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by

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Abstract

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We are interested in finding for which positive integers D we have rational solutions for the equation $x^3 + y^3 = D$. The aim of this thesis is to compute the value of the L-function $L(E_D, 1)$, for $E_D : x^3 + y^3 = D$. For the case of p prime $p \equiv 1 \mod 9$, two formulas have been computed by Rodriguez-Villegas and Zagier in [17]. We have computed several formulas that relate $L(E_D, 1)$ to the trace of a modular function at a CM point. This offers a criterion for when the integer D is the sum of two rational cubes. Furthermore, when $L(E_D, 1)$ is nonzero we get a formula for the number of elements in the Tate-Shafarevich group of E_D.

To my parents, *Liliana and Daniel*

Who always supported me

To *Adrian Zanoschi*, my high school mathematics teacher

Contents

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Chapter 1

Introduction

In this thesis we are interested in finding which positive integers D can be written as the sum of two rational cubes:

$$x^3 + y^3 = D, \ x, y \in \mathbb{Q} \tag{1.1}$$

Despite the simplicity of the problem, an elementary approach to solving the Diophantine equation fails. However, we can restate the problem in the language of elliptic curves. After making the equation homogeneous, we get the equation $x^3 + y^3 = Dz^3$ that has a rational point at $\infty = [1 : -1 : 0]$. Moreover, after a change of coordinates $X = 12D\dfrac{z}{x+y}$, $Y = 36D\dfrac{x-y}{x+y}$ the equation becomes:

$$E_D : Y^2 = X^3 - 432D^2,$$

which defines an elliptic curve over $\mathbb{Q}$ written in its Weierstrass affine form.

Thus the problem reduces to finding if $E_D(\mathbb{Q})$, the set of rational points of the elliptic curve E_D, is non-trivial:

$$D = x^3 + y^3 \text{ has solutions in } \mathbb{Q} \Longleftrightarrow E_D(\mathbb{Q}) \neq \{O\}$$

By the *Mordell-Weil Theorem*, the set of rational points $E_D(\mathbb{Q})$ is a finitely generated abelian group. For simplicity, we will assume that D is cube free and $D \neq 1, 2$ (trivial cases) throughout the paper. It is known that $E_D(\mathbb{Q})$ has trivial torsion for $D \neq 1, 2$ (see [20]). Thus, (1.1) has a solution iff $E_D(\mathbb{Q})$ has positive rank. From the *Birch and Swynnerton-Dyer(BSD)* conjecture, this is equivalent conjecturally to the vanishing of $L(E_D, 1)$.

Without assuming BSD, from the work of Coates-Wiles [2], or more generally Gross-Zagier [7] and Kolyvagin [12], when $L(E_D, 1) \neq 0$, we have rank $E_D(\mathbb{Q}) = 0$, thus no rational solutions in (1.1).

For the case of prime numbers, Sylvester conjectured that the answer is affirmative in the case of D a prime number $\equiv 4, 7, 8 \mod 9$. In the cases of D prime $\equiv 2, 3, 5 \mod 9$ we have $L(E_D, 1) \neq 0$ and D is not the sum of two cubes. This follows either from a 3-descent argument (given in the 19th century by Sylvester, Lucas and Pepin) or from the theorem of Coates-Wiles [2].

We define an invariant S_D of E_D as follows:

$$S_D = \frac{L(E_D, 1)}{\Omega_{D,\infty} R_{E_D}},$$

where the denominator contains easily computable arithmetic invariants:

- $\Omega_{D,\infty} = \dfrac{\sqrt{3}}{18\pi\sqrt[3]{D}}\Gamma\left(\dfrac{1}{3}\right)^3$ is the real period,
- R_{E_D} is the regulator of the elliptic curve E_D.

The definition is made such that in the case of $L(E_D, 1) \neq 0$ we expect to get from the full BSD conjecture:

$$S_D = \#\text{Ш}(E_D) \prod_{p|6D} c_p, \tag{1.2}$$

where $\#\text{Ш}$ is the order of the Tate-Shafarevich group and c_p are the the Tamagawa numbers corresponding to the elliptic curve E_D.

Note that from the work of Rubin [18], when $L(E_D, 1) \neq 0$ we have $\#\text{Ш}(E_D)$ is finite. Furthermore, using the Cassels-Tate pairing, Cassels proved in [1] that when Ш is finite, then its order $\#\text{Ш}$ is a square. Thus we expect S_D to be an *integer square*. Current work in Iwasawa theory shows that for semistable elliptic curves at the good primes p we have $\mathrm{ord}_p(\#\text{Ш}[p^\infty]) = \mathrm{ord}_p(S_D)$, where $\text{Ш}[p^\infty]$ is the p^∞-torsion part of Ш (see [5]). However, this cannot be applied at the place 3 in our case.

The goal of the current thesis is to compute several formulas for S_D. By computing the value of S_D, we can determine when we have solutions in (1.1) and, assuming the full BSD conjecture, we can find in certain cases the order of Ш:

(i) $S_D \neq 0 \implies$ no solutions in (1.1)

(ii) $S_D \neq 0 \overset{BSD}{\Longrightarrow} S_D = \#\text{Ш}$

(iii) $S_D = 0 \overset{BSD}{\Longleftrightarrow}$ we have solutions in (1.1)

In [17], Rodriguez-Villegas and Zagier computed formulas for $L(E_p, 1)$ in the case of primes $p \equiv 1 \mod 9$. In this case it is predicted by BSD that the rank of $E_p(\mathbb{Q})$ is either 0 or 2. They compute two formulas for S_p. In the current paper, we are extending their results to all integers D.

Before stating the results, we will make a few remarks on the nature of the problem. First, note that each of the elliptic curves E_D is a cubic twist of E_1. This means that over $\mathbb{Q}$ the two elliptic curves are not isomorphic; however, they are isomorphic over $\mathbb{Q}[\sqrt[3]{D}]$ as can be easily shown by rewriting $E_D : 1 = \left(\frac{x}{\sqrt[3]{D}}\right)^3 + \left(\frac{y}{\sqrt[3]{D}}\right)^3$.

In the case of quadratic twists of elliptic curves, an important tool in computing the values of the L-functions is the work of Waldspurger [23]. For example, this is used to obtain Tunnell's Theorem for congruent numbers in [22]. However, the cubic twist case proves to be significantly more difficult.

Another important observation is that E_D is an elliptic curve with complex multiplication by $\mathcal{O}_K = \mathbb{Z}[\omega]$, the ring of integers of the number field $K = \mathbb{Q}[\sqrt{-3}]$ and $\omega = \frac{-1+\sqrt{-3}}{2}$. Then from CM theory there is a Hecke character $\chi_{E_D} : K^\times \setminus \mathbb{A}_K^\times \to \mathbb{C}^\times$ such that:

$$L(E_D, s) = L(s, \chi_{E_D}).$$

In order to compute the value of S_D and thus the value of the L-function we resort to automorphic methods to compute the value of $L(s, \chi_{E_D})$ and get the following result:

Theorem 1.1. *For all integers D, S_D is an integer and we have the formula:*

$$S_D = \mathrm{Tr}_{H_{3D}/K}\left(D^{1/3}\frac{\Theta_K(D\omega)}{\Theta_K(\omega)}\right), \tag{1.3}$$

where:

- *H_{3D} is the ring class field associated to the order $\mathcal{O}_{3D} = \mathbb{Z} + 3D\mathcal{O}_K$,*
- *$\omega = \frac{-1+\sqrt{-3}}{2}$ is a third root of unity, and*
- *$\Theta_K(z) = \sum_{a,b\in\mathbb{Z}} e^{2\pi i z(a^2+b^2-ab)}$ is the theta function of weight one associated to the number field $K = \mathbb{Q}[\sqrt{-3}]$.*

Note that using the formula (1.3) we can show that an integer D cannot be written as the sum of two cubes by computationally checking whether $L(E_D, 1) \neq 0$.

Furthermore, assuming BSD, we have $S_D = \#\text{Ш}$, thus we can compute the expected order of Ш explicitly. The formula (1.3) above proves that the term S_D is, as expected, an integer.

To compute the value of $L(s, \chi_{E_D})$, we look at the Hecke character adelically and using Tate's thesis, we integrate Tate's zeta function $Z(s, \chi_{E_D}, \Phi_K)$, for Φ_K a Schwartz-Bruhat function for $\mathcal{S}(\mathbb{A}_K)$. The proof is based on the following surprising fact: after integrating the Schwartz-Bruhat functions Φ_K, we recover a Siegel-Eisenstein series for $\Phi_{\mathbb{Q}}^\circ$:

$$\sum_{k\in K^\times} \frac{k}{|k|_{\mathbb{C}}^{2s}} \Phi_K(k\alpha_f) = E(g_\alpha, 2s-2, \Phi_{\mathbb{Q}}^\circ)$$

Furthermore, for $s = 1$ the Eisenstein series $E(g_\alpha, 0, \Phi_{\mathbb{Q}}^\circ)$ is equal to the value of the theta function $\Theta_K(g_\alpha)$ by the Siegel-Weil theorem [15] (up to a constant). Finally, the L-function at 1 is expressed as a linear combination of theta functions at CM-points. We further show using Shimura's reciprocity law that they are all Galois conjugates over K.

A different result is obtained by making a different choice for the Schwartz-Bruhat functions Φ_K above. This is presented in the following theorem in Section 8:

Theorem 1.2. *For all integers D, S_D is an integer and we have the formula:*

$$S_D = c_D \operatorname{Tr}_{H_\circ[\sqrt{D}]/K[\sqrt{D}]} \left(D^{-1/6} \frac{\theta_{1/2}^2(3D\omega)}{\Theta_K(\omega)} \right), \tag{1.4}$$

where:

- $\theta_{1/2}(z) = \sum_{n\in\mathbb{Z}} e^{2\pi i n^2 z}(-1)^n$ *is a theta function of weight* $1/2$
- $H_\circ$ *is the ray class field for the modulus* $12D$
- $c_D = D^{1/2} \prod_{p|D} (1 - (-1)^{(p-1)/2} p^{-1})$

The hope is to extend this result to show that S_D is an integer square up to Tamagawa numbers. In the following theorem we compute S_D as the absolute value of an element of K:

Theorem 1.3. *In the case of $D = \prod_{p_i \equiv 1 \mod 3} p_i^{e_i}$, S_D is an integer and we have:*

$$S_D = \left| \operatorname{Tr}_{H_{\mathcal{O}}/H_0} \frac{\theta_1(z_0)}{\theta_0(z_0)} D^{-1/3} \right|^2 \tag{1.5}$$

where:

- $\theta_1(z) = \sum_{n\in\mathbb{Z}} (-1)^n e^{\pi i (n+1/D-1/6)^2 z}$ *a* $1/2$*-weight modular form*
- $z_0 = \frac{-b+\sqrt{-3}}{2}$ *a CM-point, with* $b^2 \equiv -3 \mod 4D^2$,
- $H_{\mathcal{O}}$ *is the ray class field of modulus* $3D$ *and* H_0 *is an intermediate field* $K \subset H_0 \subset H_{\mathcal{O}}$ *that is the fixed field of a certain Galois group* G_0.

The idea of the proof of Theorem 1.3 is based on factoring each weight one theta function $\Theta_K(z)$ into a product of theta functions of weight 1/2. The method we are using is a factorization lemma of Rodriguez-Villegas and Zagier from [16] applied to the formula in Theorem 1.1 . This gives us the absolute value of a linear combination of theta functions evaluated at CM points. Finally, using Shimura reciprocity law,, we show that all the factors are Galois conjugates to each other.

Using similar methods, we obtain a more general formula for all integers D prime to 6.

Theorem 1.4. *Using the same notation as in Theorem 1.3, we have for all integers D prime to 6:*

$$S_D = \frac{\sqrt{D}}{\#\,\mathrm{Cl}(\mathcal{O}_{3D})} \sum_{r=0}^{D-1} \left| \mathrm{Tr}_{H_{\mathcal{O}}/H_1} \frac{\theta_r(Dz_0)}{\theta_0(z_0)} D^{-1/3} \right|^2, \tag{1.6}$$

where:

- $\theta_r(z) = \sum_{n\in\mathbb{Z}} (-1)^n e^{\pi i (n+r/D-1/6)^2 z}$ *a 1/2-weight modular form*
- $z_0 = \frac{-b+\sqrt{-3}}{2}$ *a CM-point*
- $H_{\mathcal{O}}$ *the ray class field of modulus* $3D$ *and* H_1 *is a subfield of* $H_{\mathcal{O}}$

Note that most of the proofs are presented for D a product of primes. However, the proofs easily go through for general D, for $(D, 6) = 1$.

Also note that in Appendix A we present some properties of the theta function Θ_K and in Appendix B we work with Shimura reciprocity in the setting of Shimura curves to provide a different proof for finding the Galois conjugates of the ratio of theta functions from Theorem 1.1.

Further results. The following result is announced without being included in the thesis.

The current approach inspired the answer to a different related problem. More precisely, for a family of cubic twists of characters $\varphi' = \varphi\varepsilon^*$ by χ_D, we became interested in computing the special value of the L-function $L(1, \chi_D\varphi')$. Here ε^* is a certain Hecke character of $\mathbb{A}_K^\times$. This family of characters does not correspond to a family of characters of elliptic curves. However, the special values of the L-functions suggest arithmetic properties. We showed the following theorem:

Theorem 1.5.

$$L(1, \chi_D\varphi') = c_D \left| \mathrm{Tr}_{H_{3D}/K} \frac{\theta_*(D\omega)}{\theta_*(\omega)} \right|^2,$$

where we take θ_* *to be a theta function of half-integral weight and* c_D *is a constant depending on* D.

The idea is based on a double integration of the Eisenstein series $E(s, g, \phi_1 \otimes \phi_2)$ over $\mathbb{A}_K^\times$ viewed as a subset of $\mathrm{GL}_2(\mathbb{A}_Q)$. The computation is inspired by the Rallis inner product.

Chapter 2

Background.

Let $K = \mathbb{Q}[\sqrt{-3}]$. Note that K is a PID and has the ring of integers $\mathcal{O}_K = \mathbb{Z}[\omega]$, where $\omega = \frac{-1+\sqrt{-3}}{2}$ is a fixed root of unity. We will denote K_v the localization of K at the place v. We will denote by $K_p := \prod_{v|p} K_v \cong \mathbb{Q}_p[\sqrt{-3}]$.

The L-function

Our goal is to compute several formulas for the special value of the L-function $L(E_D, 1)$ of the elliptic curve $E_D : x^3+y^3 = Dz^3$. The elliptic curve E_D has complex multiplication (CM) by $\mathcal{O}_K$. Then $L(E_D, s)$ is the L-function of a Hecke character that is computed explicitly in Ireland and Rosen [10]. We have:

$$L(E_D, s) = L(s, \chi_D \varphi),$$

where χ_D and φ are classical Hecke characters such that $\varphi\chi_D$ is the Hecke character corresponding to the elliptic curve E_D. The Hecke character φ is the Hecke character corresponding to E_1 and χ_D is the Hecke character corresponding to the cubic twist. More precisely, the Hecke characters are defined to be:

- $\varphi : I(3) \to K^\times$ is defined on the ideals prime to 3 by $\varphi(\mathcal{A}) = \alpha$, where α is the unique generator of the ideal $\mathcal{A}$ such that $\alpha \equiv 1 \mod 3$.
- $\chi_D : I(3D) \to \{1, \omega, \omega^2\}$ is the cubic character defined below in Section 2; it is defined on the space $I(3D)$ of all fractional ideals of $\mathcal{O}_K$ prime to $3D$. Moreover, it is well-defined over $\mathrm{Cl}(\mathcal{O}_{3D})$ the ring class group corresponding to the order $\mathcal{O}_{3D} = \mathbb{Z}+3D\mathcal{O}_K$.

The L-function can be expanded:

$$L(E_D, s) = \sum_{\mathcal{A}\in I(3D)} \frac{\chi_D(\mathcal{A})\varphi(\mathcal{A})}{(\mathrm{Nm}\,\mathcal{A})^s} = \sum_{\alpha\in\mathcal{O}_K, \alpha\equiv 1 (\mathrm{mod}\ 3)} \frac{\chi_D(\alpha)\alpha}{\mathbf{N}\alpha^s}.$$

Ring class fields.

Recall that an order $\mathcal{O}$ of K is a subring of $\mathcal{O}_K$ that is a finitely generated $\mathbb{Z}$-module and such that $\mathcal{O}\otimes_{\mathbb{Z}}\mathbb{Q} = K$. As K is a quadratic number field, each order is of the form $\mathcal{O} = \mathbb{Z} + f\mathcal{O}_K$ and we call $f = [\mathcal{O}_K : \mathcal{O}]$ the conductor of $\mathcal{O}$. We can also write $\mathcal{O}$ using a $\mathbb{Z}$-basis in the form $\mathcal{O} = [1, f\omega]_{\mathbb{Z}}$.

We define the class group $\mathrm{Cl}(\mathcal{O})$ of the order $\mathcal{O}$ of conductor f is defined to be:

$$\mathrm{Cl}(\mathcal{O}) := I_{\mathcal{O}}(f)/P_{\mathcal{O}}(f),$$

where $I_{\mathcal{O}}(f)$ is the set of fractional $\mathcal{O}$-ideals prime to the conductor f, and $P_{\mathcal{O}}(f)$ the subgroup of $I_{\mathcal{O}}(f)$ of principal fractional $\mathcal{O}$-ideals.

We define the ring class field to be the abelian extension $H_{\mathcal{O}}$ of K corresponding to the Galois group $\mathrm{Cl}(\mathcal{O})$ from class field theory, meaning:

$$\mathrm{Gal}(H_{\mathcal{O}}/K) \cong \mathrm{Cl}(\mathcal{O}).$$

We denote by $I(N)$ the group of fractional ideals in K prime to N. We denote the subgroup $P_{\mathbb{Z},N} = \{(\alpha)\colon\ \alpha \in K$ such that $\alpha \equiv a \mod N$ for some integer a such that $gcd(a, N) = 1\}$. Furthermore, let $\mathcal{O}_N := \mathbb{Z} + N\mathcal{O}_K$ be the order of K of conductor N. Then we can define the ring class field of the order $\mathcal{O}_N$ to be

$$\mathrm{Cl}(\mathcal{O}_N) := I(N)/P_{\mathbb{Z},N}$$

Note that K has class number one and thus by the Strong Approximation theorem we have:

$$\mathbb{A}_K^\times = K^\times \mathbb{C}^\times \prod_{v\nmid\infty} \mathcal{O}_{K_v}^\times.$$

We would like to describe $\mathrm{Cl}(\mathcal{O}_N)$ adelically. We do this below:

Lemma 2.1. *For N a positive integer, we can think of the ring class group adelically as:*

$$\mathrm{Cl}(\mathcal{O}_N) \cong U(N) \setminus \mathbb{A}_{K,f}^\times/K^\times,$$

where $U(N) = \prod_p (\mathbb{Z} + N\mathbb{Z}_p[\omega])^\times$.

Proof. From the Strong approximation theorem, as K is a PID, we have:

$$\mathbb{A}_K^\times \cong K^\times \mathbb{C}^\times \prod_{v\nmid\infty} \mathcal{O}_{K_v}^\times.$$

Taking the quotient by $K^\times\mathbb{C}^\times$, we get:

$$\mathbb{A}_{K,f}^\times/K^\times \cong \prod_{v\nmid\infty} \mathcal{O}_{K_v}^\times / \left(K^\times \cap \prod_{v} \mathcal{O}_{K_v}^\times \right) \cong \prod_{v\nmid\infty} \mathcal{O}_{K_v}^\times / \langle -\omega \rangle \,,$$

where $\langle -\omega \rangle$ is the group of sixth roots of unity.

Furthermore, note that $U(N) \cong \prod\limits_{v \nmid N} \mathcal{O}_{K_v}^\times \prod\limits_{p|N} (\mathbb{Z} + N\mathbb{Z}_p[\omega])^\times$. Moreover note that

$$\langle -\omega \rangle \, U(N) = U(N).$$

Thus we have:

$$\mathbb{A}_{K,f}^\times / K^\times U(N) \cong \prod_{v \nmid \infty} \mathcal{O}_{K_v}^\times / \langle -\omega \rangle \, U(N) \cong \prod_{v|N} \mathcal{O}_{K_v}^\times / \prod_{p|N} (\mathbb{Z} + N\mathbb{Z}_p[\omega])^\times \cong$$

$$\cong \prod_{p|N} \prod_{v|p} \mathcal{O}_{K_v}^\times / (\mathbb{Z} + N\mathbb{Z}_p[\omega])^\times$$

Finally, we need to show an isomorphism between

$$\mathrm{Cl}(\mathcal{O}_N) = I(N)/P_{\mathbb{Z}}(N)$$

and

$$\prod_{v|N} \mathcal{O}_{K_v}^\times / \prod_{p|N} (\mathbb{Z} + N\mathbb{Z}_p[\omega])^\times.$$

We construct the map:

$$I(N) \to \prod_{v|N} \mathcal{O}_{K_v}^\times \to \prod_{v|p} \mathcal{O}_{K_v}^\times / \prod_{p|N} (\mathbb{Z} + N\mathbb{Z}_p[\omega])^\times$$

Let $(k_0) \in I(N)$ be an ideal. Then we can map $k_0 \to (k_0)_{v|N}$. After taking the projection map, we want to look at the kernel of the composition $I(N) \to \prod_{v|p} \mathcal{O}_{K_v}^\times / \prod\limits_{p|N} (\mathbb{Z} + N\mathbb{Z}_p[\omega])^\times$. This consists of ideals $(k_0) \in I(N)$ such that $k_0 \equiv a_p \mod N\mathbb{Z}_p[\omega]$, where $a_p \in \mathbb{Z}$ and $(a_p, p) = 1$.

By the Chinese remainder theorem, we can find $a \in \mathbb{Z}$ such that $a \equiv a_p \mod N$ for all $p|N$. Then we have $k_0 \equiv a \mod N\mathbb{Z}_p[\omega]$ for all $a \in \mathbb{Z}$. Thus $(k_0) \in P_{\mathbb{Z}}(N)$ and $P_{\mathbb{Z}}(N)$ is the kernel of the above map. Thus we get:

$$I(N)/P_{\mathbb{Z}}(N) \cong \prod_{v|p} \mathcal{O}_{K_v}^\times / \prod_{p|N} (\mathbb{Z} + N\mathbb{Z}_p[\omega])^\times,$$

which proves our claim.

□

Another easy result that we will use is the following straight forward application of the Chinese remainder theorem. This map will be important in our proof:

Lemma 2.2. *For any $(l_{1,v})_{v|N} \in \prod_{v|N} \mathcal{O}_{K_v}^{\times}$, we can find $k_1 \in \mathcal{O}_K$ such that for all $v|N$ we have:*

$$l_{1,v} \equiv k_1 \mod N\mathcal{O}_{K_v},$$

Proof. For any $v|N$ we can find $a_{1,v} \in \mathcal{O}_K$ such that $l_{1,v} \equiv a_{1,v} \mod N\mathcal{O}_{K_v}$. We will pick for $N = \prod_{v|N} \mathfrak{p}_v^{e_v}$, where $\mathfrak{p}_v$ is the prime corresponding to the place v:

$$k_1 = \sum_{v|N} a_{1,v} m_v \frac{N}{\mathfrak{p}_v^{e_v}},$$

where $m_v \in \mathcal{O}_K$, $m_v \frac{N}{\mathfrak{p}_v^{e_v}} \equiv 1 \mod \mathfrak{p}_v^{e_v}$. We can find such an inverse since $\mathcal{O}_K$ is a PID, thus $\mathcal{O}_K/N\mathcal{O}_K \cong \prod_{v|N} \mathcal{O}_K/\mathfrak{p}_v^{e_v}\mathcal{O}_K$.

□

Characterization of ideals in ring class fields

Recall that a primitive ideal is an ideal not divisible by any integral ideal. It is easy to prove:

Lemma 2.3. *Any primitive ideal of $\mathcal{O}_K$ can be be written in the form $\mathcal{A} = [a, \frac{-b+\sqrt{-3}}{2}]$ as a $\mathbb{Z}$-module, where b is an integer (determined modulo $2a$) such that $b^2 \equiv -3 \mod 4a$ and $\operatorname{Nm} \mathcal{A} = a$. This implies that for $\mathcal{A} = (\alpha)$, we have $\|\alpha\| = a$.*

Conversely, given an integer satisfying the above congruence and $\mathcal{A}$ defined as above, we get that $\mathcal{A}$ is an ideal in $\mathcal{O}_K$ of norm a.

The cubic character

In the following we will define the cubic character χ_D and check that it is well defined on the class group $\mathrm{Cl}(\mathcal{O}_{3D})$. Let $\omega = \frac{-1+\sqrt{-3}}{2}$ be a fixed cube root of unity. Then we can define the cubic residue character following Ireland and Rosen [10].

Definition 2.1. *For $\alpha \in \mathbb{Z}[\omega]$ such that α is prime to 3, we define a cubic residue character $\chi_\alpha : I(3\alpha) \to \{1, \omega, \omega^2\}$ on the fractional ideals of K prime to 3α. For every prime ideal $\mathfrak{p}$ of $\mathbb{Z}[\omega]$, the character is defined to be:*

$$\chi_\alpha(\mathfrak{p}) = \omega^j,$$

for $j \in \{0, 1, 2\}$ such that ω^j is the unique third root of unity for which:

$$\alpha^{(\operatorname{Nm}\mathfrak{p}-1)/3} \equiv \omega^j \mod \mathfrak{p}, \text{ for } \operatorname{Nm}\mathfrak{p} \neq 3.$$

It is further defined multiplicatively on the fractional ideals of $I(3\alpha)$.

Notation: We will also denote $\chi_D(\cdot) =: \left(\frac{D}{\cdot}\right)_3$.

First let us check that this definition makes sense. Since K is a PID, any prime ideal $\mathfrak{p}$ has a generator of the form $\pi = a + b\omega \in \mathbb{Z}_p[\omega]$. Then the norm $\mathbf{N}\mathfrak{p} = a^2 - ab + b^2$ which is congruent to $0, 1 \mod 3$. Then, if $\mathfrak{p}$ is prime to 3, we must have $\mathbf{N}\mathfrak{p} \equiv 1 \mod 3$, implying that 3 divides $\mathbf{N}\mathfrak{p} - 1$.

Furthermore, the group $(\mathbb{Z}[\omega]/\mathfrak{p}\mathbb{Z}[\omega])^\times$ has $\mathrm{Nm}\,\mathfrak{p} - 1$ elements, thus we have $\alpha^{\mathrm{Nm}\,\mathfrak{p}-1} \equiv 1 \mod \mathfrak{p}$. Then since $\mathrm{Nm}\,\mathfrak{p} - 1$ is divisible by 3, we can factor out:

$$\mathfrak{p} | (\alpha^{(\mathrm{Nm}\,\mathfrak{p}-1)/3} - 1)(\alpha^{(\mathrm{Nm}\,\mathfrak{p}-1)/3} - \omega)(\alpha^{(\mathrm{Nm}\,\mathfrak{p}-1)/3} - \omega^2)$$

Finally since $K = \mathbb{Q}[\sqrt{-3}]$ is an UFD, $\mathfrak{p}$ divides exactly one of these terms, say

$$(\alpha^{(\mathrm{Nm}\,\mathfrak{p}-1)/3} - \omega^i).$$

Thus we can take $\chi_\alpha(\mathfrak{p}) = \omega^i$ and it is well-defined.

Following Ireland and Rosen, it is natural to look at the primary elements of K:

Definition 2.2. *For a prime ideal $\mathfrak{p}$ of K we call π* ***primary*** *if π generates $\mathfrak{p}$ a prime ideal and $\pi \equiv 2 \mod 3$.*

Lemma 2.4. *For any ideal $\mathcal{A}$ prime to 3, we can find a generator $\alpha \in \mathbb{Z}[\omega]$ such that $\alpha \equiv 2 \mod 3$.*

Proof. Since K is a PID, we can find a generator $\alpha_0 = a + b\omega$ be a generator of $\mathcal{A}$. Then note that $\pm\alpha_0, \pm\alpha_0\omega, \pm\alpha_0\omega^2$ also generate the ideal $\mathcal{A}$ and exactly one of them is $\equiv 2 \mod 3$. □

Remark 2.1. Note that from the definition of χ_{π_1} we have $\chi_{\pi_1}(\pi_2) = \chi_{-\pi_1}(\pi_2)$, as

$$\pi_1^{(\mathrm{Nm}\,\pi_2-1)/3} = (-\pi_1)^{(\mathrm{Nm}\,\pi_2-1)/3}$$

when $\mathrm{Nm}\,\pi_2$ is odd and $\pi_1^{(\mathrm{Nm}\,2-1)/3} \equiv (-\pi_1)^{(\mathrm{Nm}\,2-1)/3} \equiv 1 \mod 2$ when $\pi_2 = 2$. Moreover $\chi_{\pi_1}(\pi_2) = \chi_{\pi_1}(-\pi_2)$, as $\chi_{\pi_1}(-1) = 1$. Then we actually have for any choices of $\pm$:

$$\chi_{\pm\pi_1}(\pm\pi_2) = \chi_{\pm\pi_2}(\pm\pi_1)$$

Theorem 2.1. ***(Cubic reciprocity law)****. For $\pi_1, \pi_2 \equiv 2 \mod 3$ primary generators of primes $\mathfrak{p}_1, \mathfrak{p}_2$, $N\pi_1 \neq N\pi_2$ and $N\pi_1, N\pi_2 \neq 3$, then:*

$$\left(\frac{\pi_1}{\pi_2}\right)_3 = \left(\frac{\pi_2}{\pi_1}\right)_3$$

Corollary 2.1. *For $\pi_i, \pi_i' \equiv 2 \mod 3$, we have*

$$\chi_{\pm\pi_1\ldots\pi_n}(\pm\pi_1'\ldots\pi_n') = \chi_{\pm\pi_1'\ldots\pi_n'}(\pm\pi_1\ldots\pi_n)$$

Proof. We will first show that $\chi_{\pi_1\dots\pi_n}(\pi'_i) = \chi_{\pi_i}(\pi_1\dots\pi_n)$. By definition, we have:

$$\chi_{\pi_1\dots\pi_n}(\pi'_i) \equiv (\pi_1\dots\pi_n)^{(\mathrm{Nm}\,\pi'_i-1)/3} \mod \pi'_i$$

Thus, we have:

$$\chi_{\pi_1\dots\pi_n}(\prod_{i=1}^{m}\pi'_i) = \prod_{i=1}^{m}\chi_{\pi_1\dots\pi_n}(\pi'_i) = \prod_{i=1}^{m}\prod_{j=1}^{n}\chi_{\pi_j}(\pi'_i)$$

Using the cubic reciprocity, we have $\chi_{\pi_j}(\pi'_i) = \chi_{\pi'_i}(\pi_j)$, thus we get $\prod_{i=1}^{m}\prod_{j=1}^{n}\chi_{\pi_j}(\pi'_i) = \prod_{i=1}^{m}\prod_{j=1}^{n}\chi_{\pi'_i}(\pi_j)$, which furthermore implies:

$$\chi_{\pi_1\dots\pi_n}(\prod_{i=1}^{m}\pi'_i) = \chi_{\pi'_1\dots\pi'_m}(\prod_{j=1}^{n}\pi_i).$$

□

Note that we can always write the elements of $\mathbb{Z}[\omega]$ that are congruent to $\pm 1 \mod 3$ as a product of primary elements up to sign. Using the above corollary for α and D, we get:

Corollary 2.2. *If $\alpha \equiv \pm 1 \mod 3$ and D an integer prime to 3, then we have:*

$$\chi_D(\alpha) = \chi_\alpha(D)$$

Proof. Since $\alpha, D \equiv \pm 1 \mod 3$, we can write each of them in the form $\alpha = \pm\prod_{i=1}^{n}\pi_i$ and $D = \pm\prod_{j=1}^{m}\pi'_j$.

Then using the previous Corollary and Remark 2.1, we have

$$\chi_{\pm\prod\limits_{i=1}^{n}\pi_i}(\pm\prod_{j=1}^{m}\pi_j) = \chi_{\pm\prod\limits_{j=1}^{m}\pi_j}(\pm\prod_{i=1}^{n}\pi_i).$$

□

Lemma 2.5. *Let α be prime to 3 and $\mathfrak{p}$ a prime ideal prime to 3. Then the cubic residue can also be rewritten as:*

$$\chi_\alpha(\mathfrak{p}) \equiv \left(\frac{\overline{\alpha}}{\alpha}\right)^{(\mathrm{Nm}\,\pi-1)/3} \mod \pi$$

Proof. We have by definition $\chi_\alpha(\mathfrak{p}) \equiv \alpha^{(\mathrm{Nm}\,\pi-1)/3} \equiv \omega^i \mod \mathfrak{p}$. Taking the complex conjugate we have $\overline{\alpha}^{(\mathrm{Nm}\,\pi-1)/3} \equiv \omega^{2i} \mod \mathfrak{p}$. Then by taking the ratio we get:

$$\left(\frac{\overline{\alpha}}{\alpha}\right)^{(\mathrm{Nm}\,\pi-1)/3} \equiv \frac{\omega^{2i}}{\omega^i} \mod \mathfrak{p}$$

Thus we have $\chi_\alpha(\mathfrak{p}) \equiv \alpha^{(\mathrm{Nm}\,\pi-1)/3} \equiv \omega^i \equiv \left(\frac{\overline{\alpha}}{\alpha}\right)^{(\mathrm{Nm}\,\pi-1)/3} \mod \mathfrak{p}$ which finishes the proof of the lemma.

□

Corollary 2.3. *Let $D = \prod_{i=1}^{m} \mathfrak{p}_i$. For $\alpha \in \mathcal{P}_{\mathbb{Z},3D}$, we have $\chi_D(\alpha) = 1$. Thus χ_D is well defined on $\mathrm{Cl}(\mathcal{O}_{3D})$.*

Proof. Recall from the previous Lemma that if $\alpha \equiv \pm 1 \mod 3$, then we have:

$$\chi_\alpha(\mathfrak{p}) \equiv \left(\frac{\overline{\alpha}}{\alpha}\right)^{(\mathrm{Nm}\,\mathfrak{p}-1)/3} \mod \mathfrak{p}$$

Let $\mathfrak{p}|D$. Since $\alpha \in \mathcal{P}_{\mathbb{Z},3D}$, we have $\alpha \equiv a \mod 3D$ for some $a \in \mathbb{Z}$ and $(a, 3D) = 1$. Thus $\alpha \equiv a \mod \mathfrak{p}$, which also $\overline{\alpha} \equiv \overline{a} \mod \mathfrak{p}$, which implies:

$$\chi_\alpha(\mathfrak{p}) \equiv \left(\frac{\overline{\alpha}}{\alpha}\right)^{(\mathrm{Nm}\,\mathfrak{p}-1)/3} \equiv \left(\frac{\overline{a}}{a}\right)^{(\mathrm{Nm}\,\mathfrak{p}-1)/3} \equiv 1 \mod \mathfrak{p}$$

Thus we get $\chi_\alpha(\mathfrak{p}) = 1$ for all $\mathfrak{p}|D$. Thus we have $\chi_\alpha(D) = 1$. Moreover, using Corollary 2.2, we have $\chi_D(\alpha) = \prod_{i=1}^{m} \chi_{\mathfrak{p}_i}(\alpha) = \prod_{i=1}^{m} \chi_\alpha(\mathfrak{p}_i) = 1$.

□

Remark 2.2. For any fractional ideal $\mathcal{A}$ of K, when we write $\chi_D(\mathcal{A})$ we will mean:

$$\chi_D(\mathcal{A}) := \chi_D(\alpha),$$

where α is the unique generator of $\mathcal{A}$ such that $\alpha \equiv 1 \mod 3$.

Relating χ_D to the Galois conjugates of $D^{1/3}$.

There is another way to look at the cubic character using the Galois conjugates of $D^{1/3}$. We have the following lemma:

Lemma 2.6. *Let D be an integer prime to 3. Then for a prime ideal $\mathfrak{p}$ of K prime to $3D$, we have:*

$$D^{1/3}\chi_D(\mathfrak{p}) = (D^{1/3})^{\sigma_\mathfrak{p}},$$

where $\sigma_\mathfrak{p} \in \mathrm{Gal}(\mathbb{C}/K)$ is the Galois action corresponding to the ideal $\mathfrak{p}$ in the Artin correspondence.

Proof. It is enough to prove the claim for $\sigma_i \in \mathrm{Gal}(F/K)$, where $L = K[D^{1/3}, D^{1/3}\omega, D^{1/3}\omega^2]$. Let $\sigma_\mathfrak{p} = \left(\frac{L/K}{\mathfrak{p}}\right)$ the Frobenius element corresponds to $\mathfrak{p}$ the prime ideal of $\mathcal{O}_K$. Then using the definition of the Frobenius element for $D^{1/3} \in L$, we get:

$$(D^{1/3})^{\sigma_\mathfrak{p}} \equiv (D^{1/3})^{\mathrm{Nm}\,\mathfrak{p}} \mod \mathfrak{p}\mathcal{O}_L$$

Furthermore, note that $(D^{1/3})^{\mathrm{Nm}\,\mathfrak{p}} = D^{1/3}D^{(\mathrm{Nm}\,\mathfrak{p}-1)/3} \equiv D^{1/3}\chi_D(\mathfrak{p}) \mod \mathfrak{p}\mathcal{O}_L$.Since the Galois conjugates of $D^{1/3}$ are the roots of $x^3 - D$, the Galois conjugates of $D^{1/3}$ must be:

$$(D^{1/3})^{\sigma_\mathfrak{p}} \in \{D^{1/3}, D^{1/3}\omega, D^{1/3}\omega^2\}$$

and from the congruences above we get:

$$(D^{1/3})^{\sigma_\mathfrak{p}} = D^{1/3}\chi_D(\mathfrak{p})$$

□

Corollary 2.4. *Let D be an integer prime to 3 and $\mathcal{A}$ an ideal of K prime to $3D$. Moreover, let $\sigma_A \in \mathrm{Gal}(K^{ab}/K)$ be the Galois action corresponding to the ideal $\mathcal{A}$ through the Artin map. Then for the cubic character χ_D, we have:*

$$(D^{1/3})^{\sigma_\mathcal{A}} = D^{1/3}\chi_D(\mathcal{A}). \tag{2.1}$$

Proof. Let $\mathcal{A} = \prod\limits_j \mathfrak{p}_j^{f_j}$ the prime decomposition of $\mathcal{A}$ in K. Note that $\chi_D(\mathfrak{p}_i) \in K$, thus it is preserved by the Galois action. Applying the above Lemma we get:

$$((D^{1/3})^{\sigma_{\mathfrak{p}_i}})^{\sigma_{\mathfrak{p}_j}} = (D^{1/3}\chi_D(\mathfrak{p}_i))^{\sigma_{\mathfrak{p}_i}} = D^{1/3}\chi_D(\mathfrak{p}_j)\chi_D(\mathfrak{p}_j)$$

Using this step repeatedly, we get $(D^{1/3})^{\sigma_\mathcal{A}} = D^{1/3}\chi_D(\mathcal{A}) = D^{1/3}\chi_D(\mathcal{A})$.

□

Remark 2.3. Note that for the complex conjugate character $\overline{\chi_D}$ we have a similar result:

$$(D^{2/3})^{\sigma_\mathcal{A}} = D^{2/3}\overline{\chi_D(\mathcal{A})}. \tag{2.2}$$

Hecke characters

There are two equivalent ways of defining a Hecke character: classically and adelically. We define the **classical Hecke character** over K to be $\widetilde{\chi} : I(f) \to \mathbb{C}^\times$ a character from the set of fractional ideals prime to f, where f is a nonzero ideal of $\mathcal{O}_K$. We further say that $\widetilde{\chi}$ has infinity type $\widetilde{\chi}_\infty$ if it is characterized by the condition that on the set of principal ideals $P(f)$ prime to f it satisfies the condition:

$$\widetilde{\chi}((\alpha)) = \widetilde{\epsilon}(\alpha)\widetilde{\chi}_\infty^{-1}(\alpha),$$

where:

- $\widetilde{\varepsilon} : (\mathcal{O}_K/f\mathcal{O}_K)^\times \to \mathbb{T}$ is called the $(\mathcal{O}_K/f\mathcal{O}_K)^\times$-type character i.e. $\tilde{\varepsilon}$ is a character taking values in a finite group $\mathbb{T}$.
- $\widetilde{\chi}_\infty$ is an infinity type continuous character i.e. $\widetilde{\chi}_\infty : \mathbb{C}^\times \to \mathbb{C}^\times$ is a continuous character.

We define the **idelic Hecke character** to be a continuous character $\chi : \mathbb{A}^\times/K^\times \to \mathbb{C}^\times$.

There is a unique correspondence between the idelic and the classical Hecke characters. The correspondence can be explicitly constructed in the following way:

- $\tilde{\chi}(\mathcal{O}_v^\times \varpi_v) := \chi(\mathfrak{p}_v)$, $v \nmid f$
- $\tilde{\chi}_\infty$ is determined by χ_∞
- $\tilde{\chi}_v$ with $v|f$ is determined by Weak Approximation Theorem.

Converting the characters.

We want to compute a formula for $L(s, \chi)$, where $\chi : \mathbb{A}_K^\times/K^\times \to \mathbb{C}^\times$ is the Hecke character defined by $\chi = \chi_{3D}\varphi$. Here $\chi_{3D}\varphi$ are the adelic correspondent Hecke characters of the classical Hecke characters:

1. $\chi_{3D} : I(3D) \to \{1, \omega, \omega^2\}$ is the cubic character.
2. $\varphi : I(3) \to \mathbb{C}^\times$ is the Hecke character defined by $\chi((\alpha)) = \alpha$ for $\alpha \equiv 1 \mod 3$.

By abuse of notation, I will use φ, χ_{3D} both for the classical and the adelic Hecke characters. This should be clear from the context. We can rewrite the two characters adelically:

1. $\varphi : \mathbb{A}_K^\times \to \mathbb{C}^\times$ such that:

$$\begin{cases} \varphi_v(p) = -p, \ \varphi_v(\mathcal{O}_{K_v}^\times) = 1, & \text{for } v = p, p \equiv 2 \mod 3, \\ \varphi_v(\varpi_v) = \varpi_v, \ \varphi_v(\mathcal{O}_{K_v}^\times) = 1, & \text{for } v|p, p \equiv 1 \mod 3, \\ \varphi_\infty(x_\infty) = x_\infty^{-1}, & v = \infty \end{cases}$$

   For places $v|p$ with $p \equiv 1 \mod 3$, ϖ_v is a uniformizer of $\mathcal{O}_{K_v}$ such that $\varpi_v \equiv 1 \mod 3$. Also, at the place $v = \sqrt{-3}$, φ_v is determined from the Weak approximation theorem.

2. Note that χ_{3D} is trivial on $P_{\mathbb{Z},3D}$, thus χ_{3D} is a character on $Cl(\mathcal{O}_{3D})$. We will define the character by making it trivial on $\mathbb{C}^\times$, $U(3D)$ and $K^\times$. Then we can define using Lemma 2.2:

$$\chi_{3D}(l) = \chi_{3D}(l_1) = \chi_{3D}((k_1)).$$

   More precisely, this will be:

$$\begin{cases} \chi_{3D,v}(\varpi_v) = \chi_{3D}((\omega_v)), \ \chi_{3D,v}(\mathcal{O}_{K_v}^\times) = 1, & \text{if } v \nmid 3D \\ \chi_{3D,\infty}(x_\infty) = 1, & v = \infty \\ \chi_{3D,v}(\varpi_v) \ \text{ can be determined from the Weak approximation theorem,} & \text{if } v|3D \end{cases}$$

We can generally compute $\chi_f(l_f)$ in the following way:

Lemma 2.7. *If $\chi = \chi_{3D}\varphi$, let $l_f = kl_1$, $k \in K^\times, l_1 \in \prod_v \mathcal{O}_{K_v}^\times$. Note that this decomposition is unique up to a unit of $\mathcal{O}_K^\times$ and pick k such that $l_{1,3} \equiv 1 \mod 3$. Moreover take $k_1 \in K^\times$ such that $l_1 \equiv k_1 \mod 3D\mathcal{O}_{K_v}$. Then:*

$$\chi_f(l_f) = k\overline{\tilde{\chi}_{3D}((k_1))}$$

Proof. We start by writing:

$$\chi_f(l_f) = \chi_f(k)\chi_f(l_1) = \chi_\infty(k)^{-1}\chi_{v|3D}(l_{1,v})$$

Moreover, from the Chinese remainder theorem, we can find $k_1 \in K^\times$ such that $k_1 \equiv l_{1,v} \mod 3D\mathcal{O}_{K_v}$. As we have $k_1^{-1}l_1 \in 1 \mod 3D\mathcal{O}_{K_v}$ and χ is trivial on $(\mathbb{Z}+3D\mathcal{O}_{K_v})^\times$ for $v|3D$, we get $\chi_v(k_1) = \chi(l_{1,v})$. This implies:

$$\chi_f(l_f) = k\chi_{v|3D}(k_1) = k\chi_{v\nmid 3D}(k_1)^{-1}\chi_\infty(k_1)^{-1}$$

Note that if we write $k_1 = u\prod_v \omega_v^{e_v}$, where $u \in \mathcal{O}_{K_v}^\times$, we get:

$$\prod_{v\nmid 3D}\chi_v(k_1) = \prod_{v\nmid 3D}\chi_v(\omega_v)^{e_v} = \prod_{v\nmid 3D}\tilde{\chi}(\mathfrak{p}_v)^{e_v} = \tilde{\chi}((k_1))$$

This moreover implies:

$$\chi_f(l_f) = k\tilde{\chi}((k_1))^{-1}k_1 = kk_1^{-1}k_1\overline{\tilde{\chi}_{3D}((k_1))} = k\overline{\tilde{\chi}_{3D}((k_1))}$$

$\square$

Chapter 3

$L(E_D, 1)$ and Tate's zeta function

In this section we will compute the value of $L(E_D, 1) = L(1, \chi_D \varphi)$, working with χ_D, φ as automorphic Hecke characters. We will show the following result:

Theorem 3.1. *For H_{3D} the ring class field for the order $\mathcal{O}_{3D} = \mathbb{Z} + 3D\mathcal{O}_K$ and $S_D := \dfrac{18\pi D^{-1/3}}{\sqrt{3}\Gamma\left(\frac{1}{3}\right)^3} L(E_D, 1)$, we have $S_D \in \mathbb{Z}$ and*

$$S_D = \mathrm{Tr}_{H_{3D}/K}\left(D^{1/3}\frac{\Theta_K(D\omega)}{\Theta_K(\omega)}\right) \tag{3.1}$$

zeta functions.

We will compute the formula (3.1) using Tate's zeta function. We start by recalling some background and notation.

Schwartz-Bruhat functions.

We take $V = K$ a quadratic vector space over $\mathbb{Q}$ and $V_{\mathbb{A}_K} = \mathbb{A}_{\mathbb{Q}} \otimes_{\mathbb{Q}} K$. Then we can define the Schwartz-Bruhat functions $\Phi = \prod\limits_v \Phi_v$, $\Phi_v \in \mathcal{S}(V_{\mathbb{A}_K})$ to be:

$$\begin{cases} \Phi_v = \mathrm{char}_{\mathcal{O}_{K_v}}, & \text{if } v \nmid 3D \\ \Phi_p = \sum\limits_{(a,D)=1} \mathrm{char}_{(a+D\mathcal{O}_{K_p})}, & \text{if } v | 3D, v \nmid 3, \mathcal{O}_{K_p} = \prod_{v|p} \mathcal{O}_{K_v} \\ \Phi_p = \mathrm{char}_{(1+3\mathcal{O}_{K_v})}, & \text{if } v = \sqrt{-3} \\ \Phi_\infty(z) = ze^{-\pi q(\det z)}, & \text{where } z \in \mathbb{C} \end{cases}$$

Here $q(z) = |z|^2$ the usual absolute value on $\mathbb{C}$.

Remark 3.1. Note that $\mathrm{char}_{(a+p\mathcal{O}_{K_p})}(m) = \prod\limits_{v|p} \mathrm{char}_{(a+D\mathcal{O}_{K_v})}(m) = \prod\limits_{v|p} \mathrm{char}_{(1+p\mathcal{O}_{K_v})}(a^{-1}m)$ and each $\mathrm{char}_{(1+p\mathcal{O}_{K_v})}$ is a locally constant function with compact support. We are taking a linear combination of these Schwartz-Bruhat functions, thus we do get a Schwartz-Bruhat function.

Haar measure.

We will pick the self-dual additive Haar measure dx_v for all places v of K. We also take the usual multiplicative Haar measure:

$$\begin{cases} d_v^\times x_v = \frac{dx_v}{|x_v|_v}, \text{ normalized such that} \operatorname{vol}(\mathcal{O}_{K_v}^\times) = 1, & \text{if } v \nmid \infty \\ d^\times z = \frac{dz}{|z|_{\mathbb{C}}}, dz \text{ usual Lebesgue measure}, & z \in \mathbb{C}, |z|_{\mathbb{C}} = x^2 + y^2, \text{for } z = x + yi \end{cases}$$

Tate's zeta function.

We recall Tate's zeta function. For a Hecke character $\chi_v : K_v^\times \to \mathbb{C}^\times$ and a Schwartz-Bruhat function $\Phi_v \in \mathcal{S}(K_v)$, it is defined locally to be:

$$Z_v(s, \chi_v, \Phi_v) = \int\limits_{K_v^\times} \chi_v(\alpha_v)|\alpha_v|_v^s \Phi_v(\alpha_v) d^\times \alpha_v,$$

where $d^\times \alpha_v$ is the multiplicative Haar measure defined above.

We define globally $Z(s, \chi, \Phi) = \prod\limits_v Z_v(s, \chi_v, \Phi_v)$. As a global integral, this is:

$$Z(s, \chi, \Phi) = \int\limits_{\mathbb{A}_K^\times} \chi(\alpha)|\alpha|^s \Phi(\alpha) d^\times \alpha,$$

Tate's zeta function $Z(s, \chi, \Phi)$ has meromorphic continuation to all $s \in \mathbb{C}$ and in our case is entire.

Lemma 3.1. *For all s and Φ Schwartz-Bruhat functions chosen as above, we have:*

$$L_f(s, \chi_D \varphi) = Z_f(s, \chi_D \varphi, \Phi) V_{3D},$$

where $V_{3D} = \operatorname{vol}(1 + 3\mathbb{Z}_3[\omega]) \operatorname{vol}(\mathbb{Z} + D \prod\limits_{p|D} \mathbb{Z}_p[\omega])^\times = \dfrac{1}{6} \prod\limits_{p|D} \dfrac{1}{(p - (\frac{p}{3}))}$

Proof. From Tate's thesis, we have $L_f(s, \chi_D \varphi) = Z_f(s, \chi_D \varphi) \dfrac{\prod\limits_{p|3D} L_p(s, \chi_{D,p} \varphi_p)}{\prod\limits_{p|3D} Z_f(s, \chi_{D,p} \varphi_p, \Phi_p)}$. Since $\chi_D \varphi$ is ramified at $3D$, we have $L_p(s, \chi_{D,p} \varphi_p) = 1$. We need to compute the integral:

$$Z_p(s, \chi_D \varphi, \Phi_p) = \int\limits_{\mathbb{Q}_p[\omega]^\times} \chi_{D,p}(\alpha_p) \varphi_p(\alpha_p) |\alpha_p|_p^s \Phi_p(\alpha_p) d^\times \alpha_p$$

From the choice of the Schwartz-Bruhat function $\Phi_p = \mathrm{char}_{(\mathbb{Z}+3D\mathbb{Z}_p[\omega])^\times}$ for $p|D$, the integral reduces to $Z_p(s, \chi_D\varphi, \Phi_p) = \int\limits_{(\mathbb{Z}+3D\mathbb{Z}_p[\omega])^\times} \chi_{D,p}(\alpha_p)\varphi_p(\alpha_p)|\alpha_p|_p^s d^\times\alpha_p$. Note that for $p \neq 3$, all the characters χ_D, φ and $|\cdot|_p$ are unramified, thus we just get the volume $\mathrm{vol}\left((\mathbb{Z}+3D\mathbb{Z}_p[\omega])^\times\right)$.

For $p = 3$, we have $\Phi_p = \mathrm{char}_{(1+3\mathbb{Z}_3[\omega])}$. Similarly, we get $\mathrm{vol}\left((1+3\mathbb{Z}_3[\omega])^\times\right)$.

We compute the volumes. For D a product of primes, we have

$$\mathrm{vol}\left((\mathbb{Z}+3D\mathbb{Z}_p[\omega])^\times\right) = \mathrm{vol}\left((\mathbb{Z}+p\mathbb{Z}_p[\omega])^\times\right) = (p-1)\,\mathrm{vol}\,(1+p\mathbb{Z}_p[\omega]) = \frac{1}{(p-\left(\frac{p}{3}\right))}$$

Note that $\mathrm{vol}\,(1+p\mathbb{Z}_p[\omega]) = \frac{1}{p^2-1}\mathrm{vol}(\mathbb{Z}_p[\omega]^\times)$ when p is nonsplit and $\mathrm{vol}\,(1+p\mathbb{Z}_p[\omega]) = \frac{1}{(p-1)^2}\mathrm{vol}(\mathbb{Z}_p^\times)^2$ when p is split. This is computed by writing:

- p nonsplit: $\mathrm{vol}(\mathbb{Z}_p[\omega]^\times) = \sum \mathrm{vol}(a + b\omega + p\mathbb{Z}_p[\omega])$, where the sum is taken over all $a + b\omega$ prime to p and $0 \leq a, b \leq p-1$. We count $p^2 - 1$ of them and we get $\mathrm{vol}(\mathbb{Z}_p[\omega]^\times) = (p^2-1)\,\mathrm{vol}(1+p\mathbb{Z}_p[\omega])$.
- p split: $\mathrm{vol}(\mathbb{Z}_p[\omega]^\times) = \sum \mathrm{vol}(a + b\omega + p\mathbb{Z}_p[\omega])$. We count similarly $p^2 - 2p + 1$ such terms, as p splits and we have to discard the divisors of p.

For $p = 3$, we have $\mathrm{vol}\,(1+3\mathbb{Z}_3[\omega]) = \frac{1}{6}$.

We compute:

- $\mathbb{Z}_3[\omega] = \mathbb{Z}_3[\sqrt{-3}] = \{a_0 + a_1\sqrt{-3} + a_2(-3) + \ldots, 0 \leq a_i \leq 2\}$
- $\mathrm{vol}(\mathbb{Z}_3[\omega])^\times = 1$
- $(\mathbb{Z}_3[\omega])^\times = \bigcup(a_0 + a_1\sqrt{-3})(1 + 3\mathbb{Z}_3[\omega])$, where $a_0 + a_1\sqrt{-3}$ is prime to 3. Then we have 6 possibilities and thus $\mathrm{vol}(1+3\mathbb{Z}_3[\omega]) = \frac{1}{6}$.

$\square$

By plugging in $s = 1$ in the above Lemma, we get:

Corollary 3.1. *The finite part of the L-function at $s = 1$ equals:*

$$L_f(1, \chi_D\varphi) = \frac{1}{6}\prod_{p|D}\frac{1}{(p-\left(\frac{p}{3}\right))}Z_f(1, \chi_D\varphi, \Phi),$$

Computing the finite part of Tate's zeta function $Z_f(s, \chi_D\varphi, \Phi)$.

In this section we will compute the value of $Z_f(s, \chi_D\varphi, \Phi)$. We begin by rewriting Tate's zeta function $Z_f(s, \chi_D\varphi, \Phi)$ as a linear combination of Hecke characters:

Lemma 3.2. *For all $s \in \mathbb{C}$ and the Schwartz-Bruhat functions $\Phi_f \in \mathcal{S}(\mathbb{A}_{K,f})$, we have:*

$$Z_f(s, \chi_D\varphi, \Phi_f) = V_{3D} \sum_{\alpha_f \in U(3D)\backslash \mathbb{A}_{K,f}^\times / K^\times} I(s, \alpha_f, \Phi_f)\chi_D(\alpha)\varphi(\alpha),$$

where $I(s, \alpha_f, \Phi_f) = \sum\limits_{k \in K^\times} \frac{k}{|k|_{\mathbb{C}}^{2s}} \Phi_f(k\alpha_f)$.

Proof. By definition, we have $Z_f(s, \chi_D\varphi, \Phi_f) = \int\limits_{\mathbb{A}_{K,f}^\times} \chi_D(\alpha_f)\varphi(\alpha_f)|\alpha_f|_f^s \Phi_f(\alpha_f) d^\times \alpha_f$. We rewrite the integral by taking a quotient by $K^\times$:

$$Z_f(s, \chi_D\varphi, \Phi_f) = \int\limits_{\mathbb{A}_{K,f}^\times / K^\times} \sum_{k \in K^\times} \chi_{D,f}(k\alpha'_f)\varphi_f(k\alpha'_f)|k\alpha_f|_f^s \Phi_f(k\alpha'_f) d^\times \alpha'_f$$

Note that from the definition of Hecke characters, we have

$$\chi_{D,f}(k\alpha'_f) = \chi_{D,\infty}^{-1}(k)\chi_{D,f}(\alpha'_f) = \chi_{D,f}(\alpha'_f),$$

$$\varphi_f(k\alpha'_f) = \varphi_\infty^{-1}(k)\varphi_f(\alpha'_f) = k\varphi_f(\alpha'_f)$$

and

$$|k\alpha'_f|_f^s = |k|_\infty^{-s}|\alpha_f|_f^s = |k|_{\mathbb{C}}^{-2s}|\alpha'_f|_f^s,$$

where $|\cdot|_{\mathbb{C}}$ is the usual absolute value over $\mathbb{C}$. Then the integral reduces to:

$$Z_f(s, \chi_D\varphi, \Phi_f) = \int\limits_{\mathbb{A}_{K,f}^\times / K^\times} \left(\sum_{k \in K^\times} \frac{k}{|k|_{\mathbb{C}}^{2s}} \chi_{D,f}(\alpha'_f)\Phi_f(k\alpha'_f) \right) \varphi_f(\alpha'_f)|\alpha'_f|_f^s \; d^\times \alpha'_f$$

Furthermore, note that our choice of Schwartz-Bruhat functions $\Phi_f(k\alpha'_f)$ are invariant over $U(3D)$. Similarly:

- $|\cdot|_f$ is trivial on units, thus on $U(3D)$
- χ_D is invariant on $U(3D)$ by definition
- φ is trivial on all the units at all the unramified places. At 3 it is invariant under $1 + 3\mathbb{Z}_3[\omega]$, thus it is trivial on all of $U(3D)$

Thus we can take the quotient by $U(3D)$ as well. Note that the integral is now a finite sum:

$$Z_f(s, \chi_D\varphi, \Phi_f) = \mathrm{vol}(U(3D)) \sum_{\alpha''_f \in U(3D)\backslash \mathbb{A}^\times_{K,f}/K^\times} \left(\sum_{k\in K^\times} \frac{k}{|k|^{2s}_{\mathbb{C}}} \chi_{D,f}(\alpha''_f)\Phi_f(k\alpha''_f) \right) \varphi_f(\alpha''_f)|\alpha''_f|^s_f$$

Moreover, note that $\mathrm{vol}(U(3D)) = \mathrm{vol}(1 + 3\mathbb{Z}_3\omega) \prod_{p|D} \mathrm{vol}(\mathbb{Z} + D\mathbb{Z}_p[\omega]) = V_{3D}$.

By denoting $I(s, \alpha_f, \Phi_f) = \sum_{k\in K^\times} \frac{k}{|k|^s_{\mathbb{C}}} \Phi_f(k\alpha_f)$, we get the conclusion of the Lemma.

□

Combining the Corollary 3.1 and Lemma 3.2, we get:

Corollary 3.2. *For all $s \in \mathbb{C}$ and the Schwartz-Bruhat functions $\Phi_f \in \mathcal{S}(\mathbb{A}_{K,f})$ chosen above, we have:*

$$L_f(s, \chi_D\varphi) = \sum_{\alpha_f \in U(3D)\backslash \mathbb{A}^\times_{K,f}/K^\times} I(s, \alpha_f, \Phi_f)\chi_D(\alpha)\varphi(\alpha),$$

Adelic representatives for $\mathrm{Cl}(\mathcal{O}_{3D})$.

From the Strong approximation theorem, we can write $\alpha_f \in \mathbb{A}^\times_K = \mathbb{C}^\times K^\times \prod_{v\nmid\infty} \mathcal{O}^\times_{K_v}$ in the form $\alpha_f = \gamma_\infty k_\alpha \beta_f$, where $k_\alpha \in K^\times$, $\gamma_\infty \in \mathbb{C}^\times$ and $\beta_f \in \prod_{v\nmid\infty} \mathcal{O}^\times_{K_v}$. Then we can take representatives in $\alpha_f \in U(3D) \setminus \mathbb{A}^\times_{K,f}/K^\times$ such that $\alpha_f \in \prod_{v\nmid\infty} \mathcal{O}^\times_{K,v}$. Moreover, since we are taking the quotient by the cube roots of six $\{\pm 1, \pm\omega, \pm\omega^2\}$, we can pick α_f such that $\alpha_3 \equiv 1 \mod 3$. This can be done by replacing α_f by $\pm\alpha_f\omega^i$ for some i, $0 \le i \le 2$.

Furthermore, note that representatives α_f, α'_f are in the same class in $U(3D)$ iff $\alpha_f\alpha_f'^{-1} \equiv a \mod D\mathbb{Z}_p[\omega]$, for some integer a such that $(a, D) = 1$.

Moreover, we can define an ideal $\mathcal{A}_\alpha$ that is generated by $k_\alpha \in \mathcal{O}_K$ such that

$$\alpha_p \equiv k_\alpha \mod 3D\mathbb{Z}_p[\omega].$$

Note that this ideal is unique only as a class in $\mathrm{Cl}(\mathcal{O}_{3D})$.

Connection to the Eisenstein series.

Using the above representatives, note that φ_f and $|\cdot|_f$ are trivial for the representatives l_f and the Corollary 3.2 becomes:

$$L_f(s, \chi_D\varphi) = \sum_{\alpha_f \in U(3D)\backslash \mathbb{A}^\times_{K,f}/K^\times} I(s, \alpha_f, \Phi_f)\chi_D(\alpha_f)$$

We will now connect $I(s, \alpha_f, \Phi_f)$ to an Eisenstein series. We define the following classical Eisenstein series of weight 1:

$$E_\varepsilon(s, z) = \sum_{m,n} \frac{\varepsilon(n)}{(3mz+n)|3mz+n|^s},$$

where the sum is taken over all $m, n \in \mathbb{Z}$ except for the pair $(0, 0)$, and $\varepsilon = \left(\frac{\cdot}{3}\right)$ is the quadratic character associated to the field extension $K/\mathbb{Q}$.

Note that the Eisenstein series does not converge absolutely. However, we can still compute its value at 0 using the Hecke trick in order for it to converge. We will compute its Fourier expansion in the following section.

Recall that for $\alpha_f \in \prod_{v \nmid \infty} \mathcal{O}_{K_v}^\times$, we have the corresponding ideal class $[\mathcal{A}_\alpha]$ in $\mathrm{Cl}(\mathcal{O}_{3D})$. Such a representative is $\mathcal{A}_{\alpha_f} = (k_\alpha)$, where $k_\alpha \in \mathcal{O}_K$ is chosen such that $k_\alpha \equiv \alpha_p \mod 3D\mathbb{Z}_p[\omega]$ for $p|3D$. Note that we can pick $\mathcal{A}_\alpha$ to be a primitive ideal.

We can further write $\mathcal{A}_\alpha$ as a $\mathbb{Z}$-lattice $\mathcal{A}_\alpha = [a, \frac{-b+\sqrt{-3}}{2}]_\mathbb{Z}$, where $a = \mathrm{Nm}\,\mathcal{A}_\alpha$ and b is chosen (not uniquely) such that $b^2 \equiv -3 \mod 4a$. Then we can take the corresponding CM point $z_{\mathcal{A}_\alpha} := \frac{-b+\sqrt{-3}}{2a}$

Using this notation, we have the following equality:

Lemma 3.3. *For $\alpha_f \in \prod_{v \nmid \infty} \mathcal{O}_{K_v}^\times$ and any choice of $z_{\mathcal{A}_\alpha}$ as above, we have:*

$$I(s, \alpha_f, \Phi_f) = \frac{1}{2} \frac{(\mathrm{Nm}\,\mathcal{A}_\alpha)^{1-s}}{k_\alpha} E_\varepsilon(s, z_{\overline{\mathcal{A}_\alpha}})$$

Remark 3.2. Note that the variable $z_{\overline{\mathcal{A}_\alpha}}$ on the left hand side is not uniquely defined. However, the function is going to be invariant on the class $[\mathcal{A}_\alpha]$ in $\mathrm{Cl}(\mathcal{O}_{3D})$.

Proof. Recall that $I(s, \alpha_f, \Phi_f) = \sum_{k \in K^\times} \frac{k}{|k|_\mathbb{C}^{2s}} \Phi_f(k\alpha_f)$. We need to compute $\Phi_f(k\alpha_f)$. Note that for all finite places v we have $\Phi_v(k\alpha_v) \neq 0$ only for $k\alpha_v \in \mathcal{O}_{K_v}$, and since $\alpha_v \in \mathcal{O}_{K_v}^\times$, we must have $k \in \mathcal{O}_{K_v}$ for all $v \nmid \infty$. This implies $k \in \mathcal{O}_K$ and for all $v \nmid 3D$ we get $\Phi_v(k\alpha_v) = 1$ for $k \in \mathcal{O}_K$. Thus we can rewrite:

$$I(s, \alpha_f, \Phi_f) = \sum_{k \in \mathcal{O}_K} \frac{k}{|k|_\mathbb{C}^{2s}} \Phi_{3D}(k\alpha_{3D}),$$

where $\Phi_{3D} = \prod_{v|3D} \Phi_v$ and $\alpha_{3D} = (\alpha_v)_{v|3D}$.

We can further compute $\Phi_v(k\alpha_v)$ for $v|3D$. Recall that for $p|D$ we defined $\Phi_p = \mathrm{char}_{(\mathbb{Z}+3D\mathbb{Z}_p[\omega])^\times}$ and $\Phi_3 = \mathrm{char}_{(1+3\mathbb{Z}_3[\omega])^\times}$. Then we have $\Phi_{3D}(k\alpha_{3D}) \neq 0$ iff $k\alpha_p \in a + 3D\mathbb{Z}_p[\omega]$ for some integer $(a, p) = 1$ and for $p = 3$ we need $k\alpha_3 \in 1 + 3\mathcal{O}_{K_3}$.

Recall that we can define k_α such that $k_\alpha \equiv \alpha_p \mod 3D\mathbb{Z}_p[\omega]$ for all $p|3D$. Then the we have $kk_\alpha \in a + 3D\mathbb{Z}_p[\omega]$ for $(a, p) = 1$ and $kk_\alpha \in 1 + 3\mathbb{Z}_3[\omega]$ as well. Furthermore, for

$k \in \mathcal{O}_K$ we actually have $\Phi_{3D}(k\alpha_{3D}) = \Phi_{3D}(kk_\alpha)$. Then we can rewrite $I(s, \alpha_f, \Phi_f)$ using k_α in the form:

$$I(s, \alpha_f, \Phi_f) = \sum_{k \in \mathcal{O}_K} \frac{k}{|k|_{\mathbb{C}}^{2s}} \Phi_{3D}(kk_\alpha),$$

We can rewrite this further:

$$I(s, \alpha_f, \Phi_f) = \frac{|k_\alpha|_{\mathbb{C}}^{2s}}{k_\alpha} \sum_{k \in \mathcal{O}_K} \frac{kk_\alpha}{|kk_\alpha|_{\mathbb{C}}^{2s}} \Phi_{3D}(kk_\alpha),$$

Finally, we will make this explicit. Note that we must have $kk_\alpha \in \mathcal{A}_\alpha$, where $\mathcal{A}_\alpha = (k_\alpha)$, we well as $kk_\alpha \in a_p + D\mathbb{Z}_p[\omega]$ for some integer a_p, $(a_p, p) = 1$ as well as $kk_\alpha \in 1 + 3\mathbb{Z}_3[\omega]$. By the Chinese remainder theorem, we can find an integer a such that $a \equiv a_p \mod D$ and $a \equiv 1 \mod 3$. Then we have $kk_\alpha \in a + D \prod_{p|3D} \mathbb{Z}_p[\omega] \cap \mathcal{O}_K$, thus $kk_\alpha \in P_{\mathbb{Z},3D} \cap P_{1,3}$. Here $P_{\mathbb{Z},3D} = \{k \in K : k \equiv a \mod 3D\mathcal{O}_K \text{ for some integer } a, (a, 3D) = 1\}$ and $P_{1,3} = \{k \in K : k \equiv 1 \mod 3\}$. We rewrite:

$$I(s, \alpha_f, \Phi_f) = \frac{|k_\alpha|_{\mathbb{C}}^{2s}}{k_\alpha} \sum_{k \in \mathcal{A}_\alpha \cap P_{\mathbb{Z},D} \cap P_{1,3}} \frac{k}{|k|_{\mathbb{C}}^{2s}},$$

Finally, we want to write the elements of $\mathcal{A}_\alpha \cap P_{\mathbb{Z},D} \cap P_{1,3}$ explicitly.

Recall that we can write $\mathcal{A}_\alpha$ as a $\mathbb{Z}$-lattice $\mathcal{A}_\alpha = [a, \frac{b+\sqrt{-3}}{2}]$. Then all of the elements of $\mathcal{A}$ are of the form $ma + n\frac{b+\sqrt{-3}}{2}$ for some integers $m, n \in \mathbb{Z}$. Moreover, note that the intersection of $\mathcal{A}$ and $P_{\mathbb{Z},3D} = \{k \in \mathcal{O}_K : k \equiv n \mod 3D, \text{for some integer } n, (n, 3D) = 1\}$ is $\{ma + 3Dn\frac{b+\sqrt{-3}}{2} : m, n \in \mathbb{Z}\}$. Further taking the intersection with $P_{1,3}$, we must have $ma \equiv 1$, thus we must have $m \equiv 1 \mod 3$. Thus we can rewrite $I(s, \alpha_f, \Phi_f)$ in the form:

$$I(s, \alpha_f, \Phi_f) = \frac{a^s}{k_\alpha} \sum_{m,n \in \mathbb{Z}, m \equiv 1 (\text{mod } 3)} \frac{ma + n\frac{-b+\sqrt{-3}}{2}}{|ma + n\frac{-b+\sqrt{-3}}{2}|_{\mathbb{C}}^{2s}},$$

Here we have also used the fact that $|k_\alpha|_{\mathbb{C}} = a$. Note that we can further rewrite this as:

$$I(s, \alpha_f, \Phi_f) = a^{s-1}\overline{k_\alpha} \sum_{m,n \in \mathbb{Z}, m \equiv 1 (\text{mod } 3)} \frac{1}{(ma + n\frac{-b-\sqrt{-3}}{2})|ma + n\frac{-b+\sqrt{-3}}{2}|_{\mathbb{C}}^{2s-2}},$$

Furthermore, by changing $n \to -n$ and taking out a factor of a^{1-2s}, we have:

$$I(s, \alpha_f, \Phi_f) = a^{-s}\overline{k_\alpha} \sum_{m,n \in \mathbb{Z}, m \equiv 1 (\text{mod } 3)} \frac{1}{(m + n\frac{b+\sqrt{-3}}{2a})|m + n\frac{b+\sqrt{-3}}{2a}|_{\mathbb{C}}^{2s-2}},$$

Note that for $Re(s) > 1$ the integral converges absolutely, thus we can rewrite it in the form:

$$I(s,\alpha_f,\Phi_f) = \frac{1}{2}a^{-s}\overline{k_\alpha}\sum_{m,n\in\mathbb{Z},m\equiv 1(\mathrm{mod}\ 3)}\frac{1}{(m+n\frac{b+\sqrt{-3}}{2a})|m+n\frac{b+\sqrt{-3}}{2a}|_{\mathbb{C}}^{2s-2}}$$
$$+\frac{1}{2}a^{-s}\overline{k_\alpha}\sum_{m,n\in\mathbb{Z},m\equiv 2(\mathrm{mod}\ 3)}\frac{1}{(-m+n\frac{b+\sqrt{-3}}{2a})|-m+n\frac{b+\sqrt{-3}}{2a}|_{\mathbb{C}}^{2s-2}}$$

Changing $n \to -n$ in the second sum, we get:

$$I(s,\alpha_f,\Phi_f) = \frac{1}{2}a^{-s}\overline{k_\alpha}\sum_{m,n\in\mathbb{Z},m\equiv 1(\mathrm{mod}\ 3)}\frac{1}{(m+n\frac{b+\sqrt{-3}}{2a})|m+n\frac{b+\sqrt{-3}}{2a}|_{\mathbb{C}}^{2s-2}}-$$
$$-\frac{1}{2}a^{-s}\overline{k_\alpha}\sum_{m,n\in\mathbb{Z},m\equiv 2(\mathrm{mod}\ 3)}\frac{1}{(m+n\frac{b+\sqrt{-3}}{2a})|m+n\frac{b+\sqrt{-3}}{2a}|_{\mathbb{C}}^{2s-2}}$$

Thus we can write for $Re(s) > 1$ we can rewrite:

$$I(s,\alpha_f,\Phi_f) = \frac{1}{2}a^{-s}\overline{k_\alpha}\sum_{m,n\in\mathbb{Z},m\equiv 1(\mathrm{mod}\ 3)}\frac{\varepsilon(m)}{(m+n\frac{b+\sqrt{-3}}{2a})|m+n\frac{b+\sqrt{-3}}{2a}|_{\mathbb{C}}^{2s-2}}$$

On the right hand side we can recognize the Eisenstein series $E_\varepsilon(2s-2, z_{\overline{\mathcal{A}_\alpha}})$, thus we get:

$$I(s,\alpha_f,\Phi_f) = \frac{1}{2}a^{-s}\overline{k_\alpha}E_\varepsilon(2s-2, z_{\overline{\mathcal{A}_\alpha}}) = \frac{1}{2}\frac{a^{1-s}}{k_\alpha}E_\varepsilon(2s-2, z_{\overline{\mathcal{A}_\alpha}})$$

By analytic continuation, we can extend the equality to all $s \in \mathbb{C}$.

□

Using this Lemma, we can rewrite the Corollary 3.2 in the form:

Corollary 3.3. *For all s, we have:*

$$L_f(s,\chi_D\varphi) = \frac{1}{2}\sum_{\mathcal{A}\in \mathrm{Cl}(\mathcal{O}_{3D})} E_\varepsilon(2s-2, z_{\mathcal{A}})\overline{\chi_D(\mathcal{A})}\frac{(\mathrm{Nm}\,\mathcal{A})^{1-s}}{\overline{k_{\mathcal{A}}}},$$

Proof. Recall that in the Corollary 3.2 we got

$$L_f(s,\chi_D\varphi) = \sum_{\alpha_f\in U(3D)\backslash \mathbb{A}_{K,f}^\times/K^\times} I(s,\alpha_f,\Phi_f)\chi_D(\alpha)\varphi(\alpha),$$

We can rewrite $I(s,\alpha_f,\Phi_f) = \frac{1}{2}\frac{a^{1-s}}{k_\alpha}E_\varepsilon(2s-2, z_{\mathcal{A}_\alpha})$ and $\varphi(\alpha) = 1$, $\chi_D(\alpha) = \chi_D(k_\alpha) = \chi_D(\mathcal{A}_{\alpha)}$. Then we get:

$$L_f(s, \chi_D\varphi) = \sum_{\alpha_f \in U(3D)\backslash \mathbb{A}_{K,f}^\times / K^\times} \frac{1}{2}\frac{a^{1-s}}{k_\alpha} E_\varepsilon(2s-2, z_{\mathcal{A}_\alpha})\chi_D(\mathcal{A}_\alpha)$$

Finally, consider $\mathcal{A}$ as representatives of $\mathrm{Cl}(\mathcal{O}_{3D})$. Note that by changing $\mathcal{A} \to \overline{\mathcal{A}}$ we just invert the classes of $\mathrm{Cl}(\mathcal{O}_{3D})$. Thus we get the result of the Corollary:

$$L_f(s, \chi_D\varphi) = \sum_{\mathcal{A} \in \mathrm{Cl}(\mathcal{O}_{3D})} \frac{1}{2}\frac{a^{1-s}}{\overline{k_\mathcal{A}}} E_\varepsilon(2s-2, z_\mathcal{A})\overline{\chi_D(\mathcal{A})}.$$

□

Fourier expansion of the Eisenstein series $E_\varepsilon(s, z)$ at $s = 0$.

We want to connect the Eisenstein series $E_\varepsilon(s, z)$ to the theta function $\Theta_K(z)$. In order to do this, we will compute the Fourier expansion of $E_\varepsilon(s, z)$ at $s = 0$.

We will use the Hecke trick to compute the Fourier expansion of the Eisenstein series:

$$E_\varepsilon(s, z) = {\sum_{c,d}}' \frac{\varepsilon(d)}{(3cz+d)|3cz+d|^{2s}}$$

We will follow closely the proof of Pacetti [14]. This is also done by Hecke in [8]. We rederive the formula:

$$E_1(z, s) = {\sum_{d}}' \frac{\varepsilon(d)}{d^{1+2s}} + 2\sum_{c=1}^{\infty}\sum_{r=0}^{2}\sum_{d\in\mathbb{Z}} \frac{\varepsilon(r)}{(3cz+(3d+r))|3cz+(3d+r)|^{2s}}$$

We divide by 3^{2s+1} and get:

$$E_1(z, s) = 2L(\varepsilon, 1+2s) + 2\sum_{c=1}^{\infty}\sum_{r=0}^{2} \frac{\varepsilon(r)}{3^{2s+1}} \sum_{d\in\mathbb{Z}} \frac{\varepsilon(r)}{(\frac{3cz+r}{3}+d)|\frac{3cz+r}{3}+d|^{2s}}$$

We define for z in the upper-half plane:

$$H(z, s) = \sum_{m\in\mathbb{Z}} \frac{1}{(z+m)|z+m|^{2s}}$$

Following Shimura (Lemma 1, p. 84, [19]), for $z = x + yi$ and $s > 0$ we have the Fourier expansion:

$$H(z, s) = \sum_{n=-\infty}^{\infty} \tau_n(y, s+1, s)e^{2\pi i n x},$$

where $\tau_n(y, s+1, s)\dfrac{i\Gamma(s+1)\Gamma(s)}{(2\pi)^{2s+1}} = \begin{cases} n^{2s}e^{-2\pi ny}\sigma(4\pi ny, s+1, s), & \text{if } n > 0 \\ |n|^{2s}e^{-2\pi|n|y}\sigma(4\pi|n|y, s, s+1), & \text{if } n < 0 \\ \Gamma(2s)(4\pi y)^{-2s}, & \text{if } n = 0, \end{cases}$

where $\gamma(Y, \alpha, \beta) = \int\limits_0^\infty (t+1)^{\alpha-1}t^{\beta-1}e^{-Yt}dt$

For any $s > 0$, $H(z, s)$ converges, thus we can compute the limits of each of its Fourier coefficients:

- $n = 0$: $\lim\limits_{s\to 0} \frac{(2\pi)^{2s+1}}{i\Gamma(s+1)} \frac{\Gamma(2s)}{\Gamma(s)} (4\pi y)^{-2s} = -2\pi i \lim\limits_{s\to 0} \frac{\Gamma(2s)}{\Gamma(s)}$
- $n < 0$: $\lim\limits_{s\to 0} \dfrac{(2\pi)^{2s+1}}{i\Gamma(s+1)\Gamma(s)}|n|^{2s}e^{-2\pi|n|y} \int\limits_0^\infty (t+1)^{s-1}t^s e^{-4\pi|n|yt}dt =$

  $= -2\pi i e^{-2\pi|n|y} \lim\limits_{s\to 0} \frac{1}{\Gamma(s)} \int\limits_0^\infty (t+1)^{s-1}t^s e^{-4\pi|n|yt}dt$
- $n > 0$: $\lim\limits_{s\to 0} \dfrac{(2\pi)^{2s+1}}{i\Gamma(s+1)\Gamma(s)}n^{2s}e^{-2\pi ny} \int\limits_0^\infty (t+1)^s t^{s-1}e^{-4\pi nyt}dt$

We get, following [14]:

$$\lim_{s\to 0} H(s, z) = -\pi i - 2\pi i \sum_{n=1}^{\infty} q^n$$

Finally, note that:

$$E_1(s, z) = 2L(\varepsilon, s) + 2\sum_{c=1}^{\infty}\sum_{r=0}^{2} \frac{\varepsilon(r)}{3^{2s+1}} H\left(\frac{3dz + r}{3}, s\right)$$

Using the Fourier expansion of $H(z, s)$, we get:

$$E_1(s, z) = 2L(\varepsilon, s) + 2\sum_{c=1}^{\infty}\sum_{r=0}^{2} \frac{\varepsilon(r)}{3^{2s+1}} \sum_{n\in\mathbb{Z}} \tau_n(yn, s+1, s)e^{2\pi in\frac{3xc+r}{3}}$$

Taking the limit as $s \to 0$, and the Fourier expansion above, we get:

$$E_1(s, z) = 2L(\varepsilon, s) + 2\sum_{c=1}^{\infty}\sum_{r=0}^{2} \frac{\varepsilon(r)}{3} \left(-\pi i - 2\pi i \sum_{n=1}^{\infty} e^{2\pi inzc}\omega^{nr}\right)$$

We compute separately the inner sum:

$$\sum_{r=0}^{2} \frac{\varepsilon(r)}{3}\left(-\pi i + \sum_{n=1}^{\infty} e^{2\pi i n z c}\omega^{nr}\right) =$$

$$= -2\pi i \sum_{n=1}^{\infty} e^{2\pi i n z c}\varepsilon(n) \sum_{r=0}^{2} \omega^{nr}\varepsilon(rn) = -\frac{2\pi i}{3} G(\varepsilon) \sum_{n=1}^{\infty} e^{2\pi i n z c}\varepsilon(n),$$

where $G(\varepsilon) = \sum_{r=0}^{2} \varepsilon(r)\omega^r = \sqrt{-3}$ is the Gaussian quadratic sum corresponding to ε. Then we get:

$$E_1(0, z) = 2L(\varepsilon, 1) - \frac{4\pi i\sqrt{-3}}{3} \sum_{c=1}^{\infty}\sum_{n=1}^{\infty} e^{2\pi i n z c}\varepsilon(n) = 2L(\varepsilon, 1) + \frac{4\pi\sqrt{3}}{3} \sum_{N=1}^{\infty}\left(\sum_{m|N} \varepsilon(m)\right) e^{2\pi i N z}$$

Since ε is a quadratic character, we can compute $L(1, \varepsilon) = \frac{\pi\sqrt{3}}{9}$ (see Kowalski [13]). This gives us the Fourier expansion:

$$E_1(0, z) = \frac{2\pi\sqrt{3}}{9}\left(1 + 6\sum_{N=1}^{\infty}\left(\sum_{m|N}\varepsilon(m)\right) e^{2\pi i N z}\right)$$

Connection to the theta function $\Theta_K(z)$.

Recall the theta function Θ_K associated to the number field K:

$$\Theta_K(z) = \sum_{a,b\in\mathbb{Z}} e^{2\pi i(a^2-ab+b^2)z}.$$

Equivalently, we can rewrite the theta function in the form: $\Theta_K(z) = 1 + 6\sum_{\mathcal{A}} e^{2\pi i \operatorname{Nm}\mathcal{A} z}$, where we sum over all ideals $\mathcal{A}$. Thus we have the Fourier expansion for Θ_K:

$$\Theta_K(z) = 1 + 6\sum_{n\geq 1} c(n)q^n,$$

where $c(n)$ is the number of ideals of norm n. We will show the following version of Siegel-Weil theorem:

Theorem 3.2. *For $E_\varepsilon(s, z)$ defined in the previous section and ε the quadratic character corresponding to to the extension $K/\mathbb{Q}$, we have:*

$$E_\varepsilon(0, z) = 2L(0, \varepsilon)\Theta_K(z)$$

The proof consists of comparing the Fourier expansions of the two sides. This is mainly going to be based on the lemma below:

Lemma 3.4. *For $n \geq 1$ then for the ideals in $\mathcal{O}_K$ we have:*

$$\sum_{d|n} \varepsilon(d) = \#\textit{ideals of norm } n$$

Proof. We first show the result for powers of primes p^e. We consider three cases:

If $p \equiv 1 \mod 3$, then there are two ideals of norm p: $(a+b\omega)$ and $(a-b\omega)$ such that $a^2-ab+b^2=p$. Then we have $k+1$ ideals of norm p^k: $(a+b\omega)^i(a+b\omega)^{k-i}$ for $0 \leq i \leq k$. Moreover, since $\varepsilon(p)=1$, we have $(1+\varepsilon(p)+\ldots\varepsilon(p^k)) = k+1$.

If $p \equiv 2 \mod 3$, then there are no ideals of norm p. Thus, if k is even, we have exacly one ideal of norm p^k: $\mathcal{A} = (p^{k/2})$. In this case $(1+\varepsilon(p)+\ldots\varepsilon(p^k)) = 1-1+\cdots+1 = 1$. If k is odd, we have no ideals of norm p^{2k+1}. Moreover $(1+\varepsilon(p)+\ldots\varepsilon(p^k)) = 1-1+\cdots-1 = 0$.

If $p=3$, then we have exactly one ideal of norm 3^k, namely the ideal $(\sqrt{-3}^k)$. Moreover $\varepsilon(3)=0$, thus $(1+\varepsilon(3)+\ldots\varepsilon(3^k)) = 1$.

It is easy to extend the result to all integers. As ε is a character, we have:

$$\sum_{d|n} \varepsilon(d) = \prod_{p_i|n} (1+\varepsilon(p_i)+\cdots+\varepsilon(p_i)^{c_i}),$$

where $n = \prod_i p_i^{c_i}$, $e_i \geq 1$ and p_i are primes. If we have any ideal $\mathcal{A}$ of norm n, then $\mathcal{A} = \prod_{p_v} \mathfrak{p}_v^{e_v}$, and we must have $n = \prod_v \operatorname{Nm} \mathfrak{p}_v^{e_v}$. Moreover, we have #ideals of norm n = $\prod_{p_i|n}$ #ideals of norm $(\operatorname{Nm} p_i)^{c_i}$, which finishes the proof. ∎

We are ready to state the proof of the theorem. Using the above Lemma we can rewrite the Fourier expansion of Θ_K as:

$$\Theta_K(z) = 1 + 6\sum_{N=1}^{\infty}\left(\sum_{m|N}\varepsilon(m)\right)e^{2\pi i N z}$$

Multiplying by a factor of $\frac{2\pi\sqrt{3}}{9}$, we recognize the Eisenstein series $E_\varepsilon(0,z)$. Thus it implies $E_\varepsilon(0,z) = \frac{2\pi\sqrt{3}}{9}\Theta_K(z)$. Note that this is the same as:

$$E_\varepsilon(0,z) = 2L(1,\varepsilon)\Theta_K(z)$$

Final formula for $L(1,\chi_D\varphi)$.

Applying Corollary 3.3 for $s=1$ we get:

$$L_f(1,\chi_D\varphi)=\frac{1}{2}\sum_{\mathcal{A}\in\mathrm{Cl}(\mathcal{O}_{3D})}\frac{1}{\overline{k}_{\mathcal{A}}}E_\varepsilon(0,Dz_{\mathcal{A}})\overline{\chi_D(\mathcal{A})}$$

Furthermore, from Theorem 3.2 this is the same as:

$$L_f(1,\chi_D\varphi)=\frac{\pi\sqrt{3}}{9}\sum_{\mathcal{A}\in\mathrm{Cl}(\mathcal{O}_{3D})}\frac{1}{\overline{k}_{\mathcal{A}}}\Theta_K(Dz_{\mathcal{A}})\overline{\chi_D(\mathcal{A})} \tag{3.2}$$

We need one more step before rewriting the formula as a trace. This is going to be the following lemma:

Lemma 3.5. *For* $\mathcal{A}=\left[a,\dfrac{-b+\sqrt{-3}}{2}\right]$ *a primitive ideal of norm* $\mathrm{Nm}\,\mathcal{A}=a$*, with generator* $\mathcal{A}=(k_{\mathcal{A}})$*, where* $k_{\mathcal{A}}\equiv 1 \mod 3$*, we have:*

$$\Theta_K\left(\frac{-b+\sqrt{-3}}{2a}\right)=\overline{k_{\mathcal{A}}}\Theta_K\left(\frac{-1+\sqrt{-3}}{2}\right)$$

Proof. Since $\mathcal{A}=\left[a,\frac{-b+\sqrt{-3}}{2}\right]_{\mathbb{Z}}$ as a $\mathbb{Z}$-lattice, we can write its generator $k_{\mathcal{A}}$ in the form $k_{\mathcal{A}}=ma+3n\dfrac{-b+\sqrt{-3}}{2}$ for some integers m,n. Moreover, since $k_{\mathcal{A}}$ is the generator of a primitive ideal, we have $\gcd(m,3n)=1$. Then we can find through the Euclidean algorithm integers A,B such that $mA+3nB=1$, which makes $\begin{pmatrix}A & B\\ -3n & m\end{pmatrix}$ a matrix in $\Gamma_0(3)$. Since Θ is a modular form of weight 1 for $\Gamma_0(3)$, we have:

$$\Theta_K\left(\frac{A\frac{-b+\sqrt{-3}}{2a}+B}{-3n\frac{-b+\sqrt{-3}}{2a}+m}\right)=\left(m-3n\frac{-b+\sqrt{-3}}{2a}\right)\Theta_K\left(\frac{-b+\sqrt{-3}}{2a}\right)$$

Noting that $-3n\frac{-b+\sqrt{-3}}{2a}+m=k_{\mathcal{A}}/a=1/\overline{k}_{\mathcal{A}}$, we can compute

$$\frac{A\frac{-b+\sqrt{-3}}{2a}+B}{-3n\frac{-b+\sqrt{-3}}{2a}+m}=\frac{(A\frac{-b+\sqrt{-3}}{2}+Ba)\overline{k}_{\mathcal{A}}}{a}.$$

This is $(aB+A\frac{-b+\sqrt{-3}}{2})(ma+3n\frac{b+\sqrt{-3}}{2})/a$. After expanding, we get:

$$-3nA\frac{b^2+3}{4a}+abB+\frac{b(-mA+3nB)}{2}+\frac{\sqrt{-3}}{2}$$

Note that $mA+3nB=1$ implies that mA and $3nB$ have different parities. Also we chose b odd, since $b^2+3\equiv 0 \mod 4a$. Then we note that $-3nA\frac{b^2+3}{4a}+abB+\frac{b(-mA+3nB)+1}{2}\in\mathbb{Z}$ and thus using the period 1 of Θ_K we get:

$$\Theta_K\left(\frac{A\frac{-b+\sqrt{-3}}{2a}+B}{-3n\frac{-b+\sqrt{-3}}{2a}+m}\right)=\Theta_K\left(\frac{-1+\sqrt{-3}}{2}\right)$$

This finishes the proof.

□

Since the Lemma above tells us that $\Theta_K(\tau_{\mathcal{A}})=\overline{k_{\mathcal{A}}}\Theta_K(\omega)$, where $\tau_{\mathcal{A}}=\frac{-b+\sqrt{-3}}{2a}$, we can rewrite (3.2) as:

Proposition 3.1.

$$L_f(1,\varphi\chi_D)=\frac{\pi\sqrt{3}}{9}\Theta_K(\omega)\sum_{\mathcal{A}\in\mathrm{Cl}(\mathcal{O}_{3D})}\frac{\Theta_K(D\tau_{\mathcal{A}})}{\Theta_K(\tau_{\mathcal{A}})}\overline{\chi_D(\mathcal{A})} \tag{3.3}$$

Turning the formula into a trace.

We will rewrite (3.3) as a trace. First, let $f(z) = \dfrac{\Theta_K(Dz)}{\Theta_K(z)}$. This is a modular function for $\Gamma_0(3D)$. We will prove in Chapter 4 the following proposition (see Proposition 4.1):

Proposition 3.2. *Take $\mathcal{A}$ representative ideals for $\mathrm{Cl}(\mathcal{O}_{3D})$. We can take all $\mathcal{A}$ to be primitive and we can write them in the form $\mathcal{A} = [a, \frac{-b+\sqrt{-3}}{2}]_{\mathbb{Z}}$. Then the Galois conjugates of $f(\omega)$ are:*

$$f(\omega)^{\sigma_{\mathcal{A}}^{-1}} = \frac{\Theta\left(D\frac{-b+\sqrt{-3}}{2a}\right)}{\Theta\left(\frac{-b+\sqrt{-3}}{2a}\right)}$$

We will also rewrite the character χ_D to include a trace. In the Chapter 2 we have also showed in Corollary 2.4 that $(D^{1/3})^{\sigma_{\mathcal{A}}^{-1}} = D^{1/3}\overline{\chi_D(\mathcal{A})}$.

Then the formula (3.3) becomes:

$$L_f(E_D, 1) = \frac{\pi\sqrt{3}}{9} D^{-1/3}\Theta_K(\omega) \sum_{\mathcal{A}\in Cl(\mathcal{O}_{3D})} \left(D^{1/3}\frac{\Theta_K(D\omega)}{\Theta_K(\omega)}\right)^{\sigma_{\mathcal{A}^{-1}}} \tag{3.4}$$

Moreover, we also have $D^{1/3} \in H_{3D}$. See Cohn [3] for a proof. Thus we can rewrite the sum on the left hand side as $\mathrm{Tr}_{H_{3D}/K}\left(D^{1/3}\frac{\Theta_K(D\omega)}{\Theta_K(\omega)}\right)$. We can compute the extra terms as well.

- Rodriguez-Villegas and Zagier in [17] cite $\Theta_K\left(\frac{-9+\sqrt{-3}}{18}\right) = -3\Gamma\left(\frac{1}{3}\right)^3/(2\pi)^2$. We will use several properties of Θ_K proved in Appendix A. We can rewrite $\Theta_K\left(\frac{-9+\sqrt{-3}}{18}\right)$ as $\Theta_K\left(\frac{-3+\sqrt{-3}}{18} - \frac{1}{3}\right)$ and using Lemma 9.1 we get:

$$\Theta_K\left(\frac{-3+\sqrt{-3}}{18} - \frac{1}{3}\right) = (1-\omega^2)\Theta_K\left(\frac{-3+\sqrt{-3}}{6}\right) + \omega^2\Theta_K\left(\frac{-3+\sqrt{-3}}{18}\right)$$

  Using $\Theta_K\left(\frac{-3+\sqrt{-3}}{6}\right) = 0$, we get $\Theta_K\left(\frac{-9+\sqrt{-3}}{18}\right) = \omega^2\Theta_K\left(\frac{-3+\sqrt{-3}}{18}\right)$.

Furthermore, the functional equation $\Theta(-1/3z) = -\sqrt{-3}z\Theta(z)$ for $z = \frac{3+\sqrt{-3}}{2}$, we get $-\sqrt{-3}\frac{3+\sqrt{-3}}{2}\Theta(\omega) = \Theta_K\left(\frac{-3+\sqrt{-3}}{18}\right)$. Note that $-\sqrt{-3}\frac{3+\sqrt{-3}}{2} = 3\omega$, thus we get $\Theta_K\left(\frac{-9+\sqrt{-3}}{18}\right) = 3\Theta(\omega)$.

This gives us the value $\Theta(\omega) = \Gamma\left(\frac{1}{3}\right)^3/(2\pi)^2$

- $L_\infty(s, \chi_D\varphi) = L_\infty(s, \varphi_\infty)$, where $\varphi_\infty(z) = z^{-1}$. Then we can compute:

$$L_\infty(s, \varphi_\infty) = L_\infty(s - 1/2, |\cdot|_\infty^{1/2}\varphi_\infty) = 2(2\pi)^s\Gamma(s).$$

This gives us $L_\infty(1, \chi_D\varphi) = 2$.

- The real period Ω_D of the elliptic curve E_D. The real period of E_1 is $\dfrac{\Gamma\left(\frac{1}{3}\right)^3}{9\pi}$. Then to compute the real period of E_D we twist by a factor of $D^{-1/3}$ and get:

$$\Omega_D = D^{-1/3}\frac{\Gamma\left(\frac{1}{3}\right)^3}{18\pi}$$

Multiplying all the terms, we get:

$$L(E_D, 1) = 2\frac{\pi\sqrt{3}}{9}D^{-1/3}\frac{\Gamma\left(\frac{1}{3}\right)^3}{(2\pi)^2}\operatorname{Tr}_{H_{3D}/K}\left(D^{1/3}\frac{\Theta_K(D\omega)}{\Theta_K(\omega)}\right)$$

This gives us the first part of Theorem 3.1:

$$L(E_D, 1) = \frac{\sqrt{3}\Gamma\left(\frac{1}{3}\right)^3}{18\pi}D^{-1/3}\operatorname{Tr}_{H_{3D}/K}\left(D^{1/3}\frac{\Theta_K(D\omega)}{\Theta_K(\omega)}\right) \tag{3.5}$$

S_D is an integer.

In the previous section we have showed that $S_D \in K$. Moreover, it is easy to see that note that $S_D \in \mathbb{Q}$. To show this, it is enough to check the invariance of $D^{1/3}\Theta(D\omega)/\Theta(\omega)$ under complex conjugation:

$$\overline{D^{1/3}\Theta(D\omega)/\Theta(\omega)} = D^{1/3}\Theta(-D + D\omega)/\Theta(-1 + \omega) = D^{1/3}\Theta(D\omega)/\Theta(\omega).$$

Now we would like to show that $S_D \in \mathbb{Z}$. First we look at the Fourier expansion of $f(z) = \Theta(Dz)/\Theta(z)$:

$$\Theta(z) = 1 + 6\sum_{N\in\mathbb{Z}_{\geq 1}} c(N)q^N,$$

where $c(N) = \#$ ideals with norm N in K and, $q = e^{2\pi i z}$. Then we also have the Fourier expansion of $\Theta(Dz)$:

$$\Theta(Dz) = 1 + 6 \sum_{N \in \mathbb{Z}_{\geq 1}} c(N) q^{DN},$$

By taking their ratio we get $\dfrac{\Theta(Dz)}{\Theta(z)} = \displaystyle\sum_{n \in \mathbb{Z}} a_n q^n$, $a_n \in \mathbb{Z}$. This is easy to see just by straight computation. The minimal polynomial of $D^{1/3} f(\omega)$ is:

$$\prod_{\mathcal{A} \in \mathrm{Cl}(\mathcal{O}_{3D})} (X - D^{1/3} \chi_D(\mathcal{A})(f(\omega))^{\sigma_{\mathcal{A}}}) \in \mathbb{Z}[\omega, D^{1/3}](X, q)$$

This implies that $\mathrm{Tr}_{H_{3D}/K} D^{1/3} f(\omega) \in \mathbb{Z}[\omega, D^{1/3}]$. We already know that $\mathrm{Tr}_{H_{3D}/K} D^{1/3} f(\omega) \in \mathbb{Q}$, thus $\mathrm{Tr}_{H_{3D}/K} D^{1/3} f(\omega) \in \mathbb{Z}$.

Chapter 4

Shimura reciprocity law in the classical setting.

Let $\mathcal{F}$ be the field of modular functions over $\mathbb{Q}$. From CM theory (see [21], for example), it is known that if $\tau \in K \cap \mathcal{H}$ and $f \in \mathcal{F}$, then we have $f(\tau) \in K^{ab}$, where K^{ab} is the maximal abelian extension of K. Shimura reciprocity law gives us a way to compute the Galois conjugates $f(\tau)^\sigma$ of $f(\tau)$ when acting with $\sigma \in \mathrm{Gal}(K^{ab}/K)$. We will follow the exposition of Stevenhagen [21]. For more details also see Gee [6].

We recall that $\mathcal{F} = \bigcup_{N \geq 1} \mathcal{F}_N$, where $\mathcal{F}_N$ is the space of modular functions of level N. Moreover, we can think of $\mathcal{F}_N$ as the function field of the modular curve $X(N) = \Gamma(N) \setminus \mathcal{H}^*$ over $\mathbb{Q}(\zeta_N)$, where $\zeta_N = e^{2\pi i/N}$ and $\mathcal{H}^* = \mathcal{H} \cup \mathbb{P}^1(\mathbb{Q})$. We can compute explicitly $\mathcal{F}_N = \mathbb{Q}(j, j_N)$, where j is the j-invariant and $j_N(z) = j(Nz)$. In particular, we have $\mathcal{F}_1 = \mathbb{Q}(j)$.

When working over $\mathbb{Q}$, one has an isomorphism:

$$\mathrm{Gal}(\mathcal{F}_N/\mathcal{F}_1) \cong \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})/\{\pm 1\}.$$

More precisely, if we denote by γ_σ the Galois action corresponding to the matrix $\gamma \in \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$ under the isomorphism above, it is enough to define the Galois action for $\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$ and for $G_N = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix}, d \in (\mathbb{Z}/N\mathbb{Z})^\times \right\}$. We state explicitly the two actions below.

- **Action of $\alpha \in \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$ on $\mathcal{F}_N$.**

$$(f(\tau))^{\sigma_\alpha} = f^\alpha(\tau) := f(\alpha\tau),$$

  where α is acting on the upper half plane via fractional linear transformations.

- **Action of $\begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix} \in (\mathbb{Z}/N\mathbb{Z})^\times$ on $\mathcal{F}_N$.** Note that for $f \in \mathcal{F}_N$ we have a Fourier expansion $f(z) = \sum_{n \geq 0} a_n q^{n/N}$ with coefficients $a_n \in \mathbb{Q}(\zeta_N)$, $q = e^{2\pi i z}$. If we denote

$u_d := \begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix}$, then the action of σ_{u_d} is given by

$$(f(\tau))^{\sigma_{u_d}} = f^{u_d}(\tau) := \sum_{n\geq 0} a_n^{\sigma_d} q^{n/N},$$

where σ_d is the Galois action in $\mathrm{Gal}(\mathbb{Q}(\zeta_N)/\mathbb{Q})$ that sends $\zeta_N \to \zeta_N^d$.

As the restriction maps between the fields $\mathcal{F}_N$ are in correspondence with the natural maps between the groups $\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})/\{\pm 1\}$ we can take the projective limit to get the isomorphism:

$$\mathrm{Gal}(\mathcal{F}/\mathcal{F}_1) \cong \mathrm{GL}_2(\widehat{\mathbb{Z}})/\{\pm 1\}.$$

To further get all the automorphisms of $\mathcal{F}$ we need to consider the action of $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q},f})$. We get the exact sequence:

$$1 \to \{\pm 1\} \to \mathrm{GL}_2(\mathbb{A}_{\mathbb{Q},f}) \to \mathrm{Aut}(\mathcal{F}) \to 1$$

For this to make sense, we need to extend the action from $\mathrm{GL}_2(\widehat{\mathbb{Z}})$ to $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q},f})$. We do this by using the action of $\mathrm{GL}_2(\mathbb{Q})^+$:

- **Action of $\alpha \in \mathrm{GL}_2(\mathbb{Q})^+$ on $\mathcal{F}$.**

$$f^{\alpha}(\tau) = f(\alpha\tau),$$

where α acts by fractional linear transformations.

We extend the action of $\mathrm{GL}_2(\widehat{\mathbb{Z}})$ to $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$ by writing the elements $\gamma \in \mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$ in the form $\gamma = u\alpha$, where $u \in \mathrm{GL}_2(\widehat{\mathbb{Z}})$ and $\alpha \in \mathrm{GL}_2(\mathbb{Q})^+$. Note that this decomposition is not uniquely determined. However, by combining the two actions of u and α, a well defined action is given by:

$$f^{u\alpha} = (f^u)^{\alpha}.$$

We want to look at the action of $\mathrm{Gal}(K^{ab}/K)$ inside $\mathrm{Aut}(\mathcal{F})$. From class field theory we have the exact sequence:

$$1 \to K^{\times} \to \mathbb{A}_{K,f}^{\times} \xrightarrow{[\cdot,K]} \mathrm{Gal}(K^{ab}/K) \to 1,$$

where $[\cdot, K]$ is the Artin map.

We are going to embed $\mathbb{A}_{K,f}^{\times}$ into $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q},f})$ such that the Galois action of $\mathbb{A}_{K,f}^{\times}$ through the Artin map and the action of the matrices in $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q},f})$ are compatible. We do this by constructing a matrix $g_\tau(x)$ for the idele $x \in \mathbb{A}_{K,f}^{\times}$.

Let $\mathcal{O}$ be the order of K generated by τ i.e. $\mathcal{O} = \mathbb{Z}[\tau]$. We define the matrix $g_\tau(x)$ to be the unique matrix in $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ such that $x \begin{pmatrix} \tau \\ 1 \end{pmatrix} = g_\tau(x) \begin{pmatrix} \tau \\ 1 \end{pmatrix}$. We can compute it explicitly. To do that, consider the minimal polynomial of τ:

$$p(X) = X^2 + BX + C$$

Then if we write $x_p \in \mathbb{Q}_p^\times$ in the form $x_p = s_p\tau + t_p \in \mathbb{Q}_p^\times$ with $s_p, t_p \in \mathbb{Q}_p$, we can compute:

$$g_\tau(x_p) = \begin{pmatrix} t_p - s_pB & -s_pC \\ s_p & t_p \end{pmatrix}$$

Shimura reciprocity law is going to make the following diagram commute:

$$\begin{tikzcd} 1 \arrow[r] & K^\times \arrow[r] & \mathbb{A}_{K,f}^\times \arrow[r, "{[\cdot, K]}"] \arrow[d, "g_\tau"] & \mathrm{Gal}(K^{ab}/K) \arrow[r] & 1 \\ 1 \arrow[r] & \{\pm 1\} \arrow[r] & \mathrm{GL}_2(\mathbb{A}_{\mathbb{Q},f}) \arrow[r] & \mathrm{Aut}(\mathcal{F}) \arrow[r] & 1 \end{tikzcd}$$

We make the statement explicit below:

Theorem 4.1. ***(Shimura reciprocity law)*** *For $f \in \mathcal{F}$ and $x \in \mathbb{A}_{K,f}^\times$, we have:*

$$(f(\tau))^{[x,K]} = f^{g_\tau(x^{-1})}(\tau),$$

where $[x, K]$ is the Galois action corresponding to the idele x via the Artin map, g_τ is defined above and the action of $g_\tau(x)$ is the action in $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q},f})$.

Remark 4.1. Note that the elements of $K^\times$ have trivial action. This can be easily seen by embedding $K^\times \hookrightarrow \mathrm{GL}_2(\mathbb{Q})^+$ given by $k \hookrightarrow g_\tau(k)$. Noting that τ is fixed by the action of the torus $K^\times$, we have:

$$f^{g_\tau(k^{-1})}(\tau) = f(g_\tau(k^{-1})\tau) = f(\tau)$$

Remark 4.2. We can also rewrite the theorem for ideals in K. Let $f \in \mathcal{F}_N$ and $\mathcal{O} = \mathbb{Z}[\tau]$ of conductor M. Going through the Artin map, we can restate Shimura reciprocity in this case in the form:

$$f(\tau)^{\sigma_\mathcal{A}} = f^{g_\tau(\mathcal{A})^{-1}}(\tau), \tag{4.1}$$

where $\sigma_\mathcal{A}$ is the Galois action corresponding to the ideal $\mathcal{A}$ through the Artin map and

$$g_\tau(\mathcal{A}) := g_\tau((\alpha)_{p \mid \mathrm{Nm}(\mathcal{A})}).$$

Note that $g_\tau(\mathcal{A})$ is unique up to multiplication by roots of unity in K. However, these have trivial action on f. This can be easily seen by multiplying by an element of $(\pm\omega^j)_v \in K^\times$ and noticing that we get trivial action at the unramified places $p \nmid MN$.

Remark 4.3. Note that the action of $g_\tau(\mathcal{A})$ is the same as the action of $g_\tau((\alpha)_{p|MN})^{-1}$.

Remark 4.4. Note that the maps above are based on the map between the ideals $\mathcal{A}$ prime to MN and the ideles:

$$I(MN) \to \mathbb{A}_{K,f}^\times / K^\times$$
$$\mathcal{A} = \prod_v \mathfrak{p}_v^{e_v} \to (\varpi_v)_v^{e_v},$$

where ϖ_v is the uniformizer of the ideal $\mathfrak{p}_v$ at the place $v \nmid \infty$.

Applying Shimura reciprocity law to $K = \mathbb{Q}[\sqrt{-3}]$.

Lemma 4.1. *The function* $f(z) = \dfrac{\Theta_K(Dz)}{\Theta_K(z)}$ *is a modular function of level* $3D$ *with integer Fourier coefficients at the cusp* ∞.

Proof. Since $\Theta_K(z)$ is a modular form of weight 1 for $\Gamma_0(3)$, it can be easily seen that $\Theta(Dz)$ is a modular form of weight 1 for $\Gamma(3D)$. Furthermore, their ratio is modular function for $\Gamma_0(3D)$. We check this below. For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(3D)$, we have:

$$f(\gamma z) = \frac{\Theta\left(\begin{pmatrix} D & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} z\right)}{\Theta\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} z\right)} = \frac{\Theta\left(\begin{pmatrix} a & bD \\ c/D & d \end{pmatrix} (Dz)\right)}{\Theta\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} z\right)} = \frac{(cz+d)\Theta(Dz)}{(cz+d)\Theta(z)} = f(z)$$

To find the Fourier expansion of $f(z)$ at ∞, it is enough to write the Fourier expansions of $\Theta(Dz)$ and $\Theta(z)$:

$$\frac{\Theta(Dz)}{\Theta(z)} = \frac{1 + \sum\limits_{N\geq 1} c(N) q^{ND}}{1 + \sum\limits_{N\geq 1} c(N) q^{N}} = \sum_{M \geq 0} a_M q^M$$

We can compute the Fourier coefficients explicitly from the equality:

$$1 + \sum_{N\geq 1} c(N) q^{ND} = (1 + \sum_{N \geq 1} c(N) q^N)(\sum_{M\geq 0} a_M q^M)$$

Note that we have $a_0 = 1$ and $a_M = -a_{M-1}c(1) - a_{M-2}c(2) - \cdots - a_1 c(M-1) - a_0 c(M)$ if $D \nmid M$ and $a_M = c(M/D) - a_{M-1}c(1) - a_{M-2}c(2) - \cdots - a_1 c(M-1) - a_0 c(M)$ if $D|M$. By induction, since $c(N) \in \mathbb{Z}$, we get all the coefficients $a_M \in \mathbb{Z}$. □

$f(\omega)$ **is in the ring class field** H_{3D}**.**

From CM-theory, we have that if $f \in \mathcal{F}_{3D}$ and τ generating $\mathcal{O}_K$, we have $f(\tau) \in H_{3D,\mathcal{O}_K}$ the ray class field of conductor $3D$. We claim that $f(\omega) \in H_{3D}$. Recall that we have $\mathrm{Gal}(K^{ab}/H_{3D}) \cong U(3D) \setminus \mathbb{A}_{K,f}^{\times}/K^{\times}$. Thus in order to show that $f(\omega) \in H_{3D}$, we need to check that $f(\omega)$ is invariant under the action of $U(3D)$.

Lemma 4.2. *For* $\omega = \frac{-1+\sqrt{-3}}{2}$ *and* $f(z) = \frac{\Theta_K(Dz)}{\Theta_K(z)}$ *we have* $f(\omega) \in H_{3D}$.

Proof. In order for $f(\omega) \in H_{3D}$, we need to show that it is invariant under $\mathrm{Gal}(K^{ab}/H_{3D})$. Using Shimura reciprocity law, we need to show:

$$f(\omega) = f^{r_\omega(s)}(\omega),$$

for all $s \in K^{\times}U(3D)$. From Remark 4.1, the action of $K^{\times}$ is trivial. Thus it is enough to show the result for all elements $l = (A_p + B_p\omega)_p \in U(3D)$. By the definition of $U(3D)$, this implies that $A_p + B_p\omega \in (\mathbb{Z}_p[\omega])^{\times}$ for all p and $A_3 \equiv 1 \mod 3$, $B_3 \equiv 1 \mod 3$, $B_p \equiv 0 \mod D$ for all $p|D$. Since the action for $p \nmid 3D$ is trivial, s has the same action $l_D = (A_p + B_p\omega)_{p|3D} \in U(3D)$. Moreover, this has the same action as $l_0 = (A + B\omega)_{p|3D}$, where $A + B\omega \in \mathcal{O}_K$ and $A \equiv A_p \mod 3D\mathbb{Z}_p$ and $B \equiv B_p \mod 3D\mathbb{Z}_p$ for all $p|3D$.

Note further that we can pick A, B such that $(A + B\omega)$ generates a primitive ideal $\mathcal{A}$ in $\mathcal{O}_K$. Moreover, from above we have $3D|B$ and $A \equiv 1 \mod 3$. Recall that we can rewrite any primitive ideal in the form $\mathcal{A} = [a, \frac{-b+\sqrt{-3}}{2}]_{\mathbb{Z}}$, where $a = \mathrm{Nm}\,\mathcal{A}$ and $b^2 \equiv -3 \mod 4a$. Then the generator is $A + B\omega = ta + s\frac{-b+\sqrt{-3}}{2}$ for $t, s \in \mathbb{Z}$, $3D|s$.

Now observe that $f(\omega) = f(\tau)$, where $\tau = \frac{-b+\sqrt{-3}}{2}$, thus from Shimura reciprocity law, we have:

$$(f(\tau))^{\sigma_{l^{-1}}} = f^{r_\tau(l)}(\omega).$$

Here $r_\tau(l) = \begin{pmatrix} A_p - bB_p & -B_p c \\ B_p & A_p \end{pmatrix}_p$ and $r_\tau(l)$ has the same action as $r_\tau(l_0)$, where $l_0 = (A + B\omega)_{p|3D}$ and $A + B\omega = ta + s\frac{-b+\sqrt{-3}}{2}$. Then we need to compute the action of:

$$(f(\tau))^{\sigma_{l^{-1}}} = f^{r_\tau(l_0)}(\tau).$$

Note that $r_\tau(l_0) = \begin{pmatrix} ta-sb & -sc \\ s & ta \end{pmatrix}_{p|3D}$, where $c = \frac{b^2+3}{4}$. Then we can rewrite the action of $r_\tau(l_0)$:

$$f^{r_\tau(l_0)}(\tau) = f^{\begin{pmatrix} ta-sb & -sc/a \\ s & t \end{pmatrix}_{p|3D}\begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}_{p|3D}}(\tau) = f^{\begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}_{p|3D}}\left(\begin{pmatrix} ta-sb & -sc/a \\ s & t \end{pmatrix}\tau\right)$$

Since $a|c$, the matrix $\begin{pmatrix} ta-sb & -sc/a \\ s & t \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ and we can rewrite:

$$f\left(\begin{pmatrix} ta-sb & -sc/a \\ s & t \end{pmatrix} z\right) = \frac{\Theta_K\left(\begin{pmatrix} D & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} ta-sb & -sc/a \\ s & t \end{pmatrix} z\right)}{\Theta_K\left(\begin{pmatrix} ta-sb & -sc/a \\ s & t \end{pmatrix} z\right)}. = \frac{\Theta_K\left(\begin{pmatrix} ta-sb & -scD/a \\ s/D & t \end{pmatrix}(Dz)\right)}{\Theta_K\left(\begin{pmatrix} ta-sb & -sc/a \\ s & t \end{pmatrix} z\right)}.$$

Note that since $3D|s$, we actually have $\left(\begin{smallmatrix} ta-sb & -scD/a \\ s/D & t \end{smallmatrix}\right), \left(\begin{smallmatrix} ta-sb & -sc/a \\ s & t \end{smallmatrix}\right) \in \Gamma_0(3)$ and we can apply the properties of the modular form Θ_K:

$$\frac{\Theta_K\left(\left(\begin{smallmatrix} ta-sb & -scD/a \\ s/D & t \end{smallmatrix}\right)(Dz)\right)}{\Theta_K\left(\left(\begin{smallmatrix} ta-sb & -sc/a \\ s & t \end{smallmatrix}\right)z\right)}. = \frac{(sz+t)^{-1}\Theta_K\left(Dz\right)}{(sz+t)^{-1}\Theta_K\left(z\right)}. = f(z)$$

Finally, note that since $(a, 3D) = 1$ and f has rational coefficients, the action of $\left(\begin{smallmatrix} 1 & 0 \\ 0 & a \end{smallmatrix}\right)_{p|3D}$ is trivial. This finishes the proof that $f(\omega)$ is invariant under the Galois action coming from $U(3D)$, thus $f(\omega) \in H_{3D}$.

$\square$

Remark 4.5. A different proof is shown in the in Appendix B, where we reinterpret the classical Shimura reciprocity law in the setting of Shimura curves following Hida [9].

Galois conjugates of $f(\omega)$.

Let $\mathcal{A} = \left[a, \frac{-b+\sqrt{-3}}{2}\right]_{\mathbb{Z}}$ be a primitive ideal prime to $3D$. For $\tau_1 = \frac{-b+\sqrt{-3}}{2}$, let $\mathcal{O}_D = \mathbb{Z}+D\tau\mathbb{Z}$.

Lemma 4.3. *Let $f \in \mathcal{F}_N$ be a modular function of level N with rational Fourier coefficients in its Fourier expansion. Let $\tau_1 = \frac{-b+\sqrt{-3}}{2}$ be a CM point and let $\mathcal{A} = \left[a, \frac{-b+\sqrt{-3}}{2}\right]$ be a primitive ideal prime to N. Then we have the Galois action:*

$$f(\tau)^{\sigma_{\mathcal{A}}^{-1}} = f(\tau/a)$$

Proof. From Shimura reciprocity (4.1), we have:

$$f(\tau)^{\sigma_{\mathcal{A}}^{-1}} = f^{g_\tau(\mathcal{A})}(\tau).$$

Note that the minimum polynomial of τ is $p_{\tau(X)} = X^2 + bX + \frac{b^2+3}{4}$. Now let $\alpha = ta + s\frac{-b+\sqrt{-3}}{2} = ta + s\tau$ be a generator of $\mathcal{A}$. Then we have $g_\tau(\mathcal{A}) = \left(\begin{smallmatrix} ta-sb & -s\frac{b^2+3}{4} \\ -s & ta \end{smallmatrix}\right)_{p|a}$. We can rewrite the matrix in the form:

$$g_\tau(\mathcal{A}) = \left(\begin{smallmatrix} ta-sb & \frac{b^2+3}{4a} \\ -s & t \end{smallmatrix}\right)_{p|a}\left(\begin{smallmatrix} 1 & 0 \\ 0 & a \end{smallmatrix}\right)_{p|a}$$

As $\left(\begin{smallmatrix} ta-sb & -\frac{b^2+3}{4a} \\ -s & t \end{smallmatrix}\right)_{p|a} \in \mathrm{SL}_2(\mathbb{Z}_p)$ for $p \nmid ND$, it has a trivial action. Then:

$$f^{g_\tau(\mathcal{A})}(\tau) = f^{\left(\begin{smallmatrix} 1 & 0 \\ 0 & a \end{smallmatrix}\right)_{p|a}}(\tau)$$

We rewrite the matrix $\left(\begin{smallmatrix} 1 & 0 \\ 0 & a \end{smallmatrix}\right)_{p|a} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & a \end{smallmatrix}\right)_{p\nmid a}\left(\begin{smallmatrix} 1 & 0 \\ 0 & a \end{smallmatrix}\right)_{\mathbb{Q}}$, where $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1/a \end{smallmatrix}\right)_p \in \mathrm{GL}_2(\widehat{\mathbb{Z}})$ and $\left(\begin{smallmatrix} 1 & 0 \\ 0 & a \end{smallmatrix}\right)_{\mathbb{Q}} \in \mathrm{GL}_2(\mathbb{Q})^+$.

Note that the action of $\left(\begin{smallmatrix}1 & 0\\ 0 & 1/a\end{smallmatrix}\right)_p$ is only given by $\left(\begin{smallmatrix}1 & 0\\ 0 & 1/a\end{smallmatrix}\right)_{p|NM}$. However, since f has rational Fourier coefficients in its Fourier expansion, this action is trivial. Thus we are left with:

$$f^{g_\tau(\mathcal{A})}(\tau) = f^{\left(\begin{smallmatrix}1 & 0\\ 0 & a\end{smallmatrix}\right)_{\mathbb{Q}}}(\tau)$$

This is just $f^{g_\tau(\mathcal{A})}(\tau) = f(\tau/a)$.

$\square$

Proposition 4.1. *Take the primitive ideals* $\mathcal{A} = \left[a, \frac{-b+\sqrt{-3}}{2}\right]_{\mathbb{Z}}$ *to be the representatives of the ring class field* H_{3D} *such that all norms* $\operatorname{Nm}\mathcal{A}$ *are relatively prime to each other and* $b^2 \equiv -3 \mod 4a$ *for all the* $a = \operatorname{Nm}\mathcal{A}$ *chosen.*

Then the only Galois conjugates of $f(\omega) = \dfrac{\Theta_K(D\omega)}{\Theta_K(\omega)}$ *are the following:*

$$\left(\frac{\Theta_K(D\omega)}{\Theta_K(\omega)}\right)^{\sigma_{\mathcal{A}}^{-1}} = \frac{\Theta_K\left(D\frac{-b+\sqrt{-3}}{2a}\right)}{\Theta_K\left(\frac{-b+\sqrt{-3}}{2a}\right)}$$

Proof. Note that $\dfrac{\Theta_K(D\omega)}{\Theta_K(\omega)} = \dfrac{\Theta_K\left(D\frac{-b+\sqrt{-3}}{2}\right)}{\Theta_K\left(\frac{-b+\sqrt{-3}}{2}\right)}$ and apply lemma 4.3 to $\tau = \frac{-b+\sqrt{-3}}{2}$ and $f(z) = \dfrac{\Theta_K(Dz)}{\Theta_K(z)}$. These are the only Galois conjugates we showed that $f(\tau) \in H_{3D}$. $\square$

Chapter 5

Writing S_D as a square.

In this section we will show the following result:

Theorem 5.1. *For* $D = \prod_{p_i \equiv 1 \mod 3} p_i^{e_i}$*, let* $\tau = \frac{-b+\sqrt{-3}}{2}$ *such that* $b^2 \equiv -3 \mod 12D^2$*. Moreover, let* $b^* \equiv b^{-1} \mod D$*.*

Let $H_{\mathcal{O}}$ *be the ray class field of conductor* $3D$ *and let* $H_0 \subset H_{\mathcal{O}}$ *be the subfield of* $H_{\mathcal{O}}$ *that is the fixed field of* $G_0 = \{r \in (\mathbb{Z}/D\mathbb{Z})^\times, r \equiv 1 \mod 6 : \mathcal{A}_r^\circ = \left(1 + b^*(1-r)\frac{-b+\sqrt{-3}}{2}\right)\}$*. Then we have*

$$S_D = |\operatorname{Tr}_{H_{\mathcal{O}}/H_0}(f_1(\tau)D^{2/3})|^2$$

and $S_D \in \mathbb{Z}$*.*

The main tool in proving Theorem 5.1 is a Factorization Formula of Rodriguez-Villegas and Zagier [16]. We will apply the Factorization Formula (5.1) to the formula for the L-function $L(E_D, 1)$ in Theorem 1.1.

Factorization Formula

We recall the version of Factorization Formula ([16], Theorem, page 7) simplified to the case of $\alpha = p = 0$:

Theorem 5.2. ***(Factorization formula.)*** *For* $a \in \mathbb{Z}_{>0}$*,* $\mu, \nu \in \mathbb{Q}$*,* $z = x + yi \in \mathbb{C}$*, we have:*

$$\sum_{m,n\in\mathbb{Z}} e^{2\pi i(m\nu+n\mu)} e^{\pi(imn - \frac{|mz-n|^2}{2y})/a} = \sqrt{2ay}\,\theta\begin{bmatrix}a\mu\\ \nu\end{bmatrix}(a^{-1}z)\cdot\theta\begin{bmatrix}\mu\\ -a\nu\end{bmatrix}(-a\bar{z}), \tag{5.1}$$

where $\theta\begin{bmatrix}\mu\\ \nu\end{bmatrix}(z) = \sum_{n\in\mathbb{Z}+\mu} e^{\pi i n^2 z + 2\pi i\nu n}$ *is a theta function of half integral weight.*

Using the formula above, we will prove the following Proposition:

Proposition 5.1. *For $a \equiv a_1 \equiv 1 \mod 6$, $D \equiv 1 \mod 6$ and $b^2 \equiv -3 \mod 4D^2a^2a_1$, $b \equiv 1 \mod 16$, we have:*

$$\frac{3}{2}\Theta\left(D\frac{-b+\sqrt{-3}}{2a}\right) - \frac{1}{2}\Theta\left(D\frac{-b+\sqrt{-3}}{6a}\right) =$$
$$= \sum_{r\in\mathbb{Z}/D\mathbb{Z}} \frac{\sqrt[4]{3}}{D\sqrt{a_1}} e^{\pi i(a-1)/6}\theta_{ar}\left(\frac{-b+\sqrt{-3}}{2a^2a_1}\right)\theta_r\left(\frac{b+\sqrt{-3}}{2a_1}\right), \tag{5.2}$$

where $\theta_s(z) = \sum\limits_{n\in\mathbb{Z}} e^{\pi i(n+s/D-1/6)^2 z}(-1)^n$ is a theta function of weight $1/2$ for s non-negative integer.

Remark 5.1. Throughout the paper we will use the notation $r \in \mathbb{Z}/D\mathbb{Z}$ to mean any representatives r for the residues $\mod D$.

Remark 5.2. Also note that $\theta_0(z) = \eta(z/3)$, where η is the Dedekind eta function, while $\sum\limits_{r\in\mathbb{Z}/D\mathbb{Z}} \theta_r(z) = \eta\left(\frac{z}{3D^2}\right)$.

We start the proof of Proposition 5.1 by applying the Factorization Formula (5.1) several times for $\mu := \frac{\mu+r}{D}$, where $r \in \mathbb{Z}/D\mathbb{Z}$, and for $z := z/D$. Summing up the formulas, we are going to get the result of the next lemma.

Lemma 5.1. *We have the following factorization formula:*

$$\sum_{r\in\mathbb{Z}/D\mathbb{Z}} \frac{\sqrt{2ay}}{\sqrt{D}}\theta\begin{bmatrix} a(\mu+r)/D \\ \nu \end{bmatrix}\left(D\frac{z}{a}\right)\theta\begin{bmatrix} (\mu+r)/D \\ -a\nu \end{bmatrix}(-aD\overline{z}) =$$

$$= \sum_{m,n\in\mathbb{Z}} e^{2\pi i(m\nu+n\mu)} e^{\pi(mni - \frac{|n-mz|^2}{2y})\frac{D}{a}}$$

Proof. Plugging in $\mu := \frac{\mu+r}{D}$, $z := z/D$ in (5.1), we get:

$$\sqrt{2ay}\theta\begin{bmatrix} a\frac{(\mu+r)}{D} \\ \nu \end{bmatrix}\left(\frac{z}{a}\right)\theta\begin{bmatrix} \frac{(\mu+r)}{D}) \\ -a\nu \end{bmatrix}(-a\overline{z}) = \sum_{m,n\in\mathbb{Z}} e^{2\pi i(m\nu+n(\mu+r)/D)} e^{\pi(mni-\frac{|n-mz|^2}{2y})\frac{1}{a}}$$

We sum for r in $\mathbb{Z}/D\mathbb{Z}$:

$$\sum_{r\in\mathbb{Z}/D\mathbb{Z}} \sqrt{2ay}\theta\begin{bmatrix} a\frac{(\mu+r)}{D} \\ \nu \end{bmatrix}\left(\frac{z}{a}\right)\theta\begin{bmatrix} \frac{(\mu+r)}{D}) \\ -a\nu \end{bmatrix}(-a\overline{z}) = \sum_{r\in\mathbb{Z}/D\mathbb{Z}}\sum_{m,n\in\mathbb{Z}} e^{2\pi i(m\nu+n(\mu+r)/D)} e^{\pi(mni-\frac{|n-mz|^2}{2y})\frac{1}{a}}$$

We change the two sums on the RHS:

$$\sum_{r\in\mathbb{Z}/D\mathbb{Z}}\sum_{m,n\in\mathbb{Z}} e^{2\pi i(m\nu+n(\mu+r)/D)}e^{\pi(mni-\frac{|n-mz|^2}{2y})\frac{1}{a}} = \sum_{m,n\in\mathbb{Z}}\sum_{r\in\mathbb{Z}/D\mathbb{Z}} e^{2\pi i(m\nu+n(\mu+r)/D)}e^{\pi(mni-\frac{|n-mz|^2}{2y})\frac{1}{a}}$$

Note that the LHS can be rewritten as $\sum_{m,n\in\mathbb{Z}} e^{2\pi i(m\nu+n(\mu)/D)}e^{\pi(mni-\frac{|n-mz|^2}{2y})\frac{1}{a}} \sum_{r\in\mathbb{Z}/D\mathbb{Z}} e^{2\pi inr/D}$ and note further that:

$$\sum_{r\in\mathbb{Z}/D\mathbb{Z}} e^{2\pi inr/D} = \sum_{r=0}^{D-1} e^{2\pi inr/D} = \begin{cases} 0, & \text{for } D\nmid n \\ D, & \text{for } D|n \end{cases}$$

Thus we are only summing over the n's that are multiples of D, and the RHS of the formula becomes:

$$\sum_{r\in\mathbb{Z}/D\mathbb{Z}}\sum_{m,n\in\mathbb{Z}} e^{2\pi i(m\nu+n(\mu+r)/D)}e^{\pi(mni-\frac{|n-mz|^2}{2y})\frac{1}{a}} = D\sum_{m,n'\in\mathbb{Z}} e^{2\pi i(m\nu+n'(\mu+r))}e^{\pi(mn'i-\frac{|n'-m(z/D)|^2}{2(y/D)})\frac{D}{a}}$$

Going back to our initial equality, we can change the variable z to $z' := z/D$ and get:

$$\sqrt{2aDy'}\sum_{r\in\mathbb{Z}/D\mathbb{Z}}\theta\begin{bmatrix} a\frac{(\mu+r)}{D} \\ \nu \end{bmatrix}\left(D\frac{z'}{a}\right)\theta\begin{bmatrix} \frac{(\mu+r)}{D}) \\ -a\nu \end{bmatrix}\left(-aD\overline{z'}\right) =$$

$$= D\sum_{m,n'\in\mathbb{Z}} e^{2\pi i(m\nu+n'(\mu+r))}e^{\pi(mn'i-\frac{|n'-mz'|^2}{2y'})\frac{D}{a}}$$

□

Corollary 5.1. *Another version of the factorization lemma above is:*

$$\sum_{r\in\mathbb{Z}/D\mathbb{Z}}\frac{\sqrt{2ay}}{\sqrt{D}}\theta\begin{bmatrix} a\mu+ar/D \\ \nu \end{bmatrix}\left(D\frac{z}{a}\right)\theta\begin{bmatrix} \mu+r/D \\ -a\nu \end{bmatrix}(-aD\overline{z}) = \sum_{m,n\in\mathbb{Z}} e^{2\pi i(m\nu+nD\mu)}e^{\pi(mni-\frac{|n-mz|^2}{2y})\frac{D}{a}} \tag{5.3}$$

Proof. We apply the previous factorization lemma for $\mu := D\mu$.

□

We will apply Corollary 5.1 for $\mu = -1/6$ and $\nu = 1/2$, D odd, $z = \frac{-b+\sqrt{-3}}{2aa_1}$, where $b^2 \equiv -3 \mod 4a^2a_1D$ and $b \equiv 1 \mod 16$. This gives us:

$$\sum_{r\in\mathbb{Z}/D\mathbb{Z}}\sqrt{2ay/D}\theta\begin{bmatrix} -\frac{a}{6}+\frac{ar}{D} \\ 1/2 \end{bmatrix}\left(D\frac{-b+\sqrt{-3}}{2a^2a_1D}\right)\theta\begin{bmatrix} -\frac{1}{6}+\frac{r}{D} \\ -a/2 \end{bmatrix}\left(D\frac{b+\sqrt{-3}}{2Da_1}\right) =$$

$$= \sum_{m,n\in\mathbb{Z}} e^{2\pi i(m/2-nD/6)} e^{\pi(mni - \frac{|nDaa_1 - m\frac{-b+\sqrt{-3}}{2}|^2}{Daa_1\sqrt{3}})\frac{D}{a}}$$

We will analyze first the LHS of the equation. Note that from the definition of $\theta\begin{bmatrix}\mu\\ \nu\end{bmatrix}$ we have:

$$\theta\begin{bmatrix}-\frac{1}{6}+\frac{r}{D}\\ -a/2\end{bmatrix}(z) = \sum_{n\in\mathbb{Z}} e^{\pi i\left(n+\frac{r}{D}-\frac{1}{6}\right)^2 z} e^{-a\pi i\left(n+\frac{r}{D}-\frac{1}{6}\right)} = e^{-a\pi ir/D} e^{a\pi i/6}\theta_r(z).$$

Similarly, as $a \equiv 1 \mod 6$, we have $\theta\begin{bmatrix}-\frac{a}{6}+\frac{ar}{D}\\ 1/2\end{bmatrix}(z) = \theta\begin{bmatrix}-\frac{1}{6}+\frac{ar}{D}\\ 1/2\end{bmatrix}(z)$. Then from the definition of θ_r we get:

$$\theta\begin{bmatrix}-\frac{a}{6}+\frac{ar}{D}\\ 1/2\end{bmatrix}(z) = \sum_{n\in\mathbb{Z}} e^{\pi i\left(n+\frac{ar}{D}-\frac{1}{6}\right)^2 z} e^{\pi i\left(n+\frac{ar}{D}-\frac{1}{6}\right)} = e^{\pi iar/D} e^{-\pi i/6}\theta_{ar}(z)$$

Also, since $D \equiv 1 \mod 6$ we also have: $e^{-2\pi inD/6} = e^{-2\pi in/6}$. We can also compute: $\frac{\sqrt{2ay}}{\sqrt{D}} = \frac{\sqrt{2a\frac{\sqrt{3}}{2aa_1D}}}{\sqrt{D}} = \frac{\sqrt[4]{3}}{D\sqrt{a_1}}$. Thus we can rewrite the formula:

$$\frac{\sqrt[4]{3}}{D\sqrt{a_1}} \sum_{r\in\mathbb{Z}/D\mathbb{Z}} e^{\pi i(a-1)/6}\theta_{ar}\left(\frac{-b+\sqrt{-3}}{2a^2a_1}\right)\theta_r\left(\frac{b+\sqrt{-3}}{2a_1}\right) =$$

$$= \sum_{m,n\in\mathbb{Z}} e^{2\pi i(m/2-n/6)} e^{\pi(mni - \frac{|nDaa_1 - m\frac{-b+\sqrt{-3}}{2}|^2}{Daa_1\sqrt{3}})\frac{D}{a}} \tag{5.4}$$

Now we are going to analyze below the RHS of the equation (5.4):

$$\sum_{m,n\in\mathbb{Z}} e^{2\pi i(m/2-n/6)} e^{\pi(mni - \frac{|nDaa_1 - m\frac{-b+\sqrt{-3}}{2}|^2}{Daa_1\sqrt{3}})\frac{D}{a}}$$

First note that we have the following lemma:

Lemma 5.2. *For $b \equiv 1 \mod 16, b \equiv 0 \mod 3, b^2 \equiv -3 \mod 4a^2a_1D$, we have:*

$$e^{2\pi i(m/2+n/2)} e^{\pi(mni - \frac{|naa_1D - m\frac{-b+\sqrt{-3}}{2}|^2}{aa_1D\sqrt{3}})\frac{D}{a}} = e^{2\pi i\frac{|naa_1D - m\frac{-b+\sqrt{-3}}{2}|^2}{aa_1D} D\frac{-b+\sqrt{-3}}{6a}}$$

Proof. We only need to show that:

$$2\pi i\left(\frac{m}{2}+\frac{n}{2}+\frac{Dmn}{2a}\right) \equiv -2\pi i\frac{|naa_1D - m\frac{-b+\sqrt{-3}}{2}|^2}{aa_1D} D\frac{b}{6a} \mod 2\pi i\mathbb{Z}.$$

After dividing by $2\pi i$, we compute the RHS of the identity:

$$\frac{|naa_1D - m\frac{-b+\sqrt{-3}}{2}|^2}{aa_1D} D\frac{b}{6a} = \left(Dm^2\frac{b(b^2+3)}{24a^2a_1} - D\frac{b^2mn}{6a} + \frac{Dba_1n^2}{6}\right)$$

Thus our claim turns into:

$$\left(\frac{m}{2} + \frac{n}{2} + \frac{Dmn}{2a}\right) \equiv \left(Dm^2\frac{b(b^2+3)}{24a^2a_1} - D\frac{b^2mn}{6a} + \frac{Dba_1n^2}{6}\right) \mod \mathbb{Z}$$

Equivalently:

$$\frac{m}{2} + \frac{n}{2} \equiv \left(D\frac{m^2}{2}\frac{b}{3}\frac{(b^2+3)}{4a^2a_1} - D\frac{(b^2+3)mn}{6a} + \frac{n^2}{2}\frac{b}{3}Da_1\right) \mod \mathbb{Z}$$

We have $b^2 \equiv -3 \mod 4aa_1^2$, $b \equiv 1 \mod 16$, $b \equiv 0 \mod 3$. Note that this implies that b is odd and that $b^2+3 \equiv 4 \mod 8$, as well as $b^2+3 \equiv 0 \mod 3$. Then, since a, a_1, D are odd, we get:

- $m/2 \equiv m^2/2 \equiv D\frac{m^2}{2}\frac{b}{3}\frac{(b^2+3)}{4a^2a_1} \mod \mathbb{Z}$
- $n/2 \equiv n^2/2 \equiv \frac{n^2}{2}\frac{b}{3}Da_1 \mod \mathbb{Z}$
- $-D\frac{(b^2+3)mn}{6a} \in \mathbb{Z}$

This finishes the proof.

□

Lemma 5.3. *Under the same conditions as above we have:*

$$\sum_{m,n\in\mathbb{Z}} e^{2\pi i n/3} e^{2\pi i \frac{|m\cdot\frac{b+\sqrt{-3}}{2}+naa_1|^2}{aa_1}\cdot z} = \frac{3}{2}\Theta(3z) - \frac{1}{2}\Theta(z),$$

where $z \in \mathcal{H}$, $\mathcal{A}\mathcal{A}_1 = [aa_1, \frac{-b+\sqrt{-3}}{2}]$ *and* $b \equiv 0 \mod 3$, $b^2 \equiv -3 \mod 4aa_1$.

Proof. Note first that by changing $m \to -m$ and $-m\cdot\frac{-b+\sqrt{-3}}{2} + naa_1$ to its conjugate, we have $\sum_{m,n\in\mathbb{Z}} e^{2\pi i n/3} e^{2\pi i \frac{|-m\cdot\frac{-b+\sqrt{-3}}{2}+naa_1|^2}{aa_1}z} = \sum_{m,n\in\mathbb{Z}} e^{2\pi i n/3} e^{2\pi i \frac{|m\cdot\frac{b+\sqrt{-3}}{2}+naa_1|^2}{aa_1}z}$.

We can split the sum in three terms, depending on $n \mod 3$:

$\sum_{m,3|n\in\mathbb{Z}} e^{\frac{2\pi i|m\cdot\frac{b+\sqrt{-3}}{2}+naa_1|^2}{aa_1}\cdot z} + \omega \sum_{m,n\in\mathbb{Z}, n\equiv 1 \mod 3} e^{2\pi i\frac{|m\cdot\frac{b+\sqrt{-3}}{2}+naa_1|^2}{aa_1}\cdot z} +$

$$+\omega^2 \sum_{m,n\in\mathbb{Z}, n\equiv 2 \mod 3} e^{2\pi i \frac{|m\cdot\frac{b+\sqrt{-3}}{2}+naa_1|^2}{aa_1}\cdot z}$$

Note that the first term equals $\sum_{m,n\in\mathbb{Z}} e^{2\pi i \frac{|m\cdot\frac{b+\sqrt{-3}}{2}+n3aa_1|^2}{3aa_1}\cdot 3z} = \Theta_K(3z)$.

Also note that the two terms

$$\sum_{m,n\in\mathbb{Z}, n\equiv 1 \mod 3} e^{2\pi i \frac{|m\cdot\frac{b+\sqrt{-3}}{2}+naa_1|^2}{aa_1}\cdot z}$$

and

$$\sum_{m,n\in\mathbb{Z}, n\equiv 2 \mod 3} e^{2\pi i \frac{|m\cdot\frac{b+\sqrt{-3}}{2}+naa_1|^2}{aa_1}\cdot z}$$

equal each other, by changing in the latter $n \to -n$ and $m \to -m$. Thus we got so far:

$$\Theta(3z) + (\omega+\omega^2) \sum_{m,n\in\mathbb{Z}, n\equiv 1 \mod 3} e^{2\pi i \frac{|m\cdot\frac{b+\sqrt{-3}}{2}+naa_1|^2}{aa_1}\cdot z}$$

Furthermore, we have:

$$\sum_{m,n\in\mathbb{Z}, n\equiv 1 \mod 3} e^{2\pi i \frac{|m\cdot\frac{b+\sqrt{-3}}{2}+naa_1|^2}{aa_1}\cdot z} = \frac{1}{2} \sum_{m,n\in\mathbb{Z},(n,3)=1} e^{2\pi i \frac{|m\cdot\frac{b+\sqrt{-3}}{2}+naa_1|^2}{aa_1}\cdot z}$$

Finally, this is just:

$$\frac{1}{2}\sum_{m,n\in\mathbb{Z}} e^{2\pi i \frac{|m\cdot\frac{b+\sqrt{-3}}{2}+naa_1|^2}{aa_1}\cdot z} - \frac{1}{2}\sum_{m,n\in\mathbb{Z}} e^{2\pi i \frac{|m\cdot\frac{b+\sqrt{-3}}{2}+3naa_1|^2}{3aa_1}\cdot 3z} = \frac{1}{2}(\Theta(z) - \Theta(3z))$$

Finally, we get $\sum_{m,n\in\mathbb{Z}} e^{2\pi i n/3} e^{2\pi i \frac{|m\cdot\frac{b+\sqrt{-3}}{2}+naa_1|^2}{aa_1}\cdot z} = \Theta(3z) - \frac{1}{2}(\Theta(z)-\Theta(3z)) = \frac{3}{2}\Theta(3z) - \frac{1}{2}\Theta(z)$.

$\square$

From the previous two lemmas, we get the following corollary:

Corollary 5.2. *Under the above conditions, we have:*

$$\sum_{m,n\in\mathbb{Z}} e^{2\pi i(m/2-n/6)} e^{\pi(mni - \frac{|naa_1D - m\frac{-b+\sqrt{-3}}{2}|^2}{aa_1D\sqrt{3}})\frac{D}{a}} = \frac{3}{2}\Theta\left(D\frac{-b+\sqrt{-3}}{2a}\right) - \frac{1}{2}\Theta\left(D\frac{-b+\sqrt{-3}}{6a}\right)$$

Proof. Note that we can rewrite the LHS in the form:

$$\sum_{m,n\in\mathbb{Z}} e^{2\pi i(m/2-n/6)} e^{\pi(mni-\frac{|naa_1D-m\frac{-b+\sqrt{-3}}{2}|^2}{aa_1D\sqrt{3}})\frac{D}{a}} = \sum_{m,n\in\mathbb{Z}} e^{2\pi i(m/2-n/2+n/3)} e^{\pi(mni-\frac{|naa_1D-m\frac{-b+\sqrt{-3}}{2}|^2}{aa_1D\sqrt{3}})\frac{D}{a}}$$

Then, from Lemma 5.2, we have:

$$\sum_{m,n\in\mathbb{Z}} e^{2\pi i(m/2-n/6)} e^{\pi(mni-\frac{|naa_1D-m\frac{-b+\sqrt{-3}}{2}|^2}{aa_1D\sqrt{3}})\frac{D}{a}} = \sum_{m,n\in\mathbb{Z}} e^{2\pi in/3} e^{\frac{|naa_1D-m\frac{-b+\sqrt{-3}}{2}|^2}{aa_1D})D\frac{-b+\sqrt{-3}}{6a}}$$

Now apply Lemma 5.3 for $z = D\frac{-b+\sqrt{-3}}{6a}$, we get:

$$\sum_{m,n\in\mathbb{Z}} e^{2\pi in/3} e^{\frac{|naa_1D-m\frac{-b+\sqrt{-3}}{2}|^2}{aa_1D}D\frac{-b+\sqrt{-3}}{6a}} = \frac{3}{2}\Theta\left(D\frac{-b+\sqrt{-3}}{2a}\right) - \frac{1}{2}\Theta\left(D\frac{-b+\sqrt{-3}}{6a}\right)$$

Finally, from (5.4) and Corollary 5.2 we get the result of Proposition 5.1:

$$\frac{3}{2}\Theta\left(D\frac{-b+\sqrt{-3}}{2a}\right) - \frac{1}{2}\Theta\left(D\frac{-b+\sqrt{-3}}{6a}\right) =$$
$$= \frac{\sqrt[4]{3}}{D\sqrt{a_1}} \sum_{r\in\mathbb{Z}/D\mathbb{Z}} e^{\pi i(a-1)/6}\theta_{ar}\left(\frac{-b+\sqrt{-3}}{2a^2a_1}\right)\theta_r\left(\frac{b+\sqrt{-3}}{2a_1}\right).$$

□

A particular case of Proposition 5.1 is going to be the following result:

Corollary 5.3. *For $b^2 \equiv -3 \mod 12a^2a_1$, $b \equiv 1 \mod 16$, we have:*

$$\frac{3}{2}\Theta\left(\frac{-b+\sqrt{-3}}{2a}\right) = \frac{\sqrt[4]{3}}{\sqrt{a_1}} e^{\pi i(a-1)\frac{1}{6}}\theta_0\left(\frac{-b+\sqrt{-3}}{2a^2a_1}\right)\theta_0\left(\frac{b+\sqrt{-3}}{2a_1}\right),$$

where $\theta_0(z) = \sum_{n\in\mathbb{Z}} e^{\pi i(n-1/6)^2z}(-1)^n$.

Proof. Applying the Proposition 5.1 for $D = 1$ we get:

$$\frac{3}{2}\Theta\left(\frac{-b+\sqrt{-3}}{2a}\right) - \frac{1}{2}\Theta\left(\frac{-b+\sqrt{-3}}{6a}\right) = \frac{\sqrt[4]{3}}{\sqrt{a_1}} e^{\pi i(a-1)\frac{1}{6}}\theta_0\left(\frac{-b+\sqrt{-3}}{2a^2a_1}\right)\theta_0\left(\frac{b+\sqrt{-3}}{2a_1}\right).$$

Furthermore, using the result from Appendix A, Lemma 9.3 that $\Theta\left(\frac{-b+\sqrt{-3}}{6a}\right) = 0$, we get the result of the Corollary.

□

We further take the ratios of the theta functions in Proposition 5.1 and Corollary 5.3 to get the following corollary.

Corollary 5.4. *Under the same conditions as above, we have:*

$$\frac{\Theta\left(D\frac{-b+\sqrt{-3}}{2a}\right)}{\Theta\left(\frac{-b+\sqrt{-3}}{2a}\right)}-\frac{1}{3}\frac{\Theta\left(D\frac{-b+\sqrt{-3}}{6a}\right)}{\Theta\left(\frac{-b+\sqrt{-3}}{2a}\right)}=\sum_{r\in\mathbb{Z}/D\mathbb{Z}}\frac{\theta_{ar}\left(\frac{-b+\sqrt{-3}}{2a^2a_1}\right)\theta_r\left(\frac{b+\sqrt{-3}}{2a_1}\right)}{\theta_0\left(\frac{-b+\sqrt{-3}}{2a^2a_1}\right)\theta_0\left(\frac{b+\sqrt{-3}}{2a_1}\right)}$$

Proof. We begin by writing the ratio of the formulas in Proposition 5.1 and Corollary 5.3:

$$\frac{\Theta\left(D\frac{-b+\sqrt{-3}}{2a}\right)}{\Theta\left(\frac{-b+\sqrt{-3}}{2a}\right)}-\frac{1}{3}\frac{\Theta\left(D\frac{-b+\sqrt{-3}}{6a}\right)}{\Theta\left(\frac{-b+\sqrt{-3}}{2a}\right)}=\frac{\frac{\sqrt[4]{3}}{a_1}e^{\pi i(a-1)/6}\sum_{r\in\mathbb{Z}/D\mathbb{Z}}\theta_{ar}\left(\frac{-b+\sqrt{-3}}{2a^2a_1}\right)\theta_r\left(\frac{b+\sqrt{-3}}{2a_1}\right)}{\frac{\sqrt[4]{3}}{a_1}e^{\pi i(a-1)/6}\theta_0\left(\frac{-b+\sqrt{-3}}{2a^2a_1}\right)\theta_0\left(\frac{b+\sqrt{-3}}{2a_1}\right)}$$

Simplifying, we get the result of the Corollary. □

Ratios of θ_r and θ_0

Now we will apply the Factorization Lemma once more to connect the theta functions θ_r to the theta function θ_0. We do this by applying the Factorization Formula (5.1) twice and comparing the results.

Note first that any primitive ideal $\mathcal{A}$ in $\mathcal{O}_K$ prime to 6 has a generator $(n_a a+m_a\frac{-b+\sqrt{-3}}{2})$ such that $a=\mathrm{Nm}(\mathcal{A})$, $b^2\equiv -3 \mod 12a$ and $n_a\equiv 1 \mod 3$. Moreover, note that $a=n_a'^2a^2+m_a^2\frac{b^2+3}{4}-m_an_aab$, thus $m_an_ab\equiv 1 \mod a$, as $a|(b^2+3)/4$.

Using this notation, we have:

Lemma 5.4. *For* $b\equiv 0 \mod 3$, $b^2\equiv -3 \mod 4D^2aa'$, $n_{a'}\equiv 1 \mod 3$, *we have:*

$$\theta_r\left(\frac{-b+\sqrt{-3}}{2aa'}\right)\theta_0\left(\frac{b+\sqrt{-3}}{2D^2aa'}\right)=\frac{1}{\sqrt{a'}}\theta_{n_{a'}r}\left(\frac{-b+\sqrt{-3}}{2a}\right)\theta_0\left(\frac{b+\sqrt{-3}}{2D^2a}\right)$$

Proof. We write the generator of $\mathcal{A}'$ in the form $(n_{a'}a'+m_{a'}\frac{-b+\sqrt{-3}}{2})$, where $b^2\equiv -3 \mod 4aa'D^2$. Moreover, we can pick $n_{a'}\equiv 1 \mod 3$. Then, using the Factorization Formula (5.1) for $\mu=-\frac{1}{6}+\frac{r}{D}$, $\nu=\frac{1}{2}$, $a:=D$ and $z=\frac{-b+\sqrt{-3}}{2aa'D}$, we have:

$$\frac{\sqrt[4]{3}}{2\sqrt{aa'}}\theta\begin{bmatrix}-\frac{1}{6}+\frac{r}{D}\\ D/2\end{bmatrix}\left(D\frac{-b+\sqrt{-3}}{2aa'D}\right)\theta\begin{bmatrix}-\frac{D}{6}\\ -1/2\end{bmatrix}\left(\frac{b+\sqrt{-3}}{2D^2aa'}\right)=$$

$$=\sum_{m,n}e^{2\pi i\frac{nr}{D}}e^{2\pi i\frac{n}{3}}e^{2\pi i(\frac{mn}{2D}+\frac{m}{2}+\frac{n}{2})}e^{2\pi i\frac{|m\frac{-b+\sqrt{-3}}{2}+naa'D|^2}{aa'D}\frac{\sqrt{-3}}{6D}}$$

Note that on the LHS we have

$$\theta\begin{bmatrix}-\frac{1}{6}+\frac{r}{D}\\ D/2\end{bmatrix}(z)=e^{-\pi i/6}e^{\pi ir}\theta_r(z)$$

and

$$\theta\begin{bmatrix}-\frac{D}{6}\\ -1/2\end{bmatrix}=e^{\pi i/6}\theta_r(z).$$

Furthermore, using Lemma 5.2, the RHS equals:

$$\sum_{m,n}e^{2\pi i\frac{nr}{D}}e^{2\pi i\frac{n}{3}}e^{2\pi i\frac{|m\frac{-b+\sqrt{-3}}{2}+naa'D|^2}{aa'D}\frac{-b+\sqrt{-3}}{6D}}.$$

Thus we got:

$$\frac{\sqrt[4]{3}}{2\sqrt{aa'}}e^{\pi ir}\theta_r\left(D\frac{-b+\sqrt{-3}}{2aa'D}\right)\theta_0\left(\frac{b+\sqrt{-3}}{2D^2aa'}\right)=\sum_{m,n}e^{2\pi i\frac{nr}{D}}e^{2\pi i\frac{n}{3}}e^{2\pi i\frac{|m\frac{-b+\sqrt{-3}}{2}+naa'D|^2}{aa'D}\frac{-b+\sqrt{-3}}{6D}}. \tag{5.5}$$

Note that if we write any element of $\mathcal{A}\mathcal{A}'D$, we can write it as an element of $\mathcal{A}D$ multiplied by the generator of $\mathcal{A}'$. Thus if we write an element of $\mathcal{A}\mathcal{A}'D$, in the form $m\frac{-b+\sqrt{-3}}{2}+naa'D$, it is going to equal an element $m_0\frac{-b+\sqrt{-3}}{2}+n_0aD\in\mathcal{A}D$ times the generator $m_{a'}\frac{-b+\sqrt{-3}}{2}+n_{a'}a'$ of $\mathcal{A}'$:

$$m\frac{-b+\sqrt{-3}}{2}+naa'D=(m_0\frac{-b+\sqrt{-3}}{2}+n_0aD)(m_{a'}\frac{-b+\sqrt{-3}}{2}+n_{a'}a')$$

This gives us:

$$\begin{cases}m=m_0m_{a'}+n_0n_{a'}-m_0m_{a'}b\\ n=n_0n_{a'}-m_0m_{a'}\frac{b^2+3}{4aa'D}\end{cases}$$

Since $b^2+3\equiv 0 \mod 4D^2$, it implies that $n\equiv n_0n_{a'} \mod D$. Then we have:

$$\sum_{m,n}e^{2\pi i\frac{nr}{D}}e^{2\pi i\frac{n}{3}}e^{2\pi i\frac{|m\frac{-b+\sqrt{-3}}{2}+naa'D|^2}{aa'D}\frac{-b+\sqrt{-3}}{6D}}=\sum_{m_0,n_0}e^{2\pi i\frac{n_0n_{a'}r}{D}}e^{2\pi i\frac{n_0n_{a'}}{3}}e^{2\pi i\frac{|m_0\frac{-b+\sqrt{-3}}{2}+n_0aD|^2}{aD}\frac{-b+\sqrt{-3}}{6D}}$$

Since we picked $n_{a'}\equiv 1 \mod 3$, this is the same as

$$\sum_{m_0,n_0}e^{2\pi i\frac{n_0n_{a'}r}{D}}e^{2\pi i\frac{n_0}{3}}e^{2\pi i\frac{|m_0\frac{-b+\sqrt{-3}}{2}+n_0aD|^2}{aD}\frac{-b+\sqrt{-3}}{6D}}.$$

Then applying the Factorization Formula (5.1) again for $\mu:=-\frac{1}{6}+\frac{n'_ar}{D}$, $\nu:=\frac{1}{2}$, $a:=D$ and $z:=\frac{-b+\sqrt{-3}}{2aD}$, we get:

$$\sum_{m_0,n_0} e^{2\pi i \frac{n_0 n_{a'} r}{D}} e^{2\pi i \frac{n_0}{3}} e^{2\pi i \frac{|m_0 \frac{-b+\sqrt{-3}}{2} + n_0 aD|^2}{aD} \frac{-b+\sqrt{-3}}{6D}} =$$

$$= \frac{\sqrt[4]{3}}{2\sqrt{a}} \theta \begin{bmatrix} -\frac{1}{6} + \frac{n'_a r}{D} \\ D/2 \end{bmatrix} \left(D \frac{-b+\sqrt{-3}}{2aD} \right) \theta \begin{bmatrix} -\frac{D}{6} \\ 1/2 \end{bmatrix} \left(\frac{-b+\sqrt{-3}}{2D^2 a}. \right)$$

Moreover, on the RHS we have the theta functions $\theta \begin{bmatrix} -\frac{1}{6} + \frac{n'_a r}{D} \\ D/2 \end{bmatrix} (z) = e^{-\pi i/6} e^{\pi i n'_a r} \theta_r(z)$ and $\theta \begin{bmatrix} -\frac{D}{6} \\ 1/2 \end{bmatrix} (z) = e^{\pi i/6} \theta_0(z)$. Thus we can rewrite the equality as:

$$\sum_{m_0,n_0} e^{2\pi i \frac{n_0 n_{a'} r}{D}} e^{2\pi i \frac{n_0}{3}} e^{2\pi i \frac{|m_0 \frac{-b+\sqrt{-3}}{2} + n_0 aD|^2}{aD} \frac{-b+\sqrt{-3}}{6D}} = \tag{5.6}$$

$$\frac{\sqrt[4]{3}}{2\sqrt{a}} e^{\pi i n'_a r} \theta_{n'_a r} \left(D \frac{-b+\sqrt{-3}}{2aD} \right) \theta_r \left(\frac{-b+\sqrt{-3}}{2D^2 a}. \right) \tag{5.7}$$

Comparing the two relations (5.5) and (5.7), we get:

$$\frac{1}{\sqrt{a'}} e^{\pi i r} \theta_r \left(\frac{-b+\sqrt{-3}}{2aa'} \right) \theta_0 \left(\frac{b+\sqrt{-3}}{2D^2 aa'} \right) = e^{\pi i n'_a r} \theta_{n_{a'} r} \left(\frac{-b+\sqrt{-3}}{2a} \right) \theta_0 \left(\frac{b+\sqrt{-3}}{2D^2 a} \right)$$

□

Lemma 5.5. *Under the same conditions as above, we have:*

$$\frac{e^{\pi i r} \theta_r \left(\frac{-b+\sqrt{-3}}{2aa'} \right)}{\theta_0 \left(\frac{-b+\sqrt{-3}}{2aa' D^2} \right)} = \frac{e^{\pi i n_{a'} r} \theta_{n_{a'} r} \left(\frac{-b+\sqrt{-3}}{2a} \right)}{\theta_0 \left(\frac{-b+\sqrt{-3}}{2aa' D^2} \right)}$$

Proof. Note that from Corollary 5.3, we have $\frac{3}{2}\Theta\left(\frac{-b+\sqrt{-3}}{2}\right) = \frac{\sqrt[4]{3}}{D\sqrt{aa'}} \theta_0 \left(\frac{-b+\sqrt{-3}}{2aa'D^2} \right) \theta_0 \left(\frac{b+\sqrt{-3}}{2aa'D^2} \right)$.

Moreover, we also have from the same corollary that

$$\frac{3}{2}\Theta\left(\frac{-b+\sqrt{-3}}{2}\right) = \frac{\sqrt[4]{3}}{D\sqrt{a}} \theta_0 \left(\frac{-b+\sqrt{-3}}{2aD^2} \right) \theta_0 \left(\frac{b+\sqrt{-3}}{2aD^2} \right),$$

thus:

$$\frac{1}{\sqrt{a'}} \theta_0 \left(\frac{-b+\sqrt{-3}}{2aa'D^2} \right) \theta_0 \left(\frac{b+\sqrt{-3}}{2aa'D^2} \right) = \theta_0 \left(\frac{-b+\sqrt{-3}}{2aD^2} \right) \theta_0 \left(\frac{b+\sqrt{-3}}{2aD^2} \right)$$

Recall from the previous Lemma that we also have:

$$\frac{1}{\sqrt{a'}} e^{\pi i r} \theta_r \left(\frac{-b+\sqrt{-3}}{2aa'} \right) \theta_0 \left(\frac{b+\sqrt{-3}}{2aa'D^2} \right) = e^{\pi i n_{a'} r} \theta_{n_{a'} r} \left(\frac{-b+\sqrt{-3}}{2a} \right) \theta_0 \left(\frac{b+\sqrt{-3}}{2aD^2} \right)$$

Dividing the two relations, we get exactly:

$$\frac{e^{\pi i r}\theta_r\left(\frac{-b+\sqrt{-3}}{2aa'}\right)}{\theta_0\left(\frac{-b+\sqrt{-3}}{2aa'D^2}\right)} = \frac{e^{\pi i n_{a'} r}\theta_{n_{a'}r}\left(\frac{-b+\sqrt{-3}}{2a}\right)}{\theta_0\left(\frac{-b+\sqrt{-3}}{2aD^2}\right)}$$

□

Applying the factorization lemma to get a square.

We would like to apply the factorization lemma for the formula in Theorem 1.1 for certain ideals that are representatives of the ring class field $\mathrm{Cl}(\mathcal{O}_{3D})$. We will pick these ideals below.

Representatives of $\mathrm{Cl}(\mathcal{O}_{3D})$.

Recall that, using Cox [4], for the ideal class group of conductor $3D$, we have:

$$\mathrm{Cl}(\mathcal{O}_{3D}) = (\mathcal{O}_{3D}/3D\mathcal{O}_K)^\times/(\mathbb{Z}/3D\mathbb{Z})^\times(\mathcal{O}_K^\times/\{\pm 1\})$$

Moreover,we can compute explicitly that for $D = \prod_{p_i \equiv 1 \mod 3} p_i$ we have $\mathrm{Cl}(\mathcal{O}_{3D}) \cong (\mathbb{Z}/D\mathbb{Z})^\times$ which also gives us $\#\,\mathrm{Cl}(\mathcal{O}_{3D}) = \phi(D)$, where ϕ is Euler's totient function.

Furthermore, we are claiming that we can take as representatives of $\mathrm{Cl}(\mathcal{O}_{3D})$ ideals with norm $\mathrm{Nm}\,\mathcal{A}_k \equiv k \mod D$ for $k \in (\mathbb{Z}/D\mathbb{Z})^\times$. We construct these ideals in the following lemma:

Lemma 5.6. *We can take as representatives of* $\mathrm{Cl}(\mathcal{O}_{3D})$ *the ideals:*

$$\mathcal{A}_k = \left(n_k a_k + m_k \frac{-b+\sqrt{-3}}{2}\right),$$

where $\mathrm{Nm}\,\mathcal{A}_k = a_k \equiv k \mod D$ *for* $k \in (\mathbb{Z}/D\mathbb{Z})^\times$*,* $a_k \equiv 1 \mod 6$ *and* $n_k \equiv 1 \mod D$*. We can pick such an ideal if we take* $m_k \equiv b^{-1}(k+1) \mod D$*. We can further put the conditions* $n_k, m_k \equiv 1 \mod 3$ *to determine the ideal uniquely modulo* $3D$*.*

Proof. Note first that two ideals $\mathcal{A}, \mathcal{B}$ are in the same class in $\mathrm{Cl}(\mathcal{O}_{3D})$ if we can find generators α, β for $\mathcal{A}$ and $\mathcal{B}$, respectively, such that $\alpha\beta^{-1} \equiv m \mod 3D$, where m is an integer prime to $3D$. Note that this implies $\alpha\beta^{-1} \equiv \pm 1 \mod 3$.

Let us assume that $\mathcal{A}_k$ and $\mathcal{A}_l$ are in the same class in $\mathrm{Cl}(\mathcal{O}_{3D})$. Then we must have $\pm\omega^i\left(n_k a_k + m_k \frac{-b+\sqrt{-3}}{2}\right) \equiv \pm\omega^j R\left(n_l a_l + m_l \frac{-b+\sqrt{-3}}{2}\right) \mod 3D$ for some i, j and some integer R. Since we chose $n_k, m_k, n_l, m_l \equiv 1 \mod 3$ and b is odd, we actually have $n_k a_k + m_k \frac{-b+\sqrt{-3}}{2} \equiv n_l a_l + m_l \frac{-b+\sqrt{-3}}{2} \equiv \omega \mod 3$, which determines the choice of $\pm\omega^i = \pm\omega^j$ on both sides. We further need the condition:

$$n_k a_k + m_k \frac{-b+\sqrt{-3}}{2} \equiv R(n_l a_l + m_l \frac{-b+\sqrt{-3}}{2}) \mod D$$

Note that this is equivalent to:

$$k + b^{-1}(k+1)\frac{-b+\sqrt{-3}}{2} \equiv R(l + b^{-1}(l+1)\frac{-b+\sqrt{-3}}{2}) \mod D$$

Furthermore, this can be rewritten as:

$$\frac{kb + (k+1)\sqrt{-3}}{2} \equiv R\frac{lb + (l+1)\sqrt{-3}}{2} \mod D$$

This implies $k \equiv lR \mod D$ and $k+1 \equiv lR + R \mod D$, thus $R \equiv 1 \mod D$ and $k \equiv l \mod D$.

Finally, we have $\#(\mathbb{Z}/D\mathbb{Z})^\times$ such ideals, all in different classes of $\mathrm{Cl}(\mathcal{O}_{3D})$, thus we have representatives in every class of $\mathrm{Cl}(\mathcal{O}_{3D})$.

$\square$

Using the factorization formula

We will pick representatives as in the above Lemma to rewrite the Proposition 5.1 and apply Corollary 5.4. We will denote by $\{s \in (\mathbb{Z}/D\mathbb{Z})^\times, s \equiv 1 \mod 6\}$ the norms of the ideals chosen in Lemma 5.6. Furthermore, we are going to choose in Proposition 5.1 all r to be even. We will use the notation $\{r \in \mathbb{Z}/D\mathbb{Z}, r \text{ even}\}$ to express this.

Lemma 5.7. *Picking representatives of $s \in (\mathbb{Z}/D\mathbb{Z})^\times$ such that $s \equiv 1 \mod 6$ and $r \in \mathbb{Z}/D\mathbb{Z}$ also such that $r \equiv 0 \mod 2$, we have*

$$\sum_{\substack{s \in (\mathbb{Z}/D\mathbb{Z})^\times \\ s \equiv 1 \mod 6}} \frac{\Theta\left(D\frac{-b+\sqrt{-3}}{2a_s}\right)}{\Theta\left(\frac{-b+\sqrt{-3}}{2a_s}\right)}\overline{\chi(\mathcal{A}_s)} = \sum_{\substack{s \in (\mathbb{Z}/D\mathbb{Z})^\times \\ s \equiv 1 \mod 6}} \sum_{\substack{r \in \mathbb{Z}/D\mathbb{Z}, \\ r \text{ even}}} \frac{\theta_{sr}\left(\frac{-b+\sqrt{-3}}{2}\right)}{\theta_0\left(\frac{-b+\sqrt{-3}}{2D^2}\right)}\overline{\chi(\mathcal{A}_r\mathcal{A}_s)} \cdot \frac{\theta_r\left(\frac{b+\sqrt{-3}}{2}\right)}{\theta_0\left(\frac{b+\sqrt{-3}}{2D^2}\right)}\chi(\mathcal{A}_r)$$

Proof. We fix $\phi(D)$ ideals $\mathcal{A}_k$ as in Lemma 5.6. Recall that we pick $\mathcal{A}_k$ such that $\mathrm{Nm}\,\mathcal{A}_k = a_k \equiv k \mod D$ for $k \in (\mathbb{Z}/D\mathbb{Z})^\times$, $a_k \equiv 1 \mod 6$ and $n_k \equiv 1 \mod D$. We can pick such an ideal if we take $\mathcal{A}_k = (n_k a_k + m_k \frac{-b+\sqrt{-3}}{2})$ with $m_k \equiv b^{-1}(k+1) \mod D$. We will try to compute:

$$S = \sum_{\substack{k \in (\mathbb{Z}/D\mathbb{Z})^\times \\ k \equiv 1 \mod 6}} \frac{\Theta\left(D\frac{-b+\sqrt{-3}}{2a_k}\right)}{\Theta\left(\frac{-b+\sqrt{-3}}{2a_k}\right)}\overline{\chi(\mathcal{A}_k)}$$

Recall that from Corollary 5.4, we have:

$$\frac{\Theta\left(D\frac{-b+\sqrt{-3}}{2a_s}\right)}{\Theta\left(\frac{-b+\sqrt{-3}}{2a_s}\right)} - \frac{1}{3}\frac{\Theta\left(D\frac{-b+\sqrt{-3}}{2a_s}\right)}{\Theta\left(\frac{-b+\sqrt{-3}}{6a_s}\right)} = \sum_{\substack{r \in \mathbb{Z}/D\mathbb{Z}, \\ r \text{ even}}} \frac{\theta_{a_s r}\left(\frac{-b+\sqrt{-3}}{2a_s^2}\right)\theta_r\left(\frac{b+\sqrt{-3}}{2}\right)}{\theta_0\left(\frac{-b+\sqrt{-3}}{2D^2a_s^2}\right)\theta_0\left(\frac{b+\sqrt{-3}}{2D^2}\right)}$$

Moreover since $r, a_r s$ are both even, we have $e^{\pi i n_{a_s^2} rs} = e^{\pi i rs} = 1$ and thus in Lemma 5.5 we have:

$$\frac{\theta_{rs}\left(\frac{-b+\sqrt{-3}}{2a_s^2}\right)}{\theta_0\left(\frac{-b+\sqrt{-3}}{2a_s^2 D^2}\right)} = \frac{\theta_{rs}\left(\frac{-b+\sqrt{-3}}{2}\right)}{\theta_0\left(\frac{-b+\sqrt{-3}}{2D^2}\right)}$$

Then our sum can be written in the form:

$$\frac{\Theta\left(D\frac{-b+\sqrt{-3}}{2a_s}\right)}{\Theta\left(\frac{-b+\sqrt{-3}}{2a_s}\right)} - \frac{1}{3}\frac{\Theta\left(D\frac{-b+\sqrt{-3}}{6a_s}\right)}{\Theta\left(\frac{-b+\sqrt{-3}}{2a_s}\right)} = \sum_{\substack{r\in\mathbb{Z}/D\mathbb{Z},\\ r \text{ even}}} \frac{\theta_{sr}\left(\frac{-b+\sqrt{-3}}{2}\right)\theta_r\left(\frac{b+\sqrt{-3}}{2}\right)}{\theta_0\left(\frac{-b+\sqrt{-3}}{2D^2}\right)\theta_0\left(\frac{b+\sqrt{-3}}{2D^2}\right)} \tag{5.8}$$

Now summing up for all $s \in (\mathbb{Z}/D\mathbb{Z})^\times$, we get the result of the lemma:

$$\sum_{\substack{s\in(\mathbb{Z}/D\mathbb{Z})^\times\\ s\equiv 1 \mod 6}} \frac{\Theta\left(D\frac{-b+\sqrt{-3}}{2a_s}\right)}{\Theta\left(\frac{-b+\sqrt{-3}}{2a_s}\right)}\overline{\chi(\mathcal{A}_s)} - \frac{1}{3}\sum_{\substack{s\in(\mathbb{Z}/D\mathbb{Z})^\times\\ s\equiv 1 \mod 6}} \frac{\Theta\left(D\frac{-b+\sqrt{-3}}{6a_s}\right)}{\Theta\left(\frac{-b+\sqrt{-3}}{2a_s}\right)}\overline{\chi(\mathcal{A}_s)} =$$

$$= \sum_{\substack{s\in(\mathbb{Z}/D\mathbb{Z})^\times\\ s\equiv 1 \mod 6}} \sum_{\substack{r\in\mathbb{Z}/D\mathbb{Z},\\ r \text{ even}}} \frac{\theta_{rs}\left(\frac{-b+\sqrt{-3}}{2}\right)}{\theta_0\left(\frac{-b+\sqrt{-3}}{2D^2}\right)}\overline{\chi(\mathcal{A}_r\mathcal{A}_s)} \cdot \frac{\theta_r\left(\frac{b+\sqrt{-3}}{2}\right)}{\theta_0\left(\frac{b+\sqrt{-3}}{2D^2}\right)}\chi(\mathcal{A}_r)$$

From Lemma 9.7 in Appendix A, we have $\displaystyle\sum_{\substack{s\in(\mathbb{Z}/D\mathbb{Z})^\times\\ s\equiv 1 \mod 6}} \frac{\Theta\left(D\frac{-b+\sqrt{-3}}{6a_s}\right)}{\Theta\left(\frac{-b+\sqrt{-3}}{2a_s}\right)}\overline{\chi(\mathcal{A}_s)} = 0$. This gives us the result of the Lemma.

□

Proposition 5.2. *Under the conditions above, we have:*

$$\sum_{\substack{s\in(\mathbb{Z}/D\mathbb{Z})^\times\\ s\equiv 1 \mod 6}} \frac{\Theta\left(D\frac{-b+\sqrt{-3}}{2a_s}\right)}{\Theta\left(\frac{-b+\sqrt{-3}}{2a_s}\right)}\overline{\chi(\mathcal{A}_s)} = \left|\sum_{\substack{s\in(\mathbb{Z}/D\mathbb{Z})^\times\\ s\equiv 1 \mod 6}} \frac{\theta_s\left(\frac{-b+\sqrt{-3}}{2}\right)}{\theta_0\left(\frac{-b+\sqrt{-3}}{2D^2}\right)}\overline{\chi(\mathcal{A}_s)}\right|^2$$

Proof. Only for the purpose of this proposition we will use the following notation for θ_r, to emphasize how it depends on D:

$$\theta_{r/D}(z) = \sum_{n\in\mathbb{Z}} e^{\pi i\left(n+\frac{r}{D}-\frac{1}{6}\right)^2 z}(-1)^n$$

Using the new notation, in the previous Lemma we have proved:

$$\sum_{\substack{s\in(\mathbb{Z}/D\mathbb{Z})^\times \\ s\equiv 1 \mod 6}} \frac{\Theta\left(D\frac{-b+\sqrt{-3}}{2a_s}\right)}{\Theta\left(\frac{-b+\sqrt{-3}}{2a_s}\right)}\overline{\chi(\mathcal{A}_s)} = \sum_{\substack{s\in(\mathbb{Z}/D\mathbb{Z})^\times \\ s\equiv 1 \mod 6}} \sum_{\substack{r\in\mathbb{Z}/D\mathbb{Z}, \\ r \text{ even}}} \frac{\theta_{sr/D}\left(\frac{-b+\sqrt{-3}}{2}\right)}{\theta_0\left(\frac{-b+\sqrt{-3}}{2D^2}\right)} \cdot \frac{\theta_{r/D}\left(\frac{b+\sqrt{-3}}{2}\right)}{\theta_0\left(\frac{b+\sqrt{-3}}{2D^2}\right)}\overline{\chi(\mathcal{A}_s)}.$$

Note that using Corollary 5.3 for $a = D^2$ can rewrite:

$$\theta_0\left(\frac{-b+\sqrt{-3}}{2D^2}\right)\theta_0\left(\frac{b+\sqrt{-3}}{2D^2}\right) = \frac{D}{\sqrt[4]{3}}\Theta\left(\frac{b+\sqrt{-3}}{2}\right)$$

Thus the equation becomes:

$$\sum_{\substack{s\in(\mathbb{Z}/D\mathbb{Z})^\times \\ s\equiv 1 \mod 6}} \frac{\Theta\left(D\frac{-b+\sqrt{-3}}{2a_s}\right)}{\Theta\left(\frac{-b+\sqrt{-3}}{2a_s}\right)}\overline{\chi(\mathcal{A}_s)} =$$

$$\frac{1}{\frac{D}{\sqrt[4]{3}}\Theta\left(\frac{b+\sqrt{-3}}{2}\right)} \sum_{\substack{s\in(\mathbb{Z}/D\mathbb{Z})^\times \\ s\equiv 1 \mod 6}} \sum_{\substack{r\in\mathbb{Z}/D\mathbb{Z}, \\ r \text{ even}}} \theta_{sr/D}\left(\frac{-b+\sqrt{-3}}{2}\right)\theta_{r/D}\left(\frac{b+\sqrt{-3}}{2}\right)\overline{\chi(\mathcal{A}_s)} \tag{5.9}$$

Let $R \equiv R' \mod D$, R even and $S \equiv 1 \mod 6$. Then we have by definition:

$$\theta_{RS}(z_1)\theta_R(z_2) = \sum_{n\in\mathbb{Z}} e^{\pi i(n+RS/D-1/6)^2 z_1} e^{\pi i n} \sum_{m\in\mathbb{Z}} e^{\pi i(m+R/D-1/6)^2 z_2} e^{\pi i m}$$

By changing $n \to n+S$ and $m \to m+1$, we change $R \to D+R$ and $R+D \equiv R' \mod 2D$. We get

$$\theta_{RS}(z_1)\theta_R(z_2) =$$

$$= \sum_{n\in\mathbb{Z}} e^{\pi i(n+R'S/D-1/6)^2 z_1} e^{\pi i n}(-1)^S \sum_{m\in\mathbb{Z}} e^{\pi i(m+R'/D-1/6)^2 z_2} e^{\pi i m}(-1) = \theta_{R'S}(z_1)\Theta_{R'}(z_2')$$

Thus we can choose in the formulas above all r to be actually odd. Furthermore, by making a change of $r \pm 2D$ we can also choose $r \equiv 1 \mod 3$. Then we can rewrite the equation as:

$$\sum_{\substack{s\in(\mathbb{Z}/D\mathbb{Z})^\times \\ s\equiv 1 \mod 6}} \frac{\Theta\left(D\frac{-b+\sqrt{-3}}{2a_s}\right)}{\Theta\left(\frac{-b+\sqrt{-3}}{2a_s}\right)}\overline{\chi(\mathcal{A}_s)} =$$

$$= \frac{1}{\frac{D}{\sqrt[4]{3}}\Theta\left(\frac{b+\sqrt{-3}}{2}\right)} \sum_{\substack{s\in(\mathbb{Z}/D\mathbb{Z})^\times \\ s\equiv 1 \mod 6}} \sum_{\substack{r\in\mathbb{Z}/D\mathbb{Z}, \\ r\equiv 1 \mod 6}} \theta_{sr/D}\left(\frac{-b+\sqrt{-3}}{2}\right)\theta_{r/D}\left(\frac{b+\sqrt{-3}}{2}\right)\overline{\chi(\mathcal{A}_s)} \tag{5.10}$$

Denote $\tau_D = \frac{-b+\sqrt{-3}}{2}$. Note that we are summing over all residues $r \mod D$. We can separate the terms, depending on whether a prime divisor p_i divides both D and r. We do this by using the Inclusion-Exclusion principle and note that the sum gets rewritten as:

$$\sum_{\substack{s\in(\mathbb{Z}/D\mathbb{Z})^\times \\ s\equiv 1 \mod 6}} \sum_{\substack{r\in\mathbb{Z}/D\mathbb{Z}, \\ r\equiv 1 \mod 6}} \theta_{sr/D}(\tau_D)\theta_{r/D}(-\overline{\tau_D})\overline{\chi(\mathcal{A}_s)} =$$

$$= \sum_{\substack{s\in(\mathbb{Z}/D\mathbb{Z})^\times \\ s\equiv 1 \mod 6}} \sum_{\substack{r\in(\mathbb{Z}/D\mathbb{Z})^\times \\ r\equiv 1 \mod 6}} \theta_{sr/D}(\tau_D)\theta_{r/D}(-\overline{\tau_D})\overline{\chi(\mathcal{A}_s)}$$

$$+\sum_{p_i|D} \sum_{\substack{s\in(\mathbb{Z}/D\mathbb{Z})^\times \\ s\equiv 1 \mod 6}} \sum_{\substack{r\in\mathbb{Z}/(D/p_i)/\mathbb{Z} \\ r\equiv 1 \mod 6}} \theta_{\frac{sr}{(D/p_i)}}(\tau_D)\theta_{\frac{r}{(D/p_i)}}(-\overline{\tau_D})\overline{\chi(\mathcal{A}_s)}-$$

$$-\sum_{p_ip_j|D} \sum_{\substack{s\in(\mathbb{Z}/D\mathbb{Z})^\times \\ s\equiv 1 \mod 6}} \sum_{\substack{r\in\mathbb{Z}/(D/p_ip_j)\mathbb{Z} \\ r\equiv 1 \mod 6}} \theta_{\frac{sr}{(D/p_ip_j)}}(\tau_D)\theta_{\frac{r}{(D/p_ip_j)}}(-\overline{\tau_D})\overline{\chi(\mathcal{A}_s)}+$$

$$\cdots$$

$$+(-1)^{n-1} \sum_{p_1\dots p_n|D} \sum_{\substack{s\in(\mathbb{Z}/D\mathbb{Z})^\times \\ s\equiv 1 \mod 6}} \sum_{\substack{r\in\mathbb{Z}/(D/p_1\dots p_n)\mathbb{Z} \\ r\equiv 1 \mod 6}} \theta_{\frac{sr}{(D/p_1\dots p_n)}}(\tau_D)\theta_{\frac{r}{(D/p_1\dots p_n)}}(-\overline{\tau_D})\overline{\chi(\mathcal{A}_s)}$$

Using Lemma 5.8 proved below, all of the terms except for the first one equal 0. Thus getting back to the equation (5.10), we get:

$$\sum_{\substack{s\in(\mathbb{Z}/D\mathbb{Z})^\times \\ s\equiv 1 \mod 6}} \frac{\Theta\left(D\frac{-b+\sqrt{-3}}{2a_s}\right)}{\Theta\left(\frac{-b+\sqrt{-3}}{2a_s}\right)}\overline{\chi(\mathcal{A}_s)} =$$

$$= \sum_{\substack{s,r\in(\mathbb{Z}/D\mathbb{Z})^\times \\ s\equiv r\equiv 1 \mod 6}} \sum_{\substack{s\in(\mathbb{Z}/D\mathbb{Z})^\times \\ s\equiv 1 \mod 6}} \frac{\theta_{sr/D}\left(\frac{-b+\sqrt{-3}}{2}\right)}{\theta_0\left(\frac{-b+\sqrt{-3}}{6D^2}\right)}\overline{\chi(\mathcal{A}_r\mathcal{A}_s)} \cdot \frac{\theta_{r/D}\left(\frac{b+\sqrt{-3}}{2}\right)}{\theta_0\left(\frac{b+\sqrt{-3}}{6D^2}\right)}\chi(\mathcal{A}_r)$$

$$= \sum_{s,r} \sum_{\substack{s\in(\mathbb{Z}/D\mathbb{Z})^\times \\ s\equiv 1 \mod 6}} \frac{\theta_{sr/D}\left(\frac{-b+\sqrt{-3}}{2}\right)}{\theta_0\left(\frac{-b+\sqrt{-3}}{2D^2}\right)}\overline{\chi(\mathcal{A}_r\mathcal{A}_s)} \cdot \overline{\frac{\theta_{r/D}\left(\frac{-b+\sqrt{-3}}{2}\right)}{\theta_0\left(\frac{-b+\sqrt{-3}}{2D^2}\right)}\overline{\chi(\mathcal{A}_r)}}$$

$$= \left|\sum_{\substack{s\in(\mathbb{Z}/D\mathbb{Z})^\times \\ s\equiv 1 \mod 6}} \frac{\theta_{s/D}\left(\frac{-b+\sqrt{-3}}{2}\right)}{\theta_0\left(\frac{-b+\sqrt{-3}}{2D^2}\right)}\overline{\chi(\mathcal{A}_s)}\right|^2$$

$\square$

Below we prove Lemma 5.8 used in the proof of Proposition 6.1:

Lemma 5.8. *If $D = p_1 \dots p_n$ and $D' = D/(p_{i_1} \dots p_{i_k})$, then:*

$$\sum_{\substack{s\in(\mathbb{Z}/D\mathbb{Z})^\times \\ s\equiv 1 \mod 6}} \sum_{\substack{r\in\mathbb{Z}/D'\mathbb{Z} \\ r\equiv 1 \mod 6}} \theta_{sr/D'}\left(\frac{-b+\sqrt{-3}}{2}\right)\theta_{r/D'}\left(\frac{b+\sqrt{-3}}{2}\right)\overline{\chi(\mathcal{A}_s)} = 0$$

Proof. Note that first that we can rewrite the sum in the form:

$$\sum_{\substack{s\in(\mathbb{Z}/D\mathbb{Z})^\times \\ s\equiv 1 \mod 6}} \sum_{\substack{r\in\mathbb{Z}/D'\mathbb{Z} \\ r\equiv 1 \mod 6}} \theta_{sr/D'}\left(\frac{-b+\sqrt{-3}}{2}\right)\theta_{r/D'}\left(\frac{b+\sqrt{-3}}{2}\right)\overline{\chi(\mathcal{A}_s)} =$$

$$= \theta_0\left(\frac{-b+\sqrt{-3}}{2D'^2}\right)\theta_0\left(\frac{b+\sqrt{-3}}{2D'^2}\right)\sum_{\substack{s\in(\mathbb{Z}/D\mathbb{Z})^\times \\ s\equiv 1 \mod 6}} \sum_{\substack{r\in\mathbb{Z}/D'\mathbb{Z} \\ r\equiv 1 \mod 6}} \frac{\theta_{rs/D'}\left(\frac{-b+\sqrt{-3}}{2}\right)}{\theta_0\left(\frac{-b+\sqrt{-3}}{2D'^2}\right)}\cdot\frac{\theta_{r/D'}\left(\frac{b+\sqrt{-3}}{2}\right)}{\theta_0\left(\frac{b+\sqrt{-3}}{2D'^2}\right)}\overline{\chi(\mathcal{A}_s)}$$

Using (5.8) for $D := D'$, we recognize the sum on the LHS to be:

$$\sum_{\substack{s\in(\mathbb{Z}/D\mathbb{Z})^\times \\ s\equiv 1 \mod 6}} \frac{\Theta\left(D'\frac{-b+\sqrt{-3}}{2a_s}\right)}{\Theta\left(\frac{-b+\sqrt{-3}}{2a_s}\right)}\overline{\chi(\mathcal{A}_s)}$$

Denote $m = D/D' = p_{i_1}\dots p_{i_k}$. Moreover, recall that from the definition of the cubic character we have:

$$D^{1/3}\overline{\chi_D(\mathcal{A}_s)} = (D^{1/3})^{\sigma_{\mathcal{A}_s}} = (D'^{1/3})^{\sigma_{\mathcal{A}_s}}(m^{1/3})^{\sigma_{\mathcal{A}_s}} = D'^{1/3}\overline{\chi_{D'}(\mathcal{A}_s)}m^{1/3}\overline{\chi_m(\mathcal{A}_s)}$$

Then we can rewrite the sum as:

$$\sum_{\substack{s\in(\mathbb{Z}/D\mathbb{Z})^\times \\ s\equiv 1 \mod 6}} \frac{\Theta\left(D'\frac{-b+\sqrt{-3}}{2a_s}\right)}{\Theta\left(\frac{-b+\sqrt{-3}}{2a_s}\right)}\overline{\chi_D(\mathcal{A}_s)} =$$

$$\sum_{s'\in(\mathbb{Z}/D'\mathbb{Z})^\times, s'\equiv 1 \mod 6} \frac{\Theta\left(D'\frac{-b+\sqrt{-3}}{2a'_s}\right)}{\Theta\left(\frac{-b+\sqrt{-3}}{2a'_s}\right)}\overline{\chi_{D'}(\mathcal{A}_s)}\sum_{\substack{s\in(\mathbb{Z}/D\mathbb{Z})^\times, \\ s\equiv s'\equiv 1 \mod 6 \\ s\equiv s' \mod D'}} \overline{\chi_m(\mathcal{A}_s)}$$

Note that as $D = p_1\cdots p_n$, we have $\{s \in (\mathbb{Z}/D\mathbb{Z})^\times, s \equiv s' \mod D'\} \cong (\mathbb{Z}/m\mathbb{Z})^\times$. Moreover, note that $\chi_m(\mathcal{A}_s)$ depends only on $s \mod m$. Thus we are summing the character $\chi_m(\mathcal{A}_s) = \chi_m(\mathcal{A}_{s''})$ over $s'' \in (\mathbb{Z}/m\mathbb{Z})^\times$.

Moreover, $\overline{\chi_m(\mathcal{A}_s)}$ is a nontrivial character as a function of s, as

$$m^{1/3}\overline{\chi_m(\mathcal{A}_s)} = (m^{1/3})^{\sigma_{\mathcal{A}_s}} = m^{1/3}$$

for all $\mathcal{A}_s$ iff $m^{1/3} \in \mathbb{Q}[\sqrt{-3}]$. As we are summing a non-trivial character over a group, the sum is just 0:

$$\sum_{s'' \in (\mathbb{Z}/m\mathbb{Z})^\times} \overline{\chi_m(\mathcal{A}_{s''})} = 0,$$

thus the whole sum is zero.

We left out the case $r \equiv 0 \mod D$. In this case we have:

$$\sum_{\substack{s \in (\mathbb{Z}/D\mathbb{Z})^\times \\ s \equiv 1 \mod 6}} \frac{\theta_0\left(\frac{-b+\sqrt{-3}}{2}\right)\theta_0\left(\frac{b+\sqrt{-3}}{2}\right)}{\frac{D}{\sqrt[4]{3}}\Theta\left(\frac{b+\sqrt{-3}}{2}\right)}\overline{\chi(\mathcal{A}_s)} = \sum_{\substack{s \in (\mathbb{Z}/D\mathbb{Z})^\times \\ s \equiv 1 \mod 6}} \frac{1}{D}\overline{\chi(\mathcal{A}_s)} = 0$$

□

Chapter 6

Shimura reciprocity applied to θ_r

We will first rewrite below the formula of Proposition 5.2. We define the function $f_r(z) = \dfrac{\theta_r(z)}{\theta_0(z)}$, where $\theta_r(z) = \sum_{n\in\mathbb{Z}} e^{\pi i\left(n+\frac{r}{D}-\frac{1}{6}\right)^2 z} e^{\pi i n}$ and $\theta_0(z) = \sum_{n\in\mathbb{Z}} e^{\pi i\left(n-\frac{1}{6}\right)^2 z} e^{\pi i n}$. Then we can rewrite Proposition 5.2 as:

$$\sum_{s\in(\mathbb{Z}/D\mathbb{Z})^\times} \frac{\Theta\left(D\frac{-b+\sqrt{-3}}{2a_s}\right)}{\Theta\left(\frac{-b+\sqrt{-3}}{2a_s}\right)}\overline{\chi(\mathcal{A}_s)} = \left|\sum_{s\in(\mathbb{Z}/D\mathbb{Z})^\times} f_s(\tau)\overline{\chi(\mathcal{A}_s)}\right|^2 \left|\frac{\theta_0\left(\frac{-b+\sqrt{-3}}{2}\right)}{\theta_0\left(\frac{-b+\sqrt{-3}}{2D^2}\right)}\right|^2$$

Note that from the Corollary 5.3, we can compute

$$\frac{3}{2}\Theta\left(\frac{-b+\sqrt{-3}}{2}\right) = \frac{\sqrt[4]{3}}{1}\theta_0\left(\frac{-b+\sqrt{-3}}{2}\right)\theta_0\left(\frac{b+\sqrt{-3}}{2}\right)$$

as well as

$$\frac{3}{2}\Theta\left(\frac{-b+\sqrt{-3}}{2}\right) = \frac{\sqrt[4]{3}}{D}\theta_0\left(\frac{-b+\sqrt{-3}}{2D^2}\right)\theta_0\left(\frac{b+\sqrt{-3}}{2D^2}\right).$$

Taking the ratio of the two relations, gives us $\left|\dfrac{\theta_0\left(\frac{-b+\sqrt{-3}}{2}\right)}{\theta_0\left(\frac{-b+\sqrt{-3}}{2D^2}\right)}\right|^2 = D$. Thus we get:

$$\sum_{s\in(\mathbb{Z}/D\mathbb{Z})^\times} \frac{\Theta\left(D\frac{-b+\sqrt{-3}}{2a_s}\right)}{\Theta\left(\frac{-b+\sqrt{-3}}{2a_s}\right)}\overline{\chi(\mathcal{A}_s)} = D\left|\sum_{s\in(\mathbb{Z}/D\mathbb{Z})^\times} f_s(\tau)\overline{\chi(\mathcal{A}_s)}\right|^2.$$

By further multiplying by $D^{1/3}$, we have:

$$\sum_{s\in(\mathbb{Z}/D\mathbb{Z})^\times} \frac{\Theta\left(D\frac{-b+\sqrt{-3}}{2a_s}\right)}{\Theta\left(\frac{-b+\sqrt{-3}}{2a_s}\right)}D^{1/3}\overline{\chi(\mathcal{A}_s)} = \left|\sum_{s\in(\mathbb{Z}/D\mathbb{Z})^\times} f_s(\tau)\overline{\chi(\mathcal{A}_s)}D^{2/3}\right|^2. \tag{6.1}$$

Our goal in this section is to show that all the terms $f_s(\tau)\overline{\chi(\mathcal{A}_s)}D^{2/3}$ are Galois conjugates of each other.

θ_r as an automorphic form

We will first look closer at the function θ_r. We will rewrite θ_r as an automorphic theta function $\Theta : \mathrm{SL}_2(\mathbb{A}_\mathbb{Q}) \to \mathbb{C}$:

$$\Theta(g) = \sum_{m\in\mathbb{Q}} r(g)\Phi(m),$$

where $\Phi \in \mathcal{S}(\mathbb{A}_\mathbb{Q})$ is a Schwartz-Bruhat function and r is the Weil representation defined by:

- $r\left(\begin{pmatrix} a & 0 \\ 0 & a^{-1}\end{pmatrix}\Phi\right)(x) = \chi_0(a)|a|^{1/2}\Phi(ax)$
- $r\left(\begin{pmatrix} 1 & b \\ 0 & 1\end{pmatrix}\Phi\right)(x) = \psi(bx^2)\Phi(x)$
- $r\left(\begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix}\Phi\right)(x) = \gamma\widehat{\Phi}(x),$

where $\psi_p(x) = e^{-2\pi i\,\mathrm{Frac}_p(x)}$ and $\psi_\infty(x) = e^{2\pi i x}$, γ is an 8th root of unity, and χ_0 is a quadratic character. For precise definitions see Chapter 8.

We define the following Schwartz-Bruhat functions for θ. Let $\Phi^r = \prod_v \Phi^r_v$, where:

$$\begin{cases} \Phi_p^{(r)} = \mathrm{char}_{\mathbb{Z}_p}, & \text{if } p\nmid D \\ \Phi_p^{(r)} = \mathrm{char}_{\mathbb{Z}_p-\frac{r}{D}}, & \text{if } p|D, p\nmid 2,3 \\ \Phi_3^{(r)} = \mathrm{char}_{\mathbb{Z}_3+\frac{1}{3}}, & \\ \Phi_2^{(r)}(n) = e^{\pi i\,\mathrm{Frac}_2(n)}\,\mathrm{char}_{\mathbb{Z}_2+\frac{1}{2}}(n), & \\ \Phi_\infty^{(r)}(x) = e^{-2\pi q(x)}. & \end{cases}$$

We define the theta function:

$$\Theta_{\Phi^{(r)}}(g) = \sum_{n\in\mathbb{Q}} r(g)\Phi^{(r)}(n)$$

Note that $\Phi_f^{(r)}(n) \neq 0$ for $n \in \mathbb{Q}$ implies $n - \frac{r}{D} + \frac{1}{6} \in \mathbb{Z}_p$ for all p. This implies $n - \frac{r}{D} + \frac{1}{6} \in \mathbb{Z}$, thus $n \in \mathbb{Z} + \frac{r}{D} - \frac{1}{6}$ Also note that for $g_z = \left(\begin{smallmatrix} y^{1/2} & y^{-1/2}x \\ 0 & y^{-1/2}\end{smallmatrix}\right)$, we have $r(g_z)\Phi_\infty(n) = r\left(\left(\begin{smallmatrix} y^{1/2} & 0 \\ 0 & y^{-1/2}\end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & xy^{-1} \\ 0 & 1\end{smallmatrix}\right)\Phi\right)(n) = y^{1/2}e^{2\pi i(x+yi)n^2}$. Then we can compute:

$$\Theta_{\Phi^{(r)}}(g_z, 1_f) = \sum_{n \in \mathbb{Z} + \frac{r}{D} - \frac{1}{6}} e^{2\pi i z n^2} e^{\pi i \operatorname{Frac}_2(n)} = y^{1/2}\theta_r(2z)$$

Note that $\theta_r(2z) = y^{-1/2}\Theta_{\Phi^{(r)}}(g_z, 1_f)$ and $\theta_0(2z) = y^{-1/2}\Theta_{\Phi^{(0)}}(g_z, 1_f)$, which implies:

$$\frac{\theta_r(z)}{\theta_0(z)} = \frac{\Theta_{\Phi^{(r)}}(g_{z/2}, 1_f)}{\Theta_{\Phi^{(0)}}(g_{z/2}, 1_f)}$$

Galois conjugates of $f_r(\tau_D)$.

We will compute below the Galois conjugates of $f_r\left(\frac{-b+\sqrt{-3}}{2}\right)$ using the Shimura reciprocity law. We first recall the function $f_r(z) = \dfrac{\theta_r(z)}{\theta_0(z)}$. Note that we can rewrite the function using $\Theta_{\Phi^{(r)}}$ and $\Theta_{\Phi^{(0)}}$:

$$f_r(z) = \frac{\Theta_{\Phi^{(r)}}(g_{z/2}, 1_f)}{\Theta_{\Phi^{(0)}}(g_{z/2}, 1_f)}$$

We will first check that $f_r(z)$ is a modular function. We begin by checking that $\theta_r(z)$ is a modular form of weight 1/2 in the Lemma below.

Lemma 6.1. *For $r \in \mathbb{Z}$, the transformation of the function $\theta_r(z)$ under $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(72D^2)$ is:*

$$-2i \operatorname{sgn}(d)\gamma_\infty^4 \sqrt{\frac{1}{cz+d}}\,\theta_r\left(\frac{az+b}{cz+d}\right) = \theta_r(z)$$

Proof. Recall that $\theta_r(z) = \Theta_{\Phi^{(r)}}(g_{z/2})$. We will compute $\theta_r\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} z\right)$, for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(72D^2)$. Note first that:

$$\theta_r\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} z\right) = \Theta_{\Phi^{(r)}}\left(\begin{pmatrix} 1/\sqrt{2} & 0 \\ 0 & \sqrt{2} \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} g_z\right) = \Theta_{\Phi^{(r)}}\left(\begin{pmatrix} a & b/2 \\ 2c & d \end{pmatrix}\begin{pmatrix} 1/\sqrt{2} & 0 \\ 0 & \sqrt{2} \end{pmatrix} g_z\right)$$

As $\Theta_{\Phi^{(r)}}$ is invariant under $\mathrm{SL}_2(\mathbb{Q})$, we can rewrite $\Theta_{\Phi^{(r)}}$ as:

$$\Theta_{\Phi^{(r)}}\left(\begin{pmatrix} a & b/2 \\ 2c & d \end{pmatrix}\begin{pmatrix} 1/\sqrt{2} & 0 \\ 0 & \sqrt{2} \end{pmatrix} g_z, 1\right) = \Theta_{\Phi^{(r)}}\left(\begin{pmatrix} 1/\sqrt{2} & 0 \\ 0 & \sqrt{2} \end{pmatrix} g_z, \begin{pmatrix} a & b/2 \\ 2c & d \end{pmatrix}^{-1}\right)$$

We will compute separately the two terms, using the Weil representations. For the RHS, note that we have to compute $r\begin{pmatrix} a & b/2 \\ 2c & d \end{pmatrix}^{-1}\Phi_f^{(r)} = r\begin{pmatrix} d & -b/2 \\ -2c & a \end{pmatrix}\Phi_f^{(r)}$. We will show:

$$\Theta_{\Phi^{(r)}}\left(\begin{pmatrix}1/\sqrt{2} & 0\\ 0 & \sqrt{2}\end{pmatrix} g_z, \begin{pmatrix}a & b/2\\ 2c & d\end{pmatrix}^{-1}\right) = \prod_{p|6D}\gamma_p^2\Theta_{\Phi^{(r)}}\left(\begin{pmatrix}1/\sqrt{2} & 0\\ 0 & \sqrt{2}\end{pmatrix} g_z,\right)$$

We rewrite the matrix as:

$$\begin{pmatrix}d & -b/2\\ -2c & a\end{pmatrix} = \begin{pmatrix}1/a & -b/2\\ 0 & a\end{pmatrix}\begin{pmatrix}1 & 0\\ -2c/a & 1\end{pmatrix}$$

At $p \nmid 6D$, the action of $\begin{pmatrix}d & -b/2\\ -2c & a\end{pmatrix}$ is trivial, as it belongs to $\mathrm{SL}_2(\mathbb{Z}_p)$ and $\Phi_p^{(r)}$ is the characteristic function of $\mathbb{Z}_p$. For $p|6D$, we compute:

First we compute $r\begin{pmatrix}1 & 0\\ -2c/d & 1\end{pmatrix}\Phi_p^{(r)}(x) = \gamma_p^2\Phi_p^{(r)}(x)$. We rewrite the matrix as

$$\begin{pmatrix}1 & 0\\ -2c/d & 1\end{pmatrix} = \begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}1 & 2c/d\\ 0 & 1\end{pmatrix}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$$

and compute the Weil representation action:

- $r\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\Phi_p^{(r)}(x) = \gamma_p\widehat{\Phi_p^{(r)}}(x)$. Note that we can compute:

$$\widehat{\Phi_p^{(r)}}(x) = \int_{\mathbb{Q}_p} e^{-2\pi i\,\mathrm{Frac}_p(2xy)}\,\mathrm{char}_{\mathbb{Z}_p+\frac{r}{D}}(y)dy$$

$$= \int_{\mathbb{Z}_p} e^{-2\pi i\,\mathrm{Frac}_p(2x(y+r/D))}dy = e^{-2\pi i\,\mathrm{Frac}_p(2rx/D}\,\mathrm{char}_{\mathbb{Z}_p}(x)$$

- $r\begin{pmatrix}1 & 2c/d\\ 0 & 1\end{pmatrix}\widehat{\Phi^{(r)}}_p(x) = e^{-2\pi i\,\mathrm{Frac}_p(2c/dx^2}e^{-2\pi i\,\mathrm{Frac}_p(2rx/D}\,\mathrm{char}_{\mathbb{Z}_p}(x)$. As $v_p(c/d)\geq 0$, we have $e^{-2\pi i\,\mathrm{Frac}_p(2c/dx^2} = 1$, thus the action is trivial on $\widehat{\Phi_p^{(r)}}(x)$

- $r\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\widehat{\Phi_p^{(r)}}(x) = \gamma_p\widehat{\widehat{\Phi_p^{(r)}}}(x)$. By the choice of the self-dual Haar measure, this equals $\gamma_p\Phi_p^{(r)}(-x)$.

- $r\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\Phi_p^{(r)}(-x) = \Phi_p^{(r)}(x)$

Now we also want to compute the action of $r\begin{pmatrix}1/a & -b/2\\ 0 & a\end{pmatrix}\Phi_p^{(r)}(x)$. We rewrite the matrix as:

$$\begin{pmatrix}1/a & -b/2\\ 0 & a\end{pmatrix} = \begin{pmatrix}1/a & 0\\ 0 & a\end{pmatrix}\begin{pmatrix}1 & -ba/2\\ 0 & 1\end{pmatrix}$$

and compute the action:

- $r\begin{pmatrix}1 & -ba/2\\ 0 & 1\end{pmatrix}\Phi_p^{(r)}(x) = e^{2\pi i\,\mathrm{Frac}_p(ba/2x^2)}\,\mathrm{char}_{\mathbb{Z}_p+r/D}(x)$.

  As $D^2|ba/2$, we have $e^{2\pi i\,\mathrm{Frac}_p(ba/2x^2)}$, thus we have trivial action.

- $r\begin{pmatrix}1/a & 0\\ 0 & a\end{pmatrix}\Phi_p^{(r)}(x) = \chi_0(a)|a|_p^{1/2}\Phi_p^{(r)}(x/a)$. As $a \equiv 1 \mod D$, we get $\Phi_p^{(r)}(x/a) = \Phi_p^{(r)}(x)$, as well as $\chi_0(a) = |a|_p^{1/2} = 1$.

For $p = 3$ the computation is similar. For $p = 2$, we compute first the action of $\begin{pmatrix}1 & 0\\ -2c/d & 1\end{pmatrix}$ to get:

$$r\begin{pmatrix}1 & 0\\ -2c/d & 1\end{pmatrix}\Phi_p^{(r)}(x) = \gamma_p^2\Phi_p^{(r)}(x).$$

The computation is done below:

- $r\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\Phi_2^{(r)}(x) = \gamma_2\widehat{\Phi_p^{(r)}}(x)$. Note that we can compute:

  $$\widehat{\Phi_2^{(r)}}(x) = \int\limits_{\mathbb{Q}_2} e^{-2\pi i\,\mathrm{Frac}_2(2xy)}\,\mathrm{char}_{\mathbb{Z}_2+\frac{1}{2}}(y)e^{\pi i\,\mathrm{Frac}_2(y)}dy$$
  $$= e^{\pi i/2}\int\limits_{\mathbb{Z}_2} e^{-2\pi i\,\mathrm{Frac}_2(2x(y+1/2))}e^{\pi i\,\mathrm{Frac}_2(y)}dy$$
  $$= e^{\pi i/2}e^{-2\pi i\,\mathrm{Frac}_2(x)}\int\limits_{\mathbb{Z}_2} e^{-2\pi i\,\mathrm{Frac}_2((2x-1/2)y)}dy$$
  $$= e^{\pi i/2}e^{-2\pi i\,\mathrm{Frac}_2(x)}\,\mathrm{char}_{\frac{1}{2}(\mathbb{Z}_2+1/2)}(x)$$

- $r\begin{pmatrix}1 & 2c/d\\ 0 & 1\end{pmatrix}\widehat{\Phi^{(r)}}_2(x) =$

  $= e^{-2\pi i\,\mathrm{Frac}_2(2c/dx^2}\widehat{\Phi^{(r)}}_2(x)$. As $v_p(2c/d) \geq 4$, we have $e^{-2\pi i\,\mathrm{Frac}_p(2c/dx^2} = 1$, thus the action is trivial on $\widehat{\Phi_p^{(r)}}(x)$

- $r\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\widehat{\Phi_p^{(2)}}(x) = \gamma_2\widehat{\widehat{\Phi_2^{(r)}}}(x)$. By the choice of the self-dual Haar measure, this equals $\gamma_2\Phi_2^{(r)}(-x)$.

- $r\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\Phi_2^{(r)}(-x) = \Phi_2^{(r)}(x)$

We compute similarly the action of $r\begin{pmatrix}1/a & -b/2\\ 0 & a\end{pmatrix}\Phi_2^{(r)}(x)$:

- $r\begin{pmatrix}1 & -ba/2\\ 0 & 1\end{pmatrix}\Phi_2^{(r)}(x) = e^{2\pi i\,\mathrm{Frac}_p(ba/2x^2)}e^{\pi i\,\mathrm{Frac}_2(x)}\,\mathrm{char}_{\mathbb{Z}_2+1/2}(x)$. As $4|ba/2$, we have $e^{2\pi i\,\mathrm{Frac}_2(ba/2x^2)} = 1$, thus we have trivial action.
- $r\begin{pmatrix}1/a & 0\\ 0 & a\end{pmatrix}\Phi_2^{(r)}(x) = \chi_0(a)|a|_2^{1/2}\Phi_2^{(r)}(x/a)$. As $a \equiv 1 \mod 8$, we get $\Phi_2^{(r)}(x/a) = \Phi_2^{(r)}(x)$, as well as $\chi_0(a) = |a|_2^{1/2} = 1$.

This finishes the computation of the finite part. We got:

$$\Theta_{\Phi^{(r)}}\left(\begin{pmatrix}1/\sqrt{2} & 0\\ 0 & \sqrt{2}\end{pmatrix}g_z, \begin{pmatrix}a & b/2\\ 2c & d\end{pmatrix}^{-1}\right) = 2^{-1/4}y^{1/2}\sum_{m\in\mathbb{Z}+\frac{r}{D}-\frac{1}{6}} e^{\pi i m^2 z}(-1)^m = 2^{-1/4}y^{1/2}\theta_r(z) \tag{6.2}$$

We will compute now the infinite part. Note first that $r(g_z)\Phi_\infty(m) = y^{1/4}e^{2\pi i z|m|^2}$ We rewrite the matrix:

$$\begin{pmatrix}a & b\\ c & d\end{pmatrix} = \begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}1 & b/d\\ 0 & 1\end{pmatrix}\begin{pmatrix}1/d & 0\\ 0 & d\end{pmatrix}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}1 & -c/d\\ 0 & 1\end{pmatrix}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$$

We compute the Weil representation action:

- $F_1(m) := r\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}e^{2\pi i z m^2} = \gamma_\infty\sqrt{2}\frac{1}{\sqrt{-iz}}e^{-2\pi i\frac{1}{z}}$
- $F_2(m) := r\begin{pmatrix}1 & -c/d\\ 0 & 1\end{pmatrix}F_1(m) = e^{-2\pi i\frac{c}{d}m^2}F_1(m) = \sqrt{2}\gamma_\infty\frac{1}{\sqrt{-iz}}e^{-2\pi i\frac{cz+d}{dz}}$
- $F_3(m) := r\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}F_3(m) = \gamma_\infty\widehat{F_3(m)} = 2\gamma_\infty^2\frac{1}{\sqrt{-iz}}e^{2\pi i\frac{dz}{cz+d}}\sqrt{i\frac{dz}{cz+d}}$
  $= 2\gamma_\infty^2\sqrt{\dfrac{-d}{cz+d}}e^{-2\pi i\frac{dz}{cz+d}m^2}$
- $F_4(m) := r\begin{pmatrix}1/d & 0\\ 0 & d\end{pmatrix}F_3(m) = \mathrm{sgn}(d)d^{-1/2}F_3(m/d)$
  $= 2\,\mathrm{sgn}(d)d^{-1/2}F_3(m/d)\gamma_\infty^2\sqrt{\dfrac{-d}{cz+d}}e^{2\pi i\frac{z}{d(cz+d)}m^2}$
  $= 2\,\mathrm{sgn}(d)F_3(m/d)\gamma_\infty^2\sqrt{\dfrac{-1}{cz+d}}e^{2\pi i\frac{z}{d(cz+d)}m^2}$

- $F_5(m) := r\begin{pmatrix}1 & b/d\\ 0 & 1\end{pmatrix} F_4(m) = e^{2\pi i \frac{b}{d} m^2} F_4(m) = 2\,\mathrm{sgn}(d)\gamma_\infty^2\sqrt{\frac{-1}{cz+d}} e^{2\pi i\left(\frac{b}{d}+\frac{z}{d(cz+d)}\right)m^2} = 2\,\mathrm{sgn}(d)\gamma_\infty^2\sqrt{\frac{-1}{cz+d}} e^{2\pi i\left(\frac{(bc+1)z+bd}{d(cz+d)}\right)m^2} = 2\,\mathrm{sgn}(d)\gamma_\infty^2\sqrt{\frac{-1}{cz+d}} e^{2\pi i\left(\frac{az+b}{cz+d}\right)m^2}$
- $r\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix} F_5(m) = -F_5(-m) = -2\,\mathrm{sgn}(d)\gamma_\infty^2\sqrt{\frac{-1}{cz+d}} e^{2\pi i\left(\frac{az+b}{cz+d}\right)m^2}$

We still have to compute the action of $r\begin{pmatrix}1/\sqrt{2} & 0\\ 0 & \sqrt{2}\end{pmatrix}$ on

$$-2y^{1/2}\,\mathrm{sgn}(d)\gamma_\infty^2\sqrt{\frac{-1}{cz+d}} e^{2\pi i\left(\frac{az+b}{cz+d}\right)m^2}.$$

This gives us just:

$$-2^{3/4} i y^{1/2}\,\mathrm{sgn}(d)\gamma_\infty^2\sqrt{\frac{1}{cz+d}} e^{\pi i\left(\frac{az+b}{cz+d}\right)m^2}$$

Thus we have:

$$\Theta\left(\begin{pmatrix}1/\sqrt{2} & 0\\ 0 & \sqrt{2}\end{pmatrix}\begin{pmatrix}a & b\\ c & d\end{pmatrix} z, 1\right) = -2^{3/4} i y^{1/2}\,\mathrm{sgn}(d)\gamma_\infty^2\sqrt{\frac{1}{cz+d}} \sum_{m\in\mathbb{Z}+\frac{r}{D}-\frac{1}{6}} e^{\pi i\left(\frac{az+b}{cz+d}\right)m^2}(-1)^m$$

Note that this is exactly:

$$-2^{3/4} i y^{1/2}\,\mathrm{sgn}(d)\gamma_\infty^2\sqrt{\frac{1}{cz+d}}\theta_r\left(\frac{az+b}{cz+d}\right) \tag{6.3}$$

From (6.2) and (6.3) we get that:

$$-2i\,\mathrm{sgn}(d)\gamma_\infty^4\sqrt{\frac{1}{cz+d}}\theta_r\left(\frac{az+b}{cz+d}\right) = \theta_r(z)$$

□

Lemma 6.2. *$f \in \mathcal{F}$ is a modular function for* $\Gamma(72D^2)$.

Proof. We need to check that for $\begin{pmatrix}a & b\\ c & d\end{pmatrix} \in \Gamma(72D^2)$ we have:

$$f_r\left(\begin{pmatrix}a & b\\ c & d\end{pmatrix} z\right) = f_r(z).$$

Using the previous lemma, for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(72D^2)$ we have $-2i\,\mathrm{sgn}(d)\gamma_\infty^4\sqrt{\frac{1}{cz+d}}\theta_r(\gamma z) = \theta_r(z)$. Applying the same computation for $r = 0$, we get $-2i\,\mathrm{sgn}(d)\gamma_\infty^4\sqrt{\frac{1}{cz+d}}\theta_0(\gamma z) = \theta_0(z)$. Thus we have:

$$\frac{\theta_r(\gamma z)}{\theta_0(\gamma z)} = \frac{\frac{1}{-2i\,\mathrm{sgn}(d)\gamma_\infty^4}\sqrt{cz+d}\theta_r(z)}{\frac{1}{-2i\,\mathrm{sgn}(d)\gamma_\infty^4}\sqrt{cz+d}\theta_0(z)} = \frac{\theta_r(z)}{\theta_0(z)}$$

□

Lemma 6.3. *The modular function f_r has rational Fourier coefficients in its Fourier expansion at the cusp ∞.*

Proof. Note that $\theta_r(z) = q^{(D-r)^2/72}(1+\sum_{M\geq 1} a_M q^{M/(72D^2)})$, where $a_m \in \mathbb{Z}$ and $\theta_0(z) = q^{1/72}(1+\sum_{M\geq 1} b_M q^{M/72})$

Then we can compute $f_r(z) = q^{((D-r)^2-1)/72}(1+\sum_m a_m q^{m/72D^2})$ with $a_m \in \mathbb{Z}$. □

From CM-theory we have $f(\tau) \in H_{\mathcal{O}}$, where $H_{\mathcal{O}}$ is the ray class field of modulus $72D^2$. In order to compute its Galois conjugates over K we can use Shimura reciprocity law. In its generality:

Shimura reciprocity law. For $\tau \in K \cap \mathcal{H}$ with minimal polynomial $X^2+BX+C = 0$, we have its Galois conjugates:

$$f_r(\tau)^{\sigma_x^{-1}} = f_r^{g_\tau(x)}(\tau),$$

for $x \in \mathbb{A}_{K_f}^\times$, $g_\tau(x) = \begin{pmatrix} t-sB & -sC \\ s & t \end{pmatrix}$.

In our case, we want to compute the Galois conjugates of $f_r(\tau)$, where $\tau = \frac{-b+\sqrt{-3}}{2}$. Note that it has the minimum polynomial $X^2 + bX + \frac{b^2+3}{4}$. Thus we have to compute the action of all $g_\tau((x_p)_p) = \prod_p \begin{pmatrix} t_p - s_p b & -s_p\frac{b^2+3}{4} \\ s_p & t_p \end{pmatrix}_p$.

We will compute all these actions. However, we claim that it is enough to compute the action of the ideals $\mathcal{A}$ through the correspondence:

$$I(3) \to \mathbb{A}_{K,f}^\times/K^\times$$
$$\mathcal{A} = (A+B\omega) \longrightarrow (A+B\omega)_{p|6D},$$

where $A + B\omega \equiv 1 \mod 3$ is the generator of the ideal $\mathcal{A}$.

More precisely, in order to find the Galois conjugates over K, we will compute the action of all Galois actions corresponding to $(A_p + B_p\omega)_p \in \mathbb{A}_K^\times$ and we will prove that the Galois action from Shimura reciprocity law is:

Proposition 6.1. *For $\mathcal{A} = (n_a a + m_a \frac{-b+\sqrt{-3}}{2})$, where $b^2 \equiv -3 \mod 4Db^2$ is an ideal prime to $6D$, we have:*

$$f_1(\tau)^{\sigma_A} = f_{n_a}(\tau)$$

and $f_r(\tau)$ are all the Galois conjugates of $f(\tau)$, where $r \in (\mathbb{Z}/D\mathbb{Z})^\times$. Moreover, this implies that $f_1(\tau) \in H_{6D}$.

Proof. First we note that we do not have to consider the action of all $(x_p)_p \in \mathbb{A}_K^\times$. By applying the Strong Approximation Theorem for GL_1 and the number field K that is a PID, we have:

$$\mathbb{A}_K^\times = K^\times \times \prod_{v \nmid \infty} \mathcal{O}_{K_v}^\times \times \mathbb{C}^\times$$

This implies:

$$\mathbb{A}_{K_f}^\times = K^\times \times \prod_{v \nmid \infty} \mathcal{O}_{K_v}^\times$$

Then any $x = (x_v) \in \mathbb{A}_{K,f}^\times$ can be written as $x = k(l_v)$, where $k \in K^\times$, $(l_v)_v \in \prod_{v \nmid \infty} \mathcal{O}_{K_v}^\times$.

Since $\operatorname{Nm} k > 0$, we have the embedding:

$$k \in K^\times \hookrightarrow \mathrm{GL}_2(\mathbb{Q})^+$$

We also have the embedding:

$$(l_v)_v \in \prod_{v \nmid \infty} \mathcal{O}_{K_v}^\times \hookrightarrow \prod_p \mathrm{GL}_2(\mathbb{Z}_p)$$

Thus if we know the Galois action of $K^\times$ and of $\widehat{\mathcal{O}}_K^\times$, we will know the Galois action of $\mathbb{A}_{K,f}^\times$.

We recall the way the action of $g_\tau(x)$ is defined for. For $\alpha \in \mathrm{GL}_2(\mathbb{Q})^+$, f^α is defined by $f^\alpha(\tau) = f(\alpha\tau)$. In our case we only need to look at the action of $K^\times$. Recall that $k \in K^\times$ embeds into $\mathrm{GL}_2(\mathbb{Q})^+$ under the map:

$$k = t + s\frac{-b+\sqrt{-3}}{2} \hookrightarrow g_\tau(k) = \begin{pmatrix} t - sb & -sc \\ s & t \end{pmatrix}$$

Then the Galois action from Shimura reciprocity is:

$$f(\tau)^{k^{-1}} = f^{g_\tau(k)}(\tau) = f(g_\tau(k)\tau)$$

Note that $t + s\tau \to \left(\begin{smallmatrix} t-sb & -sc \\ s & t \end{smallmatrix}\right)$ is the torus that preserves τ, thus we have:

$$f(\tau)^{k^{-1}} = f(g_\tau(k)\tau) = f(\tau)$$

Now all we have left is to compute the action of $\prod \mathcal{O}_{K_v}^\times$. Note that for all $v \nmid 6D$ the action is trivial. For $v|6D$ we project the action of $(g_\tau(x_v))_v \twoheadrightarrow g_\tau(x') \in \mathrm{GL}_2(\mathbb{Z}/6D^2\mathbb{Z})$.

Remark 6.1. Note that we have for $(\pm\omega^i)_p \hookrightarrow \mathbb{A}_{K_f}^\times$ acting trivially. Thus we have for $x \in \mathbb{A}_{K_f}^\times$:

$$(f_r(\tau))^{\sigma_{\pm\omega^i x}} = ((f_r(\tau))^{\sigma_{\pm\omega^i}})^{\sigma_x} = (f_r^{g_\tau(\pm\omega^i)}(\tau))^{\sigma_x} = f_r(\tau)^{\sigma_x}$$

Lemma 6.4. *For $x \in \prod_v \mathcal{O}_{K_v}^\times$ we can find ω^i, $i = 0, \pm 1$ such that:*

$$(x_2 \pm \omega^i)_2 = (t_2 + s_2\omega)$$

with $v_2(t_2) = 0$, $v_2(s_2) \geq 1$ and

$$(x_3 \pm \omega^i)_3 = (t_3 + s_3\omega)$$

with $t_3 + s_3 \equiv 1 \mod 3$.

Proof. Note first that if $v_2(s) \geq 1$, then we must have $v_2(t_2) = 0$, as we need $x_2\omega^i \in (\mathbb{Z}_2[\omega])^\times$. Thus we must find $x\omega^i$ such that $v_2(s) \geq 1$. We write $x_2 = t_2' + s_2'\omega$. Then:

$$x_2\omega = t_2'\omega + s_2'\omega^2 = (t_2' - s_2')\omega + s_2'$$
$$x_2\omega^2 = t_2'\omega^2 + s_2' = (-t_2')\omega + (s_2' - t_2')$$

One of $t_2', s_2', t_2' - s_2'$ must have positive valuation. Assume this is not true: $v_2(t_2') = v_2(s_2') = 0$. Then $s_2', t_2' \equiv 1 \mod 2$, thus $s_2' - t_2' \equiv 0 \mod 2$ and has positive valuation. Thus we can always pick $x\omega^i$ as claimed above at the place 2.

Now since take $x_3\omega^i = s_3'\omega + t_3' = s_3'\frac{-3+\sqrt{-3}}{2} + (t_3' + s_3')$. Then, since x_3 is a unit in $\mathbb{Z}_3[\omega]$, we must have $v_3(s_3' + t_3') = 0$, thus $s_3' + t_3' \equiv \pm 1 \mod 3$. We pick $x_3\omega$ or $-x_3\omega$ to get the condition $s_3' + t_3' \equiv 1 \mod 3$.

Since from the remark above x and $\pm\omega^i x$ act the same, we can consider the Galois action of $\sigma_{x\omega^i}$ as in the lemma above. We compute it below.

Let $x_p \in \prod_v \mathcal{O}_{K_v}^\times$ chosen as above. Then:

$$x_p = t_p + s_p\frac{-b + \sqrt{-3}}{2} \hookrightarrow g_\tau(x_p) = \begin{pmatrix} t_p - s_p b & -s_p c \\ s_p & t_p \end{pmatrix}$$

Elements of $\prod_p \mathrm{GL}_2(\mathbb{Z}_p)$ project to $\mathrm{GL}_2(\mathbb{Z}/6D^2\mathbb{Z})$, which is the action we care about. From Chinese remainder theorem, we can find $k_0 \in K$ such that $k_0 \equiv x_p \mod 6D^2\mathbb{Z}_p$ for all $p|6D$. Note that k_0 is independent of the choice of τ.

Then we only need to compute the action of:

$$f_r(\tau)^{\sigma_x^{-1}} = f^{g_\tau(x)}(\tau) = f_r^{g_\tau(x_v)_{v|6D}}(\tau) = f^{g_\tau(t+s\tau)_{v|6D}}(\tau)$$

We will now compute $f_r^{g_\tau(x)_{p|6D}}(\tau)$. Note that, for $c' = \frac{b^2+3}{4}$, we have the map :

$$k_0 = s\tau + t \to g_\tau(k_0) = \begin{pmatrix} t - sb' & -sc \\ s & t \end{pmatrix}$$

Let $\mathrm{Nm}(k_0) = a$. We write the action:

$$f(\tau)^{\sigma_x} = f^{\begin{pmatrix} t-sb & -sc/a \\ s & t/a \end{pmatrix}_{p|6D} \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}_{p|6D}}$$

Note that $\begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}_{p|6D}$ acts trivially on f_r as both functions $\theta \begin{bmatrix} -\frac{1}{6} + \frac{r}{D} \\ \frac{1}{2} \end{bmatrix} e^{-\pi i(r/D-1/6)}$ and $\theta \begin{bmatrix} -\frac{1}{6} \\ \frac{1}{2} \end{bmatrix} e^{\pi i/6)}$ have rational Fourier coefficients.

Thus we need to compute the action:

$$f_r^{\begin{pmatrix} t-sb & -sc/a \\ s & t/a \end{pmatrix}_{p|6D}}(\tau)$$

Note that $\begin{pmatrix} t-sb & -sc/a \\ s & ta \end{pmatrix} \begin{pmatrix} t-sb & -sca^* \\ s & ta^* \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}/6D^2\mathbb{Z})$ and we can lift it to an element of $\mathrm{SL}_2(\mathbb{Z})$.

□

□

Lift from $\mathrm{SL}_2(\mathbb{Z}/6D^2)$ to $\mathrm{SL}_2(\mathbb{Z})$.

Lemma 6.5. *We can always lift a matrix in* $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$ *to* $\mathrm{SL}_2(\mathbb{Z})$.

Proof. Take $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$, $A, B, C, D \in \mathbb{Z}$. We can further assue $(C, D) = 1$. Let $AD - BC = k \in \mathbb{Z}$. Then we can take:

$A_0 = A + NA_1$
$B_0 = B + NB_1$
$C_0 = C + NC_1$

$D_0 = D + ND_1$

We want to have the condition:

$1 = A_0D_0 - B_0C_0 = AB - CD + N(AD_1 + A_1D - BC_1 - B_1C) + N^2(A_1D_1 - B_1C_1) = 1 + Nk + N(AD_1 + A_1D - BC_1 - B_1C) + N^2(A_1D_1 - B_1C_1)$

For example, pick $D_1 = C_1 = 0$. Then we only need:

$$(A_1D - B_1C) = -k$$

Note that since $(C, D) = 1$, we can find $mC + nD = 1$. Then $(-kn)D - kmC = -k$, thus pick $A_1 = -kn$ and $B_1 = km$.

We look at such a matrix$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ such that:

$$\begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix} \equiv \begin{pmatrix} s - tb & -s\frac{b^2+3}{4}a^* \\ s & t \end{pmatrix} \mod 6D^2$$

Conditions obtained:

- $v_2(s) \geq 0$ and $v_2(t) = 0$ imply $b_0, c_0 \equiv 0 \mod 2$, $a_0, d_0 \equiv 1 \mod 2$.
- From the choice $3|b$ we also have $a_0 \equiv d_0 \mod 3$ and $b_0 \equiv 0 \mod 3$. Since we picked $k_0 = t_0 + s_0\omega \equiv s\frac{-b+\sqrt{-3}}{2} + t$ with $s_0 + t_0 \equiv 1 \mod 3$, we must have $t \equiv t_0 + s_0 \mod 3$, thus $d_0 \equiv t_0 \equiv 1 \mod 3$.
- From the choice of $t + s\frac{-b+\sqrt{-3}}{2}$ unit in $\prod_{v|6D} \mathcal{O}_{K_v}^\times$, we have $(t, D) = 1$. Otherwise note that the norm is $t^2 - tsb + s^2\frac{b^2+3}{4}$ is divisible by $p|D$, a contradiction.

$\square$

We will find the action using the following lemma:

Lemma 6.6. *For* $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ *such that* $v_p(d) = 0$ *and* $d \equiv 1 \mod 6$, *we have:*

$$\Theta_{\Phi^{(r)}}\left(\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} z\right) = \Theta_{\Phi^{(d^{-1}r)}}(z/2)$$

Here by d^{-1} *we mean* $d^{-1} \mod D$.

Proof. We compute:

$$\Theta_{\Phi^{(r)}}\left(\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} z\right) = \Theta_{\Phi^{(r)}}\left(\begin{pmatrix} a & b/2 \\ 2c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} z\right)$$

Moreover, it equals:

$$\Theta_{\Phi^{(r)}}\left[z/2, \begin{pmatrix} d & -b/2 \\ -2c & a \end{pmatrix}\right]$$

Note that for $p \nmid 6D$ we have $\begin{pmatrix} d & -b/2 \\ -2c & a \end{pmatrix}_p$ in $\mathrm{SL}_2(\mathbb{Z}_p)$, thus acts trivially.
For $p|3D$, we have $\Phi_r = \mathrm{char}_{\mathbb{Z}_p - \frac{1}{6} + \frac{r}{D}}$. For now, we will call $\mu_r := -\frac{1}{6} + \frac{r}{D}$.
If $v_p(d) = 0$, we rewrite:

$$\begin{pmatrix} d & -b/2 \\ -2c & a \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -2c/d & 1 \end{pmatrix} \begin{pmatrix} d & -b/2 \\ 0 & d^{-1} \end{pmatrix}$$

We can further write it in the form:

$$\begin{pmatrix} d & -b/2 \\ -2c & a \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 2c/d & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} d & 0 \\ 0 & d^{-1} \end{pmatrix} \begin{pmatrix} 1 & -b/(2d) \\ 0 & 1 \end{pmatrix}$$

- $r \begin{pmatrix} 1 & -b/(2d) \\ 0 & 1 \end{pmatrix} \Phi_p(x) = e^{-2\pi i \,\mathrm{Frac}_p(-b/(2d)x^2)} \Phi_p(x) = \Phi_p(x)$
- $r \begin{pmatrix} d & 0 \\ 0 & d^{-1} \end{pmatrix} \Phi_p(x) = |d|_p \chi_p(d) \Phi_p(dx) = \Phi_p^{(d^{-1}r)}(x)$

  Note that $\Phi_p(dx) \neq 0$ iff $dx \in \mathbb{Z}_p + \mu_r$ iff $x \in d^{-1}\mathbb{Z}_p + d^{-1}\mu_r = \mathbb{Z}_p + d^{-1}\mu_r$. Note that $d^{-1}\mu_r = d^{-1}r/D - d^{-1}/6$. Since we picked $d \equiv 1 \mod 6$, this is the same as $\mu_{d^{-1}r}$.
- $r \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \Phi_p^{(d^{-1}r)}(x) = e^{2\pi i \,\mathrm{Frac}_p(2d^{-1}xr/D)} \,\mathrm{char}_{\mathbb{Z}_p + 1/2}(x)$
- $r \begin{pmatrix} 1 & 2c/d \\ 0 & 1 \end{pmatrix} (e^{2\pi i \,\mathrm{Frac}_p(2xd^{-1}r/D)} \,\mathrm{char}_{\mathbb{Z}_p+1/2}(x))$

  $= e^{2\pi i \,\mathrm{Frac}_p(2c/dx^2)} (e^{2\pi i \,\mathrm{Frac}_p(2xd^{-1}r/D)} \,\mathrm{char}_{\mathbb{Z}_p}(x) = (e^{2\pi i \,\mathrm{Frac}_p(2xd^{-1}r/D)} \,\mathrm{char}_{\mathbb{Z}_p}(x))$
- $r \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} (e^{2\pi i \,\mathrm{Frac}_p(2xd^{-1}r/D)} \,\mathrm{char}_{\mathbb{Z}_p}(x)) = \Phi_p^{(d^{-1}r)}(-x)$
- $r \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \Phi_p^{(d^{-1}r)}(-x) = \Phi_p^{(d^{-1}r)}(x)$

In here we have used the Fourier transform:

$$\widehat{\Phi_3^{(r)}}(x) = \int_{\mathbb{Q}_p} \Phi_p^{(r)}(y) e^{-2\pi i \,\mathrm{Frac}_p(2xy)} dy = \int_{\mathbb{Z}_p + \frac{r}{D}} \Phi_p^{(r)}(y) e^{2\pi i 2xy} dy = \int_{\mathbb{Z}_p} e^{-2\pi i \,\mathrm{Frac}_p(2x(y+r/D))} dy$$

$$= \int\limits_{\mathbb{Z}_p} e^{-2\pi i \,\mathrm{Frac}_p(2xy))} e^{-2\pi i \,\mathrm{Frac}_p(xr/D)} dy = e^{-2\pi i \,\mathrm{Frac}_p(2xr/D)} \int\limits_{\mathbb{Z}_p} e^{-2\pi i \,\mathrm{Frac}_p(2xy)} dy$$

$$= e^{-2\pi i \operatorname{Frac}_p(2xr/D)} \operatorname{char}_{\mathbb{Z}_p - 1/2}(x) = e^{-2\pi i \operatorname{Frac}_p(2xr/D)} \operatorname{char}_{\mathbb{Z}_p}(x)$$

Similarly we get $\widehat{\Phi_3^{(r)}}(x) = e^{-2\pi i \operatorname{Frac}_3(x/3)} \operatorname{char}_{\mathbb{Z}_p}(x)$

Note that the only difference for $p = 3$ in the action of $\begin{pmatrix} d & -b/2 \\ -2c & a \end{pmatrix}$ is that it does not modify r/D. Instead, it leaves $\Phi^{(r)}$ unchanged.

At the place $p = 2$, we have $\Phi_2 = e^{\pi i \operatorname{Frac}_2(x)} \operatorname{char}_{\mathbb{Z}_2 - 1/2}(x)$. We can compute:

- $r \begin{pmatrix} 1 & -b/(2d) \\ 0 & 1 \end{pmatrix} \Phi_p(x) =$

  $= e^{-2\pi i \operatorname{Frac}_2(-b/(2d)x^2)} e^{\pi i x} \operatorname{char}_{\mathbb{Z}_2 - 1/2}(x)$

  $= e^{2\pi i b/8d} e^{\pi i x} \operatorname{char}_{\mathbb{Z}_2 - 1/2}(x) \Phi_p(x)$

  Note that we picked $2|b$. Then we have $x \in \mathbb{Z}_2 - 1/2$ iff $x = n - 1/2$ for $n \in \mathbb{Z}_2$. Then $-b/2d(n-1/2)^2 = -b/(2d)n^2 + b/(2d)n - b/(8d) \in \mathbb{Z}_2 - b/8d$.

- $r \begin{pmatrix} d & 0 \\ 0 & d^{-1} \end{pmatrix} e^{2\pi i b/8d} e^{\pi i x} \operatorname{char}_{\mathbb{Z}_2 - 1/2}(x) =$

  $= e^{2\pi i b/8d} e^{\pi i d x} \operatorname{char}_{\mathbb{Z}_2 - 1/2}(dx) = e^{2\pi i b/8d} e^{\pi i x} \operatorname{char}_{\mathbb{Z}_2 - 1/2}(x)$

  Note that we have used above $v_2(d) = 0$.

- $r \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \Phi_2^{(r)}(x) = e^{2\pi i b/8d} e^{\pi i x} \operatorname{char}_{\mathbb{Z}_2 - 1/2}(x) = e^{2\pi i b/8d} e^{2\pi i \operatorname{Frac}_2(x+1/4)} \operatorname{char}_{\frac{1}{2}\mathbb{Z}_2 - 1/4}(x)$

  Below we compute the Fourier transform:

  $\int\limits_{\mathbb{Q}_2} e^{\pi i \operatorname{Frac}_2(y)} \operatorname{char}_{\mathbb{Z}_2 - 1/2}(y) e^{-2\pi i \operatorname{Frac}_2(2xy)} dy = \int\limits_{\mathbb{Z}_2 - 1/2} e^{2\pi i \operatorname{Frac}_2(y/2 + 2xy)}$

  $= \int\limits_{\mathbb{Z}_2} e^{2\pi i \operatorname{Frac}_2(y/2 + 1/4 + 2xy + x)} dy$

  $= e^{2\pi i \operatorname{Frac}_2(x+1/4)} \int\limits_{\mathbb{Z}_2} e^{2\pi i \operatorname{Frac}_2(y(1/2+2x))} dy$

  $= e^{2\pi i \operatorname{Frac}_2(x+1/4)} \operatorname{char}_{\frac{1}{2}\mathbb{Z}_2 - 1/4}(x)$

- $r \begin{pmatrix} 1 & 0 \\ 2c/d & 1 \end{pmatrix} e^{2\pi i b/8d} e^{2\pi i \operatorname{Frac}_2(x+1/4)} \operatorname{char}_{\frac{1}{2}\mathbb{Z}_2 - 1/4}$

  $= e^{2\pi i b/8d} e^{2\pi i \operatorname{Frac}_2(2c/dx^2)} e^{2\pi i \operatorname{Frac}_2(x+1/4)} \operatorname{char}_{\frac{1}{2}\mathbb{Z}_2 - 1/4}$

Note that we have the assumptions $2|c$ and $2 \nmid d$. We have $x+1/4 = 1/2n, n \in \mathbb{Z}_2$. Note $x^2 = (n-1/2)/4 = (n^2-n+1/4)/4$ and then $e^{2\pi i \operatorname{Frac}_2(2c/dx^2)} = e^{2\pi i \operatorname{Frac}_2(c/2d(n^2-n+1/4))} = e^{2\pi i \operatorname{Frac}_2(c/8d)}$. Here we have used the fact that $c/2d(n^2-n) \in \mathbb{Z}_2$. Thus we get: $e^{2\pi i \operatorname{Frac}_2((c+b)/8d)} e^{2\pi i \operatorname{Frac}_2(x+1/4)} \operatorname{char}_{\frac{1}{2}\mathbb{Z}_2-1/4}(x)$

- $r \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} e^{2\pi i \operatorname{Frac}_2((c+b)/8d)} e^{2\pi i \operatorname{Frac}_2(x+1/4)} \operatorname{char}_{\frac{1}{2}\mathbb{Z}_2-1/4}(x) =$
$= e^{2\pi i \operatorname{Frac}_2((c+b)/8d)} e^{2\pi i \operatorname{Frac}_2(-x)} \operatorname{char}_{\mathbb{Z}_2-1/2}(-x)$
- $r \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} e^{2\pi i \operatorname{Frac}_2((c+b)/8d)} e^{2\pi i \operatorname{Frac}_2(-x)} \operatorname{char}_{\mathbb{Z}_2-1/2}(-x) =$
$= e^{2\pi i \operatorname{Frac}_2((c+b)/8d)} e^{2\pi i \operatorname{Frac}_2(x)} \operatorname{char}_{\mathbb{Z}_2-1/2}(x) = e^{2\pi i \operatorname{Frac}_2((c+b)/8d)} \Phi_2(x)$

□

Finally we are ready to prove Proposition 6.1. We have showed so far that:

$$f_r(\tau)^{\sigma_x^{-1}} = f_r^{g_\tau(x)}(\tau) = f_r^{(g_\tau(k_0))_{p|6D}}(\tau) = f^{\begin{pmatrix} t-sb & -sc \\ s & t \end{pmatrix}_{p|6D}}(\tau) = f^{\begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix}_{p|6D}}(\tau)$$

From the above lemma we get immediately:

$$\left(\frac{\Theta^{(r)}(\tau/2)}{\Theta^{(0)}(\tau/2)}\right)^{\sigma_x} = f_r^{\begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix}}(\tau) = \frac{\Theta_{\Phi^{(r)}}\left(\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix}\tau\right)}{\Theta_{\Phi^{(0)}}\left(\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix}\tau\right)} = \frac{\Theta_{\Phi^{(d^{-1}r)}}(\tau/2)}{\Theta_{\Phi^{(0)}}(\tau/2)} = f_{d^{-1}r}(\tau)$$

For $\mathcal{A} \in Cl(\mathcal{O}_{3D})$, $\mathcal{A} = (k_\mathcal{A}) = (n_a a + m_a \frac{-b+\sqrt{-3}}{2})$, where $a = \operatorname{Nm}\mathcal{A}$, we take the map:

$$x = (k_\mathcal{A})_{p|6D} \leftrightarrow \mathcal{A}$$

This gives us:

$$x^{-1} \leftrightarrow \mathcal{A}^{-1}$$

Then we have:

$$f_r(\tau)^{\sigma_{\mathcal{A}^{-1}}} = f_r(\tau)^{\sigma_{x^{-1}}} = f_r^{g_\tau(x_p)_{p|6D}}(\tau) = f_r^{g_\tau(k_\mathcal{A})_{p|6D}}(\tau) = f_{n_\mathcal{A}^{-1}r}(\tau)$$

This implies for $r \equiv n_\mathcal{A} \mod D$ that we have $f_{n_\mathcal{A}}(\tau)^{\sigma_{\mathcal{A}^{-1}}} = f_1(\tau)$, or equivalently:

$$f_1(\tau)^{\sigma_\mathcal{A}} = f_{n_\mathcal{A}}(\tau)$$

Remark 6.2. This implies that for $\mathcal{A}_r = (1 \cdot r + b^*(r-1)\frac{-b+\sqrt{-3}}{2})$ we have:

$$f_1(\tau)^{\sigma_{\mathcal{A}_r}} = f_1(\tau)$$

Also it implies that $\mathfrak{a}_r = (r^{-1}) = (r \cdot r^{-2} + 0\frac{-b+\sqrt{-3}}{2})$ we have:

$$f_1(\tau)^{\sigma_{\mathfrak{a}_r}} = f_r(\tau)$$

The square is invariant under the Galois action.

We are finally ready to prove Theorem 5.1.

We define $\mathcal{A}_r^\circ = \left(1 + b^*(1-r)\frac{-b+\sqrt{-3}}{2}\right)$. Note $n_r = r^{-1}$. Note that $\mathcal{A}_r^\circ = \mathcal{A}_r(r^{-1})$, thus $\mathcal{A}_r$ and $\mathcal{A}_r^\circ$ are in the same class in $Cl(\mathcal{O}_{3D})$. This implies:

$$\overline{\chi_D(\mathcal{A}_r)} = \overline{\chi_D(\mathcal{A}_r^\circ)}$$

Moreover, from the definition of χ_D we have: $(D^{2/3})^{\sigma_{\mathcal{A}_r^\circ}} = D^{2/3}\overline{\chi_D(\mathcal{A}_r^\circ)}$

Moreover, from Proposition 6.1:

$$f_1(\tau)^{\sigma_{\mathcal{A}_r{}^\circ}} = f_{n_{\mathcal{A}_r^\circ}}(\tau) = f_r(\tau)$$

Then we can rewrite the term in Proposition 6.1:

$$\kappa := \sum_{r\in(\mathbb{Z}/D\mathbb{Z})^\times} f_r(\tau)D^{2/3}\overline{\chi(\mathcal{A}_r)} = \sum_{r\in(\mathbb{Z}/D\mathbb{Z})^\times} f_r(\tau)D^{2/3}\overline{\chi(\mathcal{A}_r^\circ)} = \sum_{r\in(\mathbb{Z}/D\mathbb{Z})^\times} f_1(\tau)^{\sigma_{\mathcal{A}_r{}^\circ}}(D^{2/3})^{\sigma_{\mathcal{A}_r^\circ}}$$

$$= \sum_{r\in(\mathbb{Z}/D\mathbb{Z})^\times} (f_1(\tau)D^{2/3})^{\sigma_{\mathcal{A}_r{}^\circ}}$$

We want to write κ as a Galois trace of a modular function at a CM-point. Note that the ideals $\{\mathcal{A}_r^{\circ,(r\in\mathbb{Z}/D\mathbb{Z})^\times}\}$ for a group, as we have $\mathcal{A}_r^\circ\mathcal{A}_s^\circ = \mathcal{A}_{rs}^\circ$. Then take $G_0 = \{r \in (\mathbb{Z}/D\mathbb{Z})^\times : \mathcal{A}_r^\circ\} \cong (\mathbb{Z}/D\mathbb{Z})^\times$ that is a subgroup of $\mathrm{Gal}(H_\mathcal{O}/K)$, where $H_\mathcal{O}$ is the ray class field of conductor $3D$.

We define fixed field of G_0 in H:

$$H_0 = \{h \in H_\mathcal{O} : \sigma(h) = h, \forall\sigma \in G_0\}$$

From abelian Galois theory this implies $\mathrm{Gal}(H_\mathcal{O}/H_0) \cong G_0$. Then we got:

$$\kappa = \mathrm{Tr}_{H_\mathcal{O}/H_0}(f_1(\tau)D^{2/3}) \tag{6.4}$$

Thus we have proved so far that:

$$S_D = |\kappa|^2,$$

where $\kappa \in H_0$. We claim that actually $|\kappa|^2 \in \mathbb{Q}$. To prove this, it is enough to show that $|\kappa|^2 \in K^\times$, as

Lemma 6.7. *We have $\kappa^3 \in K$.*

Proof. We will show that the Galois conjugates of κ over K are $\kappa\omega$ and $\kappa\omega^2$.

Take $\mathcal{A} \in \mathrm{Cl}(\mathcal{O})$. Then we have:

$$\kappa^{\sigma_{\mathcal{A}}} = \sum_{r \in (\mathbb{Z}/D\mathbb{Z})^\times} (f_1(\tau) D^{2/3})^{\sigma_{\mathcal{A}_r^\circ \mathcal{A}}}$$

We can write $\mathcal{A} = \mathcal{A}_s^\circ(m)$. Then we have:

$$\kappa^{\sigma_{\mathcal{A}}} = \sum_{r \in (\mathbb{Z}/D\mathbb{Z})^\times} (f_1(\tau) D^{2/3})^{\sigma_{\mathcal{A}_{rs}^\circ (m)}}$$

Note that (m) acts trivially on $D^{2/3}$, but acts as $\mathcal{A}_m^\circ$ on $f_1(\tau)$. Then we have:

$$\kappa^{\sigma_{\mathcal{A}}} = \sum_{r \in (\mathbb{Z}/D\mathbb{Z})^\times} (f_1(\tau))^{\sigma_{\mathcal{A}_{rsm}^\circ}} D^{2/3} \overline{\chi(\mathcal{A}_{rs}^\circ)}$$

$$= \chi(\mathcal{A}_m^\circ) \sum_{r \in (\mathbb{Z}/D\mathbb{Z})^\times} (f_1(\tau))^{\sigma_{\mathcal{A}_{rsm}^\circ}} D^{2/3} \overline{\chi(\mathcal{A}_{rsm}^\circ)}$$

$$= \chi(\mathcal{A}_m^\circ)\kappa$$

$\square$

Remark 6.3. Recall that $|\kappa|^2 \in \mathbb{Q}$. Let $\kappa^3 = a + b\sqrt{-3} \in K$. Then $|\kappa|^6 = a^2 + 3b^2$ and we must have $a^2 + 3b^2 = m^3$ for some $m \in \mathbb{Q}$. With this notation we have $|\kappa|^2 = m = \sqrt[3]{a^2 + 3b^2}$.

Chapter 7

A general formula for S_D.

In this section, we will show the following result:

Theorem 7.1. *For primitive ideals $\mathcal{A}$ that are representatives of $\mathrm{Cl}(\mathcal{O}_{3D})$ such that their norms are prime to each other, let b such that $b \equiv 1 \mod 16$, $b^2 \equiv -3 \mod 12a^2$, $a = \mathrm{Nm}\,\mathcal{A}$. Then we can rewrite:*

$$S_D = \frac{\sqrt{D}}{\#\,\mathrm{Cl}(\mathcal{O}_{3D})} \sum_{r \in \mathbb{Z}/D\mathbb{Z}} \left| \mathrm{Tr}_{H_{\mathcal{O}}/H_1} \left(\frac{\theta_r\left(D\frac{-b+\sqrt{-3}}{2}\right)}{\theta_0\left(\frac{-b+\sqrt{-3}}{2}\right)} D^{-1/3} \right) \right|^2, \tag{7.1}$$

where H_1 is a subfield of $H_{\mathcal{O}}$ the ray class field of modulus $3D$ defined below.

The proof is similar to the proof of Theorem 1.3. The proof is based on the Factorization formula proved in Corollary 5.1 and using the Shimura reciprocity law to compute the Galois conjugates of $\frac{\theta_r\left(D\frac{-b+\sqrt{-3}}{2}\right)}{\theta_0\left(\frac{-b+\sqrt{-3}}{2}\right)}$.

S_D as a sum of squares.

Recall the Factorization formula that we have proved in Corollary 5.1:

$$\sum_{r \in \mathbb{Z}/D\mathbb{Z}} \frac{\sqrt{2ay}}{\sqrt{D}} \theta \begin{bmatrix} a\mu + ar/D \\ \nu \end{bmatrix} \left(D\frac{z}{a}\right) \theta \begin{bmatrix} \mu + r/D \\ -a\nu \end{bmatrix} (-aD\overline{z}) = \sum_{m,n \in \mathbb{Z}} e^{2\pi i(m\nu + nD\mu)} e^{\pi(mni - \frac{|n-mz|^2}{2y})\frac{D}{a}} \tag{7.2}$$

Let $\mathcal{A}, \mathcal{A}_1$ be primitive ideals prime to $3D$, let $a_1 = \mathrm{Nm}\,\mathcal{A}_1$ and $a_2 = \mathrm{Nm}\,\mathcal{A}_2$ and choose b such that $b^2 \equiv -3 \mod 12a^2a_1^2$. By applying the result above for $z = \frac{-b+\sqrt{-3}}{2aa_1^2}$, $\mu = 1/2$ and $\nu = -1/6$, we get:

$$\sum_{r \in \mathbb{Z}/D\mathbb{Z}} \frac{\sqrt[4]{3}}{a_1\sqrt{D}} e^{\pi i(a-1)/6} \theta_{ar}\left(D\frac{-b+\sqrt{-3}}{2a^2a_1^2}\right) \theta_r\left(D\frac{b+\sqrt{-3}}{2a_1^2}\right) =$$

$$\sum_{m,n\in\mathbb{Z}} e^{2\pi i(m/2-nD/6+mnD/(2a))} e^{2\pi i \frac{|naa_1 - m\frac{-b+\sqrt{-3}}{2}|^2}{aa_1}\frac{\sqrt{-3}}{6a}D}$$

Note that since $(a_1, D) = 1$, we can rewrite the formula as:

$$\frac{\sqrt[4]{3}}{a_1\sqrt{D}} e^{\pi i(a-1)/6} \sum_{r\in\mathbb{Z}/D\mathbb{Z}} \theta_{aa_1r}\left(D\frac{-b+\sqrt{-3}}{2a^2a_1^2}\right)\theta_{a_1r}\left(D\frac{b+\sqrt{-3}}{2a_1^2}\right) =$$

$$= \sum_{m,n\in\mathbb{Z}} e^{2\pi i(m/2-nD/6+mnD/(2a))} e^{2\pi i \frac{|naa_1 - m\frac{-b+\sqrt{-3}}{2}|^2}{aa_1}\frac{\sqrt{-3}}{6a}D} \tag{7.3}$$

If $D \equiv 1 \mod 3$, we can show as before that $m/2 + n/2 + \frac{mnD}{2a} \equiv \frac{|naa_1 - m\frac{-b+\sqrt{-3}}{2}|^2}{aa_1}\frac{-b}{6a}D$ mod $\mathbb{Z}$.

Then on the RHS of (7.3) we obtain:

$$\sum_{m,n\in\mathbb{Z}} e^{2\pi i(\pm n/3)} e^{2\pi i \frac{|naa_1 - m\frac{-b+\sqrt{-3}}{2}|^2}{aa_1}\frac{-b+\sqrt{-3}}{6a}D} = \frac{3}{2}\Theta\left(D\frac{-b+\sqrt{-3}}{2a}\right) - \frac{1}{2}\Theta\left(D\frac{-b+\sqrt{-3}}{6a}\right).$$

If $D \equiv 2 \mod 3$, we will change $n \to -n$ and $m \to -m$. Then we have:

$$\sum_{m,n\in\mathbb{Z}} e^{2\pi i(-m/2-n/6+mnD/(2a))} e^{2\pi i \frac{|naa_1 - m\frac{-b+\sqrt{-3}}{2}|^2}{aa_1}\frac{\sqrt{-3}}{6a}D} =$$

$$= \frac{3}{2}\Theta\left(D\frac{b+\sqrt{-3}}{2a}\right) - \frac{1}{2}\Theta\left(D\frac{b+\sqrt{-3}}{6a}\right).$$

Recall that we have from Corollary 5.3:

$$\frac{3}{2}\Theta\left(\frac{-b+\sqrt{-3}}{2a}\right) = \frac{\sqrt[4]{3}}{a_1} e^{\pi i(a-1)\frac{1}{6}}\theta_0\left(\frac{-b+\sqrt{-3}}{2a^2a_1^2}\right)\theta_0\left(\frac{b+\sqrt{-3}}{2a_1^2}\right),$$

Taking the ratio, we get the following lemma:

Lemma 7.1. *For* $b^2 \equiv -3 \mod 12a^2a_1^2$ *and* $b \equiv 1 \mod 16$, *we have:*

$$\frac{\Theta\left(D\frac{-b+\sqrt{-3}}{2a}\right)}{\Theta\left(\frac{-b+\sqrt{-3}}{2a}\right)} - \frac{1}{3}\frac{\Theta\left(D\frac{-b+\sqrt{-3}}{6a}\right)}{\Theta\left(\frac{-b+\sqrt{-3}}{2a}\right)} = \frac{1}{\sqrt{D}}\sum_{r\in\mathbb{Z}/D\mathbb{Z}} \frac{\theta_{aa_1r}\left(D\frac{-b+\sqrt{-3}}{2a^2a_1^2}\right)}{\theta_0\left(\frac{-b+\sqrt{-3}}{2a^2a_1^2}\right)}\frac{\theta_{a_1r}\left(D\frac{b+\sqrt{-3}}{2a_1^2}\right)}{\theta_0\left(\frac{b+\sqrt{-3}}{2a_1^2}\right)}$$

Denote by $f_r(z_{\mathcal{A}}) = \dfrac{\theta_r\left(D\frac{-b+\sqrt{-3}}{2a}\right)}{\theta_0\left(\frac{-b+\sqrt{-3}}{2a}\right)}$. Then the result above becomes:

$$\frac{\Theta\left(D\frac{-b+\sqrt{-3}}{2a}\right)}{\Theta\left(\frac{-b+\sqrt{-3}}{2a}\right)}-\frac{1}{3}\frac{\Theta\left(D\frac{-b+\sqrt{-3}}{6a}\right)}{\Theta\left(\frac{-b+\sqrt{-3}}{2a}\right)}=\frac{1}{\sqrt{D}}\sum_{r\in\mathbb{Z}/D\mathbb{Z}}f_{aa_1r}(z_{\mathcal{A}^2\mathcal{A}_1^2})\overline{f_{a_1r}(z_{\mathcal{A}_1^2})}$$

Now we will take the ideals $\mathcal{A}$ as representatives of $\mathrm{Cl}(\mathcal{O}_{3D})$ and sum over all possible classes $\mathcal{A}_1$:

$$\#\,\mathrm{Cl}(\mathcal{O}_{3D})\left(\frac{\Theta\left(D\frac{-b+\sqrt{-3}}{2a}\right)}{\Theta\left(\frac{-b+\sqrt{-3}}{2a}\right)}-\frac{1}{3}\frac{\Theta\left(D\frac{-b+\sqrt{-3}}{6a}\right)}{\Theta\left(\frac{-b+\sqrt{-3}}{2a}\right)}\right)=$$
$$\frac{1}{\sqrt{D}}\sum_{[\mathcal{A}_1]\in\mathrm{Cl}(\mathcal{O}_{3D})}\sum_{r\in\mathbb{Z}/D\mathbb{Z}}f_{aa_1r}(z_{\mathcal{A}^2\mathcal{A}_1^2})\overline{f_{a_1r}(z_{\mathcal{A}_1^2})}$$

Furthermore twisting by the character $\overline{\chi_D(\mathcal{A})}=\chi_D(\mathcal{A}^2)$ and summing up over all representatives $\mathcal{A}$ of $\mathrm{Cl}(\mathcal{O}_{3D})$, we get:

$$\#\,\mathrm{Cl}(\mathcal{O}_{3D})\left(\sum_{\mathcal{A}\in\mathrm{Cl}(\mathcal{O}_{3D})}\frac{\Theta\left(D\frac{-b+\sqrt{-3}}{2a}\right)}{\Theta\left(\frac{-b+\sqrt{-3}}{2a}\right)}\overline{\chi_D(\mathcal{A})}-\frac{1}{3}\sum_{\mathcal{A}\in\mathrm{Cl}(\mathcal{O}_{3D})}\frac{\Theta\left(D\frac{-b+\sqrt{-3}}{6a}\right)}{\Theta\left(\frac{-b+\sqrt{-3}}{2a}\right)}\overline{\chi_D(\mathcal{A})}\right)=$$
$$\frac{1}{\sqrt{D}}\sum_{[\mathcal{A}]\in\mathrm{Cl}(\mathcal{O}_{3D}}\sum_{[\mathcal{A}_1]\in\mathrm{Cl}(\mathcal{O}_{3D})}\sum_{r\in\mathbb{Z}/D\mathbb{Z}}f_{aa_1r}(z_{\mathcal{A}^2\mathcal{A}_1^2})\overline{f_{a_1r}(z_{\mathcal{A}_1^2})}\chi_D(\mathcal{A}^2)$$

From Appendix A, Lemma 9.7, we have $\sum_{\mathcal{A}\in\mathrm{Cl}(\mathcal{O}_{3D})}\frac{\Theta\left(D\frac{-b+\sqrt{-3}}{6a}\right)}{\Theta\left(\frac{-b+\sqrt{-3}}{2a}\right)}\overline{\chi_D(\mathcal{A})}=0$, thus we can rewrite the LHS as:

$$\#\,\mathrm{Cl}(\mathcal{O}_{3D})\sum_{\mathcal{A}\in\mathrm{Cl}(\mathcal{O}_{3D})}\frac{\Theta\left(D\frac{-b+\sqrt{-3}}{2a}\right)}{\Theta\left(\frac{-b+\sqrt{-3}}{2a}\right)}\overline{\chi_D(\mathcal{A})}$$

For the RHS, note that we can distribute the character $\chi_D(\mathcal{A}^2)$ as $\chi_D(\mathcal{A}^2\mathcal{A}_1^2)\overline{\chi_D(\mathcal{A}_1^2)}$ and we can exchange the sums and rewrite:

$$\frac{1}{\sqrt{D}}\sum_{r\in\mathbb{Z}/D\mathbb{Z}}\sum_{[\mathcal{A}]\in\mathrm{Cl}(\mathcal{O}_{3D})}\sum_{[\mathcal{A}_1]\in\mathrm{Cl}(\mathcal{O}_{3D})}f_{aa_1r}(z_{\mathcal{A}^2\mathcal{A}_1^2})\chi_D(\mathcal{A}^2\mathcal{A}_1^2)\overline{f_{ar}(z_{\mathcal{A}_1^2})\chi_D(\mathcal{A}_1^2)}=$$
$$=\frac{1}{\sqrt{D}}\sum_{r\in\mathbb{Z}/D\mathbb{Z}}\left|\sum_{[\mathcal{A}]\in\mathrm{Cl}(\mathcal{O}_{3D})}f_{a_1r}(z_{\mathcal{A}^2})\chi_D(\mathcal{A}^2)\right|^2.$$

This gives us the result of the following proposition:

Proposition 7.1. *For $\mathcal{A}$ primitive ideals that are representatives of* $\mathrm{Cl}(\mathcal{O}_{3D})$ *such that their norms are prime to each other, let* b *such that* $b \equiv 1 \mod 16$, $b^2 \equiv -3 \mod 12a^2$, $a = \mathrm{Nm}\,\mathcal{A}$. *Then we can rewrite:*

$$S_D = \frac{\sqrt{D}}{\#\,\mathrm{Cl}(\mathcal{O}_{3D})} \sum_{r \in \mathbb{Z}/D\mathbb{Z}} \left| \sum_{[\mathcal{A}] \in \mathrm{Cl}(\mathcal{O}_{3D})} \frac{\theta_{ar}\left(D\frac{-b+\sqrt{-3}}{2a^2}\right)}{\theta_0\left(\frac{-b+\sqrt{-3}}{2a^2}\right)} \chi_D(\mathcal{A}^2) D^{-1/3} \right|^2 \tag{7.4}$$

Galois conjugates of $f_{ar}(z_A)$.

We let $f_r(z) = \frac{\theta_r(Dz)}{\theta_0(z)}$. We note that $f(\frac{-b+\sqrt{-3}}{2}) \in H_{\mathcal{O}}$, the ray class field of modulus $3D$. Then we compute two actions:

- The action of the element $\mathcal{A}^2 = \left[a^2, \frac{-b+\sqrt{-3}}{2}\right]$ in the ring class field H_{3D} is going to be $f(z/a^2)$. This follows from Lemma 4.3:

$$f_r(\tau)^{\sigma_{\mathcal{A}^2}} = f_r(z/a^2)$$

- The action of the ideal $\mathcal{A}_k^\circ \in \mathcal{P}_{\mathbb{Z},3D}$ such that $\mathcal{A}_k^\circ \sim (k + 3D\mathbb{Z}[\omega])$ as ideal classes in $\mathrm{Gal}(H_{\mathcal{O}}/K)$. Then the action is going to be:

$$f_r(\tau)^{\sigma_{\mathcal{A}_k^\circ}} = f_{ak}(\tau), \tau = \frac{-b+\sqrt{-3}}{2}$$

  The proof follows closely the proof of Proposition 6.1

Then we can rewrite the formula (7.4) as:

$$S_D = \frac{\sqrt{D}}{\#\,\mathrm{Cl}(\mathcal{O}_{3D})} \sum_{r \in \mathbb{Z}/D\mathbb{Z}} \left| \sum_{[\mathcal{A}] \in \mathrm{Cl}(\mathcal{O}_{3D})} \left(f_r(D\tau)D^{-1/3}\right)^{\sigma_{\mathcal{A}^2}\sigma_{\mathcal{A}_a^\circ}{}^{-1}} \right|^2$$

We denote:

$$G_1 = \{\mathcal{A}^2\mathcal{A}_a^\circ, [\mathcal{A}] \in \mathrm{Cl}(\mathcal{O}_{3D})\}.$$

Note that this is a group and G_0 is a subgroup of $\mathrm{Gal}(H_{\mathcal{O}}/K)$. Thus from Galois theory we can find H_1 to be the fixed field of G_1 in $H_{\mathcal{O}}$ and thus $\mathrm{Gal}(H_{\mathcal{O}}/H_1) \cong G_1$. Then our formula becomes the result of Theorem 7.1:

$$S_D = \frac{\sqrt{D}}{\#\,\mathrm{Cl}(\mathcal{O}_{3D})} \sum_{r \in \mathbb{Z}/D\mathbb{Z}} \left|\mathrm{Tr}_{H_{\mathcal{O}}/H_1}\left(f_r(D\tau)D^{-1/3}\right)\right|^2 \tag{7.5}$$

Chapter 8

Another formula for $L(E_D, 1)$.

In this chapter we will show:

$$L(E_D,1) = c_D \operatorname{Tr}_{H_\circ[\sqrt{D}]/K[\sqrt{D}]} \frac{\theta_{1/2}\left(3D\omega\right)^2}{\Theta_K\left(\omega\right)} D^{-1/6},$$

where $\theta_{1/2}(z) = \sum\limits_{n\in\mathbb{Z}} e^{2\pi i n^2 z}(-1)^n$, $H_\circ$ is the ray class field for the modulus $12D$ and the constant is $c_D = \frac{\pi}{48} D^{1/6} \prod\limits_{p|D}(1-(-1)^{(p-1)/2}p^{-1})\Theta_K(\omega)L_\infty(1,\chi_D\varphi)$.

Relation to the first formula proved.

In Section 3 we have computed a formula for $L(E_D,1)$ by looking at Tate's zeta function $Z_f(s,\chi_D\varphi,\Phi_K) = \int\limits_{\mathbb{A}_{K,f}^\times} \Phi_K(\alpha_f)|\alpha_f|_f^s \chi_D(\alpha_f)\varphi(\alpha_f)d^\times\alpha_f$. We showed that this integral is a linear combination of Eisenstein series, based on the observation:

$$\sum_{k\in K^\times} \frac{k}{|k|_{\mathbb{C}}^{2s}} \Phi_K(k\alpha_f) = E(g_\alpha, 2s-2, \Phi'_K),$$

where Φ'_K is a different Schwartz-Bruhat function also for $\mathcal{S}(\mathbb{A}_K)$. In this section we will make a different choice of the Schwartz-Bruhat function $\Phi^\circ_K \in \mathcal{S}(\mathbb{A}_K)$ in order to get a similar identity:

$$\sum_{k\in K^\times} \frac{k}{|k|_{\mathbb{C}}^{2s}} \Phi^\circ_K(k\alpha_f) = E(g_\alpha, 2s-2, \phi_1\otimes\phi_2),$$

where ϕ_1, ϕ_2 are Schwartz-Bruhat functions in $\mathcal{S}(\mathbb{A}_\mathbb{Q})$.

Computing the L-function.

We take the Schwartz-Bruhat function $\Phi^\circ \in \mathcal{S}(\mathbb{A}_K)$ defined by $\Phi^\circ = \Phi^\circ_\infty \prod_p \Phi^\circ_p$, where at the infinite place we define $\Phi^\circ_\infty = e^{-\pi|z|^2}$ and at the finite places we define:

$$\Phi^\circ_p = \begin{cases} \mathrm{char}_{\mathbb{Z}_p[\omega]} & \text{for } p \nmid 6D \\ \mathrm{char}_{(\mathbb{Z}+3D\mathbb{Z}_p[\omega])^\times} & \text{for } p | 3D \\ \mathrm{char}_{(1+4\mathbb{Z}_2[\omega])} + \mathrm{char}_{(1+2\omega+4\mathbb{Z}_2[\omega])} & \text{at } p = 2 \end{cases}$$

Proposition 8.1. *For Φ°_f defined above and primitive ideals $\mathcal{A}$ that are taken to be representatives of the ideal class group $\mathrm{Cl}(\mathcal{O}_{3D})$, we have:*

$$Z_f(s, \chi_{D,f}\varphi_f, \Phi^\circ_f) = \frac{1}{72}\#(\mathcal{O}_K/D\mathcal{O}_K)^\times \sum_{[\mathcal{A}] \in K^\times \backslash \mathbb{A}^\times_{K,f}/U} E(2s-2, 3Dz_{\mathcal{A}}, \phi_1 \otimes \phi_2)\overline{\chi_D(\mathcal{A})}\frac{\varphi(\mathcal{A})}{a^{2s-1}}$$

Proof. We start by recalling the definition of Tate's zeta function:

$$Z_f(s, \chi_D\varphi, \Phi^\circ) = \int_{\mathbb{A}^\times_{K,f}} \Phi^\circ_f(\alpha_f)|\alpha_f|^s_f \chi_{D,f}(\alpha_f)\varphi_f(\alpha_f) d^\times \alpha_f.$$

We will first take a quotient by $K^\times$ in the integral. This gives us:

$$Z_f(s, \chi_D\varphi, \Phi^\circ) = \int_{K^\times \backslash \mathbb{A}^\times_{K,f}} \sum_{k \in K^\times} \Phi^\circ_f(k\alpha_f)|k\alpha_f|^s_f \chi_{D,f}(k\alpha_f)\varphi_f(k\alpha_f) d^\times \alpha_f$$

Using the properties of Hecke characters, we can rewrite

$$|k|^s_f \chi_{D,f}(k)\varphi_f(k) = |k|^{-s}_\infty \chi^{-1}_{D,\infty}(k)\varphi^{-1}_\infty(k) = |k|^{-s}_\infty k = \|k\|^{-2s}_{\mathbb{C}} k,$$

where $\|\cdot\|_{\mathbb{C}}$ is the usual absolute value over $\mathbb{C}$. Thus we get:

$$Z_f(s, \chi_D\varphi, \Phi^\circ) = \int_{K^\times \backslash \mathbb{A}^\times_{K,f}} \left(\sum_{k \in K^\times} \Phi^\circ_f(k\alpha_f)k\|k\|^{-2s} \right) |\alpha_f|^s_f \chi_{D,f}(\alpha_f)\varphi_f(\alpha_f) d^\times \alpha_f$$

Moreover, we want to take the quotient by $U = \prod_{p|6D} (1 + 12D\mathbb{Z}_p[\omega])^\times \prod_{p\nmid 6D} (\mathbb{Z}_p[\omega])^\times$. For this we need invariance under U:

- Φ°_f is invariant under U
- χ_D is invariant under U

- φ is invariant under U
- $|\cdot|$ is invariant under U

We take the quotient by U:

$$Z_f(s,\chi_D\varphi,\Phi^\circ) = \mathrm{vol}(U) \sum_{K^\times\backslash\mathbb{A}^\times_{K,f}/U} \left(\sum_{k\in K^\times} \Phi^\circ_f(k\alpha'_f)k\|k\|^{-2s}\right) |\alpha'_f|^s_f \chi_{D,f}(\alpha'_f)\varphi_f(\alpha'_f)$$

Note that $K^\times \backslash \mathbb{A}^\times_{K,f}/U$ is a finite set. Furthermore, recall that from the Strong Approximation theorem, we have $\mathbb{A}^\times_K = K^\times\mathbb{C}^\times \prod_v \mathcal{O}^\times_{K_v}$. Then we can rewrite the quotient $\mathbb{A}^\times_{K,f}/K^\times \cong \prod_v \mathcal{O}^\times_{K_v}/\langle\pm\omega\rangle$ and we can pick representatives for $U \backslash \mathbb{A}^\times_{K,f}/K^\times$ elements $\alpha'_f \in \prod_v \mathcal{O}^\times_{K_v}$. Also note that φ and $|.|$ are trivial when evaluated at the elements α'_f in $\prod_v \mathcal{O}^\times_{K_v}$. Then we get:

$$Z_f(s,\chi_D\varphi,\Phi^\circ) = \mathrm{vol}(U) \int_{K^\times\backslash\mathbb{A}^\times_{K,f}/U} \left(\sum_{k\in K^\times} \Phi^\circ_f(k\alpha'_f)k\|k\|^{-2s}\right) \chi_{D,f}(\alpha'_f)$$

Furthermore, note that $\Phi_f(k\alpha) \neq 0$ implies $k\alpha_v \in \mathcal{O}_{K_v}$, thus $k \in \mathcal{O}_{K_v}$ for all finite places v. Thus we get $k \in \mathcal{O}_K$. Moreover, for $k \in \mathcal{O}_K$ we have $\Phi_v(k\alpha_v) = 1$ for all $v \nmid 6D$. Thus we can compute:

$$Z_f(s,\chi_D\varphi,\Phi^\circ) = \mathrm{vol}(U) \sum_{K^\times\backslash\mathbb{A}^\times_{K,f}/U} \left(\sum_{k\in\mathcal{O}_K{}^\times} \Phi^\circ_{6D}(k\alpha'_f)k\|k\|^{-2s}\right) \chi_{D,f}(\alpha'_f)$$

Moreover, we can pick $k_1 \equiv \alpha'_v \mod 24D\mathcal{O}_{K_v}$ for $v|6D$. The condition is lax enough that we can take k_1 such that (k_1) is a primitive ideal. Then, for $k \in \mathcal{O}_K$ we have $\Phi^\circ_{6D}(kk_1) = \Phi^\circ_{6D}(k\alpha'_f)$. Moreover, $\chi_D(\alpha_f) = \chi_D((k_1))$. Then we compute:

$$Z_f(s,\chi_D\varphi,\Phi^\circ) = \mathrm{vol}(U) \sum_{K^\times\backslash\mathbb{A}^\times_{K,f}/U} \left(\sum_{k\in\mathcal{O}_K} \Phi^\circ_{6D}(kk_1)k\|k\|^{-2s}\right) \chi_{D,f}((k_1))$$

Furthermore, we can rewrite it:

$$Z_f(s,\chi_D\varphi,\Phi^\circ) = \mathrm{vol}(U) \sum_{K^\times\backslash\mathbb{A}^\times_{K,f}/U} \left(\sum_{k\in K^\times} \Phi^\circ_{6D}(kk_1)\frac{kk_1}{\|kk_1\|^{2s}}\right) \frac{\|k_1\|^{2s}}{k_1}\chi_{D,f}((k_1))$$

We can compute the volume of U. From the choice of the normalized multiplicative Haar measure, we have $\mathrm{vol}((\mathbb{Z}_p[\omega])^\times) = 1$ for all p. Then we can compute:

- $\mathrm{vol}(1+4\mathbb{Z}_2[\omega]) = \frac{1}{12}$,
- $\mathrm{vol}(1+3\mathbb{Z}_3[\omega]) = \frac{1}{6}$,
- $\mathrm{vol}(1+p\mathbb{Z}_p[\omega]) = \frac{1}{p^2-1} = \#(\mathcal{O}_K/p\mathcal{O}_K)^\times$, if p nonsplit,
- $\mathrm{vol}(1+p\mathbb{Z}_p[\omega]) = \frac{1}{(p-1)^2} = \#(\mathcal{O}_K/p\mathcal{O}_K)^\times$, if p split.

This gives us $\mathrm{vol}(U) = \frac{1}{72}\#(\mathcal{O}_K/D\mathcal{O}_K)^\times$ and the formula becomes:

$$Z_f(s,\chi_D\varphi,\Phi^\circ) = \frac{1}{72}\#(\mathcal{O}_K/D\mathcal{O}_K)^\times \sum_{K^\times\backslash\mathbb{A}_{K,f}^\times/U}\left(\sum_{k\in K^\times}\Phi_{6D}^\circ(kk_1)\frac{kk_1}{\|kk_1\|^{2s}}\right)\frac{\|k_1\|^{2s}}{k_1}\chi_{D,f}((k_1))$$

□

For $k_1 \in K^\times$, $s \in \mathbb{C}$, we denote the term:

$$I(k_1,s) := \sum_{k\in\mathcal{O}_K}\Phi_{6D}^\circ(kk_1)\frac{kk_1}{\|kk_1\|^{2s}} \tag{8.1}$$

We will show that this gives us the value of an Eisenstein series in Lemma 8.1 below.

Lemma 8.1. *For $k_1 \in \mathcal{O}_K^\times$ such that $\mathcal{A} = (k_1)$ is a primitive ideal depending on k, we write $\mathcal{A} = [a, \frac{-B+\sqrt{-3}}{2}]$ as a $\mathbb{Z}$-module. Then we have:*

$$I(k_1,s) = L_{\mathbb{Q}}(2s-1,\chi_0)\frac{1}{2}\prod_{p|D}(1-(-1)^{(p-1)/2}p^{1-2s})(-1)^{(A-1)/2}\frac{A}{\|A\|^{2s}}E_{\varepsilon_0}(2s-2,3Dz_{\overline{\mathcal{A}}}),$$

where $z_{\overline{\mathcal{A}}} = \frac{B+\sqrt{-3}}{2a}$ and $E_{\varepsilon_0}(s,z) = \sum_{(m,n)=1,2\nmid m}\frac{(-1)^{(m-1)/2}}{\|m+nz\|^s(m+nz)}$ is an Eisenstein series.

Proof. Note first that we can rewrite:

$$I(k_1,s) = \sum_{k\in\mathcal{A}=(k_1)}\Phi_{6D}^\circ(k)\frac{k}{\|k\|^{2s}}$$

Note that for $\Phi_{6D}^\circ(k) \neq 0$ we must have $k \in \mathcal{P}_{\mathbb{Z},6D} := (\mathbb{Z}+6D\mathcal{O}_K)^\times$. Then we have:

$$I(k_1,s) = \sum_{k\in\mathcal{A}=(k_1)\cap P_{\mathbb{Z},6D}}\Phi_{6D}^\circ(k)\frac{k}{\|k\|^{2s}}$$

Let $(k_1) = \mathcal{A} = [A, \frac{-B+\sqrt{-3}}{2}]_{\mathbb{Z}}$ with $B \equiv 1 \mod 4$, $A = \mathrm{Nm}\,\mathcal{A}$ and $B^2 \equiv -3 \mod 4A$. Then all $k \in \mathcal{A}$ can be written in the form $k = mA + n\frac{-B+\sqrt{-3}}{2}$ for $m,n \in \mathbb{Z}$. Moreover, since $k \in P_{\mathbb{Z},6D}$, we have:

$$k = mA + n \cdot 6D\frac{-B+\sqrt{-3}}{2}, (m, 6D) = 1$$

Moreover, since $\Phi_2^\circ(k) \neq 0$ for $k \in (\mathbb{Z} + 2\mathbb{Z}_2[\omega])^\times$, we have:

$$\Phi_2(k) = \begin{cases} 1, & \text{for } mA \equiv 1 \mod 4 \\ 0, & \text{otherwise} \end{cases}$$

Then we have:

$$I(k_1, s) = \sum_{\substack{m,n, \\ (m,6D)=1 \\ mA\equiv 1 \mod 4}} \frac{mA + 2n \cdot 3D\frac{-B+\sqrt{-3}}{2}}{\|mA + 2n \cdot 3D\frac{-B+\sqrt{-3}}{2}\|^{2s}}$$

Note that we can rewrite this as:

$$I(k_1, s) = \frac{1}{2} \sum_{\substack{m,n, \\ (m,6D)=1 \\ mA\equiv 1 \mod 4}} \frac{mA + 2n \cdot 3D\frac{-B+\sqrt{-3}}{2}}{\|mA + 2n \cdot 3D\frac{-B+\sqrt{-3}}{2}\|^{2s}} - \frac{1}{2} \sum_{\substack{m,n, \\ (m,6D)=1 \\ mA\equiv 3 \mod 4}} \frac{mA + 2n \cdot 3D\frac{-B+\sqrt{-3}}{2}}{\|mA + 2n \cdot 3D\frac{-B+\sqrt{-3}}{2}\|^{2s}}$$

Note that this is precisely:

$$I(k_1, s) = \frac{1}{2} \sum_{\substack{m,n, \\ (m,6D)=1}} (-1)^{(mA-1)/2} \frac{mA + 2n \cdot 3D\frac{-B+\sqrt{-3}}{2}}{\|mA + 2n \cdot 3D\frac{-B+\sqrt{-3}}{2}\|^{2s}}$$

We can split the product $(-1)^{(mA-1)/2} = (-1)^{(m-1)/2}(-1)^{(A-1)/2}$ and we get:

$$I(k_1, s) = (-1)^{(A-1)/2} \frac{1}{2} \sum_{\substack{m,n \\ (m,6D)=1}} (-1)^{(m-1)/2} \frac{mA + n \cdot 3D\frac{-B+\sqrt{-3}}{2}}{\|mA + n \cdot 3D\frac{-B+\sqrt{-3}}{2}\|^{2s}}$$

We rewrite further:

$$I(k_1, s) = (-1)^{(A-1)/2} \frac{1}{2} \sum_{\substack{(m,n)=1, \\ (m,6D)=1}} (-1)^{(m-1)/2} \frac{mA + n \cdot 3D\frac{-B+\sqrt{-3}}{2}}{\|mA + n \cdot 3D\frac{-B+\sqrt{-3}}{2}\|^{2s}} \sum_{(m,6D)=1} \frac{(-1)^{(m-1)/2} m}{m^{2s}}$$

Note that the far right term is an L-function:

$$\sum_{(m,6D)=1} \frac{(-1)^{(m-1)/2} m}{m^{2s}} = \prod_{p \nmid 6D} (1 - (-1)^{(p-1)/2} p^{1-2s})^{-1} = L(2s-1, \chi_0) \prod_{p|D} (1 - (-1)^{(p-1)/2} p^{1-2s})$$

Thus we need to compute:

$$\sum_{(m,2)=1} \frac{(-1)^{(m-1)/2}}{m^{2s-1}} = L_{\mathbb{Q}}(2s-1, \chi_0)$$

Here $\chi_0(m) = \left(\frac{m}{4}\right)$ and we can compute the value of the L-function $L(\chi_0, 1)$ of a Dirichlet character (see for example [13]). We get:

$$L_{\mathbb{Q}}(1, \chi_0) = \frac{2\pi}{8} = \frac{\pi}{4}$$

Then we have:

$$I(k_1, s) = L_{\mathbb{Q}}(2s-1, \chi_0) \prod_{p|D} (1 - (-1)^{(p-1)/2} p^{1-2s})(-1)^{(A-1)/2} \times$$

$$\times \frac{1}{2} \sum_{(m,n)=1, 2\nmid m} \frac{(-1)^{(m-1)/2}(mA + n \cdot 3D\frac{-B+\sqrt{-3})}{2}}{\|mA + n \cdot 3D\frac{-B+\sqrt{-3}}{2}\|^{2s}}$$

We can rewrite this as:

$$I(k_1, s) = L_{\mathbb{Q}}(2s-1, \chi_0) \prod_{p|D} (1 - (-1)^{(p-1)/2} p^{1-2s})(-1)^{(A-1)/2} \frac{A}{\|A\|^{2s}} \times$$

$$\times \frac{1}{2} \left(\sum_{(m,n)=1, 2\nmid m} (-1)^{(m-1)/2} \frac{m + n \cdot 3D\frac{-B+\sqrt{-3}}{2a}}{\|m + n \cdot 3D\frac{-B+\sqrt{-3}}{2a}\|^{2s}} \right)$$

Note that this is:

$$I(k_1, s) = L_{\mathbb{Q}}(2s-1, \chi_0) \frac{1}{2} \prod_{p|D} (1 - (-1)^{(p-1)/2} p^{1-2s})(-1)^{(A-1)/2} \frac{A}{\|A\|^{2s}} E_{\varepsilon_0}(2s-2, 3Dz_{\overline{\mathcal{A}}}),$$

where $z_{\overline{\mathcal{A}}} = \frac{B+\sqrt{-3}}{2a}$.

□

In the next section we will show that actually $E_{\varepsilon_0}(2s-2, z)$ is a particular case of a Siegel-Eisenstein series.

A particular Eisenstein series.

In this section we will connect the automorphic Eisenstein series with a classical Eisenstein series. Our goal is to prove Lemma 8.2. We start by recalling some details about the Weil representation.

Weil representation for $\widetilde{\mathrm{SL}}_2(\mathbb{Q}) \times O_1(\mathbb{Q})$.

In order to define half-integral weight theta functions, we will define the Weil representation for the symplectic space $W = \mathbb{Q} \oplus \mathbb{Q}$ and the quadratic space $V = \mathbb{Q}$, $q(x) = x^2$. We follow Gelbart [**Ge**] to define the Weil representation for $\mathrm{SL}_2(\mathbb{Q})$ as a cross section of $\widetilde{\mathrm{SL}}_2(\mathbb{Q})$.

We define locally at the place v:

- $r \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \phi(x) = \psi(bx^2)\phi(x)$
- $r \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \phi(x) = (a,a)_v \frac{\gamma(q,\psi_a)}{\gamma(q,\psi)} \phi(ax) = (a,a)\varepsilon_v(a)|a|_v^{1/2}\phi(ax)$
- $r \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \phi(x) = \gamma(q,\psi)\widehat{\phi}(x)$,

where:

- $\psi(x) = e^{2\pi i x_\infty} \prod_p e^{-2\pi i \,\mathrm{Frac}_p(x_p)}$
- $\gamma_p(q,\psi_a) = \lim\limits_{m\to\infty} \int\limits_{p^{-m}\mathbb{Z}_p} \psi_p(ay^2)dy = \lim\limits_{m\to\infty} \int\limits_{p^{-m}\mathbb{Z}_p} e^{-2\pi i \,\mathrm{Frac}_p(ay^2)}dy = |a|_p^{1/2}\varepsilon_p(a)$, where $\varepsilon_p : \mathbb{Q}_p^\times \to \{\pm 1, \pm i\}$ and $\varepsilon_p(a)^2 = (a,-1)_p$
- $\gamma_\infty(q,\psi_a) = e^{2\pi i \,\mathrm{sgn}(a)/4}$

For now we are not interested in extending the Weil representation to all of GL_2. Instead, we will extend the Weil representation to:

$$(\mathrm{GL}_2(\mathbb{A}_\mathbb{Q}))^2 = \{g \in \mathrm{GL}_2(\mathbb{A}_\mathbb{Q}) : \det g \in \mathbb{A}_\mathbb{Q}^{\times 2}\}$$

This is done by defining:

- $r \begin{pmatrix} 1 & 0 \\ 0 & c^2 \end{pmatrix} \phi(x) = |c|^{-1/2}\phi(c^{-1}x)$

Note that this implies:

- $r \begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix} \phi(x) = (c,c)\varepsilon(c)\phi(x)$

This is done by checking:

$$r \begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix} \phi(x) = r \begin{pmatrix} c & 0 \\ 0 & 1/c \end{pmatrix} r \begin{pmatrix} 1 & 0 \\ 0 & c^2 \end{pmatrix} \phi(x) = r \begin{pmatrix} c & 0 \\ 0 & 1/c \end{pmatrix} |c|^{-1/2}\phi(c^{-1}x) = (c,c)\varepsilon(c)\phi(x).$$

Defining the Eisenstein series.

Recall first the general definition of the Eisenstein series:

$$E(s, g, \phi_1 \otimes \phi_2) = \sum_{\gamma \in P_1(\mathbb{Q}) \backslash \mathrm{SL}_2(\mathbb{Q})} f_s(\gamma g),$$

for $g \in \mathrm{GL}_2(\mathbb{A})$, where $P_1(\mathbb{Q})$ is the subset of upper triangular matrices in $\mathrm{SL}_2(\mathbb{Q})$ and $f_s(g) = r(g)\phi_1 \otimes \phi_2(0)\delta(g)^s$, where δ is the modulus character for $\mathrm{GL}_2(\mathbb{A})$ and r is the Weil representation.

Remark 8.1. Note that the definition above makes sense: we can take as representatives of $P_1(\mathbb{Q}) \setminus \mathrm{SL}_2(\mathbb{Q})$ the matrices $\left(\begin{smallmatrix} 1 & 0 \\ -b/a & 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. Then we have:

$$\sum_{\left(\begin{smallmatrix} a & b \\ -b & a \end{smallmatrix}\right) \in P(\mathbb{Q}) \backslash \mathrm{GL}_2(\mathbb{Q})} f_s(\left(\begin{smallmatrix} a & b \\ -b & a \end{smallmatrix}\right) g) =$$

$$= \sum_{\left(\begin{smallmatrix} 1 & 0 \\ -b/a & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) \in P_1(\mathbb{Q}) \backslash \mathrm{SL}_2(\mathbb{Q})} \chi_1(c/a)\chi_2(a) f_s(\left(\begin{smallmatrix} 1 & 0 \\ -b/a & 1 \end{smallmatrix}\right) g) + f_s(\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) g)$$

$$= E(s, g, \phi_1 \otimes \phi_2),$$

as χ_1, χ_2 are trivial on $\mathbb{Q}^\times$. Then we can rewrite:

$$E(s, g, \phi_1 \otimes \phi_2) = \sum_{\left(\begin{smallmatrix} a & b \\ -b & a \end{smallmatrix}\right) \in P(\mathbb{Q}) \backslash \mathrm{GL}_2(\mathbb{Q})} f_s(\left(\begin{smallmatrix} a & b \\ -b & a \end{smallmatrix}\right) g),$$

Note that for $g_z = \left(\left(\begin{smallmatrix} y^{1/2} & y^{-1/2}x \\ 0 & y^{-1/2} \end{smallmatrix}\right)_\infty, 1_f\right)$, we have:

$$f_s(g) = r(g_\infty)\phi_{1,\infty} \otimes \phi_{2,\infty}(0) r(g_f)\phi_{1,f} \otimes \phi_{2,f}(0)\delta_\infty(g_\infty)^s \delta_f(g_f)^s$$

We will fix the Schwartz-Bruhat functions $\phi_1 \otimes \phi_2$, such that $\phi_1, \phi_2 \in \mathcal{S}(\mathbb{A}_\mathbb{Q})$. More precisely, we take $\phi_1 = \phi_2 = \prod_p \phi_{1,p}$, where p goes over all archimedean and non-archimedean places of $\mathbb{Q}$, and locally we define:

$$\phi_{1,p} = \phi_{2,p} = \begin{cases} \mathrm{char}_{\mathbb{Z}_p}, & p \neq 2 \\ \mathrm{char}_{\mathbb{Z}_2}(x) e^{\pi i \,\mathrm{Frac}_2(x)} & p = 2 \\ e^{-2\pi x^2}, & p = \infty \end{cases}$$

Below we specialize the Siegel-Eisenstein in order to obtain the value of the classical Eisenstein series $E_{\varepsilon_0}(s, z)$. We have:

Lemma 8.2. *For $g_z = \left(\begin{pmatrix} y^{1/2} & y^{-1/2}x \\ 0 & y^{-1/2}\end{pmatrix}_\infty, 1_f\right)$, where $y>0$, we have:*

$$E(s, g_z, \phi_1\otimes\phi_2) = y^{(s+1)/2}E_{\varepsilon_0}(s,z),$$

where $E_{\varepsilon_0}(s,z) = \sum_{(a,b)=1, 2\nmid a} \frac{a-bz}{|a-bz|^s|a-bz|^2}(-1)^{\frac{a-1}{2}}$.

Proof. We denote $c := a^2+b^2$ and $a' = a/\sqrt{c}, b' = b/\sqrt{c}$. We compute the Weil representation action at all the places.

Place ∞. At ∞, we compute:

- $r\left(\begin{pmatrix} a & b \\ -b & a\end{pmatrix} g_z\right)\phi_{1,\infty}\otimes\phi_{2,\infty}(0) = r\left(\begin{pmatrix} a' & b' \\ -b' & a'\end{pmatrix} g_z\right)\phi_{1,\infty}\otimes\phi_{2,\infty}(0)$
  $= r\left(\begin{pmatrix} a' & b' \\ -b' & a'\end{pmatrix}\begin{pmatrix} y^{1/2} & xy^{-1/2} \\ 0 & y^{-1/2}\end{pmatrix}\right)\phi_{1,\infty}\otimes\phi_{2,\infty}(0)$
  $= r\begin{pmatrix} a'y^{1/2} & (a'x+b')y^{-1/2} \\ -b'y^{1/2} & (-b'x+a')y^{-1/2}\end{pmatrix}\phi_{1,\infty}\otimes\phi_{2,\infty}(0)$
  $= r\left(\begin{pmatrix} y^{1/2}/|a'-b'z| & * \\ 0 & |a'-b'z|/y^{1/2}\end{pmatrix}\begin{pmatrix} (-b'x+a')/|a'-b'z| & b'y/|a'-b'z| \\ -by/|a-bz| & (-bx+a)/|a-bz|\end{pmatrix}\right)\phi_{1,\infty}\otimes\phi_{2,\infty}(0)$
  $= \dfrac{(-b'x+a')+b'yi}{|a'-b'z|} r\begin{pmatrix} y^{1/2}/|a'-b'z| & * \\ 0 & |a'-b'z|/y^{1/2}\end{pmatrix}\phi_{1,\infty}\otimes\phi_{2,\infty}(0)$
  $= \dfrac{a'-b'z}{|a'-b'z|}(y^{1/2}/|a'-b'z|, -1)\dfrac{y^{1/2}}{|a'-b'z|}\phi_{1,\infty}\otimes\phi_{2,\infty}(0)$
  $= \dfrac{(a'-b'z)y^{1/2}}{|a'-b'z|^2}$
  $= \dfrac{(a-bz)y^{1/2}}{|a-bz|^2}|a^2+b^2|^{1/2}$

- $\delta_\infty\left(\begin{pmatrix} a & b \\ -b & a\end{pmatrix} g_z\right) = \delta_\infty\left(\begin{pmatrix} a' & b' \\ -b' & a'\end{pmatrix}\begin{pmatrix} y^{1/2} & xy^{-1/2} \\ 0 & y^{-1/2}\end{pmatrix}\right)$
  $= \delta_\infty\begin{pmatrix} a'y^{1/2} & (a'x+b')y^{-1/2} \\ -b'y^{1/2} & (-b'x+a')y^{-1/2}\end{pmatrix}$
  $= \delta_\infty\left(\begin{pmatrix} y^{1/2}/|a'-b'z| & * \\ 0 & |a'-b'z|/y^{1/2}\end{pmatrix}\begin{pmatrix} (-b'x+a')/|a'-b'z| & b'y/|a'-b'z| \\ -b'y/|a'-b'z| & (-b'x+a')/|a'-b'z|\end{pmatrix}\right)$
  $= \dfrac{y^{1/2}}{|a'-b'z|} = \dfrac{y^{1/2}}{|a-bz|}|a^2+b^2|^{1/2}$

Places $p\neq 2$. At $p\neq 2$, we compute:

- If $v_p(c)=0$, then $\begin{pmatrix} a & b \\ -b & a\end{pmatrix}\in \mathrm{GL}_2(\mathbb{Z}_p)$, thus it acts trivially on $\phi_1\otimes\phi_2$
  Also $\delta\begin{pmatrix} a & b \\ -b & a\end{pmatrix} = 1$ for $\begin{pmatrix} a & b \\ -b & a\end{pmatrix}\in\mathrm{GL}_2(\mathbb{Z}_p)$.

- $a^2 + b^2 = p$, then $v_p(a) = v_p(b) = 0$ and $(p, -1) = 1$. Then we have:
  $r(\left(\begin{smallmatrix} a & b \\ -b & a \end{smallmatrix}\right))\phi_{1,f} \otimes \phi_{2,f}(x, y) = |p|_p^{-1/2} r(\left(\begin{smallmatrix} a & b/p \\ -b & a/p \end{smallmatrix}\right))\phi_{1,f} \otimes \phi_{2,f}((ax - by)/p, (bx + ay)/p) =$
  $= |p|_p^{-1/2} r(\left(\begin{smallmatrix} p/a & b/p \\ 0 & a/p \end{smallmatrix}\right) \left(\begin{smallmatrix} 1 & 0 \\ -b/a & 1 \end{smallmatrix}\right))\phi_{1,f} \otimes \phi_{2,f}((ax - by)/p, (bx + ay)/p)|_{(x,y)=(0,0)} =$
  $= \ |p|_p^{-1/2}(p/a, -1)_p |p/a|_p \int\limits_{\mathbb{Q}_p \oplus \mathbb{Q}_p} e^{2\pi i - b/a(x^2+y^2)} \operatorname{char}_{\mathbb{Z}_p}(\frac{ax - by}{p}) \operatorname{char}_{\mathbb{Z}_p}(\frac{bx + ay}{p}) dx\, dy \ =$
  $|p|_p^{1/2} = |a^2 + b^2|_p^{1/2}$
  Note that $\frac{ax-by}{p}, \frac{bx+ay}{p} \in \mathbb{Z}_p$ implies $x, y \in \mathbb{Z}_p$, hence the statement above
  For δ_p, we compute:
  $\delta_p(\left(\begin{smallmatrix} p/a & b \\ 0 & a \end{smallmatrix}\right) \left(\begin{smallmatrix} 1 & 0 \\ -b/a & 1 \end{smallmatrix}\right)) = |p|_p^{1/2} = p^{-1/2} = |a^2 + b^2|_p^{1/2}$

Place $p = 2$. At $p = 2$, we have $\phi_1 = \phi_2 = \operatorname{char}_{\mathbb{Z}_2}(x)e^{\pi i x}$. Then we have $\widehat{\phi}_1(x) = \widehat{\phi}_2(x) = \operatorname{char}_{\frac{1}{2}(\mathbb{Z}_2 + 1/2)}(x)$.

The self-dual Haar measure gives us:

$$\operatorname{vol}(\mathbb{Z}_2) = 1/\sqrt{2}, \operatorname{vol}(\mathbb{Z}_2[i]) = 1/2$$

Note that if $v_2(a^2 + b^2) = 0$, then we have $v_2(a) \geq 1$ and $v_2(b) = 0$, or $v_2(b) \geq 1$ and $v_2(a) = 0$. In this case note:

- If $a(2x) + b(2y) \in \mathbb{Z}_2 + 1/2, -b(2x) + a(2y) \in \mathbb{Z}_2 + 1/2$ implies $2x, 2y \in \mathbb{Z}_2 + 1/2$, thus $x, y \in \frac{1}{2}(\mathbb{Z}_2 + 1/2)$. Moreover, if $x, y \in \frac{1}{2}(\mathbb{Z}_2 + 1/2)$, we have $a(2x) + b(2y) \in \mathbb{Z}_2 + 1/2, -b(2x) + a(2y) \in \mathbb{Z}_2 + 1/2$. Thus $\widehat{\phi}_1(ax + by)\widehat{\phi}_2(ay - bx) = \widehat{\phi}_1(x)\widehat{\phi}_2(y)$.

- If $ax + by \in \mathbb{Z}_2, -bx + ay \in \mathbb{Z}_2$ implies $(a^2 + b^2)x \in a\mathbb{Z}_2 + b\mathbb{Z}_2 = \mathbb{Z}_2$, thus $x \in \mathbb{Z}_2$. Moreover, this implies $y \in \mathbb{Z}_2$. Moreover, $e^{\pi i(ax+by)}e^{\pi i(ay-bx)} = e^{\pi i x}e^{\pi i y}$.

We compute:

- $v_2(a) = 0, v_2(b) \geq 1$:
  $r\left(\begin{smallmatrix} a & b \\ -b & a \end{smallmatrix}\right)\phi_{1,2} \otimes \phi_{2,2}(x, y)|_{(0,0)} = r\left(\begin{smallmatrix} a & b/c \\ -b & a/c \end{smallmatrix}\right)\phi_{1,2} \otimes \phi_{2,2}(ax - by, -bx + ay)|_{(0,0)} =$
  $= r\left(\begin{smallmatrix} c/a & b/c \\ 0 & a/c \end{smallmatrix}\right) r\left(\begin{smallmatrix} 1 & 0 \\ -bc/a & 1 \end{smallmatrix}\right)\phi_{1,2} \otimes \phi_{2,2}(ax - by, -bx + ay)|_{(0,0)}$
  $= r\left(\begin{smallmatrix} c/a & b/c \\ 0 & a/c \end{smallmatrix}\right) \int\limits_{\mathbb{Q}_2 \oplus \mathbb{Q}_2} e^{-2\pi i \frac{bc}{a}(x^2+y^2)} \widehat{\phi}_{1,2}(ax - by)\widehat{\phi_{2,2}}(-bx + ay) dx\, dy$
  $= (c/a, -1)_2 |c/a|_2 \int\limits_{\mathbb{Q}_2 \oplus \mathbb{Q}_2} e^{-2\pi i \frac{bc}{a}(x^2+y^2)} \widehat{\phi}_{1,2}(x)\widehat{\phi}_{2,2}(y) dx\, dy$

$$= \frac{1}{4}(a,-1)_2 \int\limits_{\mathbb{Z}_2\oplus\mathbb{Z}_2} e^{-2\pi i \frac{bc}{4a}((x+1/2)^2+(y+1/2)^2)} dx\, dy$$

$$= \frac{1}{4}(a,-1)_2 \int\limits_{\mathbb{Z}_2\oplus\mathbb{Z}_2} e^{-2\pi i \frac{bc}{4a}(x^2+x+y^2+y+1/2)} dx\, dy$$

$$= \frac{1}{4}(a,-1)_2 e^{-2\pi i \frac{bc}{2a}} \int\limits_{\mathbb{Z}_2\oplus\mathbb{Z}_2} e^{-2\pi i \frac{bc}{4a}(x^2+x+y^2+y)} dx\, dy$$

$$= \frac{1}{4}(a,-1)_2 \int\limits_{\mathbb{Z}_2} e^{-2\pi i \frac{bc/(2a)}{2}(x^2+x)} dx \int\limits_{\mathbb{Z}_2} e^{-2\pi i \frac{bc/(2a)}{2}(y^2+y)} dx \frac{1}{4}(-1)^{(a-1)/2}$$

Note that:

$$\int\limits_{\mathbb{Z}_2} e^{-2\pi i \frac{bc/2}{2a}(x^2+x)} dx = \frac{1}{2}\int\limits_{\mathbb{Z}_2} e^{-2\pi i \frac{bc/2}{2a}((2x+1)^2+(2x+1))} dx + \frac{1}{2}\int\limits_{\mathbb{Z}_2} e^{-2\pi i \frac{bc/2}{2a}(4x^2+2x)} dx = 1$$

- $v_2(b) = 0, v_2(a) \geq 1$:

$$r\left(\begin{smallmatrix} a & b \\ -b & a \end{smallmatrix}\right) \phi_{1,2} \otimes \phi_{2,2}(x,y)|_{(0,0)} = r\left(\begin{smallmatrix} b & -a \\ a & b \end{smallmatrix}\right) \widehat{\phi}_{1,2} \otimes \widehat{\phi}_{2,2}(x,y)|_{(0,0)}$$

$$= r\begin{pmatrix} b & -a/c \\ a & b/c \end{pmatrix} \widehat{\phi}_{1,2} \otimes \widehat{\phi}_{2,2}(ax+by, -bx+ay)|_{(0,0)}$$

$$= r\begin{pmatrix} c/b & -a/c \\ 0 & b/c \end{pmatrix} r\left(\begin{smallmatrix} 1 & 0 \\ ac/b & 1 \end{smallmatrix}\right) \widehat{\phi}_{1,2}(ax-by)\widehat{\phi}_{2,2}(-bx+ay) dx\, dy =$$

$$= r\begin{pmatrix} c/b & -a/c \\ 0 & b/c \end{pmatrix} \int\limits_{\mathbb{Q}_2\oplus\mathbb{Q}_2} e^{2\pi i \frac{ac}{b}(x^2+y^2)} \phi_{1,2}(ax-by)\phi_{2,2}(-bx+ay) dx\, dy$$

$$= (c/b,-1)_2 |c/b|_2 \int\limits_{\mathbb{Q}_2\oplus\mathbb{Q}_2} e^{2\pi i \frac{ac}{b}(x^2+y^2)} \phi_{1,2}(ax-by)\phi_{2,2}(-bx+ay) dx\, dy$$

$$= (b,-1)_2 \int\limits_{\mathbb{Q}_2\oplus\mathbb{Q}_2} e^{2\pi i \frac{ac}{b}(x^2+y^2)} \phi_{1,2}(x)\phi_{2,2}(y) dx\, dy$$

$$= (b,-1)_2 \frac{1}{4} \int\limits_{\mathbb{Z}_2\oplus\mathbb{Z}_2} e^{2\pi i \frac{ac}{b}(x^2+y^2)} e^{\pi i x} e^{\pi i y} dx\, dy$$

$$= (b,-1)_2 \frac{1}{4} \int\limits_{\mathbb{Z}_2\oplus\mathbb{Z}_2} e^{\pi i x} e^{\pi i y} dx\, dy = 0$$

- $v_2(a) = 0$, $v_2(b) = 0$, then we have $v_2(a^2+b^2) = 1$. Note that if $ax+by \in \mathbb{Z}_2 + 1/2$ and $ay - bx \in \mathbb{Z}_2 + 1/2$, then we have $x \in \frac{1}{2}\mathbb{Z}_2, y \in \frac{1}{2}\left(\mathbb{Z}_2 + \frac{1}{2}\right)$, or $x \in \frac{1}{2}\left(\mathbb{Z}_2 + \frac{1}{2}\right), y \in \frac{1}{2}\mathbb{Z}_2$. Thus we have:

$$\widehat{\phi}_1(ax+by)\widehat{\phi}_2(ay-bx) = \mathrm{char}_{\frac{1}{2}\left(\mathbb{Z}_2+\frac{1}{2}\right)}(x)\,\mathrm{char}_{\frac{1}{2}\mathbb{Z}_2}(y) + \mathrm{char}_{\frac{1}{2}\left(\mathbb{Z}_2+\frac{1}{2}\right)}(y)\,\mathrm{char}_{\frac{1}{2}\mathbb{Z}_2}(x)$$

We compute:
$r\left(\begin{smallmatrix} a & b \\ -b & a \end{smallmatrix}\right)\phi_{1,2}\otimes\phi_{2,2}(x,y)|_{(0,0)} = r\left(\begin{smallmatrix} b & -a \\ a & b \end{smallmatrix}\right)\widehat{\phi}_{1,2}\otimes\widehat{\phi}_{2,2}(x,y)|_{(0,0)}$
$= r\begin{pmatrix} b & -a/c \\ a & b/c \end{pmatrix}\widehat{\phi}_{1,2}\otimes\widehat{\phi}_{2,2}(ax+by, ay-bx)|_{(0,0)}$
$= r\begin{pmatrix} b & -a/c \\ a & b/c \end{pmatrix}\left(\mathrm{char}_{\frac{1}{2}\left(\mathbb{Z}_2+\frac{1}{2}\right)}(x)\,\mathrm{char}_{\frac{1}{2}\mathbb{Z}_2}(y) + \mathrm{char}_{\frac{1}{2}\left(\mathbb{Z}_2+\frac{1}{2}\right)}(y)\,\mathrm{char}_{\frac{1}{2}\mathbb{Z}_2}(x)\right)|_{(0,0)}$
Note that:
$\mathcal{FT}(\mathrm{char}_{\frac{1}{2}\left(\mathbb{Z}_2+\frac{1}{2}\right)}(x)\,\mathrm{char}_{\frac{1}{2}\mathbb{Z}_2}(y)) = \mathrm{char}_{\mathbb{Z}_2}(x)e^{\pi i x}\,\mathrm{char}_{\mathbb{Z}_2}(y)$
Then we can compute:

$$\int\limits_{\mathbb{Q}_2\oplus\mathbb{Q}_2} e^{2\pi i\frac{-ac}{b}(x^2+y^2)}\,\mathrm{char}_{\mathbb{Z}_2}(x)e^{\pi i x}\,\mathrm{char}_{\mathbb{Z}_2}(y)dx\,dy =$$

$$= (c/a,-1)_2|c/a|_2\int\limits_{\mathbb{Z}_2\oplus\mathbb{Z}_2} e^{2\pi i\frac{-ac}{b}(x^2+y^2)}e^{\pi i x}dx\,dy = 0$$

We compute similarly for $\mathcal{FT}(\mathrm{char}_{\frac{1}{2}\left(\mathbb{Z}_2+\frac{1}{2}\right)}(y)\,\mathrm{char}_{\frac{1}{2}\mathbb{Z}_2}(x)) = \mathrm{char}_{\mathbb{Z}_2}(y)e^{\pi i y}\,\mathrm{char}_{\mathbb{Z}_2}(x)$ and get 0 in the integral.

- $a=0$. We can pick $b=1$. Then we compute:
  $r\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)\phi_{1,2}\otimes\phi_{2,2}(x,y)|_{(0,0)} = \widehat{\phi}_{1,2}\otimes\widehat{\phi}_{2,2}(x,y)|_{(0,0)} = \mathrm{char}_{\frac{1}{2}(\mathbb{Z}_2+1/2)}(0)\,\mathrm{char}_{\frac{1}{2}(\mathbb{Z}_2+1/2)}(0) = 0$

We compute now the Eisenstein series $E(s, g_z, \phi_1\otimes\phi_2)$:

$$E(s, g_z, \phi_1\otimes\phi_2) =$$

$$= \sum_{(a,b)=1,2|b} y^{1/2}|a^2+b^2|_\infty^{1/2}(-1)^{(a-1)/2}\frac{a-bz}{|a-bz|^2}\prod_p |a^2+b^2|_p^{1/2}\frac{y^{s/2}}{|a-bz|^s}|a^2+b^2|_\infty^{s/2}\prod_p |a^2+b^2|_p^{s/2}$$

$$= \sum_{(a,b)=1,2|b}\frac{a-bz}{|a-bz|^s|a-bz|^2}y^{(s+1)/2}(-1)^{(a-1)/2}$$

$\square$

From Proposition 8.1 and Lemma 8.2, we get the natural Corollary:

Corollary 8.1. *Using the notation established, we have:*

$$Z_f(s, \chi_{D,f}\varphi_f, \Phi_f^\circ) =$$

$$= \frac{1}{72}\#(\mathcal{O}_K/D\mathcal{O}_K)^\times \frac{1}{2} L_{\mathbb{Q}}(2s-1, \chi_0) \prod_{p|D}(1-(-1)^{(p-1)/2}p^{1-2s})$$

$$\times \sum_{[\mathcal{A}]\in K^\times\backslash\mathbb{A}_{K,f}^\times/U} E(2s-2, 3Dg_{z_{\mathcal{A}}}, \phi_1\otimes\phi_2)\overline{\chi_D(\mathcal{A})}\frac{\varphi(\mathcal{A})}{a^{2s-1}}(-1)^{(a-1)/2}y_{\mathcal{A}}^{-s+1/2},$$

Here we let $y_{\mathcal{A}} := \frac{3D\sqrt{3}}{2a}$.

From $Z_f(s, \chi_D\varphi, \Phi_K^\circ)$ to $L(E_D, s)$

We are interested in the value of the L-function at 1. We compute in the following Lemma:

Lemma 8.3. *For all s and for the choice of Schwartz-Bruhat function Φ_K° as above, we have:*

$$L_f(E_D, 1) = \frac{(1+2^{1-s})^{-1}}{\frac{1}{12}\#(\mathcal{O}_K/D\mathcal{O}_K)^\times} Z_f(s, \chi_D\varphi, \Phi_K^\circ)$$

Proof. From Tate's thesis, we have $L_f(s, \chi_D\varphi) = Z_f(s, \chi_D\varphi)\dfrac{\prod_{p|6D} L_p(s, \chi_{D,p}\varphi_p)}{\prod_{p|3D} Z_f(s, \chi_{D,p}\varphi_p, \Phi_p)}$. Since $\chi_D\varphi$ is ramified at $3D$, we have $L_p(s, \chi_{D,p}\varphi_p) = 1$. At 2 we have $L_p(s, \chi_{D,2}\varphi_2) = (1 - \chi_D(2)\varphi(2)2^{-s})^{-1} = (1+2^{1-s})^{-1}$.

We need to compute the integral:

$$Z_p(s, \chi_D\varphi, \Phi_p) = \int_{\mathbb{Q}_p[\omega]^\times} \chi_{D,p}(\alpha_p)\varphi_p(\alpha_p)|\alpha_p|_p^s\Phi_p(\alpha_p)d^\times\alpha_p$$

From the choice of the Schwartz-Bruhat function $\Phi_p = \mathrm{char}_{(\mathbb{Z}+3D\mathbb{Z}_p[\omega])^\times}$ for $p|D$, the integral reduces to $Z_p(s, \chi_D\varphi, \Phi_p) = \int_{(\mathbb{Z}+3D\mathbb{Z}_p[\omega])^\times} \chi_{D,p}(\alpha_p)\varphi_p(\alpha_p)|\alpha_p|_p^s d^\times\alpha_p$. Note that for $p \neq 3$, all the characters χ_D, φ and $|\cdot|_p$ are unramified, thus we just get the volume $\mathrm{vol}\left((\mathbb{Z}+3D\mathbb{Z}_p[\omega])^\times\right)$.

For $p = 3$, we have $\Phi_p = \mathrm{char}_{(\pm1+3\mathbb{Z}_3[\omega])}$. Similarly, we get $\mathrm{vol}\left((\pm1+3\mathbb{Z}_3[\omega])^\times\right)$.

For $p = 2$, we get $Z_2(s, \chi_{D,2}\varphi) = 2\,\mathrm{vol}(1+4\mathbb{Z}_2[\omega])$.

We already computed the volumes.

- p nonsplit, $p|D$: $\mathrm{vol}(\mathbb{Z}_p[\omega]^\times) = (p^2 - 1)\,\mathrm{vol}(1 + p\mathbb{Z}_p[\omega])$.
- p split, $p|D$: $\mathrm{vol}(\mathbb{Z}_p[\omega]^\times) = (p - 1)^2\,\mathrm{vol}(1 + p\mathbb{Z}_p[\omega])$..
- $p = 3$, we have $\mathrm{vol}\,(\pm 1 + 3\mathbb{Z}_3[\omega]) = \frac{1}{3}$.
- $p = 2$, we have $2\,\mathrm{vol}\,(\pm 1 + 2\mathbb{Z}_2[\omega]) = \frac{2}{12} = \frac{1}{6}$.

□

Taking $s = 1$ in the Corollary 8.1 above, we get:

Corollary 8.2. *Using the notation established, we have:*

$$Z_f(1, \chi_{D,f}\varphi_f, \Phi_f^\circ) = \frac{1}{72}\#(\mathcal{O}_K/D\mathcal{O}_K)^\times \frac{\pi}{8} \prod_{p|D}(1 - (-1)^{(p-1)/2}p^{-1})$$

$$\times \sum_{[\mathcal{A}]\in K^\times\backslash\mathbb{A}_{K,f}^\times/U} E(0, 3Dg_{z_\mathcal{A}}, \phi_1 \otimes \phi_2)\overline{\chi_D(\mathcal{A})}\frac{\varphi(\mathcal{A})}{a} y_\mathcal{A}^{-1/2}(-1)^{(a-1)/2}$$

Furthermore, from the Lemma above we have:

Corollary 8.3.

$$L_f(1, \chi_{D,f}\varphi_f, \Phi_f^\circ) =$$

$$= \frac{\pi}{92}\prod_{p|D}(1 - (-1)^{(p-1)/2}p^{-1}) \sum_{[\mathcal{A}]\in K^\times\backslash\mathbb{A}_{K,f}^\times/U} E(0, 3Dg_{z_\mathcal{A}}, \phi_1 \otimes \phi_2)\overline{\chi_D(\mathcal{A})}\frac{\varphi(\mathcal{A})}{a} y_\mathcal{A}^{-1/2}(-1)^{(a-1)/2}$$

Siegel-Weil for $E(s, g, \phi_1 \otimes \phi_2)$.

The Siegel-Weil theorem connects the value of a Siegel-Eisenstein series at $s = 0$ with the value of a theta lift (see [15] for an exposition). In our case, we have:

$$E(0, g, \phi_1 \otimes \phi_2) = 2\Theta_{\phi_1\otimes\phi_2}(g), \tag{8.2}$$

where we define theta lift for $g \in \mathrm{SL}_2(\mathbb{A}_\mathbb{Q})$:

$$\Theta_{\phi_1\otimes\phi_2}(g) := \int\limits_{O(V_\mathbb{Q})\backslash O(V_{\mathbb{A}_\mathbb{Q}})} \theta(g, h_1, \phi_1 \otimes \phi_2)dh_1$$

Note that $r(h_1)\phi_{1,p} \otimes \phi_{2,p} = \phi_{1,p} \otimes \phi_{2,p}$ for all places $p \neq 2$. At 2, we are sending $(x, y) \to (ax + by, -bx + ay)$, where $a^2 + b^2 = 1$. Thus exact one of a, b is divisible by 2. Either way, we get $(\mathbb{Z}_2 + 1/2) \oplus (\mathbb{Z}_2 + 1/2)$ gets sent isomorphically to $(\mathbb{Z}_2 + 1/2) \oplus (\mathbb{Z}_2 + 1/2)$. Thus we get:

$$\Theta_{\phi_1\otimes\phi_2}(g) = \mathrm{vol}(O(V_{\mathbb{Q}}) \setminus O(V_{\mathbb{A}_{\mathbb{Q},f}}))\theta(g, 1, \phi_1 \otimes \phi_2)$$

We can easily compute $\mathrm{vol}(O(V_{\mathbb{Q}}) \setminus O(V_{\mathbb{A}_{\mathbb{Q}}})) = \mathrm{vol}(\mathbb{Q}^\times \setminus \mathbb{A}^\times_{\mathbb{Q},f} = \mathrm{vol}(\prod_p \mathbb{Z}_p^\times) = 1$ from the choice of the self-dual Haar measure.

Remark 8.2. We can further compute $\theta(g_z, 1, \phi_1 \otimes \phi_2)$ explicitly. We get immediately:

$$\theta\left(g_z, 1, \phi_1 \otimes \phi_2\right) = y^{-1/2} \sum_{m,n\in\mathbb{Z}} e^{2\pi i(m^2+n^2)z} e^{\pi i m} e^{\pi i n} = y^{-1/2}\theta_{1/2}(z)^2,$$

where $\theta_{1/2}(z) := \sum\limits_{n\in\mathbb{Z}} e^{2\pi i n^2 z}(-1)^n$ is a theta function of weight $1/2$. As an automorphic form, this is $\theta_{1/2}(z) = y^{-1/2}\theta_{\phi_1}(g_z)$.

Applying Siegel-Weil and the remark above in our case, we get for $z_{\mathcal{A}} = \frac{-b+\sqrt{-3}}{2a}$, $y = \frac{3D\sqrt{3}}{2a}$:

$$y^{1/2}E(0, g_{3Dz_{\mathcal{A}}}, \phi_1 \otimes \phi_2) = 2y^{1/2}\Theta_{\phi_1\otimes\phi_2}(g_{3Dz_{\mathcal{A}}}) = 2\theta_{1/2}(z)^2$$

This gives us in Corollary 8.2:

Corollary 8.4. *Using the same notation as above, we have:*

$$L_f(1, \chi_D\varphi) = \frac{\pi}{48}\prod_{p|D}(1-(-1)^{(p-1)/2}p^{-1}) \sum_{[\mathcal{A}]\in K^\times\setminus\mathbb{A}^\times_{K,f}/U} \theta_{1/2}(3Dz_{\mathcal{A}})^2\overline{\chi_D(\mathcal{A})}\frac{\varphi(\mathcal{A})}{a}(-1)^{(a-1)/2}$$

We showed in section 3 in Lemma 3.5 that for a primitive ideal $\mathcal{A}$, we have $\dfrac{\varphi(\mathcal{A})}{a} = \dfrac{\Theta(\omega)}{\Theta(z_{\mathcal{A}})}$. This gives us:

$$L_f(1, \chi_D\varphi) =$$
$$= D^{-1/3}\Theta(\omega)\frac{\pi}{48}\prod_{p|D}(1-(-1)^{(p-1)/2}p^{-1}) \sum_{[\mathcal{A}]\in K^\times\setminus\mathbb{A}^\times_{K,f}/U} \frac{\theta_{1/2}(3Dz_{\mathcal{A}})^2}{\Theta_K(z_{\mathcal{A}})}D^{1/3}\overline{\chi_D(\mathcal{A})}(-1)^{(a-1)/2}$$

From class field theory, we can find $H_\circ$ a finite abelian extension of K such that

$$\mathrm{Gal}(H_\circ/K) \cong K^\times \setminus \mathbb{A}^\times_{K,f}/U.$$

This is going to be the ray class field for the modulus $12D$.

Finally, we get the following theorem:

Theorem 8.1.

$$L_f(1, \chi_D\varphi) = D^{1/6}\Theta(\omega)\prod_{p|D}(1-(-1)^{(p-1)/2}p^{-1})\frac{\pi}{48}\,\mathrm{Tr}_{H_\circ[\sqrt{D}]/K[\sqrt{D}]}\frac{\theta_{1/2}(3D\omega)^2}{\Theta_K(\omega)}D^{-1/6}$$

The only step needed is to apply the Shimura reciprocity law in order to show that all elements $\frac{\theta_{1/2}(3Dz_{\mathcal{A}})^2}{\Theta_K(z_{\mathcal{A}})} D^{1/3}\overline{\chi_D(\mathcal{A})}D^{-1/2}$ are Galois conjugate. Take $f(z) = \frac{\theta_{1/2}(3Dz)^2}{\Theta_K(z)}$. This is a modular function of level $12D$ with rational coefficients at the cusp ∞. Thus $f(\omega) \in H_\circ$. The proof that all terms are conjugate is a straightforward application of Lemma 4.3.

Chapter 9

Appendix A: Properties of Θ_K

In this appendix we would like to present a few properties of Θ_K. First, we have a functional equation for the theta function (see [11]):

$$\Theta_K(-1/3z) = \frac{3}{\sqrt{-3}} z\Theta_K(z). \tag{9.1}$$

Furthermore, we can compute the transformation of $\Theta_K(z \pm 1/3)$ in the lemma below:

Lemma 9.1. *We have the following relations:*

(i) $\Theta\left(z + \frac{1}{3}\right) = (1-\omega)\Theta(3z) + \omega\Theta(z)$

(ii) $\Theta\left(z - \frac{1}{3}\right) = (1-\omega^2)\Theta(3z) + \omega^2\Theta(z)$

Proof. We will rewrite the Fourier expansion of $\Theta(z)$ for $z := z + 1/3$:

$$\Theta\left(z+\frac{1}{3}\right) = \sum_{m,n\in\mathbb{Z}} e^{2\pi i(m^2+n^2-mn)\left(z+\frac{1}{3}\right)}.$$

We split the sum in two parts, depending on whether or not the ideal $(m + n\omega)$ is prime to $(\sqrt{-3})$. Then we have:

$$\Theta\left(z+\frac{1}{3}\right) = \sum_{m,n\in\mathbb{Z},(\sqrt{-3})|(m+n\omega)} e^{2\pi i(m^2+n^2-mn)\left(z+\frac{1}{3}\right)} + \sum_{m,n\in\mathbb{Z},(\sqrt{-3})\nmid(m+n\omega))} e^{2\pi i(m^2+n^2-mn)\left(z+\frac{1}{3}\right)}.$$

Note that on the RHS we can rewrite the first term as:

$$\sum_{m,n\in\mathbb{Z},(\sqrt{-3})|(m+n\omega)} e^{2\pi i(m^2+n^2-mn)\left(z+\frac{1}{3}\right)} = \sum_{m,n\in\mathbb{Z}} e^{2\pi i(m^2+n^2-mn)(3z+1)} = \Theta(3z+1) = \Theta(3z)$$

Also note that when $3 \nmid m^2+n^2-mn$, then we have $m^2+n^2-mn \equiv 1 \mod 3$. Then the second term on the RHS can be rewritten as:

$$\sum_{m,n\in\mathbb{Z},(\sqrt{-3})\nmid(m+n\omega))} e^{2\pi i(m^2+n^2-mn)\left(z+\frac{1}{3}\right)} = \sum_{m,n\in\mathbb{Z},(\sqrt{-3})\nmid(m+n\omega))} e^{2\pi i(m^2+n^2-mn)z}\omega.$$

We rewrite this:

$$\sum_{m,n\in\mathbb{Z},(\sqrt{-3})\nmid(m+n\omega))} e^{2\pi i(m^2+n^2-mn)\left(z+\frac{1}{3}\right)} =$$

$$= \omega\sum_{m,n\in\mathbb{Z}} e^{2\pi i(m^2+n^2-mn)z} - \omega \sum_{m,n\in\mathbb{Z},(\sqrt{-3})\mid(m+n\omega))} e^{2\pi i(m^2+n^2-mn)z}$$

Finally we recognize the two terms as theta functions Θ_K:

$$\sum_{m,n\in\mathbb{Z},(\sqrt{-3})\nmid(m+n\omega))} e^{2\pi i(m^2+n^2-mn)\left(z+\frac{1}{3}\right)} = \omega\Theta(z) - \omega\Theta(3z)$$

Now going back to our initial computation, we get:

$$\Theta\left(z+\frac{1}{3}\right) = \Theta(3z) + \omega\Theta(z) - \omega\Theta(3z) = (1-\omega)\Theta(3z) + \omega\Theta(z)$$

This finishes the proof of the first formula. We get the second formula by applying the first formula for $z := z - 1/3$. We get $\Theta(z) = (1-\omega)\Theta(3z-1) + \omega\Theta(z-1/3)$ and this is easily rewritten to give us the second formula.

□

Properties of $\Theta_K((-b+\sqrt{3})/6)$.

Lemma 9.2. $\Theta_K\left(\frac{-3+\sqrt{-3}}{6}\right) = 0$

Proof. We apply the functional equation 9.1 for $z = \frac{-3+\sqrt{-3}}{6}$:

$$\Theta\left(\frac{-3+\sqrt{-3}}{6}\right) = (-\sqrt{-3})\frac{-3+\sqrt{-3}}{6}\Theta\left(\frac{3+\sqrt{-3}}{6}\right).$$

Since $\Theta\left(\frac{-3+\sqrt{-3}}{6}\right) = \Theta\left(\frac{3+\sqrt{-3}}{6}\right)$, we get the result of the lemma. □

Lemma 9.3. *For the primitive ideal $\mathcal{A} = [a, \frac{-b+\sqrt{-3}}{a}]_{\mathbb{Z}}$ prime to 3, where $a = \operatorname{Nm}\mathcal{A}$, $b \equiv 0 \mod 3$ and $b^2 \equiv -3 \mod 4a$, we have:*

$$\Theta_K\left(\frac{-b+\sqrt{-3}}{6a}\right) = 0.$$

Proof. The proof is similar to that of Lemma 3.5. We can write the generator of primitive ideal $\mathcal{A} = \left[a, \frac{-b+\sqrt{-3}}{2}\right]$ in the form $k_{\mathcal{A}} = ma + n\dfrac{-b+\sqrt{-3}}{2}$ for some integers m, n. Note that $(m, 3) = 1$, thus we can find through the Euclidean algorithm integers A, B such that $mA + 3nB = 1$, which makes $\begin{pmatrix} A & B \\ -3n & m \end{pmatrix}$ a matrix in $\Gamma_0(3)$. Since Θ is a modular form of weight 1 for $\Gamma_0(3)$, we have:

$$\Theta_K\left(\frac{A\frac{-b+\sqrt{-3}}{6a} + B}{-3n\frac{-b+\sqrt{-3}}{6a} + m}\right) = \left(m - n\frac{-b+\sqrt{-3}}{2a}\right)\Theta_K\left(\frac{-b+\sqrt{-3}}{6a}\right).$$

Noting that $-3n\frac{-b+\sqrt{-3}}{6a} + m = k_{\mathcal{A}}/a = 1/\overline{k}_{\mathcal{A}}$, we can compute

$$\frac{A\frac{-b+\sqrt{-3}}{6a} + B}{-n\frac{-b+\sqrt{-3}}{2a} + m} = \frac{(A\frac{-b+\sqrt{-3}}{2} + 3Ba)\overline{k}_{\mathcal{A}}}{3a}.$$

This is $(3aB + A\frac{-b+\sqrt{-3}}{2})(ma + n\frac{b+\sqrt{-3}}{2})/(3a)$. After expanding, we get:

$$-nA\frac{b^2+3}{4a} + abB/3 + \frac{b(-mA+3nB)}{6} + \frac{\sqrt{-3}}{6}$$

Note that $mA + 3nB = 1$ implies that mA and $3nB$ have different parities. Also we chose b odd, since $b^2 + 3 \equiv 0 \mod 4a$. Finally, recall $3|b$ and thus using the period 1 of Θ_K we get:

$$\Theta_K\left(\frac{A\frac{-b+\sqrt{-3}}{6a} + B}{-3n\frac{-b+\sqrt{-3}}{2a} + m}\right) = \Theta_K\left(\frac{-3+\sqrt{-3}}{6}\right)$$

From the previous Lemma, we have $\Theta_K\left(\frac{-3+\sqrt{-3}}{6}\right)$, thus $\Theta_K\left(\frac{-b+\sqrt{-3}}{6a}\right) = 0$ which finishes the proof.

□

About $\Theta_K(D(-3+\sqrt{-3})/6)$.

In this section we will show that for D a product of split primes $p \equiv 1 \mod 3$ and for the representative ideals $\mathcal{A} = [a, \frac{-b+\sqrt{-3}}{2}]$ of $\mathrm{Cl}(\mathcal{O}_{3D})$with $b \equiv 0 \mod 3$, we have:

$$\sum_{\mathcal{A}\in\mathrm{Cl}(\mathcal{O}_{3D})} \frac{\Theta\left(\frac{-b+\sqrt{-3}}{6a}\right)}{\Theta\left(\frac{-b+\sqrt{-3}}{2a}\right)}\overline{\chi_D(\mathcal{A})}D^{1/3} = 0$$

We will first show that the LHS is equal to the trace of $\frac{\Theta_K\left(D\frac{-b+\sqrt{-3}}{6}\right)}{\Theta(\omega)}D^{1/3}$ with $b \equiv 0 \mod 3$. We will show this by using Shimura reciprocity law. Note first that:

Lemma 9.4. *The modular function* $f_0(z) = \dfrac{\Theta(Dz/3)}{\Theta(z)}$ *is a modular function for* $\Gamma(3D)$ *and* $f_0(z)$ *has rational Fourier coefficients at the cusp* ∞.

Proof. The proof that f_0 is invariant under $\Gamma(3D)$ is straightforward. The proof that the Fourier coefficients are rational is also similar to the proof of Lemma 4.1. $\square$

Lemma 9.5. *For* f_0 *as above and* $\tau = \frac{-b_0+\sqrt{-3}}{2}$, *we have* $f_0(\tau) \in H_{3D}$.

Proof. To show that $f(\tau) \in H_{3D}$, we need to look at action of $U(3D)$. We follow closely the proof of Lemma 4.2. We rewrite the primitive ideal $\mathcal{A} = (A + B\omega)$ as $\mathcal{A} = [a, \frac{-b+\sqrt{-3}}{2}]_{\mathbb{Z}}$ with $b \equiv b_0 \mod 3$. The only difference is computing:

$$f_0(\left(\begin{smallmatrix} ta-sb & -sc/a \\ s & t \end{smallmatrix}\right) z) = \frac{\Theta_K \left(\left(\begin{smallmatrix} D & 0 \\ 0 & 3 \end{smallmatrix}\right) \left(\begin{smallmatrix} ta-sb & -sc/a \\ s & t \end{smallmatrix}\right) z \right)}{\Theta_K \left(\left(\begin{smallmatrix} ta-sb & -sc/a \\ s & t \end{smallmatrix}\right) z \right)} = \frac{\Theta_K \left(\left(\begin{smallmatrix} ta-sb & -scD/(3a) \\ 3s/D & t \end{smallmatrix}\right) (Dz) \right)}{\Theta_K \left(\left(\begin{smallmatrix} ta-sb & -sc/a \\ s & t \end{smallmatrix}\right) z \right)}.$$

Note that we still have $\left(\begin{smallmatrix} ta-sb & -scD/(3a) \\ 3s/D & t \end{smallmatrix}\right), \left(\begin{smallmatrix} ta-sb & -sc/a \\ s & t \end{smallmatrix}\right) \in \Gamma_0(3)$, thus we simply get $f_0(z)$ and all the arguments from Lemma 4.2 follow.

$\square$

Lemma 9.6. *For* $\mathcal{A} = \left[a, \frac{-b+\sqrt{-3}}{2}\right]$ *a primitive ideal ideal with* $a = \operatorname{Nm} \mathcal{A}$ *and* $b^2 \equiv -3 \mod 4a$, *we have:*

$$\frac{\Theta\left(D\frac{-b+\sqrt{-3}}{6a}\right)}{\Theta\left(\frac{-b+\sqrt{-3}}{2a}\right)} = \left(\frac{\Theta\left(D\frac{-b+\sqrt{-3}}{6}\right)}{\Theta(\omega)}\right)^{\sigma_{\mathcal{A}}^{-1}}$$

Proof. Note that $f_0(z)$ satisfies the properties of Lemma 4.3, thus applying its result for $f_0\left(\frac{-b+\sqrt{-3}}{2}\right)$ gives us the result. $\square$

From the previous two lemmas, we immediately get the following Corollary:

Corollary 9.1. *For* $\mathcal{A} = \left[a, \frac{-b+\sqrt{-3}}{2}\right]$ *primitive ideals that are representatives of* $\mathrm{Cl}(\mathcal{O}_{3D})$ *as above, we have:*

$$\mathrm{Tr}_{H_{3D}/K} \frac{\Theta(D\frac{-b+\sqrt{-3}}{6})}{\Theta\left(\frac{-b+\sqrt{-3}}{2}\right)} D^{1/3} = \sum_{\mathcal{A} \in \mathrm{Cl}(\mathcal{O}_{3D})} \frac{\Theta\left(\frac{-b+\sqrt{-3}}{6a}\right)}{\Theta\left(\frac{-b+\sqrt{-3}}{2a}\right)} \overline{\chi_D(\mathcal{A})} D^{1/3}$$

Traces of theta functions

We will show the following lemma:

Lemma 9.7. *For $D \equiv 1 \mod 3$, $b_0 \equiv 0 \mod 3$ as before, we have:*

$$\mathrm{Tr}_{H_{3D}/K} \frac{\Theta\left(D\frac{-3+\sqrt{-3}}{6}\right)}{\Theta(\omega)} D^{1/3} = \sum_{\mathcal{A}\in\mathrm{Cl}(\mathcal{O}_{3D})} \frac{\Theta_K\left(D\frac{-b_0+\sqrt{-3}}{6a}\right)}{\Theta_K\left(D\frac{-b_0+\sqrt{-3}}{6a}\right)} \overline{\chi_D(\mathcal{A})} D^{1/3} = 0.$$

Proof. The method will be to apply Lemma 9.1 two times. We first apply Lemma 9.1 (i) for $z = \frac{1-2D}{6D}$ to get:

$$\Theta\left(\frac{1+\sqrt{-3}}{6D}\right) = (1-\omega)\Theta\left(\frac{1+\sqrt{-3}}{2D}\right) + \omega\Theta\left(\frac{1-2D+\sqrt{-3}}{6D}\right)$$

This can be rewritten as:

$$\frac{\Theta\left(\frac{1+\sqrt{-3}}{6D}\right)}{\Theta(\omega)} = (1-\omega)\frac{\Theta\left(\frac{1+\sqrt{-3}}{2D}\right)}{\Theta(\omega)} + \omega\frac{\Theta\left(\frac{1-2D+\sqrt{-3}}{6D}\right)}{\Theta(\omega)}$$

By taking the inverses and denoting $B_1 := -1+2D$, $a_1 := (B_1^2+3)/4$, we have:

$$3D\frac{\Theta\left(D\frac{-1+\sqrt{-3}}{2}\right)}{\Theta(\omega/3)} = (1-\omega)\frac{\Theta\left(\frac{1+\sqrt{-3}}{2D}\right)}{\Theta(\omega)} + 3D\omega\frac{\Theta\left(D\frac{B_1+\sqrt{-3}}{2a}\right)}{\Theta\left(\frac{B_1+\sqrt{-3}}{6a}\right)}$$

Note that $B_1 \equiv 1-2D \equiv 1 \mod 3$. Furthermore, noting that $\Theta(\omega/3) = (1-\omega)\Theta(\omega)$ and $\Theta\left(\frac{B_1+\sqrt{-3}}{6a}\right) = (1-\omega^2)\Theta\left(\frac{B_1+\sqrt{-3}}{2a}\right)$, we get:

$$\frac{3D}{1-\omega}\frac{\Theta\left(D\frac{-1+\sqrt{-3}}{2}\right)}{\Theta(\omega)} =$$

$$= (1-\omega)\frac{\Theta\left(\frac{1+\sqrt{-3}}{2D}\right)}{\Theta(\omega)} + \frac{3D\omega}{1-\omega^2}\frac{\Theta\left(D\frac{B_1+\sqrt{-3}}{2a_1}\right)}{\Theta\left(\frac{B_1+\sqrt{-3}}{2a}\right)}$$

Multiplying by $D^{1/3}$ and rewriting the first term on the RHS, we have:

$$\frac{3D}{1-\omega}\frac{\Theta\left(D\frac{-1+\sqrt{-3}}{2}\right)}{\Theta(\omega)}D^{1/3} =$$

$$= (1-\omega)(1-\omega^2)\frac{\Theta\left(\frac{1+\sqrt{-3}}{2D}\right)}{(1-\omega^2)\Theta(\omega)}D^{1/3} + \frac{3D\omega}{1-\omega^2}\chi_D(\mathcal{A}_1)^{-1}\frac{\Theta\left(D\frac{B_1+\sqrt{-3}}{2a_1}\right)}{\Theta\left(\frac{B_1+\sqrt{-3}}{2a}\right)}D^{1/3}\chi_D(\mathcal{A}_1)$$

By taking the trace from H_{3D} to K and denoting by $\mathcal{A}_1 := \left(\frac{B_1+\sqrt{-3}}{2}\right)$, we have:

$$\frac{3D}{1-\omega}\operatorname{Tr}_{H_{3D}/K}\frac{\Theta\left(D\omega\right)}{\Theta(\omega)}D^{1/3} =$$

$$= 3\operatorname{Tr}_{H_{3D}/K}\frac{\Theta\left(-D\omega^2\right)}{\Theta(-\omega^2/3)}D^{1/3} + \frac{3D\omega}{1-\omega^2}\overline{\chi_D(\mathcal{A}_1)}^{-1}\operatorname{Tr}_{H_{3D}/K}\frac{\Theta\left(D\frac{B_1+\sqrt{-3}}{2a_1}\right)}{\Theta\left(\frac{B_1+\sqrt{-3}}{2a}\right)}D^{1/3}\overline{\chi_D(\mathcal{A}_1)}$$

Note that by definition we have $\chi_D(\mathcal{A}_1) = \chi_D\left(\frac{B_1+\sqrt{-3}}{2}\omega\right)$. We can compute the value of the character using Lemma 2.5. For each $\mathfrak{p}|D$, we have:

$$\chi_{\mathfrak{p}}\left(\frac{B_1+\sqrt{-3}}{2}\omega\right) = \left(\frac{(1-2D-\sqrt{-3})\omega^2}{(1-2D+\sqrt{-3}}) \omega\right)^{(\operatorname{Nm}\mathfrak{p}-1)/3} = \left(\frac{-1}{1}\right)^{(\operatorname{Nm}\mathfrak{p}-1)/3} = 1.$$

Thus we get $\chi_D(\mathcal{A}_1) = 1$, and we can rewrite the equation above as:

$$\frac{3D}{1-\omega}\operatorname{Tr}_{H_{3D}/K}\frac{\Theta\left(D\omega\right)}{\Theta(\omega)}D^{1/3} =$$

$$= 3\operatorname{Tr}_{H_{3D}/K}\frac{\Theta\left(-D\omega^2\right)}{\Theta(-\omega^2/3)}D^{1/3} + \frac{3D\omega}{1-\omega^2}\operatorname{Tr}_{H_{3D}/K}\frac{\Theta\left(D\frac{B_1+\sqrt{-3}}{2a_1}\right)}{\Theta\left(\frac{B_1+\sqrt{-3}}{2a}\right)}D^{1/3}\overline{\chi_D(\mathcal{A}_1)}.$$

Furthermore, using Lemma 9.6, we have

$$\frac{\Theta\left(D\frac{B_1+\sqrt{-3}}{2a_1}\right)}{\Theta\left(\frac{B_1+\sqrt{-3}}{2a}\right)}D^{1/3}\overline{\chi_D(\mathcal{A}_1)} = \left(\frac{\Theta\left(D\omega\right)}{\Theta\left(\omega\right)}D^{1/3}\overline{\chi_D(\mathcal{A}_1)}\right)^{\sigma_{\mathcal{A}}^{-1}},$$

thus:

$$\operatorname{Tr}_{H_{3D}/K}\frac{\Theta\left(D\frac{B_1+\sqrt{-3}}{2a_1}\right)}{\Theta\left(\frac{B_1+\sqrt{-3}}{2a}\right)}D^{1/3}\overline{\chi_D(\mathcal{A}_1)} = \operatorname{Tr}_{H_{3D}/K}\frac{\Theta\left(D\omega\right)}{\Theta\left(\omega\right)}D^{1/3}$$

Denoting $S := \operatorname{Tr}_{H_{3D}/K}\frac{\Theta(D\omega)}{\Theta(\omega)}D^{1/3}$, we get the relation:

$$\frac{3D}{1-\omega}S = 3\operatorname{Tr}_{H_{3D}/K}\frac{\Theta\left(\frac{1+\sqrt{-3}}{2D}\right)}{\Theta(-\omega^2/3)}D^{1/3} + \frac{3D\omega}{1-\omega^2}S.$$

This implies:

$$\frac{3D}{1-\omega^2}S = 3\,\mathrm{Tr}_{H_{3D}/K}\,\frac{\Theta\left(\frac{1+\sqrt{-3}}{2D}\right)}{\Theta(-\omega^2/3)}D^{1/3}.$$

This is equivalent to:

$$\frac{D}{1-\omega^2}S = \mathrm{Tr}_{H_{3D}/K}\,\frac{\Theta\left(\frac{1+\sqrt{-3}}{2D}\right)}{\Theta(-\omega^2/3)}D^{1/3}.$$

Note that if we apply the transformation $z \to -1/3z$ given by the functional equation (9.1) to both theta functions on the RHS we get:

$$\frac{1}{1-\omega^2}S = \frac{1}{3}\,\mathrm{Tr}_{H_{3D}/K}\,\frac{\Theta\left(D\frac{-1+\sqrt{-3}}{6}\right)}{\Theta(\omega)}D^{1/3}.$$

This is equivalent to:

$$(1-\omega)S = \mathrm{Tr}_{H_{3D}/K}\,\frac{\Theta\left(D\frac{-1+\sqrt{-3}}{6}\right)}{\Theta(\omega)}D^{1/3}. \tag{9.2}$$

We will apply now Lemma 9.1 (ii) for $z = D\frac{-b_1+\sqrt{-3}}{2a}$, where $b_1 \equiv 1 \mod 3$. We denote by b_0 an integer $b_0 \equiv 0 \mod 3$ such that $b_0 \equiv b_1 \mod 4a$. Then we have:

$$\Theta\left(D\frac{-b_0+\sqrt{-3}}{6a}\right) = (1-\omega^2)\Theta\left(D\frac{-1+\sqrt{-3}}{2a}\right) + \omega^2\Theta\left(D\frac{-b_1+\sqrt{-3}}{6a}\right).$$

This can be rewritten as:

$$\frac{\Theta\left(D\frac{-b_0+\sqrt{-3}}{6a}\right)}{\Theta\left(\frac{-b_0+\sqrt{-3}}{2a}\right)}D^{1/3}\chi_D(\mathcal{A}) =$$

$$= (1-\omega^2)\frac{\Theta\left(D\frac{-b_0+\sqrt{-3}}{2a}\right)}{\Theta\left(\frac{-b_0+\sqrt{-3}}{2a}\right)}D^{1/3}\chi_D(\mathcal{A}) + \omega^2\frac{\Theta\left(D\frac{-b_1+\sqrt{-3}}{6a}\right)}{\Theta\left(\frac{-b_0+\sqrt{-3}}{2a}\right)}D^{1/3}\chi_D(\mathcal{A}).$$

By taking the sums, we get:

$$M = (1-\omega^2)S + \omega^2(1-\omega)S = 0$$

Remark 9.1. The above lemma is also true for $D \equiv 2 \mod 3$ with very small adjustments in the proof.

□

Chapter 10

Appendix B: Shimura reciprocity law over Shimura curves

We can look at the modular curve $X_0(3D)$ as a Shimura curve:

$$X_0(3D)(\mathbb{C}) = \mathrm{GL}_2(\mathbb{Q})^+ \setminus \mathcal{H}^\pm \times \mathrm{GL}_2(\mathbb{A}_{\mathbb{Q},f})/V_0(3D),$$

where $V_0(3D) = \left\{\gamma \in \mathrm{GL}_2(\mathbb{A}_{\mathbb{Q},f}) : \gamma \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \mod 3D\widehat{\mathbb{Q}}\right\}$

Defining the modular function $f(z)$.

Θ_K as an automorphic form.

We will reinterpret the theta function Θ_K as a theta lift. Recall the Weil representation for $\mathrm{SL}_2(\mathbb{A}_{\mathbb{Q}})$ acting on $\mathcal{S}(\mathbb{A}_K)$ the Schwartz-Bruhat space for $\mathbb{A}_K$. For $\Phi \in (\mathbb{A}_K)$, we have:

- $r\left(\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \Phi\right)(x) = (a, -3)|a|\Phi(ax), a \in k^\times$
- $r\left(\begin{pmatrix} 1 & b \\ & 1 \end{pmatrix} \Phi\right)(x) = \psi(bq(x)\Phi(x), b \in k$
- $r\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \Phi\right)(x) = \gamma(V, q)\widehat{\Phi}(x)$

Here $\gamma(V, q)$ is the Weil factor that is a 4th root of unity and $(\cdot, -3\cdot)$ is the Hilbert symbol. Furthermore, we denote by $\widehat{\Phi}$ the Fourier transform of Φ with respect to ψ, defined to be:

$$\widehat{\Phi}(x) := \int_{\mathbb{A}_K} \Phi(y)\psi(\langle x, y\rangle)dy,$$

where $\langle x, y\rangle := q(x+y) - q(x) - q(y)$. In our case, for $x = a_1 + b_1\sqrt{-3}$ and $y = a_2 + b_2\sqrt{-3}$ we get $\langle x, y\rangle = 2a_1a_2 + 6b_1b_2$. In the integral above we choose the self-dual Haar measure i.e. the measure for which $\hat{\hat{\Phi}}(x) = \Phi(-x)$.

Using the Weil representation, we can relate the theta function Θ_K to the automorphic Θ:

$$\Theta_\Phi(g) = \sum_{k\in K} r(g)\Phi(k), g \in \mathrm{SL}_2(\mathbb{A}_\mathbb{Q})$$

We choose the Schwartz-Bruhat functions:

$$\Phi_v = \begin{cases} \mathrm{char}_{\mathcal{O}_{K_v}}, v \nmid \infty \\ e^{-2\pi|\cdot|^2}, v = \infty \end{cases}$$

Note that Θ_Φ is that it an automorphic form and it is invariant under $\mathrm{SL}_2(\mathbb{Q})$.

Then for $g_z = \begin{pmatrix} y^{1/2} & xy^{-1/2} \\ 0 & y^{-1/2} \end{pmatrix}$ and $z = x + yi$ we can easily compute:

$$\Theta_\Phi(g_z, 1_f) = y^{1/2}\Theta_K(z) \tag{10.1}$$

The classical definition of $f(z)$.

Lemma 10.1. *Define the modular function:*

$$f(z) = \frac{\Theta(Dz)}{\Theta(z)}.$$

Then f is a modular function for $\Gamma_0(3D) = \left\{\gamma \in \mathrm{SL}_2(\mathbb{Z}) : \gamma \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \mod 3D\right\}$ i.e. $f \in \mathcal{F}_{0,3D}$.

Proof. We will show that f is invariant under $\Gamma_0(3D)$. Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(3D)$. We have:

$$f\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} z\right) = \frac{\Theta\left(\begin{pmatrix} D & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} z\right)}{\Theta\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} z\right)} = \frac{\Theta\left(\begin{pmatrix} a & bD \\ c/D & d \end{pmatrix}\begin{pmatrix} D & 0 \\ 0 & 1 \end{pmatrix} z\right)}{\Theta\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} z\right)}$$

Since $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(3D)$, we have $c/D \equiv 0 \mod 3$ and $\begin{pmatrix} a & Db \\ c/D & d \end{pmatrix} \in \Gamma_0(3)$. Since Θ is a modular form of weight 1 for $\Gamma_0(3)$, we have:

$$f\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} z\right) = \frac{\Theta\left(\begin{pmatrix} a & bD \\ c/D & d \end{pmatrix}(Dz)\right)}{\Theta\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} z\right)} = \frac{(c/D \cdot Dz + d)^{-1}\Theta(Dz)}{(cz+d)^{-1}\Theta(z)} = f(z)$$

$\square$

Note also that both $\Theta(Dz)$ and $\Theta(z)$ have Fourier expansions in $q^{1/D}$ with rational Fourier coefficients.

Rewriting $f(z)$ to be defined on the Shimura curve.

We rewrite $f(z)$ to be defined on the Shimura curve $X_0(3D)$. We take for $[z, 1]$:

$$f[z,1] := \frac{\Theta\left[\begin{pmatrix} D & 0 \\ 0 & 1 \end{pmatrix} z, 1\right]}{\Theta[z,1]}.$$

We want to extend the definition for $[z, g] \in \mathcal{H}^{\pm} \times \mathrm{GL}_2(\mathbb{A}_{\mathbb{Q},f})$. We claim there is $z_0 \in \mathcal{H}^{\pm}$ such that:

$$[z_0, 1] \sim [z, g]$$

in $X_0(3D)$. This is equivalent to having some $\gamma \in \mathrm{GL}_2(\mathbb{Q})^+$ and $g_f \in V_0(3D)$ such that $z = \gamma z_0$, $g = \gamma g_f$. This will follow from the following theorem (see Bump):

Strong approximation theorem. *Let M be a number field and K_0 open compact subgroup of $\mathrm{GL}_n(\mathbb{A}_M, f)$ such that the image of K_0 under the determinant map is $\prod_{v\nmid\infty} \mathcal{O}_v^\times$. Then:*

$$\#\, \mathrm{GL}_n(M)\, \mathrm{GL}_n(M_\infty) \setminus \mathrm{GL}_n(\mathbb{A}_M)/K_0 = \#\, \mathrm{Cl}(\mathcal{O}_M)$$

We apply strong approximation for $\mathbb{Q}$ for $n = 2$ and $V_0(3D)$. Note that $V_0(3D) \twoheadrightarrow \prod_p \mathbb{Z}_p^\times$ is an open compact subgroup of $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$ Since $\mathbb{Q}$ has class number one, we have:

$$\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}}) = \mathrm{GL}_2(\mathbb{R})\, \mathrm{GL}_2(\mathbb{Q}) V_0(3D),$$

we can write $g = \gamma g_f$ for $\gamma \in \mathrm{GL}_2(\mathbb{Q})$ and $g_f \in V_0(3D)$. Moreover, we can change g_f so that we have $\gamma \in \mathrm{GL}_2(\mathbb{Q})^+$ (note that $\gamma_- \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \in V_0(3D)$ and $\mathrm{GL}_2(\mathbb{Q}) = \mathrm{GL}_2(\mathbb{Q})^+ \cup \gamma_-\, \mathrm{GL}_2(\mathbb{Q})^+$).

Definition 10.1. *For $[z,g] \sim [z_0,1]$ on $X_0(3D)$ (i.e. for $z=\gamma z_0, g=\gamma g_f$), we can define:*

$$f[z,g] := f[z_0,1] = \frac{\Theta\left[\begin{pmatrix} D & 0 \\ 0 & 1 \end{pmatrix} z_0, 1\right]}{\Theta[z_0,1]}$$

Checking that the function is well-defined.

We will check that the above definition makes sense. In order for this definition to be well defined, we need to have $f[z_0,1] = f[z_1,1]$ for all $[z_0,1] \sim [z_1,1]$ in $X_0(3D)$. For this to happen we need to have:

$$\begin{aligned} z_0 &= \gamma z_1, \\ 1 &= \gamma g_u, \end{aligned}$$

where $\gamma \in \mathrm{GL}_2(\mathbb{Q})^+$ and $g_u \in V_0(3D)$. This implies $\gamma \in \mathrm{GL}_2(\mathbb{Q})^+ \cap V_0(3D) = \Gamma_0(3D)$. Thus we need:

$$f(z_0) = f(\gamma z_0)$$

This is true due to the Lemma above, thus the function is well defined on $X_0(3D)$.

Rewriting the definition to include $[z,g]$.

If we want to further rewrite the definition, for $z=\gamma z_0, g=\gamma g_f$, where $\gamma \in \mathrm{GL}_2(\mathbb{Q})^+, g_f \in V_0(3D)$, note that in $X_0(3D)$:

$$\begin{aligned} \left[\begin{pmatrix} D & 0 \\ 0 & 1 \end{pmatrix} z_0, 1\right] &\sim \left[\gamma^{-1}z, \begin{pmatrix} 1 & 0 \\ 0 & D \end{pmatrix}\right] \sim \left[z, \gamma \begin{pmatrix} 1 & 0 \\ 0 & D \end{pmatrix}\right] \\ &\sim \left[z, g g_f^{-1} \begin{pmatrix} 1 & 0 \\ 0 & D \end{pmatrix}\right] \sim \left[z, g \begin{pmatrix} 1 & 0 \\ 0 & D \end{pmatrix} g_f'\right], \end{aligned}$$

Here $g_f^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in V_0(3D)$. Then we can write $g_f' = \begin{pmatrix} a & Db \\ c/D & d \end{pmatrix}$.

We define $R(V_0(3D)) := V_0(3D) \cap R(\mathbb{A}_f)$.

Lemma 10.2. $R(\mathbb{A}_f) = R(\mathbb{Q})R(V_0(3D))$

Proof. Let $g \in R(\mathbb{A}_f)$. By strong approximation, we can write $g = \gamma g_v$ for $\gamma \in \mathrm{GL}_2(\mathbb{Q})^+$ and $g_v \in V_0(3D)$. First we will show that we can write $\det g = \det \gamma_0 \det g_{v_0}$, for $\gamma_0 \in R(\mathbb{Q})$ and $g_{v_0} \in R(V_0(3D))$. Since $g \in R(\mathbb{A}_f)$, we have $\det g = q(x)$ for $x \in \mathbb{A}_{K,f}^\times$. We can apply strong approximation to K:

$$\mathbb{A}_{K,f}^{\times} = K^{\times} U_0(3D),$$

thus we can write $x = ku_0$. Then $q(x) = q(k)q(u_0)$. Then we can pick:

$$\begin{pmatrix} 1 & 0 \\ 0 & (\det g) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & q(k) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & q(u_0) \end{pmatrix},$$

Note that $\begin{pmatrix} 1 & 0 \\ 0 & q(k) \end{pmatrix} \in R(\mathbb{Q})$ and $\begin{pmatrix} 1 & 0 \\ 0 & q(u_0) \end{pmatrix} \in R(V_0(3D))$. Then we can write:

$$\begin{aligned} g = \gamma g_v &= \gamma \begin{pmatrix} 1 & 0 \\ 0 & \det \gamma^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \det g \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \det g_v^{-1} \end{pmatrix} g_v \\ &= \gamma \begin{pmatrix} 1 & 0 \\ 0 & \det \gamma^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & q(k) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & q(u_0) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \det g_v^{-1} \end{pmatrix} g_v \end{aligned}$$

We take

$$\gamma_0 = \gamma \begin{pmatrix} 1 & 0 \\ 0 & \det \gamma^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & q(k) \end{pmatrix} \in R(\mathbb{Q})$$

and

$$g_{u_0} = \begin{pmatrix} 1 & 0 \\ 0 & q(u_0) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \det g_v^{-1} \end{pmatrix} g_v \in R(V_0(3D)).$$

$\square$

Lemma 10.3. *For $[z, g] \in X_0(3D)$, we can find $\gamma \in \mathrm{GL}_2(\mathbb{Q})^+, g_u \in V_0(3D)$ such that $\gamma g g_u \in R(\mathbb{A}_f)$.*

Proof. We can write $g = \gamma_0 g_{u,0}$ for $\gamma_0 \in \mathrm{GL}_2(\mathbb{Q})^+$ and $g_{u,0} \in V_0(3D)$. We can rewrite:

$$\gamma g g_u = \begin{pmatrix} 1 & 0 \\ 0 & \det \gamma_0^{-1} \end{pmatrix} \gamma_0 g_{u,0} \begin{pmatrix} 1 & 0 \\ 0 & \det g_{u,0}^{-1} \end{pmatrix},$$

for $\gamma = \begin{pmatrix} 1 & 0 \\ 0 & \det \gamma_0^{-1} \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Q})^+$, $g_u = \begin{pmatrix} 1 & 0 \\ 0 & \det g_{u,0}^{-1} \end{pmatrix} \in V_0(3D)$. Then we get $\gamma g g_u \in R(\mathbb{A}_f)$, since it has determinant 1.

$\square$

In the following we use the notation:

$$g' := \begin{pmatrix} D & 0 \\ 0 & 1 \end{pmatrix} g \begin{pmatrix} 1/D & 0 \\ 0 & 1 \end{pmatrix}$$

Lemma 10.4. *For $g \in R(\mathbb{A}_f)$, we have:*

$$f[z, g] = \frac{\Theta\,[Dz, g']}{\Theta[z, g]}$$

Proof: From Lemma 10.2, we can write $g = \gamma g_u$ for $\gamma \in R(\mathbb{Q})$ and $g_u \in R(V_0(3D))$. Then $[z, g] \sim [\gamma^{-1}z, \gamma^{-1}g] \sim [z_0, 1]$, where $z_0 := \gamma^{-1}z$. Then we can apply the definition:

$$f[z, g] := f[z_0, 1] = \frac{\Theta\,[Dz_0, 1]}{\Theta[z_0, 1]} = \frac{\Theta\,[\gamma' Dz_0, \gamma']}{\Theta[\gamma z_0, \gamma]} = \frac{\Theta\,[Dz, \gamma']}{\Theta[z, \gamma]}$$

We need to show that $\Theta\,[Dz, \gamma']$ is invariant under g'_u and $\Theta[z, \gamma]$ invariant under g_u. From Lemma 10.3, we have that $g_u \in V_0(3D)$ acts trivially on $\Phi_f(x)$, thus:

$$\Theta\,[z, \gamma g_u] = \Theta[z, g]$$

We need to show: $\Theta\,[Dz, \gamma' g'_u] = \Theta\,[Dz, \gamma']$.
We rewrite

$$g'_u = \begin{pmatrix} a & Db \\ c/D & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & ad - bc \end{pmatrix} \begin{pmatrix} a & Db \\ c/(D(ad-bc)) & d/(ad-bc) \end{pmatrix}.$$

Note that we have $g'_u \in \mathrm{GL}_2(\widehat{\mathbb{Z}})$, thus $v_p(ad - bc) = 0$ for all p. Then

$$g'_u := \begin{pmatrix} a & Db \\ c/(D(ad-bc)) & d/(ad-bc) \end{pmatrix}$$

is an element of $\mathrm{SL}_2(\widehat{\mathbb{Z}})$. Using Lemma 10.12, we have $r(g'_u)\Phi_f = \Phi_f$, thus:

$$\Theta\,[Dz, \gamma'] = \Theta\,[Dz, g']$$

This gives us the conclusion.

Lemma 10.5. *If we pick different $[z_1, g_1], [z_2, g_2]$ that are equivalent to $[z, g]$ in $X_0(3D)$ and such that $g_1, g_2 \in R(\mathbb{A}_f)$, we still have:*

$$f[z_1, g_1] = f[z_2, g_2]$$

Proof: By above Lemma we have:

$$f[z_i, g_i] = \frac{\Theta\,[Dz_i, g'_i]}{\Theta[z_i, g_i]}$$

Since $g_i \in R(\mathbb{A}_f)$ by Lemma 10.2 we can find $\gamma_i \in R(\mathbb{Q})$ and $g_{u,i} \in R(V_0(3D))$ such that $g_i = \gamma_i g_{u,i}$. We denote $z_i^* = \gamma_i^{-1} z_i$.

Then $\gamma_i' \in R(\mathbb{Q})$ as well and:

$$f[z_i, g_i] = \frac{\Theta\left[\gamma'^{-1} D z_i, \gamma'^{-1} g_i'\right]}{\Theta\left[\gamma_i^{-1} z_i, \gamma_i^{-1} g_i'\right]} = \frac{\Theta\left[D z_i^*, g_{u,i}'\right]}{\Theta\left[z_i^*, g_{u,i}'\right]}$$

Note that $g_{u,i}, g_{u,i}' \in R(V_0(3D))$ act trivially on $\Phi_f(x)$, thus we actually have:

$$f[z_i, g_i] = \frac{\Theta\left[D z_i^*, 1\right]}{\Theta\left[z_i^*, 1\right]} = f[z_i^*, 1]$$

Since we have $[z_i, g_i] \sim [z_i^*, 1] \sim [z, g]$, we have $[z_1^*, 1] \sim [z_2^*, 1]$ and we have $f(z_1^*) = f(z_2^*)$.

Conclusion about well-definedness. The last few lemmas imply that it is well-defined if we take:

$$f[z, g] := f[z_1, g_1] = \frac{\Theta[D z_1, g_1']}{\Theta[z_1, g_1]},$$

for any representative $[z_1, g_1] \sim [z, g]$ such that $g_1 \in R(\mathbb{A}_f)$.

Shimura reciprocity law.

We will consider the CM point $\omega = \frac{-1+\sqrt{-3}}{2}$ for K and the value of $f(z)$ at $z = \omega$. Since ω is a CM point and $f \in \mathcal{F}$, from CM theory we have that $f(\omega) \in K^{ab}$ is an algebraic integer. Moreover, we can apply Shimura reciprocity law to $f(\omega)$.

In the following we follow the notation of Hida [9].

We have the torus embedding:

$$\begin{aligned} r_\omega : & T_\omega \to \mathrm{GL}_2 \\ & A + B\omega \to \begin{pmatrix} A & B \\ -B & A - B \end{pmatrix} \end{aligned}$$

Note that ω is the unique element of the upper-half plane $\mathcal{H}$ that is fixed by the action of $r_\omega(T_\omega \mathbb{Q})$. We check this action below:

$$\begin{pmatrix} A & B \\ -B & A - B \end{pmatrix} z = z,$$

iff $Az + B = -Bz^2 + (A - B)z$, or equivalently $B(z^2 + z + 1) = 0$ for all $B \in \mathbb{Q}$ i.e. $z = \omega, -1 - \omega$, of which $\omega \in \mathcal{H}$.

Using the Artin map $\mathbb{A}_K^\times \to \mathrm{Gal}(K^{ab}/K), s \to \sigma_s$, we apply Shimura reciprocity law to $f(\omega)$:

$$f[\omega, 1]^{\sigma_{s^{-1}}} = f[z, r_\omega(s)],$$

$f(\omega)$ is in the ring class field H_{3D}.

We claim that $f(\omega) \in H_{3D}$. For this to be true, we need to check that $f(\omega)$ is invariant under the action of $U(3D)$.

Lemma 10.6. $f(\omega) \in H_{3D}$

Proof: In order for $f(\omega) \in H_{3D}$, we need to show that it is invariant under

$$\mathrm{Gal}(K^{ab}/H_{3D}).$$

Using Shimura reciprocity law, we need to show:

$$f[\omega, 1] = f[\omega, r_\omega(s)],$$

for all $s \in U(3D)$. Take as representatives: $s = (A_p + B_p\omega)_p \in U(3D)$. This implies that $A_p + B_p\omega \in (\mathbb{Z}_p[\omega])^\times$, $3D|B_p$ for $p|3D$ and $A_3 \equiv 1 \mod 3$.

Using the definition of $f(z)$ above, invariance under $U(3D)$ is equivalent to:

$$\frac{\Theta\left[\omega, \begin{pmatrix} A & B \\ -B & A-B \end{pmatrix}_p \begin{pmatrix} 1 & 0 \\ 0 & D \end{pmatrix}\right]}{\Theta\left[\omega, \begin{pmatrix} A & B \\ -B & A-B \end{pmatrix}_p\right]} = \frac{\Theta\left[\omega, \begin{pmatrix} 1 & 0 \\ 0 & D \end{pmatrix}\right]}{\Theta\left[\omega, 1\right]}$$

It is enough to show:

$$\Theta\left[\omega, \begin{pmatrix} A & B \\ -B & A-B \end{pmatrix}_p \begin{pmatrix} 1 & 0 \\ 0 & D \end{pmatrix}\right] = \Theta\left[\omega, \begin{pmatrix} 1 & 0 \\ 0 & D \end{pmatrix}\right].$$

Note that the statement $\Theta\left[\omega, \begin{pmatrix} A & B \\ -B & A-B \end{pmatrix}_p\right] = \Theta\left[\omega, 1\right]$ is a particular case.

We rewrite:

$$\Theta\left[\omega, \begin{pmatrix} A & B \\ -B & A-B \end{pmatrix}_p \begin{pmatrix} 1 & 0 \\ 0 & D \end{pmatrix}\right] = \Theta\left[\omega, \begin{pmatrix} 1 & 0 \\ 0 & D \end{pmatrix}\begin{pmatrix} A & BD \\ -B/D & A-B \end{pmatrix}_p\right].$$

Note that showing that $\begin{pmatrix} A & BD \\ -B/D & A-B \end{pmatrix}_p$ has trivial action in the Weil representation is enough to give us our result.

We analyze this action. For $p \nmid D$, we trivially have $\begin{pmatrix} A & BD \\ -B/D & A-B \end{pmatrix}_p \in \mathrm{GL}_2(\mathbb{Z}_p)$. For $p|D$, since $3D|B_p$ we have $B/D \in \mathbb{Z}_p$, thus $\begin{pmatrix} A & BD \\ -B/D & A-B \end{pmatrix}_p \in \mathrm{GL}_2(\mathbb{Z}_p)$ as well.

Denote $m_p = A_p^2 + B_p^2 - A_pB_p \in \mathbb{Z}_p^\times$. Note that $A + B\omega^2 \in (\mathbb{Z}_p[\omega])^\times$. We have:

$$\begin{aligned} r\left(\begin{pmatrix} A & BD \\ -B/D & A-B \end{pmatrix}_p, h_p\right)\Phi_p(x) &= r\begin{pmatrix} A & BD/m \\ -B/D & (A-B)/m \end{pmatrix}_p \Phi_p((A+B\omega^2)x) \\ &= r\begin{pmatrix} A & BD/m \\ -B/D & (A-B)/m \end{pmatrix}_p \Phi_p(x) \\ &= \Phi_p(x) \end{aligned}$$

Since $(A+B\omega)$ is a unit in $\mathbb{Z}_p[\omega]$, we have $\begin{pmatrix} A & BD/m \\ -B/D & (A-B)/m \end{pmatrix}_p \in \mathrm{SL}_2(\mathbb{Z}_p)$, thus acts trivially on Φ_p. This finishes the proof.

One more case: p=3 Here we have to be careful about the Fourier action. Should still work since $3|B$:

$$r\left(\begin{pmatrix} A & BD \\ -B/D & A-B \end{pmatrix}_3, h_p\right)\Phi_p(x) = r\begin{pmatrix} A & BD/m \\ -B/D & (A-B)/m \end{pmatrix}_3 \Phi_p((A+B\omega^2)x)$$

$$= r\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & B/(DA) \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} A & 0 \\ 0 & A^{-1} \end{pmatrix}\begin{pmatrix} 1 & BD/(Am) \\ 0 & 1 \end{pmatrix}\Phi_3(x)$$

Works because $3|B$ (needs a careful calculation of $\widehat{\Phi}_3(x)$).

Recall that $\Phi_3(x) = \mathrm{char}_{\mathbb{Z}_3[\omega]}$. We compute:

- $r\begin{pmatrix} 1 & BD/(Am) \\ 0 & 1 \end{pmatrix}\Phi_3(x) = e^{2\pi i\,\mathrm{Frac}_3(BD/(Am)x^2)}\Phi_3(x)dx = \Phi_3(x)$
- $r\begin{pmatrix} A & 0 \\ 0 & A^{-1} \end{pmatrix}\Phi_3(x) = (A,-3)_3|A|_3\Phi_3(Ax) = \Phi_3(x)$
- $r\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\Phi_3(x) = \int\limits_{\mathbb{Q}_3[\omega]} e^{2\pi i(2aa'+6bb')}\,\mathrm{char}_{\mathbb{Z}_3[\omega]}(a'+b'\sqrt{-3})da'db' = \mathrm{char}_{\mathbb{Z}_3[\omega]}(a+3b\sqrt{-3})$
- $r\begin{pmatrix} 1 & B/(DA) \\ 0 & 1 \end{pmatrix}\mathrm{char}_{\mathbb{Z}_3[\omega]}(a+b/\sqrt{-3}) = e^{2\pi i\,\mathrm{Frac}_3((a^2+3b^2)\frac{B}{DA})}\,\mathrm{char}_{\mathbb{Z}_3[\omega]}(a+3b\sqrt{-3}) = e^{2\pi i\,\mathrm{Frac}_3((3a^2+(3b)^2)\frac{B/3}{DA})}\,\mathrm{char}_{\mathbb{Z}_3[\omega]}(a+3b\sqrt{-3}) = \mathrm{char}_{\mathbb{Z}_3[\omega]}(a+3b\sqrt{-3})$
- $r\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\mathrm{char}_{\mathbb{Z}_3[\omega]}(a+3b\sqrt{-3}) = \mathrm{char}_{\mathbb{Z}_3}(-a-b\sqrt{-3})$
- $r\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\Phi_3(-x) = \Phi_3(x)$

Galois conjugates of $f(\omega)$.

We will look now at the Galois action of $\mathrm{Gal}(H_{3D}/K)$ and compute the Galois conjugates of $f(\omega)$. This is done in the following Lemma:

Proposition 10.1. *For $\mathcal{A} = [a, \frac{b+\sqrt{-3}}{2}]$ a primitive ideal in $I(3D)$, we have:*

$$f(\omega)^{\sigma_{\mathcal{A}}} = \frac{\Theta\left(D\frac{b+\sqrt{-3}}{2a}\right)}{\Theta\left(\frac{b+\sqrt{-3}}{2a}\right)}$$

Proof: Under the isomorphism $\mathrm{Cl}(\mathcal{O}_{3D}) \cong K^\times \backslash \mathbb{A}_{K,f}^\times / U(3D)$, we take as a representative for $\mathcal{A}$ the idele $s = (A + B\omega)_{p|3D}$, where $A + B\omega \in \mathcal{O}_K$ is a generator of $\mathcal{A}$ with $A, B \in \mathbb{Z}$.

Using Shimura reciprocity law, we have:

$$f[\omega, 1]^{\sigma_s} = f[\omega, r_\omega(s^{-1})] = \frac{\Theta\left[\omega, \begin{pmatrix} A & B \\ -B & A-B \end{pmatrix}_{p|3D}^{-1} \begin{pmatrix} 1 & 0 \\ 0 & D \end{pmatrix}\right]}{\Theta\left[\omega, \begin{pmatrix} A & B \\ -B & A-B \end{pmatrix}_{p|3D}^{-1}\right]}$$

By multiplying by $\begin{pmatrix} A & B \\ -B & A-B \end{pmatrix}$ on both sides, we get:

$$f[\omega, 1]^{\sigma_s} = \frac{\Theta\left[\omega, \begin{pmatrix} A & B \\ -B & A-B \end{pmatrix}_{p\nmid 3D} \begin{pmatrix} 1 & 0 \\ 0 & D \end{pmatrix}\right]}{\Theta\left[\omega, \begin{pmatrix} A & B \\ -B & A-B \end{pmatrix}_{p\nmid 3D}\right]} = \frac{\Theta\left[\omega, \begin{pmatrix} 1 & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} A & BD \\ -B/D & A-B \end{pmatrix}_{p\nmid 3D}\right]}{\Theta\left[\omega, \begin{pmatrix} A & B \\ -B & A-B \end{pmatrix}_{p\nmid 3D}\right]}$$

We will compute the numerator. The denominator can be computed similarly for $D = 1$. We compute using the Weil representation for $p \nmid 3D$:

- $p \nmid 3Da$:

$$r\left(\begin{pmatrix} A & BD \\ -B/D & A-B \end{pmatrix}_p, h_p\right) \Phi_p(x) = r\begin{pmatrix} A & B/a \\ -B & (A-B)/a \end{pmatrix} \Phi_p((A + B\omega^2)^{-1}x)|a|_p^{-1}$$
$$= \Phi_p(x)$$

- $p|a$:

$$F_0(x) = r\left(\begin{pmatrix} A & BD \\ -B/D & A-B \end{pmatrix}_p, h_p\right) \Phi_p(x)$$

$$= r\begin{pmatrix} A & BD/a \\ -B/D & (A-B)/a \end{pmatrix}_p \Phi_p((A+B\omega^2)^{-1}x)|a|_p^{-1}$$

We rewrite $\begin{pmatrix} A & BD \\ -B/D & A-B \end{pmatrix}$ as a product:

$$\left(\begin{smallmatrix} 1 & BD/(A-B) \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} a/(A-B) & 0 \\ 0 & (A-B)/a \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & Ba/D(A-B) \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$$

We denote $\alpha = A + B\omega^2$ compute:

- $F_1(x) = |a|_p^{-1} r\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\Phi_p(\alpha^{-1}x) = \Phi_p(\alpha x)|a^{-1}\alpha|_p$
- $F_2(x) := r\begin{pmatrix} 1 & Ba/D(A-B) \\ 0 & 1 \end{pmatrix} F_1(x) = |a^{-1}\alpha|_p e^{-2\pi i \operatorname{Frac}_p\left(\frac{Ba}{D(A-B)}q_p(x)\right)}\Phi_p(\alpha x) = |a^{-1}\alpha|_p\Phi_p(\alpha x)$.
- $F_3(x) := r\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} F_2(x) = |a|_p^{-1}\Phi_p(\alpha^{-1}x)$
- $F_4(x) := r\begin{pmatrix} a/(A-B) & 0 \\ 0 & (A-B)/a \end{pmatrix} F_3(x)$
  $= |a|_p^{-1}(a/(A-B), -3)_p|a/(A-B)|_p\Phi_p(a\alpha^{-1}x) = \Phi_p(\alpha x)$
- $F_5(x) := r\begin{pmatrix} 1 & BD/(A-B) \\ 0 & 1 \end{pmatrix}\Phi_p(\alpha x)$.

Note that $\Phi_p(\alpha x) = |a|_p r\begin{pmatrix} 1 & 0 \\ 0 & 1/a \end{pmatrix}\Phi_p(x)$. Then we can write:

$$\Theta\left[\omega, \left(\begin{smallmatrix} A & B \\ -B & A-B \end{smallmatrix}\right)^{-1}_{p|3D}\left(\begin{smallmatrix} 1 & 0 \\ 0 & D \end{smallmatrix}\right)\right] = \Theta\left[\omega, \left(\begin{smallmatrix} 1 & 0 \\ 0 & D \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & BD/(A-B) \\ 0 & 1 \end{smallmatrix}\right)_{p|a}\left(\begin{smallmatrix} 1 & 0 \\ 0 & a^{-1} \end{smallmatrix}\right)_{p|a}\right]$$

Lemma 10.7. *For $p|a$, we have:*

$$r\begin{pmatrix} 1 & BD/(A-B) \\ 0 & 1 \end{pmatrix}\Phi_p(\alpha x) = r\begin{pmatrix} 1 & -D(b+1)/2 \\ 0 & 1 \end{pmatrix}\Phi_p(\alpha x)$$

Proof: For this to be true it is enough to show that $D\frac{B}{(A-B)} \equiv -D\frac{b+1}{2} \mod a$. Since $(a, D) = 1$, this is equivalent to: $\frac{B}{A-B} \equiv -\frac{b+1}{2} \mod a$. Moreover, since $(a, 2) = 1$, it is equivalent to:

$$\frac{A+B}{A-B} \equiv -b \mod a$$

Recall that we have $b^2 \equiv -3 \mod 4a$. Moreover, $A^2+B^2-AB=a$, thus $A^2+B^2 \equiv AB \mod a$. This is equivalent to $4A^2+4B^2 \equiv 4AB \mod a$, or $(A+B)^2 \equiv -3(A-B)^2 \mod a$, or $\left(\dfrac{A+B}{A-B}\right)^2 \equiv -3 \mod a$. Note that we have used $v_p(A-B)=0$.

Thus we must have $\dfrac{A+B}{A-B} \equiv \pm b \mod a$. Write $(X+Y\sqrt{-3}) = [a, \frac{b+\sqrt{-3}}{2}]$. Then we must have: $X+Y\sqrt{-3} = ma + n\frac{b+\sqrt{-3}}{2}$, thus $X = ma+nb/2 \equiv nb/2 \mod a$ and $Y=n/2$.Moreover, we must have $(n,a)=1$, since $\mathcal{A}$ is a primitive ideal. Thus $X/Y \equiv b \mod a$. Also note that $X=A+B/2, Y=B/2$ and:

$$\frac{A+B}{A-B} = \frac{X+3Y}{X-Y} = \frac{X/Y+3}{X/Y-1} \equiv \frac{b+3}{b-1} \equiv -b \mod a.$$

Thus we got in our proposition:

$$\Theta\left[\omega, \begin{pmatrix} A & B \\ -B & A-B \end{pmatrix}^{-1}_{p|3D} \begin{pmatrix} 1 & 0 \\ 0 & D \end{pmatrix}\right] = \Theta\left[\omega, \begin{pmatrix} 1 & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} 1 & -D(b+1)/2 \\ 0 & 1 \end{pmatrix}_{p|a} \begin{pmatrix} 1 & 0 \\ 0 & a^{-1} \end{pmatrix}_{p|a}\right]$$

We multiply both the infinite and the finite part by $\begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}\begin{pmatrix} 1 & D(b+1)/2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} D & 0 \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Q})$. The action on ω is:

$$\begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}\begin{pmatrix} 1 & D(b+1)/2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} D & 0 \\ 0 & 1 \end{pmatrix} \omega = D\frac{b+\sqrt{-3}}{2a}$$

We get:

$$\Theta\left[\omega, \left(\begin{smallmatrix} A & B \\ -B & A-B \end{smallmatrix}\right)^{-1}_{p|3D} \left(\begin{smallmatrix} 1 & 0 \\ 0 & D \end{smallmatrix}\right)\right] = |a|_f^{-1}\Theta\left[D\frac{b+\sqrt{-3}}{2a}, \left(\begin{smallmatrix} 1 & 0 \\ 0 & a^{-1} \end{smallmatrix}\right)_{p\nmid a} \left(\begin{smallmatrix} 1 & D(b+1)/2 \\ 0 & 1 \end{smallmatrix}\right)_{p\nmid a}\right]$$

Note that for $p \nmid a$, we have: $\begin{pmatrix} 1 & 0 \\ 0 & a^{-1} \end{pmatrix}_p, \begin{pmatrix} 1 & D(b+1)/2 \\ 0 & 1 \end{pmatrix}_p \in \mathrm{SL}_2(\mathbb{Z}_p)$ and act trivially on $\Phi_p(x)$. Thus we get:

$$\Theta\left[\omega, \begin{pmatrix} A & B \\ -B & A-B \end{pmatrix}^{-1}_{p|3D} \begin{pmatrix} 1 & 0 \\ 0 & D \end{pmatrix}\right] = |a|_f^{-1}\Theta\left[D\frac{b+\sqrt{-3}}{2a}, 1\right]$$

Applying this also to $D=1$, we get:

$$\Theta\left[\omega, \begin{pmatrix} A & B \\ -B & A-B \end{pmatrix}^{-1}_{p|3D}\right] = |a|_f^{-1}\Theta\left[\frac{b+\sqrt{-3}}{2a}, 1\right]$$

Taking the ratio of the two theta functions gives us the result.

Some lemmas needed.

We have used above several facts:

Lemma 10.8. *Let* $F(x) = \Phi_p(\alpha x)$. *Then*

$$\widehat{F}(x) = |\alpha|_p^{-1} \Phi_p(\alpha^{-1} x)$$

Proof: We have by definition:

$$\widehat{F}(x) = \int_{\mathbb{Q}_p[\omega]} \psi(\langle x, y \rangle) \Phi_p(\alpha y) dy$$

We make the change of variable $y' := \alpha y$ and get:

$$\widehat{F}(x) = |\alpha|_p^{-1} \int_{\mathbb{Q}_p[\omega]} \psi(\langle x, \alpha^{-1} y' \rangle) \Phi_p(y') dy' = |\alpha|_p^{-1} \int_{\mathbb{Z}_p[\omega]} \psi(\langle \alpha^{-1} x, y' \rangle) dy'$$

Note that the integral is 0 iff $\psi(\langle \alpha^{-1} x, y' \rangle)$ is non-trivial on $\mathbb{Z}_p[\omega]$. Thus it is nonzero exactly for $\alpha^{-1} x \in \mathbb{Z}_p[\omega]$, in which case the integral equals $|\alpha|_p^{-1}$.

Lemma 10.9. *If* $q_p(\alpha) = a$ *and* $v_p(a) \leq v_p(m)$, *then:*

$$e^{-2\pi i \operatorname{Frac}_p(mq(x))} \Phi_p(\alpha x) = \Phi_p(\alpha x)$$

Proof: We have $\Phi_p(\alpha x) \neq 0$ iff $v_p(\alpha x) \geq 0$. Thus when $\Phi_p(\alpha x) \neq 0$, we have $v_p(\alpha x) \geq 0$, then $v_p(q_p(\alpha x)) \geq 0$. Then we have:

$$\operatorname{Frac}_p(mq(x)) = \operatorname{Frac}_p\left(\frac{m}{a} q_p(\alpha x)\right) = 0$$

Thus we have either both sides equal to 0, or $e^{-2\pi i \operatorname{Frac}_p(mq(x))} = 1$ and both sides are equal to $\Phi_p(\alpha x)$.

Lemma 10.10. *For* $\mathcal{A} = (X + Y\sqrt{-3}) = \left[a, \frac{b+\sqrt{-3}}{2}\right]$ *a primitive ideal, we have* $X/Y \equiv b \mod a$.

Proof: We must have: $X + Y\sqrt{-3} = ma + n\frac{b+\sqrt{-3}}{2}$, thus $X = ma + nb/2 \equiv nb/2 \mod a$ and $Y = n/2$. Moreover, we have $(n, a) = 1$, since $\mathcal{A}$ is a primitive ideal. Thus $X/Y \equiv b \mod a$.

Lemma 10.11. *For* $g_p \in \prod \mathrm{SL}_2(\mathbb{Z}_p)$, *we have:*

$$r(g_p)\Phi_p(x) = \Phi_p(x)$$

Lemma 10.12. *For* $g_f \in R(V_0(3D))$, *we have:*

$$r(g_f)\Phi_f(x) = \Phi_f(x)$$

Bibliography

[1] J. W. S. Cassels. "Arithmetic on curves of genus 1, IV. Proof of the Hauptvermutung". In: *J. reine angew. Math. 211* 211 (1962), pp. 95–112.

[2] J. Coates and A. Wiles. "On the conjecture of Birch and Swinnerton-Dyer". In: *Invent. Math.* 32 (1977), pp. 233–251.

[3] H. Cohn. *Introduction to the Construction of Class Fields*. Dover Publications, 1985.

[4] D. A. Cox. *Primes of the form $x^2 + ny^2$: Fermat, class field theory, and complex multiplication*. Vol. 34. John Wiley & Sons, 2011.

[5] C. Skinner D. Jetchev and X. Wan. "The Birch and Swinnerton-Dyer formula for elliptic curves of analytic rank one". In: *prerpint, arXiv:1512.06894* (2015).

[6] A.C.P. Gee. "Class fields by Shimura reciprocity". In: *thesis* (2001).

[7] B. Gross and D. Zagier. "Heegner points and derivatives of L-series". In: *Invent. Math.* 84.2 (1986), pp. 225–320.

[8] E. Hecke. *Mathematische Werke*. Auftrage der Akademie der Wissenschaften zu Gottingen Vandenhoeck & Ruprecht, 1959.

[9] H. Hida. *Elliptic curves and arithmetic invariants.* New York: Springer, 2013.

[10] M. Rosen K. Ireland. *A classical introduction to modern number theory*. Second ed. Vol. 84. Grad. Texts in Math, 1985.

[11] G. Koehler. *Eta products and theta series identities*. Springer Science & Business Media, 2011, pp. 526–534.

[12] V. A. Kolyvagin. "Finiteness of $E(\mathbb{Q})$ and $\text{Ш}(E;\mathbb{Q})$ for a subclass of Weil curves". In: *Math.USSR Izvestiya* 32.3 (1989).

[13] E. Kowalski. *Automorphic forms, L-functions and number theory.*

[14] A. Pacetti. "A formula for the central value of certain Hecke L-functions". In: *Journal of Number Theory* 113.2 (2005), pp. 339–379.

[15] D. Prasad. *A brief survey on the theta correspondence.* Tiruchirapalli, India., Jan. 1998.

[16] F. Rodriguez-Villegas and D. Zagier. "Square root of central values of Hecke L-functions". In: *Advances in Number Theory* (1993). Ed. by F. Q. Gouvea and N. Yui, pp. 81–99.

[17] F. Rodriguez-Villegas and D. Zagier. *Which primes are sums of two cubes?* Vol. 15. Amer. Math. Soc., Providence, RI.: Number theory (Halifax, NS, 1994), CMS Conf. Proc., 1994, pp. 295–306.

[18] K. Rubin. “The "main conjectures" of Iwasawa theory for imaginary quadratic fields”. In: *Invent. Math.* 103 (1991), pp. 25–68.

[19] G. Shimura. “On the holomorphy of certain Dirichlet series”. In: *Proc. London Math. Soc.* 31.3 (1975), pp. 79–98.

[20] J. Silverman. *The arithmetic of elliptic curves.* Vol. 106. Springer-Verlag, New York: Graduate texts in math., 1986.

[21] P. Stevenhagen. “Hilbert’s 12th problem, complex multiplication and Shimura reciprocity”. In: *Department of Mathematics, Computer Science, Physics and Astronomy, University of Amsterdam* (1998).

[22] J.B. Tunnell. “A Classical Diophantine Problem and Modular Forms of Weight 3/2”. In: *Invent. math.* 72.323-334 (1983).

[23] J.-L. Waldspurger. “Sur les coefficients de Fourier des formes modulaires de poids demi-entier”. In: *J. de Math. pures et appliques* 60 (1981), pp. 375–484.