[g(t)]$ separately.

Recall that the definition of linearization. Let $g_s = g + sh$ where $s \in (-\epsilon, \epsilon)$ and $h$ is a symmetric 2 tensor. The linearization of operator $\mathcal{M}[g(t)]$ is

$$D\mathcal{M}[g(t)] := \frac{\partial}{\partial s}\bigg|_{s=0} \mathcal{M}[g(t)] \tag{3.5.11}$$

For the first part $\mathcal{F}[g(t)]$, we recall that the definition of Bach tensor.

**Lemma 3.5.3.** *The leading term of Bach tensor is*

$$B_{ij} = \frac{1}{n-2}R_{ij} - \frac{1}{2(n-1)}\nabla_i\nabla_j R - \frac{1}{2(n-1)(n-2)}(\Delta R)g_{ij} + \textit{Lower order terms}$$

*Proof.* To see this, we use an alternate form of Bach tensor in terms of Schouten tensor and Weyl curvature.

$$B_{ij} = \Delta A_{ij} - \nabla^k \nabla_i A_{jk} + A^{kl} W_{ikjl}$$

The last term only contains the derivative of metric up to second order, therefore, we only consider the first two term. With Ricci identity and contracted Bianchi identity, one can calculate the following results.

$$\begin{aligned}
\Delta A_{ij} - \nabla^k \nabla_i A_{jk} &= \frac{1}{n-2}\left(\Delta R_{ij} - \frac{1}{2(n-1)}(\Delta R) g_{ij}\right) \\
&\quad - \frac{1}{n-2}\left(\nabla_i \nabla^k R_{jk} - \frac{1}{2(n-1)}\nabla_i \nabla_j R\right) + \text{Lower order terms} \\
&= \frac{1}{n-2}\Delta R_{ij} + \frac{1}{2(n-1)}\nabla_i \nabla_j R - \frac{1}{2(n-1)(n-2)}(\Delta R) g_{ij} \\
&\quad + \text{Lower order terms}
\end{aligned}$$

The lower order terms come from the Ricci identity when we exchange the covariant derivatives. □

From above lemma (3.5.3), it is clear that the term $\frac{1}{2(n-1)(n-2)}(\Delta R) g_{ij}$ is taken care of by the modification. As for the first two terms, the DeTurck's trick will exactly cancels most of them. Recall that our vector filed is given in (3.2.10). Schematically, we are doing the same thing as Ricci flow case. We refer the details in [16] Thm 2.43. By adding the DeTurck term, the linearization of Ricci flow is

$$\left.\frac{\partial}{\partial s}\right|_{s=0} (-2R_{ij} + \mathcal{L}_W g) = \Delta_L h_{ij} + \text{Lower order terms} \tag{3.5.12}$$

where $\Delta_L$is called Lichnerowicz Laplacian, and defined by (3.2.3).

Similar to the Ricci curvature, the linearization of $\mathcal{F}$ is

$$\frac{\partial}{\partial s}\Big|_{s=0}\mathcal{F}[g(t)] = -\Delta\Delta_L h + \sum_{k=0}^{2} M_k * \nabla^k h \tag{3.5.13}$$

In this result, we replace the lower order terms by $\sum_{k=0}^{2} M_k * \nabla^k h$ where $M_k$'s are curvature quantities. One may easily verified this results. Also, in (3.5.13), the $k$ is not for contraction, it is just the index, and the aster symbol means contraction of tensor quantities, we will use this symbol a lot through this thesis.

The linearization of $\mathcal{P}[g(t)]$ denoted by $\mathcal{P}'_g(h) = \frac{\partial}{\partial s}\big|_{s=0}\mathcal{P}[g(t)]$. To find $\mathcal{P}'_g(h)$ we compute the linearization of the both sides in equation ((3.5.2)) with $g = g_s$. We first have the following lemma:

**Lemma 3.5.4.** *Let $g_s = g + sh$ where $s \in (-\epsilon, \epsilon)$ and $h$ is a symmetric 2 tensor. The linearization of rough Laplacian is*

$$\frac{\partial}{\partial s}\Big|_{s=0}\Delta_{g_s} = -h_{ij}\nabla^i\nabla^j - \nabla^i h_{ij}\nabla^j + \nabla^i Tr_g(h)\nabla_i \tag{3.5.14}$$

*Proof.* Rough Laplacian is defined by

$$\Delta_g = g^{ij}\nabla_i\nabla_j$$

therefore, the variation will be

$$\begin{aligned}
\frac{\partial}{\partial s}\Big|_{s=0}(g^{ij}\nabla_i\nabla_j) &= -h^{ij}\nabla_i\nabla_j + g^{ij}\frac{\partial}{\partial s}\Gamma^k_{ij}\nabla_k \\
&= -h^{ij}\nabla_i\nabla_j + \frac{1}{2}g^{ij}g^{kl}(\nabla_i h_{jl} + \nabla_j h_{il} - \nabla_l h_{ij})\nabla_k \\
&= -h_{ij}\nabla^i\nabla^j - \nabla^i h_{ij}\nabla^j + \nabla^i Tr_g(h)\nabla_i
\end{aligned}$$

in which we used (B.1.1) and (B.3.1). □

With this lemma, for the left hand side, we have

$$\begin{aligned}
&\frac{d}{ds}\Big|_{s=0}[(n-1)\Delta_{g(t)}+s_0]\mathcal{P}[g(t)] \\
&= -(n-1)h_{ij}\nabla_i\nabla_j\mathcal{P}(g) - \frac{n-1}{2}\left(2\nabla_i h_{ij} - \nabla_j Tr_g(h)\right)\nabla_j\mathcal{P}(g) \\
&\quad + ((n-1)\Delta_g + s_0)\mathcal{P}'_g(h)
\end{aligned} \tag{3.5.15}$$

For the right hand side, we have

$$\frac{d}{ds}\Big|_{s=0}\Big[-(n-2)A_{g(t)}\cdot B_{g(t)} + \nabla^2_{g(t)}B_{g(t)}\Big] = \sum_{l=0}^{4} P_l * \nabla^l h \tag{3.5.16}$$

where $P_l$'s are curvature quantities depend on metric $g$ up to fourth order derivatives by using the fact of divergence of Bach tensor is of fourth order in (2.2.1). Again, the index $l$ is used to differ terms not for contraction.

Combine with these two results, we have

$$\begin{aligned}
\mathcal{P}'_g(h) = [(n-1)\Delta_g + s_0]^{-1}\Bigg(&(n-1)h_{ij}\nabla_i\nabla_j\mathcal{P}(g) \\
&+ \frac{n-1}{2}\left(2\nabla_i h_{ij} - \nabla_j Tr_g(h)\right)\nabla_j\mathcal{P}(g) + \sum_{l=0}^{4} P_l * \nabla^l h\Bigg)
\end{aligned} \tag{3.5.17}$$

where we have assumed the invertibility of the elliptic operator $(n-1)\Delta_g + s_0$.

We summarize the result as the following lemma.

**Lemma 3.5.5.** *Suppose that $g(t)$, $t \in [0,T]$, is a family of $C^{4+\alpha,0}$-metrics such that the operator $(n-1)\Delta_{g(t)} + s_0$ is invertible for all $t$. Then for the linearization defined by using $g_s(t) = g(t) + sh(t)$ for a family of symmetric 2-tensor $h(t)$ we have*

$$D\mathcal{M}(g(t)) = \frac{\partial}{\partial t}h + \Delta_g\Delta_L h - \sum_{k=0}^{2} M_k(g) * \nabla^k_g h - 2(n-2)\mathcal{P}'_g(h)g - 2(n-2)\mathcal{P}(g)h \tag{3.5.18}$$

*where operator $\mathcal{P}_g'(h)$ is defined by* (3.5.17).

**Short Time Existence Theorem**

The following proof for the short time existence of conformal Bach flow is very close to the proof for that of conformal Ricci flow in [38, §3.3.2-3.3.3]. First we need to solve the linearized flow

$$\begin{cases} \dfrac{\partial}{\partial t}h + \Delta_g \Delta_L h - \displaystyle\sum_{k=0}^{2} M_k(g) * \nabla_g^k h - 2(n-2)\mathcal{P}_g'(h)g - 2(n-2)\mathcal{P}(g)h = \gamma \\ h(\cdot, 0) = 0 \end{cases} \tag{3.5.19}$$

for each $\gamma \in C^{\alpha,0}(M \times [0,T])$, namely,

**Proposition 3.5.6.** *Suppose that $g(t)$, $t \in [0,T]$, is a family of $C^{4+\alpha,0}$-metrics such that the elliptic operator $(n-1)\Delta_{g(t)} + s_0$ is invertible for all $t$. Then for each $\gamma \in C^{\alpha,0}(M \times [0,T])$ the initial value problem* (3.5.19) *has a unique solution $h \in C^{4+\alpha,1}(M \times [0,T])$. Moreover there is a constant $C$ such that*

$$\|h\|_{C^{4+\alpha,1}} \leq C \|\gamma\|_{C^{\alpha,0}}. \tag{3.5.20}$$

*for all $\gamma \in C^{\alpha,0}(M \times [0,T])$.*

*Proof.* To use contractive mapping theorem we consider the Banach space

$$E_1([0,T^*]) = \{\tilde{h} \in C^{4+\alpha,0}(M \times [0,T^*]) : \tilde{h}(\cdot, 0) = 0\}$$

where $T^* \in (0,T]$ is a small constant to be chosen below. By Lemma (3.4.2), for a given $\tilde{h} \in E_1([0, T^*])$ we can solve the system of linear parabolic equations

$$\begin{cases} \dfrac{\partial}{\partial t} h + \Delta_g \Delta_L h - \displaystyle\sum_{k=0}^{2} M_k(g) * \nabla_g^k h - 2(n-2)\mathcal{P}(g)h = \tilde{\gamma} \\ h(\cdot, 0) = 0, \end{cases} \tag{3.5.21}$$

where $\tilde{\gamma} = \gamma + 2(n-2)\mathcal{P}'_g(\tilde{h}) \in C^{\alpha,0}$.

Hence we define a map which maps all element to the solution

$$\Psi : E_1([0, T^*]) \to E_1([0, T^*]), \quad \Psi(\tilde{h}) = h \in C^{4+\alpha,1}$$

With this construction, the fixed point of this mapping will be exactly the solution. Let $\tilde{h}_i \in E_1([0, T^*])$, $i = 1, 2$. Note that if we set

$$v = \Psi(\tilde{h}_1) - \Psi(\tilde{h}_2)$$

then $v$ satisfies

$$\begin{cases} \dfrac{\partial}{\partial t} v + \Delta_g \Delta_L v - \displaystyle\sum_{k=0}^{3} M_k(g) * \nabla_g^k v - 2(n-2)\mathcal{P}(g)v = 2(n-2)(\mathcal{P}'_g(\tilde{h}_1) - \mathcal{P}'_g(\tilde{h}_2)), \\ v(\cdot, 0) = 0. \end{cases}$$

From (3.5.17) and Lemma 3.5.1 we have

$$\left\| \mathcal{P}'_g(\tilde{h}_1) - \mathcal{P}'_g(\tilde{h}_2)) \right\|_{C^{2+\alpha,0}} \leq C \|\tilde{h}_1 - \tilde{h}_2\|_{C^{4+\alpha,0}}, \tag{3.5.22}$$

then it follows from Lemma 3.4.2 and $g \in C^{4+\alpha,0}$ that

$$\|v\|_{C^{6+\alpha,1}} \leq C \left\| \tilde{h}_1 - \tilde{h}_2 \right\|_{C^{4+\alpha,0}}. \tag{3.5.23}$$

Hence by Lemma 3.4.1 we have

$$\|v(\cdot,t_1)-v(\cdot,t_2)\|_{C^{4+\alpha}} \le C\cdot|t_1-t_2|\cdot\left\|\tilde{h}_1-\tilde{h}_2\right\|_{C^{4+\alpha,0}}. \tag{3.5.24}$$

In particular, using $v(\cdot,0)=0$ we get

$$\left\|\Psi(\tilde{h}_1)-\Psi(\tilde{h}_2)\right\|_{C^{4+\alpha,0}} \le CT^*\left\|\tilde{h}_1-\tilde{h}_2\right\|_{C^{4+\alpha,0}} \tag{3.5.25}$$

for all $\tilde{h}_i\in E_1([0,T^*])$, $i=1,2$.

To apply contractive mapping theorem to $\Psi$ we observe that

$$\left\|\Psi(\tilde{h})\right\|_{C^{4+\alpha,0}} \le \|\Psi(0)\|_{C^{4+\alpha,0}} + CT^*\left\|\tilde{h}\right\|_{C^{4+\alpha,0}} \tag{3.5.26}$$

by (3.5.25) and for some constant $C_0$

$$\|\Psi(0)\|_{C^{4+\alpha,1}} \le C_0\|\gamma\|_{C^{\alpha,0}} \tag{3.5.27}$$

by Lemma 3.4.2 and the definition of $\Psi$.

Let $R=2C_0\|\gamma\|_{C^{\alpha,0}}$. Thus when $T^*$ is chosen so that $CT^*\le\frac{1}{2}$, the map

$$\Psi: B_R=\{\tilde{h}\in E_1([0,T^*]):\left\|\tilde{h}\right\|_{C^{4+\alpha,0}}\le R\}\to B_R \tag{3.5.28}$$

is a contractive mapping. We get a fixed point of $\Psi$ on $B_R$ which gives the existence of the solution of equations (3.5.19) on time interval $[0,T^*]$.

Note that if $\gamma\in C^{2+\alpha,0}(M\times[0,T])$, then from (3.5.23) the solution $h\in C^{6+\alpha,1}(M\times[0,T])$.

To see the uniqueness of solutions to (3.5.19), suppose that $h_1$ and $h_2$ are two solutions, it follows from 3.5.25 using $\tilde{h}_i=h_i=\Psi(\tilde{h}_i)$ and $CT^*\le\frac{1}{2}$ that

$h_1 - h_2 = 0$.

Because (3.5.19) is linear, there will be no short time blowup, one may extend its solution from $[0, T^*]$ to $[0, T]$ by steps over time interval of length $T^*$. Note that when we extend the solution to $[T^*, 2T^*]$, we need to make some simple adjustment to the equations so that the initial condition at $T^*$ for new equations is 0. Then the estimate (3.5.20) follows from the estimates (3.5.26) and (3.5.23). □

In summary we have established that the linear operator defined by (3.5.18)

$$D\mathcal{M}(g) : \left\{ h \in C^{4+\alpha,1}(M \times [0,T]),\ h(\cdot, 0) = 0 \right\} \to C^{\alpha,0}(M \times [0,T])$$

is an isomorphism, provided that $g = g(t)$ satisfies the assumptions in Proposition 3.5.6. By the comment before the proof of the uniqueness in Proposition 3.5.6 we also have

$$D\mathcal{M}(g) : \left\{ h \in C^{6+\alpha,1}(M \times [0,T]),\ h(\cdot, 0) = 0 \right\} \to C^{2+\alpha,0}(M \times [0,T]) \quad (3.5.29)$$

is an isomorphism.

Now we apply the implicit function theorem 3.3.1 to the nonlinear map defined by (3.5.9)

$$\mathcal{M} : \left\{ g \in C^{4+\alpha,1}(M \times [0,T]),\ g(0) = g_0 \right\} \to C^{\alpha,0}(M \times [0,T]). \quad (3.5.30)$$

Here $g_0$ will be defined later and we choose the metric $\tilde{g} = g_0$ which is used in our DeTurck's term (3.2.10) as the background metric.

We begin with showing that operator $\mathcal{M}$ is continuously differentiable.

**Proposition 3.5.7.** *Suppose that $M^n$ is a closed manifold with metrics $g(t) \in C^{4+\alpha,1}(M \times [0,T])$. Suppose that the elliptic operator $(n-1)\Delta_{g(t)} + s_0$ is invertible* *for all $t$. Then there is a constant $\delta_0 > 0$ such that we have the following estimate of the norm of the difference of the linear operator*

$$\|D\mathcal{M}(g_1) - D\mathcal{M}(g_2)\|_{L(C^{4+\alpha,1},C^{\alpha,0})} \leq C\,\|g_1 - g_2\|_{C^{4+\alpha,1}} \tag{3.5.31}$$

*for all $g_i \in C^{4+\alpha,1}(M \times [0,T])$ which satisfies $\|g_i - g\|_{C^{4+\alpha,1}} \leq \delta_0$, $i = 1, 2$.*

*Proof.* This is another straightforward telescoping argument. For any symmetric 2 tensor $h \in C^{1,4+\alpha}(M \times [0,T])$ and $h(0,\cdot) = 0$, we have:

$$\begin{aligned}
&(D\mathcal{M}(g_1) - D\mathcal{M}(g_2))(h) \\
&= (\Delta_{g_1}\Delta_{L,g_1} - \Delta_{g_2}\Delta_{L,g_2})h - \sum_{k=0}^{3}(M_k(g_1) *_{g_1} \nabla^k_{g_1} h - M_k(g_2) *_{g_2} \nabla^k_{g_2} h) \\
&- 2(n-2)\left(\mathcal{P}'_{g_1}(h)g_1 - \mathcal{P}'_{g_2}(h)g_2\right) - 2(n-2)\left(\mathcal{P}(g_1) - \mathcal{P}(g_2)\right)h.
\end{aligned}$$

Now we will estimate every term in this expression. For the first term, we have the following estimate:

$$\begin{aligned}
&\|(\Delta_{g_1}\Delta_{L,g_1} - \Delta_{g_2}\Delta_{L,g_2})h\|_{C^{\alpha,0}} \\
\leq &\|(\Delta_{g_1} - \Delta_{g_2})\Delta_{L,g_1}h\|_{C^{\alpha,0}} + \|\Delta_{g_2}(\Delta_{L,g_1} - \Delta_{L,g_2})h\|_{C^{\alpha,0}} \\
\leq &C\,\|g_1 - g_2\|_{C^{4+\alpha,0}} \cdot \|h\|_{C^{4+\alpha,0}}
\end{aligned} \tag{3.5.32}$$

For each $k = 0, 1, 2, 3$ we have

$$\begin{aligned}
&\left\|M_k(g_1) *_{g_1} \nabla^k_{g_1} h - M_k(g_2) *_{g_2} \nabla^k_{g_2} h\right\|_{C^{\alpha,0}} \\
\leq &\left\|(M_k(g_1) *_{g_1} - M_k(g_2)) *_{g_2})\nabla^k_{g_1} h\right\|_{C^{\alpha,0}} + \left\|M_k(g_2) *_{g_2} (\nabla^k_{g_1} h - \nabla^k_{g_2} h)\right\|_{C^{\alpha,0}} \\
\leq &C\,\|g_1 - g_2\|_{C^{4+\alpha,0}} \cdot \|h\|_{C^{4-k+\alpha,0}} + C\,\|g_1 - g_2\|_{C^{4-k+\alpha,0}} \cdot \|h\|_{C^{4-k+\alpha,0}}
\end{aligned} \tag{3.5.33}$$

From a proof similar to that on the bottom of [38, p.426], we recall the definition of $\mathcal{P}'_g(h)$ from (3.5.17)

$$\begin{aligned}[(n-1)\Delta_g + s_0]\mathcal{P}'_g(h) = (n-1)h_{ij}\nabla^i\nabla^j\mathcal{P}(g) \\ + \frac{n-1}{2}\left(2\nabla_i h_{ij} - \nabla_j Tr_g(h)\right)\nabla_j\mathcal{P}(g) \\ + \sum_{l=0}^{4} P_l * \nabla^l h\end{aligned} \tag{3.5.34}$$

It is not difficult to evaluate such term. Telescoping again and with Lemma 3.4.2, we have:

$$\left\|\mathcal{P}'_{g_1}(h) - \mathcal{P}'_{g_2}(h)\right\|_{C^{2+\alpha,0}} \leq C\left\|g_1 - g_2\right\|_{C^{4+\alpha,0}} \cdot \left\|h\right\|_{C^{4+\alpha,0}} \tag{3.5.35}$$

hence

$$\begin{aligned}&\left\|\mathcal{P}'_{g_1}(h)g_1 - \mathcal{P}'_{g_2}(h)g_2\right\|_{C^{2+\alpha,0}} \\ \leq &\left\|\left(\mathcal{P}'_{g_1}(h) - \mathcal{P}'_{g_2}(h)\right)g_1\right\|_{C^{2+\alpha,0}} + \left\|\mathcal{P}'_{g_2}(h)(g_1 - g_2)\right\|_{C^{2+\alpha,0}} \\ \leq &C\left\|g_1 - g_2\right\|_{C^{4+\alpha,0}} \cdot \left\|h\right\|_{C^{4+\alpha,0}} + C\left\|g_1 - g_2\right\|_{C^{2+\alpha,0}} \cdot \left\|h\right\|_{C^{4+\alpha,0}}\end{aligned} \tag{3.5.36}$$

where we have used (3.5.22) with $\tilde{h}_2 = 0$ to get the last term in the second inequality. Using Lemma 3.5.1 we have

$$\left\|\left(\mathcal{P}(g_1) - \mathcal{P}(g_2)\right)h\right\|_{C^{2+\alpha,0}} \leq C\left\|g_1 - g_2\right\|_{C^{4+\alpha,0}} \cdot \left\|h\right\|_{C^{2+\alpha,0}} \tag{3.5.37}$$

The lemma now follows from combining together the inequalities $\square$

The invertibility condition for the operator $\mathcal{M}$ requires the interior estimate for strong parabolic system, but we don't have such condition for arbitrary metric $g(t)$. By the calculations before, we know that for any metric with constant scalar curvature, the conformal Bach flow will be a strong parabolic system. In order to apply the inverse function theorem, we need to choose a suitable initial data.

**Proposition 3.5.8.** *There exists a metric $\bar{g} \in C^{4+\alpha,1}(M \times [0,T])$ such that*

$$\|\mathcal{M}(\bar{g})\|_{C^{0,\alpha}} \leq C \tag{3.5.38}$$

*Proof.* Let $\bar{g}(t) = g_0 + th$, then we have:

$$\mathcal{M}(\bar{g}(t)) = h - \mathcal{F}(g_0 + th) = h - \mathcal{F}(g_0) + \mathcal{F}(g_0) - \mathcal{F}(g_0 + th) \tag{3.5.39}$$

Let $h = \mathcal{F}(g_0)$, since $g_0$ is the initial metric in our geometric flow, its constant scalar curvature is a constant, $R[g_0] = s_0$. In fact, $h = -2(n-2)\Big(B - \frac{1}{2(n-1)(n-2)}\Delta R g + pg\Big)$ because the Deturck's term also vanishes. With the same calculation, the linearization at $\bar{g}$ is a strongly parabolic system. Furthermore, with small $t$, we have a nice control for the initial data such that:

$$\|\mathcal{M}(\bar{g})\|_{C^{\alpha,0}} \leq CT \quad \text{for} \quad t \in [0,T] \tag{3.5.40}$$

□

With this choice of initial data, we are ready to prove the short time existence of the conformal Bach flow.

**Theorem 3.5.9.** *Let $M^n$ be a closed manifold. Suppose that $g_0$ is a $C^{8+\alpha}$-Riemannian metric on $M$ with constant scalar curvature $R = s_0$ and that the elliptic operator $(n-1)\Delta_{g_0} + s_0$ is invertible. Then there exists a unique $C^{4+\alpha,1}$-solution $g(t)$, $t \in [0,T]$, of the decoupled DeTurck CBF (3.5.9) for some $T > 0$.*

*Proof.* Let

$$\mathcal{M} : \{g \in C^{4+\alpha,1}(M \times [0,T]), g(0) = g_0\} \to C^{\alpha,0}(M \times [0,T]) \tag{3.5.41}$$

be a map defined in (3.5.10). We define $\bar{g} \in C^{4+\alpha,1}(M \times [0,T])$ where metrics

$$\bar{g}(t) = \bar{g}(t) = g_0 + t\mathcal{F}(g_0). \tag{3.5.42}$$

as we defined in Proposition 3.5.8. We will apply inverse function theorem 3.3.1 to map $\mathcal{M}$ around $\bar{g}$ to prove the existence of a solution of equation $\mathcal{M}(g) = 0$. Note that linear operator $(n-1)\Delta_{\bar{g}(t)} + s_0$ is a small perturbation of $(n-1)\Delta_{g_0} + s_0$ when $t$ is small. Since operator $(n-1)\Delta_{g_0} + s_0$ is invertible, we conclude that $(n-1)\Delta_{\bar{g}(t)} + s_0$ is invertible for each $t \in [0,T]$ when $T$ is small enough. In general metric $\bar{g}(t)$ does not have constant scalar curvature for $t > 0$.

Note that

$$\|\mathcal{M}(\bar{g})\|_{C^{\alpha,0}} \leq CT \quad \text{for} \quad t \in [0,T] \tag{3.5.43}$$

where $C$ depends on $\|g_0\|_{C^{10+\alpha}}$. Hence $\bar{g}(t)$ is an approximate solution when $T$ is small.

By (3.5.29) we have that for sufficient small $T$

$$\left\| [D\mathcal{M}(g)]^{-1} \right\|_{L(C^{\alpha,0}, C^{4+\alpha,1})} \leq \bar{C} \tag{3.5.44}$$

for some constant $\bar{C}$. Let $B_\epsilon(\bar{g}) = \{g \in C^{4+\alpha,1}(M \times [0,T]) : \|g - \bar{g}\|_{C^{4+\alpha,1}} \leq \epsilon\}$. By the perturbation theory of bounded linear operators and Proposition 3.5.7 there is a constant $C_0$ and a small number $\epsilon > 0$ such that the operator norm

$$\left\| [D\mathcal{M}(g)]^{-1} \right\|_{L(C^{\alpha,0}, C^{4+\alpha,1})} \leq C_0 \tag{3.5.45}$$

for $g \in B_\epsilon(\bar{g})$. By Proposition 3.5.7 we can choose $\epsilon$ even smaller if necessarily such that for the constant $C_0$ above we have

$$\|D\mathcal{M}(g_1) - D\mathcal{M}(g_2)\|_{L(C^{4+\alpha,1}, C^{\alpha,0})} \leq \frac{1}{2C_0} \tag{3.5.46}$$

for all $g_1, g_2 \in B(\bar{g}, \delta_0)$. We may choose an even smaller $T$ if necessary to get

$$\|\mathcal{M}(\bar{g})\|_{C^{\alpha,0}} \leq \frac{\epsilon}{2C_0} \tag{3.5.47}$$

Hence the short time existence of DeTurck conformal Bach flow is proved by the inverse function theorem 3.3.1. Furthermore, by Lemma 3.2.1, we finish the proof of the short time existence of conformal Bach flow. $\square$

**Remark 3.5.10.** *We already proved the short time existence for conformal Bach flow. As for conformal gradient flow* (2.8.4)*, the proof is the identical, and since it is divergence free, the elliptic equation in conformal gradient flow system is easier than conformal Bach flow.*

## 3.6 Uniqueness

The proof is standard as the proof of the uniqueness of Ricci flow on closed manifolds (see, for example, [16, p.117-118]) or the uniqueness proof in [51, p.254-255]. The basic idea is that given two solutions $(g_i(t), p_i(t)), i = 1, 2$, of conformal Bach flow (3.0.1), from Lemma (3.2.1) we have two diffeomorphisms $\varphi_i(t)$, $i = 1, 2$ which are solutions of the parabolic equations (3.2.11) corresponding to $g_i(t)$. Then the pushing-forward metrics $(\varphi_i(t))_* g_i(t)$ are solutions of DeTurck conformal Bach flow (3.2.11) satisfying the same the initial condition, hence by the uniqueness in Theorem 3.5.9, we have

$$(\varphi_1(t))_* g_1(t) = (\varphi_2(t))_* g_2(t) = g_*(t)$$

Then $\varphi_i(t)$ are the solutions of ordinary differential equation (3.2.12) for metric $g_*(t)$ with initial condition

$$\varphi_1(0) = \varphi_2(0) = \mathrm{Id}_M$$

Hence $\varphi_1(t) = \varphi_2(t)$ and

$$g_1(t) = \varphi_1(t)^* g_*(t) = \varphi_2(t)^* g_*(t) = g_2(t)$$

## 3.7 Regularity

In previous theorem, we found that given a suitable initial data, there exist a solution in $C^{4+\alpha,1}(M \times [0,T])$ to the conformal Bach flow for some small $T$. The regularity theorem implies this solution is actually smooth.

**Theorem 3.7.1.** *If the metric $g_0$ in Theorem 3.5.9 is smooth, then the solution $g(t)$ is smooth in space and time.*

*Proof.* Let metric $\tilde{g}$ in ((3.2.10)) to be $g_0$. Using local coordinates $\{x^i\}_{i=1}^n$ we may rewrite the DeTurck modified conformal Bach flow (3.2.11) in a schematic way

$$\begin{cases} \dfrac{\partial}{\partial t} g_{ij} + (g^{kl}\partial_k \partial_l)(g^{pq}\partial_p \partial_q) g_{ij} + A_{ij}(g_0, g) - 2(n-2)p g_{ij} = 0, \\ ((n-1)\Delta_{g(t)} + s_0)p = -(n-2)A(g(t)) \cdot B(g(t)) + \nabla^2_{g(t)} B(g(t)), \end{cases} \tag{3.7.1}$$

where $A_{ij}(g_0, g)$ depends on $\{g_{pq}\}$ up to their third derivatives. We want to prove

$\partial^k g_{ij}$ is in $C^{4+\alpha,1}(M \times [0,T])$ for each $k \in \mathbb{N}$ by a bootstrap argument. Below we only consider the the base case $k = 1$, as an example.

By Theorem 3.5.9, if $g \in C^{4+\alpha,1}$, the right hand side of the second equation in (3.7.1) is in $C^{\alpha,0}$ and hence $p \in C^{4+\alpha,0}$. Since $A_{ij}(g_0, g) - 2(n-2)pg_{ij} \in C^{1+\alpha,0}$, it follows from Lemma 3.4.2 and the smoothness of $g_0$ that $g \in C^{5+\alpha,1}$.

After we have improved the spatial regularity to smoothness, we can use the equation (3.7.1) to improve the regularity in time to smoothness. The theorem is proved. $\square$

# Chapter 4

# Integral Estimate and Long Time Behavior

In this chapter, we will introduce the integral estimate of Riemann curvature tensor when the metric is evolving under the conformal Bach flow and discuss the long time behavior of the solution to conformal Bach flow.

In Sec.4.1, we will calculate some evolution equations for curvature quantity. Since conformal Bach flow is a fourth order partial differential equation system, no maximum principle is readily available, and so we cannot expect to easily bound curvature quantities pointwise. Instead of that, integral estimate is a well developed approach, we will develop a integral estimate in Sec.4.3 follow the ideas from[32, Sec.3] and [51, Sec.5]. We will also discuss how the volume changes under this flow in Sec.4.4. Also, a special case in dimension 4 will be mentioned. Once we assume the Sobolev inequality, such global estimate can be turned into point-wise estimate. In Sec.4.5, we will see point-wise estimate of curvatures allows us to characterize the finite time singularities.

We first introduce some notations in our paper. We denote any possible contraction between curvatures by $Rm * Rm$. For example, this can represent

$3R_{ikjl}R_{pqrs}$ or $R_{ij}R^i_k$. Another useful notation in curvature estimate is $P^m_s(Rm)$, which is defined as:

$$P^k_s(Rm) = \sum C_{i_1 i_2 \cdots i_s} \nabla^{i_1} Rm * \nabla^{i_2} Rm * \cdots * \nabla^{i_s} Rm$$

where $i_1 + i_2 + \cdots + i_s = m$.

Before we proceeding, we recall that the conformal Bach flow (3.0.1) is

$$\begin{cases} \dfrac{\partial}{\partial t} g = 2(n-2)(B + pg) \\ -(n-1)\Delta p - s_0 p = (n-2)A \cdot B - \nabla^2 B \end{cases}$$

From now on, we will drop the modified term which will simplify our calculation. Also, we recall the definition of Bach tensor with Schouten tensor (2.1.11)

$$\begin{aligned} B_{ij} &= \Delta A_{ij} - \nabla^k \nabla^i A_{jk} + A^{kl} W_{ikjl} \\ &= \frac{1}{n-2}\Delta R_{ij} + \frac{2}{n-2} R^{kl} R_{ikjl} - \frac{n-4}{(n-2)^2} R_{ik} R^k_j \\ &\quad - \frac{n}{(n-1)(n-2)^2} s_0 R_{ij} - \frac{1}{(n-2)^2} |Ric|^2 g_{ij} + \frac{1}{(n-1)(n-2)} s_0^2 g_{ij} \end{aligned} \tag{4.0.1}$$

One can verify this equation by using Ricci identity (A.6.1) and the definition of Weyl tensor (A.9.3). If we preserve the scalar curvature to be constant, the highest order term in Bach curvature is $\Delta R_{ij}$, the rest term will be quadratic.

And we recall that the gradient of $L^2$ norm of Weyl curvature.

$$\begin{aligned} \mathcal{B} = &- \frac{4(n-3)}{n-2} \Delta R_{ij} + \frac{2(n-3)}{(n-1)(n-2)} \Delta R g_{ij} + \frac{2(n-3)}{n-1} \nabla_i \nabla_j R \\ &- \frac{4(n-4)}{n-2} R^{pq} R_{ipjq} + 4 R_{ip} R^p_j - \frac{4}{(n-1)(n-2)} R R_{ij} \\ &- 2 R_{pqri} R^{pqr}{}_j + \frac{1}{2} |W|^2 g_{ij} \end{aligned} \tag{4.0.2}$$

Since these two tensors are similar, in this chapter, we will consider the following differential equation system.

$$\begin{cases} \frac{\partial}{\partial t} g = C_n \Big( \Delta Ric + Rm * Rm + pg \Big) \\ -(n-1)p - s_0 p = (n-2) A \cdot B - \nabla^2 B \end{cases} \tag{4.0.3}$$

where $C_n = 2(n-2)$ and $s_0$ is a constant for scalar curvature. Here we simply use $A \cdot B$, because schematically, $\mathcal{B}$ and $B$ are the same. And the double divergence term $\nabla^2 B$ will vanish when $n = 4$ in conformal Bach flow and vanish for any dimension in conformal gradient flow. But this term will not hurt our result, we decide to keep it. We will see in Sec. 4.2, $\nabla^2 B$ only contains the same type of curvature quantity as $A \cdot B$.

## 4.1 Evolution Equation

In this section, we will derive some evolution equations for Riemann curvature tensor. Our main goals are deriving the evolution equations for the $L^2$-norm of $Rm$.

**Commutation of Derivatives**

First, we will introduce a basic formula.

**Lemma 4.1.1.** *Let $T$ be a tensor quantity, we have*

$$[\nabla^k, \Delta] T = \nabla^k \Delta T - \Delta \nabla^k T = \sum_{j=0}^{k} \nabla^j Rm * \nabla^{k-j} T \tag{4.1.1}$$

*Proof.* We will use induction argument to prove this formula. The base case $k = 1$ directly coming from the Ricci identity (A.6.1). Now suppose this formula is true for $k-1$, then we have

$$
\begin{aligned}
&\nabla^k \Delta T - \Delta \nabla^k T \\
&= \nabla \nabla^{k-1} \Delta T - \Delta \nabla^k T \\
&= \nabla\Big(\Delta \nabla^{k-1} T + \sum_{j=0}^{k-1} \nabla^j Rm * \nabla^{k-1-j} T\Big) - \Delta \nabla^k T \\
&= \nabla \Delta \nabla^{k-1} T + \sum_{j=0}^{k} \nabla^j Rm * \nabla^{k-j} T - \Delta \nabla^k T \\
&= \Delta \nabla \nabla^{k-1} T + \sum_{j=0}^{1} \nabla^j Rm * \nabla^{1-j} \nabla^{k-1} T + \sum_{j=0}^{k} \nabla^j Rm * \nabla^{k-j} T - \Delta \nabla^k T \\
&= \sum_{j=0}^{k} \nabla^j Rm * \nabla^{k-j} T
\end{aligned}
$$

□

**Evolution Equation of** $Rm$

We first introduce a fundamental formula in curvature estimate.

**Lemma 4.1.2.** *Given* $(M^n, g(t))$ *to be the solution of conformal Bach flow* (3.0.1), *the evolution equation of Riemann curvature tensor is:*

$$\frac{\partial}{\partial t} Rm = -\Delta^2 Rm + P_2^2(Rm) + P_3^0(Rm) + T(\nabla^2 p) + 2(n-2)pRm \tag{4.1.2}$$

*where*

$$T(\nabla^2 p) = (n-2)\Big[\nabla_i \nabla_l p g_{jk} - \nabla_i \nabla_j p g_{kl} - \nabla_k \nabla_l p g_{ij} + \nabla_k \nabla_j p g_{il}\Big]$$

*Here, we ignore all of coefficients by using aster symbol.*

*Proof.* To see this, we recall that the variation of Riemann curvature tensor

(B.4.1)(also see [16, Page 120 Eq.2.67]), we have:

$$\frac{\partial}{\partial t}R_{ijkl} =\frac{1}{2}\left(\nabla_i\nabla_k h_{jl} - \nabla_i\nabla_l h_{jk} - \nabla_j\nabla_k h_{il} + \nabla_j\nabla_l h_{ik}\right) + \frac{1}{2}\left(R_{ijkp}h_{pl} + R_{ijpl}h_{pk}\right)$$

where $h_{ij} = 2(n-2)(B_{ij} + pg_{ij})$. We will consider three different types of terms, i.e.

$$h = \Delta Ric + Rm * Rm + 2(n-2)pg = h_1 + h_2 + h_3$$

First, we know that the leading term of Bach tensor is $h_1 = 2Ric$, therefore, if $\frac{\partial}{\partial t}g = h_1$ we have:

$$\begin{aligned}\frac{\partial}{\partial t}R_{ijkl} &= \nabla_i\nabla_k\Delta R_{jl} - \nabla_i\nabla_l\Delta R_{jk} - \nabla_j\nabla_k\Delta R_{il} + \nabla_j\nabla_l\Delta R_{ik}\\ &\quad + R_{ijkp}\Delta R^p_l + R_{ijpl}\Delta R^p_k\\ &= \Delta\Big(\nabla_i\nabla_k R_{jl} - \nabla_i\nabla_l R_{jk} - \nabla_j\nabla_k R_{il} + \nabla_j\nabla_l R_{ik}\Big) + Rm * \nabla^2 Rm\\ &= \Delta\Big(-\Delta Rm + Rm * Rm\Big) + Rm * \nabla^2 Rm\\ &= -\Delta^2 Rm + P^2_2(Rm)\end{aligned} \tag{4.1.3}$$

where we use the result from [16, Page 120 Eq.2.64] to get an extra Laplacian. We also don't specify $\Delta$ and $\nabla^2$ when it is of the lower order term. And we treat $s_0$ as a curvature quantity $Rm$ for our convenient.

Second, in the Bach tensor we have some quadratic terms $h_2 = Rm * Rm$, therefore, if $\frac{\partial}{\partial t}g = h_2$ we have:

$$\frac{\partial}{\partial t}R_{ijkl} = P^2_2(Rm) + P^0_3(Rm) \tag{4.1.4}$$

For the last type, we don't simplify any term with pressure function. Combining with the previous results, we have:

$$\frac{\partial}{\partial t}Rm = -\Delta^2 Rm + P_2^2(Rm) + P_3^0(Rm) + T(\nabla^2 p) + 2(n-2)pRm$$

where

$$T(\nabla^2 p) = (n-2)\Big[\nabla_i\nabla_l p g_{jk} - \nabla_i\nabla_j p g_{kl} - \nabla_k\nabla_l p g_{ij} + \nabla_k\nabla_j p g_{il}\Big]$$

□

**Evolution Equation for $\nabla^k Rm$**

Next step is raising the covariant derivatives. The key lemma is the following, about commuting the time derivative and the covariant derivative.

**Lemma 4.1.3.** *Let $T$ be a tensor quantity, we have*

$$\frac{\partial}{\partial t}\nabla^k T = \nabla^k \frac{\partial}{\partial t}T + \sum_{j=0}^{k-1}\nabla^j\Big(\nabla(\frac{\partial}{\partial t}g) * \nabla^{k-1-j}Rm\Big) \tag{4.1.5}$$

The extra term $\nabla(\frac{\partial}{\partial t}g)$ comes from the derivative of Christoffel symbols with respect to time. With this lemma, we have the following evolution equation.

**Lemma 4.1.4.** *Given $(M^n, g(t))$ to be the solution of conformal Bach flow* (3.0.1), *let $k \in \mathbb{N}$, the evolution equation of the $k$-th order covariant derivative of Riemann curvature tensor is:*

$$\begin{aligned}\frac{\partial}{\partial t}\nabla^k Rm = &-\Delta^2\nabla^k Rm + P_2^{k+2}(Rm) + P_3^k(Rm)\\ &+ \nabla^k T(\nabla^2 p) + 2(n-2)\sum_{j=1}^{k}\nabla^j p\nabla^{k-j}Rm\end{aligned} \tag{4.1.6}$$

*where*

$$T(\nabla^2 p) = (n-2)\Big[\nabla_i\nabla_l p g_{jk} - \nabla_i\nabla_j p g_{kl} - \nabla_k\nabla_l p g_{ij} + \nabla_k\nabla_j p g_{il}\Big]$$

*Here, we ignore all of coefficients by using aster symbol.*

*Proof.* From now on, we simply write Bach tensor as some contractions of Riemann curvature and its covariant derivatives

$$\frac{\partial}{\partial t} g = \nabla^2 Rm + Rm * Rm + 2(n-2)pg$$

With a direct calculation, we have:

$$\begin{aligned}
\frac{\partial}{\partial t}\Big(\nabla^k Rm\Big) &= \nabla^k\Big(-\Delta^2 Rm + P_2^2(Rm) + P_3^0(Rm) + T(\nabla^2 p) + 2(n-2)pRm\Big) \\
&\quad + \sum_{j=0}^{k-1}\nabla^j\Big[\nabla\Big(\nabla^2 Rm + Rm * Rm + 2(n-2)pg\Big) * \nabla^{k-1-j} Rm\Big] \\
&= -\nabla^k\Delta^2 Rm + P_2^{k+2}(Rm) + P_3^k(Rm) + \nabla^k T(\nabla^2 p) \\
&\quad + 2(n-2)\nabla^k(pRm) + 2(n-2)\sum_{j=0}^{k-1}\nabla^j(\nabla p \nabla^{k-1-j} Rm)
\end{aligned}$$

For the last line, we commute the double Laplacian and the $k$-th order covariant derivative, all of extra terms are absorbed by $P_2^{k+2}(Rm)$. Now we need to rearrange the index to simplify the terms with $p$. First, we notice that

$$\nabla p \nabla^{k-1-j} Rm = \nabla(p\nabla^{k-1-j} Rm) - p\nabla^{k-j} Rm$$

therefore,

$$\begin{aligned}\sum_{j=0}^{k-1}\nabla^j\nabla p\nabla^{k-1-j}Rm &= \sum_{j=0}^{k-1}\nabla^j\Big[\nabla(p\nabla^{k-1-j}Rm) - p\nabla^{k-j}Rm\Big]\\ &= \sum_{j=0}^{k-1}\nabla^{j+1}(p\nabla^{k-1-j}Rm) - \sum_{j=0}^{k-1}\nabla^j(p\nabla^{k-j}Rm)\\ &= \sum_{j=1}^{k}\nabla^j(p\nabla^{k-j}Rm) - \sum_{j=0}^{k-1}\nabla^j(p\nabla^{k-j}Rm)\\ &= \nabla^k(pRm) - p\nabla^k Rm\end{aligned}$$

In the end, we have:

$$\begin{aligned}\frac{\partial}{\partial t}\Big(\nabla^k Rm\Big) &= -\nabla^k\Delta^2 Rm + P_2^{k+2}(Rm) + P_3^k(Rm) + \nabla^k T(\nabla^2 p)\\ &\quad + 4(n-2)\nabla^k(pRm) - 2(n-2)p\nabla^k Rm\end{aligned}$$

□

**Evolution Equation of $|Rm|^2$ and $|\nabla^k Rm|^2$**

**Lemma 4.1.5.** *Given $(M^n, g(t))$ to be the solution of conformal Bach flow* (3.0.1), *let $k \in \mathbb{N}$, the evolution equations of $|Rm|^2$ and $|\nabla^k Rm|^2$ are*

$$\begin{aligned}\frac{\partial}{\partial t}|Rm|^2 &= -\Delta^2|Rm|^2 + 2|\Delta Rm|^2 + 4|\nabla^2 Rm|^2 + 8\langle\nabla Rm, \Delta\nabla Rm\rangle\\ &\quad + P_3^2(Rm) + P_4^0(Rm) + 2T(\nabla^2 p)Rm - 4(n-2)p|Rm|^2\end{aligned} \tag{4.1.7}$$

*and*

$$\begin{aligned}\frac{\partial}{\partial t}|\nabla^k Rm|^2 &= -\Delta^2|\nabla^k Rm|^2 + 2|\Delta\nabla^k Rm|^2 + 4|\nabla^{k+2}Rm|^2\\ &\quad + 8\langle\nabla^{k+1}Rm, \nabla\Delta\nabla^k Rm\rangle + P_3^{2k+2}(Rm) + P_4^{2k}(Rm)\\ &\quad + 2\nabla^k Rm\nabla^k T(\nabla^2 p) + 8(n-2)\nabla^k Rm\nabla^k(pRm)\\ &\quad - 2(k+6)(n-2)p|\nabla^k Rm|^2\end{aligned} \tag{4.1.8}$$

*where*

$$T(\nabla^2 p) = \nabla_i \nabla_l p g_{jk} - \nabla_i \nabla_j p g_{kl} - \nabla_k \nabla_l p g_{ij} + \nabla_k \nabla_j p g_{il}$$

*Here, we ignore all of coefficients by using aster symbol.*

**Remark 4.1.6.** *From these two results, we can see that after integrating over a closed manifold, the first terms in* (4.1.7) *and* (4.1.8) *are vanished due to the divergence theorem. And we will see in next section that the following three terms will contribute negative terms after integration by parts and commuting derivatives.*

*Proof.* From Lemma 4.1.2, we have:

$$\begin{aligned}
\frac{\partial}{\partial t}|Rm|^2 &= \frac{\partial}{\partial t}\langle Rm, Rm\rangle + \frac{\partial}{\partial t}g^{-1} * Rm * Rm \\
&= 2\langle Rm, \frac{\partial}{\partial t}Rm\rangle - \frac{\partial}{\partial t}g * Rm * Rm \\
&= -2\langle Rm, \Delta^2 Rm + P_2^2(Rm) + P_3^0(Rm) + T(\nabla^2 p) + 2(n-2)pRm\rangle \\
&\quad - \Big(\nabla^2 Rm + Rm * Rm + 2(n-2)pg\Big) * Rm * Rm \\
&= -2\langle Rm, \Delta^2 Rm\rangle + P_3^2(Rm) + P_4^0(Rm) \\
&\quad + 2T(\nabla^2 p)Rm - 4(n-2)p|Rm|^2
\end{aligned}$$

Since we have:

$$\begin{aligned}
\Delta^2|Rm|^2 &= 2\langle Rm, \Delta^2 Rm\rangle + 2|\Delta Rm|^2 + 4|\nabla^2 Rm|^2 \\
&\quad + 4\langle \nabla Rm, \Delta\nabla Rm\rangle + 4\langle \nabla Rm, \nabla\Delta Rm\rangle \\
&= 2\langle Rm, \Delta^2 Rm\rangle + 2|\Delta Rm|^2 + 4|\nabla^2 Rm|^2 \\
&\quad + 8\langle \nabla Rm, \nabla\Delta Rm\rangle + P_3^2(Rm)
\end{aligned}$$

Combine with these two results, we have:

$$\begin{aligned}\frac{\partial}{\partial t}|Rm|^2 &= -2\langle Rm, \Delta^2 Rm\rangle + P_3^2(Rm) + P_4^0(Rm) \\ &\quad + 2T(\nabla^2 p)Rm - 4(n-2)p|Rm|^2 \\ &= -\Delta^2|Rm|^2 + 2|\Delta Rm|^2 + 4|\nabla^2 Rm|^2 + 8\langle \nabla Rm, \Delta\nabla Rm\rangle \\ &\quad + P_3^2(Rm) + P_4^0(Rm) + 2T(\nabla^2 p)Rm - 4(n-2)p|Rm|^2\end{aligned}$$

From Lemma 4.1.4, we have:

$$\begin{aligned}\frac{\partial}{\partial t}|\nabla^k Rm|^2 &= 2\langle \nabla^k Rm, \frac{\partial}{\partial t}\nabla^k Rm\rangle + k\frac{\partial}{\partial t}g^{-1} * \nabla^k Rm * \nabla^k Rm \\ &= -2\langle \nabla^k, \Delta^2\nabla^k Rm\rangle + P_3^{2k+2}(Rm) + P_4^{2k}(Rm) \\ &\quad + 2\nabla^k Rm\nabla^k T(\nabla^2 p) + 8(n-2)\nabla^k Rm\nabla^k(pRm) \\ &\quad - 2(k+6)(n-2)p|\nabla^k Rm|^2\end{aligned}$$

Another similar argument for $\Delta|\nabla^k Rm|^2$ is

$$\begin{aligned}\Delta|\nabla^k Rm|^2 &= -2\langle \nabla^k Rm, \Delta^2\nabla^k Rm\rangle + 2|\Delta\nabla^k Rm|^2 + 4|\nabla^{k+2}Rm|^2 \\ &\quad + 8\langle \nabla^{k+1}Rm, \nabla\Delta\nabla^k Rm\rangle + P_3^{2k+2}(Rm)\end{aligned}$$

Finally, we have:

$$\begin{aligned}\frac{\partial}{\partial t}|\nabla^k Rm|^2 &= -\Delta^2|\nabla^k Rm|^2 + 2|\Delta\nabla^k Rm|^2 + 4|\nabla^{k+2}Rm|^2 \\ &\quad + 8\langle \nabla^{k+1}Rm, \nabla\Delta\nabla^k Rm\rangle + P_3^{2k+2}(Rm) + P_4^{2k}(Rm) \\ &\quad + 2\nabla^k Rm\nabla^k T(\nabla^2 p) + 8(n-2)\nabla^k Rm\nabla^k(pRm) \\ &\quad - 2(k+6)(n-2)p|\nabla^k Rm|^2\end{aligned}$$

□

## 4.2 Integral Estimate of Pressure Function

In this section, we will derive some integral estimate for pressure function $p$. We remark that in this section, instead of assuming elliptic operator $(n-1)\Delta_g + s_0$ is invertible, we assume that $L^2$-norm of $p$ is bounded, which is a more desirable condition.

**Lemma 4.2.1.** *Given $(M^n, g(t))$ to be the solution of conformal Bach flow* (3.0.1), *for any positive constant $\epsilon$, we can find two constant $C_1$ and $C_2$ such that $C_1 = C_1(n, s_0, \epsilon)$. $C_2 = C_2(n)$, we have the following integral estimate for pressure function $p$:*

$$\int_M |\nabla p|^2 d\mu \leq C_1 \int_M |p|^2 d\mu + \epsilon \|Rm\|_\infty^2 \int_M |\nabla^2 Rm|^2 d\mu + C_2 \|Rm\|_\infty^4 \int_M |Rm|^2 d\mu \tag{4.2.1}$$

*Proof.* We first look at the elliptic equation:

$$[-(n-1)\Delta - s_0]p = \nabla^i \nabla^j B_{ij} + A^{ij} B_{ij}$$

From (2.2.1) and (2.2.2), we know that the bouble divergence term $\nabla^i \nabla^j B_{ij}$ is actually a second order term, schematically,

$$\nabla^i \nabla^j B_{ij} + A^{ij} B_{ij} = \nabla Rm * \nabla Rm + \nabla^2 Rm * Rm + Rm^{*3} = P_2^2(Rm) + P_3^0(Rm)$$

Therefore, the classical energy estimate is following:

$$-(n-1)p\Delta p - s_0 p^2 = p\nabla^i \nabla^j B_{ij} + pA^{ij} B_{ij}$$

Integrate this equation, apply integration by parts, we have:

$$(n-1)\int_M |\nabla p|^2 d\mu = s_0 \int_M |p|^2 d\mu + \int_M p \cdot P_2^2(Rm) d\mu + \int_M p \cdot P_3^0(Rm) d\mu$$

Then we apply Hölder inequality with exponent $\frac{1}{2}$ and Young's inequality with constant $a$ and $b$ which are decided later, we have:

$$\begin{aligned}\int_M |\nabla p|^2 d\mu \leq & \frac{s_0}{n-1}\int_M |p|^2 d\mu + \frac{1}{n-1}\left(\int_M |p|^2 d\mu\right)^{\frac{1}{2}}\left(\int_M P_4^4(Rm) d\mu\right)^{\frac{1}{2}} \\ & + \frac{1}{n-1}\left(\int_M |p|^2 d\mu\right)^{\frac{1}{2}}\left(\int_M P_6^0(Rm) d\mu\right)^{\frac{1}{2}} \\ \leq & \frac{s_0 + \frac{1}{2a} + \frac{1}{2b}}{n-1}\int_M |p|^2 d\mu + \frac{a}{2}\int_M P_4^4(Rm) d\mu + \frac{b}{2}\int_M P_6^0(Rm) d\mu\end{aligned}$$

Now we directly apply Proposition D.1.6 to the last two terms with $(s,k) = (4,2)$ and $(s,k) = (6,0)$, we have the following estimate.

$$\begin{aligned}\int_M |\nabla p|^2 d\mu \leq & \frac{s_0 + \frac{1}{2a} + \frac{1}{2b}}{n-1}\int_M |p|^2 d\mu + \frac{a}{2}\int_M P_4^4(Rm) d\mu + \frac{b}{2}\int_M P_6^0(Rm) d\mu \\ \leq & \frac{s_0 + \frac{1}{2a} + \frac{1}{2b}}{n-1}\int_M |p|^2 d\mu + \frac{a}{2}C_a \|Rm\|_\infty^2 \int_M |\nabla^2 Rm|^2 d\mu \\ & + \frac{b}{2}C_b \|Rm\|_\infty^4 \int_M |Rm|^2 d\mu\end{aligned}$$

Now for any positive constant $\epsilon > 0$, we choose constant $a$ to satisfies

$$\frac{a}{2}C_a \leq \epsilon$$

and let $C_1(s_0, a, b, n, \epsilon) = \frac{s_0 + \frac{1}{2a} + \frac{1}{2b}}{n-1}$ and $C_2(b,n) = \frac{b}{2}C_b$, we conclude that

$$\int_M |\nabla p|^2 d\mu \leq C_1 \int_M |p|^2 d\mu + \epsilon \|Rm\|_\infty^2 \int_M |\nabla^2 Rm|^2 d\mu + C_2 \|Rm\|_\infty^4 \int_M |Rm|^2 d\mu$$

$\square$

Now we estimate the derivatives of pressure function. Note that we have the following commutator equation:

$$\nabla^{k-1}\Delta p = \Delta\nabla^{k-1}p + \nabla^{k-1}(p \cdot Rm) \tag{4.2.2}$$

**Lemma 4.2.2.** *Given* $(M^n, g(t))$ *to be the solution of conformal Bach flow* (3.0.1)*, let* $k \in \mathbb{N}$ *and* $k \geq 2$*, we can find three positive constants* $C_i = C_i(n, k, s_0, \|Rm\|_\infty)$ *for* $i = 1, 2, 3$*, such that we have the following integral estimate for pressure function* $p$*:*

$$\int_M |\nabla^k p|^2 d\mu \leq C_1 \int_M |\nabla^{k+1} Rm|^2 d\mu + C_2 \int_M |Rm|^2 d\mu + C_3 \int_M |p|^2 d\mu \tag{4.2.3}$$

*Proof.* With the same idea, we can raise the order of derivative of $p$. Let $k \in \mathbb{N}$,and $k > 1$, we have the following elliptic equation

$$-(n-1)\Delta\nabla^{k-1}p - s_0\nabla^{k-1}p = \nabla^{k-1}\nabla^2 B + \nabla^{k-1}(A \cdot B) + \nabla^{k-1}(p \cdot Rm)$$

Multiply this equation by $\nabla^{k-1}p$ and integrate over the manifold, we have:

$$\begin{aligned}
&(n-1)\int_M |\nabla^k p|^2 d\mu - s_0 \int_M |\nabla^{k-1}p|^2 d\mu \\
= &\int_M \nabla^{k-1}p \cdot P_2^{k+1}(Rm) d\mu + \int_M \nabla^{k-1}p \cdot P_3^{k-1}(Rm) + \int_M \nabla^{k-1}p\nabla^{k-1}(p \cdot Rm) d\mu
\end{aligned}$$

Therefore, we have the following estimate by Hölder inequality, Young's inequality and Proposition D.1.6. The first term can be written as:

$$
\begin{aligned}
\int_M \nabla^{k-1}p \cdot P_2^{k+1}(Rm)d\mu &\leq \left(\int_M |\nabla^{k-1}p|^2 d\mu\right)^{\frac{1}{2}} \left(\int_M |P_4^{2k+2}(Rm)|d\mu\right)^{\frac{1}{2}} \\
&\leq \frac{1}{2}\int_M |\nabla^{k-1}p|^2 d\mu + \frac{1}{2}\int_M |P_4^{2k+2}(Rm)|d\mu \\
&\leq \frac{1}{2}\int_M |\nabla^{k-1}p|^2 d\mu + \frac{C(n,k)}{2}\|Rm\|_\infty^2 \int_M |\nabla^{k+1}Rm|^2 d\mu
\end{aligned} \tag{4.2.4}
$$

With the same argument, the second term can be written as:

$$
\begin{aligned}
\int_M \nabla^{k-1}p \cdot P_2^{k+1}(Rm)d\mu &\leq \left(\int_M |\nabla^{k-1}p|^2 d\mu\right)^{\frac{1}{2}} \left(\int_M |P_6^{2k-2}(Rm)|d\mu\right)^{\frac{1}{2}} \\
&\leq \frac{1}{2}\int_M |\nabla^{k-1}p|^2 d\mu + \frac{1}{2}\int_M |P_6^{2k-2}(Rm)|d\mu \\
&\leq \frac{1}{2}\int_M |\nabla^{k-1}p|^2 d\mu + \frac{C(n,k)}{2}\|Rm\|_\infty^4 \int_M |\nabla^{k-1}Rm|^2 d\mu
\end{aligned} \tag{4.2.5}
$$

For the last term, we use integration by parts first

$$
\left|\int_M \nabla^{k-1}p\nabla^{k-1}(p \cdot Rm)d\mu\right| = \left|\int_M \nabla^{2k-2}p \cdot p \cdot Rmd\mu\right| \tag{4.2.6}
$$

we use a telescoping formula

$$
\nabla^{2k-2}p \cdot Rm = \nabla^{k-1}(\nabla^{k-1}p \cdot Rm) - \nabla^{k-2}(\nabla^{k-1}p \cdot \nabla Rm) \tag{4.2.7}
$$

therefore, the last term can be written as

$$
\begin{aligned}
&\left|\int_M \nabla^{k-1}p\nabla^{k-1}(p\cdot Rm)d\mu\right| \\
&=\left|\int_M \nabla^{k-1}(\nabla^{k-1}p\cdot Rm)\cdot pd\mu\right|+\left|\int_M \nabla^{k-2}(\nabla^{k-1}p\cdot \nabla Rm)\cdot pd\mu\right| \\
&=\left|\int_M \nabla^{k-1}p\cdot Rm\cdot \nabla^{k-1}pd\mu\right|+\left|\int_M \nabla^{k-1}p\cdot \nabla Rm\cdot \nabla^{k-2}pd\mu\right| \\
&\leq 2\left|\int_M \nabla^{k-1}p\cdot Rm\cdot \nabla^{k-1}pd\mu\right|+\left|\int_M \nabla^{k}p\cdot Rm\cdot \nabla^{k-2}pd\mu\right| \\
&\leq 2\left\|Rm\right\|_\infty\int_M |\nabla^{k-1}p|^2d\mu+\frac{1}{2}\int_M |\nabla^k p|^2d\mu+\frac{1}{2}\left\|Rm\right\|_\infty^2\int_M |\nabla^{k-2}p|^2d\mu
\end{aligned}
\tag{4.2.8}
$$

where we use Young's inequality at the last line. Now we collect all of these results, we have:

$$
\begin{aligned}
(n-\frac{3}{2})\int_M |\nabla^k p|^2 d\mu \leq (1+s_0+2\left\|Rm\right\|_\infty)&\int_M |\nabla^{k-1}p|^2d\mu \\
+\frac{1}{2}\left\|Rm\right\|_\infty^2&\int_M |\nabla^{k-2}p|^2d\mu \\
+\frac{1}{2}C(n,k)\left\|Rm\right\|_\infty^2&\int_M |\nabla^{k+1}Rm|^2d\mu \\
+\frac{1}{2}C(n,k)\left\|Rm\right\|_\infty^4&\int_M |\nabla^{k-1}Rm|^2d\mu
\end{aligned}
\tag{4.2.9}
$$

From here, we iterate this result, in the meantime, we use Proposition D.1.4 to raise any $\int_M |\nabla^j Rm|d\mu$ to $\int_M |\nabla^{k+1}Rm|d\mu$, in the end, for any $\epsilon\geq 0$, we can find three positive constant $C_i=C_i(n,k,s_0,\left\|Rm\right\|_\infty)$ for $i=1,2,3$, such that

$$
\int_M |\nabla^k p|^2d\mu\leq C_1\int_M |\nabla^{k+1}Rm|^2d\mu+C_2\int_M |Rm|^2d\mu+C_3\int_M |p|^2d\mu \tag{4.2.10}
$$

□

## 4.3 Integral Estimate of Curvatures

After deriving some key evolution equations and an estimate of pressure function, we are proceeding to the classical integral estimate of curvatures. In this section, we rely on interpolation inequalities which are derived in Appendix D.1, we also refer most of interpolation inequalities to [25, Sec 12]. Such integral estimates were showed on several papers to deal with higher order geometric flows [30, Sec.3][51, Sec.5][37, Sec.4].

**Integral Estimate for** $\int_M |Rm|^2 d\mu$

First, we have the following lemma to convert $\int_M |\Delta Rm|^2 d\mu$ to $\int_M |\nabla^2 Rm|^2 d\mu$

**Lemma 4.3.1.** *Let $(M^n, g)$ be a closed n-dimensional manifold and $T$ is any tensor defined on $M$, we have the following inequality:*

$$\int_M |\nabla^2 T|^2 d\mu = \int_M |\Delta T|^2 d\mu + \int_M \nabla T * \nabla T * Rm d\mu + \int_M \nabla^2 T * T * Rm d\mu \quad (4.3.1)$$

*Proof.* This equation comes from the integration by parts and Ricci identity (A.6.1).

$$\begin{aligned}
\int_M \langle \nabla^2 T, \nabla^2 T \rangle d\mu &= \int_M \langle \nabla_i \nabla_j T, \nabla_i \nabla_j T \rangle d\mu \\
&= \int_M \langle \nabla_i \nabla_j T, \nabla_j \nabla_i T \rangle d\mu + \int_M \nabla^2 T * T * Rm d\mu \\
&= -\int_M \langle \nabla_j T, \nabla_i \nabla_j \nabla_i T \rangle d\mu + \int_M \nabla^2 T * T * Rm d\mu \\
&= -\int_M \langle \nabla_j T, \nabla_j \nabla_i \nabla_i T \rangle d\mu + \int_M \nabla T * \nabla T * Rm d\mu \\
&\quad + \int_M \nabla^2 T * T * Rm d\mu \\
&= \int_M |\Delta T|^2 d\mu + \int_M \nabla T * \nabla T * Rm d\mu + \int_M \nabla^2 T * T * Rm d\mu
\end{aligned}$$

□

With this lemma, combine with our previous result in Lemma 4.1.2, we have

**Lemma 4.3.2.** *Given* $(M^n, g(t))$ *to be the compact solution to conformal Bach flow* (3.0.1)*, we have the following integral estimate for Riemann curvature tensor:*

$$\frac{\partial}{\partial t}\int |Rm|^2 d\mu + \frac{3}{2}\int_M |\nabla^2 Rm|^2 d\mu \leq C(n)(\|Rm\|_\infty + \|Rm\|_\infty^2)\int_M |Rm|^2 d\mu + C(n)\int_M |p|^2 d\mu \tag{4.3.2}$$

*where* $C(n)$ *is a constant depends on dimension of manifold.*

*Proof.* Under the conformal Bach flow, we have:

$$Tr_g\Big(\frac{\partial}{\partial t}g_{ij}\Big) = 2(n-2)\Big(Tr(B_{ij}) + pTr(g_{ij})\Big) = 2n(n-2)p$$

Therefore, from Lemma 4.1.5, we have

$$\begin{aligned}\frac{\partial}{\partial t}\int |Rm|^2 d\mu &= 2\int_M \langle Rm, \frac{\partial}{\partial t}Rm\rangle d\mu + \frac{1}{2}\int_M |Rm|^2 Tr(\frac{\partial}{\partial t}g) d\mu\\ &= -2\int_M |\Delta Rm|^2 d\mu + \int_M P_3^2(Rm) d\mu + \int_M P_4^0(Rm) d\mu\\ &\quad + \int_M Rm \cdot T(\nabla^2 p) d\mu + (n-4)(n-2)\int_M p|Rm|^2 d\mu\end{aligned}$$

Now we apply Lemma 4.3.1 and interpolation inequality D.1.6, we have the following estimate:

$$\begin{aligned}\frac{\partial}{\partial t}\int |Rm|^2 d\mu \leq &-2\int_M |\nabla^2 Rm|^2 d\mu + C_1\|Rm\|_\infty \int_M |\nabla Rm|^2 d\mu\\ &+ C_2\|Rm\|_\infty^2 \int_M |Rm|^2 d\mu + \int_M Rm\cdot T(\nabla^2 p) d\mu\\ &+ (n-4)(n-2)\int_M p|Rm|^2 d\mu\end{aligned}$$

Also, we have the following observation, from the definition of $T(\nabla^2 p)$ in Lemma

4.1.2, we have

$$\int_M Rm \cdot T(\nabla^2 p) d\mu = (n-2) \int_M R_{ijkl} \nabla^i \nabla^l p g^{jk} + R_{ijkl} \nabla^j \nabla^j p g^{il} d\mu$$
$$= -(n-2) \int_M R_{il} \nabla^i \nabla^l p + R_{jk} \nabla^j \nabla^j p d\mu$$

After using integration by part, this term actually vanishes since we preserve the constant scalar curvature under this flow.

For the rest terms, we use interpolation inequality (D.1.4) and Hölder inequality again, we have:

$$\int_M |\nabla Rm|^2 d\mu \leq \epsilon \int_M |\nabla^2 Rm|^2 d\mu + C(n, \epsilon) \int_M |Rm|^2 d\mu$$

we can simply choose $\epsilon = \frac{1}{2}$. And

$$\int_M p|Rm|^2 d\mu \leq \left( \int_M |p|^2 d\mu \right)^{\frac{1}{2}} \left( \int_M |Rm|^4 d\mu \right)^{\frac{1}{2}}$$
$$\leq \frac{1}{2} \int_M |p|^2 d\mu + \frac{1}{2} \|Rm\|_\infty^2 \int_M |Rm|^2 d\mu$$

Combine all of these results, we have:

$$\frac{\partial}{\partial t} \int |Rm|^2 d\mu + \frac{3}{2} \int_M |\nabla^2 Rm|^2 d\mu \leq C(n)(\|Rm\|_\infty + \|Rm\|_\infty^2) \int_M |Rm|^2 d\mu$$
$$+ C(n) \int_M |p|^2 d\mu$$

here we don't specify the constants but both of them only depend on the dimension of manifold. □

**Estimate for** $\int_M |\nabla^k Rm|^2 d\mu$

Now we are ready to estimate the higher order derivatives of Riemann curvature, we have the following proposition.

**Lemma 4.3.3.** *Given $(M^n, g(t))$ to be the compact solution to conformal Bach flow* (3.0.1)*, let $k \in \mathbb{N}$, we can find two constants $C_1$ and $C_2$ depend on $s_0, n, k$ and $\|Rm\|_\infty$ such that we have the following integral estimate for $k^{th}$ order derivative of Riemann curvature tensor:*

$$\frac{\partial}{\partial t}\int_M |\nabla^k Rm|^2 d\mu + \int_M |\nabla^{k+2} Rm|^2 d\mu \leq C_1 \int_M |Rm|^2 d\mu + C_2 \int_M |p|^2 d\mu \tag{4.3.3}$$

*Proof.* From Proposition 4.1.5, we have:

$$\begin{aligned}\frac{\partial}{\partial t}\int_M |\nabla^k Rm|^2 d\mu = &-2\int_M |\Delta\nabla^k Rm|^2 d\mu + \int_M P_3^{2k+2}(Rm) d\mu + \int_M P_4^{2k}(Rm) d\mu \\ &+ 2\int_M \nabla^k Rm * \nabla^k[T(\nabla^2 p)] d\mu + \int_M \nabla^k Rm * \nabla^k(pRm) d\mu\end{aligned}$$

Now we can estimate the right hand side term by term. First, we have the leading term, with Lemma 4.3.1,

$$-2\int_M |\Delta\nabla^k Rm|^2 d\mu = -2\int_M |\nabla^{k+2} Rm|^2 d\mu + \int_M P_3^{2k+2}(Rm) d\mu + \int_M P_4^{2k}(Rm) d\mu \tag{4.3.4}$$

The last two terms are absorbed and we use interpolation inequality in Proposition D.1.6, we have the following two estimates:

$$\int_M P_3^{2k+2}(Rm) d\mu \leq C(n,k)\, \|Rm\|_\infty \int_M |\nabla^{k+1} Rm|^2 d\mu \tag{4.3.5}$$

and

$$\int_M P_4^{2k}(Rm) d\mu \leq C(n,k)\, \|Rm\|_\infty^2 \int_M |\nabla^{k} Rm|^2 d\mu \tag{4.3.6}$$

We don't specify the constants in these two estimates but both of them depend on the dimension and the order of covariant derivatives. For the next term, we first use integration by parts twice, then Young's inequality gives us desirable result.

$$\begin{aligned}\int_M \nabla^k Rm * \nabla^k[T(\nabla^2 p)]d\mu &= \int_M \nabla^{k+2} Rm * \nabla^{k-2}[T(\nabla^2 p)]d\mu \\ &\leq \left(\int_M |\nabla^{k+2} Rm|^2 d\mu\right)^{\frac{1}{2}} \left(\int_M |\nabla^k p|^2 d\mu\right)^{\frac{1}{2}} \\ &\leq \frac{1}{2}\int_M |\nabla^{k+2} Rm|^2 d\mu + \frac{1}{2}\int_M |\nabla^k p|^2 d\mu\end{aligned} \tag{4.3.7}$$

For the last term, we have

$$\int_M \nabla^k Rm * \nabla^k (pRm) d\mu = \int_M \nabla^{2k} Rm * pRm d\mu \tag{4.3.8}$$

with a telescoping argument, we have:

$$\begin{aligned}\nabla^{2k} Rm * Rm &= \sum_{j=0}^{k} \nabla^{2k-j} Rm * \nabla^j Rm - \sum_{j=0}^{k-1} \nabla^{k+j} Rm * \nabla^{k-j} Rm \\ &= \nabla^k(\nabla^k Rm * Rm) - \nabla^{k-1}(\nabla^k Rm * \nabla Rm)\end{aligned} \tag{4.3.9}$$

therefore, we rewrite (4.3.8) to be

$$\begin{aligned}&\left|\int_M \nabla^k Rm * \nabla^k (pRm) d\mu\right| \\ &\leq \left|\int_M \nabla^k(\nabla^k Rm * Rm) * p d\mu\right| + \left|\int_M \nabla^{k-1}(\nabla^k Rm * \nabla Rm) * p d\mu\right| \\ &\leq \left|\int_M \nabla^k Rm * Rm * \nabla^k p d\mu\right| + \left|\int_M \nabla^k Rm * \nabla Rm * \nabla^{k-1} p d\mu\right|\end{aligned} \tag{4.3.10}$$

We have the following estimates

$$
\begin{aligned}
&\left|\int_M \nabla^k Rm * Rm * \nabla^k p d\mu\right| \\
&\leq \|Rm\|_\infty \left|\int_M \nabla^k Rm * \nabla^k p d\mu\right| \\
&\leq \frac{1}{2}\|Rm\|_\infty^2 \int_M |\nabla^k Rm|^2 d\mu + \frac{1}{2}\int_M |\nabla^k p|^2 d\mu
\end{aligned} \tag{4.3.11}
$$

and

$$
\begin{aligned}
&\left|\int_M \nabla^k Rm * \nabla Rm * \nabla^{k-1} p d\mu\right| \\
&= \left|\int_M \nabla^{k+1} Rm * Rm * \nabla^{k-1} p d\mu\right| + \left|\int_M \nabla^k Rm * Rm * \nabla^k p d\mu\right| \\
&\leq \|Rm\|_\infty \int_M |\nabla^{k+1} Rm * \nabla^{k-1} p| d\mu + \|Rm\|_\infty \int_M |\nabla^k Rm * \nabla^k p| d\mu \\
&\leq \frac{1}{2}\|Rm\|_\infty^2 \left(\int_M |\nabla^{k+1} Rm|^2 d\mu + \int_M |\nabla^k Rm|^2 d\mu\right) \\
&\quad + \frac{1}{2}\left(\int_M |\nabla^{k-1} p|^2 d\mu + \int_M |\nabla^k p|^2 d\mu\right)
\end{aligned} \tag{4.3.12}
$$

Collecting the results from (4.3.5)-(4.3.7),(4.3.11) and (4.3.12), we have

$$
\begin{aligned}
&\frac{\partial}{\partial t}\int_M |\nabla^k Rm|^2 d\mu + \frac{3}{2}\int_M |\nabla^{k+2} Rm|^2 d\mu \\
&\leq \Big(C(n,k)\|Rm\|_\infty + \frac{1}{2}\|Rm\|_\infty^2\Big)\int_M |\nabla^{k+1} Rm|^2 d\mu \\
&\quad + \Big(C(n,k)+1\Big)\|Rm\|_\infty^2 \int_M |\nabla^k Rm|^2 d\mu + \frac{3}{2}\int_M |\nabla^k p|^2 d\mu + \frac{1}{2}\int_M |\nabla^{k-1} p|^2 d\mu
\end{aligned} \tag{4.3.13}
$$

With Lemma 4.2.2 and Proposition D.1.4, we have the following estimate:

$$
\frac{\partial}{\partial t}\int_M |\nabla^k Rm|^2 d\mu + \int_M |\nabla^{k+2} Rm|^2 d\mu \leq C_1 \int_M |Rm|^2 d\mu + C_2 \int_M |p|^2 d\mu \tag{4.3.14}
$$

where we choose a suitable $\epsilon$ such that the top order term on the right hand side can be absorbed by $\frac{3}{2}\int_M |\nabla^{k+2}Rm|^2 d\mu$ on the left. The remaining constants $C_1$ and $C_2$ depend on $s_0, n, k, \|Rm\|_\infty$. $\square$

## 4.4 Volume Estimate

In this section, we will derive a volume estimate of manifold under the conformal Bach flow and conformal gradient flow. In general, the volume growth only depends on the pressure function due to the trace free property for both Bach tensor and gradient of $L^2$ norm of Weyl curvature. But for the conformal gradient flow, a more desirable volume estimate is that using the gradient property, that is $L^2$ norm of Weyl curvature is non-increasing. Furthermore, in dimension 4, this monotonicity combine with Gauss-Bonnet-Chern formula (2.3.3) provides a better estimate for us.

**Volume Estimate**

In this subsection, we will discuss the volume estimate for manifold deforming under conformal Bach flow and conformal gradient flow. In the following lemma, we will see that the volume growth only depends on the $L^2$ norm of pressure function. Furthermore, in dimension 4, we obtain a volume estimate without any assumption on pressure function.

**Lemma 4.4.1.** *Given* $(M^n, g(t))$ *to be a closed solution to conformal Bach flow* (3.0.1) *or conformal gradient flow* (2.8.4)*, we have the following volume estimate:*

$$vol(t) \leq \Big(\frac{n(n-2)t}{2}\int_M p^2 d\mu\Big)^{\frac{1}{2}} \tag{4.4.1}$$

*Proof.* By the variation formula of Riemann volume form (B.2.1), for both flows, we have

$$\frac{d}{dt}vol(t) = \frac{d}{dt}\int_M d\mu = n(n-2)\int_M p d\mu$$

by Hölder inequality, we have

$$\frac{d}{dt}\int_M d\mu \le n(n-2)\Big(\int_M p^2 d\mu\Big)^{\frac{1}{2}}\Big(\int_M d\mu\Big)^{\frac{1}{2}}$$

Therefore, we have:

$$vol(t) \le \Big(\frac{n(n-2)t}{2}\int_M p^2 d\mu\Big)^{\frac{1}{2}} \tag{4.4.2}$$

□

From here, we can see that if we assume the $L^2$ norm of pressure function is bounded, then we have the volume control, volume growth is at most proportional to $t^{\frac{1}{2}}$.

Next, we will discuss the volume estimate for conformal gradient flow with monotonicity property of $L^2$ norm of Weyl curvature.

**Volume Estimate for Conformal Gradient flow**

**Lemma 4.4.2.** *Given $(M^n, g(t))$ to be the closed solution to conformal Bach flow* (2.8.4)*, we have the following volume estimate*

$$vol(t) \le e^{Ct}\Big(vol(0) + \frac{1}{2}\int_M |W[g(0)]|^2 d\mu\Big) \tag{4.4.3}$$

*where $C = \frac{n^2(n-2)^2}{2s_0^2}\|Rm\|_\infty^2$, $vol(0)$ is the initial volume, and $W[g(0)]$ is the initial Weyl curvature.*

*Proof.* With the same calculation, we have

$$\begin{aligned}
\frac{d}{dt}vol(t) &= n(n-2)\int_M p d\mu \\
&\leq \frac{n(n-2)}{|s_0|}\int_M Ric\cdot grad\mathcal{F}_W d\mu \\
&\leq \left(\frac{n^2(n-2)^2}{s_0^2}\int_M |Ric|^2 d\mu\right)^{\frac{1}{2}}\left(\int_M |grad\mathcal{F}_W|^2 d\mu\right)^{\frac{1}{2}} \\
&\leq \frac{n^2(n-2)^2}{2s_0^2}\int_M |Ric|^2 d\mu + \frac{1}{2}\int_M |grad\mathcal{F}_W|^2 d\mu \\
&\leq \frac{n^2(n-2)^2\,\|Rm\|_\infty^2}{2s_0^2}\int_M d\mu + \frac{1}{2}\int_M |grad\mathcal{F}_W|^2 d\mu \\
&= C(n, s_0, \|Rm\|_\infty)vol(t) + \frac{1}{2}\int_M |grad\mathcal{F}_W|^2 d\mu
\end{aligned}$$

Therefore, we obtain a differential inequality, and we have:

$$\begin{aligned}
\frac{d}{dt}\Big[e^{-Ct}vol(t)\Big] &= e^{-Ct}\Big[\frac{d}{dt}vol(t) - Cvol(t)\Big] \\
&\leq \frac{1}{2}e^{-Ct}\int_M |grad\mathcal{F}_W|^2 d\mu \\
&\leq \frac{1}{2}\int_M |grad\mathcal{F}_W|^2 d\mu
\end{aligned}$$

Integrating this inequality from 0 to $t$, we have:

$$\begin{aligned}
e^{-Ct}vol(t) - vol(0) &\leq \frac{1}{2}\int_0^t\int_M |grad\mathcal{F}_W|^2 d\mu ds \\
&= \frac{1}{2}\int_M |W[g(0)]|^2 d\mu - \frac{1}{2}\int_M |W[g(t)]|^2 d\mu \\
&\leq \frac{1}{2}\int_M |W[g(0)]|^2 d\mu
\end{aligned}$$

In the end, we conclude that the volume estimate is

$$vol(t) \leq e^{Ct}\Big(vol(0) + \frac{1}{2}\int_M |W[g(0)]|^2 d\mu\Big) \tag{4.4.4}$$

□

This result shows that under the conformal gradient flow, the volume growth is related to initial metric and $\|Rm[g(t)]\|_{\infty}$. In next lemma, we will investigate the case when dimension is 4. In dimension 4, conformal Bach flow and conformal gradient flow coincide. Also, we are going to use Gauss-Bonnet-Chern theorem to handle the curvature term showed up in previous lemma.

**Volume Estimate in Dimension 4**

**Lemma 4.4.3.** *Given* $(M^4, g(t))$ *to be the closed solution to conformal Bach flow* (3.0.1)*, suppose that the initial metric* $g_0$ *satisfies*

$$\int_M \sigma_2[g(0)] \geq 0$$

*then we have the following volume estimate*

$$vol(t) \leq e^{\frac{8}{3}t}\Big(vol(0) + \frac{1}{2}\int_M |W[g(0)]|^2 d\mu\Big) \tag{4.4.5}$$

*Proof.* The calculation is similar to the previous lemma. We replace the Ricci curvature by traceless Ricci curvature and we have:

$$\begin{aligned}
\frac{d}{dt}vol(t) &= 8\int_M p d\mu \\
&\leq \frac{8}{|s_0|}\int_M E \cdot grad\mathcal{F}_W d\mu \\
&\leq \left(\frac{64}{s_0^2}\int_M |E|^2 d\mu\right)^{\frac{1}{2}}\left(\int_M |grad\mathcal{F}_W|^2 d\mu\right)^{\frac{1}{2}} \\
&\leq 32\int_M \frac{|E|^2}{s_0^2} d\mu + \frac{1}{2}\int_M |grad\mathcal{F}_W|^2 d\mu
\end{aligned}$$

Recall the Gauss-Bonnet-Chern's formula(2.3.3):

$$32\pi^2\chi(M^4) = \int_M |W|^2 + 4\sigma_2(A_g) d\mu \tag{4.4.6}$$

where $\sigma_2(A_g) = \frac{1}{24}R^2 - \frac{1}{2}|E|^2$. By the property of conformal Bach flow in dimension 4, $\int_M \sigma_2 d\mu$ is non decreasing, that is:

$$\int_M \sigma_2[g(t)]d\mu \geq \int_M \sigma_2[g(0)]d\mu \geq 0 \tag{4.4.7}$$

Therefor, we have:

$$\int_M \frac{1}{24}R^2 - \frac{1}{2}|E|^2 d\mu \geq 0 \tag{4.4.8}$$

for all $t$, and we conclude that

$$\int_M \frac{|E|^2}{s_0^2} d\mu \leq \frac{1}{12} vol(t) \tag{4.4.9}$$

and

$$\frac{d}{dt} vol(t) \leq \frac{8}{3} vol(t) + \frac{1}{2}\int_M |grad\mathcal{F}_W|^2 d\mu \tag{4.4.10}$$

By the same argument before, we have the volume estimate

$$vol(t) \leq e^{\frac{8}{3}t}\Big(vol(0) + \frac{1}{2}\int_M |W[g(0)]|^2 d\mu\Big) \tag{4.4.11}$$

In dimension 4, the volume growth depends only on initial metric $g_0$ and $t$. $\square$

In dimension 4, one direct result from the volume estimate is the following consequence of Sobolev constant.

**Corollary 4.4.4.** *Given* $(M^4, g(t))$ *to be the compact solution to conformal Bach flow* (3.0.1), *suppose that the initial metric* $g_0$ *satisfies*

$$\int_M \sigma_2[g(0)] \geq 0$$

*and*

$$Y[M,g]>0$$

*The Sobolev constant $C_s[g(t)]$ remains bounded.*

*Proof.* To see this, we recall that in dimension 4, we derived the following inequality (2.5.4)

$$C_s(g)\leq \frac{\max\{6, R_g V^{\frac{1}{2}}\}}{Y_{[g]}} \tag{4.4.12}$$

Under the conformal Bach flow, the volume has an upper bound by our previous estimate. We only need to show that Yamabe constant has a lower bound, then we can conclude that Sobolev constant has an upper bound.

To see this, under conformal Bach flow, we have

$$\int_M \sigma_2[g(0)] \leq \int_M \sigma_2[g(t)]d\mu \leq \int_M \frac{1}{24}R^2_{g(t)}d\mu = \frac{1}{24}\frac{\left(\int_M R_g d\mu\right)^2}{\int_M d\mu} \tag{4.4.13}$$

therefore, we have a uniform lower bounded for Yamabe constant such that

$$\frac{1}{24}Y^2_{[g]} \geq \int_M \sigma_2[g(0)] \geq 0 \tag{4.4.14}$$

and we conclude that Sobolev constant has an upper bound. □

## 4.5 Long Time Behavior

In this section, we will discuss the long time behavior of conformal gradient flow under some assumptions. Since this is the gradient flow for $L^2$ norm of Weyl curvature in any dimension, which is more interesting than conformal Bach flow.

We are going to prove a long time behavior theorem, under certain conditions, the conformal gradient flow will exist for a long time. Our proof follows [15, §7.2].

We will use a proof by contradiction, if the solution to conformal gradient flow only exists on a finite time interval $[0, T)$ and $T < \infty$, and all of three conditions in Theorem 4.5.1 hold, we will see that when $t$ approaches to $T$, the metric $g(t)$ converges to a smooth metric $g(T)$ such that the flow passes $T$.

To show this, we first realize that Sobolev inequalities helps us convert all global bounds for curvature and its derivatives to their point-wise bounds. That is $|\nabla^k Rm|_{g(t)}$ and $|\nabla^k p|_{g(t)}$ are bounded. Such bounds induce the bounds for $|\tilde{\nabla}^k Rm|_{\tilde{g}}$ and $|\tilde{\nabla}^k p|_{\tilde{g}}$ for some fixed background metric $\tilde{g}$. Under these conditions, Lemma 4.5.4 shows that $g(t)$ converges to a continuous metric $g(T)$, and next lemma show that this metric $g(T)$ is actually smooth which allows us to extend the solution pass $T$ and contradicts to our assumption.

**Theorem 4.5.1.** *Let $(M^n, g(t), p(t))$, $t \in [0, T)$, $x \in M$ be a smooth solution of conformal gradient flow on a closed manifold $M^n$ with constant scalar curvature $s_0$. We assume that there is a constant $K > 0$ such that the curvature of $g(t)$ and potential function $p(t)$ satisfy the following conditions*

*(a)* $\sup_{x \in M} |Rm(x,t)|_{g(t)} \leq K$ *for* $t \in [0, T)$

*(b) the best Sobolev constant satisfies* $C_S(g(t)) \leq K$

*(c) the elliptic operator* $(n-1)\Delta_{g(t)} + s_0$ *is uniformly invertible, that is*

$$\left\| (n-1)\Delta_{g(t)} + s_0 \right\|_{L(C^\alpha, C^{2+\alpha})} \leq C_E$$

*for some constant* $C_E$.

*Then $(M^n, g(t), p(x,t))$ can be extend to a solution of conformal gradient flow on $[0, T+\delta)$ for some $\delta > 0$.*

Our proof follows [15, Prop.6.48; Lemma 6.49; §7.2]. Before we prove this main theorem, we state some key elements in this proof as following lemmas. First, we show that under these conditions, all curvatures and their derivatives are bounded in $L^\infty$ sense. After that, we will show that the metric $g(t)$ converges to a smooth metric $g(T)$, which allows us to extend the flow.

**Lemma 4.5.2.** *Given* $(M^n, g(t))$ *to be the compact solution to conformal gradient flow* (2.8.4)*, which satisfies all condition in Theorem 4.5.1. Let* $k \in \mathbb{N}$*, we can find constants* $C_k = C_k(n, k, g_0, vol(0), \|Rm\|_\infty, T)$ *such that such that we have the following estimate for* $k^{th}$ *order derivative of Riemann curvature tensor:*

$$\int_M |\nabla^k Rm|^2 d\mu \leq C_k \tag{4.5.1}$$

*Proof.* Recall that by our volume estimate for conformal gradient flow Lemma 4.4.1, we simply use a constant $C = C(n, g_0, vol(0), \|Rm\|_\infty, T)$ to be the volume bound. Therefore, we immediately obtain

$$\int_M |Rm|^2 d\mu \leq \|Rm\|_\infty^2 C \equiv C_0$$

For higher order derivatives, recall the integral estimates we derived in Lemma 4.3.3:

$$\frac{\partial}{\partial t}\int_M |\nabla^k Rm|^2 d\mu + \int_M |\nabla^{k+2} Rm|^2 d\mu \leq C_1 \int_M |Rm|^2 d\mu + C_2 \int_M |p|^2 d\mu \tag{4.5.2}$$

which depends on $\int_M |Rm|^2 d\mu$ and $\int_M |p|^2 d\mu$. We assume that the elliptic operator is invertible, that means there is a constant $C$ such that:

$$\begin{aligned}\int_M |p|^2 d\mu \leq \|p\|_{W^{2,2}} \leq C_E \int_M |P_2^2(Rm)^2 d\mu + C_E \int_M |P_3^0(Rm)^2 d\mu \\ \leq C_E C_1(n) \|Rm\|_\infty^2 \int_M |\nabla^2 Rm|^2 d\mu \\ + C_E C_2(n) \|Rm\|_\infty^4 \int_M |Rm|^2 d\mu\end{aligned} \tag{4.5.3}$$

where we use the interpolation inequality (D.1.6). Combine (4.5.2) and (4.5.3), using the interpolation inequality (D.1.4) to raise derivative, for any $k \geq 1$, we have the following estimate:

$$\frac{\partial}{\partial t} \int_M |\nabla^k Rm|^2 d\mu \leq C(n, k, g_0, vol(0), \|Rm\|_\infty, T) \tag{4.5.4}$$

Therefore, we find constant $C_k$ for $k \in \mathbb{N}$ such that

$$\int_M |\nabla^k Rm|^2 d\mu \leq C_k \tag{4.5.5}$$

□

**Remark 4.5.3.** *With this result, all of $L^2$ norms of pressure function and its derivatives are immediately bounded.*

In the next lemma, we prove that under certain condition, when $t$ approached to the maximal time $T$, the metric converges to a continuous metric which is equivalent to the initial metric.

**Lemma 4.5.4.** *Let $M^n$ be a closed Riemannian manifold. For $t \in [0, T)$, where $T \leq \infty$, let $g(t)$ be a one-parameter family of metric on $M^n$ depends on space and time smoothly. If there is a constant $C > 0$ such that*

$$\int_0^T \left| \frac{\partial}{\partial t} g(x,t) \right|_{g(t)} dt \leq C$$

*for all $x \in M$, then*

$$e^{-C}g(x,0) \leq g(x,t) \leq e^{C}g(x,0) \tag{4.5.6}$$

*for all $x \in M$ and $t \in [0,T)$. Furthermore, as $t$ approaches to $T$, the metrics $g(t)$ converge uniformly to a continuous metric $g(T)$ such that for all $x \in M$,*

$$e^{-C}g(x,0) \leq g(x,T) \leq e^{C}g(x,0) \tag{4.5.7}$$

*Proof.* For any $(x,t_0) \in M^n \times [0.T)$, let $V \in T_xM$ be any vector in tangent space. The length of this vector $|V|_{g(t)} = g_{(x,t_0)}(V,V)$, we have

$$\begin{aligned}
\left|\log\Big(\frac{g_{(x,t_0)}(V,V)}{g_{(x,0)}(V,V)}\Big)\right| &= \left|\int_0^{t_0} \frac{\partial}{\partial t}\log\Big[g_{(x,t)}(V,V)\Big]dt\right| \\
&= \left|\int_0^{t_0} \frac{\frac{\partial}{\partial t}g_{(x,t)}(V,V)}{g_{(x,t)}(V,V)}dt\right| \\
&\leq \int_0^{t_0}\left|\frac{\partial}{\partial t}g_{(x,t)}(\frac{V}{|V|},\frac{V}{|V|})\right|dt \\
&\leq \int_0^{t_0}\left|\frac{\partial}{\partial t}g(x,t)\right|dt \\
&\leq C
\end{aligned}$$

we have

$$e^{-C}g(x,0) \leq g(x,t_0) \leq e^{C}g(x,0) \tag{4.5.8}$$

This shows that for any $t \in [0,T)$, $g(t)$ is equivalent to $g(0)$. Therefore, we can find a constant $C'$ such that

$$\int_0^T \left|\frac{\partial}{\partial t}g(x,t)\right|_{g(0)} dt \leq C' \tag{4.5.9}$$

Now we define

$$g(x,T) = g(x,0) + \int_0^T g(x,t)dt \tag{4.5.10}$$

with this definition, we have:

$$\lim_{t\to T}|g(x,T)-g(x,t)|_{g(0)} \le \lim_{t\to T}\int_t^T \left|\frac{\partial}{\partial t}g(x,s)\right|_{g(0)} ds \tag{4.5.11}$$

from where we see that metric $g(t)$ converges to $g(T)$ for any $x \in M$. Since $M$ is compact, we conclude that this is a uniformly convergence, so $g(T)$ is continuous, furthermore, it is also equivalent to $g(0)$. $\square$

Now we are ready to apply previous lemma to conformal gradient flow.

**Proposition 4.5.5.** *Let $(M^n, g(t), p(t))$, $t \in [0,T)$, $x \in M$ be a smooth solution of conformal gradient flow on a closed manifold $M^n$ with constant scalar curvature $s_0$. We assume that the conditions in Theorem 4.5.1 hold, then metric $g(t)$ converges uniformly to a continuous metric $g(T)$ when $t$ approaches to $T$, furthermore, $g(T)$ is equivalent to $g(0)$.*

*Proof.* We want to directly apply Lemma 4.5.4, it is suffice to verified that $|\frac{\partial}{\partial t}g|$ is bounded point-wisely. Recall that $(M^n, g)$ is a closed Riemannian manifold, the Sobolev constant $C_S$ is the best constant such that for any function $u \in C_0^1(M)$, we have (2.5.1)

$$\|u\|_{L^{\frac{2n}{n-2}}} \le C_S\Big(\|\nabla u\|_{L^2} + Vol^{-\frac{1}{n}}\|u\|_{L^2}\Big) \tag{4.5.12}$$

We also have the following multiplicative Sobelev inequality (2.5.3). For $u \in C_0^1(M)$, $n < p \le \infty$, $0 \le m \le \infty$, we have

$$\|u\|_\infty \le C_S \cdot C(n,m,p)\,\|u\|_m^{1-\alpha}\Big(\|\nabla u\|_p + \|u\|_p\Big)^\alpha \tag{4.5.13}$$

where $0 < \alpha \le 1$ and $\frac{1}{\alpha} = \left(\frac{1}{n} - \frac{1}{p}\right)m + 1$.

Our aim is to derive the $L^\infty$ bound for all derivatives of curvatures, which requires the $L^p$ bound for curvatures, in which $p$ is strictly greater than the dimension of manifold.

To derive this estimate, we will iterate the Sobolev inequality. For example, see [51, Page 266], we have

$$\begin{aligned}
\|u\|_8^2 &= \Big(\int |u^2|^4\Big)^{\frac{1}{4}} \\
&\le C_s\Big(\left\|\nabla|u^2|\right\|_2 + Vol^{-1}\left\|u^2\right\|_2\Big) \\
&\le C_s \cdot C\Big(\int |u\nabla u|^2\Big)^{\frac{1}{2}} + C_s \cdot Vol^{-1}\|u\|_4^{\frac{1}{2}} \\
&\le C_s \cdot C\left(\|u\|_4 + \|\nabla u\|_4\right) + C_s \cdot Vol^{-1}\|u\|_4^{\frac{1}{2}} \\
&\le C(C_s)\|u\|_{W^{2,3}}
\end{aligned}$$

As for the $L^p$ estimate of derivative of $u$, we need to combine with the Kato's inequality: $|\nabla|\nabla u||^2 \le |\nabla^2 u|^2$, the prove will be identical.

Therefore, these assumptions combine with the integral estimate for curvatures in Lemma 4.5.2, we have:

$$\left|\frac{\partial}{\partial t}g(x,t)\right|_{g(t)} = 2n(n-2)|B+pg|_{g(t)} \le C \tag{4.5.14}$$

in which $C$ is a constant depends on $n, K, T, C_S$ and $C_k$ in Lemma 4.5.2. Then the result follows. $\square$

Now we want to prove that $g(T)$ is actually smooth. We start with the following lemma, then the smoothness of $g(T)$ will be a direct consequence.

**Lemma 4.5.6.** *Let $(M^n, g(t), p(x,t))$, $t \in [0,T)$, $x \in M$ be a smooth solution of conformal gradient flow on a closed manifold $M^n$ with constant scalar curvature $s_0$. We assume that the conditions in Theorem 4.5.1 hold, Let $\mathcal{U} \subset M$ be a local* *coordinate patch, for all $k \in \mathcal{N}$, $(x,t) \in \mathcal{U} \times [0,T)$, we have:*

$$|\partial^k g| \le C \tag{4.5.15}$$

*for some constant $C$ depends on $k, n, s_0, K, C_S, T$. And we simply choose the norm to be Euclidean norm.*

*Proof.* For our convenience, we define some constant $\tilde{C}_k$ such that for every $k \in \mathbb{N}$, we have $|\nabla^k Rm|_{g(t)} \le \tilde{C}_k$ and $|\nabla^k p|_{g(t)} \le \tilde{C}_k$.

In local coordinate system, we have:

$$0 = \nabla_i g_{jk} = \frac{\partial}{\partial x^i} g_{jk} - \Gamma^l_{ij} g_{lk} - \Gamma^l_{ik} g_{jl} \tag{4.5.16}$$

that is

$$\frac{\partial}{\partial x^i} g_{jk} = \Gamma^l_{ij} g_{lk} + \Gamma^l_{ik} g_{jl} \tag{4.5.17}$$

Since $g$ in uniformly bounded on the time interval $[0,T)$, we have:

$$\left| \frac{\partial}{\partial x} g \right| \le C|\Gamma| \tag{4.5.18}$$

for some constant C. To obtain an estimate for Christoffel symbol, we look at its time derivative, and we adopt the aster symbol $*$ from previous proof. By the variation of Christoffel symbol (B.3.1), we have:

$$\frac{\partial}{\partial t}\Gamma^k_{ij} = \frac{1}{2} g^{kl}\Big(\nabla_i B_{jl} + \nabla_j B_{il} - \nabla_l B_{ij}\Big) + \frac{1}{2} g^{kl}\Big(\nabla_i p g_{jl} + \nabla_j p g_{il} - \nabla_l p g_{ij}\Big) \tag{4.5.19}$$

therefore, we have

$$\frac{\partial}{\partial t}\Gamma = C(\nabla B + \nabla p g) = C(P^3_2(Rm) + P^1_3(Rm) + \nabla p g) \tag{4.5.20}$$

The right hand side is clearly bounded since we have $L^\infty$ bound for all derivatives of $Rm$ and $p$. Also, for any two tensor quantities $A$ and $B$

$$|A * B| \leq |A| \cdot |B| \tag{4.5.21}$$

Therefore,

$$\left|\frac{\partial}{\partial t}\Gamma\right| \leq C(g(0), \tilde{C}_0, \tilde{C}_1, \tilde{C}_2, \tilde{C}_3) \tag{4.5.22}$$

Integrating over time interval, we have

$$|\Gamma| \leq C(g(0), \tilde{C}_0, \tilde{C}_1, \tilde{C}_2, \tilde{C}_3, T) \tag{4.5.23}$$

Therefore, $\partial g$ is bounded, where $\partial$ is the ordinary spatial derivative. Inductively, let $k \in \mathbb{N}$ and $\alpha$ is a multi-index of length $k$, $\beta_1$, and $|\beta_2|$ are multi-index with length $j$ and $k-j$, we have:

$$\frac{\partial^k}{\partial x^{|\alpha|}} g = \sum_{j=1}^{k-1} \frac{\partial^j}{\partial x^{|\beta_1|}} \Gamma * \frac{\partial^{k-j}}{\partial x^{|\beta_2|}} g \tag{4.5.24}$$

and

$$\frac{\partial}{\partial t}\frac{\partial^k}{\partial x^{|\alpha|}}\Gamma = P_2^{3+k}(Rm) + P_3^{k+1}(Rm) + \nabla^{k+1} p g \tag{4.5.25}$$

With the same argument,

$$\left|\frac{\partial^k}{\partial x^{|\alpha|}}\Gamma\right| \leq C(g(0), \tilde{C}_0, \cdots, \tilde{C}_{k+3}, T) \tag{4.5.26}$$

and the result follows. $\square$

Now we are ready to prove the main theorem. In Lemma 4.5.4, we show that when $t$ approaches to $T$, the metric $g(t)$ converges to a continuous metric $g(T)$. We are going to show that this metric is in fact smooth, and use it as a new initial condition, we are able to extend the flow forward.

*Proof of Theorem 4.5.1.* Suppose that all conditions hold. Suppose that the solution to conformal gradient flow exists on a finite interval $[0, T)$. Fix a local coordinate around arbitrary point $x \in M$, let $t \in [0, T)$,by Proposition 4.5.5, a continuous metric $g(T)$ exists and defined as

$$g(x,T) = g(x,t) + \int_t^T \frac{\partial}{\partial s} g(x,s) ds \tag{4.5.27}$$

For any $k \in \mathbb{N}$, let $\alpha$ be a multi-index with length $k$, by Lemma 4.5.6, $\partial^k g$ is uniformly bounded, so is $\partial^k Rm$, therefore, we have:

$$\frac{\partial^k}{\partial x^\alpha} g(x,T) = \frac{\partial^k}{\partial x^\alpha} g(x,t) + \int_t^T \frac{\partial^k}{\partial x^\alpha} \frac{\partial}{\partial s} g(x,s) ds \tag{4.5.28}$$

Since $k$ is arbitrary, it shows that $g(x,T) \in C^\infty(M^n)$, and

$$\left| \frac{\partial^k}{\partial x^\alpha} g(x,T) - \frac{\partial^k}{\partial x^\alpha} g(x,t) \right| = \left| \int_t^T \frac{\partial^k}{\partial x^\alpha} \frac{\partial}{\partial s} g(x,s) ds \right| \leq C(T-t) \tag{4.5.29}$$

for some constant $C$, which shows that $g(t) \to g(T)$ in $C^\infty$ sense when $t$ approaches to $T$. Now, with a new initial condition $g_0 = g(T)$, the conformal gradient flow can be extended by the short time existence theorem 3.5.9, this contradicts to our assumption that $T$ is the maximal time, and we complete the proof. □

**Remark 4.5.7.** *In dimension 4, by Corollary 4.4.4, we don't need to assume the Sobolev constant since it comes naturally.*

# Chapter 5

# Compactness and Singularity Models

In this chapter, we will prove a compactness theorem for conformal gradient flow, we will follow Hamilton's compactness theorem for solutions of the Ricci flow [26, Theorem 1.2]. Some similar results can be found in [51, §7] and [37, §7].

In Sec.5.1, we will introduce some definitions and state Cheeger Gromov compactness theorem. Some results in Ricci flow will also be stated. In Sec.5.2, we will prove our main result. In Sec.5.3, we will discuss some singularity models obtained by re-scaling metric.

Throughout this chapter, we will follow the notations and conventions in [14, Chapter 3]. We use $m$ for $m^{th}$ order derivatives. We use subscript $k$ for any quantities depending on metric $g_k$, for example, $|\cdot|_k$, $\nabla_k$ and $Rm_k$ denote the norm, covariant derivative and Riemann curvature tensor with respect to $g_k$. All other quantities without any subscript will be with respect to a fixed background metric $g$.

## 5.1 Introduction

In this section, we will introduce the Cheeger Gromov compactness theorem and some historical results.

**Convergence on Compact Sets**

We first introduce what is convergence on compact sets in manifold. Most of definitions come form [14, Chapter 3].

**Definition 5.1.1** (Definition 3.1, Page 128, [14]). *Let $V \subset M^n$ be a compact set in manifold $M^n$, fix an arbitrary background metric $g$ on $V$, let $\{g_k\}_{k\in\mathbb{N}}$ be a sequence of metric on $V$, we say that $\{g_k\}_{k\in\mathbb{N}}$ uniformly converges to a metric $g_\infty$ on this in $C^p$ sense if for any $\epsilon > 0$, exists an index $k_0$ depending on $\epsilon$ such that for all $k \geq k_0$, we have*

$$\sup_{0\leq\alpha\leq p}\sup_{x\in V}|\nabla_g^\alpha(g_k - g_\infty)|_g \leq \epsilon \tag{5.1.1}$$

For a non-compact case, especially such manifold comes from re-scaling, we must ensure that this convergence still makes sense. In order to define convergence on such manifold, we use an exhaustion on manifold.

**Definition 5.1.2** (Page 128, [14]). *Given a manifold $M^n$, we say that a sequence of open sets $\{U_k\}_{k\in\mathbb{N}}$ to be an exhaustion of $M^n$ if for any compact set $V \subset M^n$, there exists an index $k_0 \in \mathbb{N}$ such that for any $k \geq k_0$, $V \subset U_k$.*

With an exhaustion, we have the following definition about the $C^\infty$ convergence.

**Definition 5.1.3** (Definition 3.2, Page 129, [14]). *Let $\{U_k\}_{k\in\mathbb{N}}$ be an exhaustion on a manifold $M^n$, and $g_k$ are Riemannian metric on $U_k$. We say that $(U_k, g_k)$*

*uniformly converges to in* $C^\infty$ *sense on a compact set in manifold if for any compact set* $V \subset M^n$*, and any* $p > 0$*, there exists an index* $k_0$ *depending on* $p$ *and* $V$ *such that* $\{g_k\}_{k \geq k_0}$ *converges in* $C^p$ *sense to* $g_\infty$ *on* $V$*.*

**Pointed Manifolds and Solutions**

We introduce the so called pointed manifold and the pointed solution to conformal gradient flow.

**Definition 5.1.4** (Definition 3.3, Page 129, [14])**.** *A complete pointed Riemannian manifold is a 4-tuple* $(M^n, g, O, F)$ *where* $O \in M^n$ *is a chosen point called base point and* $F$ *is a frame at point* $O$*.*

**Remark 5.1.5.** *In the rest of this thesis, we define a complete pointed Riemannian manifold to be a 5-tuple* $(M^n, g, p, O, F)$ *instead of 4-tuple. Also, we say that this tuple is a complete pointed solution to conformal gradient flow if* $(M^n, g(t))$ *is a solution to conformal gradient flow* (3.0.1)*.*

**Cheeger-Gromov Compactness**

We first introduce what is Cheeger Gromov convergence. This definition states that a sequence of metrics is convergent after diffeomorphisms.

**Definition 5.1.6** (Page 120, [23])**.** *[Cheeger-Gromov Convergence] Let* $\{(M_k^n, g_k)\}_{k\in\mathbb{N}}$ *be a sequence of smooth compact Riemannian n-manifolds. We say that this sequence converges to a limit manifold* $(M_\infty, g_\infty)$ *in Cheeger Gromov sense if there exists a sequence of diffeomorphisms* $\Phi_k : M_\infty \to M_k$ *such that the sequence of the pull-back metrics* $\{\Phi_k^*(g_k)\}_{k\in\mathbb{N}}$ *converges to* $g_\infty$ *in* $C^{k+\alpha}$*.*

The next theorem states that under certain condition, this convergence happens subsequentially. This theorem consider a space of smooth compact Riemannian n-manifolds under certain conditions on curvatures, diameters and volume, and such space is pre-compact.

**Theorem 5.1.7** (Theorem 8.25,8.28, [24]). *[Cheeger-Gromov Compactness Theorem] Let $\mathcal{M}_{CG}$ be a space of smooth compact Riemannian n-manifolds satisfies:*

*(a)* $|sec(M^n)| \leq K$

*(b)* $diam(M^n) \leq D$

*(c)* $\tau(x) \geq \tau_0$

*for some constants $K, D, v_0$. Here, $sec(M^n)$ is the sectional curvature, $diam(M^n)$ is the diameter and $\tau(x)$ is the injective radius for any point in $M^n$. Then for any sequence $\{(M_k, g_k)\}_{k\in\mathbb{N}}$, there is a sub-sequence and a $C^{1+\alpha}$-Riemannian manifold $(M_\infty, g_\infty)$ with a $C^{2+\alpha}$ atlas of coordinate charts such that there exists a sequence of $C^{2+\alpha}$ diffeomorphisms $\Phi_k : M_\infty \to M_k$ such that the pull-back metrics $\Phi_k^*(g_k) \to g_\infty$ in $C^{1+\alpha}$ sense with respect to a fixed $C^{2+\alpha}$ atlas on $M_\infty$.*

Historically, this theorem showed up on several papers [22][23][45] in 1980's, all of them can trace back to the original idea of Gromov [24] and Cheeger [12].

On can modify the last condition by assuming the volume lower bound since the following Cheeger's theorem.

**Theorem 5.1.8** (Corollary 2.2, [12]). *Let $\mathcal{M}_{CG}$ be a space of smooth compact Riemannian n-manifolds satisfies:*

*(a)* $|sec(M^n)| \leq K$

*(b)* $diam(M^n) \leq D$

*(c)* $\tau(x) \geq \tau_0$

*for some constants $K, D, v_0$. Then there is a constant $C = C(K, D, v_0, n)$ such that the injective radius has a lower bound, $\tau \geq C$.*

**Cheeger-Gromov Compactness(Pointed Manifolds)**

Next, we state a definition about the convergence complete pointed manifolds.

**Definition 5.1.9** (Definition 3.5, Page. 129, [14])**.** *[Cheeger-Gromov Convergence for Pointed Manifolds] Let* $\{(M_k^n, g_k, O_k, F_k)\}_{k\in\mathbb{N}}$ *be a sequence of complete pointed Riemannian n-manifolds. We say that this sequence converges to a limit space* $(M_\infty, g_\infty, O_\infty, F_\infty)$ *in Cheeger Gromov sense if*

*(a) there exists an exhaustion* $\{U_k\}_{k\in\mathbb{N}}$ *of* $M_\infty$ *with* $O_k \in U_k$

*(b) there exists a sequence of diffeomorphisms* $\Phi_k : U_k \to V_k := \Phi_k(U_k) \subset M_k$ *such that* $\Phi_k(O_\infty) = O_k$

*such that* $\left(U_k, \Phi_k^*\left[g_k\big|_{V_k}\right]\right)$ *converges in* $C^\infty$ *to* $(M_\infty, g_\infty)$ *uniformly on compact set in* $M_\infty$*.*

The corresponding definition for sequence of complete pointed solution to conformal Bach flow is given by following.

**Definition 5.1.10** (Definition 3.6, Page. 130, [14])**.** *[Cheeger-Gromov Convergence for Pointed Manifolds Solutions] Let* $\{(M_k^n, g_k(t), p_k(t), O_k, F_k)\}_{k\in\mathbb{N}}$*,* $t \in (\alpha, \omega)$*, be a sequence of complete pointed Riemannian n-manifolds. We say that this sequence converges to a limit space* $(M_\infty, g_\infty(t), p_\infty(t), O_\infty, F_\infty)$ *,* $t \in (\alpha, \omega)$*, in Cheeger Gromov sense if*

*(a) there exists an exhaustion* $\{U_k\}_{k\in\mathbb{N}}$ *of* $M_\infty$ *with* $O_k \in U_k$

*(b) there exists a sequence of diffeomorphisms* $\Phi_k : U_k \to V_k := \Phi_k(U_k) \subset M_k$ *such that* $\Phi_k(O_\infty) = O_k$

*such that* $\left(U_k, \Phi_k^*\left[g_k(t)\big|_{V_k}\right]\right)$ *converges in* $C^\infty$ *to* $(M_\infty, g_\infty(t))$ *and* $p_k(t)$ *converges in* $C^\infty$ *to* $p_\infty(t)$ *uniformly on compact set in* $M_\infty \times (\alpha, \omega)$*.*

**Cheeger-Gromov Compactness(Hamliton's Results)**

In this subsection, we will introduce the convergence theorem proved by Hamilton in [26].

**Theorem 5.1.11** (Theorem 2.3 [26]). *Let $\{(M_k^n, g_k, O_k, F_k)\}_{k\in\mathbb{N}}$ be a sequence of complete pointed Riemannian n-manifolds that satisfies*

*(a) Uniformly bounded covariant derivative of Rm*

$$|\nabla_k^m Rm_k|_k \leq C_m$$

*on every $M_k^n$, for all $m \in \mathbb{N}$ and $C_m \geq 0$ is a sequence of constants independent of $k$.*

*(b) Injective radius lower bound*

$$inj_{g_k}(O_k) \geq \tau_0$$

*for some constant $\tau_0$*

*Then there exists a subsequence $\{k_j\}_{j\in\mathbb{N}}$ such that the sequence $\{(M_{k_j}^n, g_{k_j}, O_{k_j}, F_{k_j})\}_{j\in\mathbb{N}}$ converges to a complete pointed Riemannian manifold $(M_\infty, g_\infty, O_\infty, F_\infty)$.*

We will use this theorem to prove our result in next section.

## 5.2 Compactness

In this section, we will prove a compactness theorem to the pointed solution of conformal Bach flow.

**Theorem 5.2.1.** *Let $\{M_k^n, g_k(t), p_k(t), O_k, F_k\}_{k\in\mathbb{N}}$ be a sequence of pointed solutions to conformal Bach flow on a time interval $(\alpha, \omega)$ such that for all $t \in (\alpha, \omega)$*

*(a) Uniformly bounded curvature*

$$\sup_{x\in M_k} |Rm_k|_k \leq C$$

*(b) Uniformly bounded $L^2$ norm of pressure function $p(t)$*

$$\int_{M_k} |p_k(t)|^2 d\mu \leq C$$

*(c) Uniformly bounded Sobolev constant*

$$C_S(g_k(t)) \leq C$$

*Then there exists a subsequence $\{M^n_{k_j}, g_{k_j}(t), p_{k_j}(t), O_{k_j}, F_{k_j}\}_{j\in\mathbb{N}}$ converges in $C^\infty$ Cheeger-Gromov sense to a complete pointed solution $(M^n_\infty, g_\infty, p_\infty, O_\infty, F_\infty)$ to conformal Bach flow.*

**Remark 5.2.2.** *In this theorem, we don't assume all covariant derivative of Riemann curvature bounded since that combine with our integral estimate in Lemma 4.3.3, volume estimate in Lemma 4.4.1 and Sobolev inequality, we are able to obtain the covariant derivative bound for Riemann curvature.*

**Remark 5.2.3.** *In this theorem, we don't assume injective radius because the assumption on Sobolev constant directly implies the lower bound of injetive radius. We will see this in next subsection.*

We also outline our proof here. In Hamilton's compactness theorem 5.1.11, it only focuses at a fixed time. In order to prove our main theorem, we first see that at a single time slice, say $t = 0$, this convergence holds. Next, we need to show such convergence still holds for all $t \in (\alpha, \omega)$, for which we need to extend bounds on covariant derivatives and time derivative of metric $g$ at $t = 0$ to the whole time interval. This is proved in Lemma 5.2.5. Once we achieve this result, Arzela-Ascoli theorem shows that such bounds imply a subsequence converges to a solution of conformal Bach flow. This is proved in Lemma 5.2.7.

**Injective Radius**

In this subsection, we will sketch a proof to show that the upper bound of Sobolev constant implies the lower bound of injective radius. We refer our proof to several different chapters in [16]. The main theorem we use is the following.

**Theorem 5.2.4** (Theorem 5.4.2, Page. 199, [16])**.** *[Cheeger-Gromov-Taylor Theorem] Let $(M^n, g)$ be a complete Riemannian manifold with $|sec(M)| \leq 1$. For any positive constants $c, r_0$, there is a positive constant $\tau_0$ if $p \in M^n$ satisfies*

$$\frac{vol(B_p(r))}{r^n} \geq c \tag{5.2.1}$$

*for all $r \in (0, r_0]$, then the injective radius at point $p$ has a lower bound, $inj_g(p) \geq \tau_0$.*

In order to apply this theorem to obtain the lower bound of injective radius, we first realize that the Sobolev inequality (2.5.1) implies the log Sobolev inequality [16, Page 184, Lemma 5.8, Remark 5.10]. The proof is just the Hölder inequality. Next, log Sobolev inequality is equivalent to the lower bound of Perelman's entropy $\mathcal{W}$, this is a direct result from the definition of this entropy functional, we refer this result to [16, Page 190, §4.2, §4.3]. $\mathcal{W}$ functional lower bound is equivalent to volume ratio lower bound[16, Page 195, Prop 5.37], by Cheeger-Gromov-Taylor theorem, we have the injective radius lower bound.

**Uniformly Bounded for Derivative of Metric for All Time**

**Lemma 5.2.5.** *Let $(M^n, g)$ be a Riemannian manifold and let $U$ be a compact* *subset of M. Let $\{g_k\}_{k\in\mathbb{N}}$ be a collections of Riemannian metrics that are solutions to conformal Bach flow on $U\times[\alpha,\omega]$ and $t_0\in[\alpha,\omega]$. Let us denote $\nabla$ and $\nabla_k$ to be the covariant derivative with respect to the background metric $g$ and the sequential metric $g_k$, and the norms $|\cdot|$ and $|\cdot|_k$ follow the same rule. Suppose that:*

*(a)* $g_k(t_0)$ *is equivalent to* $g(t_0)$ *for all* $k\in\mathbb{N}$. *That is for* $x\in M$ *and for all* $V\in T_xM$,

$$C^{-1}g(V,V)\leq g_k(t_0)(V,V)\leq Cg(V,V)$$

*for some constant* $C$ *independent of* $U,k$.

*(b) The* $m^{th}$ *order covariant derivatives of metric* $g_k$ *with respect to the metric* $g$ *are all uniformly bounded at* $t=t_0$ *on the compact set* $U$. *That is*

$$|\nabla^m g_k(t_0)|\leq C_m$$

*for all* $m\in\mathbb{N}$ *and some constants* $C_m$ *independent of* $k$.

*(c) The* $m^{th}$ *order covariant derivatives of Riemann curvature* $Rm_k$ *with respect to the metric* $g_k$ *are all uniformly bounded on* $U\times[\alpha,\omega]$.

$$|\nabla_k^m Rm_k|_k\leq C'_m$$

*for all* $m\in\mathbb{N}$ *and some constants* $C'_m$ *independent of* $k$.

*(d) Sobolev constant* $C_S(g_k(t))$ *and* $L^2$ *norm of pressure function* $p_k(t)$ *are uniformly bounded.*

$$C_S(g_k(t))\leq C \qquad \int_M |p_k|^2 d\mu\leq C$$

*for some constant* $C$ *independent of* $k$.

*Then we have the following conclusions:*

(a) *metric $g_k$'s are uniformly bounded with respect to $g$ on $U \times [\alpha, \omega]$, that is:*

$$B^{-1}(\alpha, \omega) g \leq g_k(t) \leq B(\alpha, \omega) g$$

*for some constants*

$$B(\alpha, \omega) = Ce^{\bar{C}(\omega - \alpha)}$$

*independent of $k$.*

(b) *the time derivative and covariant derivative of metric $g_k(t)$ with respect to $g$ are uniformly bounded on $U \times [\alpha, \omega]$, that is for every pair $(p, q)$ where $p, q \in \mathbb{N}$, there is a constant $\tilde{C}_{p,q}$ independent of $k$ such that*

$$\left| \frac{\partial^q}{\partial t^q} \nabla^p g_k(t) \right| \leq \tilde{C}_{p,q}$$

*Proof.* For the conclusion (a), for any $t_1 \in [\alpha, \omega]$, we have:

$$\left| \int_{t_0}^{t_1} \frac{\partial}{\partial t} \ln g_k(t) dt \right| \leq \int_{t_0}^{t_1} \left| \frac{\partial}{\partial t} \ln g_k(t) \right| dt = \int_{t_0}^{t_1} \left| \frac{\frac{\partial}{\partial t} g_k(t)}{g_k(t)} \right| dt \tag{5.2.2}$$

Note that we have:

$$\frac{\partial}{\partial t} g_k(t) = C(n) \Big( \nabla_k^2 Rm_k + Rm_k * Rm_k + p_k g_k \Big) \tag{5.2.3}$$

where $C(n)$ is a constant depends on dimension $n$. Therefore, the right hand side of (5.2.2) is bounded by a constant $\bar{C}$ which depends on all constants in assumption and independent of $k$. This argument is the same as Proposition

4.5.5. And we have:

$$\left| \int_{t_0}^{t_1} \frac{\partial}{\partial t} \ln g_k(t) dt \right| \leq \bar{C}(t_1 - t_0) \tag{5.2.4}$$

which implies

$$g_k(t_0) e^{-\bar{C}(t_1 - t_0)} \leq g_k(t_1) \leq g_k(t_0) e^{\bar{C}(t_1 - t_0)} \tag{5.2.5}$$

But $g_k(t_0)$ is equivalent to $g$ by our assumption, then the result follows. We define a constant

$$B(\alpha, \omega) = C e^{\bar{C}(\omega - \alpha)}$$

such that we have:

$$B^{-1}(\alpha, \omega) g \leq g_k(t) \leq B(\alpha, \omega) g$$

For conclusion (b), we will prove it after the following lemma. □

**Lemma 5.2.6** (Page 134, Eq. 3.7 [14])**.** *Let $g_k$ and $g$ be metrics, and $\nabla_k$, $\Gamma_k$ are covariant derivative and Christoffel symbol with respect to $g_k$, $\nabla$, $\Gamma$ are covariant derivative and Christoffel symbol with respect to $g$. Under the same assumptions in Lemma 5.2.5, the tensors $\nabla g_k$ and $\Gamma_k - \Gamma$ are equivalent.*

*Proof.* With a direct calculation, in local coordinate, we have:

$$\nabla_a (g_k)_{bc} = \frac{\partial}{\partial x^a} (g_k)_{bc} - \Gamma^d_{ab} (g_k)_{dc} - \Gamma^d_{ac} (g_k)_{bd}$$

Then we have:

$$\begin{aligned}
&(g_k)^{ec} \Big[ \nabla_a (g_k)_{bc} + \nabla_b (g_k)_{ac} - \nabla_c (g_k)_{ab} \Big] \\
= {}& 2(\Gamma_k)^e_{ab} - \Gamma^e_{ab} - (g_k)^{ec} \Gamma^d_{ac} (g_k)_{bd} - \Gamma^e_{ab} \\
& - (g_k)^{ec} \Gamma^d_{bc} (g_k)_{ad} + (g_k)^{ec} \Gamma^d_{ac} (g_k)_{bd} + (g_k)^{ec} \Gamma^d_{bc} (g_k)_{ad} \\
= {}& 2(\Gamma_k)^e_{ab} - 2\Gamma^e_{ab}
\end{aligned}$$

Therefore, we have:

$$\Big|\Gamma_k - \Gamma\Big|_k \leq \frac{3}{2}|\nabla g_k|_k \tag{5.2.6}$$

For another direction, we have:

$$\nabla_a (g_k)_{bc} = (g_k)_{eb}\Big[(\Gamma_k)^e_{ac} - \Gamma^e_{ac}\Big] + (g_k)_{ec}\Big((\Gamma_k)^e_{ab} - \Gamma^e_{ab}\Big) \tag{5.2.7}$$

Then we have:

$$\Big|\Gamma_k - \Gamma\Big|_k \geq \frac{1}{2}|\nabla g_k|_k \tag{5.2.8}$$

In fact, since $\nabla g = 0$, we can write the conclusion as

$$\frac{1}{2}|\nabla g_k - \nabla g|_k \leq \Big|\Gamma_k - \Gamma\Big|_k \leq \frac{3}{2}|\nabla g_k - \nabla g|_k \tag{5.2.9}$$

□

*Proof for Lemma 5.2.5 part b.*

Now we continue the proof to Lemma 5.2.5. We will prove the second statement inductively. For our convenience, we define constants $\hat{C}_m$ to be the bounds

$$|\nabla^m_k Rm_k|_k \leq \hat{C}_m \qquad \text{and} \qquad |\nabla^m_k p_k|_k \leq \hat{C}_m \tag{5.2.10}$$

for $m \in \mathbb{N}$, and $C_m$ is independent of $k$.

We first prove the case for $\Big|\nabla g_k(t)\Big|$, which is equivalent to $\Big|\nabla g_k(t)\Big|_k$ because we already proved the first statement. Base on previous lemma, we only need to bound $\Gamma_k - \Gamma$. We have:

$$\Big|\frac{\partial}{\partial t}(\Gamma_k - \Gamma)\Big|_k = C(n)\Big|\nabla_k B_k + (\nabla_k p_k) g_k\Big|_k \leq C(n)(\hat{C}_3 + \hat{C}_1\hat{C}_0 + \hat{C}_1) = \bar{C}_0 \tag{5.2.11}$$

Therefore, we have:

$$\begin{aligned}
\Big|(\Gamma_k)(t_1)-\Gamma\Big|_k &\leq \left|\int_{t_0}^{t_1}\frac{\partial}{\partial t}(\Gamma_k-\Gamma)dt\right| - \Big|(\Gamma_k)(t_0)-\Gamma\Big|_k \\
&\leq \int_{t_0}^{t_1}\left|\frac{\partial}{\partial t}(\Gamma_k-\Gamma)\right|dt + \Big|(\Gamma_k)(t_0)-\Gamma\Big|_k \\
&\leq \bar{C}_0(t_1-t_0)+3C_1 \\
&\leq \bar{C}_0(\alpha-\omega)+3C_1
\end{aligned} \tag{5.2.12}$$

and

$$|\nabla g_k(t)| \leq B(\omega,\alpha)^{\frac{3}{2}}|\nabla g_k(t)|_k \leq 2B(\omega,\alpha)^{\frac{3}{2}}\Big|\Gamma_k-\Gamma\Big|_k \leq \tilde{C}_{1,0} \tag{5.2.13}$$

where the $\frac{3}{2}$ comes from the type of tensor, we view $\nabla g_k$ to be a $(2,1)$-tensor.

For higher order case, we are going to derive a Grönwall type differential inequality for $|\nabla^N g_k(t)|_k^2$. Recall that we have the following telescoping identity for any tensor T:

$$\nabla^N T = \sum_{i=1}^{N}\nabla^{N-i}(\nabla-\nabla_k)\nabla_k^{i-1}T + \nabla_k^N T \tag{5.2.14}$$

Also, from here, we simply use constant $D_m$ for

$$\left|\nabla_k^m\frac{\partial}{\partial t}g_k(t)\right|_k = \Big|\nabla_k^m(B_k+p_k g_k)\Big|_k \leq D_m \tag{5.2.15}$$

We first claim the follow inequality in true.

$$|\nabla^m\partial_t g_k(t)| \leq C'_m|\nabla^m g_k(t)| + C''_m \tag{5.2.16}$$

When $m = 1$, we have:

$$\begin{aligned} |\nabla \partial_t g_k(t)| &= |(\nabla - \nabla_k)\partial_t g_k(t) + \nabla_k \partial_t g_k(t)| \\ &\leq B(\omega,\alpha)^{\frac{3}{2}} \left( |\Gamma - \Gamma_k|_k |\partial_t g_k(t)|_k + |\nabla_k \partial_t g_k(t)|_k \right) \\ &\leq B(\omega,\alpha)^{\frac{3}{2}} D_0 |\nabla g_k(t)|_k + B(\omega,\alpha)^{\frac{3}{2}} D_1 \\ &\leq B(\omega,\alpha)^{\frac{3}{2}} C D_0 |\nabla g_k(t)| + B(\omega,\alpha)^{\frac{3}{2}} D_1 \end{aligned} \tag{5.2.17}$$

Inductively, suppose that (5.2.16) holds for $m = 1, 2, 3, \cdot, N-1$, for $N$, we have:

$$\begin{aligned} |\nabla^N \partial_t g_k(t)| &\leq |\sum_{i=1}^{N} \nabla^{N-i}(\nabla - \nabla_k)\nabla_k^i \partial_t g_k(t)| + |\nabla_k^N \partial_t g_k(t)| \\ &\leq B(\omega,\alpha)^{C_N} |\nabla^{N-1}(\nabla - \nabla_k)\nabla_k \partial_t g_k(t)|_k \\ &+ B(\omega,\alpha)^{C_N} |\sum_{i=2}^{N} \nabla^{N-i}(\nabla - \nabla_k)\nabla_k^i \partial_t g_k(t)|_k + B(\omega,\alpha)^{C_N} D_1 \end{aligned} \tag{5.2.18}$$

With the induction hypothesis, the results follows. Therefore, we have the following differential inequality:

$$\begin{aligned} \frac{\partial}{\partial t} |\nabla^m g_k|^2 &= 2\langle \partial_t \nabla^m g_k, \nabla^m g_k \rangle \\ &\leq 2|\partial_t \nabla^m g_k|^2 + |\nabla^m g_k| \\ &\leq (1 + 2(C'_m)^2)|\nabla^m g_k| + 2(C''_m)^2 \end{aligned} \tag{5.2.19}$$

The Gröwall type differential inequality give us the desired result. This complete the prove for $q = 0$. For $q > 0$, the proof is identical by exchange the time derivatives and covariant derivatives. □

**Subsequentially Convergence**

Once we extend all the bounds over the whole interval, the following Arzela Ascoli theorem allows us to extract a subsequence converges to a limit manifold.

**Lemma 5.2.7** (Lemma 3.14, Page 137, [14])**.** *Let $X$ be a $\sigma$-compact, locally compact Hausdorff space. If $\{f_k\}_{k\in\mathbb{N}}$ is an equi-continuous, point-wise bounded sequence of continuous functions $f_k : X \to \mathbb{R}$, then there exists a sub-sequence which converges uniformly on compact sets to a continuous function $f_\infty : X \to \mathbb{R}$*

In this theorem, $\sigma$-compact means the space is a countable union of compact sets, hence, a complete Riemannian manifold satisfies such assumption.

**Corollary 5.2.8** (Corollary 3.15, Page 137, [14])**.** *Let $M^n$ be a Riemannian manifold let $U \subset M^n$ be a compact set. Let $m \in \mathbb{N}$ and $g$ a fixed background metric on $U$. If $\{g_k\}_{k\in\mathbb{N}}$ is a sequence of Riemannian metric on $U$ such that*

$$\sup_{0\le i\le m+1} \sup_{x\in U} |\nabla^i g_k| \le C < \infty \tag{5.2.20}$$

*and if $g_k$ is equivalent to $g$, then there is a sub-sequence $\{g_{k_j}\}_{j\in\mathbb{N}}$ such that $g_{k_j}$ converges to a Riemannian metric $g_\infty$ in $C^m$ sense.*

Now we are ready to proof the main theorem 5.2.1

*Proof.* Throughout this proof, we don't specify the sub-sequence $\{g_{k_j}\}_{j\in\mathbb{N}}$. We also drop the local frame $F_k$ for our convenience.

Under the assumption, at $t = 0$, we have a sub-sequence $\{(M_k^n, g_k(0), p_k(0), O_k)\}_{k\in\mathbb{N}}$ converges to $\{(M_\infty^n, g_\infty, p_\infty, O_\infty)\}$. We shall extend this convergence to all time interval $(\alpha, \omega)$ such that $g_\infty(0) = g_\infty$.

Since $\{(M_k^n, g_k(0), p_k(0), O_k)\}_{k\in\mathbb{N}}$ converges to $\{(M_\infty^n, g_\infty, p_\infty, O_\infty)\}$, there are diffeomorphisms $\Phi_k : U_k \to V_k$ where $U_k \subset M_\infty$ and $V_k \subset M_k$ such that the pull back $\Phi^*(g_k(0))$ converges to $g_\infty$ uniformly on a compact set.

We are going to apply Lemma 5.2.5 with $t_0 = 0$ and the background metric is $g_\infty$ and $\Phi^* g_k(t)$ is the sequence of metrics.

We apply Arzela Ascoli theorem and its corollary to $\{\Phi^* g_k(t) + dt^2\}_{k\in\mathbb{N}}$, and fix the back ground metric $g_\infty + dt^2$ on $M_\infty \times (\alpha, \omega)$. Therefore we extract another subsequence $\{\Phi^* g_k(t) + dt^2\}_{k\in\mathbb{N}}$ which converges to $g_\infty(t) + dt^2$. Since all derivatives of metric converges, $\{p_k\}_{k\in\mathbb{N}}$ also converges to $p_\infty$.

Then we conclude that $\{(M_k^n, g_k(t), p_k(t), O_k)\}_{k\in\mathbb{N}}$ sub-sequentially converges to $(M_\infty, g_\infty, p_\infty, O_\infty)$ which is also a solution to conformal Bach flow. □

## 5.3 Singularity Models

As our first corollary of the compactness Theorem 5.2.1, we show that under suitable conditions, we can obtain a singularity model for the conformal gradient flow. Similar argument can be found in [49, Theorem 1.5] and [37, Theorem 1.4]

Before we discussing such singularity model, we first look at the scale property of conformal Bach flow.

**Proposition 5.3.1.** *Suppose $(M^n, g(t), p(t))$ is a compact solution to conformal Bach flow, if $\tilde{g} = \lambda g(\frac{t}{\lambda^2})$ for some positive constant $\lambda$, by the scale property of curvatures, $(M^n, \lambda g(\frac{t}{\lambda^2}), \lambda^{-2} p(t))$ is also a solution to conformal Bach flow.*

*Proof.* To see this, we have the following scale properties for curvatures. Under the scaling, $\tilde{g} = \lambda g$, we have: $\tilde{Rm} = \lambda Rm$, $\tilde{Ric} = \lambda Ric$, $\tilde{R} = \lambda^{-1} R$, and $\tilde{B} = \lambda^{-1} B$, then we conclude that $(M^n, \lambda g(\frac{t}{\lambda^2}), \lambda^{-2} p(t))$ is a solution to conformal Bach flow. □

**Remark 5.3.2.** *We also notice that under this scaling,*

$$\int_M \tilde{p}^2 d\tilde{\mu} = \lambda^{\frac{n}{2}-4} \int_M p^2 d\mu$$

By this remark, we see that the $L^2$ norm of pressure function is decreasing when the dimension is less than 7. We restrict our singularity model in 4 dimension since our integral estimate relies on the $L^2$ norm of $p$.

**Theorem 5.3.3.** *Let $(M^4, g(t))$ be a closed solution to conformal Bach flow on a max time interval $[0,T)$ with $T<\infty$. Suppose that*

*(a)* $Y[M,g]>0$,

*(b)* $\int_M \sigma_2[g(0)]d\mu > 0$,

*(c)* $\int_M p^2 d\mu \leq K$ *for* $t\in[0,T)$,

*for some constant $K$. Let $\{(x_i,t_i)\}_{i\in\mathbb{N}}$ be a sequence of points in $M^4\times[0,T)$ such that $t_i\to T$ and $|Rm(x_i,t_i)| = \sup_{M^4} Rm(x,t_i)$. Let $\lambda_i = |Rm(x_i,t_i)|$ and $\lambda_i\to\infty$. Then the sequence of solutions to conformal Bach flow $\{(M^4, g_i(t), p_i(t), x_i)\}_{i\in\mathbb{N}}$ with*

$$g_i(t) = \lambda_i g(t_i + \frac{t}{\lambda_i^2}) \qquad p_i(t) = \frac{1}{\lambda_i^2}p(t) \qquad \text{for} \quad t\in[-\lambda_i^2 t_i, \lambda_i^2(T-t_i)) \tag{5.3.1}$$

*sub-sequentially converges to a limit space $(M_\infty, g_\infty(t), p_\infty(t), x_\infty)$ on time interval $(-\infty, a]$ in Cheeger Gromov sense, which is a non flat, non compact, complete pointed manifold with zero scalar curvature. Here $a = \lim_{i\to\infty}\lambda_i^2(T-t_i)\geq 0$. And the limit space satisfies the following flow:*

$$\frac{\partial}{\partial t}g_\infty = B_\infty \tag{5.3.2}$$

*for $t\in(-\infty,0]$. Furthermore, on the limit space $M_\infty$, it satisfies $E_\infty\cdot B_\infty = 0$*

*Proof.* First we see that under the assumption on $\sigma_2[g(0)]$, by Corollary 4.4.4, the Sobolev constant has an upper bound, and this bound is scale invariant.

Under the scaling, we have $|Rm|_{g_i} \leq 1$ for all points on manifold. Furthermore, if $\int_M p^2 d\mu \leq K$, after scaling, we have $\int_M p^2 d\mu = 0$.

We simply choose time interval to be $(-1, 0]$, and there is an integer $k$ such that when $i \geq k$, $\lambda_i^2 t_i \geq 1$. We take the sub-sequence $\{(M^n, g_i(t), p_i(t), x_i)\}_{i \geq k}$ and relabel it as $\{(M^n, g_i(t), p_i(t), x_i)\}_{i \in \mathbb{N}}$. This sequence consists complete pointed solutions to conformal Bach flow. We then replace $(\alpha, \omega)$ in Theorem 5.2.1 by $(-1, 0]$. There is no hurt we take the closed interval since we can find a subsequence for $(-1, -\epsilon_i)$ and send $\epsilon_i$ to zero by taking a further subsequence. Thus we apply the compactness theorem and we obtain a subsequence converges to a limit space

$$(M_\infty, g_\infty(t), p_\infty(t), x_\infty)$$

This limit space is clearly non flat since re-scaling, we know that at $x_\infty \in M_\infty$, we have $|Rm_\infty(x_\infty)|_{g_\infty} = 1$.

On this limit space, the conformal Bach flow will be:

$$\begin{cases} \dfrac{\partial}{\partial t} g_\infty = B_\infty + p_\infty g_\infty \\ -3\Delta_\infty p_\infty = A_\infty B_\infty \end{cases} \tag{5.3.3}$$

where $B_\infty = \frac{1}{\lambda_\infty} B$, $A_\infty = A$, $\Delta_\infty = \frac{1}{\lambda_\infty}\Delta$. Therefore, the conformal Bach flow on limit space is modified as follows.

$$\begin{cases} \dfrac{\partial}{\partial t} g_\infty = B_\infty + p_\infty g_\infty \\ \Delta_\infty p_\infty = 0 \end{cases} \tag{5.3.4}$$

We see that $p_\infty$ is a harmonic function. By our assumption, the $L^2$ norm of $p_\infty$ is zero, by mean value property of harmonic function, for any ball with radius $r$

centered at arbitrary point $x \in M_\infty$, we have:

$$|p_\infty(x)| \leq \frac{1}{Vol(B_x(r))} \int_{B_x(r)} |p_\infty(y)| dy = \frac{1}{Vol(B_x(r))^{\frac{1}{2}}} \Big( \int_{B_x(r)} |p_\infty(y)|^2 dy \Big)^{\frac{1}{2}} \tag{5.3.5}$$

Here we use Hölder inequality on the right hand side and we see that $p_\infty = 0$. Therefore, the conformal Bach flow on the limit space is

$$\frac{\partial}{\partial t} g_\infty = B_\infty \tag{5.3.6}$$

and by re-scaling, the product of Ricci tensor and Bach tensor will vanish. $\square$

Next, we state another theorem about s special case of singularity models. We will see that under a low energy assumption, the limit space is Bach flat.

**Theorem 5.3.4.** *Let $(M_0^4, g(t))$ be a closed solution to conformal Bach flow on a max time interval $[0,T)$ with $T < \infty$. Suppose that there exists some constant $K > 0$ and a small constant $\epsilon > 0$ such that*

*(a)* $Y[M_0, g] > 0$,

*(b)* $\int_M \sigma_2[g(0)] d\mu > 0$,

*(c)* $\int_{M_0} p^2 d\mu \leq K$ *for* $t \in [0,T)$,

*(d)* $\int_{M_0} |W|^2 + 2|E|^2 d\mu \leq \epsilon$ *for* $t \in [0,T)$,

*By the same re-scaling in Theorem 5.3.3, we denote the limit space as $M$, this limit space will be Bach flat.*

*Proof.* Throughout this proof, we denote the manifold before re-scaling by $M_0$ and the limit space by $M$ instead of $M_\infty$. All of the curvature quantities are referred to the limit space $M$. By re-scaling, on the limit space $M$, we have the following equation:

$$E^{ij}B_{ij} = 0 \tag{5.3.7}$$

Recall that in dimension 4, the Bach tensor is

$$B_{ij} = \frac{1}{2}\Delta R_{ij} - \frac{1}{12}\Delta R g_{ij} - \frac{1}{6}\nabla_i\nabla_j R + R^{kl}R_{ikjl} - \frac{1}{4}|Ric|^2 g_{ij} + \frac{1}{12}R^2 g_{ij} - \frac{1}{3}RR_{ij} \tag{5.3.8}$$

Since the limit space is scalar flat, we have:

$$B_{ij} = \frac{1}{2}\Delta E_{ij} + E^{kl}R_{ikjl} - \frac{1}{4}|E|^2 g_{ij} \tag{5.3.9}$$

Therefore, we have:

$$E^{ij}B_{ij} = \frac{1}{2}E^{ij}\Delta E_{ij} + E^{ij}E^{kl}R_{ikjl} = 0 \tag{5.3.10}$$

Here we don't specify the Ricci tensor and traceless Ricci tensor since they are the same. Furthermore, by Riemann curvature decomposition, we have:

$$R_{ikjl} = W_{ikjl} + \frac{1}{2}(E \owedge g)_{ikjl} \tag{5.3.11}$$

thus, the second term will be:

$$\begin{aligned} E^{ij}E^{kl}R_{ikjl} &= E^{ij}E^{kl}W_{ikjl} + \frac{1}{2}E^{ij}E^{kl}(E \owedge g)_{ikjl} \\ &= E^{ij}E^{kl}W_{ikjl} - E_{ij}E^{ik}E^j_k \end{aligned} \tag{5.3.12}$$

Combine all of these results, we have:

$$-\frac{1}{2}E^{ij}\Delta E_{ij} = E^{ij}E^{kl}W_{ikjl} - E_{ij}E^{ik}E_k^j \tag{5.3.13}$$

We denote the quantity $E_{ij}E^{ik}E_k^j$ by $TrE^3$. And in dimension 4, we have a sharp inequality(Page 129, [10]):

$$TrE^3 \geq -\frac{1}{\sqrt{3}}|E|^3 \tag{5.3.14}$$

Therefore, we have:

$$-E^{ij}\Delta E_{ij} = 2E^{ij}E^{kl}W_{ikjl} - 2E_{ij}E^{ik}E_k^j \leq 2|E|^2|W| + \frac{2}{\sqrt{3}}|E|^3 \tag{5.3.15}$$

To derive the Bach flat result, we only need to show that the limit space is Ricci flat. To see this, we have:

$$-|E|\Delta|E| = -\frac{1}{2}\Delta|E|^2 + |\nabla|E||^2 = -E^{ij}\Delta E_{ij} - |\nabla E|^2 + |\nabla|E||^2 \tag{5.3.16}$$

Here we use Kato's inequality $|\nabla|E||^2 \leq |\nabla E|^2$, then we have:

$$-|E|\Delta|E| \leq -E^{ij}\Delta E_{ij} \tag{5.3.17}$$

Let $\phi$ be a cutoff function which will be chosen later. With integration by parts, we have:

$$\int_M \phi^2|\nabla|E||^2 d\mu + 2\int_M \phi\nabla\phi|E|\nabla|E| = -\int_M \phi^2|E|\Delta|E| d\mu \tag{5.3.18}$$

Therefore,

$$\int_M \phi^2|\nabla|E||^2 d\mu + 2\int_M \phi\nabla\phi|E|\nabla|E| \le \int_M 2\phi^2|E|^2|W| + \frac{2}{\sqrt{3}}\phi^2|E|^3 d\mu \quad (5.3.19)$$

Use the Yamabe quotient, we apply the Hölder inequality:

$$\begin{aligned}\left(\int_M |\phi|E||^4 d\mu\right)^{\frac{1}{2}} &\le Y\int_M |\nabla\phi|E| + \nabla|E|\phi|^2 d\mu \\ &\le Y\int_M |\nabla\phi|^2|E|^2 d\mu + 2\int_M \phi^2|E|^2|W|d\mu + \frac{2}{\sqrt{3}}\int_M \phi^2|E|^3 d\mu\end{aligned} \quad (5.3.20)$$

For the second term, by the same argument:

$$2\int_M \phi^2|E|^2|W|d\mu \le 2\left(\int_M |\phi|E||^4 d\mu\right)^{\frac{1}{2}}\left(\int_M |W|^2 d\mu\right)^{\frac{1}{2}} \quad (5.3.21)$$

For the last term, we have:

$$\frac{2}{\sqrt{3}}\int_M \phi^2|E|^3 d\mu \le \frac{2}{\sqrt{3}}\left(\int_M |\phi|E||^4 d\mu\right)^{\frac{1}{2}}\left(\int_M |E|^2 d\mu\right)^{\frac{1}{2}} \quad (5.3.22)$$

We conclude that:

$$\left(\int_M |\phi|E||^4 d\mu\right)^{\frac{1}{2}} \le C\int_M |\nabla\phi|^2|E|^2 d\mu \quad (5.3.23)$$

where $C$ is a constant defined by:

$$C = \frac{Y}{1 - 2\left\|W\right\|_2 - \frac{2}{\sqrt{3}}\left\|E\right\|_2} \quad (5.3.24)$$

Therefore, if

$$2\int_M |W|^2 d\mu + \frac{2}{\sqrt{3}}\int_M |E|^2 d\mu \leq \epsilon \tag{5.3.25}$$

such that $1 - 2\|W\|_2 - \frac{2}{\sqrt{3}}\|E\|_2 > 0$, we choose the cut-off function $\phi$ as:

$$\phi = \begin{cases} 1 \quad \text{for} \quad x \in B_r \\ 0 \quad \text{for} \quad x \in M \backslash B_{2r} \\ |\nabla \phi| \leq \frac{1}{r} \quad \text{for} \quad x \in B_{2r} \backslash B_r \end{cases} \tag{5.3.26}$$

with $\phi \in [0, 1]$, we have:

$$\left( \int_M |\phi |E||^4 d\mu \right)^{\frac{1}{2}} \leq C \int_M |\nabla \phi|^2 |E|^2 d\mu \leq \frac{C}{r^2} \cdot \frac{2}{\sqrt{3}}\epsilon \tag{5.3.27}$$

By taking $r$ to $\infty$, we have $|E| \equiv 0$ on $M$, and $M$ is Bach flat. □

In fact, this type of singularity will not happen. To see this, we first introduce a gap theorem for non-compact complete Bach flat manifolds in dimension 4, which is proved in [28].

**Theorem 5.3.5** (Theorem 1, [28])**.** *Let $(M^4, g)$ be a non-compact complete Bach-flat Riemannian 4-manifold with zero scalar curvature and the Yamabe constant $Y[M] > 0$. Then there exists a small number $c_0$ such that if*

$$\int_M |W|^2 + 2|E|^2 d\mu \leq c_0$$

*then $M^4$ is flat.*

With this gap theorem, we have the following corollary.

**Corollary 5.3.6.** *Under the low energy assumption in Theorem 5.3.4, the singularity will not happen, that is, the curvature tensor $|Rm|$ will remain bounded as*

*$t$ approaches to the maximal time $T$.*

*Proof.* By the rigidity theorem 5.3.5, the limit space satisfies all conditions in the gap theorem, and has to be flat, but we know that at a specific point $x \in M$, we have $|Rm| = 1$, which leads to a contradiction. $\square$

# Chapter 6

# Further Remarks

In this thesis, we study a fourth order geometric flow. Its shot time existence, uniqueness and regularity are established. We also derive integral estimates and volume estimates for this flow. Based on these results, we characterize the finite time singularity. A compactness property of the solutions to this flow is proved, and we study a singularity model obtained by re scaling the metric.

We remark some possible improvement in this chapter. In [50, Theorem 1.3], a long time existence of the geometric flow is proposed by assuming the small initial energy, i.e, a small initial $L^2$ norm instead of the point-wise bound of curvature. To obtain a similar result, we need a refinement for our integral estimate in Chapter 4, without assuming the $L^\infty$ norm of curvature. Such condition is desirable since under our flow, the Weyl functional is non increasing. The monotonicity property might help us prove the gap theorem we mentioned in Chapter 1. Another remark is about the singularity model, in our theorem 5.3.3 and 5.3.4, we assume that $L^2$ norm of pressure function is bounded, we may ask to questions here. One is why this quantity remains bounded. Notice that we don't discuss the invertibility of the elliptic operator in this theorem, we might need to investigate some details about it. The other question is that what if the $L^2$ norm of pressure function blows up?

We might rescale the metric based this quantity instead of the Riemann curvature tensor.

# Appendix A

# Formulae in Riemannian Geometry

In this chapter, we will list some basic formulae in Riemannian geometry and we will use the same convention as [16].

Let $(M^n, g)$ be an n-dimensional Riemannian manifold. Let $x \in M$ be a fixed point, let $\{x^i\}_{i=1}^n$ be a local coordinate system.

## A.1 Christoffel Symbol

Under the local coordinate system, the Levi-Civita connection is given by

$$\nabla_i \partial_j = \Gamma_{ij}^k \partial_k \tag{A.1.1}$$

where $\Gamma_{ij}^k$ is called Christoffel symbol and defined as follows

$$\Gamma_{ij}^k = \frac{1}{2} g^{kl} \Big( \partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij} \Big) \tag{A.1.2}$$

## A.2 Normal Coordinate

For any point $x \in M$, we consider the normal coordinate system $\{\partial_i\}_{i=1}^n$ around this fixed point $x$. We have $g_{ij} = \delta_{ij}$ and $\partial_k g_{ij} = 0$, therefore, the Christoffel symbol vanishes, i.e., $\Gamma_{ij}^k = 0$.

## A.3 Covariant Derivative

Let $T_{i_1 i_2 \cdots i_m}^{j_1 j_2 \cdots j_n}$ be a tensor object on manifold, the covariant derivative is given by

$$\nabla_k T_{i_1 i_2 \cdots i_m}^{j_1 j_2 \cdots j_n} = \partial_k T_{i_1 i_2 \cdots i_m}^{j_1 j_2 \cdots j_n} + \sum_{s=1}^{n} T_{i_1 i_2 \cdots i_m}^{j_1 j_2 \cdots p \cdots j_n} \Gamma_{kp}^{j_s} - \sum_{t=1}^{m} T_{i_1 i_2 \cdots q \cdots i_m}^{j_1 j_2 \cdots j_n} \Gamma_{k i_t}^{q} \tag{A.3.1}$$

It is clear that if we choose a normal coordinate, the covariant derivative is the same as the normal derivative in calculus.

## A.4 Riemann Curvature Tensor and Symmetry

To measure the deviation of a metric from a flat metric, we define the Riemann curvature operator:

$$R(u,v)w = -\nabla_u \nabla_v w + \nabla_v \nabla_u w + \nabla_{[u,v]} w \tag{A.4.1}$$

In local coordinate system, we have:

$$R(\partial_i, \partial_j)\partial_k = \nabla_j \nabla_i \partial_k - \nabla_i \nabla_j \partial_k = {R_{ijk}}^l \partial_l \tag{A.4.2}$$

where

$$R_{ijk}{}^{l} = \partial_j \Gamma^l_{ik} - \partial_i \Gamma^l_{jk} + \Gamma^l_{js}\Gamma^s_{ik} - \Gamma^l_{ks}\Gamma^s_{ij} \tag{A.4.3}$$

Note that we choose to upper the last index as our convention.

We can lower the index by:

$$R_{ijkl} = g_{lm} R_{ijk}{}^{m} \tag{A.4.4}$$

The symmetries of Riemann curvature tensor are the following equalities. It is symmetric for switching the first pair of indices and the last ones, and it is anti-symmetric for flipping each pair of indices.

$$R_{ijkl} = R_{klij} = -R_{jikl} = -R_{ijlk} \tag{A.4.5}$$

## A.5 Bianchi Identities

The following two formulae are the first and second Bianchi identities.

$$R_{ijkl} + R_{iljk} + R_{iklj} = 0 \tag{A.5.1}$$

Fix the first index and the cyclic permutation sum for the last three indices is zero.

$$\nabla_m R_{ijkl} + \nabla_l R_{ijmk} + \nabla_k R_{ijlm} = 0 \tag{A.5.2}$$

Fix the first two indices and the cyclic permutation sum of the last three indices is zero. From here, by contracting indices, we have two direct consequences. The divergence of Riemann curvature tensor is

$$\nabla^j R_{ikjl} = \nabla_i R_{kl} - \nabla_k R_{il} \tag{A.5.3}$$

The divergence of Ricci curvature tensor is

$$\nabla^j R_{ij} = \frac{1}{2}\nabla_i R \tag{A.5.4}$$

## A.6 Ricci Identity

Ricci identity is a general case of Bianchi identity, it tells us how to exchange covariant derivatives. For arbitrary tensor object $T_{k_1 k_2 \cdots k_n}$, we have

$$\nabla_j \nabla_i T_{k_1 k_2 \cdots k_n} - \nabla_i \nabla_j T_{k_1 k_2 \cdots k_n} = -\sum_{s=1}^{n} R_{ijk_s}{}^{l} T_{k_1 \cdots k_{s-1} l k_{s+1} k_n} \tag{A.6.1}$$

## A.7 Ricci Curvature and Scalar Curvature

We define the Ricci curvature by contracting the second and fourth indices in Riemann curvature.

$$R_{ij} = g^{kl} R_{ikjl} \tag{A.7.1}$$

And the scalar curvature is the trace of Ricci curvature.

$$R = g^{ij} R_{ij} \tag{A.7.2}$$

## A.8 Kulkarni Nomizu Product

In the mathematical field of differential geometry, the Kulkarni-Nomizu product is defined for two (0,2)-tensors and gives as a result a (0,4)-tensor.

$$(T \owedge S)_{ikjl} = T_{ij} S_{kl} + T_{kl} S_{ij} - T_{il} S_{kj} - T_{kj} S_{il} \tag{A.8.1}$$

From here, we have the following formulae:

$$(g \owedge g)_{ikjl} = 2g_{ij}g_{kl} - 2g_{il}g_{kj} \tag{A.8.2}$$

$$(T \owedge g)_{ikjl}g^{kl} = 2T_{ij} + Tr_g(T)g_{ij} \tag{A.8.3}$$

## A.9 Traceless Ricci Curvature and Weyl Curvature

We define the traceless Ricci curvature by subtracting the trace part in Ricci curvature.

$$E_{ij} = R_{ij} - \frac{1}{n}Rg_{ij} \tag{A.9.1}$$

We also introduce the Schouten tensor, which is a second-order tensor introduced by Jan Arnoldus Schouten.

$$A_{ij} = \frac{1}{n-2}\Big(R_{ij} - \frac{R}{2(n-1)}g_{ij}\Big) \tag{A.9.2}$$

The Weyl curvature tensor equals the Riemann curvature tensor minus the Kulkarni Nomizu product of the Schouten tensor with the metric.

$$W_{ikjl} = R_{ikjl} - (A \owedge g)_{ikjl} \tag{A.9.3}$$

**Proposition A.9.1.** *The divergence of Weyl curvature is*

$$\nabla^j W_{ikjl} = (n-3)\Big(\nabla_i A_{kl} - \nabla_k A_{il}\Big) \tag{A.9.4}$$

*Proof.* Follow the previous result, combine with the divergence of Riemann curvature tensor (A.5.3) and Ricci curvature tensor (A.5.4), we have

$$\begin{aligned}
\nabla^j W_{ikjl} &= \nabla^j R_{ikjl} - \nabla^j \Big(A \owedge g\Big)_{ikjl} \\
&= \nabla_i R_{kl} - \nabla_k R_{il} - \frac{1}{2(n-2)} \nabla_i R g_{kl} - \nabla_i A_{kl} - \nabla_k A_{il} + \frac{1}{2(n-2)} \nabla_k R g_{il} \\
&= (n-3)\Big(\nabla_i A_{kl} - \nabla_k A_{il}\Big)
\end{aligned}$$

□

## A.10 Riemann Curvature Decomposition

With Weyl curvature and traceless Ricci curvature, we have the following Riemann curvature decomposition.

**Proposition A.10.1.** *Let $R_{ikjl}$ be the Riemann curvature, we have the following decomposition:*

$$R_{ikjl} = W_{ikjl} + (A \owedge g)_{ikjl} \tag{A.10.1}$$

$$R_{ikjl} = W_{ikjl} + \frac{1}{n-2}(E \owedge g)_{ikjl} + \frac{1}{2n(n-1)} R (g \owedge g)_{ikjl} \tag{A.10.2}$$

*where $W_{ikjl}$ is the Weyl curvature. Note, these two are orthogonal decomposition.*

**Proposition A.10.2.** *We also have the following quadratic decomposition:*

$$|Rm|^2 = |W|^2 + \frac{4}{n-2}|Ric|^2 - \frac{2}{(n-1)(n-2)} R^2 \tag{A.10.3}$$

*and*

$$|Rm|^2 = |W|^2 + \frac{4}{n-2}|E|^2 + \frac{2}{n(n-1)} R^2 \tag{A.10.4}$$

## A.11 Bach Tensor

We define Bach tensor as:

$$B_{ij} = \frac{1}{n-3}\nabla^k\nabla^l W_{ikjl} + \frac{1}{n-2}R^{kl}W_{ikjl} \tag{A.11.1}$$

With Schouten tensor, we have the following equivalent formula which shows that the leading term in Bach tensor contains $\Delta R_{ij}$:

$$B_{ij} = \Delta A_{ij} - \nabla^k\nabla_i A_{jk} + A^{kl}W_{ikjl} \tag{A.11.2}$$

**Remark A.11.1.** *$\nabla^k\nabla_i A_{ij}$ is symmetric about $i$ and $j$.*

From here, one can calculate Bach tensor in terms of Riemann curvature, Ricci curvature and scalar curvature.

$$\begin{aligned} B_{ij} &= \Delta A_{ij} - \nabla^k\nabla_i A_{jk} + A^{kl}W_{ikjl} \\ &= \frac{\Delta R_{ij}}{n-2} - \frac{\Delta R g_{ij}}{2(n-1)(n-2)} - \frac{\nabla^k\nabla_i R_{jk}}{n-2} + \frac{\nabla^k\nabla_i R g_{jk}}{2(n-1)(n-2)} \\ &+ \frac{1}{n-2}R^{kl}\Big(R_{ikjl} - (A \owedge g)_{ikjl}\Big) \end{aligned} \tag{A.11.3}$$

For the third term, we use Ricci identity A.6.1 as follows:

$$\begin{aligned} \nabla^k\nabla_i R_{jk} &= \nabla_i\nabla^k R_{ik} - R_{ikj}{}^{l}R_{lk} - R_{ikk}{}^{l}R_{jl} \\ &= \frac{1}{2}\nabla_i\nabla_j R - R^{kl}R_{ikjl} + R_{ik}R^k_j \end{aligned} \tag{A.11.4}$$

For the last term, we have:

$$
\begin{aligned}
&\frac{1}{n-2}R^{kl}\Big(R_{ikjl} - (A \owedge g)_{ikjl}\Big)\\
=& \frac{1}{n-2}R^{kl}R_{ikjl} - \frac{1}{(n-2)^2}\Big(RR_{ij} - \frac{1}{2(n-1)}R^2 g_{ij}\Big)\\
&- \frac{1}{(n-2)^2}\Big(|Ric|^2 g_{ij} - \frac{1}{2(n-1)}R^2 g_{ij}\Big) + \frac{2}{(n-2)^2}\Big(R_{ik}R_j^k - \frac{1}{2(n-1)}RR_{ij}\Big)
\end{aligned}
\tag{A.11.5}
$$

Combine all of these results, we obtain:

$$
\begin{aligned}
B_{ij} =& \frac{1}{n-2}\Delta R_{ij} - \frac{1}{2(n-1)(n-2)}\Delta R g_{ij} - \frac{1}{2(n-1)}\nabla_i\nabla_j R\\
&+ \frac{2}{n-2}R^{kl}R_{ikjl} - \frac{n-4}{(n-2)^2}R_{ik}R_j^k - \frac{1}{(n-2)^2}|Ric|^2 g_{ij}\\
&+ \frac{1}{(n-1)(n-2)^2}R^2 g_{ij} - \frac{n}{(n-1)(n-2)^2}RR_{ij}
\end{aligned}
\tag{A.11.6}
$$

Specially, in dimension 4, we have:

$$
B_{ij} = \frac{1}{2}\Delta R_{ij} - \frac{1}{12}\Delta R g_{ij} - \frac{1}{6}\nabla_i\nabla_j R + R^{kl}R_{ikjl} - \frac{1}{4}|Ric|^2 g_{ij} + \frac{1}{12}R^2 g_{ij} - \frac{1}{3}RR_{ij}
\tag{A.11.7}
$$

We will show that in dimension 4, Bach tensor is the gradient of $L^2$ norm of Weyl curvature in Appendix C.

**Proposition A.11.2.** *The divergence of Bach tensor is given by:*

$$
\nabla^j B_{ij} = \frac{n-4}{(n-2)^2}R^{jk}C_{ijk}
\tag{A.11.8}
$$

*where $C_{ijk}$ is Cotton tensor defined by*

$$
C_{ijk} = (n-2)\Big(\nabla_i A_{jk} - \nabla_j A_{ik}\Big)
\tag{A.11.9}
$$

*Proof.* First, we derive some useful identities about the Schouten tensor. Recall that Schouten tensor is

$$A_{ij} = \frac{1}{n-2}\Big(R_{ij} - \frac{1}{2(n-2)} R g_{ij}\Big) \tag{A.11.10}$$

The trace of Schouten tensor is

$$Tr_g(A) = \frac{1}{2(n-1)} R \tag{A.11.11}$$

The divergence of Schouten tensor is

$$\nabla^j A_{ij} = \frac{1}{2(n-1)} \nabla^i R \tag{A.11.12}$$

In terms of Schouten tensor and Weyl curvature, we have:

$$\begin{aligned}
\nabla^j B_{ij} &= \nabla^j \nabla^k \nabla_k A_{ij} - \nabla^j \nabla^k \nabla_j A_{ik} + \nabla^j \Big(A^{kl} W_{ikjl}\Big) \\
&= \nabla^j \nabla^k \nabla_k A_{ij} - \nabla^k \nabla^j \nabla_k A_{ij} + \nabla^j \Big(A^{kl} W_{ikjl}\Big) \\
&= -R_{kjk}{}^{l} \nabla_l A^j_i - R_{kji}{}^{l} \nabla^k A^j_l - R_{kj}{}^{jl} A_{il} + \nabla^j \Big(A^{kl} W_{ikjl}\Big) \\
&= -\nabla^j A^{kl} R_{ikjl} + \nabla^j \Big(A^{kl} W_{ikjl}\Big)
\end{aligned}$$

We relabel some redundant indices and apply the Ricci identity (A.6.1) in this equation. Next, we use the Riemann curvature decomposition (A.10.1) to this result, combine with the divergence of Weyl curvature (A.9.4), we have:

$$\begin{aligned}
\nabla^j B_{ij} &= -\nabla^j A^{kl} R_{ikjl} + \nabla^j \Big(A^{kl} W_{ikjl}\Big) \\
&= -\nabla^j A^{kl} R_{ikjl} + \nabla^j A^{kl} W_{ikjl} + A^{kl} \nabla^j W_{ikjl} \\
&= -\nabla^j A^{kl} \Big(A \owedge g\Big)_{ikjl} + (n-3) A^{kl} \Big(\nabla_i A_{kl} - \nabla_k A_{il}\Big)
\end{aligned}$$

For the first term, we have:

$$\begin{aligned}
\nabla^j A^{kl}\Big(A \owedge g\Big)_{ikjl} &= \nabla^j A^{kl}\Big(A_{ij}g_{kl} + A_{kl}g_{ij} - A_{il}g_{kj} - A_{kj}g_{il}\Big) \\
&= \frac{1}{2(n-1)}A_{ik}\nabla^i R + A^{kl}\nabla_i A_{kl} - \frac{1}{2(n-1)}A_{il}\nabla^l R - A_{kj}\nabla^j A_{ik} \\
&= A^{kl}\nabla_i A_{kl} - A^{kl}\nabla_l A_{ik}
\end{aligned}$$

In the last line, we changed the redundant index. Then we conclude that the divergence of Bach tensor is

$$\begin{aligned}
\nabla^j B_{ij} &= -A^{kl}\nabla_i A_{kl} + A^{kl}\nabla_l A_{ik} + (n-3)A^{kl}\Big(\nabla_i A_{kl} - \nabla_k A_{il}\Big) \\
&= (n-4)A^{kl}\Big(\nabla_i A_{kl} - \nabla_k A_{il}\Big) \\
&= \frac{n-4}{(n-2)^2}R^{kl}C_{ikl}
\end{aligned}$$

□

# Appendix B

# Variations of Curvatures

In this Chapter, we will give the details of calculation of the first variation formulae of metric, Christoffel symbol and curvatures.

Let $(M^n, g)$ be an n-dimensional Riemannian manifold, and $g(t) = g + th$ be a one-parameter family of metrics. where $h$ is a symmetric 2-tensor. That is $\frac{\partial g}{\partial t} = h$.

For any point $x \in M$, we consider the normal coordinate system $\{\partial_i\}_{i=1}^n$ around this fixed point $x$. We have $g_{ij} = \delta_{ij}$ and $\partial_k g_{ij} = 0$, therefore, the Christoffel symbol vanishes, i.e., $\Gamma_{ij}^k = 0$. In this chapter, we will do calculations with the normal coordinate.

## B.1 Riemannian Metric

**Proposition B.1.1.** *Variation of metric inverse*

$$\dot{g}^{ij} = -h^{ij} \tag{B.1.1}$$

*Proof.* By a direct calculation, we differentiate the equation $g^{ik}g_{kj} = \delta^i_j$, we have:

$$\begin{aligned}
\dot{g}_{ik}g_{kj} &= -g^{ik}h_{kj} \\
\dot{g}_{ik} &= -g^{kj}h^i_j \\
\dot{g}_{ik} &= -h^{ik}
\end{aligned}$$

□

## B.2 Volume Form

**Proposition B.2.1** (Equation 2.11, Page 104, [16])**.** *Variation of Riemann volume form*

$$\frac{\partial}{\partial t}d\mu = \frac{1}{2}Tr_g(h)d\mu \tag{B.2.1}$$

*where $Tr_g$ is the trace with respect to $g$.*

*Proof.* Recall that the Riemann volume form is given by [34] (Proposition15.31.)

$$d\mu = \sqrt{det(g_{ij})}dx^1 \wedge dx^2 \cdots \wedge dx^n \tag{B.2.2}$$

In order to derive (B.2.1), we need to calculate the variation of determinant of a matrix $A$, which is given by the original definition of determinant and chain rule. For a fixed element $A_{pq}$, we have:

$$\begin{aligned}
\frac{\partial}{\partial A_{pq}}det(A) &= \frac{\partial}{\partial A_{pq}}\left(\sum_{\sigma\in S_n} sgn(\sigma)\sum_{i=1}^{n} A_{i,\sigma_i}\right) \\
&= \sum_{\sigma\in S_n} sgn(\sigma)\sum_{i\neq p,\sigma_i\neq q} A_{i,\sigma_i}
\end{aligned}$$

which is exactly the adjugate element at index $(p, q)$. Therefore, we have:

$$\begin{aligned}\frac{\partial}{\partial t} det(g) &= \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial g}{\partial g_{ij}} \frac{\partial g_{ij}}{\partial t} \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} adj_{ij} h_{ij} \\ &= det(g) g^{-1} h\end{aligned}$$

At last line, we use the definition of the inverse of matrix. With this result, we have:

$$\begin{aligned}\frac{\partial}{\partial t} d\mu &= \frac{\partial}{\partial t}\sqrt{det(g_{ij})} dx^1 \wedge dx^2 \cdots \wedge dx^n \\ &= \frac{1}{2}\frac{1}{\sqrt{det(g_{ij})}} det(g_{ij}) g^{ij} h_{ij} dx^1 \wedge dx^2 \cdots \wedge dx^n \\ &= \frac{1}{2} Tr_g(h) d\mu\end{aligned}$$

□

## B.3 Christoffel symbols

**Proposition B.3.1** (Lemma 2.27, Page 108, [16]). *Variation of Christoffel symbol*

$$\frac{\partial}{\partial t}\Gamma_{ij}^{k} = \frac{1}{2} g^{kl}(\nabla_i h_{jl} + \nabla_j h_{il} - \nabla_l h_{ij}) \tag{B.3.1}$$

*Proof.* We have:

$$\begin{aligned}\frac{\partial}{\partial t}\Gamma_{ij}^{k} &= \frac{1}{2} g^{kl} \frac{\partial}{\partial t}\Big(\nabla_j g_{il} + \nabla_i g_{jl} - \nabla_l g_{ij}\Big) \\ &= \frac{1}{2} g^{kl}\Big(\nabla_j h_{il} + \nabla_i h_{jl} - \nabla_l h_{ij}\Big)\end{aligned}$$

□

## B.4 Riemann Curvature Tensor

**Proposition B.4.1** (Equation 2.66 2.67, Page 120, [16])**.** *Variation of Riemann curvature tensor*

$$\frac{\partial}{\partial t}R_{ijkl} = \frac{1}{2}\left\{\begin{array}{c} \nabla_j\nabla_i h_{kl} + \nabla_j\nabla_k h_{il} - \nabla_j\nabla_l h_{ik} \\ -\nabla_i\nabla_j h_{kl} - \nabla_i\nabla_k h_{jl} + \nabla_i\nabla_l h_{jk} \end{array}\right\} + R_{ijkp}h_l^p \tag{B.4.1}$$

*Proof.* With a direct calculation,

$$\begin{aligned} \frac{\partial}{\partial t}g_{lp}R_{ijk}{}^{p} &= g_{lp}\left(\partial_j\frac{\partial}{\partial t}\Gamma_{ik}^p - \partial_i\frac{\partial}{\partial t}\Gamma_{jk}^p\right) + R_{ijkp}h_l^p \\ &= \frac{1}{2}g_l^p\Big(\nabla_j\nabla_i h_{kp} + \nabla_j\nabla_k h_{ip} - \nabla_j\nabla_p h_{ik} \\ &\quad - \nabla_i\nabla_j h_{kp} - \nabla_i\nabla_k h_{jp} + \nabla_i\nabla_p h_{jk}\Big) + R_{ijkp}h_l^p \\ &= \frac{1}{2}\Big(\nabla_j\nabla_i h_{kl} + \nabla_j\nabla_k h_{il} - \nabla_j\nabla_l h_{ik} \\ &\quad - \nabla_i\nabla_j h_{kl} - \nabla_i\nabla_k h_{jl} + \nabla_i\nabla_l h_{jk}\Big) + R_{ijkp}h_l^p \end{aligned}$$

With Ricci identity (A.6.1), we have:

$$\begin{aligned} \frac{\partial}{\partial t}g_{lp}R_{ijk}{}^{p} = \frac{1}{2}&\Big(\nabla_j\nabla_k h_{il} - \nabla_j\nabla_l h_{ik} - \nabla_i\nabla_k h_{jl} + \nabla_i\nabla_l h_{jk}\Big) \\ &+ \frac{1}{2}\Big(R_{ijkp}h_l^p - R_{ijlp}h_k^p\Big) \end{aligned} \tag{B.4.2}$$

∎

## B.5 Ricci Curvature Tensor

**Proposition B.5.1** (Equation 2.31, Page 109, [16])**.** *Variation of Ricci curvature*

$$\frac{\partial}{\partial t}R_{ik} = -\frac{1}{2}\left(\Delta h_{ik} + \nabla_i\nabla_k Tr_g(h) - \nabla^p\nabla_i h_{kp} - \nabla^p\nabla_k h_{ip}\right) \tag{B.5.1}$$

*Proof.* This follows from the formula (B.4.1) by contracting the first and third indices. We use rough Laplacian $\Delta = g^{pq}\nabla_p\nabla_q$ here. □

## B.6 Scalar Curvature

**Proposition B.6.1** (Lemma 2.7, Page 99, [16])**.** *Variation of scalar curvature*

$$\frac{\partial}{\partial t}R = -\Delta Tr_g(h) + \nabla^p\nabla^q h_{pq} - h^{pq}R_{pq} \tag{B.6.1}$$

*Proof.* This follows from formula (A.7.2) and (B.1.1),

$$\begin{aligned}\frac{\partial}{\partial t}R &= \frac{\partial}{\partial t}g^{ij}R_{ij}\\ &= -h^{ij}R_{ij} - \frac{1}{2}g^{ij}\left(\Delta h_{ij} + \nabla_i\nabla_j Tr_g(h) - \nabla^p\nabla_i h_{jp} - \nabla^p\nabla_j h_{ip}\right)\\ &= -\Delta Tr_g(h) + \nabla^p\nabla^q h_{pq} - h^{pq}R_{pq}\end{aligned} \tag{B.6.2}$$

□

# Appendix C

# Gradient of Energy Functionals

In this chapter, we will calculate the gradients of some $L^2$ norms of curvatures. All of the results can be found in [6]. Let $(M^n, g)$ be an n dimensional closed manifold, $T$ a tensor on this manifold, we define the energy functional to be

$$\mathcal{F}_T = \int_M |T|^2 d\mu$$

Let $g$ be a family of metric defined as $g = g_0 + th$, where $h$ is a symmetric 2 tensor. The gradient of this energy functional will be denoted by $grad\mathcal{F}_T$ and defined in the following way.

$$\frac{\partial}{\partial t}\mathcal{F}_T = \int_M \langle grad\mathcal{F}_T, h\rangle d\mu$$

With this manner, we define the following energy functionals, and we will go through details and derive their gradient.

(a)

$$\mathcal{F}_{Rm} = \int_M |Rm|^2 d\mu$$

(b)

$$\mathcal{F}_{Ric} = \int_M |Ric|^2 d\mu$$

(c)

$$\mathcal{F}_{R} = \int_M |R|^2 d\mu$$

(d)

$$\mathcal{F}_{W} = \int_M |W|^2 d\mu$$

## C.1 Riemann Curvature

**Proposition C.1.1** (Prop 4.70, Page 134, [6])**.**

$$grad\mathcal{F}_{Rm} = -4\Delta R_{ij} + 2\nabla_i\nabla_j R - 4R^{pq}R_{ipjq} + 4R_{ip}R^p_j - 2R_{pqri}R^{pqr}{}_j + \frac{1}{2}|Rm|^2 g_{ij} \tag{C.1.1}$$

*Proof.* With our previous calculation about the variation of Riemann curvature (B.4.1), and the variation of Riemannian volume form (B.2.1)

$$\begin{aligned}
\frac{\partial}{\partial t}\mathcal{F}_{Rm} &= 2\int_M \langle R_{ijkl}, \frac{\partial}{\partial t}R_{ijkl}\rangle d\mu + \int_M \langle \frac{1}{2}|Rm|^2 g_{ij} - 4R_{pqri}R^{pqr}{}_j, h_{ij}\rangle d\mu \\
&= \int_M \langle R_{ijkl}, (\nabla_j\nabla_i - \nabla_i\nabla_j)h_{kl}\rangle d\mu + 2\int_M \langle R_{pqri}R^{pqr}{}_j, h_{ij}\rangle d\mu \\
&+ \int_M \langle R_{ijkl}, \nabla_j\nabla_k h_{il}\rangle d\mu + \int_M \langle R_{ijkl}, -\nabla_j\nabla_l h_{ik}\rangle d\mu \\
&+ \int_M \langle R_{ijkl}, -\nabla_i\nabla_k h_{jl}\rangle d\mu + \int_M \langle R_{ijkl}, \nabla_i\nabla_l h_{jk}\rangle d\mu \\
&+ \int_M \langle \frac{1}{2}|Rm|^2 g_{ij} - 4R_{pqri}R^{pqr}{}_j, h_{ij}\rangle d\mu
\end{aligned}$$

We apply integration by parts, Ricci identity (A.6.1) and Bianchi identity (A.5.1) to this equation.

For the first term, we have

$$
\begin{aligned}
&\int_M \langle R_{ijkl}, (\nabla_j\nabla_i - \nabla_i\nabla_j)h_{kl}\rangle d\mu \\
=&\int_M \langle \nabla^i\nabla^j R_{ijkl} - \nabla^j\nabla^i R_{ijkl}, h_{kl}\rangle d\mu \\
=&\int_M \langle -R^{j\ \ ip}_{\ i}R_{pjkl} - R_j^{\ ijp}R_{ipkl} - R^{ji\ \ p}_{\ \ k}R_{ijpl} - R^{ji\ \ p}_{\ \ l}R_{ijkp}, h_{kl}\rangle d\mu = 0
\end{aligned} \tag{C.1.2}
$$

At the last line, we simplified all of terms by contracting indices.

The next four terms are the same, we only look at one of them.

$$
\begin{aligned}
&\int_M \langle R_{ijkl}, \nabla_j\nabla_k h_{il}\rangle d\mu \\
=&\int_M \langle \nabla_k\nabla_j R_{ijkl}, h_{il}\rangle d\mu \\
=&\int_M \langle \nabla_k\Big(-\nabla_k R_{ijlj} - \nabla_l R_{ijjk}\Big), h_{il}\rangle d\mu \\
=&\int_M \langle -\Delta R_{ij} + \nabla^p\nabla_j R_{ip}, h_{ij}\rangle d\mu \\
=&\int_M \langle -\Delta R_{ij} + \nabla^j\nabla_p R_{ip} - R^{pq}R_{ipjq} + R_{ip}R_j^{\ p}, h_{ij}\rangle d\mu \\
=&\int_M \langle -\Delta R_{ij} + \frac{1}{2}\nabla_i\nabla_j R - R^{pq}R_{ipjq} + R_{ip}R_j^{\ p}, h_{ij}\rangle d\mu
\end{aligned} \tag{C.1.3}
$$

Combine (C.1.2) and (C.1.3), we have

$$
grad\mathcal{F}_{Rm} = -4\Delta R_{ij} + 2\nabla_i\nabla_j R - 4R^{pq}R_{ipjq} + 4R_{ip}R_j^p - 2R_{pqri}R^{pqr}{}_j + \frac{1}{2}|Rm|^2 g_{ij}
$$

□

## C.2 Ricci Curvature

**Proposition C.2.1** (Prop 4.66, Page 133, [6])**.**

$$
grad\mathcal{F}_{Ric} = -\Delta R_{ij} - \frac{1}{2}\Delta R g_{ij} + \nabla_i\nabla_j R - 2R^{pq}R_{ipjq} + \frac{1}{2}|Ric|^2 g_{ij} \tag{C.2.1}
$$

*Proof.* Follow the same calculation, use (B.5.1), we have

$$
\begin{aligned}
\frac{\partial}{\partial t}\int_M |Ric|^2 d\mu &= \int_M \langle R_{ij}, \frac{\partial}{\partial t} R_{ij}\rangle d\mu + \int_M \langle \frac{1}{2}|Ric|^2 g_{ij} - 2R_{ip}R_j^p, h_{ij}\rangle d\mu \\
&= \int_M \langle R_{ij}, -\Delta h_{ij}\rangle d\mu + \int_M \langle R_{ij}, -\nabla_i\nabla_j Tr_g(h)\rangle d\mu \\
&\quad + \int_M \langle R_{ij}, \nabla^p\nabla_i h_{jp}\rangle d\mu + \int_M \langle R_{ij}, \nabla^p\nabla_j h_{ip}\rangle d\mu \\
&\quad + \frac{1}{2}\int_M \langle |Ric|^2 g_{ij}, h_{ij}\rangle d\mu
\end{aligned}
$$

Use integration by parts again, the result follows. □

## C.3 Scalar Curvature

**Proposition C.3.1** (Prop 4.66, Page 133, [6])**.**

$$
grad\mathcal{F}_R = -2\Delta R g_{ij} + 2\nabla_i\nabla_j R - 2RR_{ij} + \frac{1}{2}R^2 g_{ij} \tag{C.3.1}
$$

*Proof.* Follow the same calculation, use (B.6.1), we have

$$
\begin{aligned}
\frac{\partial}{\partial t}\int_M |R|^2 d\mu &= 2\int_M \langle R, \frac{\partial}{\partial t} R\rangle d\mu + \frac{1}{2}\int_M \langle |R|^2 g_{ij}, h_{ij}\rangle d\mu \\
&= 2\int_M \langle R, -\Delta Tr_g(h)\rangle d\mu + 2\int_M \langle R, \nabla^i\nabla^j h_{ij}\rangle d\mu \\
&\quad - 2\int_M \langle R, R^{ij} h_{ij}\rangle d\mu + \frac{1}{2}\int_M \langle |R|^2 g_{ij}, h_{ij}\rangle d\mu
\end{aligned}
$$

then the result follows. □

## C.4 Weyl Curvature

Now, we are ready to derive the gradient for $L^2$ norm of Weyl curvature.

**Proposition C.4.1.**

$$
\begin{aligned}
grad\mathcal{F}_W = &-\frac{4(n-3)}{n-2}\Delta R_{ij} + \frac{2(n-3)}{(n-1)(n-2)}\Delta R g_{ij} + \frac{2(n-3)}{n-1}\nabla_i\nabla_j R \\
&- \frac{4(n-4)}{n-2}R^{pq}R_{ipjq} + 4R_{ip}R_j^p - \frac{4}{(n-1)(n-2)}RR_{ij} \\
&- 2R_{pqri}R^{pqr}{}_j + \frac{1}{2}|W|^2 g_{ij}
\end{aligned}
\tag{C.4.1}
$$

*Proof.* From the Riemann curvature decomposition (A.10.3), we have:

$$
\begin{aligned}
grad\int_M |W|^2 d\mu = & \, grad\int_M |Rm|^2 - \frac{4}{n-2}|Ric|^2 + \frac{2}{(n-1)(n-2)}R^2 d\mu \\
= & -4\Delta R_{ij} + 2\nabla_i\nabla_j R - 4R^{pq}R_{ipjq} + 4R_{ip}R_j^p - 2R_{pqri}R^{pqr}{}_j \\
& + \frac{1}{2}|Rm|^2 g_{ij} - \frac{4}{n-2}\Big( - \Delta R_{ij} - \frac{1}{2}\Delta R g_{ij} + \nabla_i\nabla_j R \\
& - 2R^{pq}R_{ipjq} + \frac{1}{2}|Ric|^2 g_{ij}\Big) + \frac{2}{(n-1)(n-2)}\Big( - 2\Delta R g_{ij} \\
& + 2\nabla_i\nabla_j R - 2RR_{ij} + \frac{1}{2}R^2 g_{ij}\Big) \\
= & -\frac{4(n-3)}{n-2}\Delta R_{ij} + \frac{2(n-3)}{(n-1)(n-2)}\Delta R g_{ij} + \frac{2(n-3)}{n-1}\nabla_i\nabla_j R \\
& - \frac{4(n-4)}{n-2}R^{pq}R_{ipjq} + 4R_{ip}R_j^p - \frac{4}{(n-1)(n-2)}RR_{ij} \\
& - 2R_{pqri}R^{pqr}{}_j + \frac{1}{2}|W|^2 g_{ij}
\end{aligned}
\tag{C.4.2}
$$

□

**Proposition C.4.2** (Equation 4.77, Page 135, [6])**.** *In dimension 4, we the gradient of $\mathcal{F}_W$ is Bach tensor.*

*Proof.* In dimension 4, we have:

$$
\begin{aligned}
grad\int_M |W|^2 d\mu = & -2\Delta R_{ij} + \frac{1}{3}\Delta R g_{ij} + \frac{2}{3}\nabla_i\nabla_j R + 4R_{ip}R_j^p - \frac{2}{3}RR_{ij} \\
& - 2R_{pqri}R^{pqr}{}_j + \frac{1}{2}|W|^2 g_{ij}
\end{aligned}
\tag{C.4.3}
$$

Also, in dimension 4, we have the following identity:

$$R_{ipqr}R_j^{pqr} - \frac{1}{4}|Rm|^2 g_{ij} = \frac{1}{3}RE_{ij} + 2R^{pq}W_{ipjq} \tag{C.4.4}$$

therefore, for the last two terms, we have:

$$R_{ipqr}R_j^{pqr} - \frac{1}{4}|W|^2 g_{ij} = 2R^{pq}W_{ipjq} + \frac{1}{3}RE_{ij} + \frac{1}{2}|E|^2 + \frac{1}{24}R^2 \tag{C.4.5}$$

With this result, we have:

$$\begin{aligned} grad\int_M |W|^2 d\mu = &-2\Delta R_{ij} + \frac{1}{3}\Delta R g_{ij} + \frac{2}{3}\nabla_i\nabla_j R + 4R_{ip}R_j^p - \frac{2}{3}RR_{ij} \\ &- 2\Big(2R^{pq}W_{ipjq} + \frac{1}{3}RE_{ij} + \frac{1}{2}|E|^2 g_{ij} + \frac{1}{24}R^2 g_{ij}\Big) \end{aligned} \tag{C.4.6}$$

In (A.11.5), we have:

$$\begin{aligned} R^{pq}W_{ipjq} &= R^{pq}\Big(R_{ipjq} - (A \owedge g)_{ipjq}\Big) \\ &= R^{pq}R_{ipjq} - \frac{1}{2}\Big(RR_{ij} - \frac{1}{6}R^2 g_{ij}\Big) - \frac{1}{2}\Big(|Ric|^2 g_{ij} - \frac{1}{6}R^2 g_{ij}\Big) \\ &\quad + \Big(R_{ip}R_j^p - \frac{1}{6}RR_{ij}\Big) \\ &= R^{pq}R_{ipjq} - \frac{2}{3}RR_{ij} + \frac{1}{6}R^2 g_{ij} - \frac{1}{2}|Ric|^2 g_{ij} + R_{ip}R_j^p \end{aligned} \tag{C.4.7}$$

and we conclude that

$$\begin{aligned} grad\int_M |W|^2 d\mu &= -4\Big[\frac{1}{2}\Delta R_{ij} - \frac{1}{12}\Delta R g_{ij} - \frac{1}{6}\nabla_i\nabla_j R \\ &\qquad + R^{pq}W_{ipjq} - |Ric|^2 g_{ij} + \frac{1}{12}R^2 g_{ij} - \frac{1}{3}RR_{ij}\Big] \\ &= -4B_{ij} \end{aligned} \tag{C.4.8}$$

$\square$

**Remark C.4.3.** *We use a different definition with Equation 4.77 in Page 135 [6]* *by a factor $-4$. This definition is adopted today.*

**Proposition C.4.4** (Equation 4.72, Page 134, [6])**.** *In dimension 4, we have the following identity*

$$R_{ipqr}R_j^{pqr} - \frac{1}{4}|Rm|^2 g_{ij} = \frac{1}{3}RE_{ij} + 2R^{pq}W_{ipjq} \tag{C.4.9}$$

*Proof.* This formula is a direct consequence of the following identity

$$W_{ipqr}W_j^{pqr} = \frac{1}{4}|W|^2 g_{ij} \tag{C.4.10}$$

then we have

$$\begin{aligned}
R_{ipqr}R_j^{pqr} &= \Big[W_{ipqr} + (A \owedge g)_{ipqr})\Big]\Big[W_j^{pqr} + (A \owedge g)_j^{pqr}\Big] \\
&= W_{ipqr}W_j^{pqr} + W_{ipqr}(A_j^q g^{pr} + A^{pr}g_j^q - A_j^r g^{pq} - A^{pg}g_j^r) \\
&+ W_{jpqr}(A_i^q g^{pr} + A^{pr}g_i^q - A_i^r g^{pq} - A^{pg}g_i^r) \\
&+ (A_{iq}g_{pr} + A_{pr}g_{iq} - A_{ir}g_{pq} - A_{pg}g_{ir})(A_j^q g^{pr} + A^{pr}g_j^q - A_j^r g^{pq} - A^{pg}g_j^r) \\
&= W_{ipqr}W_j^{pqr} + 4W_{ikjl}A^{kl} + 4Tr(A)A_{ij} + 2|A|^2 g_{ij} \\
&= W_{ipqr}W_j^{pqr} + 2W_{ikjl}E^{kl} + \frac{1}{3}RE_{ij} + \frac{1}{2}|E|^2 + \frac{1}{24}R^2
\end{aligned}$$

combine with the curvature decomposition, the result follows. $\square$

# Appendix D

# Interpolation Inequality

In this chapter, we will introduce some interpolation inequalities which plays an important roll in the integral estimate in geometric analysis. In general, these inequalities say that any order derivative can be controlled by higher order and lower order derivatives. This is why they are named by 'interpolation'. Most of results can be found in [25, Chapter 12].

We only consider a closed manifold as our model which allows us to use integration by parts. But all of these inequalities has the same local form with some suitable cut-off function. These local estimates can be found in [30, P.332] and [51, P.270].

## D.1 Interpolation Inequality

**Proposition D.1.1** (Theorem 12.1, Page 291, [25])**.** *Let $(M^n, g)$ be a closed $n$-dimensional manifold and $T$ is any tensor defined on $M$. Suppose that $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$, where $r \geq 1$, then there exists a constant $C = C(n, r)$, such that:*

$$\left\{ \int |\nabla T|^{2r} d\mu \right\}^{\frac{1}{r}} \leq C \left\{ \int |\nabla^2 T|^p d\mu \right\}^{\frac{1}{p}} \left\{ \int |T|^q d\mu \right\}^{\frac{1}{q}} \tag{D.1.1}$$

*Proof.* With integration by parts, we have:

$$\begin{aligned}\int |\nabla T|^{2r} d\mu &= \int \langle \nabla_i T_j, \nabla_i T_j \rangle \langle \nabla_k T_l, \nabla_k T_l \rangle^{r-1} d\mu \\ &= -\int \langle T_j, \Delta T \rangle \langle \nabla_k T_l, \nabla_k T_l \rangle^{r-1} d\mu \\ &\quad - 2(r-1) \int \langle T_j \nabla_i \nabla_k T_l, \nabla_i T_j \nabla_k T_l \rangle |\nabla T|^{2r-4} d\mu \\ &\leq n \int |T||\nabla^2 T||\nabla T|^{2r-2} d\mu + 2(r-1) \int |T||\nabla^2 T||\nabla T|^{2r-2} d\mu \\ &= C(r,n) \int |T||\nabla^2 T||\nabla T|^{2r-2} d\mu\end{aligned}$$

Note that $\frac{1}{p} + \frac{1}{q} - \frac{r-1}{r} = 1$, apply Hölder inequality to the right hand side, we have:

$$\int |\nabla T|^{2r} d\mu \leq C(r,n) \Big\{ \int |\nabla^2 T|^p d\mu \Big\}^{\frac{1}{p}} \Big\{ \int |T|^q d\mu \Big\}^{\frac{1}{q}} \Big\{ \int |\nabla T|^{2r} d\mu \Big\}^{\frac{r-1}{r}}$$

Then the result follows. □

**Proposition D.1.2** (Corollary 12.2, Page 292, [25])**.** *Let $(M^n, g)$ be a closed $n$-dimensional manifold and $T$ is any tensor defined on $M$. There exists a constant $C = C(n,p)$, such that:*

$$\Big\{ \int |\nabla T|^{2p} d\mu \Big\}^{\frac{1}{p}} \leq C \max_M |T| \Big\{ \int |\nabla^2 T|^p d\mu \Big\}^{\frac{1}{p}} \tag{D.1.2}$$

*Proof.* From (D.1.1), let $p = r$ and $q = \infty$. □

**Proposition D.1.3.** *Let $(M^n, g)$ be a closed $n$-dimensional manifold and $T$ is any tensor defined on $M$. For any $\epsilon > 0$, there exists a constant $C = C(n, \epsilon)$, such that:*

$$\int |\nabla T|^2 d\mu \leq \epsilon \int |\nabla^2 T|^2 d\mu + C \int |T|^2 d\mu \tag{D.1.3}$$

*Proof.* From (D.1.1), let $r = 1$ and $p = q = 2$. After that, apply Young's inequality with $\epsilon$. $\square$

**Proposition D.1.4** (Corollary 12.7, Page 294, [25]). *Let $(M^n, g)$ be a closed $n$-dimensional manifold and $T$ is any tensor defined on $M$. For any $\epsilon > 0$, there exists a constant $C = C(n, \epsilon)$, such that:*

$$\int |\nabla^k T|^2 d\mu \leq \epsilon \int |\nabla^{k+1} T|^2 d\mu + C \int |T|^2 d\mu \tag{D.1.4}$$

*Proof.* We prove this by induction. The base case is from previous lemma, assume that this inequality hold for all $k \leq s-1$, then we have:

$$\begin{aligned}
\int |\nabla^s T|^2 d\mu &= \int |\nabla(\nabla^{s-1} T)|^2 d\mu \\
&\leq \epsilon \int |\nabla^{s+1} T|^2 d\mu + C \int |\nabla^{s-1} T|^2 d\mu \\
&\leq \epsilon_1 \int |\nabla^{s+1} T|^2 d\mu + \epsilon_2 \int |\nabla^s T|^2 d\mu + C \int |\nabla T|^2 d\mu
\end{aligned}$$

Then the result follows. $\square$

**Proposition D.1.5** (Corollary 12.6, Page 293, [25]). *Let $(M^n, g)$ be a closed $n$-dimensional manifold and $T$ is any tensor defined on $M$. For any $k \in \mathbb{N}$, $k \geq 1$, and $1 \leq i \leq k$, there exists a constant $C = C(n, k, i)$ such that we have the following inequality:*

$$\int |\nabla^i T|^{\frac{2k}{i}} d\mu \leq C \, \|T\|_\infty^{\frac{2k}{i}-2} \int |\nabla^k T|^2 d\mu \tag{D.1.5}$$

**Proposition D.1.6.** *Let $(M^n, g)$ be a closed $n$-dimensional manifold and $T$ is any tensor defined on $M$. For any $k \in \mathbb{N}$, $k \geq 1$, and $i_1 + i_2 + \cdots + i_s = 2k$, there exists a constant $C = C(n, k)$ such that we have the following inequality:*

$$\int |\nabla^{i_1} T * \nabla^{i_2} T * \cdots * \nabla^{i_s}| d\mu \leq C \, \|T\|_\infty^{s-2} \int |\nabla^k T|^2 d\mu \tag{D.1.6}$$

*Proof.* We start with the Hölder inequality, since $\frac{i_1}{2k} + \frac{i_2}{2k} + \cdots + \frac{i_s}{2k} = 1$, we have:

$$\int |\nabla^{i_1} T * \nabla^{i_2} T * \cdots * \nabla^{i_s}| d\mu \leq \left\{ \int |\nabla^{i_1} T|^{\frac{2k}{i_1}} \right\}^{\frac{i_1}{2k}} \cdots \left\{ \int |\nabla^{i_1} T|^{\frac{2k}{i_s}} \right\}^{\frac{i_s}{2k}}$$

For each term, we apply previous Proposition (D.1.5), we have:

$$\left\{ \int |\nabla^{i_j} T|^{\frac{2k}{i_j}} \right\}^{\frac{i_j}{2k}} \leq C_j \left\| T \right\|_\infty^{1-\frac{i_j}{k}} \left\{ \int |\nabla^k T|^2 d\mu \right\}^{\frac{i_j}{2k}}$$

therefore,

$$\begin{aligned} \int |\nabla^{i_1} T * \nabla^{i_2} T * \cdots * \nabla^{i_s} T| d\mu &\leq \left\{ \int |\nabla^{i_1} T|^{\frac{2k}{i_1}} \right\}^{\frac{i_1}{2k}} \cdots \left\{ \int |\nabla^{i_1} T|^{\frac{2k}{i_s}} \right\}^{\frac{i_s}{2k}} \\ &\leq C \left\| T \right\|_\infty^{s-2} \int |\nabla^k T|^2 d\mu \end{aligned}$$

□

# Appendix E

# Sobolev Inequality

In this chapter, we will introduce the Sobolev inequality on manifold. In Sec.1, we will introduce the general Sobolev inequality in $\mathbb{R}^n$, in Sec.2, we will move to manifold, and in this case, we will focus on $L^2$ Sobolev inequality, some variants of Sobolev inequality will also be mentioned. In Sec. 3, we will show that how to obtain the $L^\infty$ control by iterating the $L^2$ Sobolev inequality.

## E.1 Sobolev Inequality in Euclidean Space

In this section, we will prove the Sobolev inequality for Euclidean space. All of the results comes from $[19, \S5.6, Page.275]$

**Motivation**

In this subsection, we assume that $1 \leq p < n$, and we want to establish the following inequality

$$\|u\|_{L^q(\mathbb{R}^n)} \leq C \|Du\|_{L^p(\mathbb{R}^n)}$$

for some constants $C > 0$, $1 \leq q < \infty$ and all function $u \in C_0^\infty(\mathbb{R}^n)$.

Since we want the inequality to be scale invariant, the choice of constant $q$

is not arbitrary. Choose arbitrary function $u(x)$, a positive constant $\lambda$, if the inequality above holds for $u(x)$, we consider a new function $u_\lambda := u(\lambda x)$, then we have:

$$\text{LHS} = \|u_\lambda\|_{L^q(\mathbb{R}^n)} = \lambda^{-\frac{n}{q}} \|u\|_{L^q(\mathbb{R}^n)} \tag{E.1.1}$$

and

$$\text{RHS} = \|Du_\lambda\|_{L^p(\mathbb{R}^n)} = \lambda^{\frac{p-n}{q}} \|Du\|_{L^p(\mathbb{R}^n)} \tag{E.1.2}$$

Therefore, the scale invariant property comes from the choice of $q$ with $q = \frac{np}{n-p}$.

**Sobolev Inequality**

Now we are ready to state the Sobolev inequality we state before. Historically, this inequality was first proved by Sobolev [47].

**Theorem E.1.1** (Theorem 1, Page. 277, [19])**.** *[Gagliardo-Nirenberg-Sobolev Inequality] Assume that $1 \le p < n$, there exists a constant $C$ depending only on $n, p$, such that*

$$\|u\|_{L^{\frac{n-p}{np}}(\mathbb{R}^n)} \le C \|u\|_{L^p(\mathbb{R}^n)} \tag{E.1.3}$$

*for all $u \in C_c^1(\mathbb{R}^n)$.*

**Remark E.1.2.** *This constant $C$ does not depend on the size of the compact support domain.*

For an open bounded subset in $\mathbb{R}^n$, we have the following Sobolev inequality.

**Theorem E.1.3** (Theorem 2, Page 279, [19])**.** *Let $U$ be a bounded, open subset of $\mathbb{R}^n$, and suppose that $\partial U$ is $C^1$. Assume that $1 \le p < n$ and $u \in W^{1,p}(U)$. Then $u \in L^{\frac{np}{n-p}}(U)$, with the following estimate:*

$$\|u\|_{L^{\frac{np}{n-p}}(U)} \le C \|u\|_{W^{1,p}(U)} \tag{E.1.4}$$

*the constant $C$ depends on $n, p$ and the domain $U$.*

**Remark E.1.4.** *This is the Sobolev inequality we want to used in manifold. Because smooth Riemannian manifolds are locally diffeomorphic to an open set in* $\mathbb{R}^{\ltimes}$

## E.2 Sobolev Inequality on Manifolds

In this section, we introduce the Sobolev constant on Riemannian manifold. We refer the following results to [35].

**Theorem E.2.1** (Lemma 2, [35]). *Given a closed Riemannian manifold $(M^n, g)$. There exists a constant $C_S(M^n, g) \geq 0$ such that*

$$\|u\|_{L^{\frac{2n}{n-2}}(M)} \leq C_S\Big( \|u\|_{L^2(M)} + \frac{1}{vol^{\frac{2}{n}}(M)} \|u\|_{L^2(M)} \Big) \tag{E.2.1}$$

**Remark E.2.2.** *The volume term in this $L^2$ Sobolev inequality will guarantee it is scale invariant.*

## E.3 Multiplicative Sobolev Inequality

In this section, we present a multiplicative Sobolev inequality. This inequality allows us to convert global bounds to point-wise bound.

**Theorem E.3.1** (Theorem 2.2, Page 62, [33]). *Let $(M^n, g)$ be a close Riemannian manifold with unit volume. For $u \in C_0^1(M)$, $4 < q \leq \infty$,$0 \leq m \leq \infty$, we have:*

$$\|u\|_\infty \leq C_S \cdot C(m, n, p) \|u\|_m^{1-\alpha} \left( \|u\|_p + \|\nabla u\|_p \right)^\alpha \tag{E.3.1}$$

*Proof.* The proof also can be found in several papers. In [30, Theorem 5.6, Page 336], author proves this inequality for surface, in [49, Theorem 19, Page 354], author proves this inequality for four dimensional manifold.

We assume that the manifold has unit volume, and we also rescale the function $u$ such that

$$C_S(\|u\|_p + \|\nabla u\|_p) = 1 \tag{E.3.2}$$

for a fixed number $p$.

Consider the function $u^{1+\omega}$ for some positive constant $\omega$, by the $L^2$ Sobolev inequality, we have:

$$\begin{aligned}
\left\|u^{1+\omega}\right\|_{\frac{2n}{n-2}} &\leq C_S\Big( \left\|u^{1+\omega}\right\|_2 + \left\|\nabla u^{1+\omega}\right\|_2 \Big) \\
&\leq C_S(1+\omega)\Big( \|u^\omega \cdot u\|_2 + \|u^\omega \cdot \nabla u\|_2 \Big) \\
&\leq C_S(1+\omega) \|u^\omega\|_q \Big( \|u\|_p + \|\nabla u\|_p \Big) \\
&= (1+\omega) \|u^\omega\|_q
\end{aligned} \tag{E.3.3}$$

in which we use Hölder inequality and $q = \frac{2p}{p-2}$. Therefore, we have:

$$\|u\|_{\frac{2n(1+\omega)}{n-2}} \leq (1+\omega)^{\frac{1}{1+\omega}} \|u\|_{\omega q}^{\frac{\omega}{1+\omega}} \tag{E.3.4}$$

We define $j$ with $jq = \frac{2n}{n-2}$, and we have:

$$\|u\|_{jq(1+\omega)} \leq (1+\omega)^{\frac{1}{1+\omega}} \|u\|_{\omega q}^{\frac{\omega}{1+\omega}} \tag{E.3.5}$$

Now we try to iterate this inequality by defining a sequence of constants. Let $\omega_0 = \frac{m}{q}$ and $\omega_{i+1} = j(1+\omega_i)$, we also set $C_i = (1+\omega_i)^{\frac{1}{1+\omega_i}}$ and $\delta_i = \frac{\omega_i}{1+\omega_i}$. With this setting, we have a sequence of inequalities

$$\|u\|_{\omega_{i+1} q} \leq C_i \|u\|_{\omega_i q}^{\delta_i} \tag{E.3.6}$$

Therefore, we have:

$$\|u\|_{\omega_i q} \leq \left( \prod_{k=0}^{i-1} C_k^{\delta_{k+1}} \right) \|u\|_m^{\delta_0 \delta_1 \cdots \delta_{i-1}} \tag{E.3.7}$$

We observe that $1 + \omega_i = j^i \omega_0 + \sum_{k=0}^{i} j^k$, therefore, we can find a constant $C = C(m, n, p)$ such that

$$\frac{1}{C} j^i \leq 1 + \omega_i \leq C j^i \tag{E.3.8}$$

with this estimate, we have:

$$log \prod_{k=0}^{i-1} C_k^{\delta_{k+1}} = \sum_{k=0}^{i-1} \frac{1}{1+\omega_i} log(1+\omega_i) \leq \sum_{k=0}^{i-1} C j^{-k} k log(j) \leq C \tag{E.3.9}$$

where we use the fact the $j > 1$. We also have:

$$\prod_{k=0}^{\infty} \delta_k = \lim_{k \to \infty} j^k \frac{\omega_0}{1+\omega_k} = \lim_{k \to \infty} \frac{\omega_0}{\omega_0 + \frac{j}{j-1}} = 1 - \alpha \tag{E.3.10}$$

Then we finish the proof. $\square$

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