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Author
Séquin, Carlo H

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Möbius Bridges

Carlo H. Séquin
CS Division, University of California, Berkeley
E-mail: sequin@cs.berkeley.edu

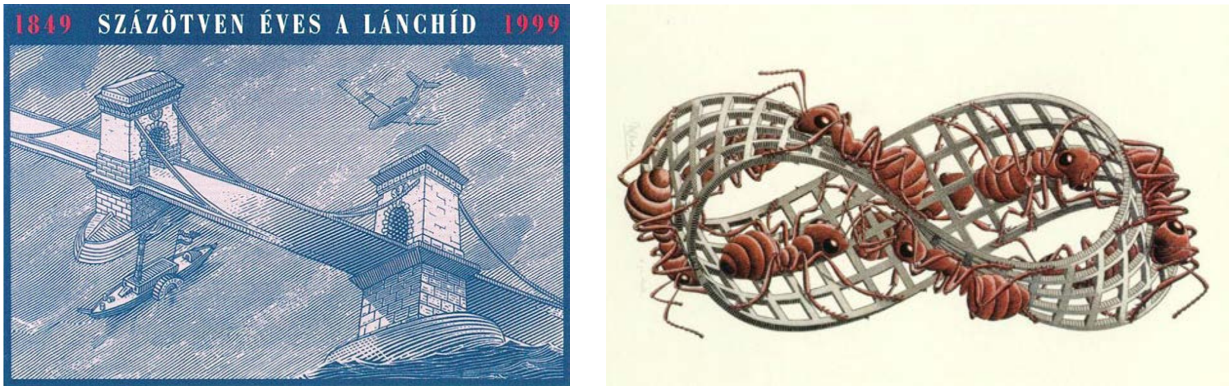
Abstract

Key concepts and geometrical constraints are discussed that allow the construction of a usable bridge that is topological equivalent to a Möbius band. A multi-year search in publications and on the internet for real-world bridges that meet these requirements has not identified a single clean construction that warrants the designation "Möbius bridge," but a few promising designs can be found. Several simple but practical designs are presented here.

1. Introduction

July 2017 marks the twentieth installment of the annual Bridges conference [1], which elucidates the connections between mathematics and art, music, architecture, and many other cultural venues. This year the conference has been held in Waterloo, Canada. In its twenty-year history it has visited many places around the globe, including Seoul in South Korea, Coimbra in Portugal, Pécz in Hungary, and Leeuwarden in the Netherlands, the hometown of M.C. Escher.

This conference series got started by Reza Sarhangi [2] at Southwestern College in Winfield, Kansas. After a few occurrences at this initial location, Reza Sarhangi and other core members of this conference started discussing the possibility of establishing some kind of a commemorative entity of the conference on the Winfield campus. Since Escher, Möbius, and Klein are among the heroes of this Math-Art community, suggestions included an *Escher Garden*, a *Möbius Bridge*, or a *Klein Bottle House*. This prompted me to study the feasibility of such entities; and over the following year, I developed some practical designs for bridges and buildings that follow the geometry of a Möbius band [3]. From the start, I ruled out optical illusions that imply some twisted geometry as, for instance, in the *Impossible Bridge* by István Orosz in Figure 1(a) [4]. Also, while M.C. Escher's illustration of a Möbius Strip (Figure 1b) [5] may serve as an inspiration, I wanted to find practical structures suitable for human use; humans are not like ants or geckos that can walk upside-down along a ceiling.



(a) (b)

Figure 1: *(a) Impossible Bridge by I. Orosz [4], (b) Möbius Strip II by M.C. Escher [5].*

While modernist architecture and industrial design in the 20th century typically follow the paradigm "form follows function" [6], here we have an extreme case of a design challenge where this principle is reversed: A specific form is given – or rather some topological constraints – and the question is then asked: can we still have this construction function as a bridge? Section 2 starts with a discussion of the key concepts that underlie the construction of a functional and usable Möbius bridge. Sections 3-5 present several conceptual approaches that lead to a variety of possible designs for true Möbius bridges. Sections 6 and 7 analyze some real-world bridges and some pending designs that show up when one searches the world wide web for pictures of Möbius bridges; however, most of them do not satisfy the requirements for an actual Möbius topology.

2. Key Considerations

To have a chance of being topologically equivalent to Möbius band, the bridge and its support structure must form a closed loop with a cross-sectional profile that readily conveys how much it is twisted. Half of the complete loop may serve as the usable part of the bridge going fairly directly from some point A to another point B. The other part of the loop could possibly serve as second redundant path between the same two points – but, more practically, it might serve as part of the suspension structure of the bridge or as a dramatic, decorative element.

If the 180° twist inherent in a Möbius loop is spread out uniformly along the whole loop, then the usable portion of the Bridge will undergo a serious amount of roll – perhaps as much as 90°. This would constitute a problem for most users of the bridge (Figure 2a). However, this longitudinal roll can be accommodated, if the profile of the walkway surface is made concave, so that there is always a portion of the walkway surface that is horizontal, regardless of the azimuthal orientation of the roadbed (Figure 2b,c).

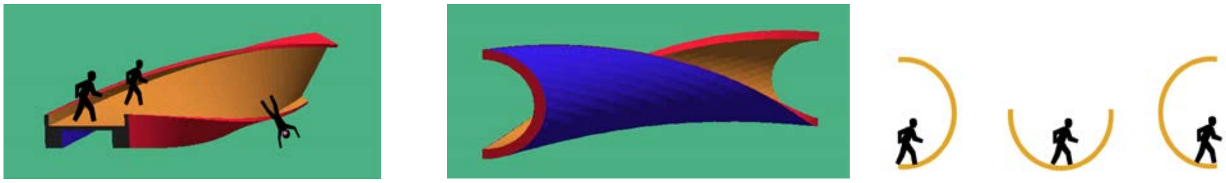

(a) (b) (c)

Figure 2: *(a) A twisted walkway, (b) concave cross section that accommodates some roll (c).*

Figure 3 shows how such a twisted walkway could be completed into a full Möbius band, possibly using the return loop as the supporting arch of a suspension bridge.

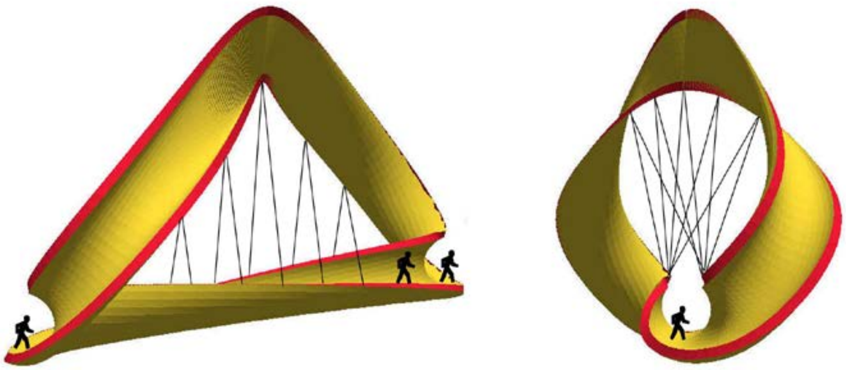

(a) (b)

Figure 3: *Möbius bridge formed by a non-planar loop: (a) side view, (b) end view.*

An easier and more practical solution is to keep the usable portion of the Bridge horizontal and relegate the full 180° of twist into the return path. This approach offers many possibilities, and the two key issues then become how to connect the return loop to the ends of the roadway, so that the Möbius configuration remains plainly visible, and how to route people and traffic onto the mostly horizontal roadbed.

To a mathematician, a Möbius band is just a 2-manifold that is single-sided, has a single border curve, and a genus of 1. Geometrically, Möbius bands come in an infinite variety of different shapes (Figure 4). Most of them do not look very useful for being molded into a practical bridge design. The one shape that I have focused on in this paper is the "classical" Möbius band produced by sweeping a thin rectangular cross section along a smooth, closed space curve, while giving it an azimuthal twist around that sweep curve of ±180° (Figure 4f). All conceivable Möbius bands can be smoothly transformed with a regular homotopy into one of these two minimally twisted ribbons. Figures 3, 4b, 4c, 4e, 4f, 5, 7b, 10, and 21 in this paper depict Möbius bands with a right-handed (positive) half twist, while Figures 1b, 7a, 9, 11, 15, and 16 exhibit the opposite direction of twisting. Figures 4a, 4d, 8, and 17 show bands with three negative half-twists, which makes them topologically equivalent to a band with a single right-handed half-twist.

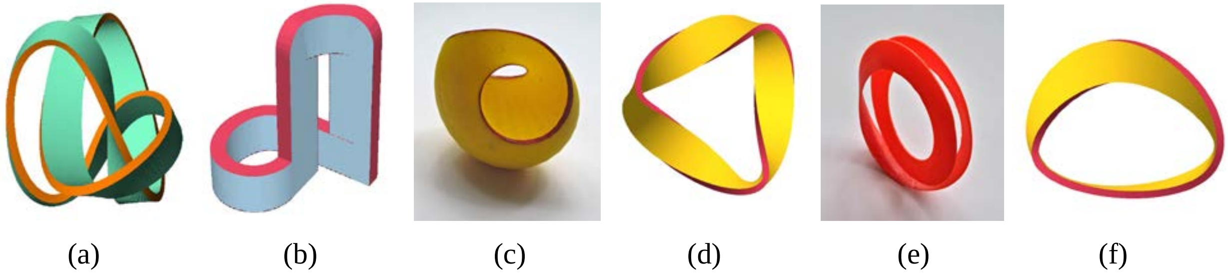

(a) (b) (c) (d) (e) (f)

Figure 4: *Various Möbius bands.*

Within this selected class of simple Möbius bands, there is still an infinite variety of geometrical realizations that all smoothly transform into one another. There is thus no clear logical way to partition the various Möbius bridge designs into different classes except for the sign of the built-in twist. Some attributes that might be used for classification are: whether the Möbius band has any punctures ("holes") (Figure 5b), whether the return path is below or above the useful roadbed, and whether it serves as part of the support structure for the latter. Other attributes might be whether the Möbius loop is close to planar and whether the roadbed is mostly straight. But all of these attributes are somewhat fuzzy: What if the roadbed is just slightly curved, or if portions of the return path lie below the roadbed, while other portions extend above it?

For this reason, I have chosen a different approach to present several different options of constructing a Möbius bridge. I start with different conceptual views of a Möbius band structure – Escher's woodcut (Figure 1b) or one version of the recycling symbol (Figure 10a) – and describe the geometrical refinement necessary to turn that geometry into a useful bridge design. This should also make it clear implicitly that there are most likely many other functional and beautiful designs for a Möbius bridge. For instance, one could use a Möbius that twists through 540° or an even higher multiple of 180°. But it seems challenging enough to accommodate even a single half-twist in a practical manner. Of course, if the twist is relegated entirely into the return path or into a mostly decorative arch, then the twist could readily be increased to three, five, or more half-twists. In most cases, this would yield no structural advantage and offer only limited aesthetic benefits.

3. Twisted Loop Returns

A first simple and elegant approach is to keep the Möbius loop mostly in a vertical plane, keeping the roadbed flat, and putting 180° of twist into the return loop. In this approach, the Möbius topology stays maximally visible, but access to the flat, untwisted roadbed becomes somewhat tricky. If the return loop is below the usable road bed, one may need some kind of (transparent?) access panels to allow users to get onto the bridge without obscuring the geometry of the Möbius loop (Figure 5a). Alternatively, if the return loop is being placed above the road, then an elegant suspension bridge may result; but some openings would have to be cut into the lower portions of this arch to give traffic access to the roadbed (Figure 5b).

(a) (b)

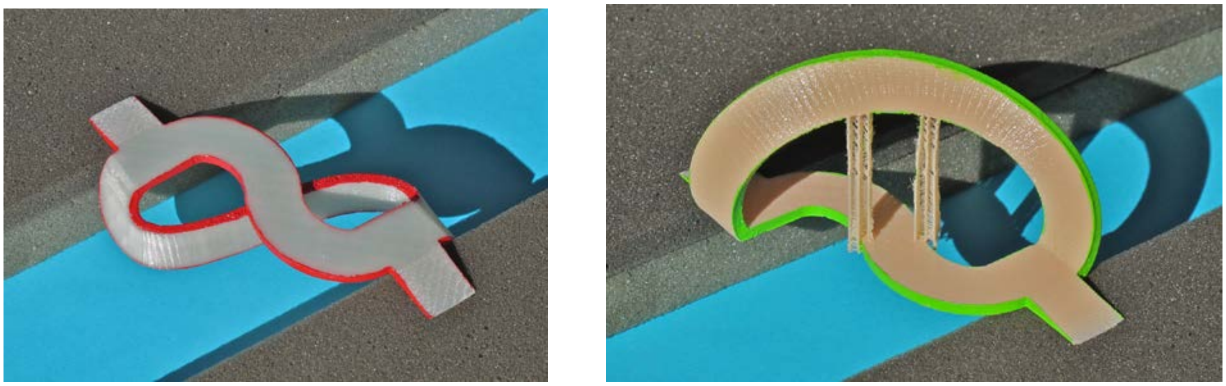
Figure 5: *Mostly planar Möbius loops: (a) the return path of the loop is below the walkway, (b) the loop return is above the walkway.*

In this model, the profile of the Möbius band is not just a simple rectangle, but has the shape of an I-beam. This has been done primarily to highlight the single contiguous edge of this structure, but also to provide some sort of railing at the edges of the walking surface. In the suspension arch, these edge enhancements may not be necessary, but they are kept to maintain a consistent cross section all around the Möbius loop.

If these edges were inflated more significantly, to the point where they become tubular structures in their own right, the suspension system could now be seen as two separate arches crossing each other. This is a design that is quite practical from an engineering point of view. It has been used in several existing bridges, for instance, in the Bill Coats Bridge in the Hermann Park in Houston, Texas (Figure 6). If the two tubular arches supporting this bridge were linked to one another, as indicated with the half-dozen horizontal connector tubes, and if the two arches actually crossed over one another, rather than passing through each other, so that a clear sense of chiral twist were established, then this bridge could be seen as a close relative of the design shown in Figure 5b. However, the discussion of bridges with interesting, multi-arch suspension systems is beyond the scope of this article.

Figure 6: *Bill Coats Bridge in the Hermann Park in Houston, Texas [7].*

The access problems shown in Figure 5 can be overcome by routing bridge traffic onto the roadbed from its sides. To avoid highly curved on-ramps, the roadbed itself could take on a more curvaceous form, rather than stretching at right angle across the given river or valley (Figure 7). This approach would also typically lead to a non-planar return loop. A solution very close to this concept has actually been chosen for a real-world design in the UK (Section 7).

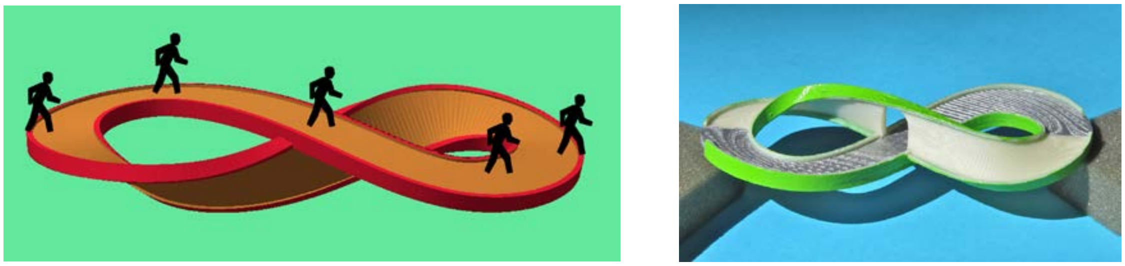


(a) (b)

Figure 7: *Non-planar Möbius loops with a meandering walkway and "lateral" entry ramps: (a) return loop supports the walkway from below, (b) return loop serves as a suspension arch.*

4. Figure-8 Configurations

Another conceptual approach to building a Möbius bridge starts with M.C. Escher's "*Möbius Strip II*" (Figure 1b) [5]. It suggests forming a smoothly turning, twisted figure-8 shape, in which half of it is kept horizontal to serve as the usable part of the bridge. This basic geometry also results in a meandering usable path with lateral entries into the roadway (Figure 8). Again, the other half of the figure-8 shape, which accommodates the required 180° of twist, can be used as a supporting structure. The supporting loop return is most naturally placed below the walkway (Figure 8a). If it is routed directly above the usable pathway, a cutout may be needed to let traffic flow through (Figure 8b). In both cases, the ribbon stands on end in the middle of the bridge span. This vertical orientation of the ribbon provides maximal strength against bending due to the weight of the bridge and the traffic on it.



(a) (b)

Figure 8: *Bridges inspired by Figure 1b: (a) the supporting branch lies below the walkway, (b) the suspension structure with cut-out is placed above the walkway.*

5. Dual-Path Structures

Section 2 mentions the possibility of using both the "forward" and the "return" branch of the Möbius loop to accommodate some traffic. A first solution is already implied by Figure 8b. The narrow, green edge in the upper branch could be modified to accommodate pedestrian traffic, while the silver roadbed accommodates bicycles and heavier traffic. The main difficulty is how to give these two kinds of traffic access to their respective branches without obscuring the overall Möbius geometry. A possible solution is shown by Figure 9, where the twisting of the Möbius band has been modified so that the top edge of the upper branch, which provides the pedestrian path, merges into the outer edges of the figure-8 loop; this enables a more natural access to this pathway. The resulting ribbon geometry then becomes a close relative of Figure 7b.

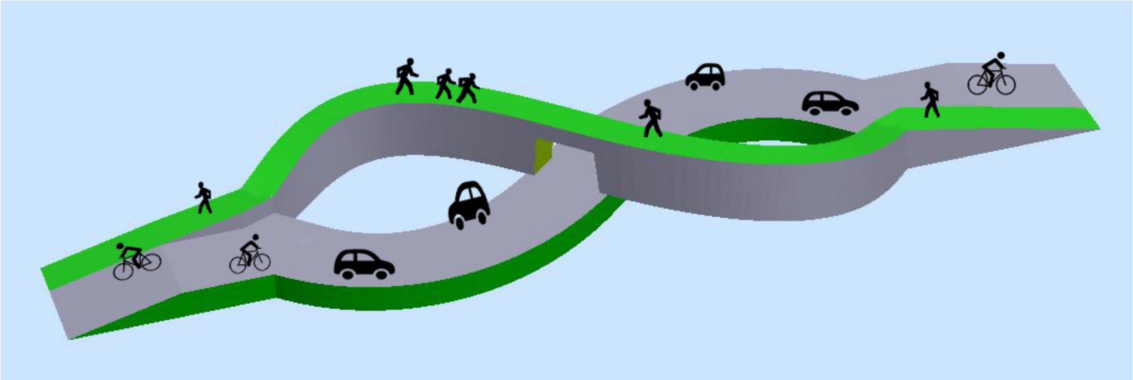
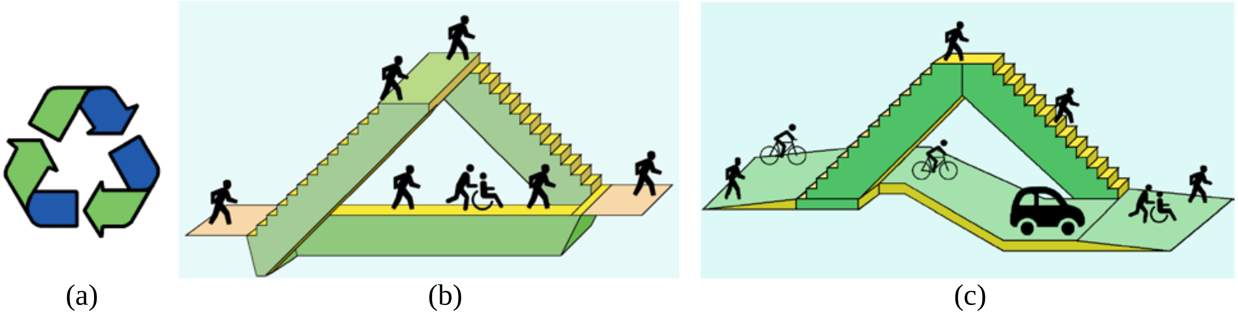


Figure 9: *Dual-use bridge: The green edge of a thick Möbius band is used for pedestrian traffic.*

Another dual-path solution has been derived from the recycling symbol – not the 3-fold-symmetric one with its 3-fold flip, but the less symmetric, singly-twisted version (Figure 10a). The three green beams and their connections directly follow the folding pattern in this recycling symbol. The horizontal branch of the triangle may be made usable for wheelchairs, while the two other triangle branches provide a more adventurous route in the style of a stair bridge, as may be found in a Japanese garden (Figure 10b). Note, that in this bridge all traffic happens mostly on the narrow edge of the Möbius band; only at the very top will pedestrians travel across the wide, flat portion of the ribbon to get from one edge to the other.

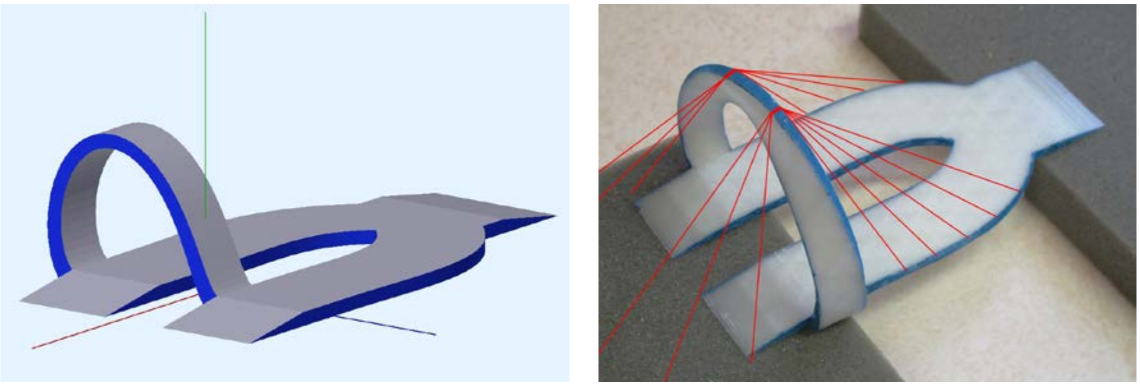


(a) (b) (c)

Figure 10: *Two-path structure derived from recycling symbol (a), path on the edge of the ribbon (b); (c) horizontal path widened to allow motorized traffic.*

Figure 10c shows a modified solution, where the horizontal branch has been widened to accommodate larger vehicular traffic. To result in a true Möbius ribbon, the green faces of the arch must now emerge out of the sides of the flat roadbed. The stair-bridge part can be kept in a single vertical plane, if the roadbed is allowed to meander between the endpoints of the stairs. The foot traffic is now relegated entirely to the narrower yellow edge of the Möbius band.

Figure 11 explores ways to make both branches available for vehicular traffic. Both branches are kept in a horizontal plane and merged into a flat loop at one end. At the other end, a decorative arch is introduced to complete the Möbius geometry. This arch itself can be completely twist-free, if it is connected to the two roadbeds at edges facing in the same direction (Figure 11a). Alternatively, a more dramatic arch may span both roadbeds; but now this arch must exhibit a twist of 180°. This arch need not be purely decorative; as one of the reviewers pointed out, it could serve as a supporting arch for a cable stay bridge (Figure 11b).



(a) (b)

Figure 11: *Two-path bridge: (a) CAD model showing a decorative arch to close the Möbius loop, (b) 3D print showing a twisted arch that supports stay cables.*

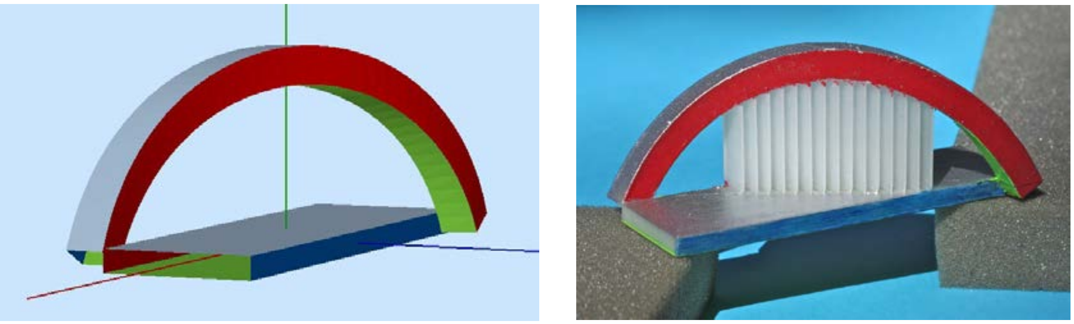
6. Analysis of a Bridge in Poland

I have been looking for an actual, functional Möbius bridge ever since I wrote my paper "To Build a Twisted Bridge" for the Bridges 2000 conference [3]. I have been wondering how long it might take until such a bridge finally gets realized somewhere in the world. More recently, a short YouTube video [8] made some of these ideas more widely accessible. This prompted Paweł Bieliński [9] to send me a reference to a suspension bridge [10] in his hometown Tczew in Poland (Figure 12), suggesting that this might possibly qualify as a Möbius bridge.

(a) (b)

Figure 12: *Wiadukt in Tczew, Poland: (a) side view [11], (b) end view [12].*

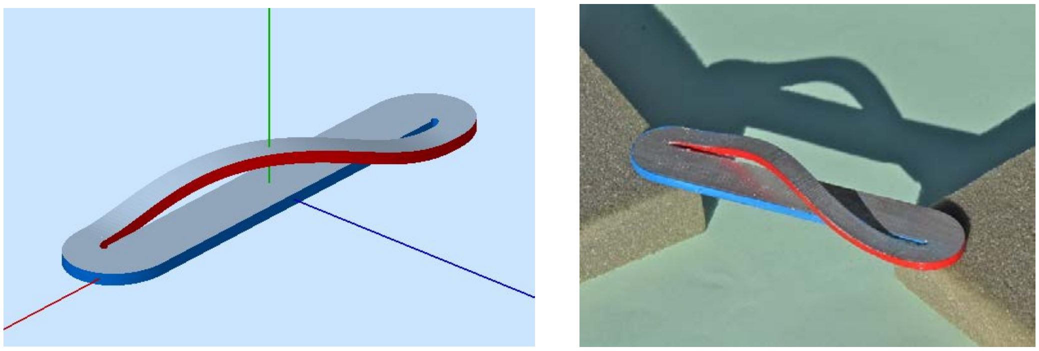
It is a beautiful bridge, and I was excited to see an actual candidate for a Möbius bridge. To check the twistiness of this bridge, I made a simple CAD model, where the roadbed and the suspension arch are both modeled with simple prismatic sweeps using rectangular cross sections (Figure 13a). I also fabricated this object on a 3D printer (Figure 13b). In the Tczew Wiadukt, the suspension arch is emerging from separate foundations in the ground and is not directly coupled to the roadbed. To analyze the twist of the whole structure, the ends of the road and of the arch have to be connected in some way, so that they form a single, smoothly connected, continuous loop. It is not immediately clear how this should be done. In the CAD model in Figure 13a, the two sides of the arch can be colored differently (red & blue), and they can connect to equally colored sides of the roadbed without any conflict. This is not a property that is expected from a Möbius band.



(a) (b)

Figure 13: *Tczew Wiadukt, simplified bridge geometry: (a) CAD model, (b) 3D print.*

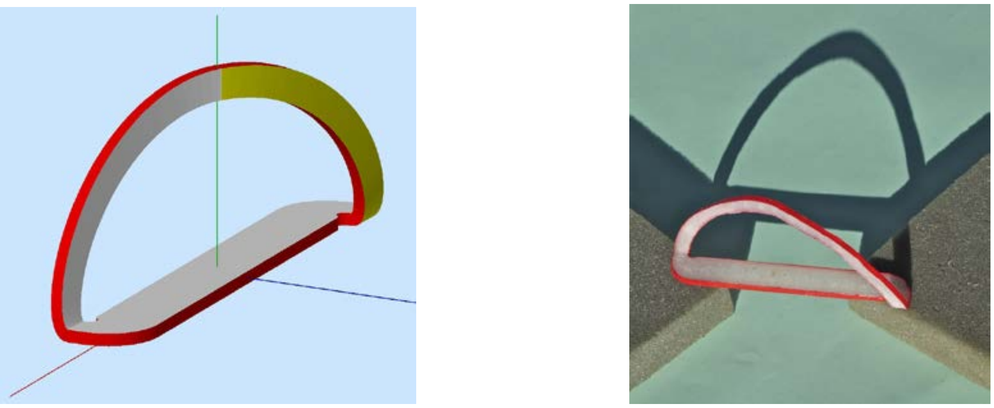
In a first attempt of casting this into an unambiguous ribbon geometry, I compressed the vertical dimension, to obtain an almost planar model. In this model, I then connected the ends of the road and of the arch with two small, horizontal, semicircular loops, resulting in the figure-8 configuration shown in Figure 14. Clearly, this is not a Möbius configuration. The wide face of the ribbon has an upper side shown in silver, and a differently colored lower side. Moreover, the two loops formed by the narrow side-faces of the ribbon can be painted separately with red and blue colors.



(a) (b)

Figure 14: *Flattened ribbon with smooth end-connections: (a) CAD model, (b) 3D print.*

To turn this into a Möbius bridge, I chose a vertical orientation for the arch profile, and I connected the arch laterally into the sides of the roadbed (Figure 15). Now all the narrow edges of the ribbon loop form a single (red) edge, and by follow the wide sides of the ribbon, one can get from the upper side of the roadbed to its lower side as one moves once through the arch. This is now a true Möbius strip.



(a) (b)

Figure 15: *A true Möbius configuration: (a) CAD model, (b) 3D print.*

Figure 16 shows a slightly different model with the same topology, but with a modified geometry that better accommodates the roadway with tapered on-ramps and also provides more solid foundations for the suspension arch. It also indicates that with a suitable color scheme, the Möbius geometry of this bridge could be made more obvious.

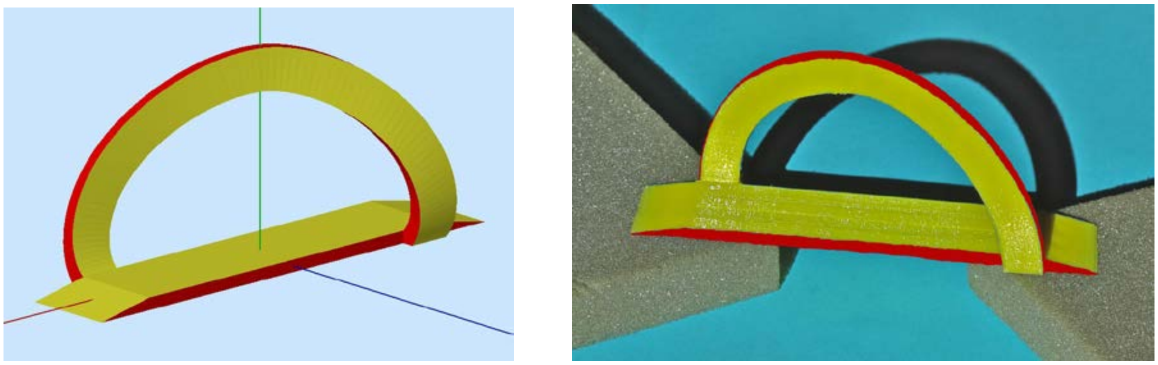

(a) (b)

Figure 16: *Usable Möbius configuration: (a) CAD model, (b) 3D print.*

In summary, the Wiadukt in Tczew is neither a Möbius band nor a twisted prismatic loop. It is simply a straight, flat road-bed with a separate untwisted arch placed diagonally across it. Figures 15 and 16 show how this could be turned into a true Möbius bridge.

7. Other "Twisted" Bridges

The quest for an actual, functional Möbius bridge continues. For a while, I have been following the design for a pedestrian bridge in Bristol, UK (Figure 17a) by Hakes Associates [13]. This clearly represents a Möbius ribbon with a meandering walkway with "lateral" access. Interestingly enough, it follows the scheme of Figure 7b on one end of the bridge and that of Figure 7a on the other end, where the return/suspension loop approaches the walkway from below. Sadly, this lovely bridge has not yet been built; it became a victim of the recession a decade ago. So far, there exist only a couple of models, and some "visuals" (Figure 17b, 17c) [14], [15].

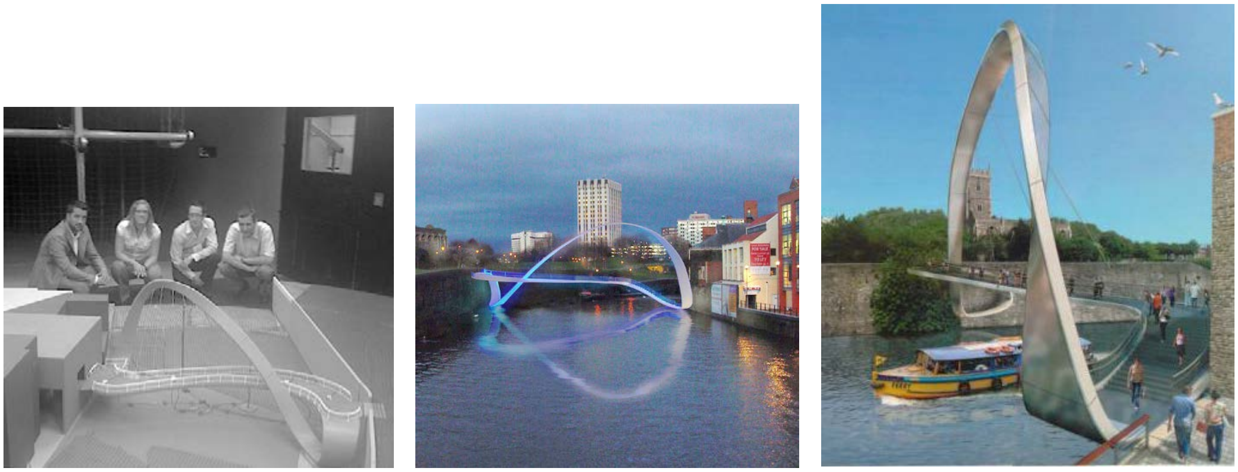

(a) (b) (c)

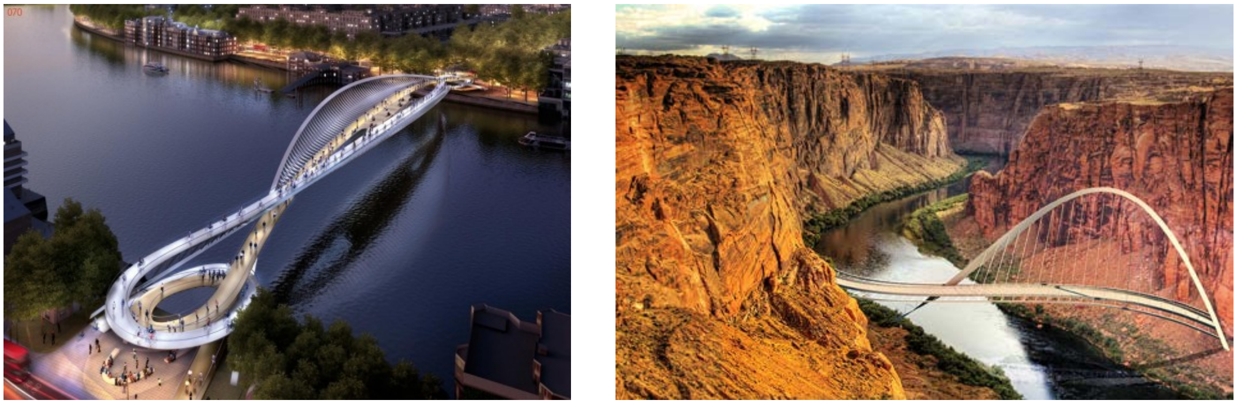
Figure 17: *Hakes' design for a pedestrian bridge in Bristol: (a) architectural model [13], (b) rendering by Hakes Associates [14], (c) another visual by Martin Booth [15].*

Searching the World Wide Web for “Möbius bridges” or “twisted bridges” returns a plethora of interesting bridge designs. Some of them definitely have a twisted look, but they still do not form a single-sided Möbius ribbon. Quite often, a more or less straight walkway is supported within a twisted prismatic frame. Good examples are the Floral Street Bridge at Covent Garden (Figure 18a) [16] and the bridge connecting the Holy-Zuid district and the Broekpolder over Vlaardingse Vaart in The Netherlands by West 8 Architects (Figure 18b) [17]. These bridges cannot be seen as Möbius structures, since the usable “forward” path and the twisted return path have quite different cross sections, and the ends of these two structures are not properly connected.

(a) (b)

Figure 18: *Twisted bridges: (a) Floral Street Bridge, Covent Garden, London [16], (b) bridge from Holy-Zuid district to the Broekpolder over Vlaardingse Vaart in The Netherlands [17].*

Other returned images depict design ideas that are often difficult to analyze, since not both ends may be shown in enough detail to determine the sidedness of the resulting ribbon geometry. The “Wild Design” proposal for a pedestrian bridge across the Thames in London by Nine Elms Vauxhall Partnership [18] (Figure 19a) has a dual-path structure (Section 5). But unless the far end of this bridge has a geometry radically different from what can be seen at the near end, the resulting surface is a two-sided, orientable ribbon. Dennis Chow’s group design exercise for an architecture class (Figure 19b) exhibits design elements found in the design by Hakes Associates (Figure 17); but the image does not make it clear how the suspension arch is connected to the roadbed and whether this results in a Möbius configuration.



(a) (b)

Figure 19: *(a) Design by Nine Elms Vauxhall Partnership [18], (b) Dennis Chow: A group design for ARCH-507, University of Southern California, 2015 [19].*

Figure 20 shows a "crazy" design that is actually under construction. This is the Meixi Lake undulating pedestrian bridge, being built over the Dragon King River in China [20]. The undulating walkways and suspension arches, designed by NEXT Architects based in Amsterdam, create an interesting ribbon configuration, in which the ribbon even splits laterally and then rejoins again. However, this does not form a single-sided surface. The topside of one wave train seems to connect at the ends of the bridge to the underside of the other wave train. Thus, the opposite sides of both wave trains could then still be painted with a second color.


(a) (b)

Figure 20: *Dragon King Bridge in Changsa, China: (a) architectural rendering by NEXT Architects, (b) construction photo by Shanghai Metal Corporation (2015) [20].*

However, I have been made aware of one real walkable Möbius structure. This is a pedestrian walkway (or rather "sculpture") in a suburb of Rotterdam, also designed by NEXT Architects, called "The Elastic Perspective" (Figure 21). This iron "boardwalk" mostly follows the hilly surface of a public park. At one location it raises off the ground to form a vertical loop that quickly descends again to the grass surface on this hill. This loop, which introduces the Möbius twist, is not passable for humans, and the structure does not actually bridge across a valley or river, or across some other obstacle.

Figure 21: *Brückentreppe in Rotterdam [21]: Six different views and two architectural plans.*

8. Discussion and Conclusions

A search for the term *Möbius bridge* on the web returns many flamboyant designs with gratuitous undulations and twists. But the construction of a practical, usable bridge that clearly exhibits the geometry of a Möbius strip needs not be very complicated. Several basic designs are being presented here. In my selection of possible designs, I have focused on the conceptual ideas by which the Möbius topology may be turned into a functional bridge. The many possible models could be classified by the number of half-twists comprised within the complete loop. This would then place them into only two different topological homotopy classes, since a Möbius band with three left-flips can be turned into a band with a single right-flip by allowing the band to cross itself in the center of a figure-8 shaped loop. From an architectural point of view, this is not an interesting classification.

Considering the actual geometry of the various models presented, it is not clear by what criterion one could sort them into different discrete classes, since many geometrical deformations can occur in a continuous space of arbitrary many design parameters. In other words, there are infinitely many possible designs with no discrete boundaries between them, and there is no hope of defining a complete list of all possible designs for a Möbius bridge.

Even though some fairly simple geometries can convincingly demonstrate the topology of a Möbius band, I still have not been able to identify any true Möbius bridge in the real world. Thus the quest to find such a bridge continues.

Acknowledgements

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