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"Densities" and Maximal Monotonicity

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Dedicated to the memory of Jean Jacques Moreau.

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We discuss "Banach SN spaces", which include Hilbert spaces, negative Hilbert spaces, and the product of any real Banach space with its dual. We introduce "L-positive" sets, which generalize monotone multifunctions from a Banach space into its dual. We introduce the concepts of "r_L–density" and its specialization "quasidensity": the closed quasidense monotone multifunctions from a Banach space into its dual form a (generally) strict subset of the maximally monotone ones, though all surjective maximally monotone and all maximally monotone multifunctions on a reflexive space are quasidense. We give a sum theorem and a parallel sum theorem for closed monotone quasidense multifunctions under very general constraint conditions. That is to say, quasidensity obeys very nice calculus rules. We give a short proof that the subdifferential of a proper convex lower semicontinuous function on a Banach space is quasidense, and deduce generalizations of the Brezis–Browder theorem on linear relations to non reflexive Banach spaces. We also prove that any closed monotone quasidense multifunction has a number of other very desirable properties.

Keywords: Banach SN space, L–positive set, r_L–density, quasidensity, multifunction, maximal monotonicity, sum theorem, subdifferential, negative alignment, monotone linear relation, Brezis–Browder theorem

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1. Introduction

This paper falls logically into four parts. In the first part, Sections 2–6, 9 and part of Section 13, we discuss "Banach SN spaces", "L–positive sets", "r_L–density", "touching functions" and the functions Φ_A and Θ_A determined by an L–positive set. The second part, Sections 7–8, 10 and 12, is about Banach SN spaces of the special form $E \times E^*$, where E is a nonzero real Banach space. This part includes a short proof of a strict generalization of Rockafellar's result (see [18]) that the subdifferential of a proper convex lower semicontinuous function on a Banach space is maximally monotone, a sum theorem, a parallel sum theorem, and generalizations of the Brezis–Browder theorem on monotone linear relations.

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The third part, Section 11, is about negative alignment conditions. The fourth part, Section 13, is about a generalization of an inequality of Zagrodny.

We now give an overview of the second part of this paper in which the notation will be familiar to the greatest number of readers, namely Sections 7–8, 10 and 12. We give the initial definitions in Section 7. The rather cumbersome definition of r_L–density in this special situation appears in (28). We use the term "quasidensity" instead of "r_L–density" in this context. There are two other conditions equivalent to quasidensity in this paper, which can be found in Theorem 7.8 and Theorem 11.6. More can be found in [29, Theorems 13.3, 13.6, 15.2 and 18.5.]

It is shown in Theorem 7.4(a) that a closed, monotone, quasidense set is maximally monotone. Theorem 7.5, Corollary 7.9 and Theorem 7.4(b) show that the closed, monotone, quasidense multifunctions do not form *too* small a class of sets. We now discuss these three results.

In Theorem 7.5, we prove that the subdifferential of a proper, convex, lower semicontinuous function is quasidense. The main nontrivial building blocks in the proof of Theorem 7.5 are Rockafellar's formula for the conjugate of a sum and the Cauchy sequence argument used in the proof of Theorem 4.4((b)$\Longrightarrow$(c)). By contrast with the proof given in Theorem 7.5, it also is possible to give a "direct proof" using none of the results of Section 4 after Definition 4.1, but using instead a separation theorem in $E \times E^*$, the Brøndsted–Rockafellar theorem and Rockafellar's formula for the subdifferential of a sum. This "direct proof" is not much harder than the most recent proof of Rockafellar's original result that we have seen in print. Since the formula for the subdifferential of a sum is very close to the formula for the conjugate of a sum, this leads one to speculate that the proof of Theorem 4.4((b)$\Longrightarrow$(c)) is somehow related to the Brøndsted–Rockafellar theorem (or, more likely, Ekeland's variational principle). We refer the reader to [28, Theorem 8.4, p. 15] for more details of this "direct proof". We do not discuss it any more in this paper because the result established in Simons–Wang, [30, Theorem 3.2] shows that the (appropriate) subdifferential of an (appropriate) proper (and not necessarily convex) lower semicontinuous function is quasidense. In other word, r_L–density and quasidensity have some interest outside the context of L–positive or monotone sets, as the case may be.

Finally, we mention the novel use of Theorem 7.5 in Lemma 9.2 to obtain results on linear sets (see below). We do not know if there are similar applications of Theorem 7.5 to the nonlinear case. It would be very intriguing if there were such applications.

In Corollary 7.9 we prove that every surjective maximally monotone multifunction is quasidense, and in Theorem 7.4(b) we prove that if E is reflexive then every maximally monotone multifunction on E is quasidense.

A useful counterexample for the nonsurjective, nonreflexive or non–subdifferential case is the *tail operator* (see Example 7.10), which is a maximally monotone linear operator from $E = \ell_1$ into $\ell_\infty = E^*$ that is not quasidense. This example brings into stark relief the difference of behavior between surjective maximally monotone multifunctions and maximally monotone multifunctions with full domain.

Section 8 is devoted to a sum theorem and a parallel sum theorem for closed, monotone quasidense multifunctions: Theorem 8.4 contains a result that implies that if S and T are closed monotone quasidense multifunctions and the effective domains $D(S)$ and $D(T)$ satisfy the Rockafellar constraint condition then $S + T$ is closed, monotone and quasidense. Theorem 8.8 contains an analogous but more technical result when we have information about the ranges $R(S)$ and $R(T)$. Since closed, monotone, quasidense monotone multifunctions are maximally monotone, Theorem 8.4 presents a stark contrast to the situation for maximally monotone multifunctions: it is still apparently not known whether the sum of two maximally monotone multifunctions satisfying the Rockafellar constraint condition is maximally monotone. Theorem 8.8 uses the concept of the "Fitzpatrick extension" of a closed, monotone quasidense multifunction, which is defined in Definition 8.5, and further developed in Section 12.

The quasidensity of subdifferentials is used in Section 9 to obtain results about closed *linear* L–positive subspaces of Banach SN spaces. These results are applied in Section 10 to monotone linear relations. Specifically, it is proved in Theorem 10.1 that if A is a closed monotone linear relation with adjoint relation $A^{\mathbf{T}}$ then A is quasidense if, and only if, $A^{\mathbf{T}}$ is monotone if, and only if $A^{\mathbf{T}}$ is maximally monotone. This extends results established in [3] and [4] by Bauschke, Borwein, Wang and Yao for general Banach spaces which, in turn, extend a result proved by Brezis and Browder in [6] for reflexive Banach spaces. It is also worthy of note that Theorem 10.4 provides a two–dimensional quadrant of examples of maximally monotone linear operators that fail to be quasidense.

It is interesting to observe that the analysis of Sections 9 and 10 uses the quasidensity of subsets of $B \times B^* = (E \times E^*) \times (E \times E^*)^* = E \times E^* \times E^* \times E^{**}$. The relatively simple notation seems to hide the actual complexity of the objects being considered.

We now discuss the third part of the paper, Section 11, in which we discuss negative alignment conditions. Theorem 11.4(c) contains a version of the Brøndsted–Rockafellar theorem for closed, monotone, quasidense sets extending part of [11, Theorem 4.2]. In Theorem 11.4(d), we prove that the projections of such a sets have convex closures. Finally, in Theorem 11.6, we give a criterion in terms of negative alignment for a closed monotone set to be quasidense. In Definition 11.8, we define maximal monotonicity of "type (ANA)". This concept was introduced, though unnamed, in [19, Theorem 4.5, pp. 367–369] as a property of subdifferentials. We prove in Theorem 11.9 that a closed monotone quasidense set is maximally monotone of type (ANA). We do not have an example of a maximally monotone set that is not of type (ANA).

There is one issue that we wish to mention briefly: "quasidensity" (see (28)) does not require E^{**} for its definition, and E^{**} is not mentioned explicitly in the statements of Theorems 8.4, 11.6 and 11.9, but our proofs of all of these results use Theorem 5.2, which does depend on (B^* hence) E^{**}, at one point or another. This raises the question whether there are proofs of any of these results that do not depend on E^{**}.

We now discuss the analysis in the first part of this paper, Sections 2–6 and 9, which provides the theoretical underpinnings for the results described above. A glance at the condition for the "quasidensity" of subsets of $E \times E^*$ in (28) should convince the reader that the sheer length of the expression in this condition would make the concept hard to study. In Sections 2–6, we show how to embed the analysis in a more general situation ("Banach SN spaces") for which the notation is much more concise. The definition of r_L–density in this more general situation can be found in Definition 4.1.

Banach SN spaces are defined in Definition 2.3. Banach SN spaces possess a quadratic form (denoted by q_L), and much of the analysis in Sections 2–4 is devoted to a study of those proper convex functions that dominate this quadratic form (denoted by $\mathcal{PC}_q(B)$). If f is such a function, the equality set is denoted by $\{B|f = q_L\}$. The nonnegative function r_L is defined to be $\frac{1}{2}\|\cdot\|^2 + q_L$.

"L–positive sets" (which generalize monotone subsets of $E \times E^*$) are defined in Section 3. In Section 4, we introduce the concept of "r_L–density". The first main result here is Theorem 4.4, in which we give two conditions equivalent to the r_L–density of a set of the form $\{B|f = q_L\}$. The proof of the nontrivial part of Theorem 4.4 is motivated by Voisei–Zălinescu, [33, Theorem 2.12, p. 1018]. Our analysis goes by way of the concept of "touching function", defined in Definition 4.2. This concept was used in [33] in the $E \times E^*$ case, though unnamed. The second main result in Section 4 is the "theorem of the touching conjugate", Theorem 4.8.

The main result in Section 5 is Theorem 5.2, in which we give a characterization of the touchingness of a function in terms of its conjugate and the function s_L defined on B^* in Definition 5.1. The rather arcane definition of s_L is obtained by working backwards from Theorem 5.2(a), but it reduces to the simple form exhibited in Lemma 7.3 in the $E \times E^*$ case.

If A is an L–positive subset of a Banach SN space, we define in Section 6 functions Φ_A and Θ_A, which will be used extensively in what follows. Φ_A is a generalization to Banach SN spaces of the "Fitzpatrick function" of a monotone set, which was originally introduced in [8], but lay dormant until it was rediscovered by Martínez-Legaz and Théra in [13]. If A is maximally L–positive, we give conditions in Definition 6.7 for an element g of $\mathcal{PC}(B^*)$ to be a "marker function" for A, and we show in Theorem 6.10 how marker functions can be used to characterize the r_L–density of A. Marker functions have implications for the Fitzpatrick extension, which are discussed in Section 12.

Section 9 is about a closed linear L–positive subspace, A, of a Banach SN space,

B, and its polar subspace, $A^0 \subset B^*$. The main result here, in Theorem 9.3, is that A is r_L–dense if, and only if, $\sup s_L(A^0) \le 0$. This is the specific result (already alluded to) that depends on Theorem 7.5 for its proof, and is applied to monotone linear relations in Section 10.

In Section 13, we show how Banach SN spaces lead to a generalization of an inequality due to Zagrodny, which was used to prove that the closure of the domain and the range of maximally monotone operator of type (NI) is convex. It was worthy of note that Zagrodny established these results before the approach via "type (ED)" was known. (As we have already mentioned, the corresponding results appear in this paper in Theorem 11.4(d).)

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2. SN maps and Banach SN spaces

We start off by introducing some Banach space notation.

Definition 2.1. If X is a nonzero real Banach space and $f\colon\ X \to]-\infty,\infty]$, we write $\operatorname{dom} f$ for the set $\{x \in X\colon\ f(x) \in \mathbb{R}\}$. $\operatorname{dom} f$ is the *effective domain* of f. We say that f is *proper* if $\operatorname{dom} f \ne \emptyset$. We write $\mathcal{PC}(X)$ for the set of all proper convex functions from X into $]-\infty,\infty]$ and $\mathcal{PCLSC}(X)$ for the set of all proper convex lower semicontinuous functions from X into $]-\infty,\infty]$. We write X^* for the dual space of X (with the pairing $\langle\cdot,\cdot\rangle\colon X \times X^* \to \mathbb{R}$). If $f \in \mathcal{PCLSC}(X)$ then, as usual, we define the *Fenchel conjugate*, f^*, of f to be the function on X^* given by

$$x^* \mapsto \sup\nolimits_X [x^* - f] \quad (x^* \in X^*). \tag{1}$$

If $g \in \mathcal{PC}(X^*)$ then we define the *Fenchel preconjugate*, ${}^*g$, of g to be the function on X given by

$$x \mapsto \sup\nolimits_{x^* \in X^*} [\langle x, x^*\rangle - g(x^*)] \quad (x \in X). \tag{2}$$

We write X^{**} for the bidual of X (with the pairing $\langle\cdot,\cdot\rangle\colon X^* \times X^{**} \to \mathbb{R}$). If $f \in \mathcal{PCLSC}(X)$ and $f^* \in \mathcal{PCLSC}(X^*)$, we define $f^{**}\colon X^{**} \to]-\infty,\infty]$ by $f^{**}(x^{**}) := \sup_{X^*} [x^{**} - f^*]$. If $x \in X$, we write $\widehat{x}$ for the canonical image of x in X^{**}, that is to say $(x, x^*) \in X \times X^* \implies \langle x^*, \widehat{x}\rangle = \langle x, x^*\rangle$. We write X_1 for the closed unit ball of X. If $Y \subset X$, we write $\mathbb{I}_Y$ for the *indicator function of* Y, defined by $\mathbb{I}_Y(x) = 0$ if $x \in Y$ and $\mathbb{I}_Y(x) = \infty$ if $x \in X \setminus Y$. If $f, g\colon X \to [-\infty,\infty]$, then we write $\{X|f = g\}$ and $\{X|f \le g\}$ for the sets $\{x \in X,\ f(x) = g(x)\}$ and $\{x \in X,\ f(x) \le g(x)\}$, respectively.

If E and F are nonzero Banach spaces then we define the projection maps π_1 and π_2 by $\pi_1(x, y) := x$ and $\pi_2(x, y) := y$ $((x, y) \in E \times F)$.

We will use the following result in Theorem 4.4:

Lemma 2.2 (A boundedness result). *Let X be a nonzero real Banach space and $f \in \mathcal{PC}(X)$. Suppose, further, that $m := \inf_{x\in X}\left[f(x) + \frac{1}{2}\|x\|^2\right] \in \mathbb{R}$, $y, z \in X$, $f(y) + \frac{1}{2}\|y\|^2 \le m + 1$ and $f(z) + \frac{1}{2}\|z\|^2 \le m + 1$. Then $\|y\| \le \|z\| + 3$.*

Proof. We have $m \le f\left(\frac{1}{2}y + \frac{1}{2}z\right) + \frac{1}{2}\|\frac{1}{2}y + \frac{1}{2}z\|^2 \le \frac{1}{2}f(y) + \frac{1}{2}f(z) + \frac{1}{8}\left[\|y\| + \|z\|\right]^2$. Thus $m + \frac{1}{8}\left[\|y\| - \|z\|\right]^2 \le \frac{1}{2}f(y) + \frac{1}{2}f(z) + \frac{1}{4}\|y\|^2 + \frac{1}{4}\|z\|^2$. Consequently, $m + \frac{1}{8}\left[\|y\| - \|z\|\right]^2 \le \frac{1}{2}\left(f(y) + \frac{1}{2}\|y\|^2\right) + \frac{1}{2}\left(f(z) + \frac{1}{2}\|z\|^2\right) \le \frac{1}{2}(m+1) + \frac{1}{2}(m+1)$. Thus $\left[\|y\| - \|z\|\right]^2 \le 8$, which gives the required result. □

We now introduce *SN maps* and *Banach SN spaces* (which were called *Banach SNL spaces* in [25]).

Definition 2.3. Let B be a nonzero real Banach space. A *SN map on B* ("SN" stands for "symmetric nonexpansive"), is a linear map $L\colon\ B \to B^*$ such that

$$\|L\| \le 1 \quad \text{and} \quad \text{for all } b, c \in B,\ \ \langle b, Lc\rangle = \langle c, Lb\rangle. \tag{3}$$

A *Banach SN space* (B, L) is a nonzero real Banach space B together with a SN map $L\colon\ B \to B^*$. From now on, we suppose that (B, L) is a Banach SN space. We define the even functions q_L and r_L on B by $q_L(b) := \frac{1}{2}\langle b, Lb\rangle$ ("q" stands for "quadratic") and $r_L := \frac{1}{2}\|\cdot\|^2 + q_L$. Since $\|L\| \le 1$, for all $b \in B$, $|q_L(b)| = \frac{1}{2}|\langle b, Lb\rangle| \le \frac{1}{2}\|b\|\|Lb\| \le \frac{1}{2}\|b\|^2$, so that

$$0 \le r_L \le \|\cdot\|^2 \ \text{ on } B. \tag{4}$$

For all $b, d \in B$, $|\frac{1}{2}\|b\|^2 - \frac{1}{2}\|d\|^2| = \frac{1}{2}\big|\|b\| - \|d\|\big|\left(\|b\| + \|d\|\right) \le \frac{1}{2}\|b - d\|\left(\|b\| + \|d\|\right)$ and, from (3),

$$|q_L(b) - q_L(d)| = \tfrac{1}{2}|\langle b, Lb\rangle - \langle d, Ld\rangle| = \tfrac{1}{2}\big|\langle b - d, L(b + d)\rangle\big| \le \tfrac{1}{2}\|b - d\|\|b + d\|.$$

Consequently, $|r_L(b) - r_L(d)| \le \|b - d\|\left(\|b\| + \|d\|\right)$, thus

$$r_L(b) \le \|b - d\|\left(\|b\| + \|d\|\right) + r_L(d) \quad \text{and} \quad r_L \text{ is continuous.} \tag{5}$$

Notation 2.4. We write

$$\mathcal{PC}_q(B) := \{f \in \mathcal{PC}(B)\colon\ f \ge q_L \text{ on } B\}$$

and

$$\mathcal{PCLSC}_q(B) := \{f \in \mathcal{PCLSC}(B)\colon\ f \ge q_L \text{ on } B\}.$$

Lemma 2.5 below will be used in Lemma 3.2(a) and Theorem 4.4.

Lemma 2.5. *Let (B, L) be a Banach SN space, $f \in \mathcal{PC}_q(B)$ and $a, c \in B$. Then*

$$-q_L(a - c) \le 2(f - q_L)(a) + 2(f - q_L)(c).$$

Proof. We have

$$\begin{aligned}-q_L(a-c) &= q_L(a+c) - 2q_L(a) - 2q_L(c)\\ &= 4q_L(\tfrac{1}{2}a + \tfrac{1}{2}c) - 2q_L(a) - 2q_L(c)\\ &\le 4f(\tfrac{1}{2}a + \tfrac{1}{2}c) - 2q_L(a) - 2q_L(c)\\ &\le 2f(a) + 2f(c) - 2q_L(a) - 2q_L(c). \qquad \square\end{aligned}$$

Remark 2.6. The following result stronger than Lemma 2.5 was proved in [23, Lemma 2.6, p. 231]: if $f \in \mathcal{PC}_q(B)$ and $a, c \in B$ then

$$-q_L(a-c) \le \left[\sqrt{(f-q_L)(a)} + \sqrt{(f-q_L)(c)}\right]^2.$$

If B is any Banach space then $(B, 0)$ is obviously a Banach SN space, $q_0 = 0$ and $r_0 = \frac{1}{2}\|\cdot\|^2$. There are many more interesting examples of Banach SN spaces. The following are extensions of the examples in [23, Example 2.3, pp. 230–231]. More examples can be derived from [23, Remark 6.7, p. 246] and [9]. The significant example which leads to results on monotonicity appeared in [23, Example 6.5, p. 245] and [25, Example 3.1, pp. 606–607]. We will return to it in Example 7.1 of this paper. We note that some of the above examples were expressed in term of the bilinear form $\lfloor\cdot,\cdot\rfloor\colon (b, c) \mapsto \langle b, Lc\rangle$ rather than the map L.

Example 2.7. Let B be a Hilbert space with inner product $(b, c) \mapsto \langle b, c\rangle$ and $L\colon B \to B$ be a nonexpansive self–adjoint linear operator. Then (B, L) is a Banach SN space. Here are three special cases of this example:

(a) $\lambda \in\,]0, 1]$ and, for all $b \in B$, $Lb = \lambda b$. Here $r_L(b) = \frac{1}{2}(1+\lambda)\|b\|^2$.

(b) $\lambda \in\,]0, 1]$ and, for all $b \in B$, $Lb = -\lambda b$. Here $r_L(b) = \frac{1}{2}(1-\lambda)\|b\|^2$.

(c) $\lambda \in\,]0, 1]$, $B = \mathbb{R}^3$ and $L(b_1, b_2, b_3) = \lambda(b_2, b_1, b_3)$. Here

$$r_L(b_1, b_2, b_3) = \tfrac{1}{2}\left(b_1^2 + 2\lambda b_1 b_2 + b_2^2 + (1+\lambda)b_3^2\right).$$

3. L–positive sets

Let $A \subset B$. We say that A is *L–positive* ([25, Section 2, pp. 604–606]) if $A \ne \emptyset$ and $a, c \in A \Longrightarrow q_L(a-c) \ge 0$. In Example 2.7(a), all nonempty subsets of B are L–positive and, in Example 2.7(b), the only L–positive subsets of B are the singletons. In Example 2.7(c) when $\lambda = 1$, the L–positive sets are explored in [27, Example 3.2(c), p. 262], [25, Example 2.3(c), p. 606] (and other places).

Definition 3.1. Let (B, L) be a Banach SN space and $f \in \mathcal{PC}(B)$. We define the function $f^@$ on B by

$$f^@(b) := f^*(Lb) = \sup\nolimits_B [Lb - f] \quad (b \in B). \tag{6}$$

Lemma 3.2 contains three fundamental properties of Banach SN spaces, and will be used in Theorem 4.4, Theorem 4.8, Lemma 6.3, Lemma 6.9, Theorem 12.2

and (48). Lemma 3.2(a) is suggested by Burachik–Svaiter, [7, Theorem 3.1, pp. 2381–2382] and Penot, [15, Proposition 4((h)$\Longrightarrow$(a)), pp. 860–861], and is equivalent to [23, Lemma 2.9, p. 232]. Lemma 3.2(b,c) are equivalent to [23, Lemma 2.12(a,b), p. 233].

Lemma 3.2. *Let (B,L) be a Banach SN space and $f \in \mathcal{PC}_q(B)$. Then:*

(a) *If $\{B|f = q_L\} \neq \emptyset$ then $\{B|f = q_L\}$ is an L–positive subset of B.*
(b) *Let $a, b \in B$ and $f(a) = q_L(a)$. Then $q_L(a) \geq \langle b, La\rangle - f(b)$.*
(c) $\{B|f = q_L\} \subset \{B|f^{@} = q_L\}$.

Proof. (a) This is immediate from Lemma 2.5. As for (b), let $\lambda \in]0,1[$. Then

$$\begin{aligned}\lambda f(b) + (1-\lambda)q_L(a) &= \lambda f(b) + (1-\lambda)f(a)\\ &\geq f(\lambda b + (1-\lambda)a) \geq q_L(\lambda b + (1-\lambda)a)\\ &= \lambda^2 q_L(b) + \lambda(1-\lambda)\langle b, La\rangle + (1-\lambda)^2 q_L(a).\end{aligned}$$

Thus $\lambda f(b) + \lambda(1-\lambda)q_L(a) \geq \lambda^2 q_L(b) + \lambda(1-\lambda)\langle b, La\rangle$, and (b) follows by dividing by λ, letting $\lambda \to 0$ and rearranging the terms.

Now let $a \in B$ and $f(a) = q_L(a)$. Taking the supremum over b in (b) and using (6), we see that $q_L(a) \geq f^{@}(a)$. On the other hand, we also have $f^{@}(a) \geq \langle a, La\rangle - f(a) = 2q_L(a) - q_L(a) = q_L(a)$. Thus $f^{@}(a) = q_L(a)$. This completes the proof of (c). $\square$

4. r_L–dense sets and touching functions

Definition 4.1. Let A be a subset of a Banach SN space (B,L). We say that A is *r_L–dense in B* if, for all $c \in B$, $\inf r_L(A - c) \leq 0$.

If B is any Banach space, r_0–density is clearly identical to norm–density. The same is true for Example 2.7(a) for all $\lambda \in]0,1]$ and Example 2.7(b) for all $\lambda \in]0,1[$. In Example 2.7(b) when $\lambda = 1$, every nonempty subset of B is r_L–dense in B.

We will also consider the following strengthening of the condition of r_L–density: we will say that A is *stably r_L–dense in B* if, for all $c \in B$, there exists $K_c \geq 0$ such that

$$\inf\{r_L(a - c)\colon\ a \in A,\ \|a - c\| \leq K_c\} \leq 0. \tag{7}$$

The concept of stable r_L–density will be used in the proof of Theorem 11.4(a).

Definition 4.2. Let (B,L) be a Banach SN space, $f \in \mathcal{PC}_q(B)$ and $c \in B$. (4) implies that $\inf_{d\in B}[(f - q_L)(d) + r_L(d - c)] \geq 0$. We say that f is *touching* if

$$f \in \mathcal{PC}_q(B) \text{ and, for all } c \in B,\ \inf_{d\in B}[(f - q_L)(d) + r_L(d - c)] \leq 0. \tag{8}$$

Lemma 4.3 (Lower semicontinuous envelope). *Let (B,L) be a Banach SN space, $h \in \mathcal{PC}_q(B)$ and $\underline{h}$ be the lower semicontinuous envelope of h. Then:*

(a) $\underline{h} \in \mathcal{PCLSC}_q(B)$.

(b) *Let* $c \in B$. *Then we have*

$$\inf_{d \in B} [(\underline{h} - q_L)(d) + r_L(d - c)] = \inf_{d \in B} [(h - q_L)(d) + r_L(d - c)] . \quad (9)$$

(c) $\underline{h}$ *is touching if, and only if,* h *is touching.*

(d) $\underline{h}^{@} = h^{@}$ *on* B.

Proof. $\underline{h}$ is the (convex) function whose epigraph is the closure of the epigraph of h. It is well known that $\underline{h}$ is also the largest lower semicontinuous function on B such that $\underline{h} \le h$ on B. It is also well known that $\underline{h}^* = h^*$ on B^*.

(a) Since $h \in \mathcal{PC}_q(B)$, $q_L \le h$ on B thus, since q_L is (continuous hence) lower semicontinuous on B, $q_L \le \underline{h}$ on B, from which $\underline{h} \in \mathcal{PCLSC}_q(B)$.

(b) Since $\underline{h} \le h$ on B, the inequality "$\le$" in (9) is obvious. As we observed in Definition 4.2, we have $\inf_{d \in B} [(h - q_L)(d) + r_L(d - c)] \ge 0$. Now let $m := \inf_{d \in B} [(h - q_L)(d) + r_L(d - c)]$, so that $m \in \mathbb{R}$ and, for all $d \in B$, $h(d) \ge q_L(d) - r_L(d - c) + m$. The function $q_L - r_L(\cdot - c) + m$ is (continuous hence) lower semicontinuous on B and so, for all $d \in B$, $\underline{h}(d) \ge q_L(d) - r_L(d - c) + m$, that is to say, $(\underline{h} - q_L)(d) + r_L(d - c) \ge m$, which gives the inequality "$\ge$" in (9).

(c) is immediate from (a), (b) and (8).

(d) is immediate since $\underline{h}^{@} = \underline{h}^* \circ L = h^* \circ L = h^{@}$ on B. □

In the first main result of this section, Theorem 4.4, we give two characterizations of r_L–density for certain sets of the form $\{B | f = q_L\}$, including the unexpected result that, for these sets, r_L–density implies stable r_L–density. Theorem 4.4 and its consequence Corollary 4.5 will be used in Theorem 4.8, Theorem 5.2, Corollary 6.4, Theorem 6.10, Theorem 7.5 and Theorem 9.3.

Theorem 4.4 (The r_L-density of certain coincidence sets). *Let* (B, L) *be a Banach SN space,* $h \in \mathcal{PC}_q(B)$ *and* $\underline{h}$ *be the lower semicontinuous envelope of* h (since q_L is continuous, $\{B | \underline{h} = q_L\}$ is closed). *Then the conditions* (a)–(c) *are equivalent:*

(a) $\{B | \underline{h} = q_L\}$ *is an* r_L*–dense* L*–positive subset of* B.

(b) $\underline{h}$ *is touching or, equivalently* (from Lemma 4.3(c)), h *is touching.*

(c) $\{B | \underline{h} = q_L\}$ *is a stably* r_L*–dense* L*–positive subset of* B.

Proof. Let $A := \{B | \underline{h} = q_L\}$. Then, for all $c \in B$,

$$\inf_{d \in B} [(\underline{h} - q_L)(d) + r_L(d - c)] \le \inf_{a \in A} [(\underline{h} - q_L)(a) + r_L(a - c)] = \inf_{a \in A} r_L(a - c).$$

It follows easily from Definitions 4.1 and 4.2, and Lemma 4.3 that (a)⟹(b).

Suppose now that (b) is satisfied and $c \in B$. Replacing d by $b+c$,

$$0 = \inf_{d \in B} \left[(\underline{h} - q_L)(d) + r_L(d-c)\right] = \inf_{b \in B} \left[\underline{h}(b+c) - \langle b, Lc\rangle - q_L(c) + \tfrac{1}{2}\|b\|^2\right].$$

Lemma 2.2 provides $N_c \ge 0$ such that

$$\underline{h}(b+c) - \langle b, Lc\rangle - q_L(c) + \tfrac{1}{2}\|b\|^2 \le 1 \quad\Longrightarrow\quad \|b\| \le N_c.$$

Thus $(\underline{h} - q_L)(d) + r_L(d-c) \le 1 \Longrightarrow \|d-c\| \le N_c$. Let $\delta \in \left]0, \tfrac{1}{2}\right[$. Let $c_0 := c$. If $n \ge 1$ and c_{n-1} is known then, from (b) and (8) with c replaced by c_{n-1}, we can choose c_n inductively so that,

$$(\underline{h} - q_L)(c_n) + r_L(c_n - c_{n-1}) \le \delta^{2n}. \tag{10}$$

Let $n \ge 1$. From Lemma 4.3(a) and (4), $(\underline{h} - q_L)(c_n) \ge 0$ and $r_L(c_n - c_{n-1}) \ge 0$, and so (10) implies that

$$(\underline{h} - q_L)(c_n) \le \delta^{2n} \tag{11}$$

and

$$r_L(c_n - c_{n-1}) \le \delta^{2n}. \tag{12}$$

Putting $n = 1$ in (10), we have $(\underline{h} - q_L)(c_1) + r_L(c_1 - c) \le \delta^2 < 1$ and so, from the choice of N_c and also setting $n = 1$ in (12),

$$\|c_1 - c\| \le N_c \quad\text{and}\quad r_L(c_1 - c) \le \delta^2. \tag{13}$$

From (12), Lemma 2.5, (11), and the fact that $\delta^{2n+2} \le \frac{1}{4}\delta^{2n}$,

$$\begin{aligned}\tfrac{1}{2}\|c_{n+1} - c_n\|^2 &= -q_L(c_{n+1} - c_n) + r_L(c_{n+1} - c_n) \le -q_L(c_{n+1} - c_n) + \delta^{2n+2}\\ &\le 2(\underline{h} - q_L)(c_{n+1}) + 2(\underline{h} - q_L)(c_n) + \delta^{2n+2}\\ &\le 2\delta^{2n+2} + 2\delta^{2n} + \delta^{2n+2} \le 3\delta^{2n},\end{aligned}$$

from which $\|c_{n+1} - c_n\| \le 3\delta^n$. Thus $\lim_{n\to\infty} c_n$ exists. Let $a := \lim_{n\to\infty} c_n$. From (11) and the lower semicontinuity of $\underline{h} - q_L$, $(\underline{h} - q_L)(a) \le 0$, from which $a \in \{B|\underline{h} = q_L\}$. Also, $\|a - c_1\| \le \sum_{n=1}^{\infty} \|c_{n+1} - c_n\| \le 3\sum_{n=1}^{\infty} \delta^n \le 6\delta$ and so, from (13), $\|a - c\| \le \|a - c_1\| + \|c_1 - c\| \le 6\delta + N_c \le N_c + 3$. Then (5) (with $b = a - c$ and $d = c_1 - c$) and (13) give

$$r_L(a-c) \le \|a - c_1\|\left(\|a - c\| + \|c_1 - c\|\right) + r_L(c_1 - c) \le 6\delta\left(N_c + 3 + N_c\right) + \delta^2.$$

Letting $\delta \to 0$, $\inf\{r_L(a-c)\colon\ a \in \{B|\underline{h} = q_L\},\ \|a-c\| \le N_c + 3\} \le 0$. Thus $\{B|\underline{h} = q_L\}$ is stably r_L–dense in B. In particular, $\{B|\underline{h} = q_L\} \ne \emptyset$. From Lemma 3.2(a), this set is also L–positive. Thus (c) holds. Since it is obvious that (c)$\Longrightarrow$(a), this completes the proof of the theorem. □

Corollary 4.5 (The lower semicontinuous case). *Let* (B, L) *be a Banach SN space and* $k \in \mathcal{PCLSC}_q(B)$. *Then the conditions* (a)–(c) *are equivalent:*

(a) $\{B|k = q_L\}$ *is an* r_L*–dense* L*–positive subset of* B.

(b) *k is touching.*

(c) *$\{B|k = q_L\}$ is a stably r_L–dense L–positive subset of B.*

Proof. This is immediate from Theorem 4.4 since $\underline{k} = k$. □

Definition 4.6. Let A be a nonempty subset of a Banach SN space (B, L). We say that A is *maximally L–positive* if A is L–positive and A is not properly contained in any other L–positive set.

The simple result contained in Lemma 4.7 connects the concepts of maximal L–positivity and r_L–density. The converse result is not true: the graph of the tail operator mentioned in the introduction is a closed maximally L–positive linear subspace of $\ell_1 \times \ell_\infty$ that is not r_L–dense (see Example 7.10). Lemma 4.7 will be used in Theorem 4.8(b), Corollary 6.5, Theorem 7.4(a) and Corollary 9.4.

Lemma 4.7 (r_L–density and maximal L–positivity). *Let (B, L) be a Banach SN space and A be a closed, r_L–dense L–positive subset of B. Then A is maximally L–positive.*

Proof. Suppose that $b \in B$ and $A \cup \{b\}$ is L–positive. Let $\varepsilon > 0$. By hypothesis, there exists $a \in A$ such that $\frac{1}{2}\|a - b\|^2 + q_L(a - b) = r_L(a - b) < \varepsilon$. Since $A \cup \{b\}$ is L–positive, $q_L(a - b) \geq 0$, and so $\frac{1}{2}\|a - b\|^2 \leq \varepsilon$. However, A is closed. Thus, letting $\varepsilon \to 0$, $b \in A$. □

We now come to the second main result in this section. It will be used in Lemmas 8.2 and 8.7.

Theorem 4.8 (The theorem of the touching conjugate). *Let (B, L) be a Banach SN space and $h \in \mathcal{PC}_q(B)$ be touching. Then:*

(a) *$h^@ \geq q_L$ on B and $\underline{h}^@ \geq q_L$ on B.*

(b) *$\{B|h^@ = q_L\} = \{B|\underline{h}^@ = q_L\} = \{B|\underline{h} = q_L\}$, and this set is nonempty, closed, stably r_L–dense in B and maximally L–positive.*

(c) *$h^@$ is touching.*

Proof. Let $c \in B$. Then, since $q_L \leq r_L$ on B, for all $d \in B$,

$$h(d) - \langle d, Lc\rangle + q_L(c) = (h - q_L)(d) + q_L(d - c) \leq (h - q_L)(d) + r_L(d - c).$$

Thus, from (8), $\inf_{d\in B}[h(d) - \langle d, Lc\rangle + q_L(c)] \leq 0$. It follows from (6) that $h^@(c) = \sup_{d\in B}[\langle d, Lc\rangle - h(d)] \geq q_L(c)$. Thus $h^@ \geq q_L$ on B, and (a) now follows since Lemma 4.3(d) implies that $\underline{h}^@ = h^@$ on B.

From Lemma 4.3(a), Lemma 3.2(c) (with $f := \underline{h}^@$), Theorem 4.4((b)$\Longrightarrow$(c)) and Lemma 4.7, $\underline{h} \in \mathcal{PCLSC}_q(B)$, $\{B|\underline{h}^@ = q_L\} \supset \{B|\underline{h} = q_L\}$ and $\{B|\underline{h} = q_L\}$ is nonempty, closed, stably r_L–dense in B and maximally L–positive. From (a), $\underline{h}^@ \geq q_L$ on B and Lemma 3.2(a) (with $f := \underline{h}^@$) implies that $\{B|\underline{h}^@ = q_L\}$ is L–positive. Thus Lemma 4.3(d) and the maximality of $\{B|\underline{h} = q_L\}$ give (b).

(a) and (b) give $h^{@} \geq q_L$ on B and $\{B | h^{@} = q_L\} \neq \emptyset$, thus we have $h^{@} \in \mathcal{PCLSC}_q(B)$. (c) follows from (b) and Corollary 4.5((a)$\Longrightarrow$(b)), with $k := h^{@}$. $\square$

5. The function s_L and a dual characterization of touching

Theorem 5.2, one of the central result of this paper, will be used in Corollary 6.4, Theorem 6.10, Theorem 7.5, Lemma 8.2, Lemma 8.7 and Theorem 9.3. We start by defining a function s_L on the dual space, B^*, of B that plays a similar role to the function q_L that we have already defined on B. The definition of s_L is anything but intuitive — it was obtained by working backwards from Theorem 5.2. In this connection, the formula obtained in Lemma 7.3 is very gratifying, and it shows that Theorem 5.2($\Longleftarrow$) extends [33, Remark 2.3] and part of [11, Theorem 4.2], and Theorem 5.2($\Longrightarrow$) extends [33, Theorem 2.12].

Definition 5.1. Let (B, L) be a Banach SN space. We define the function $s_L\colon\ B^* \to]-\infty, \infty]$ by

$$s_L(b^*) = \sup\nolimits_{c \in B} \left[\langle c, b^* \rangle - q_L(c) - \tfrac{1}{2}\|Lc - b^*\|^2\right]. \tag{14}$$

s_L is *quadratic* in the sense that $s_L(\lambda b^*) = \lambda^2 s_L(b^*)$ whenever $b^* \in B^*$ and $\lambda \in \mathbb{R} \setminus \{0\}$. Clearly, $s_0(b^*) = \sup_{c \in B} \left[\langle c, b^* \rangle - \tfrac{1}{2}\|b^*\|^2\right]$, from which $s_0(0) = 0$ and, if $b^* \in B^* \setminus \{0\}$, then $s_0(b^*) = \infty$. In Example 2.7(a), using the properties of a Hilbert space, for all $b^* \in B^* = B$ and $c \in B$,

$$\langle c, b^* \rangle - q_L(c) - \tfrac{1}{2}\|Lc - b^*\|^2 = \tfrac{1}{2}\|b^*\|^2/\lambda - \tfrac{1}{2}(1+\lambda)\|\lambda c - b^*\|^2/\lambda,$$

and so (14) implies that $s_L(b^*) = \tfrac{1}{2}\|b^*\|^2/\lambda$.

We recall that *touching* was defined in (8).

Theorem 5.2. *Let (B, L) be a Banach SN space and $h \in \mathcal{PC}_q(B)$. Then*

$$h \text{ is touching} \iff h^* \geq s_L \text{ on } B^*.$$

Proof. In what follows, for all $c \in B$, we write $h_c(b) := h(b+c) - \langle b, Lc \rangle - q_L(c)$. Following the analysis in Theorem 4.4, h is touching if, and only if, for all $c \in B$, $\inf_{b \in B} \left[h_c(b) + \tfrac{1}{2}\|b\|^2\right] \leq 0$. From Rockafellar's version of the Fenchel duality theorem (see, for instance, Rockafellar, [17, Theorem 3(a), p. 85], Zălinescu, [35, Theorem 2.8.7(iii), p. 127], or [22, Corollary 10.3, p. 52]), this is, in turn, equivalent to the statement that, for all $c \in B$, $\left[{h_c}^*(b^*) + \tfrac{1}{2}\|b^*\|^2\right] \geq 0$. But, by direct computation, ${h_c}^*(b^*) = h^*(b^* + Lc) - \langle c, b^* \rangle - q_L(c)$. Thus h is touching exactly when, for all $c \in B$, $\inf_{b^* \in B^*} \left[h^*(b^* + Lc) - \langle c, b^* \rangle - q_L(c) + \tfrac{1}{2}\|b^*\|^2\right] \geq 0$. From the substitution $b^* = d^* - Lc$, this is equivalent to the statement that, for all $c \in B$, $\inf_{d^* \in B^*} \left[h^*(d^*) - \langle c, d^* - Lc \rangle - q_L(c) + \tfrac{1}{2}\|d^* - Lc\|^2\right] \geq 0$. It now follows from (14) that this is equivalent to the statement that $h^* \geq s_L$ on B^*. $\square$

6. Φ_A and Θ_A and marker functions

Throughout this section, (B, L) will be a Banach SN space and A will be an L–positive subset of B. Some of the results of this section appear in greater generality in [22]: here we discuss only what we will need in this paper.

Definition 6.1 (The definition of Φ_A). We define $\Phi_A\colon\ B \to]-\infty, \infty]$ by

$$\text{for all } b \in B, \quad \Phi_A(b) = \sup\nolimits_A [Lb - q_L] := \sup\nolimits_{a\in A} [\langle a, Lb\rangle - q_L(a)] \tag{15}$$
$$= q_L(b) - \inf q_L(A - b). \tag{16}$$

Φ_A is clearly lower semicontinuous. If $b \in A$ then, since A is L–positive, $\inf q_L(A - b) = 0$, and so (16) gives $\Phi_A(b) = q_L(b)$. Thus

$$A \subset \{B|\Phi_A = q_L\}. \tag{17}$$

Definition 6.2 (The definition of Θ_A). We define $\Theta_A\colon\ B^* \to]-\infty, \infty]$ by

$$\Theta_A(b^*) := \sup\nolimits_{a\in A} [\langle a, b^*\rangle - q_L(a)] = \sup\nolimits_A [b^* - q_L] \quad (b^* \in B^*). \tag{18}$$

Lemma 6.3 (Various properties of Φ_A and Θ_A). *Let A be maximally L–positive. Then:*

$$\Theta_A \circ L = \Phi_A \quad \text{on } B. \tag{19}$$
$$\Phi_A \in \mathcal{PCLSC}_q(B) \quad \text{and} \quad \{B|\Phi_A = q_L\} = A. \tag{20}$$
$$\Phi_A{}^* \geq \Theta_A \quad \text{on } B^*. \tag{21}$$
$$\Phi_A{}^{@} \geq \Phi_A \quad \text{on } B. \tag{22}$$
$$\Phi_A{}^{@} \in \mathcal{PCLSC}_q(B) \quad \text{and} \quad \left\{B|\Phi_A{}^{@} = q_L\right\} = A. \tag{23}$$

Proof. From (18) and (15), for all $b \in B$, $\Theta_A(Lb) := \sup_{a\in A} [\langle a, Lb\rangle - q_L(a)] = \Phi_A(b)$. This gives (19).

If $b \in B$ and $\Phi_A(b) \leq q_L(b)$ then (16) gives $\inf q_L(A - b) \geq 0$. From the maximality, $b \in A$ and so, from (17), $\Phi_A(b) = q_L(b)$. Thus we have proved that $\Phi_A \geq q_L$ on B and $\{B|\Phi_A = q_L\} \subset A$, and (20) follows from (17).

(1), (17) and (18) imply that, for all $b^* \in B^*$, $\Phi_A{}^*(b^*) = \sup_B [b^* - \Phi_A] \geq \sup_A [b^* - \Phi_A] = \sup_A [b^* - q_L] = \Theta_A(b^*)$. This gives (21).

(22) is immediate from (21), (6) and (19).

From (20) and Lemma 3.2(c), $\left\{B|\Phi_A{}^{@} = q_L\right\} \supset \{B|\Phi_A = q_L\}$, and (23) now follows from (22). □

Corollary 6.4. *Let A be maximally L–positive (hence closed). Then the conditions* (a)–(d) *are equivalent:*

(a) *A is an r_L–dense L–positive subset of B.*

(b) *Φ_A is touching.*

(c) *A is a stably r_L–dense, L–positive subset of B.*
(d) ${\Phi_A}^* \ge s_L$ *on* B^*.

Proof. Using (20), the equivalence of (a), (b) and (c) follows from Corollary 4.5 with $k := \Phi_A$, and Theorem 5.2 gives the equivalence with (d). □

Corollary 6.5 (Automatic stable r_L–density). *Every closed, r_L–dense L–positive subset of B is stably r_L–dense.*

Proof. This is immediate from Lemma 4.7 and Corollary 6.4((a)$\Longrightarrow$(c)). □

Corollary 6.6 (Restricted converse to Lemma 4.7). *Let L be an isometry of B* onto *B^* and A be maximally L–positive. Then A is closed and stably r_L–dense in B.*

Proof. Let $b^* \in B^*$. Choose $b \in B$ such that $Lb = b^*$. Then, from (14),

$$\begin{aligned} q_L(b) - s_L(b^*) &= \inf\nolimits_{c\in B} \left[q_L(b) - \langle c, Lb\rangle + q_L(c) + \tfrac{1}{2}\|Lc - Lb\|^2\right] \\ &= \inf\nolimits_{c\in B} \left[q_L(c-b) + \tfrac{1}{2}\|c-b\|^2\right] = \inf\nolimits_{c\in B} r_L(c-b) = 0. \end{aligned}$$

Consequently, $q_L(b) = s_L(b^*)$. Thus, from (23),

$${\Phi_A}^*(b^*) = {\Phi_A}^*(Lb) = {\Phi_A}^{@}(b) \ge q_L(b) = s_L(b^*).$$

It now follows from Corollary 6.4((d)$\Longrightarrow$(c)) that A is stably r_L–dense in B, and the maximality implies that A is closed. □

Definition 6.7 (Marker functions). Let A be maximally L–positive and $g \in \mathcal{PC}(B^*)$. We say that g is a *marker function* for A if g is $w(B^*, B)$–lower semicontinuous,

$$g \le {\Phi_A}^* \quad \text{on } B^* \tag{24}$$

and

$$g \ge \Theta_A \quad \text{on } B^*. \tag{25}$$

It is clear that if g_1 and g_2 are marker functions for A, $\lambda_1, \lambda_2 > 0$ and $\lambda_1 + \lambda_2 = 1$ then $\lambda_1 g_1 + \lambda_2 g_2$ is a marker function for A.

Lemma 6.8 (Two significant cases). *Let A be maximally L–positive. Then ${\Phi_A}^*$ and Θ_A are marker functions for A.*

Proof. ${\Phi_A}^*$ and Θ_A are obviously convex and $w(B^*, B)$–lower semicontinuous.

First, let $g := {\Phi_A}^*$. From (21), g satisfies (25), and it is clear that g satisfies (24). Thus ${\Phi_A}^*$ is a marker function for A.

Next, let $g =: \Theta_A$. It is clear that g satisfies (25) and, from (21), g satisfies (24). Thus Θ_A is a marker function for A. □

Lemma 6.9. *Let A be maximally L–positive and g be a marker function for A. Then ${}^*g \in \mathcal{PCLSC}_q(B)$ and $\{B|\ {}^*g = q_L\} = A$.*

Proof. From (24) and the Fenchel–Moreau theorem (see Moreau, [14, Sections 5–6, pp. 26–39]), ${}^*g \geq{}^*({\Phi_A}^*) = \Phi_A$ on B and so, from (20),

$$^*g \geq \Phi_A \geq q_L \ \text{ on } B. \tag{26}$$

Consequently, ${}^*g \in \mathcal{PCLSC}_q(B)$. From (25) and (18), for all $b^* \in B^*$ and $a \in A$, $g(b^*) \geq \langle a, b^*\rangle - q_L(a)$. It follows that, for all $a \in A$ and $b^* \in B^*$, $\langle a, b^*\rangle - g(b^*) \leq q_L(a)$. Thus, from (2), ${}^*g(a) \leq q_L(a)$. Consequently,

$$^*g \leq q_L \ \text{ on } A. \tag{27}$$

From (26) and (27), $\{B|\ {}^*g = q_L\} \supset A$. From Lemma 3.2(a), $\{B|\ {}^*g = q_L\}$ is L–positive, and result follows from the maximality of A. □

Theorem 6.10 below will be used in Lemma 7.6 and Theorems 12.2 and 12.5.

Theorem 6.10 (Marker function characterization of r_L–density). *Let A be maximally L–positive and g be a marker function for A. Then A is r_L–dense in B if, and only if, $g \geq s_L$ on B^*.*

In particular, A is r_L–dense in B if, and only if, $\Theta_A \geq s_L$ on B^*.

Proof. From Lemma 6.9, ${}^*g \in \mathcal{PCLSC}_q(B)$ and $\{B|\ {}^*g = q_L\} = A$. From Corollary 4.5 with $k := {}^*g$, and Theorem 5.2 with $h := {}^*g$, A is r_L–dense in B if, and only if, $({}^*g)^* \geq s_L$ on B^*. The result follows since the Fenchel–Moreau theorem for the convex lower semicontinuous function g on the locally convex space $(B^*, w(B^*, B))$ implies that $({}^*g)^* = g$ on B^*. □

7. $E \times E^*$

We suppose for the rest of this paper that E is a nonzero Banach space. Example 7.1 below appeared in [23, Example 6.5, p. 245] and [25, Example 3.1, pp. 606–607].

Example 7.1. Let $B := E \times E^*$ and, for all $(x, x^*) \in B$, we define the norm on B by $\|(x, x^*)\| := \sqrt{\|x\|^2 + \|x^*\|^2}$. We represent B^* by $E^* \times E^{**}$, under the pairing

$$\big\langle (x, x^*), (y^*, y^{**}) \big\rangle := \langle x, y^*\rangle + \langle x^*, y^{**}\rangle,$$

and define $L\colon\ B \to B^*$ by $L(x, x^*) := (x^*, \widehat{x})$. Then (B, L) is a Banach SN space and, for all $(x, x^*) \in B$, we have $q_L(x, x^*) = \langle x, x^*\rangle$ and $r_L(x, x^*) = \frac{1}{2}\|x\|^2 + \frac{1}{2}\|x^*\|^2 + \langle x, x^*\rangle$. If $A \subset B$ then we say that A is *quasidense* (resp. *stably quasidense*) if A is r_L–dense (resp. *stably r_L–dense*) in $E \times E^*$ with respect to this value of r_L. So A is quasidense exactly when

$$\left.\begin{aligned} &(x, x^*) \in B \Longrightarrow \\ &\qquad \inf\nolimits_{(s,s^*)\in A} \left[\tfrac{1}{2}\|s - x\|^2 + \tfrac{1}{2}\|s^* - x^*\|^2 + \langle s - x, s^* - x^*\rangle\right] \leq 0. \end{aligned}\right\} \tag{28}$$

If $A \subset E \times E^*$ then A is L–positive exactly when A is a nonempty monotone subset of $E \times E^*$ in the usual sense, and A is maximally L–positive exactly when A is a maximally monotone subset of $E \times E^*$ in the usual sense. Any finite dimensional Banach SN space of the form described here must have *even* dimension, and there are many Banach SN spaces of finite odd dimension. See [23, Remark 6.7, p. 246].

It is worth making a few comments about the function r_L in this context. It appears explicitly in the "perfect square criterion for maximality" in the reflexive case in [20, Theorem 10.3, p. 36]. It also appears explicitly (still in the reflexive case) in Simons–Zălinescu [31], with the symbol "Δ". It was used in the nonreflexive case by Zagrodny in [34] (see Remarks 11.7 and 13.4).

The dual norm on B^* is given by $\|(y^*, y^{**})\| := \sqrt{\|y^*\|^2 + \|y^{**}\|^2}$. We define $\widetilde{L}\colon\ B^* \to B^{**}$ by $\widetilde{L}(y^*, y^{**}) = \big(y^{**}, \widehat{y^*}\big)$. Then $\big(B^*, \widetilde{L}\big)$ is a Banach SN space and, for all $(y^*, y^{**}) \in B^*$, $q_{\widetilde{L}}(y^*, y^{**}) = \langle y^*, y^{**}\rangle$. The Banach SN spaces (B, L) and $\big(B^*, \widetilde{L}\big)$ are related by the following result (see [23, eqn. (53), p. 245]):

Lemma 7.2. *$L(B)$ is $r_{\widetilde{L}}$–dense in B^*.*

Proof. Let $(x^*, x^{**}) \in B^*$. The definition of $\|x^{**}\|$ provides an element z^* of E^* such that $\|z^*\| \le \|x^{**}\|$ and $\langle z^*, x^{**}\rangle \le -\|x^{**}\|^2 + \varepsilon$, from which it follows that

$$r_{\widetilde{L}}(z^*, x^{**}) = \langle z^*, x^{**}\rangle + \tfrac{1}{2}\|z^*\|^2 + \tfrac{1}{2}\|x^{**}\|^2 \le \langle z^*, x^{**}\rangle + \|x^{**}\|^2 \le \varepsilon.$$

Thus

$$\begin{aligned} r_{\widetilde{L}}\left(L(0, x^* - z^*) - (x^*, x^{**})\right) &= r_{\widetilde{L}}\left((x^* - z^*, 0) - (x^*, x^{**})\right) \\ &= r_{\widetilde{L}}(-z^*, -x^{**}) = r_{\widetilde{L}}(z^*, x^{**}) \le \varepsilon. \end{aligned}$$

This gives the required result. □

The following result will be used many times:

Lemma 7.3. *Let $(x^*, x^{**}) \in E^* \times E^{**}$. Then*

$$s_L(x^*, x^{**}) = \langle x^*, x^{**}\rangle = q_{\widetilde{L}}(x^*, x^{**}).$$

Proof. By direct computation from (14), and using Lemma 7.2,

$$\begin{aligned} &\langle x^*, x^{**}\rangle - s_L(x^*, x^{**}) \\ &\qquad = \inf\nolimits_{(y,y^*)\in B} \left[\langle y^* - x^*, \widehat{y} - x^{**}\rangle + \tfrac{1}{2}\|L(y, y^*) - (x^*, x^{**})\|^2\right] \\ &\qquad = \inf\nolimits_{(y,y^*)\in B} r_{\widetilde{L}}\left(L(y, y^*) - (x^*, x^{**})\right) = 0. \end{aligned}$$

Thus $s_L(x^*, x^{**}) = \langle x^*, x^{**}\rangle$, which gives the desired result. □

Theorem 7.4 (Quasidensity and maximality). *Let $A \subset E \times E^*$ be monotone.*

(a) *Let A be closed and quasidense. Then A is maximally monotone.*
(b) *Let E be reflexive and A be maximally monotone. Then A is closed and stably quasidense.*

Proof. This is immediate from Lemma 4.7 and Corollary 6.6. □

In what follows, we write $G(\cdot)$ for the graph of a multifunction. We note that there is a novel application of Theorem 7.5 below to *linear L–positive sets* in Lemma 9.2.

Theorem 7.5 (A generalization of Rockafellar's theorem on subdifferentials). Let $k \in \mathcal{PCLSC}(E)$. Then $G(\partial k)$ is stably quasidense and maximally monotone.

Proof. First, fix $x_0 \in \operatorname{dom} k$. From the Fenchel–Moreau theorem, $k(x_0) = \sup_{x^* \in E^*} [\langle x_0, x^* \rangle - k^*(x^*)]$. It follows that there exists $x_0^* \in \operatorname{dom} k^*$. Let $f(x, x^*) := k(x) + k^*(x^*)$. (Cf. [33, Remark 2.13, p. 1019].) Since $f(x_0, x_0^*) \in \mathbb{R}$, $f \in \mathcal{PCLSC}(E \times E^*)$, and the Fenchel–Young inequality implies that $f \in \mathcal{PCLSC}_q(E \times E^*)$. Now, using the Fenchel–Young inequality again and Lemma 7.3, for all $(y^*, y^{**}) \in E^* \times E^{**}$,

$$\left.\begin{aligned} f^*(y^*, y^{**}) &= \sup_{x \in E,\ x^* \in E^*} [\langle x, y^* \rangle + \langle x^*, y^{**} \rangle - k(x) - k^*(x^*)] \\ &= \sup_{x \in E} [\langle x, y^* \rangle - k(x)] + \sup_{x^* \in E^*} [\langle x^*, y^{**} \rangle - k^*(x^*)] \\ &= k^*(y^*) + k^{**}(y^{**}) \geq \langle y^*, y^{**} \rangle = s_L(y^*, y^{**}), \end{aligned}\right\} \qquad (29)$$

and so Theorem 5.2 implies that f is touching, and Corollary 4.5((b)$\Longrightarrow$(c)) implies that $\{E \times E^* | f = q_L\}$ is stably quasidense and maximally monotone. The result follows since $\{E \times E^* | f = q_L\} = G(\partial k)$. □

Lemma 7.6 will be used in Theorem 11.4 to simplify certain computations. The result of Lemma 7.6 is certainly one that one would expect to be true. The proof that we give here is surprisingly sophisticated, relying as it does on Theorem 6.10. We do not know if there is a simple proof using r_L directly.

Lemma 7.6. *Let $\alpha, \beta > 0$ and $\Delta\colon\ E \times E^* \to E \times E^*$ be the "deformation" defined by $\Delta(x, x^*) := (x/\alpha, x^*/\beta)$. Let A be a closed, monotone and quasidense subset of $E \times E^*$. Then $\Delta(A)$ is closed, monotone and stably quasidense.*

Proof. From Theorem 7.4(a), A is maximally monotone and so Theorem 6.10 and Lemma 7.3 imply that $\Theta_A \geq s_L = q_{\tilde{L}}$ on $E^* \times E^{**}$ thus, from (18), for all $(x^*, x^{**}) \in E^* \times E^{**}$,

$$\begin{aligned} \Theta_{\Delta(A)}(x^*, x^{**}) &= \sup_{(s,s^*) \in A} [\langle s/\alpha, x^* \rangle + \langle s^*/\beta, x^{**} \rangle - \langle s, s^* \rangle / \alpha\beta] \\ &= \sup_{(s,s^*) \in A} [[\langle s, \beta x^* \rangle + \langle s^*, \alpha x^{**} \rangle - \langle s, s^* \rangle] / \alpha\beta \\ &= \Theta_A(\beta x^*, \alpha x^{**}) / \alpha\beta \geq q_{\tilde{L}}(\beta x^*, \alpha x^{**}) / \alpha\beta = q_{\tilde{L}}(x^*, x^{**}). \end{aligned}$$

It is obvious that $\Delta(A)$ is maximally monotone, and so Theorem 6.10 and Corollary 6.5 imply that $\Delta(A)$ is stably quasidense. $\square$

In order to simplify some notation in the sequel, if $S\colon\ E \rightrightarrows E^*$ is a multifunction, we will say that S is *closed* if its graph, $G(S)$, is closed in $E \times E^*$, and we will say that S is *quasidense* (resp. *stably quasidense*) if $G(S)$ is quasidense (resp. stably quasidense) in $E \times E^*$. If S is nontrivial and monotone, we shall write φ_S for $\Phi_{G(S)}$. We will switch freely between discussing multifunctions from E into E^* and subsets of $E \times E^*$ in what follows, depending on the context. We have:

Lemma 7.7. *Let $S\colon\ E \rightrightarrows E^*$ be closed, monotone and quasidense. Then:*

$$\varphi_S \in \mathcal{PCLSC}_q(E \times E^*) \quad \textit{and} \quad \{E \times E^* | \varphi_S = q_L]\} = G(S). \tag{30}$$

$$\varphi_S \textit{ is touching.} \tag{31}$$

$$D(S) \subset \pi_1 \operatorname{dom} \varphi_S \quad \textit{and} \quad R(S) \subset \pi_2 \operatorname{dom} \varphi_S. \tag{32}$$

$$\varphi_S{}^{@} \in \mathcal{PCLSC}_q(E \times E^*) \quad \textit{and} \quad \left\{E \times E^* | \varphi_S{}^{@} = q_L\right\} = G(S). \tag{33}$$

Proof. From Theorem 7.4(a), S is maximally monotone. (30) follows from (20); (31) follows from Corollary 6.4((a)$\Longrightarrow$(b)); (32) follows from (30); (33) follows from (23). $\square$

Theorem 7.8 (The out-of-range criterion for quasidensity). *Let $S\colon E \rightrightarrows E^*$ be maximally monotone. Then S is stably quasidense if, and only if,*

$$(w^*, w^{**}) \in (E^* \setminus R(S)) \times E^{**} \quad \Longrightarrow \quad \varphi_S{}^*(w^*, w^{**}) \ge \langle w^*, w^{**} \rangle. \tag{34}$$

Proof. “Only if” is immediate from Corollary 6.4((c)$\Longrightarrow$(d)) and Lemma 7.3. Now if $(w^*, w^{**}) \in R(S) \times E^{**}$ then we can choose $w \in S^{-1}w^*$. From (17), $\Phi_{G(S)}{}^*(w^*, w^{**}) = \varphi_S{}^*(w^*, w^{**}) \ge \langle w, w^* \rangle + \langle w^*, w^{**} \rangle - \varphi_S(w, w^*) = \langle w^*, w^{**} \rangle$, and “if” follows from (34), Corollary 6.4((d)$\Longrightarrow$(c)) and Lemma 7.3. $\square$

Corollary 7.9 (A sufficient condition for quasidensity). *Let $S\colon\ E \rightrightarrows E^*$ be maximally monotone and $R(S) = E^*$. Then S is stably quasidense.*

Proof. This is immediate from Theorem 7.8, since $(E^* \setminus R(S)) \times E^{**} = \emptyset$. $\square$

The result given in Example 7.10 below will be extended in Theorem 10.4.

Example 7.10 (The tail operator). Let $E = \ell_1$, and define $T\colon\ \ell_1 \mapsto \ell_\infty = E^*$ by $(Tx)_n = \sum_{k \ge n} x_k$. It is well known that T, being a monotone linear operator with full domain, is maximally monotone. Let $e^* := (1, 1, \dots) \in \ell_1{}^* = \ell_\infty$. Let $x \in \ell_1$, and write $\sigma = \langle x, e^* \rangle = \sum_{n \ge 1} x_n$. Clearly, $\|x\| \ge \sigma$. Since $Tx \in c_0$, we also have $\|Tx - e^*\| = \sup_n \left|(Tx)_n - 1\right| \ge \lim_n \left|(Tx)_n - 1\right| = 1$. Thus

$$\left.\begin{aligned} \langle x, Tx \rangle &= \textstyle\sum_{n \ge 1} x_n \sum_{k \ge n} x_k = \sum_{n \ge 1} x_n^2 + \sum_{n \ge 1} \sum_{k > n} x_n x_k \\ &\ge \textstyle\frac{1}{2} \sum_{n \ge 1} x_n^2 + \sum_{n \ge 1} \sum_{k > n} x_n x_k = \frac{1}{2}\sigma^2. \end{aligned}\right\} \tag{35}$$

It follows that

$$r_L\left((x,Tx)-(0,e^*)\right)=\tfrac{1}{2}\|x\|^2+\tfrac{1}{2}\|Tx-e^*\|^2+\langle x,Tx-e^*\rangle$$
$$\geq \tfrac{1}{2}\sigma^2+\tfrac{1}{2}+\langle x,Tx\rangle-\sigma\geq \tfrac{1}{2}\sigma^2+\tfrac{1}{2}+\tfrac{1}{2}\sigma^2-\sigma=\sigma^2+\tfrac{1}{2}-\sigma\geq\tfrac{1}{4}.$$

Consequently, T is not quasidense.

8. Two sum theorems and the Fitzpatrick extension

Let X and Y be nonzero Banach spaces. Lemma 8.1 below first appeared in Simons–Zălinescu [31, Section 4, pp. 8–10]. It was subsequently generalized in [24, Theorem 9, p. 882] and [26, Corollary 5.4, pp. 121–122].

Lemma 8.1. *Let $f,g\in\mathcal{PCLSC}(X\times Y)$. For all $(x,y)\in X\times Y$, let*

$$h(x,y):=\inf\nolimits_{v\in Y}\left[f(x,y-v)+g(x,v)\right]>-\infty.$$

Suppose that

$$\textstyle\bigcup_{\lambda>0}\lambda\left[\pi_1\operatorname{dom}\,f-\pi_1\operatorname{dom}\,g\right]\ \text{is a closed linear subspace of } X. \tag{36}$$

Then $h\in\mathcal{PC}(X\times Y)$ and, for all $(x^*,y^*)\in X^*\times Y^*$,

$$h^*(x^*,y^*)=\min\nolimits_{u^*\in X^*}\left[f^*(x^*-u^*,y^*)+g^*(u^*,y^*)\right]. \tag{37}$$

Lemma 8.2. *Let $f,g\in\mathcal{PCLSC}_q(E\times E^*)$ be touching. For all $(x,x^*)\in E\times E^*$, let*

$$h(x,x^*):=\inf\nolimits_{\xi^*\in E^*}\left[f(x,x^*-\xi^*)+g(x,\xi^*)\right]. \tag{38}$$

Suppose that

$$\textstyle\bigcup_{\lambda>0}\lambda\left[\pi_1\operatorname{dom}\,f-\pi_1\operatorname{dom}\,g\right]\ \text{is a closed linear subspace of } E. \tag{39}$$

Then

$$h\in\mathcal{PC}_q(E\times E^*), \tag{40}$$

$$h\ \text{is touching}, \tag{41}$$

for all $(x,x^*)\in E\times E^*$,

$$h^@(x,x^*)=\min\nolimits_{u^*\in E^*}\left[f^@(x,x^*-u^*)+g^@(x,u^*)\right]\geq\langle x,x^*\rangle, \tag{42}$$

and

$$\left\{E\times E^*|h^@=q_L\right\}\ \text{is closed, monotone and stably quasidense.} \tag{43}$$

Proof. (39) implies that π_1 dom $f\cap\pi_1$ dom $g\neq\emptyset$, and so there exists $(x_0,y_0^*,z_0^*)\in E\times E^*\times E^*$ such that $f(x_0,y_0^*)\in\mathbb{R}$ and $g(x_0,z_0^*)\in\mathbb{R}$. By hypothesis, $f\geq q_L$ and $g\geq q_L$ on $E\times E^*$. Then, for all $(x,x^*)\in E\times E^*$,

$$h(x,x^*)\geq\inf\nolimits_{\xi^*\in E^*}\left[\langle x,x^*-\xi^*\rangle+\langle x,\xi^*\rangle\right]=\langle x,x^*\rangle$$

and

$$h(x_0, y_0^* + z_0^*) \le f(x_0, y_0^*) + g(x_0, z_0^*) < \infty,$$

consequently (40) is satisfied. From Theorem 5.2 and Lemma 7.3, for all $(x^*, x^{**}) \in E^* \times E^{**}$,

$$f^*(x^*, x^{**}) \ge \langle x^*, x^{**}\rangle \quad \text{and} \quad g^*(x^*, x^{**}) \ge \langle x^*, x^{**}\rangle.$$

Thus Lemma 8.1 (with $X := E$ and $Y := E^*$) and Lemma 7.3 imply that, for all $(x^*, x^{**}) \in E^* \times E^{**}$,

$$\left.\begin{aligned} &h^*(x^*, x^{**}) \\ =\ &\min_{u^* \in E^*} [f^*(x^* - u^*, x^{**}) + g^*(u^*, x^{**})] \\ \ge\ &\inf_{u^* \in E^*} [\langle x^* - u^*, x^{**}\rangle + \langle u^*, x^{**}\rangle] = \langle x^*, x^{**}\rangle = s_L(x^*, x^{**}). \end{aligned}\right\} \tag{44}$$

Thus (41) follows from (40) and Theorem 5.2. If $(x, x^*) \in E \times E^*$ then we obtain (42) by setting $x^{**} = \widehat{x}$ in (44), and (43) follows from (40), (41) and Theorem 4.8(b). □

Remark 8.3. We do not assert in (40) that $h \in \mathcal{PCLSC}_q(E \times E^*)$.

Theorem 8.4 below has applications to the classification of maximally monotone multifunctions. See [29, Theorems 14.2 and 16.2]. Theorem 8.4 can also be deduced from Voisei–Zălinescu [33, Corollary 3.5, p. 1024].

Theorem 8.4 (Sum theorem with domain constraints). *Let $S, T\colon E \rightrightarrows E^*$ be closed, monotone and quasidense. Then* (a)$\Longrightarrow$(b)$\Longrightarrow$(c)$\Longrightarrow$(d)*:*

(a) $D(S) \cap \operatorname{int} D(T) \ne \emptyset$ *or* $\operatorname{int} D(S) \cap D(T) \ne \emptyset$.

(b) $\bigcup_{\lambda>0} \lambda\,[D(S) - D(T)] = E$.

(c) $\bigcup_{\lambda>0} \lambda\,[\pi_1 \operatorname{dom} \varphi_S - \pi_1 \operatorname{dom} \varphi_T]$ *is a closed subspace of* E.

(d) $S + T$ *is closed, monotone and stably quasidense.*

Proof. It is immediate (using (32)) that (a)$\Longrightarrow$(b)$\Longrightarrow$(c). Now suppose that (c) is satisfied. From (30) and (31), we can apply Lemma 8.2 with $f := \varphi_S$ and $g := \varphi_T$. So, in this case, (38) gives

$$h(x, x^*) := \inf_{\xi^* \in E^*} [\varphi_S(x, x^* - \xi^*) + \varphi_T(x, \xi^*)]\,.$$

Thus (43) is satisfied and, for all $(x, x^*) \in E \times E^*$, (42) is satisfied. We now prove that

$$\{E \times E^* | h^@ = q_L\} = G(S + T). \tag{45}$$

To this end, first let $(y, y^*) \in E \times E^*$ and $h^@(y, y^*) = q_L(y, y^*) = \langle y, y^*\rangle$. (42) now gives $u^* \in E^*$ such that $\varphi_S{}^@(y, y^* - u^*) + \varphi_T{}^@(y, u^*) = \langle y, y^*\rangle$. From (33), $\varphi_S{}^@(y, y^* - u^*) \ge \langle y, y^* - u^*\rangle$ and $\varphi_T{}^@(y, u^*) \ge \langle y, u^*\rangle$. Since $\langle y, y^* - u^*\rangle + \langle y, u^*\rangle = \langle y, y^*\rangle$, in fact $\varphi_S{}^@(y, y^* - u^*) = \langle y, y^* - u^*\rangle$ and $\varphi_T{}^@(y, u^*) = \langle y, u^*\rangle$, and another application of (33) implies that

$(y, y^* - u^*) \in G(S)$ and $(y, u^*) \in G(T)$, from which $(y, y^*) \in G(S + T)$. Suppose, conversely, that $(y, y^*) \in G(S + T)$. Then there exists $u^* \in E^*$ such that $(y, y^* - u^*) \in G(S)$ and $(y, u^*) \in G(T)$. From (42) and (33),

$$\begin{aligned} h^{@}(y, y^*) &\le {\varphi_S}^{@}(y, y^* - u^*) + {\varphi_T}^{@}(y, u^*) \\ &= \langle y, y^* - u^* \rangle + \langle y, u^* \rangle = \langle y, y^* \rangle \le h^{@}(y, y^*), \end{aligned}$$

thus $h^{@}(y, y^*) = \langle y, y^* \rangle = q_L(y, y^*)$. This completes the proof of (45), and (d) follows by combining (45) and (43). □

Definition 8.5 (The Fitzpatrick extension). Let the notation be as in Section 7 and A be a closed, quasidense monotone subset of $E \times E^*$. From Theorem 7.4(a), A is maximally monotone. Corollary 6.4((a)$\Longrightarrow$(d)) and Lemma 7.3 imply that ${\Phi_A}^* \ge q_{\widetilde{L}}$ on $E^* \times E^{**}$. We then write

$$A^{\mathbb{F}} := \left\{ E^* \times E^{**} | {\Phi_A}^* = q_{\widetilde{L}} \right\}. \tag{46}$$

Let $(x, x^*) \in E \times E^*$. Then, from (6) and (23), $(x, x^*) \in L^{-1}A^{\mathbb{F}} \iff {\Phi_A}^* L(x, x^*) = q_{\widetilde{L}} L(x, x^*) \iff {\Phi_A}^{@}(x, x^*) = \langle x, x^* \rangle \iff (x, x^*) \in A$. Thus

$$L^{-1}A^{\mathbb{F}} = A, \tag{47}$$

and so $A^{\mathbb{F}}$ is, in some sense, an extension of A to $E^* \times E^{**}$. We will describe $A^{\mathbb{F}}$ as the *Fitzpatrick extension of* A. It follows from this that $A^{\mathbb{F}} \ne \emptyset$, and so Lemma 3.2(a) (with $B := E^* \times E^{**}$ and $f := {\Phi_A}^*$) implies that

$$A^{\mathbb{F}} \text{ is monotone.} \tag{48}$$

In fact, as we shall see in Theorem 12.5, $A^{\mathbb{F}}$ is always maximally monotone, but we do not need this result at the moment. We digress briefly to the multifunction versions of the above concepts. If $S\colon\ E \rightrightarrows E^*$ is closed, monotone and quasidense then

$${\varphi_S}^* \ge q_{\widetilde{L}} \ \text{ on } E^* \times E^{**}. \tag{49}$$

We define the multifunction $S^{\mathbb{F}}\colon\ E^* \rightrightarrows E^{**}$ so that $G(S^{\mathbb{F}}) = G(S)^{\mathbb{F}}$. Thus $x^{**} \in S^{\mathbb{F}}(x^*)$ exactly when ${\varphi_S}^*(x^*, x^{**}) = \langle x^*, x^{**} \rangle$. It also follows from (47) that

$$x^* \in S(x) \iff \widehat{x} \in S^{\mathbb{F}}(x^*). \tag{50}$$

Finally, $S^{\mathbb{F}}$ is monotone. We will continue our development of the theory of the Fitzpatrick extension in Section 12.

By interchanging the order of the variables in the statement of Lemma 8.1, we can prove the following result in a similar fashion:

Lemma 8.6. *Let $f, g \in \mathcal{PCLSC}(X \times Y)$. For all $(x, y) \in X \times Y$, let*

$$h(x, y) := \inf\nolimits_{u \in X} \left[f(x - u, y) + g(u, y) \right] > -\infty.$$

Suppose that

$$\bigcup_{\lambda>0} \lambda\,[\pi_2 \text{ dom } f - \pi_2 \text{ dom } g] \text{ is a closed subspace of } Y.$$

Then $h \in \mathcal{PC}(X \times Y)$ *and, for all* $(x^*, y^*) \in X^* \times Y^*$,

$$h^*(x^*, y^*) = \min_{v^* \in Y^*} [f^*(x^*, y^* - v^*) + g^*(x^*, v^*)]\,.$$

Lemma 8.7. *Let* $f, g \in \mathcal{PCLSC}_q(E \times E^*)$ *be touching. For all* $(x, x^*) \in E \times E^*$, *let*

$$h(x, x^*) := \inf_{\xi \in E} [f(x - \xi, x^*) + g(\xi, x^*)]\,. \tag{51}$$

Suppose that

$$\bigcup_{\lambda>0} \lambda\,[\pi_2 \,\text{dom } f - \pi_2 \,\text{dom } g] \text{ is a closed linear subspace of } E^*. \tag{52}$$

Then

$$h \in \mathcal{PC}_q(E \times E^*), \tag{53}$$

$$h \text{ is touching}, \tag{54}$$

for all $(x, x^*) \in E \times E^*$,

$$h^@(x, x^*) = \min_{z^{**} \in E^{**}} [f^*(x^*, \widehat{x} - z^{**}) + g^*(x^*, z^{**})] \geq \langle x, x^* \rangle, \tag{55}$$

and

$$\left\{E \times E^* | h^@ = q_L\right\} \text{ is closed, monotone and stably quasidense.} \tag{56}$$

Proof. (52) implies that π_2 dom $g \cap \pi_2$ dom $f \neq \emptyset$, and so there exists $(x_0, y_0, z_0^*) \in E \times E \times E^*$ such that $f(x_0, z_0^*) \in \mathbb{R}$ and $g(y_0, z_0^*) \in \mathbb{R}$. By hypothesis, $f \geq q_L$ and $g \geq q_L$ on $E \times E^*$. Then, for all $(x, x^*) \in E \times E^*$,

$$h(x, x^*) \geq \inf_{\xi \in E} [\langle x - \xi, x^* \rangle + \langle \xi, \xi^* \rangle] = \langle x, x^* \rangle$$

and

$$h(x_0 + y_0, z_0^*) \leq f(x_0, z_0^*) + g(y_0, z_0^*) < \infty,$$

consequently (53) is satisfied. From Theorem 5.2 and Lemma 7.3, for all $(x^*, x^{**}) \in E^* \times E^{**}$,

$$f^*(x^*, x^{**}) \geq \langle x^*, x^{**} \rangle \quad \text{and} \quad g^*(x^*, x^{**}) \geq \langle x^*, x^{**} \rangle.$$

Thus Lemma 8.6 (with $X := E$ and $Y := E^*$) and Lemma 7.3 imply that for all $(x^*, x^{**}) \in E^* \times E^{**}$,

$$\left.\begin{aligned} &h^*(x^*, x^{**}) \\ =\ &\min_{z^{**} \in E^{**}} [f^*(x^*, x^{**} - z^{**}) + g^*(x^*, z^{**})] \\ \geq\ &\inf_{z^{**} \in E^{**}} [\langle x^*, x^{**} - z^{**} \rangle + \langle x^*, z^{**} \rangle] = \langle x^*, x^{**} \rangle = s_L(x^*, x^{**}). \end{aligned}\right\} \tag{57}$$

Thus (54) follows from (53) and Theorem 5.2. If $(x, x^*) \in E \times E^*$ then we obtain (55) by setting $x^{**} = \widehat{x}$ in (57), and (56) follows from (53), (54) and Theorem 4.8(b). □

If $S,T\colon\ E \rightrightarrows E^*$ then the *parallel sum*, $S \parallel T\colon\ E \rightrightarrows E^*$ is defined to be $(S^{-1}+T^{-1})^{-1}$. Theorem 8.8 below has applications to the classification of maximally monotone multifunctions. See [29, Theorems 13.6 and 18.4].

Theorem 8.8 (Sum theorem with range constraints). *Let $S,T\colon E \rightrightarrows E^*$ be closed, monotone and quasidense. Then* (a)$\Longrightarrow$(b)$\Longrightarrow$(c)$\Longrightarrow$(d)$\Longrightarrow$(e)*:*

(a) $R(S)\cap \operatorname{int} R(T) \neq \emptyset$ *or* $\operatorname{int} R(S)\cap R(T)\neq\emptyset$.

(b) $\bigcup_{\lambda>0}\lambda\,[R(S)-R(T)] = E^*$.

(c) $\bigcup_{\lambda>0}\lambda\,[\pi_2 \operatorname{dom}\varphi_S - \pi_2\operatorname{dom}\varphi_T]$ *is a closed subspace of E^*.*

(d) *Define the multifunction $P\colon\ E \rightrightarrows E^*$ by $P(y) := (S^{\mathbb{F}}+T^{\mathbb{F}})^{-1}(\widehat{y})$. Then P is closed, monotone and stably quasidense.*

(e) *If, further, $G(T)^{\mathbb{F}} = L\,(G(T))$ then $S\parallel T$ is closed, monotone and stably quasidense.*

Proof. It is immediate (using (32)) that (a)$\Longrightarrow$(b)$\Longrightarrow$(c). Now suppose that (c) is satisfied. From (30) and (31), we can apply Lemma 8.7 with $f := \varphi_S$ and $g := \varphi_T$. So, in this case, (51) gives

$$h(x,x^*) := \inf\nolimits_{\xi\in E}\left[\varphi_S(x-\xi,x^*)+\varphi_T(\xi,x^*)\right].$$

Thus (56) is satisfied and, for all $(x,x^*)\in E\times E^*$, (55) is satisfied. Let $(y,y^*)\in E\times E^*$. We prove that

$$h^{@}(y,y^*) = q_L(y,y^*) \iff y^*\in P(y). \tag{58}$$

To this end, first let $h^{@}(y,y^*) = q_L(y,y^*) = \langle y,y^*\rangle$. (55) now gives $z^{**}\in E^{**}$ such that ${\varphi_S}^*(y^*,\widehat{y}-z^{**}) + {\varphi_T}^*(y^*,z^{**}) = \langle y,y^*\rangle$. We know from (49) that ${\varphi_S}^*(y^*,\widehat{y}-z^{**}) \geq \langle y^*,\widehat{y}-z^{**}\rangle$ and ${\varphi_T}^*(y^*,z^{**})\geq\langle y^*,z^{**}\rangle$. Since $\langle y^*,\widehat{y}-z^{**}\rangle + \langle y^*,z^{**}\rangle = \langle y,y^*\rangle$, in fact ${\varphi_S}^*(y^*,\widehat{y}-z^{**}) = \langle y^*,\widehat{y}-z^{**}\rangle$ and ${\varphi_T}^*(y^*,z^{**}) = \langle y^*,z^{**}\rangle$, that is to say, $\widehat{y}-z^{**}\in S^{\mathbb{F}}(y^*)$ and $z^{**}\in T^{\mathbb{F}}(y^*)$, and so $y^*\in P(y)$. Suppose, conversely, that $y^*\in P(y)$. Then there exists $z^{**}\in T^{\mathbb{F}}(y^*)$ such that $\widehat{y}-z^{**}\in S^{\mathbb{F}}(y^*)$. From (55),

$$\begin{aligned} h^{@}(y,y^*) &\leq {\varphi_S}^*(y^*,\widehat{y}-z^{**}) + {\varphi_T}^*(y^*,z^{**})\\ &= \langle y^*,\widehat{y}-z^{**}\rangle + \langle y^*,z^{**}\rangle = \langle y,y^*\rangle \leq h^{@}(y,y^*). \end{aligned}$$

Thus $h^{@}(y,y^*) = \langle y,y^*\rangle = q_L(y,y^*)$. This completes the proof of (58), and (d) follows by combining (58) and (56).

Suppose, finally, that $G(T)^{\mathbb{F}} = L\,(G(T))$. We will prove $P = S\parallel T$, and (e) then follows from (d). To this end, first let $y^*\in P(y)$. Then we can choose $z^{**}\in T^{\mathbb{F}}(y^*)$ such that $\widehat{y}-z^{**}\in S^{\mathbb{F}}(y^*)$. Now $(y^*,z^{**})\in G\left(T^{\mathbb{F}}\right) = L\,(G(T))$, and so there exists $(z,z^*)\in G(T)$ such that $(y^*,z^{**}) = (z^*,\widehat{z})$, from which $z^{**}=\widehat{z}$, $z\in T^{-1}(z^*) = T^{-1}(y^*)$ and $\widehat{y}-\widehat{z}\in S^{\mathbb{F}}(y^*)$. But then (50) implies that $y-z\in S^{-1}y^*$. Thus $y = (y-z)+z\in S^{-1}y^*+T^{-1}y^*$, from which $y^*\in(S\parallel T)(y)$. If, conversely, $y^*\in(S\parallel T)(y)$ then there exists $z\in E$ such that $y^*\in S(y-z)$ and $y^*\in T(z)$. From (50), $\widehat{y}-\widehat{z}\in S^{\mathbb{F}}(y^*)$ and $\widehat{z}\in T^{F}(y^*)$. Thus $\widehat{y}\in(S^{\mathbb{F}}+T^{\mathbb{F}})(y^*)$, and so $y^*\in P(y)$. This completes the proof of (e). $\square$

9. Closed L–positive linear subspaces

In this section, we suppose that (B, L) is a Banach SN space and A is a closed L–positive linear subspace of B. The analysis in this section differs from that in [27, Section 5] in that (B, L) is *not* required to have a Banach SN dual. We also point out the novel use of the r_L–density of subdifferentials to prove results on linear subspaces. We define the function $k\colon\ B \to \,]{-\infty}, \infty]$ by $k := q_L + \mathbb{I}_A$. We write A^0 for the linear subspace $\{b^* \in B^*\colon\ \langle A, b^*\rangle = \{0\}\}$ of B^*. A^0 is the "polar subspace of A". The significance of A^0 lies in the following lemma:

Lemma 9.1. *$k \in \mathcal{PCLSC}_q(B)$, $\{B|k = q_L\} = A$ and*

$$\partial k(b) = \begin{cases} Lb + A^0 & (\text{if } b \in A); \\ \emptyset & (\text{if } b \in B \setminus A). \end{cases}$$

Proof. k is obviously proper. Suppose that $b, c \in A$ and $\lambda \in]0, 1[$. Then

$$\begin{aligned} 0 \le \lambda(1-\lambda)q_L(b-c) &= \lambda q_L(b) + (1-\lambda)q_L(c) - q_L\left(\lambda b + (1-\lambda)c\right) \\ &= \lambda k(b) + (1-\lambda)k(c) - k\left(\lambda b + (1-\lambda)c\right). \end{aligned}$$

This implies the convexity of k. (See [22, Lemma 19.7, pp. 80–81].) Since q_L is continuous and A is closed, k is lower semicontinuous. It is now obvious that $k \in \mathcal{PCLSC}_q(B)$ and $\{B|k = q_L\} = A$. Since $\partial k(b) = \emptyset$ if $b \in B \setminus \operatorname{dom} k$, that is to say, $b \in B \setminus A$, it only remains to show that $\partial k(b) = Lb + A^0$ if $b \in A$. So suppose that $b \in A$. Then, since $c := a - b$ runs through A as a runs through A and then $k(b) - k(c+b) = q_L(b) - q_L(c+b) = -\langle c, Lb\rangle - q_L(c)$,

$$\begin{aligned} b^* \in \partial k(b) &\iff \sup\nolimits_{a \in A}\left[k(b) + \langle a - b, b^*\rangle - k(a)\right] \le 0 \\ &\iff \sup\nolimits_{c \in A}\left[k(b) + \langle c, b^*\rangle - k(c+b)\right] \le 0 \\ &\iff \sup\nolimits_{c \in A}\left[\langle c, b^* - Lb\rangle - q_L(c)\right] \le 0. \end{aligned}$$

Since $q_L(c) \ge 0$ for all $c \in A$, this is trivially satisfied if $b^* \in Lb + A^0$. On the other hand, if $b^* \in \partial k(b)$ then it follows from the above that, for all $c \in A$ and $\lambda \in \mathbb{R}$, $\lambda\langle c, b^* - Lb\rangle - \lambda^2 q_L(c) = \langle \lambda c, b^* - Lb\rangle - q_L(\lambda c) \le 0$. Thus, from the standard quadratic arguments, for all $c \in A$, $\langle c, b^* - Lb\rangle = 0$. This is equivalent to the statement that $b^* \in Lb + A^0$. $\square$

Lemma 9.2. *Let $b \in B$. Then $\inf r_L(A - b) \le \sup s_L\left(A^0\right)$.*

Proof. Let $\varepsilon > 0$. Define $\widehat{L}\colon\ B \times B^* \to B^* \times B^{**}$ by $\widehat{L}(b, b^*) := \left(b^*, \widehat{b}\right)$. From Lemma 9.1 and Theorem 7.5, there exist $a \in A$ and $d^* \in \partial k(a) = La + A^0$ such that

$$\tfrac{1}{2}\|a - b\|^2 + \tfrac{1}{2}\|d^* - Lb\|^2 + \langle a - b, d^* - Lb\rangle = r_{\widehat{L}}\left((a, d^*) - (b, Lb)\right) < \varepsilon.$$

We write $c = b - a$ and $b^* = d^* - La \in A^0$. Then $d^* - Lb = b^* - Lc$, from which $\frac{1}{2}\|c\|^2 + \frac{1}{2}\|b^* - Lc\|^2 - \langle c, b^* - Lc\rangle < \varepsilon$, which can be rewritten

$$\tfrac{1}{2}\|c\|^2 < \langle c, b^*\rangle - 2q_L(c) - \tfrac{1}{2}\|b^* - Lc\|^2 + \varepsilon.$$

It follows from (14) that

$$r_L(a-b) = \tfrac{1}{2}\|c\|^2 + q_L(c) < \langle c, b^* \rangle - q_L(c) - \tfrac{1}{2}\|b^* - Lc\|^2 + \varepsilon \le s_L(b^*) + \varepsilon.$$

Since $b^* \in A^0$, this gives the required result. □

Theorem 9.3 and Corollary 9.4 will be used in Theorem 10.1.

Theorem 9.3. *A is r_L–dense in B if, and only if,* $\sup s_L(A^0) \le 0$.

Proof. Suppose first that A is r_L–dense in B. From Lemma 9.1, $k \in \mathcal{PCLSC}_q(B)$ and $\{B|k = q_L\} = A$. From the L–positivity of A, Corollary 4.5((a)$\Longrightarrow$(b)) and Theorem 5.2, for all $b^* \in A^0$,

$$0 = \sup\nolimits_A[-q_L] = \sup\nolimits_A[b^* - q_L] = \sup\nolimits_B[b^* - k] = k^*(b^*) \ge s_L(b^*).$$

Thus $\sup s_L(A^0) \le 0$. If, conversely, $\sup s_L(A^0) \le 0$, then it is immediate from Lemma 9.2 that A is r_L–dense in B. □

In what follows, "lin" stands for "linear span of".

Corollary 9.4. *Let $c^* \in B^*$ and* $\sup s_L(A^0 + \mathrm{lin}\{c^*\}) \le 0$. *Then $c^* \in A^0$.*

Proof. Suppose that $c^* \notin A^0$. Let $Z = \{b \in B\colon\ \langle b, c^* \rangle = 0\}$. It is well known that $Z^0 = \mathrm{lin}\{c^*\}$. Since $c^* \notin A^0$, there exists $a \in A \setminus Z$, and so the fact that Z has codimension 1 implies that $A - Z = B$, that is to say $\mathrm{dom}\ \mathbb{I}_A - \mathrm{dom}\ \mathbb{I}_Z = B$. From the Attouch–Brezis formula for the subdifferential of a sum,

$$\begin{aligned}(A \cap Z)^0 &= \partial(\mathbb{I}_{A\cap Z})(0) = \partial(\mathbb{I}_A + \mathbb{I}_Z)(0)\\ &= \partial\mathbb{I}_A(0) + \partial\mathbb{I}_Z(0) = A^0 + Z^0 = A^0 + \mathrm{lin}\{c^*\}.\end{aligned}$$

Thus, by assumption, $\sup s_L((A \cap Z)^0) \le 0$. Since $A \cap Z \subset A$, $A \cap Z$ is also a closed L–positive linear subspace of B, thus Theorem 9.3 and Lemma 4.7 imply that $A \cap Z$ is maximally L–positive. Since $A \cap Z \subset A$, it follows from this that $A \cap Z = A$, and so $A \subset Z$, which gives $c^* \in Z^0 \subset A^0$, a contradiction. □

Remark 9.5. One can use [27, Lemma 2.2, p. 260–261] instead of the Attouch–Brezis formula in the proof of Corollary 9.4.

10. Monotone linear relations and operators

In this section, we suppose that A is a linear subspace of $E \times E^*$ (commonly called a *linear relation*). The *adjoint linear subspace*, $A^{\mathbf{T}}$, of $E^{**} \times E^*$, is defined by:

$$(y^{**}, y^*) \in A^{\mathbf{T}} \iff \text{for all } (s, s^*) \in A,\ \langle s, y^* \rangle = \langle s^*, y^{**} \rangle.$$

This definition goes back at least to Arens in [1]. (We use the notation "$A^{\mathbf{T}}$" rather than the more usual "A^*" to avoid confusion with the dual space of A.) It is clear that

$$(y^{**}, y^*) \in A^{\mathbf{T}} \iff (y^*, -y^{**}) \in A^0.$$

Theorem 10.1 below extends the result obtained by combining Bauschke, Borwein, Wang and Yao [3, Theorem 3.1((iii)$\Longrightarrow$(ii))] and [4, Proposition 3.1], which in turn extends the result proved in the reflexive case by Brezis and Browder in [6]. Corollary 10.2 follows indirectly from Bauschke and Borwein, [2, Theorem 4.1((iii)$\Longleftrightarrow$(v)), pp. 10–12]. Theorem 10.4 provides more examples of maximally monotone linear operators that are not quasidense. These examples can also be derived from the decomposition results of Bauschke and Borwein, [2, Theorem 4.1((v)$\Longleftrightarrow$(vi)), pp. 10–12].

Theorem 10.1. *Suppose that A is a monotone closed linear subspace of $E \times E^*$. Then A is quasidense if, and only if, $A^{\mathbf{T}}$ is a monotone subspace of $E^{**} \times E^*$ if, and only if, $A^{\mathbf{T}}$ is a maximally monotone subspace of $E^{**} \times E^*$.*

Proof. From Theorem 9.3 and Lemma 7.3, A is quasidense if, and only if, for all $(x^*, x^{**}) \in A^0$, $\langle x^*, x^{**} \rangle \le 0$, that is to say,

$$\text{for all } (y^{**}, y^*) \in A^{\mathbf{T}}, \quad \langle y^*, -y^{**} \rangle \le 0.$$

This is clearly equivalent to the statement that $A^{\mathbf{T}}$ is a monotone subspace of $E^{**} \times E^*$. The second equivalence is immediate from Corollary 9.4. □

Corollary 10.2. *Suppose that $S\colon E \to E^*$ is a monotone linear operator. Then S is quasidense if, and only if, the adjoint linear operator $S^{\mathbf{T}}\colon E^{**} \to E^*$ is monotone.*

Proof. This is immediate from Theorem 10.1 and the observation that $G(S^{\mathbf{T}}) = G(S)^{\mathbf{T}}$. □

Example 10.3 (Heads and tails). We defined the *tail operator*, T, in Example 7.10. We define the *head operator* $H\colon\ \ell_1 \mapsto \ell_\infty = E^*$ by $(Hx)_n = \sum_{k \le n} x_k$. Using the notation of Example 7.10, for all $x \in \ell_1$,

$$\langle x, Hx \rangle = \textstyle\sum_{n \ge 1} x_n \sum_{k \le n} x_k = \sum_{k \ge 1} x_k \sum_{n \ge k} x_n = \langle x, Tx \rangle. \tag{59}$$

If $\lambda, \mu \in \mathbb{R}$, $\lambda + \mu \ge 0$ and $S := \lambda T + \mu H$ then, from (59), S is monotone. Since S is linear and has full domain, S is maximally monotone. In Theorem 10.4 below, we find for which values of λ and μ (with $\lambda + \mu \ge 0$) S is quasidense.

Theorem 10.4 (The theorem of the two quadrants). *Let $\lambda, \mu \in \mathbb{R}$, $\lambda + \mu \ge 0$ and $S := \lambda T + \mu H$. Then S is quasidense if, and only if, $\lambda - \mu \le 0$. In particular, let $G\colon\ \ell_1 \mapsto \ell_\infty = E^*$ be* Gossez's operator*, defined by $G := T - H$. Then G is not quasidense, but $-G$ is quasidense.*

Proof. See [29, Theorem 10.7]. □

11. Negative alignment conditions

The material in this section was initially motivated by a result proved for reflexive spaces by Torralba in [32, Proposition 6.17] and extended to maximally monotone sets of type (D) by Revalski–Théra in [16, Corollary 3.8, p. 513]. In Theorem 11.6, we shall give a criterion for a closed monotone set to be quasidense in terms of *negative alignment pairs*, which are defined below, though the main result of this section is Theorem 11.4. Theorem 11.4(c) is a version of the Brøndsted–Rockafellar theorem for closed monotone quasidense sets. See [21, Section 8, pp. 274–280] for a more comprehensive discussion of the history of this kind of result. In this section we shall give complete details of proofs only if they differ in some significant way from those in [21].

Definition 11.1. Let $A \subset E \times E^*$ and $\rho,\ \sigma \geq 0$. We say that (ρ, σ) is a *negative alignment pair* for A with respect to (w, w^*) if there exists a sequence $\{(s_n, s_n^*)\}_{n\geq 1}$ of elements of A such that

$$\lim_{n\to\infty} \|s_n - w\| = \rho, \quad \lim_{n\to\infty} \|s_n^* - w^*\| = \sigma \quad \text{and} \quad \lim_{n\to\infty} \langle s_n - w, s_n^* - w^*\rangle = -\rho\sigma.$$

Lemma 11.2. *Let A be a closed subset of $E \times E^*$, $(w, w^*) \in E \times E^*$, $\alpha, \beta > 0$, $\tau \geq 0$ and $(\alpha\tau, \beta\tau)$ be a negative alignment pair for A with respect to (w, w^*).*

(a) *If $(w, w^*) \in E \times E^* \setminus A$ then $\tau > 0$.*

(b) *If $\inf_{(s,s^*)\in A}\langle s - w, s^* - w^*\rangle > -\alpha\beta$ then $\tau < 1$.*

Proof. From Definition 11.1, there exists a sequence $\{(s_n, s_n^*)\}_{n\geq 1}$ of elements of A such that $\lim_{n\to\infty} \|s_n - w\| = \alpha\tau$, $\lim_{n\to\infty} \|s_n^* - w^*\| = \beta\tau$ and $\lim_{n\to\infty}\langle s_n - w, s_n^* - w^*\rangle = -\alpha\beta\tau^2$.

(a) If $\tau = 0$ then, since A is closed, $(w, w^*) \in A$.

(b) Since $-\alpha\beta\tau^2 = \lim_{n\to\infty}\langle s_n - w, s_n^* - w^*\rangle \geq \inf_{(s,s^*)\in A}\langle s - w, s^* - w^*\rangle > -\alpha\beta$, it follows that $\tau < 1$. □

Our next result contains a uniqueness theorem for negative alignment pairs for the case when A is monotone. The proof can be found in [21, Theorem 8.4(b), p. 276].

Lemma 11.3. *Let A be a monotone subset of $E \times E^*$, $(w, w^*) \in E \times E^*$ and $\alpha, \beta > 0$. Then there exists at most one value of $\tau \geq 0$ such that $(\alpha\tau, \beta\tau)$ is a negative alignment pair for A with respect to (w, w^*).*

We now give our main result on the existence of negative alignment pairs, and some simple consequences. We refer the reader to Remark 11.7 for more discussion on some of the issues raised by these results.

Theorem 11.4. *Let A be a closed, monotone, quasidense subset of $E \times E^*$, $(w, w^*) \in E \times E^*$ and $\alpha, \beta > 0$. Then:*

(a) *There exists a unique value of $\tau \geq 0$ such that $(\alpha\tau, \beta\tau)$ is a negative alignment pair for A with respect to (w, w^*).*

(b) *If $(w,w^*) \notin A$ then there exists a sequence $\{(s_n,s_n^*)\}_{n\ge 1}$ of elements of A such that, for all $n \ge 1$, $s_n \ne w$, $s_n^* \ne w^*$,*

$$\lim_{n\to\infty}\frac{\|s_n-w\|}{\|s_n^*-w^*\|}=\frac{\alpha}{\beta}\quad \textit{and}\quad \lim_{n\to\infty}\frac{\langle s_n-w,s_n^*-w^*\rangle}{\|s_n-w\|\|s_n^*-w^*\|}=-1. \tag{60}$$

(c) *If $\inf_{(s,s^*)\in A}\langle s-w,s^*-w^*\rangle > -\alpha\beta$ then there exists $(s,s^*)\in A$ such that $\|s-w\|<\alpha$ and $\|s^*-w^*\|<\beta$. If, further, $(w,w^*)\notin A$ then there exists a sequence $\{(s_n,s_n^*)\}_{n\ge 1}$ of elements of A such that, for all $n\ge 1$, $s_n\ne w$, $s_n^*\ne w^*$, $\|s_n-w\|<\alpha$, $\|s_n^*-w^*\|<\beta$, and (60) is satisfied.*

(d) *$\overline{\pi_1 A}=\overline{\pi_1\operatorname{dom}\Phi_A}$ and $\overline{\pi_2 A}=\overline{\pi_2\operatorname{dom}\Phi_A}$. Consequently, the sets $\overline{\pi_1 A}$ and $\overline{\pi_2 A}$ are convex.*

Proof. Define Δ as in Lemma 7.6 and let $(u,u^*) := \Delta(w,w^*)$. From Lemma 7.6, $\Delta(A)$ is closed, monotone and stably quasidense, and so there exists a *bounded* sequence $\{(t_n,t_n^*)\}_{n\ge1}$ of elements of $\Delta(A)$ such that

$$\left.\begin{aligned}0&=\lim_{n\to\infty} r_L(t_n-u,t_n^*-u^*)\\&=\lim_{n\to\infty}\left(\tfrac12\|t_n-u\|^2+\tfrac12\|t_n^*-u^*\|^2+\langle t_n-u,t_n^*-u^*\rangle\right)\\&\ge\limsup_{n\to\infty}\tfrac12\left(\|t_n-u\|-\|t_n^*-u^*\|\right)^2\ge 0.\end{aligned}\right\} \tag{61}$$

Thus $\lim_{n\to\infty}(\|t_n-u\|-\|t_n^*-u^*\|)=0$. Since $\{\|(t_n-u\|\}_{n\ge1}$ is bounded in $\mathbb{R}$, passing to an appropriate subsequence, there exists $\tau\in\mathbb{R}$ such that $\tau\ge 0$ and $\lim_{n\to\infty}\|t_n-u\|=\tau$, from which $\lim_{n\to\infty}\|t_n^*-u^*\|=\tau$ also. From (61), $\lim_{n\to\infty}\langle t_n-u,t_n^*-u^*\rangle=-\frac12\tau^2-\frac12\tau^2=-\tau^2$. For all $n\ge1$, let $(s_n,s_n^*) := (\alpha t_n,\beta t_n^*)\in A$. Then, since $(w,w^*)=(\alpha u,\beta u^*)$, $\lim_{n\to\infty}\|s_n-w\|=\alpha\tau$, $\lim_{n\to\infty}\|s_n^*-w^*\|=\beta\tau$ and $\lim_{n\to\infty}\langle s_n-w,s_n^*-w^*\rangle=-\alpha\beta\tau^2$. Thus $(\alpha\tau,\beta\tau)$ is a negative alignment pair for A with respect to (w,w^*), and the "uniqueness" is immediate from Lemma 11.3. This completes the proof of (a).

(b) follows from (a) and Lemma 11.2(a).

(c) follows from (a) and Lemma 11.2(a,b).

(d) If $w\in\pi_1\operatorname{dom}\Phi_A$ then there exists $w^*\in E^*$ such that $\Phi_A(w,w^*)<\infty$ thus, from (15),

$$\begin{aligned}\inf_{(s,s^*)\in A}\langle s-w,s^*-w^*\rangle&=\langle w,w^*\rangle-\sup_{(s,s^*)\in A}\left[\langle s,w^*\rangle+\langle w,s^*\rangle-\langle s,s^*\rangle\right]\\&=\langle w,w^*\rangle-\Phi_A(w,w^*)>-\infty.\end{aligned}$$

Let $n\ge1$ and $\beta>-n\inf_{(s,s^*)\in A}\langle s-w,s^*-w^*\rangle$. (c) now gives $(s,s^*)\in A$ such that $\|s-w\|<1/n$. Consequently, $w\in\overline{\pi_1 A}$. Thus we have proved that $\pi_1\operatorname{dom}\Phi_A\subset\overline{\pi_1 A}$. On the other hand, from Theorem 7.4(a) and (20), $\pi_1 A\subset\pi_1\operatorname{dom}\Phi_A$, and so $\overline{\pi_1 A}=\overline{\pi_1\operatorname{dom}\Phi_A}$. Similarly, $\overline{\pi_2 A}=\overline{\pi_2\operatorname{dom}\Phi_A}$. The convexity of the sets $\overline{\pi_1 A}$ and $\overline{\pi_2 A}$ now follows immediately. □

Remark 11.5. In multifunction terms, Theorem 11.4(d) implies that if $S\colon E \rightrightarrows E^*$ is closed monotone and quasidense then $\overline{D(S)}$ and $\overline{R(S)}$ are both convex.

Theorem 11.6 (A negative alignment criterion for quasidensity). *Let A be a closed, monotone, subset of $E \times E^*$. Then A is quasidense if, and only if, for all $(w, w^*) \in E \times E^*$, there exists $\tau \geq 0$ such that (τ, τ) is a negative alignment pair for A with respect to (w, w^*).*

Proof. Suppose first that, for all $(w, w^*) \in E \times E^*$, there exists $\tau \geq 0$ such that (τ, τ) is a negative alignment pair for A with respect to (w, w^*). Then, for all $(w, w^*) \in E \times E^*$, Definition 11.1, provides a sequence $\{(s_n, s_n^*)\}_{n\geq 1}$ of elements of A such that

$$\lim_{n\to\infty} \|s_n - w\| = \tau, \quad \lim_{n\to\infty} \|s_n^* - w^*\| = \tau \quad \text{and} \quad \lim_{n\to\infty} \langle s_n - w, s_n^* - w^*\rangle = -\tau^2.$$

But then

$$\begin{aligned} &\lim_{n\to\infty} r_L\left((s_n, s_n^*) - (w, w^*)\right) \\ &= \lim_{n\to\infty} \left[\tfrac{1}{2}\|s_n - w\|^2 + \tfrac{1}{2}\|s_n^* - w^*\|^2 + \langle s_n - w, s_n^* - w^*\rangle\right] \\ &= \tfrac{1}{2}\tau^2 + \tfrac{1}{2}\tau^2 - \tau^2 = 0. \end{aligned}$$

So A is quasidense. (The above analysis does not use the assumption that A is closed or monotone.) The converse is immediate from Theorem 11.4(a) with $\alpha = \beta = 1$. □

Remark 11.7. Using results on maximally monotone multifunctions of type (NI), [21, Remark 11.4, p. 283] shows that the conclusion of Theorem 11.4(c) may, indeed, be true even if A is *not* quasidense, and [21, Example 11.5, p. 283–284] shows that if A is not quasidense then the conclusion of Theorem 11.4(c) may fail. In both these examples, A is the graph of a single–valued, continuous linear map. Theorem 11.4(d) implies that the closures of the domain and the range of a maximally monotone multifunction of type (NI) are both convex. This result was first proved by Zagrodny in [34], before it was known that such multifunctions of are always of type (ED). See Remark 13.4.

Definition 11.8. Let A be a maximally monotone subset of $E \times E^*$. Then A is of *type (ANA)* if, whenever $(w, w^*) \in E \times E^* \setminus A$, there exists $(s, s^*) \in A$ such that $s \neq w$, $s^* \neq w^*$, and

$$\frac{\langle s - w, s^* - w^*\rangle}{\|s - w\|\|s^* - w^*\|} \quad \text{is as near as we please to } -1.$$

(ANA) stands for "almost negative alignment". See [21, Section 9, pp. 280–281] for more discussion about this concept.

Theorem 11.9. *Let A be a closed, monotone, quasidense subset of $E \times E^*$. Then A is of type (ANA).*

Proof. This is immediate from Theorem 11.4(b). □

Problem 11.10. *Does there exist a maximally monotone set that is not of type (ANA)? The tail operator does* not *provide an example, because it was proved in Bauschke–Simons,* [5, Theorem 2.1, pp. 167–168] *that* if $S\colon\ E \to E^*$ is monotone and linear then S is maximally monotone of type (ANA).

12. More on the Fitzpatrick extension

In this section, we suppose that A is a closed monotone, quasidense subset of $E \times E^*$, and we give some characterizations of $A^{\mathbb{F}}$ in terms of marker functions.

From Theorem 7.4(a), A is maximally monotone. If $b^* \in B^* = E^* \times E^{**}$ then, from (1), (19), and (6) applied to the Banach SN space $\big(B^*, \widetilde{L}\big)$,

$$\left.\begin{aligned} &{\Phi_A}^*(b^*) \\ &= \sup\nolimits_{b\in E\times E^*}\,[\langle b, b^*\rangle - \Phi_A(b)] \\ &= \sup\nolimits_{b\in E\times E^*}\Big[\big\langle Lb, \widetilde{L}b^*\big\rangle - \Theta_A(Lb)\Big] \\ &\le \sup\nolimits_{d^*\in E^*\times E^{**}}\Big[\big\langle d^*, \widetilde{L}b^*\big\rangle - \Theta_A(d^*)\Big] = {\Theta_A}^*\Big(\widetilde{L}b^*\Big) = {\Theta_A}^{@}(b^*), \end{aligned}\right\} \tag{62}$$

and, from (6), (23) and (18),

$$\left.\begin{aligned} {\Phi_A}^{*@}(b^*) &= \sup\nolimits_{d^*\in B^*}\Big[\big\langle d^*, \widetilde{L}b^*\big\rangle - {\Phi_A}^*(d^*)\Big] \\ &\ge \sup\nolimits_{a\in A}\Big[\big\langle La, \widetilde{L}b^*\big\rangle - {\Phi_A}^*(La)\Big] \\ &= \sup\nolimits_A\big[b^* - {\Phi_A}^{@}\big] = \sup\nolimits_A\,[b^* - q_L] = \Theta_A(b^*). \end{aligned}\right\} \tag{63}$$

It would have been impossible to state (62) and (63) in Section 6, since B^* did not have a Banach SN structure in that section.

Lemma 12.1. *We suppose that $g \in \mathcal{PC}(B^*)$ is a marker function for A. Then $g \le {\Theta_A}^{@}$ and $g^{@} \ge \Theta_A$ on B^*.*

Proof. The first assertion is immediate from (24) and (62). Taking conjugates in (24), $g^{@} \ge {\Phi_A}^{*@}$ on B^*, and so the second assertion follows from (63). In fact, the second assertion can be deduced from the first by taking conjugates since ${\Theta_A}^{@@} = \Theta_A$. (See [23, Lemma 4.3(c), p. 237].) □

Theorem 12.2 (Invariance of coincidence sets for marker functions). *Let $g \in \mathcal{PC}(B^*)$ be a marker function for A. Then*

$$\big\{B^*|g^{@} = q_{\widetilde{L}}\big\} = \{B^*|g = q_{\widetilde{L}}\} = \big\{B^*|\Theta_A = q_{\widetilde{L}}\big\}.$$

Proof. From Lemma 12.1, (25), Theorem 6.10 and Lemma 7.3,

$${\Theta_A}^{@} \ge g \ge \Theta_A \ge q_{\widetilde{L}} \ \text{ on } B^*.$$

Thus $\{B^*|{\Theta_A}^@ = q_{\widetilde{L}}\} \subset \{B^*|g = q_{\widetilde{L}}\} \subset \{B^*|\Theta_A = q_{\widetilde{L}}\}$. If we apply Lemma 3.2(c) to $f := \Theta_A$, we see that $\{B^*|\Theta_A = q_{\widetilde{L}}\} \subset \{B^*|{\Theta_A}^@ = q_{\widetilde{L}}\}$, and so $\{B^*|g = q_{\widetilde{L}}\} = \{B^*|\Theta_A = q_{\widetilde{L}}\}$. Taking conjugates in (25), and using Lemma 12.1, Theorem 6.10 and Lemma 7.3 again,

$$ {\Theta_A}^@ \ge g^@ \ge \Theta_A \ge q_{\widetilde{L}} \ \text{ on } B^*. $$

Using the same argument as above, $\{B^*|g^@ = q_{\widetilde{L}}\} = \{B^*|\Theta_A = q_{\widetilde{L}}\}$. □

Problem 12.3. *Is $g^@$ necessarily a marker function for A?*

Theorem 12.4 (More characterizations of the Fitzpatrick extension). *We have:*

(a) $\{B^*|{\Phi_A}^{*@} = q_{\widetilde{L}}\} = A^{\mathbb{F}} = \{B^*|\Theta_A = q_{\widetilde{L}}\}$.

(b) *If g is a marker function for A then $\{B^*|g^@ = q_{\widetilde{L}}\} = \{B^*|g = q_{\widetilde{L}}\} = A^{\mathbb{F}}$.*

Proof. From Lemma 6.8, ${\Phi_A}^*$ is a marker function for A, and so Theorem 12.2 gives $\{B^*|{\Phi_A}^{*@} = q_{\widetilde{L}}\} = \{B^*|{\Phi_A}^* = q_{\widetilde{L}}\} = \{B^*|\Theta_A = q_{\widetilde{L}}\}$. (a) follows since, from (46), the middle of these sets is actually $A^{\mathbb{F}}$. (b) follows from (a) and Theorem 12.2 as stated. □

Theorem 12.5. *$A^{\mathbb{F}}$ is a maximally monotone subset of $E^* \times E^{**}$.*

Proof. From (48), $A^{\mathbb{F}}$ is monotone. Now suppose that $b^* \in E^* \times E^{**}$ and $\inf q_{\widetilde{L}}\left(A^{\mathbb{F}} - b^*\right) \ge 0$. From (47), $L(A) \subset A^F$, thus $\inf q_{\widetilde{L}}\left(L(A) - b^*\right) \ge 0$, and so, from (18) and the fact that, for all $a \in A$,

$$ \langle a, b^*\rangle - q_L(a) - q_{\widetilde{L}}(b^*) = \left\langle La, \widetilde{L}b^*\right\rangle - q_{\widetilde{L}}(La) - q_{\widetilde{L}}(b^*) = -q_{\widetilde{L}}(La - b^*), $$

it follows that

$$ \Theta_A(b^*) - q_{\widetilde{L}}(b^*) = \sup\nolimits_A \left[b^* - q_L - q_{\widetilde{L}}(b^*)\right] = -\inf q_{\widetilde{L}}\left(L(A) - b^*\right) \le 0. $$

Thus, from Theorem 6.10, $\Theta_A(b^*) = q_{\widetilde{L}}(b^*)$, and Theorem 12.4(a) implies that $b^* \in A^{\mathbb{F}}$. This gives the desired result. □

Remark 12.6. Theorem 12.4(a) implies that $(x^*, x^{**}) \in A^F$ exactly when (x^{**}, x^*) is in the Gossez extension of A (see [10, Lemma 2.1, p. 375]), which is known to be maximally monotone, so Theorem 12.5 is to be expected.

Problem 12.7. *Is $A^{\mathbb{F}}$ necessarily an $r_{\widetilde{L}}$–dense subset of $E^* \times E^{**}$?*

13. On a result of Zagrodny

We now give a generalization to Banach SN spaces of an inequality for monotone multifunctions proved by Zagrodny. This generalization appears in Theorem 13.2; and in Theorem 13.3, we see how this result appears when applied to monotone multifunctions. There is a discussion of Zagrodny's original result in Remark 13.4. The analysis in this section does not depend on any of the results in this paper after Section 3 other than Section 7. So let (B, L) be a Banach SN space. Then

$$\text{For all } d, e \in B,\ \|e\| \le \sqrt{2r_L(e) + 2r_L(d) - 2q_L(d-e)} + \|d\|. \qquad (64)$$

To see this, it suffices to observe that

$$r_L(e) + r_L(d) - q_L(d-e) = r_L(e) + r_L(d) - q_L(e) - q_L(d) + \langle d, Le\rangle$$
$$= \tfrac{1}{2}\|e\|^2 + \tfrac{1}{2}\|d\|^2 + \langle d, Le\rangle \ge \tfrac{1}{2}\|e\|^2 + \tfrac{1}{2}\|d\|^2 - \|d\|\|e\| = \tfrac{1}{2}\left(\|e\| - \|d\|\right)^2.$$

Lemma 13.1. *Let A_0 be an L–positive subset of B and $e, d \in A_0$. Then*

$$\|e\| \le \sqrt{2r_L(e)} + \sqrt{2}\|d\| + \|d\| \le \sqrt{2r_L(e)} + \tfrac{5}{2}\|d\|.$$

Proof. Since A_0 is L–positive, $q_L(d-e) \ge 0$, and so (64) and (4) imply that

$$\|e\| \le \sqrt{2r_L(e) + 2r_L(d)} + \|d\|$$
$$\le \sqrt{2r_L(e)} + \sqrt{2r_L(d)} + \|d\| \le \sqrt{2r_L(e)} + \sqrt{2}\|d\| + \|d\|.$$

This gives the required result. □

Theorem 13.2. *Let A be an L–positive subset of B, $a \in A$ and $b \in B$. Then $\|a\| \le \sqrt{2r_L(a-b)} + \frac{5}{2}\operatorname{dist}(b, A) + \|b\|$.*

Proof. Let A_0 be the L–positive set $A - b$. Let $c \in A$. Then $e := a - b \in A_0$ and $d := c - b \in A_0$. From Lemma 13.1, $\|a-b\| \le \sqrt{2r_L(a-b)} + \frac{5}{2}\|c-b\|$. Taking the infimum over c, $\|a-b\| \le \sqrt{2r_L(a-b)} + \frac{5}{2}\operatorname{dist}(b, A)$. □

Theorem 13.3. *Let A be a monotone subset of $E \times E^*$ and $(w, w^*) \in E \times E^*$. Then there exists $M \ge 0$ such that, for all $(s, s^*) \in A$,*

$$\|(s, s^*)\| \le M + \sqrt{\|s-w\|^2 + \|s^* - w^*\|^2 + 2\langle s-w, s^*-w^*\rangle}.$$

Proof. This follows from Theorem 13.2, with $M = \frac{5}{2}\operatorname{dist}\left((w, w^*), A\right) + \|(w, w^*)\|$. □

Remark 13.4. Theorem 13.3 was motivated by (and clearly generalizes) the second assertion of Zagrodny, [34, Corollary 3.4, pp. 780–781], which is equivalent to the following: Let A be a maximally monotone subset of $E \times E^*$ of type (NI) and $(w, w^*) \in E \times E^*$. Then there exist $\varepsilon_0 > 0$ and $R > 0$ such that if $0 < \varepsilon < \varepsilon_0$, $(s, s^*) \in A$ and

$$\|s-w\|^2 + \|s^* - w^*\|^2 + 2\langle s-w, s^*-w^*\rangle \le \varepsilon$$

then $\|(s, s^*)\| \le R$. Theorem 13.3 shows that we only need to assume that A is monotone, ε can be as large as we please, and $\|(s, s^*)\|$ is bounded by a function of the form $M + \sqrt{\varepsilon}$.

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