Algebraic Defects and Boundaries in Integrable & Topological Systems

by

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Abstract

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Topological and conformal field theories and integrable systems can be described by the algebraic structures of quantum groups and quantum affine algebras. Boundary conditions and defects for these theories are described via algebraic constructions from these quantum groups or quantum affine algebras.

In the first third of this thesis we describe constructions associated with three dimensional topological field theories. First we compute some Brauer-Picard groups which characterize nontrivial invertible structures that can be assigned using the cobordism hypothesis with singularities. Then we give a construction of bimodule categories using the procedure of covariantization(transmutation) of coquasitriangular Hopf algebras. The first third closes with a reduction by taking K_0 which results in algebraic K-theory of fusion rings.

The next third describes spin chains which are governed by quantum affine algebras. The first chapter gives a proof of the asymptotic completeness of the algebraic Bethe Ansatz for the XXZ chain. Then we describe a similar reduction of taking K_0 which results in algebraic K-theory of cluster algebras.

The last third of this thesis describes the appearance of these defects in topological field theories constructed by the AKSZ procedure. In the three dimensional case, this gives perturbative perspectives on the theories covered in the first third of the thesis. The last chapter gives the connection with topological insulators as they fit into the general AKSZ paradigm.

Contents

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Part I

Introduction

Chapter 1

Background

Quantum groups and quantum affine algebras provide a useful tool for topological and conformal field theories and integrable systems. This is done via state-sum constructions [1], modular functors [2] and the algebraic Bethe ansatz [3, 4]. Each chapter covers systems of different spacetime dimensions. Let us introduce the background for each of these systems in the simplified cases without boundaries or any other codimension 1 strata (defects).

1.1 Three Dimensional Topological

The prototypical examples of systems studied by three dimensional topological field theories are the fractional quantum Hall effect at various levels. These sorts of topological field theories can be constructed to almost meet the Atiyah-Segal axioms [5] by the method of [1]. This thesis will mostly focus on the doubled versions which avoid this complication. This means the systems of concern will be more analogous to the spin Hall effect instead.

Levin Wen Model

The Levin Wen model provides a Hamiltonian realization for a Turaev-Viro model. It assigns a Hilbert space and Hamiltonian to a Riemann surface time slice. It is built from an arbitrary unitary spherical fusion category $\mathcal{C}$[6]. From this data a lattice model is built such that the bulk excitations are given by simple objects of the Drinfeld center $Z(\mathcal{C})$.

The model is defined on a trivalent graph which is dual to a triangulation of the Riemann surface. The Hilbert space assigned to this graph is given by the finite dimensional Hilbert space spanned by the basis vectors which are labellings of edges by simple objects of the category and vertices by a basis vector of the multiplicity space for the fusion of the two incoming edges into the outgoing edge. For other orientations, this is corrected by dualizing and reversing arrows. The Hamiltonian is then cooked up with projectors as a sum on vertices and plaquettes illustrated in fig. 1.1. [7]

$$
\begin{aligned}
H &= \sum_v (1 - Q_v) + \sum_p (1 - B_p) \\
B_p &= \sum_{k \in I} \frac{d_k}{D^2} B_p^k \\
D^2 &= \sum_{i \in I} d_i^2
\end{aligned}
$$

where B_p^k inserts a loop decorated by the simple object k and evaluates the resulting morphism as illustrated in fig. 1.2 and the d_k are quantum dimensions of the specified simple objects. On the basis vectors, the first term projects to ensure that the fusion of objects on the incoming edges gives the fusion of the objects on the outgoing edges.

1.1.1 Definition *The Drinfeld Center of a monoidal category $\mathcal{C}$ is the monoidal category of endo-pseudonatural transformations of the identity 2-functor on $B\mathcal{C}$ which comes from the identity functor on $\mathcal{C}$. Unwinding this definition results in first of all an isomorphism from*

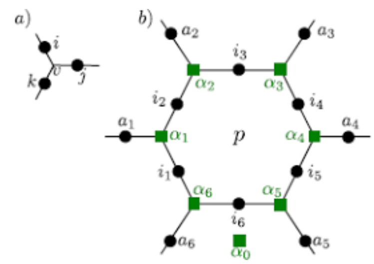

Figure 1.1: Example of a plaquette in the model where i label simple objects and α in multiplicity spaces.

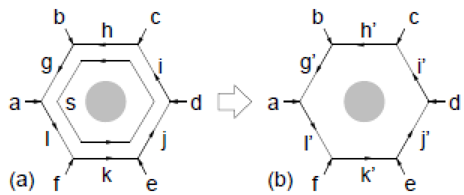

Figure 1.2: The action of a B_p^s on a plaquette

$id(pt) \to id(pt)$. That is given by a morphism of $B\mathcal{C}$ aka an object X of $\mathcal{C}$. And also the squares need to have a filling 2-morphism ϕ_Y a.k.a a morphism in $\mathcal{C}$ that does $X \otimes Y \to Y \otimes X$ which has to be natural in Y.

This provides a model for a 2+1 dimensional field theory. The particles are in the form of constraint violations are labeled by $Z(\mathcal{C})$ because the excitations are assigned to the circles. This is where the fact that the center of a spherical category gives a modular category has appeared. For further details on this relationship see [8]. Note that other lattice models provide realizations of the same topological field theory as low energy limits.

Quantum Groups

One common source for fusion categories is the positive energy representations of an affine Lie algebra. This is the one manifestly related to the Wess-Zumino-Witten model [9]. This WZW model is then holographically related to a corresponding Chern-Simons.

Another source is semisimplifed representation categories of quantum groups $U_q\mathfrak{g}$ for q a root of unity. These provide a finite Hopf algebraic perspective rather than infinite dimensional algebra. Fortunately there is the following theorem.

1.1.2 Theorem (Kazhdan-Lusztig-Finkelberg [10]) *There is an equivalence of fusion categories between the semisimplified representation category of $U_q\mathfrak{g}$ at a root of unity and the positive energy representations of $\hat{\mathfrak{g}}_k$ with k determined by the root of unity.*

This allows us to translate questions about physical models that are described by Chern-Simons theories into representation theoretic terms.

1.2 Spin Chains

Integrable spin chains with periodic boundary conditions are described by the algebraic Bethe ansatz. This is determined starting with a solution to the Yang-Baxter equation with spectral parameter.

$$R_{12}(x)R_{13}(xy)R_{23}(y) \quad = \quad R_{23}(y)R_{13}(xy)R_{12}(x)$$

where the subscripts indicate which pair of factors of $V_1 \otimes V_2 \otimes V_3$ are being used. Such a solution allows definition of a transfer matrix as a partial trace over an auxiliary space V_0.

$$\begin{aligned} T(x) &= R_{01}(x/t_1) \cdots R_{0N}(x/t_N) \\ T(x) &\in End(V_0 \otimes V_1 \cdots V_N) \\ t(x) &= \mathrm{tr}_0\, D_0 T(x) \end{aligned}$$

where $D_0 \in End(V_0)$. The Yang-Baxter equation then guarantees commutation relations $\big[t(x), t(y)\big]$. For example, one of the conserved quantities is the Hamiltonian given as

$$H \quad = \quad \frac{d}{dx} \log(t(x)) \mid_{x=0}$$

Further specializing to the case where all the $t_i = 1$ and the R matrix coming from $U_q\hat{\mathfrak{sl}}_2$ leads to the well studied periodic XXZ spin chain

$$\begin{aligned} H &= \sum_{<ij>} \sigma_i^x \sigma_j^x + \sigma_i^y \sigma_j^y + \Delta \sigma_i^z \sigma_j^z \\ \Delta &= \frac{q + q^{-1}}{2} \end{aligned}$$

with a sum over neighbors. $T(x)$ can be viewed as a matrix in V_0 but with entries in $End(V_1 \cdots V_N)$. Eigenstates for the Hamiltonian are built by applying products of matrix entries of the form $B(x_1)B(x_2) \cdots B(x_M)$ to a previously known “vacuum” state. For example, the vacuum may be all spins down. The x_i are subject to some algebraic equations called the Bethe equations.

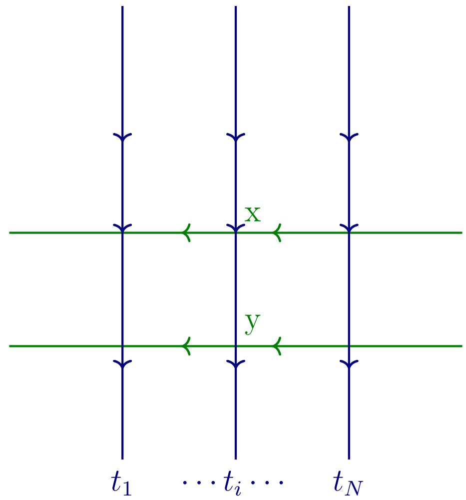

Figure 1.3: The blue line is decorated with t_i, which are called inhomegeneity parameters. Each crossing in this diagram represents a factor in the transfer matrix of $R_{0i}(x/t_i)$ or $R_{0i}(y/t_i)$. The two lines commute.

1.3 AKSZ Formalism

In contrast with the first chapter which uses a state sum perspective on topological field theories, one may also take the more usual field theoretic perspective where one is given an action functional to compute a path integral. In the case of topological field theories, this is often given as a Σ model from the spacetime X to a target M, but without using any metric structure on X and only it's smooth manifold structure. Much of this background is in chapter 9, so here we will simply give a sketch on why one should BV extend standard action functionals, reserving details for later.

The prototypical example of the necessity of the Batalin Vilkovisky formalism is a BF theory in 4 dimensions with naive action functional $\int_X B \wedge dA$. The gauge symmetry $B \to B + d\alpha$ for a 1-form α has secondary gauge symmetry by $\alpha \to \alpha + df$. Thus the familiar BRST procedure is insufficient. This is fixed by "doubling" the number of fields $\mathcal{F}$ to $\mathcal{F}_{BV}$ and then restricting to some other "half" $\mathcal{L} \neq \mathcal{F} \subset \mathcal{F}_{BV}$ of this extension other than the original fields. If $\mathcal{F}$ is thought of as functions on Y, $\mathcal{F}_{BV}$ is thought of as functions on T^*Y. By an appropriate choice for this $\mathcal{L}$, the problems of doing Feynman diagramatics before are aleviated.

Even if the BV extension is unnecessary because the critical locus is isolated or the BRST procedure sufficed for gauge symmetries, it can still be done. Though it may amount to some unnecessary extra computation for computing correlation functions on closed manifolds, the BV extension is strictly more powerful from the perspective of understanding boundaries and dualities of the original theory which is what our main goal is in this thesis.

Chapter 2

Overview of Dissertation

The goal of this thesis is to use these constructions to discuss boundaries and other codimension 1 strata (defects) for these types of systems in an algebraic manner. This is done in the following chapters.

2.1 3d Topological

In the construction of [11] we need to specify a fusion category that is assigned to the framed points of the 3-manifold. The defect structures are subsequently built from bimodule categories and functors thereof [12]. After reviewing these constructions including some of the $(\infty, 3)$-categorical machinery, this chapter includes the following results.

Brauer-Picard Groups

The invertible defects of all codimensions leads to certain computations of Brauer-Picard 3-groups for the cases of $U_q\mathfrak{sl}_2$ at various roots of unity. Performing G-extension with respect to certain finite groups involves computing some group cohomologies. Particular emphasis is placed on $G \subset O(3)$ is related to framing and $G = S_n$ in analogy with taking replicas.

K-Theory Fusion Rings

A simplification occurs if we take K_0 of the associated categories. We review this change to the world of Alg_{bim}. This is the $(\infty, 2)$ category with algebras, bimodules and bimodule intertwiners. The algebra in question will be the algebra over the rationals given by a fusion ring. The fusion ring is given by the Grothendieck ring of the fusion category and retains a lot of information while providing an easier object to study. The homotopical calculations become related to the algebraic K-theory of those unital based rings. In this thesis, we give example computations of these groups. They are particularly related to cyclotomic number fields. [13].

Transmutation

After this purely homotopical perspective, we move to an explicit procedure for constructing bimodule categories. Namely when the original category is given as representations of a coquasitriangular Hopf algebra. The bimodule category we construct is then given as representations of the transmuted algebra [14, 15]. In particular, we give new formulas for cases of the Sweedler algebra and small quantum $\mathfrak{sl}_2$.

2.2 Bethe Ansatz 1+1

In this chapter, we focus on integrable spin chains with boundaries and defects. These theories are related to quantum affine algebras and their representations.

Completeness

Here we give a proof of asymptotic completeness of Bethe vectors constructed in [16] as the length of the spin chain is fixed but inhomogeneity parameters vary.

K-theory of cluster algebras

Similar to the fusion ring case of section 2.1, a simplification occurs when taking K_0 of associated categories. In this part we consider the cluster categories of [17] and their associated cluster algebras. As before we compute some algebraic K-theoretic groups for these algebras. Namely we list their Picard groups, automorphisms as $\mathbb{Q}$-algebras and K_0 groups.

2.3 AKSZ Formalism

The Hopf algebraic and homotopic notions from the previous sections have their avatars in standard Lagrangian field theory methods by use of the AKSZ formalism. Such avatars are usually are sigma models. We review them in this section in the case without boundaries or defects. That description then allows discussion about defects and boundaries by discussing Lagrangians and Lagrangian correspondences of those target n-shifted symplectic stacks [18, 19].

2 dimensions

We review the Poisson sigma model in this section. Common targets include symplectic manifolds to give the A-model topological string [20] or Poisson Lie groups to provide deformation quantization versions of quantum groups [21]. Computations of the Picard groups for $\Sigma_{g,h}$ and G^* following [22] are included in the hope that they will be related to invertible shifted Lagrangian correspondences for their respective targets.

3 dimensions

This is the case where split Chern-Simons and Courant sigma model live. In particular this gives an interpretation of Morita equivalence of quantum tori in [23] as a boundary condition changing defect by applying the general case from [24].

4 dimensions

This is the case that is relevant for topological insulators which are described by 3+1d BF theories [25]. We quickly review the construction without BV extension to go from a fermion with flux threading to the effective BF theory. We then give the BV extension, gauge fixed action as well as writing down the BFV actions [18, 26, 27, 28]. These would be concretely realized by decorating materials like Bi_2Se_3 with defect walls and corners. The boundary fermion is seen by finding a Hodge-Dirac operator in the gauge fixed action.

Joint w/ Jon Aytac and Ivan Contreras.

Part II

Three Dimensional Topological

Chapter 3

G-Extensions and Group Cohomology

3.1 Introduction

In recent years, there has been a focus on symmetry protected topological phases [29, 30, 31, 32]. One of the questions there is to further gauge a finite symmetry group. This can be phrased in terms of the cobordism theorem [33, 34] in the case of three dimensional theories. In this framework, some obstructions become more manifest. Physically this is by viewing all symmetries as codimension 1 defects which implement that symmetry for everything in the theory at once including all the defects. This comes from understanding the action on the data assigned to the point instead of only knowing the action on the Hilbert space. If the theory was determined by a spherical fusion category assigned to the point, then this procedure of gauging is understood in the context of [12] where we must understand the $(\infty, 3)$ Morita category. It can then be asked what kind of field theory we get if the result is then assigned to the point. By being a fully extended theory, this model can be evaluated on stratified spaces with many defects. This is especially interesting in the context of quantum groups where many examples of fusion categories can be readily produced.

We will first review the way that Turaev-Viro models fit into the cobordism framework. Then how the Levin-Wen model gives a particular lattice realization. We then give the definitions of G-extensions of [12]. The next sections of equivariantization and transmutation give relations to other methods in the literature. We then say how this translates to giving anomaly inflow followed by some sample calculations in the case of some elementary quantum group categories.

The main results of these chapter are the computations of Brauer-Picard 3-groups. The emphasis is on $U_q\mathfrak{sl}_2$ because those are the cases that are physically realized via the Kazhdan-Lusztig theorem as opposed to more group-theoretical examples.

3.1.1 Remark There is some abuse of notation by calling all of these Turaev-Viro theories when those should really be the spherical case. See the adjectives framed and combed to distinguish. $\Diamond$

3.2 The essential bit of ∞ Categories

3.2.1 Theorem (Cobordism Hypothesis) *[34] n-dimensional local framed topological field theories with target a symmetric monoidal (∞, n) category $\mathcal{C}$ are in one to one correspondence with the fully dualizable objects of $\mathcal{C}$. In fact, the space of such field theories is homotopy equivalent to the space of n-dualizable objects of $\mathcal{C}$.*

3.2.2 Definition ($MonCat_{bim}$ [11]) *This is the $(\infty, 3)$ category, whose objects are finite rigid monoidal linear categories, morphisms are bimodule categories between these, 2-morphisms are bimodule functors, 3-morphisms natural transformations and from there on are equivalences. Fusion categories are fully dualizable objects in this $(\infty, 3)$ category.*

3.2.3 Theorem ([11]) *For any separable tensor category, there is a 3-dimensional 3-framed local topological field theory whose value on a point is that tensor category. In particular, there is such a field theory for any finite semisimple tensor category over a field of characteristic* *zero, and such a field theory for any fusion category of nonzero global dimension over an algebraically closed field of finite characteristic.*

This theorem is illustrated for a 2 dimensional slice by fig. 3.1 and fig. 3.2. In fig. 3.2, the different colored regions are labelled by different fully dualizable objects $\mathcal{C}_i$.

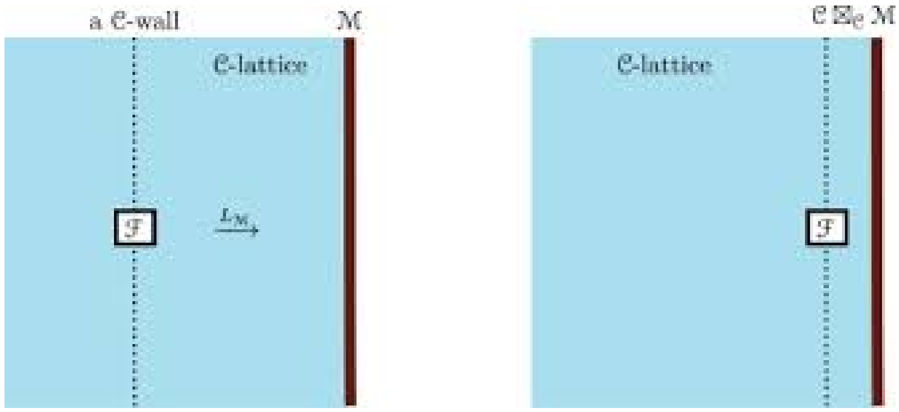

Figure 3.1: An illustration of a module category. [7]

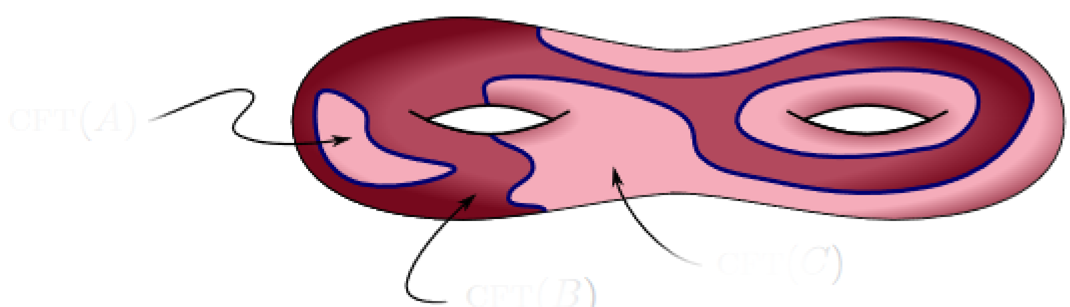

Figure 3.2: Each pink region is a material governed by $\mathcal{C}_i$ and the blue are bimodule categories between them. [35, 36]

3.2.4 Remark The 3-framing can be thought of as providing a vielbien in the context of 3D gravity. $\Diamond$

3.2.5 Conjecture ([11]) *Every pivotal fusion category in characteristic zero admits the structure of an $SO(2)$ homotopy fixed point, and therefore provides the structure of a combed 3-dimensional local field theory.*

3.2.6 Conjecture ([11]) *Every spherical fusion category in characteristic zero admits the structure of an $SO(3)$ homotopy fixed point, and therefore provides the structure of an oriented 3-dimensional local field theory.*

So to apply a symmetry transformation $g \in G$ we must apply a change of the fusion category and it's fully dualization data. This is given by a $\mathcal{C} - \mathcal{C}$ bimodule and higher coherences. This is exactly what we will see momentarily.

3.3 Brauer-Picard

3.3.1 Definition ($\boldsymbol{BiModc}(\boldsymbol{\mathcal{C}})$) *For a fusion category $\mathcal{C}$, we have the monoidal 2-category of $\mathcal{C}$ bimodule categories. The 1-morphisms are functors of bimodule categories and the 2-morphisms are natural transformations of such functors. The tensor product is given by $\boxtimes_{\mathcal{C}}$ which is defined through the universal property $Fun_{bal}(\mathcal{M} \times \mathcal{N}, \mathcal{A}) \simeq Fun(\mathcal{M} \boxtimes_{\mathcal{C}} \mathcal{N}, \mathcal{A})$ to all abelian categories $\mathcal{A}$.*

3.3.2 Definition (Bimodc) *You can also define a 3-category without picking $\mathcal{C}$, by treating the fusion categories as the objects, 1-morphisms being the bimodule categories, 2-morphisms functors of bimodule category and 3-morphisms natural transformations of above. Think of this as a truncated version of $MonCat_{bim}$ from before.*

3.3.3 Definition (Brauer-Picard 3-groupoid) *The objects are fusion categories Morita-equivalent to your starting $\mathcal{C}$. The 1-morphisms are the Morita-equivalences. These are $\mathcal{C} - \mathcal{C}$ bimodule categories. The 2-morphisms are the functors of bimodule categories that give equivalences. The 3-morphisms are natural transformations that give equivalences now taking equivalence classes. This is the connected version.*

3.3.4 Definition (Brauer-Picard 2-groupoid) *The objects are fusion categories Morita-equivalent to your starting $\mathcal{C}$. The 1-morphisms are the Morita-equivalences. These are $\mathcal{C} - \mathcal{C}$ bimodule categories. The 2-morphisms are the functors of bimodule categories that give equivalences.*

3.3.5 Definition (Brauer-Picard groupoid) *Replace each of those 1-morphisms by their equivalence classes.*

3.3.6 Definition (Brauer-Picard group) *Take the arrows only above one representative $\mathcal{C}$.*

Taking the classifying space of this 3-groupoid gives a homotopy space. Picking the connected component by the Morita equivalence class of $\mathcal{C}$ we get a 3-type. Its nontrivial homotopy groups are

- π_1 is the ordinary group $BrPic(\mathcal{C})$.

- π_2 is the isomorphism classes of invertible objects of the center. These are the invertible bulk excitations.
- π_3 is $\mathbb{C}^*$ or more generally the $\mathbb{G}_m$ of the ground field.

3.3.7 Theorem ([12]) *The Brauer-Picard group of $\mathcal{C}$ is isomorphic as a group to braided auto-equivalences of the center.*

3.3.8 Remark Just knowing the braided autoequivalence of the center (the action on the anyons) forgets the trivalent junction for the defect fusion which are interesting equivalences. ♢

3.3.9 Definition (G Extension) *A G graded category $\mathcal{C}$ is a tensor category with a decomposition into $\mathcal{C}_g$ $\forall g \in G$ such that each are full abelian subcategories and the tensor product takes $\mathcal{C}_g \times \mathcal{C}_h$ to $\mathcal{C}_{gh}$. [37] The trivial sector $\mathcal{C}_e$ is a full tensor subcategory and each other is a $\mathcal{C}_e$ bimodule category. A G-extension of a fusion category $\mathcal{D}$ is a G-graded fusion category $\mathcal{C}$ such that $\mathcal{C}_e$ is equivalent to $\mathcal{D}$. If G is finite this is still finitely many simples. Though other structures like pivotal or spherical may not carry over from $\mathcal{D}$. This is thought of as the original theory and the G-fluxes.*

3.3.10 Theorem (ENO10 Thm1.3 [12]) .

- *Equivalence classes of G-extensions of $\mathcal{C}$ are given by homotopy classes of maps of classifying spaces BG to $B\underline{\underline{BrPic(\mathcal{C})}}$*
- *Alternatively this is parameterized by c $G \to BrPic(\mathcal{C})$ ordinary group homomorphism, an M belonging to a $H^2(G, \pi_2)$ torsor (π_2 is given the structure of a G rep by c) and an α belonging to a $H^3(G, \pi_3 = \mathbb{C}^*)$ torsor. Certain obstructions $o_3(c) \in H^3(G, \pi_2)$ and $o_4(c, M) \in H^4(G, \pi_3 = \mathbb{C}^*)$ need to vanish*

So we see that what is important is G as a 3-groupoid with one object, interesting 1 arrows and boring 2 and 3 arrows. Gauging a symmetry is presented explicitly as a higher group. [31]. Note that G is often $\mathbb{Z}_2$ for parity or time reversal [38]. As far as the definition goes the 3-groupoid is natural, but computationally we have easy access to only the fundamental groups.

3.3.11 Remark One may think of the problem of computing a generalized cohomology theory $h^\bullet(BG)$ where we also compute $[BG, X]$ or at least it's stablization. [39] ♢

Of $Vect_{\mathbb{C}}$ and $SuperVect_{\mathbb{C}}$

In the case that both π_1 and π_2 were trivial, then we would have an Eilenberg-MacLane target which would mean that $[BG, K(\mathbb{C}^*, 3)]$ could be calculated with group cohomology [32]. When $\mathcal{C}$ is $Vect_{\mathbb{C}}$ (the trivial theory), π_1 is trivial because that is the classical $Br(k) =$

$\{e\}$ for an algebraically closed field (this is also true for the maximal abelian extension of $\mathbb{Q}$) and the π_2 is trivial because the only invertible vector space is the one dimensional one. This is not the general case for other $\mathcal{C}$ so we must take care of any version of group (super)cohomology we compute [40]. This is the first step that determines the possible gaugings. The next step is to use each such G extension as a new TFT. This is done for each ω cohomology class.

If we use $SuperVect_{\mathbb{C}}$ on the point, we are only using it as a fusion category without braiding so replace it with $Rep_{\mathbb{C}}(\mathbb{Z}_2)$. Using the other associator gives the double semion model which we will return to. This reduces the problem to gauging the toric code. The braided autoequivalences of the center are $\mathbb{Z}_2$ implementing electromagnetic duality. The invertible objects are $\mathbb{Z}_2 \times \mathbb{Z}_2$, the third homotopy is $\mathbb{C}^*$ as usual.

3.4 Müger style G equivariantization and quasi-trivial extensions

This takes a semisimple linear monoidal category with G acting on it via autoequivalences. It then outputs a semisimple monoidal category having a full monoidal *Rep G* subcategory. If $\mathcal{C}$ is G-braided G-crossed, then the *Rep G* is a symmetric subcategory. [41].

3.4.1 Definition (G-braided G-crossed) *A G-braided G-crossed fusion category is a fusion category equipped with a not necessarily faithful grading by G, an action of g by functors* T_g *that take* $\mathcal{C}_h \to \mathcal{C}_{ghg^{-1}}$ *and G-braidings* $X \otimes Y \simeq T_g(Y) \otimes X$ *if* X *is homogenous in degree* g*. The component with identity grading has an ordinary braiding.*

Picard 2-groupoid

3.4.2 Definition ($\boldsymbol{Modc(\mathcal{B})}$) *For a braided fusion category* $\mathcal{B}$*, we have the monoidal 2-category of left* $\mathcal{B}$ *module categories. The 1-morphisms are functors of module categories and the 2-morphisms are natural transformations of such functors.*

3.4.3 Definition (Picard categorical 2-group) *Only the invertibles in the above monoidal 2-category. This gives a particular sub full categorical 2-subgroup of the Brauer-Picard that our map from BG may or may not factor through. At the group level,* $Pic(\mathcal{C}) \simeq Aut^{br}(Z(\mathcal{C}), \mathcal{C})$

Taking the classifying space of $\underline{\underline{Pic}}(\mathcal{B})$ gives a homotopy space. The nontrivial homotopy groups of this are

- π_1 is the ordinary group $Pic(\mathcal{B})$
- π_2 is the group of isomorphism classes of invertible objects in $\mathcal{B}$
- π_3 is $\mathbb{C}^*$ or more generally the G_m of the ground field.

3.4.4 Theorem (ENO10 Thm 7.12 [12]) .

- *Equivalence classes of G-braided G-crossed categories with faithful G-grading having trivial component $\mathcal{B}$ are given by homotopy classes of maps of classifying spaces BG to $B\underline{\underline{Pic(\mathcal{B})}}$*
- *Alternatively this is parameterized by c $G \to Pic(\mathcal{B})$ ordinary group homomorphism, an M belonging to a $H^2(G, \pi_2)$ torsor (π_2 is given the structure of a G rep by c) and an α belonging to a $H^3(G, \pi_3 = \mathbb{C}^*)$ torsor. Certain obstructions $o_3(c) \in H^3(G, \pi_2)$ and $o_4(c, M) \in H^4(G, \pi_3 = \mathbb{C}^*)$ need to vanish*

De/Equivariantization

3.4.5 Definition (Equivariantization [41]) *Let β be an action of group G on $\mathcal{C}$. Then $\mathcal{C}^G$ is the category whose objects are pairs $(X, \{u_g\})$ where $u_g : {}^gX \to X$ is a system of isomorphisms such that all squares with ${}^{gh}X$ commute. The Hom sets are those $s \in Hom_{\mathcal{C}}(X, Y)$ such that the squares with u_g and v_g also commute for all g. Think of homotopy fixed points.*

3.4.6 Proposition ([41]) *If $\mathcal{C}$ is a braided G-crossed G-category, then $\mathcal{C}^G$ is braided (not G-braided).*

3.4.7 Theorem ([41]) *If $\mathcal{C}$ is a braided fusion category, $\mathcal{S} \simeq Rep\ G$ is a full monoidal subcategory, A is the corresponding commutative etale algebra object (think of the G-rep $Fun(G, k)$ with pointwise multiplication).*

Then the left A modules in $\mathcal{C}$, called ${}_A\mathcal{C}$ is a braided G-crossed fusion category.
Then $({}_A\mathcal{C})^G \simeq \mathcal{C}$ as a braided fusion category.
If $\mathcal{D}$ is a G-braided G-crossed fusion category, we can equivariantize it, then find $RepG \subset \mathcal{D}^G$ and then de-equivariantize to ${}_A(\mathcal{D}^G) \simeq \mathcal{D}$ as a braided G-crossed fusion category.

3.4.8 Proposition ([42, 43]) *Suppose $\mathcal{C}$ is a unitary fusion category, then the G de-equivariantization result is also unitary.*

3.4.9 Definition (Modularization) *Take a ribbon category and consider a collection of invertible simple objects $\mathcal{G}$ which satisfy:*

- *Closed under tensor product*
- *Every object is transparent to all of $\mathcal{C}$*
- *All dimension 1 rather than -1 (invertibility only gave ± 1)*
- *The twist factors are all 1.*

Taking the quotient with respect to this collection $\mathcal{C}//\mathcal{G}$ is the modularization. There is the essentially surjective functor $\mathcal{C} \to \mathcal{C}//\mathcal{G}$ because every object comes from $F(X)$ $X \in Obj(\mathcal{C})$ but it is very much not full (not surjective on the induced maps of hom sets)

3.4.10 Proposition ([37] 1.9) *Suppose that G is finite and a neutral-modular G-category $\mathcal{C}$ is regular. Then $\mathcal{C}$ is a modular category in the ungraded sense as well. Without the regularity assumption this may fail because neutral-modular G-category only checks the nondegeneracy of the S matrix on the $g = e$ neutral component. (In [37] this is just called modular G-category)*

3.4.11 Proposition ([44] 10.2) *A unitary G-crossed G-braided fusion extension of a unitary modular category is modular (distinguish with neutral-modular) if and only if the equivariantization is modular.*

3.4.12 Remark In [45], Kong considers the problem of giving a modular category $\mathcal{C}$ and performing condensation via giving a subcategory $\mathcal{D}$ which is also modular. This is given by the data of a connected commutative etale algebra object. So de-equivariantization provides an example of the first step of anyon condensation. This is applied at the level of 3,2,1 extended field theories not fully extended theories. There is no guarantee of a spherical $\mathcal{X}$ such that $Z(\mathcal{X}) \simeq \mathcal{C}$ or $\mathcal{D}$. $\diamondsuit$

3.4.13 Theorem ([46] 3.3,3.5) *Let $\mathcal{C}$ be a G extension of $\mathcal{D}$, then the relative center has the canonical structure of a braided G-crossed category. This then G equivariantizes to give the $(Z_{\mathcal{D}}(\mathcal{C}))^G \simeq Z(\mathcal{C})$.*

3.4.14 Corollary ([46] 3.7) *Let G be a finite group. A fusion category $\mathcal{A}$ is Morita equivalent to a G-extension of some $\mathcal{B}_e$ if and only if $Z(\mathcal{A})$ contains a Tannakian subcategory $\mathcal{E} = Rep(G)$. This implies that we may see $\mathcal{A}$ is Morita equivalent to a G-graded fusion category $\bigoplus \mathcal{B}_g$ with $Z(\mathcal{B}_e) \simeq \mathcal{E}_G'$ as braided tensor categories. This gives a procedure for translating from a G-extension to braided G-crossed extension of the center. This is the difference in perspectives between [12, 42]. So if $\mathcal{D}$ is the original theory with particles $Z(\mathcal{D})$ and we gauge it to get $\mathcal{C}$ and then can condense.*

Proof Suppose $\mathcal{A}$ is a G-extension $\bigoplus \mathcal{A}_g$. The Tannakian subcategory is given by for every representation (V, π) of G giving the object Y_π of $Z(\mathcal{A})$ defined as $V \otimes \mathbf{1}$ as an object of $\mathcal{A}$ and the half braiding $\pi(g) \otimes id_X$ $X \otimes Y_\pi \simeq V \otimes X \to V \otimes X \simeq Y_\pi \otimes X$ for $X \in \mathcal{A}_g$. The morphisms can also be checked as well as the converse. $\square$

The Müger style procedure gives us $(Z_{\mathcal{D}}(\mathcal{C}))^G$, but G-extension gives us $Z(\mathcal{C})$. This implies that even though we don't notice the difference down to codimension 2, we will notice the difference at codimension 3. For example, this can be accomplished by introducing an interval cut at some time to create point defects of the space-time at the endpoints of the interval at the instant of the introduction of the cut. Also note that the construction is conceptually simpler.

3.4.15 Definition (Nilpotent) *A fusion category $\mathcal{A}$ is called nilpotent if there is a sequence of finite groups G_i and a sequence of fusion subcategories $Vec \subset \mathcal{A}_1 \subset \cdots \mathcal{A}_n = \mathcal{A}$ such that each step is an extension by G_i. The smallest such n is called the nilpotency class of $\mathcal{A}$.*

3.4.16 Definition (Solvable) *A fusion category $\mathcal{A}$ is called solvable if there is a sequence of finite groups G_i cyclic of prime order and a sequence of fusion subcategories $Vec \subset \mathcal{A}_1 \subset \cdots \mathcal{A}_n = \mathcal{A}$ such that each step is an extension or equivariantization by G_i.*

Other Sorts of Extension

3.4.17 Proposition ([12] 7.2) *$\pi_1 B\underline{\underline{Out(\mathcal{D})}}$ is $Out(\mathcal{C})$. $\pi_2 B\underline{\underline{Out(\mathcal{D})}}$ is invertible objects of the center. π_3 is $\mathbb{C}^*$.*

3.4.18 Proposition ([12] 7.4) *$\pi_1 B\underline{Eq(\mathcal{D})}$ is $Eq(\mathcal{C})$. $\pi_2 B\underline{Eq(\mathcal{D})}$ is $Aut_{\otimes}(Id)$.*

3.4.19 Proposition (Quasi-trivial extensions [12] 7.10) *Quasi-trivial extensions $\mathcal{C}$ (where every $\mathcal{C}_g$ contains an invertible object) of a fusion category $\mathcal{D}$ up to graded equivalence are in natural bijection with $BG \to B\underline{\underline{Out(\mathcal{D})}}$. The ones that are actually trivial ($Vec_G \ltimes \mathcal{D}$) can be factored through $BG \to B\underline{Eq(\mathcal{D})} \to B\underline{\underline{Out(\mathcal{D})}}$. $\underline{\underline{Out(\mathcal{D})}}$ is a 2-subgroup of $\underline{\underline{BrPic(\mathcal{D})}}$ by only including the bimodule categories that are $\mathcal{D}$ as left module categories and the right module category structure is twisted by some autoequivalence which is determined up to conjugation. The higher structure must also respect this restriction.*

In summary, we have trivial extensions determined by $B\underline{Eq}$ which also give quasi-trivial extensions determined by maps to $B\underline{\underline{Out}}$. There are examples of G-extensions determined by maps to $B\underline{\underline{BrPic}}$. G-braided faithful G-crossed extensions also give a special kind of G-extension when the map factors through $B\underline{\underline{Pic}}$.

3.5 Hopf Algebra Transmutation

When we actually have a forgetful functor, we may may talk about Hopf algebras instead of their representation categories. For example, we may write the equivariantization as $Rep(H \rtimes kG) \simeq Rep(H)^G$ [14, 47].

If H is a co-quasitriangular Hopf algebra, then we get a module category for $D(A,r)-mod$ by taking representations of the covariantized (braided form) algebra A_r by the $A_r \to D(A) \otimes A_r$ comodule algebra structure [48] . The Heisenberg double shows up in this way [49]. We could also consider the coideal subalgebra case $A_q(K \backslash G/K) \to A_q(G)$ [50]. $A_q(K\backslash G/K)$ is intimately related with Macdonald and Koornwinder polynomials. However there q is generic so this does not do the semisimplification procedure that is needed for our setting. This procedure can be used to construct bimodule categories for related categories. This will be done in the next chapter 4. This semisimplicity concerns would need to be addressed if one were to construct these would be "Turaev-Viro-Macdonald type" theories.

The restricted quantum group as it appears in logarithmic conformal field theory gives a Hopf algebra, but sadly ruins semisimplicity [51]. But even without semisimplicity, a co-quasitriangular Hopf algebra can be covariantized and its representations applied for [52, 53] type field theories rather than TQFT's of Lurie type. This is the dual of the quasitriangular $\bar{D}$ in [51]. This perspective for defects in LCFT is described in the next chapter 4.

3.5.1 Theorem (3 of [48]) *Let $\mathcal{C}$ be a braided fusion category. Then the Drinfeld center of $\mathcal{C}$ is equivalent to the category of finite dimensional left comodules over some braided Hopf algebra ${}_R H_{\mathcal{C}}$ (Hopf algebra object). Moreover, if A is a braided bi-Galois object over ${}_R H_{\mathcal{C}}$, then the cotensor functor $A \Box -$ defines a braided autoequivalence of the Drinfeld center of $\mathcal{C}$ (trivializable on $\mathcal{C}$) if and only if A is quantum commutative.*

The braided autoequivalences that are not trivializable over $\mathcal{C}$ are not of the form $A \Box -$ for any braided bi-Galois objects over ${}_R H_{\mathcal{C}}$. In the case of $\mathcal{C}$ being the representation category of a semisimple Hopf algebra over algebraically closed field, then this gives all of $Aut^{br}(Z(\mathcal{C}), \mathcal{C})$ which is the image of $Pic \subset BrPic$.

3.6 Sphericality and Anomalies

The question is knowing that we start with the adjectives unitary and spherical, and perform G-extension we want to be able to stay unitary and spherical. That is we must take the G-graded fusion category result, forget the G grading and see that as a spherical or pivotal fusion category. If not, we can only define a framed theory or a combed field theory depending on what structure is kept. This is the main question of the subject.

3.6.1 Remark If the theory was defined after giving a 2-framing (Trivialization of $TX \bigoplus TX$) we could understand that as a bounding 4-manifold providing anomaly inflow. In that case you define your theory as a relative theory [54, 31] such as in the Crane-Yetter context. $\diamondsuit$

3.6.2 Conjecture (Etingof, Nikshcych, Ostrik [55] 2.8) *Every fusion category admits a pivotal structure.*

3.6.3 Corollary *This implies that even after G extension, we can get a combed field theory after a choice. The ENO conjecture is proven for representation categories of semisimple quasi-Hopf algebras, but again unfortunately we have lost our fiber functor so we cannot use it as a theorem.*

3.7 Quantum Group Example

The Turaev Viro model was defined originally as a state sum model, but further efforts have realized it in the cobordism framework. Assuming [11], this is given by assigning a spherical fusion category to the framed point. Knowing that and the cobordism theorem is enough to determine the entire theory [56, 57, 58]. Common sources for fusion categories are either finite groups or quantum groups at roots of unity [1]. In terms of the Brauer-Picard and equivariantization, the case of finite abelian groups is in the original [12, 59], representation categories for finite groups [60, 61] and the Asaeda-Haagerup subfactor example is computed as well in [62]. The quantum groups seem to be missing from the literature. See the appendix for the general construction as well as the explicit data and notations for the examples considered.

Because the category in this case is modular, the center is equivalent to the product $\mathcal{C} \boxtimes \mathcal{C}^{op}$ so the Brauer-Picard group is the same as braided autoequivalences of $Z(\mathcal{C}) \simeq \mathcal{C} \boxtimes \mathcal{C}^{op}$. These possibilities are checked by combinatorial criteria for where to send generating objects. The next step is determined by automorphisms of some planar algebras [63]. Conveniently many of the models of physical interest are in the rank 1 case where the calculation is significantly easier because $Aut(Temperley - Lieb) = \{id\}$.

3.7.1 Remark A Mathematica notebook for the combinatorial criteria calculation is available via GitHub upon request. ♢

Double Semion

The braided autoequivalences of the center are given by $1 \boxtimes s$ and $s \boxtimes 1$ switching. This is the combinatorial step. Then give the functors that have that as the underlying action on objects. In this case, there is nothing else to check so $\pi_1 = \mathbb{Z}_2$. The group of invertible objects of the center form a $\mathbb{Z}_2 \times \mathbb{Z}_2$. The action of π_1 on π_2 is given by switching the factors.

Double Fibonnacci

The braided autoequivalences of the center are given by $1 \boxtimes F$ and $F \boxtimes 1$ switching. This is the combinatorial step. Then give the functors that have that as the underlying action on objects. In this case, there is nothing else to check so $\pi_1 = \mathbb{Z}_2$. The group of invertible objects of the center form a trivial group.

A_1 General ℓ

The classes of simple objects will be parameterized by pairs in $[0, \ell - 2] * \omega$. We need to see where the objects $(\omega, 0)$ and $(0, \omega)$ go. These can go to various other simple objects depending on ℓ.

Table 3.1: Dimensions and Twists

obj	dim	twist
$(\omega, 0)$	$[1+1] * [1]$	$q^{3/2} * 1$
$(0, \omega)$	$[1] * [1+1]$	$1 * q^{3/2}$
$((\ell-3) * \omega, 0)$	$[1+\ell-3] * [1]$	$q^{1/2*(\ell-3)(\ell-1)} * 1$
$(0, (\ell-3) * \omega)$	$[1] * [1+\ell-3]$	$1 * q^{1/2*(\ell-3)(\ell-1)}$
$(\omega, (\ell-2)\omega)$	$[1+1] * [1+\ell-2]$	$q^{3/2} * q^{1/2*(\ell-2)(\ell)}$
$((\ell-2)\omega, \omega)$	$[1+\ell-2] * [1+1]$	$q^{1/2*(\ell-2)(\ell)} * q^{3/2}$
$((\ell-3) * \omega, (\ell-2)\omega)$	$[1+\ell-3] * [1+\ell-2]$	$q^{1/2*(\ell-3)(\ell-1)} * q^{1/2*(\ell-2)(\ell)}$
$((\ell-2)\omega, (\ell-3) * \omega)$	$[1+\ell-2] * [1+\ell-3]$	$q^{1/2*(\ell-2)(\ell)} * q^{1/2*(\ell-3)(\ell-1)}$

$$\begin{aligned} q^{3/2} == q^{1/2*(\ell-3)(\ell-1)} &\implies \ell = 0 \mod 4 \\ 1 == q^{1/2*(\ell-2)*\ell} &\implies \ell = 2 \mod 4 \\ q^{3/2} == q^{1/2*(\ell-2)(\ell)} * q^{1/2*(\ell-3)(\ell-1)} &\implies \ell = 1 \mod 2 \end{aligned}$$

$\ell = 0 \mod 4$

Based on twists and dimensions $(\omega, 0)$ and $(0, \omega)$ can go to $((\ell-3)\omega, 0)$ and $(0, (\ell-3)\omega)$ or $(\omega, 0)$ and $(0, \omega)$.

The higher π are $\pi_2 = \mathbb{Z}_2 \times \mathbb{Z}_2$ from the classes of $(\ell-2, 0)$ and $(0, \ell-2)$ and $\pi_3 = \mathbb{C}^*$.

$\ell = 1 \mod 4$

Based on twists and dimensions $(\omega, 0)$ and $(0, \omega)$ can go to $((\ell-3)\omega, (\ell-2)\omega)$ and $((\ell-2)\omega, (\ell-3)\omega)$ or $(\omega, 0)$ and $(0, \omega)$

The higher π are again $\mathbb{Z}_2 \times \mathbb{Z}_2$ and $\mathbb{C}^*$.

$\ell = 2 \mod 4$

Based on twists and dimensions $(\omega, 0)$ and $(0, \omega)$ can go to $((\ell-2)\omega, \omega)$ and $(\omega, (\ell-2)\omega)$ or $(\omega, 0)$ and $(0, \omega)$

The same π_2 and π_3 remain.

$\ell = 3 \mod 4$

Based on twists and dimensions $(\omega, 0)$ and $(0, \omega)$ can go to $((\ell-3)\omega, (\ell-2)\omega)$ and $((\ell-2)\omega, (\ell-3)\omega)$ or $(\omega, 0)$ and $(0, \omega)$

The same π_2 and π_3 remain.

3.8 Conclusion

This gives an overview of how the cobordism hypothesis and G-extensions fit together to give the act of taking symmetry defects and permeating them in order to gauge finite group symmetries. There is relation with the procedure of equivariantization when extra braiding data is given, but they give slight differences on extremely stratified spacetimes. These are potentially implementable as certain kinds of quenches because we already get codimension 2 when considering a system with boundary and then crossing with the half open interval of the future. This illustrates the difference between establishing the equality of the theories at the level of partition functions or all the way down with duality walls to say what it means to be in the same (physical) phase. We mentioned the relationship with Majid's theory of transmutation though that is only applicable in the examples when the weak Hopf algebra is actually an honest Hopf algebra which are not the categories considered here.

In the case of Walker-Wang models, the story can be described as a $(\infty, 4)$ version with which we can repeat the procedure of finding fully dualizable objects and asking what happens upon changing them with the data of a finite group. This is a much simpler case than most 3+1 dimensional systems because this $(\infty, 4)$ category is only as interesting as the $(\infty, 3)$ one discussed here (which is still extremely interesting). This means that instead of mapping BG into a homotopy 3-type we map into a homotopy 4-type with $\pi_1 = 0$. Again if we are just using a version of $Vect_{\mathbb{C}}$ shifted up we get a group cohomology classification. We can bump this game arbitrarily, but this only uses the aspects that come from 3 dimensions vs the new instruments that enter as the conductor signals a change in dimension. Proceding with this analogy matches up with [39, 64].

3.9 Appendix: Full Computation

Since the arXiv posting for the material in this section, the calculations were completed by [65]. The table is reproduced below.

Table 3.2: Brauer-Picard Groups

$\mathcal{C}$	$\pi_1 = BrPic(\mathcal{C})$	$\pi_1 = BrPic(Ad\mathcal{C})$
A_3	e	$\mathbb{Z}_2$
A_7	$\mathbb{Z}_2^2$	$D_{2\cdot 4}$
$A_{n\equiv 0(4)}$ + modular braiding	e	e
$A_{n\equiv 0(4)}$ - modular braiding	$\mathbb{Z}_2$	e
$A_{n\equiv 1}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
$A_{n\equiv 2(4)}$ + modular braiding	e	e
$A_{n\equiv 2(4)}$ - modular braiding	$\mathbb{Z}_2$	e
$A_{n\equiv 3(4)} + n \neq 3, 7$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$
D_{10}	S_3	S_3^2
D_{2N}	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$
E_6	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$
E_8	e	$\mathbb{Z}_2$

3.10 SO(3) fixed points

Homotopy fixed points for an infinite group like $SO(3)$, $O(3)$, $Spin(3)$ or $Pin^{\pm}(3)$ are not accessible via G-extension but we may ask merely for the finite subgroups thereof. This has the effect of removing some framing conditions. The relevant cohomology groups for exceptional cases are tabulated below.[1]

In particular the ones needed will be $H^2(G, \pi_2 = \mathbb{Z}_2^2)$, $H^3(G, \pi_3 = \mathbb{C}^*) \to H^4(G, \mathbb{Z})$ for the data and obstructions in $H^3(G, \pi_2 = \mathbb{Z}_2^2)$ and $H^4(G, \pi_3 = \mathbb{C}^*) \to H^5(G, \mathbb{Z})$. These are then read off from the above table and applying the universal coefficent theorem.

[1] We have also included K-Theory of these groups with $\mathbb{Z}_{(p)}$ indicating p-adics. Those are calculated by applying [66]

Table 3.3: Cohomology/K-Theory for Platonic Groups,

Group	$H^0(,\mathbb{Z})$	$H^1(,\mathbb{Z})$	$H^2(,\mathbb{Z})$	$H^3(,\mathbb{Z})$	$H^4(,\mathbb{Z})$	$H^5(,\mathbb{Z})$	$H^6(,\mathbb{Z})$	$K^0(B?)$
$T = A_4$	$\mathbb{Z}$	e	$\mathbb{Z}_3$	$\mathbb{Z}_2$	$\mathbb{Z}_6$	e	$\mathbb{Z}_2^2 \times \mathbb{Z}_3$	$\mathbb{Z} \times \mathbb{Z}^1_{(2)} \times \mathbb{Z}^2_{(3)}$
$T_d = S_4$	$\mathbb{Z}$	e	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_4$	$\mathbb{Z}_2$	$\mathbb{Z}_2^3$	$\mathbb{Z} \times \mathbb{Z}^3_{(2)} \times \mathbb{Z}^1_{(3)}$
$T_h = A_4 \times \mathbb{Z}_2$	$\mathbb{Z}$	e	$\mathbb{Z}_2 \times \mathbb{Z}_3$	$\mathbb{Z}_2$	$\mathbb{Z}_2^3 \times \mathbb{Z}_3$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^5 \times \mathbb{Z}_3$	$\mathbb{Z} \times \mathbb{Z}^3_{(2)} \times \mathbb{Z}^2_{(3)}$
$O = S_4$	$\mathbb{Z}$	e	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_4$	$\mathbb{Z}_2$	$\mathbb{Z}_2^3$	$\mathbb{Z} \times \mathbb{Z}^3_{(2)} \times \mathbb{Z}^1_{(3)}$
$O_h = S_4 \times \mathbb{Z}_2$	$\mathbb{Z}$	e	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^4 \times \mathbb{Z}_3 \times \mathbb{Z}_4$	$\mathbb{Z}_2^5$	$\mathbb{Z}_2^9$	$\mathbb{Z} \times \mathbb{Z}^7_{(2)} \times \mathbb{Z}^1_{(3)}$
$I = A_5$	$\mathbb{Z}$	e	e	$\mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_5$	e	$\mathbb{Z}_2^2$	$\mathbb{Z} \times \mathbb{Z}^1_{(2)} \times \mathbb{Z}^1_{(3)} \times \mathbb{Z}^2_{(5)}$
$I_h = A_5 \times \mathbb{Z}_2$	$\mathbb{Z}$	e	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2^3 \times \mathbb{Z}_3 \times \mathbb{Z}_5$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^5$	$\mathbb{Z} \times \mathbb{Z}^3_{(2)} \times \mathbb{Z}^1_{(3)} \times \mathbb{Z}^2_{(5)}$
$BinT = SL(2,3) \subset Spin(3)$	$\mathbb{Z}$	0	$\mathbb{Z}_3$	0	$\mathbb{Z}_{24}$	0	$\mathbb{Z}_3$	$\mathbb{Z} \times \mathbb{Z}^2_{(2)} \times \mathbb{Z}^2_{(3)}$
$BinO$	$\mathbb{Z}$	0	$\mathbb{Z}_2$	0	$\mathbb{Z}_{48}$	0	?	$\mathbb{Z} \times \mathbb{Z}^5_{(2)} \times \mathbb{Z}^1_{(3)}$
$BinI = SL(2,5)$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}_{120}$	0	0	$\mathbb{Z} \times \mathbb{Z}^2_{(2)} \times \mathbb{Z}^1_{(3)} \times \mathbb{Z}^2_{(5)}$
$T \times \mathbb{Z}_4 \subset Pin_+(3)$	$\mathbb{Z}$	e	$\mathbb{Z}_3 \times \mathbb{Z}_4$	$\mathbb{Z}_2$	$\mathbb{Z}_2^2 \times \mathbb{Z}_3 \times \mathbb{Z}_4$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^4 \times \mathbb{Z}_3 \times \mathbb{Z}_4$	$\mathbb{Z} \times \mathbb{Z}^7_{(2)} \times \mathbb{Z}^2_{(3)}$
$T \times \mathbb{Z}_2$	$\mathbb{Z}$	e	$\mathbb{Z}_2 \times \mathbb{Z}_3$	$\mathbb{Z}_2$	$\mathbb{Z}_2^3 \times \mathbb{Z}_3$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^5 \times \mathbb{Z}_3$	$\mathbb{Z} \times \mathbb{Z}^3_{(2)} \times \mathbb{Z}^2_{(3)}$
T	$\mathbb{Z}$	e	$\mathbb{Z}_3$	$\mathbb{Z}_2$	$\mathbb{Z}_6$	e	$\mathbb{Z}_2^2 \times \mathbb{Z}_3$	$\mathbb{Z} \times \mathbb{Z}^1_{(2)} \times \mathbb{Z}^2_{(3)}$
$I \times \mathbb{Z}_4$	$\mathbb{Z}$	e	$\mathbb{Z}_4$	$\mathbb{Z}_2$	$\mathbb{Z}_2^2 \times \mathbb{Z}_{3*4*5}$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^4 \times \mathbb{Z}_2$	$\mathbb{Z} \times \mathbb{Z}^7_{(2)} \times \mathbb{Z}^1_{(3)} \times \mathbb{Z}^2_{(5)}$
$I \times \mathbb{Z}_2$	$\mathbb{Z}$	e	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2^3 \times \mathbb{Z}_3 \times \mathbb{Z}_5$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^5$	$\mathbb{Z} \times \mathbb{Z}^3_{(2)} \times \mathbb{Z}^1_{(3)} \times \mathbb{Z}^2_{(5)}$
I	$\mathbb{Z}$	e	e	$\mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_5$	e	$\mathbb{Z}_2^2$	$\mathbb{Z} \times \mathbb{Z}^1_{(2)} \times \mathbb{Z}^1_{(3)} \times \mathbb{Z}^2_{(5)}$
$O \times \mathbb{Z}_4$	$\mathbb{Z}$	e	$\mathbb{Z}_2 \times \mathbb{Z}_4$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^3 \times \mathbb{Z}_4^2 \times \mathbb{Z}_3$	$\mathbb{Z}_2^4 \times \mathbb{Z}_4$	$\mathbb{Z}_2^7 \times \mathbb{Z}_4^2$	$\mathbb{Z} \times \mathbb{Z}^{15}_{(2)} \times \mathbb{Z}^1_{(3)}$
$O \times \mathbb{Z}_2$	$\mathbb{Z}$	e	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^4 \times \mathbb{Z}_3 \times \mathbb{Z}_4$	$\mathbb{Z}_2^5$	$\mathbb{Z}_2^9$	$\mathbb{Z} \times \mathbb{Z}^7_{(2)} \times \mathbb{Z}^1_{(3)}$
O	$\mathbb{Z}$	e	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_4$	$\mathbb{Z}_2$	$\mathbb{Z}_2^3$	$\mathbb{Z} \times \mathbb{Z}^3_{(2)} \times \mathbb{Z}^1_{(3)}$
$BinT \times \mathbb{Z}_2 \subset Pin_-(3)$	$\mathbb{Z}$	e	$\mathbb{Z}_2 \times \mathbb{Z}_3$	e	$\mathbb{Z}_2 \times \mathbb{Z}_8 \times \mathbb{Z}_3$	$\mathbb{Z}_2$	$\mathbb{Z}_2^2 \times \mathbb{Z}_3$	$\mathbb{Z} \times \mathbb{Z}^5_{(2)} \times \mathbb{Z}^2_{(3)}$
$BinT$	$\mathbb{Z}$	0	$\mathbb{Z}_3$	0	$\mathbb{Z}_{24}$	0	$\mathbb{Z}_3$	$\mathbb{Z} \times \mathbb{Z}^2_{(2)} \times \mathbb{Z}^2_{(3)}$
$BinO$	$\mathbb{Z}$	0	$\mathbb{Z}_2$	0	$\mathbb{Z}_{48}$	0	?	$\mathbb{Z} \times \mathbb{Z}^5_{(2)} \times \mathbb{Z}^1_{(3)}$
$BinI$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}_{120}$	0	0	$\mathbb{Z} \times \mathbb{Z}^2_{(2)} \times \mathbb{Z}^1_{(3)} \times \mathbb{Z}^2_{(5)}$

3.11 Replicas and S_n fixed points

Suppose we have a $BrPic(\mathcal{C})$ already with given π_1^{old} and π_2^{old}. We then want to study $\mathcal{C}^{\boxtimes n}$ and S_n extensions thereof. The motivation for this comes from taking replicas and then gauging the permutation symmetry of the replicas. So consider a new 3-group with:

$$\begin{aligned} \pi_1^{new} &= (\pi_1^{old}) \wr S_n \simeq (\pi_1^{old})^n \rtimes S_n \\ \pi_2^{new} &= (\pi_2^{old})^n \\ \pi_3^{new} &= \mathbb{C}^* \end{aligned}$$

Obstructions live in $H^3(G, \pi_2^{new}) = H^3(G, (\pi_2^{old})^n)$ and $H^4(G, \pi_3^{new}) = H^4(G, \mathbb{C}^*)$. The examples we found had $\pi_2^{old} \simeq \mathbb{Z}_2\ \mathbb{Z}_2^2$ or e so that reduces the computation significantly.

3.11.1 Lemma (Shapiro's) *Let H be a subgroup of G and N a representation of H. $Ind_H^G N$ is a rep of G. and $CoIndN$*

$$H^*(G, CoInd\, N) = H^*(H, N)$$

If H is finite index subgroup, then $IndN \simeq CoindN$.

In particular when $G = S_m$ and $H = S_{m-1}$ with $IndN = (\mathbb{Z}_2^m)_a$ with permutation action, we can set $N = \mathbb{Z}_2$. Similarly for $IndN = (\mathbb{Z}_2^{2m})_a$, we can set $N = \mathbb{Z}_2^2$.

$$H^*(S_m, (\mathbb{Z}_2^m)_a) \;\simeq\; H^\bullet(S_{m-1}, \mathbb{Z}_2)$$

3.11.2 Lemma *Let $A??B$ be our notation to say the result fits in an exact sequence* $0 \longrightarrow A \longrightarrow A??B \longrightarrow B \longrightarrow 0$

$$H^3(S_m, (\mathbb{Z}_2^m)_a) \simeq H^3(S_{m-1}, \mathbb{Z}_2) \;=\; \begin{cases} \mathbb{Z}_2??\mathbb{Z}_2^3 = \mathbb{Z}_4 \bigoplus \mathbb{Z}_2^2 \; or \; \mathbb{Z}_2^4 & m-1 \geq 6 \\ \mathbb{Z}_2??(\mathbb{Z}_2 \bigoplus \mathbb{Z}_2) & 6 > m-1 \geq 4 \\ 0??\mathbb{Z}_2 = \mathbb{Z}_2 & m-1 = 3 \\ 0??\mathbb{Z}_2 = \mathbb{Z}_2 & m-1 = 2 \\ 0 & m-1 = 1 \end{cases}$$

$$H^3(S_m, (\mathbb{Z}_q^m)_a) \simeq H^3(S_{m-1}, \mathbb{Z}_q) \;=\; \begin{cases} \mathbb{Z}_2??\mathbb{Z}_{(2,q)}^2 \bigoplus \mathbb{Z}_{(12,q)} & m-1 \geq 6 \\ \mathbb{Z}_2??(\mathbb{Z}_{(2,q)} \bigoplus \mathbb{Z}_{(12,q)}) & 6 > m-1 \geq 4 \\ 0??\mathbb{Z}_{(6,q)} & m-1 = 3 \\ 0??\mathbb{Z}_{(2,q)} & m-1 = 2 \\ 0 & m-1 = 1 \end{cases}$$

$$H^3(S_m, (\mathbb{Z}_2^{2m})_a) \simeq H^3(S_{m-1}, \mathbb{Z}_2^2) \;=\; \begin{cases} \mathbb{Z}_2??\mathbb{Z}_2^6 = \mathbb{Z}_4 \bigoplus \mathbb{Z}_2^5 \; or \, \mathbb{Z}_2^7 & m-1 \geq 6 \\ \mathbb{Z}_2??(\mathbb{Z}_2^2 \bigoplus \mathbb{Z}_2^2) & 6 > m-1 \geq 4 \\ 0??\mathbb{Z}_2^2 = \mathbb{Z}_2^2 & m-1 = 3 \\ 0??\mathbb{Z}_2^2 = \mathbb{Z}_2^2 & m-1 = 2 \\ 0 & m-1 = 1 \end{cases}$$

$$H^3(S_m, e) \;=\; \begin{cases} \mathbb{Z}_2 & m \geq 4 \\ 0 & m < 4 \end{cases}$$

$$H^4(S_m, \mathbb{C}^*) \;\simeq\; \begin{cases} (\mathbb{Z}_2^2 \bigoplus \mathbb{Z}_{12})??\mathbb{Z}_2^3 & m \geq 8 \\ (\mathbb{Z}_2^2 \bigoplus \mathbb{Z}_{12})??\mathbb{Z}_2^2 & 8 > m \geq 6 \\ (\mathbb{Z}_2 \bigoplus \mathbb{Z}_{12})??\mathbb{Z}_2 & 6 > m \geq 4 \\ \mathbb{Z}_6??0 = \mathbb{Z}_6 & m = 3 \\ \mathbb{Z}_2??0 = \mathbb{Z}_2 & m = 2 \end{cases}$$

$$
\begin{aligned}
H^3(S_m, \mathbb{C}^*) &\simeq \begin{cases} \mathbb{Z}_2??\mathbb{Z}_2^2 \bigoplus \mathbb{Z}_{12} & m \geq 6 \\ \mathbb{Z}_2??\mathbb{Z}_2 \bigoplus \mathbb{Z}_{12} & 6 > m \geq 4 \\ 0??\mathbb{Z}_6 & m = 3 \\ 0??\mathbb{Z}_2 & m = 2 \end{cases} \\
H^2(S_m, \mathbb{Z}_2^m) \simeq H^2(S_{m-1}, \mathbb{Z}_2) &= \begin{cases} \mathbb{Z}_2??\mathbb{Z}_2 & m-1 \geq 4 \\ \mathbb{Z}_2??0 = \mathbb{Z}_2 & m-1 = 2,3 \end{cases} \\
H^2(S_m, \mathbb{Z}_q^m) \simeq H^2(S_{m-1}, \mathbb{Z}_q) &= \begin{cases} \mathbb{Z}_2??\mathbb{Z}_{(2,q)} & m-1 \geq 4 \\ \mathbb{Z}_2??0 = \mathbb{Z}_2 & m-1 = 2,3 \end{cases} \\
H^2(S_m, \mathbb{Z}_2^{2m}) \simeq H^2(S_{m-1}, \mathbb{Z}_2^2) &= \begin{cases} \mathbb{Z}_2??\mathbb{Z}_2^2 & m-1 \geq 4 \\ \mathbb{Z}_2??0 = \mathbb{Z}_2 & m-1 = 2,3 \end{cases} \\
H^2(S_m, e) &\simeq \begin{cases} \mathbb{Z}_2 & m \geq 2 \end{cases}
\end{aligned}
$$

Proof Apply Shapiro's Lemma, the Universal Coefficient Theorem and Homological Stability of the symmetric group. □

Chapter 4

Bimodule Categories from Covariantization

In this chapter, we begin a program for this construction by supposing our fusion categories come as $RepA$ for some coquasitriangular Hopf algebra. The transmutation procedure produces a comodule algebra A_r for a certain double $D(A,r)$. Taking the representations of A_r provides a relevant module category.

In section 4.1 we review the procedure in generality. In section 4.2 we give the preliminary small example of a Sweedler algebra. Then we proceed with section 4.3 to give the examples relevant for LCFT with $u_q(\mathfrak{sl}_2)$ in the sense of [51].

4.1 Transmutation

4.1.1 Definition (Coquasitriangular Hopf Algebra) *A coquasitriangular bialgebra is a bialgebra over the field k which comes with a convolution invertible $r : A \otimes A \to k$ satisfying:*

$$\begin{aligned} r(a_1 \otimes b_1)a_2 b_2 &= b_1 a_1 r(a_2 \otimes b_2) \\ r(ab \otimes c) &= r(a \otimes c_1) r(b \otimes c_2) \\ r(a \otimes bc) &= r(a_2 \otimes b) r(a_1 \otimes c) \end{aligned}$$

4.1.2 Definition (Regular) *A coquasitriangular bialgebra is called regular if in addition to r $A \otimes A \to k$ being convolution invertible, $\tilde{r}$ $A \otimes A^{cop} \to k$ is also convolution invertible. For the first case call the inverse $\bar{r}$ and in the second call it s.*

4.1.3 Definition ($D(A,r)$) *For a regular coquasitriangular bialgebra (A,r), $D(A,r)$ is the bialgebra which as a coalgebra is $A^{cop} \otimes A$.*

This can be imagined as folding the two sides of the system so one side is orientation reversed, meaning that applying a coproduct will be opposite relative to the orientation preserving half.

However it's algebra structure is given by

$$(i(b) \otimes a)(i(b') \otimes a') = \sum r(a_1, b'_3) i(b'_2 b) \otimes a_2 a' s(a_3, b'_1)$$

where i is taking an element of A and regarding it as an element of $A^{op,cop}$.

4.1.4 Lemma *The maps $A \to D(A,r)$ and $A^{op,cop} \to D(A,r)$ by $1 \otimes a$ and $i(a) \otimes 1$ are bialgebra homomorphisms.*

4.1.5 Definition ($D(A,r)$ coaction) *The map*

$$a \to a_2 \otimes (i(a_1) \otimes a_3)$$

gives a coaction of the double, but this map $A \to A \otimes D(A,r)$ is not an algebra map.

4.1.6 Definition (Transmuted Product) *The algebra A_r is the same as A as a vector space but the algebra structure is modified to be*

$$a \cdot_r b = \sum s(a_3, b_1) a_2 r(a_1, b_2) b_3 = \sum s(a_3, b_1) b_2 r(a_2, b_3) a_1$$

gives the transmutation of a bialgebra equipped with regular r form so that it becomes a right comodule algebra for the double with coaction

Coideal Subalgebras

4.1.7 Lemma ([15]) *Taking a character χ of A_r and taking the image of the map*

$$A_r \longrightarrow A_r \otimes D(A, r) \longrightarrow D(A, r) \longrightarrow A$$

provides a coideal subalgebra in A. The first map is the comodule structure, then the character and finally $i(a_1) \otimes a_3 \to S(a_1) a_3$

This has the effect of only considering a portion of the module category $A_r - mod$. More generally if we provide a representation of A_r on H_{defect} the result lands in $End(H_{defect}) \otimes A$ instead.

(Bi)Module Categories

For a right A comodule algebra E we get

$$\begin{array}{ccc} E \otimes A - \text{mod} & \longrightarrow & E - \text{mod} \\ \downarrow & & \\ E - \text{mod} \boxtimes A - \text{mod} & & \end{array}$$

This produces module categories. For bimodule categories, use a folding trick to look at the double of A instead.

Hopf-Galois

A condition we may ask for a comodule algebra is that it be Hopf-Galois.

4.1.8 Definition (Hopf-Galois Extension [67]) *Let H be a k bialgebra and E a right H comodule algebra with comodule structure ρ. Let U be the coinvariants so $\rho(u) = u \otimes 1$. Then $U \hookrightarrow E$ is called Hopf-Galois if $E \otimes_U E \simeq E \otimes H$ by $e \otimes_U e' \to (e \otimes_k 1)(\rho(e'))$. This is well defined because*

$$\begin{aligned} e \otimes_U ue' &\to (e \otimes_k 1)(\rho(u)\rho(e')) \\ &= (e \otimes_k 1)(u \otimes_k 1)(\rho(e')) \\ &= (eu \otimes_k 1)(\rho(e')) \\ eu \otimes_U e' &\to (eu \otimes_k 1)(\rho(e')) \end{aligned}$$

4.1.9 Example *Let $E = \bigoplus E_g$ be a strongly G-graded algebra. Strongly graded means that $E_g E_h \simeq E_{gh}$ for all g and h. This is a kG-comodule algebra by letting $a \in E_g \to a \otimes g$. The coinvariants U are given by the identity component E_e and the extension $E_e \subset E$ is Hopf-Galois.*

We do not have a use for this simplifying assumption, but there may be other applications of the module categories built from Hopf-Galois comodule algebras.

4.2 Sweedler Algebra

Take the 4 dimensional Sweeler Hopf algebra with $1, g, x, gx$ and give it a coquasitriangular structure p_α [68]

$$\begin{aligned}
\Delta^2 1 &= 1 \otimes 1 \otimes 1 \\
\Delta^2 g &= g \otimes g \otimes g \\
\Delta^2 x &= x \otimes g \otimes g + 1 \otimes x \otimes g + 1 \otimes 1 \otimes x \\
\Delta^2 (xg) &= xg \otimes g^2 \otimes g^2 + 1 * g \otimes x * g \otimes g * g + 1 * g \otimes 1 * g \otimes x * g \\
&= xg \otimes 1 \otimes 1 + g \otimes xg \otimes 1 + g \otimes g \otimes xg \\
r_{ij} &= \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & \alpha & \alpha \\ 0 & 0 & \alpha & \alpha \end{pmatrix}
\end{aligned}$$

4.2.1 Lemma *The multiplication table for the transmuted product becomes:*

$$\begin{aligned}
1 \cdot_r 1 &= r(1, S(b_1))r(1, b_2)1b_3 \\
&= r(1, S(1))r(1, 1)1 * 1 = 1 \\
1 \cdot_r g &= r(1, S(b_1))r(1, b_2)1b_3 \\
&= r(1, S(g))r(1, g)1g = g \\
1 \cdot_r x &= r(1, S(b_1))r(1, b_2)1b_3 \\
&= r(1, S(x))r(1, g)1g + r(1, S(1))r(1, x)1g + r(1, S(1))r(1, 1)1x = x \\
1 \cdot_r xg &= r(1, S(b_1))r(1, b_2)1b_3 \\
&= r(1, S(xg))r(1, 1)11 + r(1, S(g))r(1, xg)11 + r(1, S(g))r(1, g)1xg \\
&= r(1, S(g)S(x))r(1, 1)11 + r(1, S(g))r(1, xg)11 + r(1, S(g))r(1, g)1xg = xg \\
g \cdot_r 1 &= r(g, S(b_1))r(g, b_2)gb_3 \\
&= r(g, S(1))r(g, 1)g * 1 = g \\
g \cdot_r g &= r(g, S(b_1))r(g, b_2)gb_3 \\
&= r(g, S(g))r(g, g)gg = 1 \\
g \cdot_r x &= r(g, S(b_1))r(g, b_2)gb_3 \\
&= r(g, S(x))r(g, g)gg + r(g, S(1))r(g, x)gg + r(g, S(1))r(g, 1)gx = gx \\
g \cdot_r xg &= r(g, S(b_1))r(g, b_2)gb_3 \\
&= r(g, S(xg))r(g, 1)g1 + r(g, S(g))r(g, xg)g1 + r(g, S(g))r(g, g)gxg \\
&= r(g, x)r(g, 1)g1 + r(g, S(g))r(g, xg)g1 + r(g, S(g))r(g, g)gxg = gxg = -x \\
x \cdot_r 1 &= x \\
x \cdot_r g &= -xg \\
x \cdot_r x &= -\alpha g + \alpha g^2 + x^2 = -\alpha g + \alpha \\
x \cdot_r xg &= \alpha - \alpha g - xxg = \alpha - \alpha g \\
xg \cdot_r 1 &= xg \\
xg \cdot_r g &= -xgg = -x \\
xg \cdot_r x &= \alpha g + \alpha g^2 + xgx = \alpha g + \alpha \\
xg \cdot_r xg &= \alpha + \alpha g - xgxg = \alpha + \alpha g
\end{aligned}$$

4.2.2 Corollary *$(x - xg)$ is now nilpotent*

Proof

$$\begin{aligned}
(x - xg) \cdot_r g &= -xg + x \\
(x - xg) \cdot_r x &= -2\alpha g \\
(x - xg) \cdot_r xg &= -2\alpha g \\
(x - xg) \cdot_r (x - xg) &= 0
\end{aligned}$$

□

The representation category for this algebra provides our desired module category for the double allowing us to construct the desired bimodule category.

4.3 Logarithmic CFT

As a more interesting example consider the case of [51]. Though the semisimplicity required for topological field theory applications is broken, this is still useful for the setup of [53]. It is interesting to see how the semisimplification procedure affects the (bi)module categories. We hope that the bimodule categories produced for the nonsemisimple case will still provide insight for bimodule categories of the semisimplified version, but that is unclear. They may still be useful to construct defects and boundaries for logarithmic minimal models.

The first class of examples are associated with $\mathfrak{g} = \mathfrak{sl}_2$ and q is a primitive $2p$'th root of unity. That is the small quantum group $u_q(\mathfrak{sl}_2)$ with generators E, F, K. This is related by an extension of Kazhdan-Lusztig-Finkelberg to $(1,p)$ models [51].

4.3.1 Definition ($(1,p)$ model) *A $(1,p)$ model is an irrational CFT with a single free boson ϕ and stress energy tensor*

$$T = \frac{1}{2}\partial\phi\partial\phi + \frac{\sqrt{2p}-\sqrt{2/p}}{2}\partial^2\phi$$

and central charge $c = 13 - 6(p + \frac{1}{p})$ using the Coloumb gas formalism.

4.3.2 Definition ($\bar{D}$) *$u_q(\mathfrak{sl}_2)$ is a Hopf algebra, but is not quasitriangular. Instead it embeds into $\bar{D}$ with generators e, f, k and the relations,counit,coproduct and antipode given as:*

$$\begin{aligned} ke &= qek \\ k\phi &= q^{-1}\phi k \\ e\phi - \phi e &= \frac{k^2 - k^{-2}}{q - q^{-1}} \\ e^p &= 0 \\ \phi^p &= 0 \\ k^{4p} &= \mathbf{1} \end{aligned} \qquad \begin{aligned} \epsilon(e) &= 0 \\ \epsilon(\phi) &= 0 \\ \epsilon(k) &= 1 \\ \Delta(e) &= \mathbf{1}\otimes e + e\otimes k^2 \\ \Delta(\phi) &= \phi\otimes\mathbf{1} + k^{-2}\otimes\phi \\ \Delta(k) &= k\otimes k \\ S(e) &= -ek^{-2} \\ S(\phi) &= -k^2\phi \\ S(k) &= k^{-1} \end{aligned}$$

The embedding $i: \; u_q(\mathfrak{sl}_2) \to \bar{D}$ is given by $E \to e$, $F \to f$ and $K \to k^2$.

4.3.3 Lemma ([51]) *$\bar{D}$ is quasitriangular with R-matrix.*

$$R = \frac{1}{4p}\sum_{m=0}^{p-1}\sum_{n,j=0}^{4p-1}\frac{(q-q^{-1})^m}{[m]!}q^{m(m-1)/2+m(n-j)-nj/2}e^m k^n \otimes \phi^m k^j$$

We will need the coproduct of various pieces of R so introduce notation:

$$\begin{aligned}
r^{mnj+}_{simp} &\equiv e^m k^n \\
r^{mnj-}_{simp} &\equiv \phi^m k^j \\
R &= \frac{1}{4p}\sum_{m=0}^{p-1}\sum_{n,j=0}^{4p-1}\frac{(q-q^{-1})^m}{[m]!}q^{m(m-1)/2+m(n-j)-nj/2}r^{mnj+}_{simp}\otimes r^{mnj-}_{simp}
\end{aligned}$$

$$\begin{aligned}
\Delta(e^m k^n) &= \Delta(e^m)\Delta(k^n) = \Delta(e^m)(k^n\otimes k^n) \\
&= \Delta(e^{m-1})(k^n\otimes ek^n + ek^n\otimes k^{2+n}) \\
&= \Delta(e^{m-2})(k^n\otimes e^2k^n + ek^n\otimes k^2ek^n + ek^n\otimes ek^{2+n} + e^2k^n\otimes k^{4+n}) \\
&= \Delta(e^{m-2})(k^n\otimes e^2k^n + (1+q^2)ek^n\otimes ek^{n+2} + e^2k^n\otimes k^{4+n}) \\
&= \Delta(e^{m-3})(k^n\otimes e^3k^n + (1+q^2+q^4)ek^n\otimes e^2k^{2+n} \\
&+ (1+q^2+q^4)e^2k^n\otimes ek^{n+4} + e^3k^n\otimes k^{6+n}) \\
&= \Delta(e^{m-4})(k^n\otimes e^4k^n + (1+q^2+q^4+q^6)ek^n\otimes e^3k^{n+2} \\
&+ (1+q^2+2*q^4+q^6+q^8)e^2k^n\otimes e^2k^{4+n} \\
&+ (1+q^2+q^4+q^6)e^3k^n\otimes ek^{n+6} + e^4k^n\otimes k^{8+n}) \\
&= \sum_x f(x,m,q)e^xk^n\otimes e^{m-x}k^{n+2x}
\end{aligned}$$

where $f(x,m,q)$ is given recursively

$$\begin{aligned}
f(0,m+1,q) &= 1 \\
f(x,m+1,q) &= f(x,m,q)+f(x-1,m,q)*q^{2m-2x} \\
\Delta(r^{mnj+}_{simp}) &\equiv \sum_x r^{mnj+}_{simp,1x}\otimes r^{mnj+}_{simp,2x} \\
r^{mnj+}_{simp,1l} &= f(x,m,q)e^xk^n \\
r^{mnj+}_{simp,2l} &= e^{m-x}k^{n+2x}
\end{aligned}$$

4.3.4 Proposition *Dualize this finite dimensional quasitriangular Hopf algebra to produce a coquasitriangular Hopf algebra with which the procedure applies straightforwardly. Parameterize a basis for A^* by $x_{a,b,c}$ which is the dual vector to $e^a\phi^bk^c$ with $0\le a\le p-1$, $0\le b\le p-1$ and $0\le c\le 4p-1$. This is a convenient basis to right the left and right actions of A on A^* given by $\triangleright$ and $\triangleleft$.*

$$\begin{aligned}
\mu^{RE}(x_{a,b,c}\otimes x_{a2,b2,c2}) &= \frac{1}{4p}\sum_{m=0}^{p-1}\sum_{n,j}^{4p-1}\frac{(q-q^{-1})^m}{[m]!}q^{m(m-1)/2+m(n-j)-nj/2} \\
&\textstyle\sum_l \quad \mu(r^{mnj+}_{simp,2l}\triangleright x_{a,b,c}\triangleleft S(r^{mnj+}_{simp,1l})\otimes x_{a2,b2,c2}\triangleleft r^{mnj-}_{simp})
\end{aligned}$$

This is suitable for a computer algebra system to write out the full $4p^3$ *dimensional algebra as a* $(4p^3)^3$ *table of structure constants.*

4.3.5 Corollary *Composition with* i^**, turns this* $\bar{D}^*$ *comodule or coideal constructed into a* $(u_q(\mathfrak{sl}_2))^*$ *comodule.*

Chapter 5

K-Theory of Fusion Rings

5.1 Introduction

Given that the machinery and presentations of module categories were rather intricate, in this part we look at a simplification by a two dimensional analog. That is, first take $K_0(\mathcal{C})$ of the desired fusion category and then look at some of it's arithmetic invariants. This is captured in K_0, K_1 and Pic of the unital based rings both with or without tensoring $\otimes_{\mathbb{Z}}\mathbb{Q}$. Before, we decorated codimension 1 defect walls with $\mathcal{C}-\mathcal{C}$ bimodule categories and codimension 2 defects with functors thereof. In the simplification, we consider $K_0(\mathcal{C})-K_0(\mathcal{C})$ bimodules and their intertwiners.

5.2 K_0 Reduction

The fusion categories have now been replaced by unital based rings $R = K_0(\mathcal{C})$.

5.2.1 Lemma ([69]) *A biexact functor F $\mathcal{A}\times\mathcal{B}\to\mathcal{C}$ induces a bilinear map $K_0\mathcal{A}\otimes K_0\mathcal{B}\to K_0\mathcal{C}$*

So when the functors defining the bimodule category structure satisfy this condition, we have bimodules between unital fusion rings. We will concentrate on the case where both sides are the same R. So the invertible bimodules are classified by their classes in $Pic(R)$ and the invertible bimodule intertwiners are specified by the units $R^* \subset R$. That is the new π_1 and π_2 replacing the homotopy 3-type of Brauer-Picard. This fits inside $K_{\leq 1}(R)$ of algebraic K-theory of R.

Fully Extended 2 Dimensional

5.2.2 Theorem (Schommer-Pries [70]) *The functors from $Bord^2 \to AlgBim_k$ are given up to equivalence by the trace-Morita classes of separable symmetric Frobenius algebras that get assigned to the point.*

5.2.3 Theorem (Eilenberg-Nakayama) *In vector spaces over a field k, an algebra A can be equipped with a symmetric Frobenius algebra structure if it is separable. It is strongly seperable when there exists a symmetric seperability idempotent.*

5.2.4 Theorem (Division Algebras) *Let k be a perfect field, such as being characteristic 0. The separable algebras over a field k are precisely finite products of matrix algebras over finite-dimensional division algebras D over k.*

5.2.5 Definition ($\mathbf{Alg}_k^\star$) *This is the sub 2-category of super k-algebras, bimodules and intertwiners with the invertible objects only.*

5.2.6 Theorem (1.28 [71]) *A super algebra is invertible if and only if it is finite dimensional central simple. Any super matrix algebra is equivalent in this 2-category to the trivial algebra k. This is the case of invertible field theories though a general seperable algebra describing a noninvertible field theory can be have many such division algebras.*

5.2.7 Theorem (1.30 [71]) *The super Brauer group of $\mathbb{C}$ is cyclic of order 2 and the super Brauer group of $\mathbb{R}$ is cyclic of order 8. In each case it is generated by the Clifford algebras $Cliff(L,+)$, where L is a line for the respective field and $+$ stands for a positive definite quadratic form. The higher homotopy groups are $\pi_1 \simeq \mathbb{Z}_2$ and $\pi_2 \simeq \mathbb{C}^*$ for the complex case and $\mathbb{Z}_2$ and $\mathbb{R}^*$ for the real case.*

These homotopy types may be modified to avoid dealing with $\mathbb{R}^*$ or $\mathbb{C}^*$ with the discrete topology as uncountable abelian groups. The trick is to look at them with the usual topology. That effectively replaces $\pi_2 = \mathbb{C}^*$ by $\pi_3 = \mathbb{Z}$ and $\pi_2 = \mathbb{R}^*$ by $\pi_2 = \mathbb{Z}_2$. In terms of spectra these are denoted $\mathcal{X}_{\text{discrete}}$ and $\mathcal{X}_{\text{continuous}}$ by [54].

5.2.8 Lemma *For a perfect field, there is no difference between separable and finite-dimensional semisimple. Therefore for our decategorification procedure we are looking at $K_0(\mathcal{C}) \otimes_{\mathbb{Z}} \mathbb{Q}$. In order to check whether or not a given associative Artinian algebra is semisimple, check whether it has trivial Jacobson radical.*

5.2.9 Lemma ([72]) *A general fusion ring R is semi-simple.*

Proof (Sketch) First show that the nilradical vanishes. Then apply a theorem that says that for the case of finitely generated commutative ring the vanishing of nil radical is equivalent to vanishing Jacobson radical. □

Having a separable algebra over $\mathbb{Q}$ in hand by $K_0(\mathcal{C}) \otimes_{\mathbb{Z}} \mathbb{Q}$, we can look at the associated division algebras that show up. This means computing specific elements of $Br(\mathbb{Q})$. Computing the torsion orders thereof gives potentially interesting arithmetic structures.

5.3 Algebraic K-theory review

$K_{\leq 1}$

The groups needed are part of $K_{\leq 1}(R)$ algebraic K theory of R.

5.3.1 Definition (K_0) *K_0 is a functor from rings to abelian groups. It sends a ring R to the group of isomorphism classes in finitely generated projective R modules with splitting short exact sequences. For commutative R, the result is a ring with operations coming from direct sum and tensor product over R. Exterior product gives a λ ring structure.*

5.3.2 Example *For a Dedekind domain, this splits as a direct sum $K_0(D) \simeq Pic(D) \bigoplus \mathbb{Z}$. In particular for the case of the ring of integers $\mathcal{O}$ in a number field K we get $K_0(\mathcal{O}) \simeq \mathbb{Z} \bigoplus Cl(\mathcal{O})$ where $Cl(\mathcal{O})$ is the (ideal) class group. For a local ring such as a field $K_0(A) \simeq \mathbb{Z}$ by rank.*

5.3.3 Proposition (Forgotten commutative Pic [73] 3.5) *For a commutative ring R over k, define $Pic_k(R)$ by equivalence classes of bimodules where the left and right k actions are the same. Then $Pic_k(R) \simeq Aut_k(R) \ltimes Pic_R(R)$ where $Pic_R(R)$ is equivalent to the usual Pic where one considers modules rather than bimodules.*

So the bimodule categories of concern give elements of $BrPic$ and upon reduction of $Pic(R)$. Quantum dimension provides a ring map $R \to O_F \to O_{\bar{Q}}$ for some cyclotomic number field F. This means we can probe $K_0(R)$ through $K_0(O_F)$.

5.3.4 Definition (K_1) $K_1(A) \equiv \frac{GL_\infty(A)}{\left[GL_\infty(A), GL_\infty(A)\right]}$

The more manageable characterizations are given as:

- For commutative rings, $K_1(A) \simeq A^* \bigoplus SK_1(A)$.
- For a Euclidean domain $SK_1(A)$ vanishes. For example $K_1(\mathbb{Z}) \simeq \mathbb{Z}_2$
- For ring of integers of number field $K_1(I) \simeq I^*$. The regulator helps characterize I^*
- A field is a Euclidean domain so we just get k^*

Brauer Groups

5.3.5 Theorem (Picard-Brauer) *The Picard-Brauer group (Picard 3-group) of Alg_R algebras over R, bimodules and intertwiners is the same as giving line 2-bundles over SpecR which has homotopy groups as $Br(R)$ $Pic(R)$ and R^*. These match with $H^2_{et}(R, G_m)_{tor}$, $H^1_{et}(R, G_m)$ and $H^0_{et}(R, G_m)$ respectively.*

5.3.6 Theorem (Global Fields) *For a global field K like a number field.*

$$0 \longrightarrow Br(K) \longrightarrow \bigoplus_{v \in S} Br(K_v) \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow 0$$

For local fields that are complete under a discrete valuation like $\mathbb{Q}_p$, we get $Br(K_v) \simeq \mathbb{Q}/\mathbb{Z}$. For the reals the place at ∞ get $\mathbb{Z}/(2)$ where we regard it as $(1/2\mathbb{Z})/\mathbb{Z}$.

5.3.7 Lemma *One way to construct finite dimensional associative division algebras over arbitrary fields is to produce quaternion algebras. To any local field there are just two quaternion algebras. This is the $\{0, \frac{1}{2}\} \subset \mathbb{Q}/\mathbb{Z}$ of each $Br(K_v)$. One can freely choose either of these two possibilities for each place with the constraint that all but finitely many are 0 (split) and there are an even number of the remaining ramified cases.*

5.3.8 Example *Rational Hamilton quaternions ramify at 2 and ∞ and split everywhere else. The rational 2 by 2 matrices split at all places.*

5.3.9 Lemma $Br(\mathcal{O}_K) = \mathbb{Z}_2^{r-1}$ *if K has $r > 0$ real places. It is 0 otherwise.*

5.4 Examples

With these ingredients we can proceed to the computations of interest.

From Quantum Groups

5.4.1 Example *The Fibonacci fusion ring is $\mathbb{Z}[\phi]$. This is a ring of integers for a number field so there is no SK_1 automatically.*

$$\begin{aligned} K_0(\mathbb{Z}[\phi]) &= \mathbb{Z} \bigoplus e = \mathbb{Z} \\ K_0(\mathbb{Z}[\phi] \otimes_{\mathbb{Z}} \mathbb{Q}) &= \mathbb{Z} \\ K_1(\mathbb{Z}[\phi]) &= \mathbb{Z}^{2+0-1} \bigoplus \mathbb{Z}_2 \\ K_1(\mathbb{Z}[\phi] \otimes_{\mathbb{Z}} \mathbb{Q}) &= (\mathbb{Q}(\phi))^* \\ Br(\mathbb{Z}[\phi]) &\simeq \mathbb{Z}_2^{2-1} \end{aligned}$$

5.4.2 Example *The Ising category is not already a number field but quantum dimension provides a ring map to $\mathbb{Z}[\sqrt{2}]$. So let's look at $K_{\leq 1}$ of that instead and then apply functoriality.*

$$\begin{aligned} K_0(\mathbb{Z}[\sqrt{2}]) &= \mathbb{Z} \bigoplus e = \mathbb{Z} \\ K_0(\mathbb{Z}[\sqrt{2}] \otimes_{\mathbb{Z}} \mathbb{Q}) &= \mathbb{Z} \\ K_1(\mathbb{Z}[\sqrt{2}]) &= \mathbb{Z}^{2+0-1} \bigoplus \mathbb{Z}_2 \\ K_1(\mathbb{Z}[\sqrt{2}] \otimes_{\mathbb{Z}} \mathbb{Q}) &= (\mathbb{Q}(\sqrt{2}))^* \\ Br(\mathbb{Z}[\sqrt{2}]) &\simeq \mathbb{Z}_2^{2-1} \end{aligned}$$

More generally the result will have a map to the ring of integers in some cyclotomic field. The K-theory of those can be tabulated for these rings of integers by standard techniques [69].

From Group Rings

$\mathbb{Z}[G]$ for a finite group G has the structure of a unital based ring with basis given by the group elements. For this class of examples, we are not concerned with having a fusion categorification, much less having it be braided.

5.4.3 Definition (Whitehead Group [74]) *The quotient of $\bar{K}_1(\mathbb{Z}[G])$ by the image of $G \subset GL(1, \mathbb{Z}[G])$.*

$$1 \longrightarrow \frac{G}{[G,G]} \longrightarrow \bar{K}_1(\mathbb{Z}[G]) \longrightarrow Wh(G) \longrightarrow 1$$

5.4.4 Example ($\mathbb{Z}G$)

$$\begin{aligned} K_{2m}(\mathbb{Z}[G]) \otimes \mathbb{Q} &\simeq 0 \\ K_{4m+1}(\mathbb{Z}[G]) \otimes \mathbb{Q} &\simeq \mathbb{Q}^r \\ K_{4m+3}(\mathbb{Z}[G]) \otimes \mathbb{Q} &\simeq \mathbb{Q}^c \\ Br(\mathbb{Z}[A]) &\simeq 0 \end{aligned}$$

where r and c count irreducible real and complex type representations. A indicates an abelian group.

Proof The result for rationalized K theory is given by [75]. Without rationalization, much of the calculations are out of reach with the exception of the few which are in [76, p.112], [77] (for K_1) or the units in dihedral groups given by [78]. □

These are the sorts of computations that are the subject of the famous Farrell-Jones conjecture.

5.4.5 Conjecture (Farrell-Jones [76]) *Let R be a ring with involution and G a group. Then for all n, the map $H_n^G(E_{VirCyc}(G), K_R) \to H_n^G(pt, K_R) \simeq K_n(R[G])$ is an isomorphism. Here $H_n^G(-, K_R)$ is an extraordinary G-homology.*

From *Rep G* of finite groups

From the category of representations *Rep G* we get a unital based ring also called *Rep G*. For the rest of this section, only the ring is meant.

5.4.6 Lemma ([79]) *The Brauer group for a character ring RepG is computed as follows. Let $\mathcal{C}_i$ be the conjugacy classes of cyclic subgroups of G. Let there be k of these. For each of these form the field $\mathbb{Q}(\mathcal{C}_i)$ by looking the field generated by $\chi(x_i)$ for $x_i \in \mathcal{C}_i$ and χ a character of an irreducible representation. It has a number of real places r_i.*

$$Br(Rep\ G) \simeq \bigoplus_{i=1}^{k} \bigoplus_{m=1}^{r_i-1} \mathbb{Z}_2$$

5.4.7 Corollary *For symmetric groups the result $Br(RepS_n) = 0$. This is also the case for the character rings of abelian groups. For D_8, the result is $\mathbb{Z}_2$.*

5.4.8 Example ($\boldsymbol{Rep_{\mathbb{C}}G}$) *When $G = A$ is a finite abelian group, then this reduces to $\mathbb{Z}G^\vee$ of the previous example for the Pontryagin dual.*

$$
\begin{aligned}
K_0(RepA) \otimes \mathbb{Q} &\simeq 0 \\
K_{4m+1}(RepA) \otimes \mathbb{Q} &\simeq \mathbb{Q}^{r^\vee} \\
K_{4m+3}(RepA) \otimes \mathbb{Q} &\simeq \mathbb{Q}^{c^\vee} \\
Br(RepG) &\simeq \bigoplus_{i=1}^{k} \bigoplus_{m=1}^{r_i-1} \mathbb{Z}_2
\end{aligned}
$$

Again there is very little that can be said for general finite groups. Mapping to known rings like with the Ising example seems to be the only recourse to computable groups.

5.5 Conclusion

In this chapter we have taken the K theory of some fusion rings in order to understand a simplified reduction of the $(\infty, 3)$ category $MonCat_{bim}$. This is akin to the $(\infty, 2)$ category Alg_{bim}, but without tensoring with the rationals. As examples we described cases when the rings were close to rings of integers in number fields, integral group rings and representation rings. This relates to some very difficult problems of K-theory of complicated rings so we only provided what limited examples we could.

We hope future work of explicit computations will be useful in the interpretation we give here. In addition to the examples above, there is an analog in the logarithmic situation. The categories are no longer semisimple. We may not define a theory in the sense of [70] by tensoring up with the rationals. These Grothendieck rings are also not semisimple [80]. In fact, the Jacobson radicals for the $(1, p)$ cases are computed in [81]. However, we may still look at the algebraic K-theoretic invariants of these Grothendieck groups. Like with the bimodule categories, there is a small hope for relation between the semisimplified and nonsemisimplified cases stemming from the physical relations between LCFT and RCFT [82].

Part III

Spin Chains

Chapter 6

Asymptotic Completeness

6.1 Introduction

The spectrum of quantum spin chains and properties of integrability are sensitive to boundary conditions. A characterization of such boundary conditions comes up in Cherednik [83] and Sklyanin [84] .

In this paper we prove that the set of eigenvectors constructed in [84] is asymptotically complete. We consider an inhomogeneous spin chain with inhomogeneities in a sector $t_1 \cdots t_N$ following $0 < Re(t_1) < \cdots << Re(t_N)$. The Bethe vectors for such a spin chain form a basis for the entire Hilbert space.

The plan of this paper is as follows. In section 6.2 we recall facts about the inhomogeneous six-vertex model with reflecting boundaries. In section 6.3 we describe the asymptotics in this sector to solutions of Bethe equations. The Bethe vectors and the proof of completeness is contained in section 6.4.

6.2 The Six Vertex Model with Reflecting Boundary

Notation

The Boltzmann weights are parameterized with the R matrix

$$R = \begin{pmatrix} b(x+\eta) & 0 & 0 & 0 \\ 0 & b(x) & b(\eta) & 0 \\ 0 & b(\eta) & b(x) & 0 \\ 0 & 0 & 0 & b(x+\eta) \end{pmatrix}$$

where $b(x) = \sinh x$, $z = e^x$, $a_i = e^{t_i}$ and $q = e^\eta$

The R matrix satisfies the Yang Baxter Equation. The Reflection matrix K must satisfy the reflection equation.

$$\begin{aligned} R_{12}(u)R_{13}(u+v)R_{23}(v) &= R_{23}(v)R_{13}(u+v)R_{12}(u) \\ R_{12}(u-v)K_1(u)R_{21}(u+v)K_2(v) &= K_2(v)R_{12}(u+v)K_1(u)R_{21}(u-v) \end{aligned}$$

Assuming that K is diagonal leads to the one parameter family of solutions.

$$K = \begin{pmatrix} b(x+\xi) & 0 \\ 0 & -b(x-\xi) \end{pmatrix}$$

6.2.1 Definition (Boundary Monodromy Matrices) .

The monodromy matrix for a single row with inhomegeneities $t_1 \cdots t_N$ on those respective columns is given by

$$T(x, t_1 \cdots t_N) = R_{0N}(x-t_N) \cdots R_{01}(x-t_1) = \begin{pmatrix} A(x) & B(x) \\ C(x) & D(x) \end{pmatrix}$$

The double row monodromy matrix takes into effect the reflection at one end. It is defined as

$$U(x, t_1 \cdots t_N, \xi_+) = T(x, \vec{t})K(x - \frac{\eta}{2}, \xi_+)\sigma_2 T(-x, \vec{t})\sigma_2 = \begin{pmatrix} \mathcal{A}(x) & \mathcal{B}(x) \\ \mathcal{C}(x) & \mathcal{D}(x) \end{pmatrix}$$

Commutation Relations and Bethe Ansatz

Sklyanin proved the reflection equation which implies the following commutation relations for the operator valued entries of the 2 by 2 double row mondodromy matrix. [84]

$$\begin{aligned}
\mathcal{A}(u)\mathcal{B}(v) &= \frac{b(u-v-\eta)b(u+v-\eta)}{b(u-v)b(u+v)}\mathcal{B}(v)\mathcal{A}(u) \\
&+ \frac{b(\eta)b(u+v-\eta)}{b(u-v)b(u+v)}\mathcal{B}(u)\mathcal{A}(v) - \frac{b(\eta)}{b(u+v)}\mathcal{B}(u)\mathcal{D}(v) \\
\mathcal{D}(u)\mathcal{B}(v) &= \frac{b(u-v+\eta)b(u+v+\eta)}{b(u-v)b(u+v)}\mathcal{B}(v)\mathcal{D}(u) - \frac{b(2\eta)b(\eta)}{b(u-v)b(u+v)}\mathcal{B}(v)\mathcal{A}(u) \\
&+ \frac{b(\eta)b(u+v+\eta)}{b(u-v)b(u+v)}\mathcal{B}(u)\mathcal{D}(v) + \frac{b(u-v+2\eta)b(\eta)}{b(u-v)b(u+v)}\mathcal{B}(u)\mathcal{A}(v)
\end{aligned}$$

The relations are simpler if we change variables to use $\tilde{\mathcal{D}}(u) = \mathcal{D}(u)b(2u) - \mathcal{A}(u)b(\eta)$ instead of $\mathcal{D}$

$$\begin{aligned}
\tilde{\mathcal{D}}(u)\mathcal{B}(v) &= \frac{b(u-v+\eta)b(u+v+\eta)}{b(u-v)b(u+v)}\mathcal{B}(v)\tilde{\mathcal{D}}(u) + \frac{b(\eta)b(2u+\eta)b(2v-\eta)}{b(u+v)b(2v)}\mathcal{B}(u)\mathcal{A}(v) \\
&- \frac{b(\eta)b(2u+\eta)}{b(u-v)b(2v)}\mathcal{B}(u)\tilde{\mathcal{D}}(v)
\end{aligned}$$

The transfer matrix associated with the above monodromy matrix illustrated in Figure 6.1 is given by

$$\begin{aligned}
t(u, \xi_+, \xi_-) = tr(K(u + \frac{\eta}{2}, \xi_+)U_-(u)) &= b(u + \xi_+ + \frac{\eta}{2})\mathcal{A}(u) - b(u - \xi_+ + \frac{\eta}{2})\mathcal{D}(u) \\
&= \frac{b(2u+\eta)}{b(2u)}b(u + \xi_+ - \frac{\eta}{2})\mathcal{A}(u) - \frac{1}{b(2u)}b(u - \xi_+ + \frac{\eta}{2})\tilde{\mathcal{D}}(u)
\end{aligned}$$

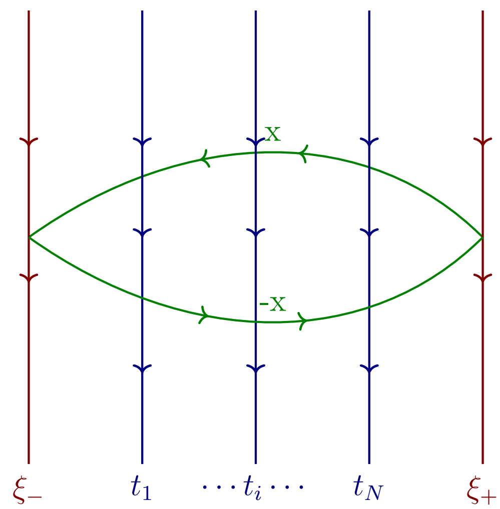

Figure 6.1: The blue lines are decorated with t_i, The two red lines with $\xi_\pm$. Each crossing in this diagram represents a factor in the transfer matrix.

On an off shell Bethe vector built up from the psuedovacuum $\Omega = e_-^{\otimes N}$ as $\mid v_1 \cdots v_m\rangle = \mathcal{B}(v_1)\mathcal{B}(v_2)\cdots\mathcal{B}(v_m)\Omega$ the result will be of the form:

$$\begin{aligned} t(u,\xi_+,\xi_-)\mathcal{B}(v_1)\mathcal{B}(v_2)\cdots\mathcal{B}(v_m)e_-^{\otimes N} &= \Lambda(u)\mathcal{B}(v_1)\mathcal{B}(v_2)\cdots\mathcal{B}(v_m)\Omega \\ &+ \sum_{j=1}^{m}\Lambda_j \mid u, v_1\cdots\hat{v}_j\cdots v_m\rangle \end{aligned} \tag{1}$$

$$\begin{aligned} \Lambda(u) = \frac{b(2u+\eta)}{b(2u)} b(u+\xi_+ - \frac{\eta}{2})\Delta_+(u)\prod_{j=1}^{m}\frac{b(u-v_j-\eta)b(u+v_j-\eta)}{b(u-v_j)b(u+v_j)} \\ -\frac{1}{b(2u)} b(u-\xi_+ + \frac{\eta}{2})\Delta_-(u)\prod_{j=1}^{m}\frac{b(u-v_j+\eta)b(u+v_j+\eta)}{b(u-v_j)b(u+v_j)} \\ \Lambda_j = res_{u=v_j}\Lambda(u) \end{aligned}$$

where $\Delta_\pm$ are the eigenvalues for $\mathcal{A}$ and $\tilde{\mathcal{D}}$ on the highest weight state respectively.

$$\begin{aligned} \Delta_+(u) &= b(u+\xi_-\frac{\eta}{2})\alpha(u)\delta(-u) \\ \Delta_-(u) &= -b(2u-\eta)b(u-\xi_+\frac{\eta}{2})\alpha(-u)\delta(u) \end{aligned}$$

For the vector $\mid v_1\cdots v_m\rangle$ to actually be an eigenvector of $t(u,\xi_+,\xi_-)$ (on shell) we must have that all the Λ_j be zero. This happens when the v_i satisfy the Bethe equations which can be realized by ensuring that the poles in $\Lambda(u)$ all cancel out.

From Equation 1, it is clear that for the vector constructed above to be an eigenstate, the v_i need to satisfy:

$$\frac{b(v_m+\xi_+-\frac{\eta}{2})b(v_m+\xi_--\frac{\eta}{2})}{b(v_m-\xi_++\frac{\eta}{2})b(v_m-\xi_-+\frac{\eta}{2})}\prod_{i=1}^{N}\frac{b(v_m-t_i+\eta)b(-v_m-t_i-\eta)}{b(v_m-t_i-\eta)b(-v_m-t_i+\eta)} = \prod_{k\neq m}\frac{b(v_m-v_k+\eta)b(v_m+v_k+\eta)}{b(v_m-v_k-\eta)b(v_m+v_k-\eta)}$$

Indeed, looking at the apparent poles when $u = v_m$ and ignoring the $-v_m$ gives the Bethe equations without extra redundancy.

$$-\frac{b(v_m+\xi_+-\frac{\eta}{2})}{b(v_m-\xi_++\frac{\eta}{2})}\frac{b(2v_m-\eta)\Delta_+(v_m)}{\Delta_-(v_m)} = \prod_{k\neq m}\frac{b(v_m-v_k+\eta)b(v_m+v_k+\eta)}{b(v_m-v_k-\eta)b(v_m+v_k-\eta)}$$

$$\begin{aligned} &- \frac{b(v_m+\xi_+-\frac{\eta}{2})}{b(v_m-\xi_++\frac{\eta}{2})}b(2v_m-\eta)\frac{b(v_m+\xi_--\frac{\eta}{2})\alpha(v_m)\delta(-v_m)}{-b(2v_m-\eta)b(v_m-\xi_-+\frac{\eta}{2})\alpha(-v_m)\delta(v_m)} \\ &= \prod_{k\neq m}\frac{b(v_m-v_k+\eta)b(v_m+v_k+\eta)}{b(v_m-v_k-\eta)b(v_m+v_k-\eta)}\end{aligned}$$

The associated eigenvalues of the transfer matrix are as in [84]:

$$\begin{aligned}\Lambda(u) &= \frac{b(2u+\eta)}{b(2u)}b(u+\xi_+-\frac{\eta}{2})b(u+\xi_-\frac{\eta}{2})\alpha(u)\delta(-u)\prod_{j=1}^{m}\frac{b(u-v_j-\eta)b(u+v_j-\eta)}{b(u-v_j)b(u+v_j)} \\ &+ \frac{1}{b(2u)}b(u-\xi_++\frac{\eta}{2})b(2u-\eta)b(u-\xi_+\frac{\eta}{2})\alpha(-u)\delta(u)\prod_{j=1}^{m}\frac{b(u-v_j+\eta)b(u+v_j+\eta)}{b(u-v_j)b(u+v_j)}\end{aligned}$$

6.3 Asymptotic Structure of Solutions to Bethe Equations

We first seek to count the number of solutions to the Bethe equations deep in this chamber $0 < Re(t_1) << Re(t_2) << Re(t_3) \cdots Re(t_N)$. This is to show that there 2^N solutions for the collections of $\{v_i\}$ as desired. In the next section, we consider the associated vectors.

6.3.1 Proposition *Let $Re(t_N) \to +\infty$.*

- *If the v_i remain finite in this limit, the $\{v_i\}$ satisfy the Bethe equations for a chain of length $N-1$*

- *If one of them v_M diverges as $t_N + O(1)$, it has asymptotic behavior of the form $e^{v_M} \to w_M e^{t_N} + o(e^{t_N})$ for fixed w_M given below and the remaining rapidities satisfy the Bethe equations for the N-1 length chain.*

$$w_M^2 = q^2 \frac{q^{4M} - e^{2\xi_+ + \xi_-} q^{2N}}{q^{4M} - e^{2\xi_+ + \xi_-} q^{2N+4}}$$

- *There may also be multiple divergences. In this case there is a decoupling between solving the system for the N-1 length chain and a system for the w_k of the divergences.*

Proof .

First we take a limit as $Re(t_N)$ goes to $+\infty$ but the v_i are finite. Then the only factor in the Bethe equations which involves t_N is

$$\frac{b(v_m - t_N + \eta) b(-v_m - t_N - \eta)}{b(v_m - t_N - \eta) b(-v_m - t_N + \eta)}$$

which tends to 1 so we see the corresponding Bethe equations for a chain of length $N-1$ and still M rapidities.

For the second part, assume that v_M goes to $+\infty$ as well and we define w_M so that $e^{v_M - t_N} = w_M$. If we can solve for w_M then we will see how to get rid of one inhomogeneity and one rapidity in the Bethe equations.

The equations for $m \neq M$ see the t_N factor go to 1 on the left hand side just as before and the $k = M$ factor go to 1 on the right hand side so we see the $M-1$ equations for a chain of length $N-1$ [1]. The $m = M$ equation then determines w_M:

$$\begin{aligned} e^{2\xi_+} q^{-1} e^{2\xi_-} q^{-1} \frac{w_M q - w_M^{-1} q^{-1}}{w_M q^{-1} - w_M^{-1} q} q^{2N} &= q^{4(M-1)} \\ w_M^2 &= q^2 \frac{q^{4M} - e^{2\xi_+ + \xi_-} q^{2N}}{q^{4M} - e^{2\xi_+ + \xi_-} q^{2N+4}} \end{aligned}$$

which is nonsingular and nonzero provided that

$$e^{2\xi_+ + 2\xi_-} q^{-1} - q^{4M-2N-5} = 0$$

so we assume that the parameters q and $\xi_\pm$ are chosen away from this bad locus. This solution is unique up to sign and fixes the behavior of e^{v_M} to be $w_M e^{t_N} + o(e^{t_N})$.

[1] This is unlike the quasiperiodic case when the left hand side gives a factor of q^2 causing the horizantal magnetic field to be modified. [3]

If we only had these possibilities we sould have acheived 2^N solutions asymptotically in this sector. However we do have other solutions when multiple v_i may also diverge with t_N. For definiteness, say they are the last J+1 in the list and call this index set $\mathbf{D}$. As before, parameterize these divergences in the same way: $e^{v_j} = w_j e^{t_N} + o(e^{t_N})$. In this situation, the Bethe equations for any of the nondivergeing rapidities give the Bethe equations of a chain of length $N-1$ with rapidities v_1 through v_{M-J-1} and no dependence on the w_j.

Considering the Bethe equations for any of the diverging rapidities gives

$$\begin{aligned}
e^{2\xi_+}q^{-1}e^{2\xi_-}q^{-1}q^{2N}\frac{w_k q - w_k^{-1}q^{-1}}{w_k q^{-1} - w_k^{-1}q} &= q^{4(M-J-1)}q^{2J}\prod_{j\in\mathbf{D}\ j\neq k}\frac{b(v_k - t_N + t_N - v_j + \eta)}{b(v_k - t_N + t_N - v_j - \eta)} \\
e^{2\xi_+}q^{-1}e^{2\xi_-}q^{-1}q^{2N}\frac{w_k q - w_k^{-1}q^{-1}}{w_k q^{-1} - w_k^{-1}q} &= q^{4(M-J-1)}q^{2J}\prod_{j\in\mathbf{D}\ j\neq k}\frac{w_k w_j^{-1}q - w_k^{-1}w_j q^{-1}}{w_k w_j^{-1}q^{-1} - w_k^{-1}w_j q}
\end{aligned}$$

This gives J quadratic equations used to solve for w_j $\forall j \in \mathbf{D}$. This system does have solutions, but we will see that the associated Bethe vectors are not independent asymptotically. These solutions would contribute to asymptotic completeness in the tensor product of a spin chain with tensorands being Verma modules.

□

6.4 Asymptotics of Bethe Vectors

Because the multiple divergences were not excluded from the system of $J+1$ equations above, we must show that the associated vectors are not included. This is done by isolating the dependence of t_N in the Bethe vectors. This then allows us to show that in the case of multiple divergences, these vectors vanish.

Isolating the contributions from t_N

First let us change parameterizations to use the following formula of [85].

$$\begin{aligned}
\bar{\mathcal{B}}^{\xi(M)}(\vec{x},\vec{t})\Omega &= \sum_{\epsilon=\pm^M}\sum_{\mathbf{J}\subset\{1\cdots M\}}\mathcal{Y}^{\xi,\epsilon,\mathbf{J}}(\vec{x},\vec{t})\prod_{i\in\mathbf{J}^c}B_N(-\epsilon_i x_i - \frac{\eta}{2}, t_N)\prod_{j\in\mathbf{J}}\hat{B}(-\epsilon_j x_j - \frac{\eta}{2},\vec{t})\Omega \\
\mathcal{Y}^{\xi,\epsilon,\mathbf{J}}(\vec{x},\vec{t}) &= \prod_{i=1}^{M}\left(\epsilon_i b(\xi - \epsilon_i x_i - \frac{\eta}{2})\prod_{r=1}^{N}\frac{b(\epsilon_i x_i - t_r - \frac{\eta}{2})}{b(\epsilon_i x_i - t_r + \frac{\eta}{2})}\right) \\
&\times \prod_{1\le i<j\le M}\frac{b(\epsilon_i x_i + \epsilon_j x_j + \eta)}{b(\epsilon_i x_i + \epsilon_j x_j)}Y^{\mathbf{J}}((-\epsilon_i x_i - \frac{\eta}{2}),\vec{t}) \\
Y^{\mathbf{J}}(\vec{x},\vec{t}) &= \prod_{i\in\mathbf{J}}\frac{b(x_i - t_N)}{b(x_i - t_N + \eta)}\prod_{(i,j)\in\mathbf{J}\times\mathbf{J}^c}\frac{b(x_i - x_j + \eta)}{b(x_i - x_j)}
\end{aligned}$$

where B_N is the operator on just the Nth site and $\hat{B}$ is the matrix element of the double row monodromy matrix with the N'th site omitted removing t_N dependence.

With all the t_N dependences now isolated, we may compute asymptotics as $t_N \to +\infty$ for either $\vec{x}$ all remaining finite or some $x_j \to \infty$ in $e^{x_j - t_N} \to w_j$.

Vanishing for Multiple Divergences

6.4.1 Theorem *Zero or one v_i diverging with t_N are the only two linearly independent possibilities. Restricting to these implies the Bethe equations give an asymptotically complete set of solutions.*

Proof Multiple divergences being linearly dependent on the zero or one case is implied by lemmas 6.4.2 and 6.4.3. We divert those to the next subsection.

Proceed by induction. Suppose for the induction step that we have already given a chain of length $N-1$ there are $\binom{N-1}{M}$ solutions for the M magnon sector. As t_N goes to infinity, the first two items of proposition 6.3.1 lends $\binom{N}{M}$ solutions of the M magnon sector of the length N chain interpreted as coming from either M magnons on an N-1 chain or M-1 magnons on an N-1 chain by $\binom{N-1}{M}+\binom{N-1}{M-1}$. Adding up all the M sectors gives the desired 2^N dimensional Hilbert space.

□

The Dominant terms

In fact we may explicitly give the dominant terms for the three cases of zero, one or many diverging x_i. This is also necessary to show how even after rescaling leaves a linearly dependent vector in the case of multiple divergences.

All Finite

If every x_i remains finite, $\mathbf{J}$ needs to be $\{1 \cdots M\}$ because for any other $\mathbf{J}$ the vector has prefactor e^{-t_N}. This leaves a sum of 2^M terms all of which do not affect the Nth site with corrections suppressed as e^{-t_N}

$$\begin{aligned}
\mathcal{B}^{\xi(M)}(\vec{x},\vec{t})\Omega \quad &\approx \quad q^{-M}\prod_{i=1}^{M}\left(\prod_{r=1}^{N-1}\frac{b(x_i-t_r)}{b(x_i-t_r-\eta)}\right)\frac{b(2x_i)}{b(\xi-x_i-\eta)b(2x_i-\eta)} \\
\textstyle\sum_{\epsilon=\pm^M} \quad & q^{-2M}\mathcal{Y}^{\xi,\epsilon,\{1\cdots M\}}(\vec{x}-\frac{\eta}{2},\{t_1\cdots t_{N-1}\})\prod_{j=1}^{M}\hat{B}(-\epsilon_j x_j+\epsilon_j\frac{\eta}{2}-\frac{\eta}{2},\vec{t})\Omega \\
= \quad & q^{-3M}\mathcal{B}^{\xi(M)}(\vec{x},\{t_1\cdots t_{N-1}\})\Omega_{N-1}\otimes e_+
\end{aligned}$$

Single Diverging Rapidity

For the single divergence x_M, there are only contributions from the $J = \{1 \cdots M-1\}$ summands. The Mth sign is also fixed to be - in this case. This leaves a sum of 2^{M-1} terms all of which flip the Nth site.

$$\bar{\mathcal{B}}^{\xi(M)}(\vec{x},\vec{t})\Omega \approx \sum_{\epsilon=\pm^{M-1}-} \mathcal{Y}^{\xi,\epsilon,\{1\cdots M-1\}}(\vec{x},\vec{t}) B_N(x_M - \frac{\eta}{2}, t_N) \prod_{j=1}^{M-1} \hat{B}(-\epsilon_j x_j - \frac{\eta}{2}, \vec{t})\Omega$$

$$\mathcal{B}^{\xi(M)}(\vec{x}+\frac{\eta}{2},\vec{t})\Omega \approx \prod_{i=1}^{M}\left(\prod_{r=1}^{N}\frac{b(x_i - t_r + \frac{\eta}{2})}{b(x_i - t_r - \frac{\eta}{2})}\right)\frac{b(2x_i+\eta)}{b(\xi - x_i - \frac{\eta}{2})b(2x_i)}$$

$$\textstyle\sum_{\epsilon=\pm^{M-1}-} \mathcal{Y}^{\xi,\epsilon,\{1\cdots M-1\}}(\vec{x},\vec{t}) B_N(x_M - \frac{\eta}{2}, t_N) \displaystyle\prod_{j=1}^{M-1} \hat{B}(-\epsilon_j x_j - \frac{\eta}{2}, \vec{t})\Omega$$

Multiple Diverging Rapidities

If multiple rapidities diverge, the dominant terms follow a similar pattern as one divergence even though in this case the vector goes to $\vec{0}$ as $t_N \to +\infty$. Again for definiteness let us say $\mathbf{D} = \{J \cdots M\}$. We now have the option of which element of $\mathbf{D}$ to insert into the B_N factor.

$$\mathcal{B}^{\xi(M)}(\vec{x},\vec{t})\Omega \approx \prod_{i=1}^{M}\left(\prod_{r=1}^{N}\frac{b(x_i - t_r)}{b(x_i - t_r - \eta)}\right)\frac{b(2x_i)}{b(\xi - x_i)b(2x_i - \eta)}$$

$$\sum_{\mathbf{J}=\{1\cdots J\cdots\hat{k}\cdots M\}} \sum_{\substack{\epsilon=\pm^M \\ \epsilon_k=-1}} \mathcal{Y}^{\xi,\epsilon,\mathbf{J}}(\vec{x}-\frac{\eta}{2},\vec{t}) B_N(x_k - \eta, t_N) \prod_{j=1,j\neq k}^{M} \hat{B}(-\epsilon_j x_j + \epsilon_j\frac{\eta}{2} - \frac{\eta}{2}, \vec{t})\Omega$$

This vector goes to 0 as e^{-st_N} with $s = \mid \mathbf{D} \mid -1$. We see what happens if we rescale that and compare to the previous two cases of zero or one divergence. We begin with the case of $\mid D \mid = 2$.

6.4.2 Lemma (2 Diverging Rapidities) *If both $v_{1,2}$ diverge as $e^{v_i} = w_i e^{t_N}$, then $\mathcal{B}^{\xi(2)}(v_1, v_2, \vec{t})\Omega$, then there exist a set of $\{z_{2,i}\}$ such that $\sum_{z_{2,i}} \mathcal{B}^{\xi(2)}(v_1, z_{2,i}, \vec{t})\Omega \propto \mathcal{B}^{\xi(2)}(v_1, v_2, \vec{t})\Omega$ and the $\{z_{2,i}\}$ are all finite in the $t_N \to +\infty$ limit.*

Proof

$$\begin{aligned}\mathcal{B}^{\xi(2)}(v_1,v_2,\vec{t})\Omega \quad &\approx \quad q^{2(N-1)}\frac{w_1-w_1^{-1}}{w_1q^{-1}-w_1^{-1}q}\frac{w_2-w_2^{-1}}{w_2q^{-1}-w_2^{-1}q}q^2\frac{1}{w_1w_2e^{2t_N}e^{-2\xi}} \\ &\sum_{k=1,2}\sum_{\epsilon=\pm}\mathcal{Y}^{\xi,\epsilon,J}(\vec{x}-\frac{\eta}{2},\vec{t})B_N(x_k-\eta,t_N)\prod_{j=1,j\neq k}^{2}\hat{B}(-\epsilon x_j-\frac{\eta}{2}(1-\epsilon),\vec{t})\Omega\end{aligned}$$

Because only linear dependence matters in this section the first line can be ignored except for the e^{2t_N}.

$$\begin{aligned}\mathcal{B}^{\xi(2)}(v_1,v_2,\vec{t})\Omega \quad \propto \quad & \frac{-1}{e^{2t_N}}b(\xi+v_1-\eta)b(\xi-v_2)\frac{b(-v_1-t_N)}{b(-v_1+\eta-t_N)} \\ & \frac{b(-v_2+v_1-\eta)}{b(-v_2+v_1)}\frac{b(v_2-\eta-t_N)}{b(v_2-t_N)}B_N(v_1-\eta,t_N)\hat{B}(-v_2,\vec{t})\Omega \\ + \quad & \frac{1}{e^{2t_N}}b(\xi+v_1-\eta)b(\xi+v_2-\eta)q^{2(N-1)}\frac{b(-v_1-t_N)}{b(-v_1+\eta-t_N)} \\ & \frac{b(-v_2+v_1-\eta)}{b(v_1-v_2)}\frac{b(v_2-\eta-t_N)}{b(v_2-t_N)}B_N(v_1-\eta,t_N)\hat{B}(v_2-\eta,\vec{t})\Omega \\ + \quad & \frac{-1}{e^{2t_N}}b(\xi+v_2-\eta)b(\xi-v_1)\frac{b(-v_2-t_N)}{b(-v_2+\eta-t_N)} \\ & \frac{b(-v_1+v_2-\eta)}{b(-v_1+v_2)}\frac{b(v_1-\eta-t_N)}{b(v_1-t_N)}B_N(v_2-\eta,t_N)\hat{B}(-v_1,\vec{t})\Omega \\ + \quad & \frac{1}{e^{2t_N}}b(\xi+v_2-\eta)b(\xi+v_1-\eta)q^{2(N-1)}\frac{b(-v_2-t_N)}{b(-v_2+\eta-t_N)} \\ & \frac{b(-v_1+v_2-\eta)}{b(v_2-v_1)}\frac{b(v_1-\eta-t_N)}{b(v_1-t_N)}B_N(v_2-\eta,t_N)\hat{B}(v_1-\eta,\vec{t})\Omega\end{aligned}$$

This simplifies upon defining an auxiliary R S_1 and S_2 to the following:

$$
\begin{aligned}
R &\equiv \frac{b(-v_1-t_N)}{b(-v_1+\eta-t_N)}\frac{b(-v_2+\eta-t_N)}{b(-v_2-t_N)}\frac{b(v_1-v_2-\eta)}{b(v_2-v_1-\eta)} \\
&\quad \frac{b(v_2-\eta-t_N)}{b(v_2-t_N)}\frac{b(v_1-t_N)}{b(v_1-\eta-t_N)}\frac{b(v_2-t_N)}{b(v_1-t_N)}e^{v_1-v_2} \\
&\approx \frac{b(-v_1-t_N)}{b(-v_1+\eta-t_N)}q^{-1}\frac{b(v_1-v_2-\eta)}{b(v_2-v_1-\eta)}\frac{b(v_2-\eta-t_N)}{b(v_1-\eta-t_N)}e^{v_1-v_2} \\
S_1 &\equiv \frac{-1}{e^{2t_N}}b(\xi+v_1-\eta)b(\xi-v_2)\frac{b(-v_1-t_N)}{b(-v_1+\eta-t_N)} \\
&\quad \frac{b(-v_2+v_1-\eta)}{b(-v_2+v_1)}\frac{b(v_2-\eta-t_N)}{b(v_2-t_N)}B_N(v_1-\eta,t_N)\hat{B}(-v_2,\vec{t})\Omega \\
S_2 &\equiv \frac{1}{e^{2t_N}}b(\xi+v_1-\eta)b(\xi+v_2-\eta)q^{2(N-1)}\frac{b(-v_1-t_N)}{b(-v_1+\eta-t_N)} \\
&\quad \frac{b(-v_2+v_1-\eta)}{b(v_1-v_2)}\frac{b(v_2-\eta-t_N)}{b(v_2-t_N)}B_N(v_1-\eta,t_N)\hat{B}(v_2-\eta,\vec{t})\Omega \\
\mathcal{B}^{\xi(2)}(v_1,v_2,\vec{t})\Omega &\propto (1+R)(S_1+S_2)
\end{aligned}
$$

This overall prefactor $(1+R)$ can be dropped leaving.

$$
\begin{aligned}
S_1+S_2 &= \frac{-1}{e^{2t_N}}\frac{b(-v_1-t_N)}{b(-v_1+\eta-t_N)}\frac{b(-v_2+v_1-\eta)}{b(-v_2+v_1)}\frac{b(v_2-\eta-t_N)}{b(v_2-t_N)} \\
&\quad B_N(v_1-\eta,t_N) \\
&\quad b(\xi+v_1-\eta)b(\xi-v_2)\hat{B}(-v_2,\vec{t})\Omega \\
&+ \frac{-1}{e^{2t_N}}\frac{b(-v_1-t_N)}{b(-v_1+\eta-t_N)}\frac{b(-v_2+v_1-\eta)}{b(v_1-v_2)}\frac{b(v_2-\eta-t_N)}{b(v_2-t_N)} \\
&\quad B_N(v_1-\eta,t_N) \\
&\quad -b(\xi+v_1-\eta)b(\xi+v_2-\eta)q^{2(N-1)}\hat{B}(v_2-\eta,\vec{t})\Omega \\
&\propto B_N(v_1-\eta,t_N)(\hat{B}(-v_2-N\eta,\vec{t})-\hat{B}(v_2-N\eta,\vec{t}))\Omega \\
(\hat{B}(-v_2-N\eta,\vec{t})-\hat{B}(v_2-N\eta,\vec{t}))\Omega &\approx -\sum_{r=0}^{N-1}q^{-r}\frac{q-q^{-1}}{1}(\frac{1}{e^{v_2-N\eta-t_{r+1}+\eta}}+\frac{1}{e^{v_2+N\eta+t_{r+1}-\eta}})e_+^r e_- e_+^{N-1-r} \\
&\approx -\frac{q-q^{-1}}{e^{v_2}}\sum_{r=0}^{N-1}q^{-r}(\frac{1}{e^{-N\eta-t_{r+1}+\eta}}+\frac{1}{e^{N\eta+t_{r+1}-\eta}})e_+^r e_- e_+^{N-1-r} \\
&= -\frac{q-q^{-1}}{e^{v_2}}\sum_{r=0}^{N-1}q^{-r}2\cosh(N\eta+t_{r+1}-\eta)e_+^r e_- e_+^{N-1-r}
\end{aligned}
$$

Compare this with the vectors we already have in the single divergence sector where b and c are of the form $t_N+O(1)$ with constant term to be determined.

$$\mathcal{B}(z_2, v_1, \vec{t})\Omega = B_N(v_1 - \eta, t_N) \sum_{r=0}^{N-1} \Bigg(\mathcal{Y}^{\xi,(-1,-1),\{1\}}((z_2, v_1) - \frac{\eta}{2}, \vec{t}) \prod_{i=1}^{r} \big(\frac{b(z_2 - 2\eta - t_i)}{b(z_2 - \eta - t_i)}\big) \frac{b(\eta)}{b(z_2 - 2\eta - t_{r+1})}$$
$$+ \mathcal{Y}^{\xi,(-1,+1),\{1\}}((z_2, v_1) - \frac{\eta}{2}, \vec{t}) \prod_{i=1}^{r} \big(\frac{b(-z_2 - \eta - t_i)}{b(-z_2 - t_i)}\big) \frac{b(\eta)}{b(-z_2 - \eta - t_{r+1})} \Bigg) e_+^r e_- e_+^{N-1-r}$$

We seek to show that

$$\mathcal{B}^{\xi(2)}(v_1, v_2, \vec{t})\Omega = \sum_{\alpha} \mathcal{B}^{\xi(2)}(z_{2,\alpha}, v_1, \vec{t})\Omega$$

for some set of regular $z_{2,\alpha}$. Matching coefficients gives the system of equations for all r:

$$e^{-t_N} \sum_{\alpha} \Bigg(\mathcal{Y}^{\xi,(-1,-1),\{1\}}((z_{2,\alpha}, v_1) - \frac{\eta}{2}, \vec{t}) \quad \prod_{i=1}^{r} \quad \big(\frac{b(z_{2,\alpha} - 2\eta - t_i)}{b(z_{2,\alpha} - \eta - t_i)}\big) \frac{b(\eta)}{b(z_{2,\alpha} - 2\eta - t_{r+1})}$$
$$+ \mathcal{Y}^{\xi,(-1,+1),\{1\}}((z_{2,\alpha}, v_1) - \frac{\eta}{2}, \vec{t}) \quad \prod_{i=1}^{r} \quad \big(\frac{b(-z_{2,\alpha} - \eta - t_i)}{b(-z_2 - t_i)}\big) \frac{b(\eta)}{b(-z_{2,\alpha} - \eta - t_{r+1})} \Bigg)$$
$$= \quad -\frac{q - q^{-1}}{e^{v_2}} q^{-r} 2\cosh(N\eta + t_{r+1} - \eta)$$

Solving this equation for the set of $z_{2,\alpha}$ then shows the desired linear dependence.

□

Now proceeding by induction for the rest when $\mid \mathbf{D} \mid > 2$ results in:

6.4.3 Lemma ($\geq$ 2 Diverging Rapidities) *Any state $e^{st_N}\mathcal{B}(\cdots, \vec{t})\Omega$ where $s \geq 0$ is one less than the number of divergences (This ensures that this vector has finite nonzero norm in the $t_N \to +\infty$ limit.) can be approximated by a linear combination of states of the form 6.4*

Proof By lemma 6.4.2 we have a base case.

$$ae^{2t_N}\mathcal{B}(x_1 + t_N, x_2 + t_N, \vec{t})\Omega = \sum b_i \mathcal{B}(y_{1i} + t_N, y_{2i}, \vec{t})\Omega$$

Therefore when including other rapidities, they come along for the ride as:

$$\mathcal{B}(x_1 + t_N, x_2 + t_N, \cdots, \vec{t})\Omega = \mathcal{B}(\cdots, \vec{t}) a^{-1} e^{-2t_N} \sum b_i \mathcal{B}(y_{1i} + t_N, y_{2i}, \vec{t})\Omega$$
$$e^{2t_N}\mathcal{B}(x_1 + t_N, x_2 + t_N, \cdots, \vec{t})\Omega = \sum a^{-1} b_i \mathcal{B}(\cdots, y_{1i} + t_N, y_{2i}, \vec{t})\Omega$$

Now take each $\mathcal{B}(\cdots, y_{1i} + t_N, y_{2i}, \vec{t})\Omega$ in the RHS and repeat the procedure if there are still $J \geq 2$ diverging rapidities. The overall factor for rescaling is e^{2t_N} for each extra divergence. The base case requires two divergences so we can reduce to the single divergence case as described in 6.4 and no further.

□

6.5 Conclusion

We have shown that as each inhomogeneity is taken to infinity the solutions to the Bethe equations break up as all remaining finite or one going to infinity in a prescribed manner. The lack of other possibilities gives the desired completeness property. This was done by looking at the asymptotics of the Bethe equations to produce the different solution sets followed by a check of linear dependence in the finite dimensional quotient. So what appear to be extra on shell vectors are actually only independent in the Verma, but not in the quotient.

There are more general solutions of the reflection equation which are not diagonal. These requires use of the Dynamical Yang Baxter Equation after the “gauge transformation.”

Chapter 7

Cluster Algebras

7.1 Introduction

The algebraic Bethe ansatz is built from decorating the sites with representations of $U_q\hat{\mathfrak{g}}$. Boundaries for such systems are determined by giving a representation of a coideal subalgebra [86]. That is a certain kind of module category. We may also talk about building bimodule categories in order to construct two-sided defects in the chain.

We would like to ask about invariants that help characterize (invertible) (bi-) module categories for these. However, that is complicated due to the nature of meromorphic tensor categories [87] as well as potential failures in the monoidal nature of K_0. So we perform 2 simplifications. First we look at the subcategory defined in [17] as cluster categories.

The next simplification is look at the "reduced" theory. That is where there used to be the monoidal category $\mathcal{C}_l$ assigned to all the bulk regions, we instead assign $\mathcal{A} = K_0(\mathcal{C}_l) \otimes \mathbb{Q}$. In this reduced theory, the codimension 1 walls are decorated by $\mathcal{A} - \mathcal{A}$ bimodules and codimension 2 phenomena have bimodule intertwiners [11]. Physical intuition says that there should be some procedure to go from a bimodule category which is being used as a defect wall in the original theory to a bimodule which is used as a defect in analogy with a 2-dimensional topological theory.[1] The naive guess of K_0 as the reduction fails unless the functor defining the module structures are bi-exact. That is an issue that has been sidestepped here by only asking about classifying (invertible) (bi)modules for these cluster algebras.

The use of the cluster category provides us with the tool of cluster algebras $\mathcal{A}$. These are significantly simpler algebras so the questions of their (bi)modules is significantly easier. That means we can focus on the problems of $Pic(\mathcal{A})/Aut(\mathcal{A})$ and $\mathcal{A} - (bi)mod$ needed for the invertible and not necessarily invertible cases respectively.

7.2 Cluster Categories and Algebras

General Cluster Algebras and Categories

7.2.1 Definition (Cluster Algebra [88]) *A cluster algebra is a commutative ring presented from a seed $(x_1 \cdots x_n)$ and a mutation rule*

$$x'_k x_k = \prod_{\substack{\alpha \in Q_1 \\ s(\alpha)=k}} x_{t(\alpha)} + \prod_{\substack{\alpha \in Q_1 \\ t(\alpha)=k}} s_{t(\alpha)}$$

to give a new cluster as $(x_1 \cdots x'_k \cdots x_n)$. Here Q is a quiver which changes upon mutation as well. This is illustrated in an example we'll need by fig. 7.2.

On occasion we will also need to make use of the affine scheme given by $Spec$ of this algebra when we phrase constructions with geometry rather than algebra.

[1] This is a mere analogy. It can not be pushed due to the failure of separable symmetric Frobenius algebra.

7.2.2 Definition (Monoidal categorification) *For a cluster algebra $\mathcal{A}$, a monoidal categorification thereof is defined to be an abelian monoidal category such that it's Grothendieck ring is isomorphic to $\mathcal{A}$ and the cluster monomials are classes of real simple objects. Cluster variables are classes of real prime simple objects. Here real simple object means that $S \otimes S$ is simple as well. Prime indicates the lack of a factorization $S \simeq S_1 \otimes S_2$.*

Cluster Categories $\mathcal{C}_{\mathfrak{g},l}$

Let us work with a $U_q\hat{\mathfrak{g}}$ for $\mathfrak{g}$ of ADE type of rank n in the Drinfeld realization. Physically this means that we are in the Jordan-Wigner perspective rather than spins. This effects important properties because the different co-product means different entanglement structure. We are interested in representations of this as a category, but it is too large. That is the purpose of the following definition.

7.2.3 Definition ($\mathcal{C}_l$) *Color the vertices of the Dynkin diagram with $\xi_i = 0/1$. Now take the full subcategory of $Rep^{fd}U_q\hat{\mathfrak{g}}$ consisting of objects such that any simple composition factor and index $i \in I$, the roots of that Drinfeld polynomial are in the set $\{q^{-2k+\xi_i}\}$ for $k \in \mathbb{Z}$. Call this $\mathcal{C}_{\mathbb{Z}}$. Define $\mathcal{C}_l$ by only allowing $k \in [0,l]$ instead.*

7.2.4 Lemma ([17]) *$K_0(\mathcal{C}_{\mathbb{Z}})$ is generated by $[V_{i,q^{2k+\xi_i}}]$ and similarly for $K_0(\mathcal{C}_l)$ but in the $\mathcal{C}_l$ case it is a polynomial ring in those $n(l+1)$ variables.*

Cluster Algebra $\mathcal{A}_{\mathfrak{g},l}$

In order to show that $\mathcal{C}_{\mathfrak{g},l}$ is a monoidal categorification, one must say what cluster algebra it is categorifying. That is given in via a quiver.

7.2.5 Definition ($Q_{\mathfrak{g},l}$) *Define a quiver for the Dynkin diagram for $\mathfrak{g}$ and l by the following procedure.*

Orient the diagram via a bipartitioning where all the vertices with $\xi_i = 0$ are colored black and those with $\xi_i = 1$ are colored white. Then orient the edges as going from black to white. Call this quiver $Q_{\mathfrak{g},0}$.

Now form a new quiver with vertex set (i,k) for $i \in I$ and $1 \le k \le l+1$ with three types of arrows.

- *$(i,k) \to (j,k)$ whenever $i \to j$ is an arrow in $Q_{\mathfrak{g},0}$ and all k*
- *$(j,k) \to (i,k+1)$ for every arrow $i \to j$ in $Q_{\mathfrak{g},0}$ and all $1 \le k \le l$*
- *$(i,k+1) \to (i,k)$ for all $i \in I$ and all $1 \le k \le l$*

7.2.6 Definition ($\mathcal{A}_{\mathfrak{g},l}$) *Define the quiver cluster algebra with $x_{i,l+1}$ all frozen variables. This defines a cluster algebra $\mathcal{A}_{\mathfrak{g},l} \subset \mathbb{Q}(x_{i,m})$, inside a field extension of $n(l+1)$ transcendental variables.*

7.2.7 Conjecture (Leclerc) *There is a ring isomorphism* $K_0(\mathcal{C}_l) \otimes \mathbb{Q} \simeq \mathcal{A}_{\mathfrak{g},l}$

7.2.8 Theorem *The conjecture is true if* $\mathfrak{g} = A_1$ *and arbitrary* l*. In this case the quiver* $Q_{A_1,l}$ *is an* A_{l+1} *with one frozen vertex at the end.*
The conjecture is also true for $l = 1$ *in this case it is 2 copies connected up nontrivially.*

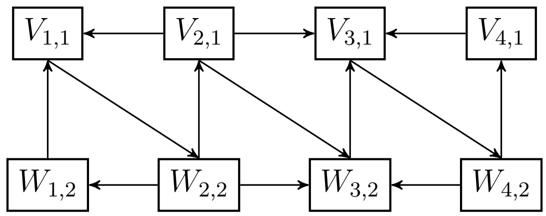

Figure 7.1: $\mathcal{A}_{A_4,1}$ with W nodes being frozen.

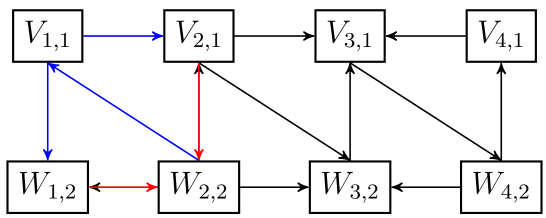

Figure 7.2: Mutating at the node $V_{1,1}$. Red-inserted, Blue-flipped, need to remove 2-cycles.

Automorphisms

7.2.9 Lemma (2.21 of [89]) *For a finite type cluster algebra* $\mathcal{A}$ *and* $\mathcal{A}^{ex}$ *the principal (unfrozen) part. Assuming* $\mathcal{A}$ *is gluing free, then the specialization map allows cluster automorphisms of* $\mathcal{A}$ *to give cluster automorphisms of* $\mathcal{A}^{ex}$*. That is* $Aut(\mathcal{A}) \subset Aut(\mathcal{A}^{ex})$*.*

The cases of $Q_{\mathfrak{g},1}$ and $Q_{A_1,l}$ will have the property that their mutable parts are orientations of a simply laced Dynkin diagram so this lemma will apply for them.

7.2.10 Lemma *There is a homomorphism* $Pic\mathcal{A} \to Pic\mathcal{A}^{ex}$

Proof Specialization gives an algebra map $\mathcal{A} \to \mathcal{A}^{ex}$ □

7.2.11 Theorem (Automorphisms of finite type[89]) *The cluster automorphisms of the principal parts for the* $\mathcal{A}_{\mathfrak{g},1}$ *are:*

This is much more specific then all automorphisms as rings but we still get a subgroup of all automorphisms. These types of automorphisms must take clusters to clusters and commute with mutations. By the lemma lemma 7.2.9, we must look at a subgroup to account for the frozen variables.

$\mathfrak{g}$	Aut
A_1	$\mathbb{Z}_2$
$A_{n>1}$	D_{n+3}
D_4	$D_4 \times S_3$
$D_{n\geq 5}$	$D_n \times \mathbb{Z}_2$
E_6	D_{14}
E_7	D_{10}
E_8	D_{16}

7.3 Bimodule Categories

When looking at the reflection equation we produce coideal subalgebras whose representations produce module categories. Similarly for two-sided defects we desire to produce comodule algebras for two copies using a folding trick. In summary we seek to produce bimodule categories.

7.3.1 Theorem ([11]) *In the context of 3-dimensional topological field theories, this happens without the affinization. That is bimodule categories produce codimension 1 strata in the cobordism hypothesis with singularities. They may or may not be invertible.*

7.3.2 Lemma *If $\mathcal{C}$ is the original category and $\mathcal{D}$ is the bimodule category. Let us also assume the $\mathcal{C} \times \mathcal{D} \to \mathcal{D}$ and vice versa are bi-exact. Then $K_0(\mathcal{D})$ is both a left and right $K_0(\mathcal{C})$ bimodule. We may extend to $\mathbb{Q}$ and potentially produce a bimodule over $K_0(\mathcal{C}) \otimes \mathbb{Q}$ but now $\mathbb{Q}$ linear.*

If $\mathcal{C}_1 \subset \mathcal{C}_2$ is a full subcategory and $\mathcal{D}$ is a bimodule category (satisfying the exactness assumptions) for $\mathcal{C}_2$ we may restrict $K_0(\mathcal{D}) \otimes \mathbb{Q}$ as a $K_0(\mathcal{C}_2) \otimes \mathbb{Q} - K_0(\mathcal{C}_2) \otimes \mathbb{Q}$ bimodule to $K_0(\mathcal{C}_1) \otimes \mathbb{Q} - K_0(\mathcal{C}_1) \otimes \mathbb{Q}$.

Here we let $\mathcal{C}_1$ be the cluster category and $\mathcal{C}_2$ be without constraints on the Drinfeld polynomials.

A major caution here is that we have not described any operation of fusing bimodule categories. Therefore any statements about the fusing of bimodules over cluster algebras (including the criterion of invertibility) does not necessarily lift to the category level. That is we are not talking about bimodule categories that describe invertible defects, but instead simply those that go to invertible objects in the simplification. This also means that the group operation on Pic/K_0 is only useful in the reduction.

7.4 Picard Groups and K_0

We may define $Pic_k(A)$ as invertible bimodules for a commutative ring A where we only demand that k be able to commute through the bimodule. This contrasts with the module definition ordinarily used. However, we may reduce the computation to two steps.

7.4.1 Theorem (Pic as bimodule [73]) *$Pic_k(A) \simeq Aut_k(A) \ltimes Pic_A(A)$ where $Pic_A(A)$ is equivalent to the usual Pic_{com} with invertible modules rather than bimodules. $Aut_k(A)$ is the automorphisms as a k algebra.*

Useful Pic_{com} Groups

Because we are considering cluster algebras, the $\mathbb{Q}$ algebras will be very special. They will be very similar to simple polynomial and Laurent polynomial rings.

7.4.2 Theorem (1.6 of [90])

$$Pic_{com}(\mathbb{Q}[t_1^{\pm} \cdots t_m^{\pm}, x_1 \cdots x_n]) = 0$$

Proof In fact true for any zero dimensional commutative ring so field is overkill. $\mathbb{Z}$ would not work because it has Krull dimension 1, but finite rings $\mathbb{Z}_n$ would.

More generally we get

7.4.3 Theorem ([90])

$$Pic_{com}(A[t_1^{\pm} \cdots t_m^{\pm}]) \simeq Pic_{com}(A) \bigoplus \coprod_{i=1}^{m} LPic_{com}(A) \bigoplus \coprod_{k=1}^{m} \coprod_{i=1}^{\binom{2^k m}{k}} N^k Pic_{com}(A)$$

$LPic_{com}(A)$ for an anodal 1 dimensional domain is trivial. Anodal means that if $b \in \bar{A}$ the integral closure and $b^3 - b^2$ and $b^2 - b$ are in A, then $b \in A$ as well.

$NPic_{com}(A) \equiv Pic_{com}(A[t])/Pic_{com}(A)$. For example, $NPic_{com}(A) = 0$ if and only if A_{red} is seminormal. Normal rings like $U[x_1 \cdots x_n]$ for a UFD U are a fortiori seminormal.

7.4.4 Theorem (2.2 of [91]) *$Pic_{com}R \simeq Pic_{com}R[x_1 \cdots x_n]$ if and only if R is seminormal. For example,*

$$Pic_{com}\mathbb{Z}[x_1 \cdots x_n] \quad \simeq \quad 0$$

Automorphisms

Here we gather useful automorphism groups for the commutative algebras that may show up as the cluster algebras. Easy ones include $Aut\mathbb{Z}[x] = \mathbb{Z}_2 \ltimes \mathbb{Z}$. However we had at the very simplest a polynomial ring in $n(l+1)$ variables. Always ≥ 2 variables. That means a complicated automorphism group. It is even complicated for $\mathcal{C}_{A_1,1}$ as described below.

7.4.5 Definition (Jonqui'ere group) *$J_n(R)$ is defined as those automorphisms of $R[x_1 \cdots x_n]$ of the form*

$$\begin{aligned} x_1 &\rightarrow F_1(x_1) = \alpha_1 x_1 + \beta \in R[x_1] \\ x_2 &\rightarrow F_2(x_1, x_2) = \alpha_2 x_2 + f(x_1) \in R[x_1, x_2] \\ x_i &\rightarrow F_i(x_1 \cdots x_i) = \alpha_i x_i + f_i(x_1 \cdots x_{i-1}) \in R[x_1 \cdots x_i] \end{aligned}$$

7.4.6 Definition *$Af_n(R)$ are those transformations of the form*

$$\begin{array}{rcl} \vec{x} & \to & A\vec{x}+\vec{b} \\ A & \in & GL_n(R) \end{array}$$

7.4.7 Theorem (Jung, van der Kulk [92]) *The group of polynomial automorphisms of $k[x,y]$ denoted $GA_2$is generated as*

$$GA_2(k) \simeq Af_2(k) \star_{Bf_2(k)} J_2(k)$$

7.4.8 Lemma ([93]) *For an integral domain D, the tame subgroup of $AutD[x,y]$ is the subgroup generated as an amalgamated product Af_2 and J_2. However this is a proper subgroup, because there exists non-tame Nagata automorphisms like the following for all $a \neq 0$ non-unit:*

$$\begin{array}{rcl} X & \to & X+a(aY-X^2) \\ Y & \to & Y+2X(aY-X^2)+a(aY-X^2)^2 \end{array}$$

As such we get more complications on $\mathbb{Q}[x_1 \cdots x_n]$ by letting $D_2 = \mathbb{Q}[x_1 \cdots x_{n-2}]$. Or on $\mathbb{Z}[x_1 \cdots x_n]$ by letting $D_2 = \mathbb{Z}[x_1 \cdots x_{n-2}]$

There are many interesting subgroups inside $GA_n(\mathbb{Q})$. In particular inside the $GL_n(\mathbb{Q})$ subgroup, we have:

7.4.9 Theorem ([94]) *Finite subgroups of $GL_n\mathbb{Q}$ with maximal order are characterized. Except for $2,4,6,7,8,9,10$ they are the orthogonal groups which have $2^n n!$.*

- *$n=2$ has a $W(G_2)$ with order $12>8$.*
- *$n=4$ has a $W(F_4)$ with order $1152>384$*
- *$n=6$ has a $W(E_6)\times\mathbb{Z}_2$ with order $103680>46080$*
- *$n=7$ has a $W(E_7)$ with order $2,903,040>645,120$*
- *$n=8$ has a $W(E_8)$ with order $696,729,600>10,321,920$*
- *$n=9$ has a $W(E_8)\times W(A_1)$ with order $1,393,459,200>185,794,560$*
- *$n=10$ has a $W(E_8)\times W(G_2)$ with order $8,360,755,200>3,715,891,200$*

When one further demands matrices over the natural numbers with natural number inverse, there are simply the permutation matrices.

Non-invertible (bi)modules

To characterize not necessarily invertible (bi)modules over cluster algebras, we consider coherent sheaves on either $SpecR$ and $SpecR \times_{Spec\mathbb{Q}} SpecR$. However we are only looking at specific (bi)modules, not accounting for maps between (bi)modules. In the discretized 1+1 picture of the Bethe ansatz, (bi)module morphisms would happen at codimension 2 at points on the walls. That means we should ignore the morphisms in these categories and only look at isomorphism classes of objects or even K_0 (split or ordinary).

7.4.10 Example *Consider a polynomial ring* $\mathbb{Q}[x_1 \cdots x_n]$*. If we restrict our attention to finitely generated projectives, we get that they are all free by Quillen-Suslin [95]. This is similarly useful for bimodules by taking* $\mathbb{Q}[x_1 \cdots x_n, y_1 \cdots y_n]$*. That is* $K_0(\mathbb{Q}[x_1, \cdots x_n]) = \mathbb{Z}$

7.5 Pic and K_0 of these cluster algebras

Now let us put all the pieces together to caluclate the associated Pic and Pic_{com} for both $K_0\mathcal{C}_l \otimes \mathbb{Q}$ and it's exchangeable parts. We can get some information about the noninvertible parts via some invariants related to K_0.

7.5.1 Example (A_1) *This has 2 clusters x and w/x and the full cluster algebra is Laurent polynomials $\mathbb{Q}[x^{\pm}]$. This has a $\mathbb{Z}_2$ for automorphisms over $\mathbb{Q}$ and a trivial $Pic_A(A)$. This means we have a $\mathbb{Z}_2$ invariant. This is the cluster algebra that shows up as the principal part for $Q_{A_1,1}$. $\mathcal{A}$ itself is a polynomial ring in $n*(l+1) = 2$ generators. It's automorphisms and Picard group are covered by section 7.4 and theorem 7.4.2. That means our coarse invariant for invertible $\mathcal{C}_{A_1,1}$ bimodule categories is valued in $Aut_{\mathbb{Q}-alg}(\mathcal{A}) \ltimes Pic_{com}(\mathcal{A}) \simeq GA_2(\mathbb{Q})$*

$$\begin{tikzcd}
Pic_{com}(\mathcal{A}) = 0 \arrow[d] & Aut_{cl}(\mathcal{A}) \arrow[r, hook] \arrow[d, hook] & Aut_{\mathbb{Q}-alg}(\mathcal{A}) = GA_2(\mathbb{Q}) \\
Pic_{com}(\mathcal{A}^{ex}) = 0 & Aut_{cl}(\mathcal{A}^{ex}) = \mathbb{Z}_2 \arrow[r, "\simeq"] & Aut_{\mathbb{Q}-alg}(\mathcal{A}^{ex})
\end{tikzcd}$$

7.5.2 Example (A_2) *This example is more interesting . It has (x_1, x_2) and 4 other clusters. Altogether $\mathcal{A} \subset \mathbb{Q}[x_1^{\pm}, x_2^{\pm}] \subset \mathbb{Q}(x_1, x_2)$. It is $\mathbb{Q}[x_1, \frac{1+x_1}{x_2}, \frac{1+x_2}{x_1}] \simeq \mathbb{Q}[u, v, w]/(uvw - u - v - 1)$ for which $Spec\mathcal{A}$ which cuts a 2 dimensional hypersurface in affine 3 space. Closure in projective space as $uvw - uz^2 - vz^2 - z^3$ which is a cubic in P^3.*
As above it's cluster automorphisms contain D_5, but it's Picard group involves some more algebraic geometry of this cubic.
This quiver shows up as the principal part of $\mathcal{A}_{A_1,2}$ and $\mathcal{A}_{A_2,1}$. The $\mathcal{A}$ for these are polynomial rings in 3 and 4 generators respectively. Our coarse invariant for bimodule categories for these is valued in $Aut_{\mathbb{Q}-alg}(\mathcal{A}) \ltimes Pic_{com}(\mathcal{A})$ where

$$\begin{tikzcd} Pic_{com}(\mathcal{A}) = 0 \arrow[d] & & Aut_{cl}(\mathcal{A}) \arrow[r, hook] \arrow[d, hook] & Aut_{\mathbb{Q}-alg}(\mathcal{A}) = GA_{3/4}(\mathbb{Q}) \\ Pic_{com}(\frac{\mathbb{Q}[u,v,w]}{(uvw-u-v-1)}) & & Aut_{cl}(\mathcal{A}^{ex}) = D_5 \arrow[r, hook] & Aut_{\mathbb{Q}-alg}(\mathcal{A}^{ex}) \end{tikzcd}$$

The difference for both cases is indicated with the 3/4 and $Aut_{cl}(\mathcal{A})$ may differ between cases.

7.5.3 Example (A_3) .
The next A_3 has 14 clusters that altogether form $\mathbb{Q}[x_1, x_3, \frac{1+x_2}{x_1} = w, \frac{1+x_2+x_1x_3}{x_2x_3} = t]$. This can be written as $\mathbb{Q}[x_1, x_3, w, t]$ quotiented by a single relation $twx_1x_3 - tx_3 - wx_1 - x_1x_3 = 0$. Upon projectivization of the associated affine scheme, it becomes a quartic hypersurface in P^4. This gives the cluster algebra $\mathcal{A}^{ex}$.
.
It's cluster automorphisms contain D_6 and computing the Picard group requires more algebraic geometry.
This quiver shows up as the principal part of $\mathcal{A}_{A_1,3}$ and $\mathcal{A}_{A_3,1}$. With the frozen part, polynomial rings in 4 or 6 variables. Our coarse invariant for bimodule categories for these is valued in $Aut_{\mathbb{Q}-alg}(\mathcal{A}) \ltimes Pic_{com}(\mathcal{A})$ where

$$\begin{tikzcd} Pic_{com}(\mathcal{A}) = 0 \arrow[d] & & Aut_{cl}(\mathcal{A}) \arrow[r, hook] \arrow[d, hook] & Aut_{\mathbb{Q}-alg}(\mathcal{A}) = GA_{4/6}(\mathbb{Q}) \\ Pic_{com}(\frac{\mathbb{Q}[x_1,x_3,w,t]}{(twx_1x_3-tx_3-wx_1-x_1x_3)}) & & Aut_{cl}(\mathcal{A}^{ex}) = D_6 \arrow[r, hook] & Aut_{\mathbb{Q}-alg}(\mathcal{A}^{ex}) \end{tikzcd}$$

The differences are indicated as above.

Proof

$$\begin{aligned} wx_1 - 1 &= x_2 \\ tx_2x_3 &= t(w * x_1 - 1)x_3 = 1 + x_2 + x_1x_3 \\ &= wx_1 + x_1x_3 \\ t(wx_1 - 1)x_3 &= wx_1 + x_1x_3 \\ twx_1x_3 - tx_3 - wx_1 - x_1x_3 &= 0 \\ twx_1x_3 - tx_3z^2 - wx_1z^2 - x_1x_3z^2 &= 0 \end{aligned}$$

□

More detailed proofs for the above examples as cluster algebras can be found in [96].

7.5.4 Theorem *In general for $\mathcal{C}_{\mathfrak{g},l} \subset RepU_q\hat{\mathfrak{g}}$ in the cases where Leclerc's conjecture is proven, we have an invariant for invertible bimodules valued in $GA_{n(l+1)}$. There is no information about invertible modules contained in this procedure. This structure also fits in diagrams of the form which gets access to more manageable parts of $GA_{n(l+1)}$*

$$\begin{tikzcd} Pic_{com}(\mathcal{A}) = 0 \arrow[d] \\ Pic_{com}(\mathcal{X}_{f.t.}) \end{tikzcd} \qquad \begin{tikzcd} Aut_{cl}(\mathcal{A}) \arrow[r, hook] \arrow[d, hook] & Aut_{\mathbb{Q}-alg}(\mathcal{A}) = GA_{n(l+1)}(\mathbb{Q}) \\ Aut_{cl}(\mathcal{A}^{ex}) = G \arrow[r, hook] & Aut_{\mathbb{Q}-alg}(\mathcal{A}^{ex}) \end{tikzcd}$$

where $\mathcal{X}_{f.t}$ is the specified cluster variety of finite type for the appropriate Dynkin diagram and G is listed in theorem 7.2.11

When asking about not necessarily invertible finitely generated projective (bi)modules.

$$K_0(\mathcal{A}) = \mathbb{Z} \longrightarrow K_0(\mathcal{A}^{ex})$$

$$K_0(\mathcal{A} \otimes_{\mathbb{Q}} \mathcal{A}) = \mathbb{Z} \longrightarrow K_0(\mathcal{A}^{ex} \otimes_{\mathbb{Q}} \mathcal{A}^{ex})$$

7.5.5 Corollary *For a "spin" system with $\mathfrak{g} = A_1$ at arbitrary $l \geq 2$, we get a $GA_{l+1}(\mathbb{Q})$ group for $Pic(\mathcal{A}_{A_1,l})$. The invertible bimodules for $\mathcal{A}^{ex}$ have $D_{l+3} = Aut_{cl}(\mathcal{A}^{ex}) \subset Aut(\mathcal{A}^{ex})$ giving $Pic(\mathcal{A}^{ex}) \simeq Aut(\mathcal{A}^{ex}) \ltimes Pic_{com}(\mathcal{A}^{ex})$. The noninvertible finitely generated projective (bi)modules have a $\mathbb{Z}$ characterization when looking at $\mathcal{A}$.*

7.5.6 Remark If we had a bimodule category fusion of defects, then the cyclic subgroup determined by our favorite defect would have a map to GA_{l+1}. In particular if it was finite, we could have a hope of landing in a finite subgroup of maximal order. When $l = 3$ or $5 \leq l \leq 9$ theorem 7.4.9 shows that they would be especially interesting. This might address questions of orders of defects in the q-fermion. ♢

7.5.7 Corollary *For $\mathcal{A}_{A_n,1}$ we can also be more specific because the exchangeable part also forms an A_n finite quiver. That is the same exchangeable part as $\mathcal{A}_{A_1,n}$ above. GA_{2n} replaces GA_{l+1} and $Aut_{cl}(\mathcal{A})$ is different because of the different frozen variables. Everything else we have described is independent of the number of variables in the polynomial ring or only involves $\mathcal{A}^{ex}$.*

7.5.8 Lemma *Let $\mathcal{A}^{ex}$ be the cluster algebra given as the homogenous coordinate ring of a Grassmannian. Then $Cl(\mathcal{A}^{ex}) = 0$.*

Proof The following correspond to the finite type cluster algebras [97]

- $Gr(2, n+3)$ for A_n
- $Gr(3, 6)$ for D_4
- $Gr(3, 7)$ for E_6
- $Gr(3, 8)$ for E_8

But these are unique factorization domains [98]. This then implies the Weil class group $Cl(Spec\mathcal{A}^{ex})$ is trivial. We also know that these examples are locally acyclic, therefore there can be at worst canonical singularities [99].

For K_0 we can apply the results of [100] for $\mathcal{A}^{ex}$ the homogenous coordinate ring of X of dimension d to give

$$K_0(\mathcal{A}^{ex}) \quad = \quad \mathbb{Z} \bigoplus Pic(\mathcal{A}^{ex}) \bigoplus_{p=1}^{d} \bigoplus_{k=1}^{\infty} H^p(X, \Omega_X^p(k))$$

□

Then [101] gives results for each of the $H^p(X, \Omega_X^p(k))$ where (k) indicates twists by L^k with L coming from the very ample hyperplane bundle of projective space.

7.6 Semiclassical Geometry

Cluster algebras often also come as Poisson algebras [102]. This means that we should classify (invertible) (bi)modules for these deformations. This deformation may or may not come from a deformation of the $\mathcal{C}_{\mathfrak{g},l} \subset Rep\ U_q\hat{\mathfrak{g}}$. Candidates for this may be possible using elliptic quantum groups [103, 104].

7.6.1 Definition (Classical Limit) *Let $\mathcal{A}$ be the starting commutative unital Poisson algebra over the commutative unital ring k. (For us $\mathbb{Q}$). Then define $\mathcal{Q}$ to be $\mathcal{A}[[\hbar]]$ [2] with product $\star$ as well as the classical limit maps cl*

$$\begin{array}{rcl} \mathcal{Q} & \to & \mathcal{A} \\ \sum_{r=0}^{\infty} \hbar a_r & \to & a_0 \\ Pic(\mathcal{Q}) & \to & Pic(\mathcal{A}) \\ K_0(\mathcal{Q}) & \to & K_0(\mathcal{A}) \end{array}$$

7.6.2 Theorem ([105]) *The map on K_0 is an isomorphism of groups. The not necessarily invertible have no changes.*

7.6.3 Theorem ([106]) *The kernel of cl on Pic is in 1-1 correspondence with outer self-equivalences of $\mathcal{Q}$. The image can be described in terms of the action of $Pic(\mathcal{A})$ on deformations through gauge transformation equivalence. This is also called a B-field transform.*

[2] $\hbar$ may not be the best notation because the system was already quantum. This is yet another deformation.

In the case where taking $\mathbb{R}$ points has given a real Poisson manifold M, one can ask the differential geometric analog[3] and with complex valued functions here we get:

7.6.4 Theorem (7.1 of [107]) *Let the deformation $\mathcal{Q}$ with $\star$ be such that:*

- *There exists a linear map from the Poisson center $\mathcal{Z}_\pi(\mathcal{A})$ to the center $Z(\mathcal{Q})$ with $f \to f + O(\hbar)$*
- *There exists a linear map $PDer(\mathcal{A}) \to Der(\mathcal{Q})$ where the vector field $X \to \mathcal{L}_X + O(\hbar)$ and for Hamiltonian vector fields, $X_H \to \frac{i}{\hbar} ad_H$*

then for the classical limit map on Pic.

$$\begin{aligned}
ker\ cl &\leftrightarrow \frac{H^1_\pi(M,\mathbb{C})}{2\pi i H^1_\pi(M,\mathbb{Z})} + \hbar H^1_\pi(M,\mathbb{C})[[\hbar]] \\
H^1_\pi(M,\mathbb{C}) = 0 &\implies ker\ cl = 0 \\
&\implies Pic(\mathcal{Q}) \hookrightarrow Diff(M) \ltimes H^2(M,\mathbb{Z})
\end{aligned}$$

7.6.5 Corollary *If in addition $\pi = \omega^{-1}$ and $H^1(M) = 0$, then $Pic(\mathcal{Q}) \hookrightarrow Diff(M) \ltimes H^2(M,\mathbb{Z})$*

Proof In the symplectic case $H^1(M) \simeq H^1_\pi(M)$. □

7.7 Conclusion

Taking inspiration from the algebraic Bethe ansatz with boundaries and defects we have considered the characterization of (invertible) (bi)modules for the associated cluster algebras. We have found interesting groups that might or might not lift to the categorical level of the spin chain.

There are many questions this raises. Among these are producing actual bimodule categories for these $\mathcal{C}_{\mathfrak{g},l}$. Our main tool for producing interesting bimodule categories is covariantization of coquasitriangular algebras [15] from chapter 4. We would also like to work in the RTT realization. The difference of coproducts means physical properties will be drastically different between these cases.

Another question is if the $\otimes\mathbb{Q}$ can be avoided. This is because the algebras were actually defined over $\mathbb{Z}$. Even for polynomial rings we get complications that can be computed as theorem 7.4.4 and lemma 7.4.8. That gives the interesting group $GA_{n(l+1)}(\mathbb{Z})$. The exchangeable part which is not a polynomial algebra will have even more complications. For example, the singularity for A_n when $n \equiv 3(\text{mod}4)$ [108]. The Picard groups for these can be tackled with theorem 7.4.3. Life in a ring is harder than life in a field. An infinitesimal formal degree 2 parameter can be used to tropicalize by defining $\xi_i = T \log x_i$. That resembles the case of equivariant K-theory of Grassmannians [109][4]. $K_0(\mathcal{C}_{A_n,l}) \otimes_{\mathbb{Z}} \mathbb{C}$ is also a quotient of

[3]See [22] and section 10.1 for the Morita theory for Poisson manifolds.

[4]The homological degee provides a caution for when one can expect convergence for numerical nonzero values of a parameter.

the homogenous coordinate ring of $Gr(n+1, n+l+2)$ so relations with the Amplitudehedron are possible [110].

In all cases we have provided a map $\mathbb{Z} \to K_0(\mathcal{A}^{ex})$. We would like to calculate the image of 1 and upon rationalization get these special elements of the Chow rings for all $\mathfrak{g}$ and l. The same is true for the bimodule case. We also described the analogous case for formal deformations. The differential geometric Pic changed in a reasonably manageable way, but we do not know about the algebraic side.

Chapter 8

Asymptotic Representations

8.1 $U_q\mathfrak{b}$

$U_q\mathfrak{b}$ is a very important Hopf subalgebra. In particular, a coideal as well. It's importance in spin chains comes from Baxter's Q operator.

8.1.1 Definition (q-character [111]) *The q-character is an injective homomorphism from $K_0(RepU_q\hat{\mathfrak{g}})$ to a Laurent polynomial ring in infinitely many variables $\mathbb{Z}[Y^{\pm}_{i,a_i}]$ for all $i \in [1, rk\mathfrak{g}]$ and $a_i \in \mathbb{C}^*$.*
It is given by first sending a representation V to the transfer matrix $t_V(z)$ using $V(z)$ as the auxiliary space. The result lands in a commutative subalgebra of $U_q b_-[[z]]$. This result is then mapped to the Laurent polynomial ring.

8.1.2 Theorem ([112, 113]) *Let $\mathcal{O}$ be the category of possibly infinite $U_q\mathfrak{b}$ representations with weight space decomposition into finite dimensional pieces. The prefundamental representations $L^+_{i,a}$ are infinite dimensional representations on which the Drinfeld currents act by $(1-za)$ or 1. Let $\left[L^+_{i,a}\right]$ be their classes in $K_0(\mathcal{O})$. In $K_0(\mathcal{O})$, the following holds:*

- *Take any finite dimensional representation V of $U_q\mathfrak{g}_{aff}$*
- *Compute it's q-character*
- *Replace all $Y_{i,a}$ by $\left[\omega_i\right]\frac{\left[L^+_{i,aq^{-1}}\right]}{\left[L^+_{i,aq}\right]}$*
- *The result is an identity in $K_0(\mathcal{O})$ relating $\left[V\right]$ and the above.*

8.1.3 Corollary (TQ relation) *Let $\mathfrak{g}_{aff} = \hat{\mathfrak{sl}}_2$ Let V be the 2-dimensional representation.*

$$\begin{aligned} \left[V\right]\left[L^+_{1,aq}\right] &= \left[-\omega_1\right]\left[L^+_{1,aq^3}\right] + \left[\omega_1\right]\left[L^+_{1,aq^{-1}}\right] \\ T(z)Q(z) &= A(z)Q(zq^2) + D(z)Q(zq^{-2}) \end{aligned}$$

This is because T uses $V(z)$ as the auxiliary space and Q uses this prefundamental representation instead.

8.1.4 Theorem ([114]) *Turning this picture on it's side allows one to decorate one of the sites with the infinite dimensional representation instead of as the auxiliary space. For simplicity, let all the rest of the sites and the auxiliary space be decorated with spin $\frac{1}{2}$ as usual. Repeating the usual procedures gives a spin chain with a defect at site j.*

$$\begin{aligned} H &= const + \sum_{i \neq j-1,j} h_{i,i+1} \\ &+ \begin{pmatrix} q^{-D} & 0 \\ 0 & r_0^{-1}q^D \end{pmatrix}_{j,j+1} h_{j-1,j+1} \begin{pmatrix} q^D & 0 \\ 0 & r_0 q^{-D} \end{pmatrix}_{j,j+1} \\ &+ \frac{q-q^{-1}}{2q} \begin{pmatrix} 0 & r_0 a^* q^{-2D} \\ r_0 a q^{2D} & 0 \end{pmatrix}_{j,j+1} \end{aligned}$$

where h_{i_1,i_2} is the usual XXZ interaction. a,$a^\dagger$ and $q^{\pm D}$ are generators of the oscillator algebra $U_q\mathfrak{b}$ and r_0 is it's parameters (change variables from a before to avoid confusion with the generators of the oscillator algebra)

8.2 Unzipping inhomogeneities

The infinite dimensional representation is given explicitly as a limit of larger and larger representations.

8.2.1 Lemma (Prop3.1 of [115]) *Let $V^k(x)$ be the spin k evaluation representations. n subscript gives the weight space decomposition and $q = e^\eta$.*

Without the evaluation parameter define i^k by:

$$\begin{aligned} i^k &\in V^{k+\frac{1}{2}} \hookrightarrow V^{\frac{1}{2}} \otimes V^k \\ i^k(v_n^{k+\frac{1}{2}}) &= q^{\frac{n-1}{2}} v_1^{\frac{1}{2}} \otimes v_n^k + q^{\frac{2+2k-n}{2}} \frac{q^{n-1} - q^{-n+1}}{q - q^{-1}} v_2^{1/2} \otimes v_{n-1}^k \end{aligned}$$

This then defines a version with evaluation paramters i_x^k $V^{k+1/2}(x) \hookrightarrow V^{1/2}(x - k\eta) \otimes V^k(x + \frac{\eta}{2})$.

Alternatively this can be permuted to give j_x^k as an intertwiner $V^{k+1/2}(x) \hookrightarrow V^k(x - \frac{\eta}{2}) \otimes V^{1/2}(x + k\eta)$. Of course this iterates to give an unzipping picture

$$\begin{aligned} V^{k+1/2}(x) &\hookrightarrow V^k(x - \frac{\eta}{2}) \otimes V^{1/2}(x + k\eta) \\ &\hookrightarrow V^{k-1/2}(x - \eta) \otimes V^{1/2}(x - \frac{\eta}{2} + (k - 1/2)\eta) \otimes V^{1/2}(x + k\eta) \\ &\hookrightarrow V^{k-1}(x - \frac{3\eta}{2}) \otimes V^{1/2}(x + k\eta - \frac{3\eta}{2}) \otimes V^{1/2}(x + k\eta - \frac{2\eta}{2}) \otimes V^{1/2}(x + k\eta) \\ &\cdots V^{k+1/2-j/2}(x - \frac{j\eta}{2}) \otimes \cdots \end{aligned}$$

Now let k be large. j_x^k then provides a site with diverging inhomogeniety parameter $x + k\eta$. The ray along which this approaches ∞ depends on η revealing the dependence on $\mid \Delta \mid \leq 1$ or $\mid \Delta \mid > 1$. In chapter 6 we saw how $t_n \to +\infty$ was effectively removing that site for the on shell Bethe vectors.

Part IV

AKSZ formalism

Chapter 9

Generalities to AKSZ Formalism

In this chapter we will briefly describe the main notions required for the AKSZ description for topological field theories. We will refer to [20] and [116] for further details. The next chapters will break this down by dimension.

9.1 Supergeometry

Q manifolds

The main idea for $AKSZ$ field theories is to extract the physical information of the TFT from geometric data, encoded in a special vector field on the space of fields (Q-structure), and a symplectic structure, defined via transgression of a symplectic structure on the target supermanifold.

9.1.1 Definition (Supermanifold) *A supermanifold $\mathcal{M}$ corresponds to a locally ringed space $(M, \mathcal{O}_M)$ such that its local model is given by*

$$(U, \mathcal{C}^\infty(U) \otimes \wedge W^*),$$

where U is an open subset of $\mathbb{R}^n$ (the body) and W is a finite dimensional vector space (the soul). Furthermore, the local isomorphism between $(M, \mathcal{O}_M)$ and $(U, \mathcal{C}^\infty(U) \otimes \wedge W^*)$ is an isomorphism of $\mathbb{Z}_2$-graded algebras.

Particular examples of this algebra of functions are:

1. If $\mathcal{M} = T[1]M$, M being an ordinary manifold, then $\mathcal{C}^\infty(\mathcal{M}) = \Omega^\bullet(M)$.
2. In a similar way, for a finite dimensional Lie algebra $\mathfrak{g}$, $\mathcal{C}^\infty(\mathfrak{g}[1]) \supset \wedge \mathfrak{g}^*$. This underlies the Chevalley-Eilenberg complex.

9.1.2 Definition (Q-structure) *A Q structure on a supermanifold is a choice of odd degree 1 vector field satisfying $[Q, Q] = 0$.* [1]

9.1.3 Example *$T[1]N$ is a Q manifold where Q is the deRham differential reinterpreted. To see this note that the functions on $T[1]N$ is $\Omega^\bullet(N)$ and, as a vector field should, the deRham differential gives a derivation of degree +1.*

9.1.4 Definition (Differential Forms) *If $\mathcal{M}$ is a graded manifold, differential forms over $\mathcal{M}$ correspond to the graded algebra $\mathcal{C}^\infty(T^*[1]M)$.*

[1] Recall this is the anticommutator because of the Koszul sign rule. That will always be implied instead of using different brackets.

9.2 Shifted Symplectic Structures

Shifted symplectic Q-manifolds

In a similar fashion as in the usual (non graded) manifolds, non degenerate closed 2 forms play a crucial role in the Hamiltonian formulation of field theories. Therefore, in the Q-manifold setting, we are also able to have an analogue version of a symplectic structure, which comes equipped with a $\mathbb{Z}$-grading.

9.2.1 Definition (Shifted symplectic structure) .
An n-shifted symplectic structure is a 2-form of internal degree n, such that the induced map from tangent to cotangent complexes is a quasi-isomorphism. It also needs to be closed under the deRham differential.

9.2.2 Example *Ordinary symplectic manifolds are 0-symplectic structures on the associated supermanifold.*

9.2.3 Example (Shifted Cotangent Bundle) *If M is an ordinary manifold one can check that $T^*[1]M$ is a 1-symplectic manifold, with the symplectic structure being the canonical (Liouville) one.*

9.2.4 Remark The case where $n = -1$ is of big relevance for the study of BV-BFV theories. It is a well known theorem, originally due to Schwarz [117] that any (-1)-symplectic dg manifold is symplectomorphic to a shifted cotangent bundle $T^*[1]M$ [2]. ♢

9.3 AKSZ Construction

9.3.1 Definition (Linear Cotangent AKSZ Theories [118]) *A Cotangent AKSZ theory corresponds to the following data:*

1. *A a d-dimensional spacetime, denoted by Σ.*
2. *The space of maps from $T[1]\Sigma$ to $T^*[d-1](E[1])$, where E is a graded vector space. This is called the space of fields $\mathcal{F}$. Observe that $\mathcal{F}$ has as local model the $\mathbb{Z}_2$-graded algebra of $\Omega^\bullet(\Sigma) \otimes R$, where R is the coordinate ring of the target.*
3. *A Hamiltonian function S for $\mathcal{F}$ (called the action) of degree d. From the AKSZ construction, this action can be constructed by transgression of a Hamiltonian function on the target, that depends on the geometric structure of the shifted cotangent bundle.*

9.3.2 Remark The target can be substantially generalized to other shifted-symplectic targets, but all our examples are in cotangent form. For the BV formalism Σ is closed, but in BV-BFV Σ is allowed to have boundary. ♢

[2] This is the symplectic analogue of Batchlelor's theorem

9.4 Shifted Lagrangian Correspondences

9.4.1 Definition (Lagrangian [19]) *A Lagrangian in a shifted symplectic manifold X is a morphism $f\ L \to X$ and a homotopy from 0 to $f^*\omega$.*[3] *It should satisfy non-degeneracy in the sense of $f^*T_X \to HoFib(f^*T_X \to L_L[n])$ should be a weak equivalence where L_M is the cotangent complex which can be realized as $T^*[1]M$ in smooth cases.*

9.4.2 Definition (Shifted Lagrangian Correspondence [19]) *For a pair X_1 and X_2 a shifted Lagrangian correspondence is a Lagrangian $L \to X_1 \times \bar{X}_2$.*

This fixes the problems with the usual symplectic "category" which lacks composition due to lack of transversality. Upon transgression this becomes the phenomenon where one side is decorated with X_1 and the other with X_2 and we are providing a defect between them. Taking one of them to be trivial reduces to the case of boundaries. The main focus will be on the case when $X_1 = X_2$.

9.4.3 Definition (Reflecting/Transmitting) *Call the cases when L is given as a product $L_1 \times L_2 \to X_1 \times \bar{X}_2$ with factorized Lagrangian structure as reflecting. This is because it upon transgression it is decorating both sides with ordinary boundaries without any communication between the pieces. At the other extreme, for $X_1 = X_2$, when the correspondence is given by a graph construction of some automorphism call that transmitting. In particular, the invisible defect being the identity map. These extreme cases can be combined to make far more general correspondences.*

In the quantum theory, think of these as giving the bimodules in an appropriate categorical level. That is the perspective that ties this chapter with the previous chapters.

9.5 Gauge Fixing Lagrangians

After transgression, the reduction from the BV extended action back to an ordinary field theory with an ordinary action principle requires a choice of Lagrangian in the -1 shifted $\mathcal{F}_{bulk,BV}$. For example, this returns to $\mathcal{F}_{bulk,BRST}$ from $\mathcal{F}_{bulk,BV}$ in the cases that the BV machinery was unnecessary. We may transgress an automorphism of the target to L_{GF} to get a new gauge-fixing.

9.5.1 Theorem ([117]) *The master equation guarantees that for a deformation of L_{GF}*

$$\int_{L_{GF,1}} e^{\frac{i}{\hbar}S} = \int_{L_{GF,2}} e^{\frac{i}{\hbar}S}$$

However the automorphism in question is not guaranteed to give something deformation equivalent. A finite dimensional picture of this is given by Airy equation where the C_i can not be deformed into each other while maintaining convergence.

[3] This contrasts with ordinary isotropic submanifolds where $f^*\omega = 0$ on the nose.

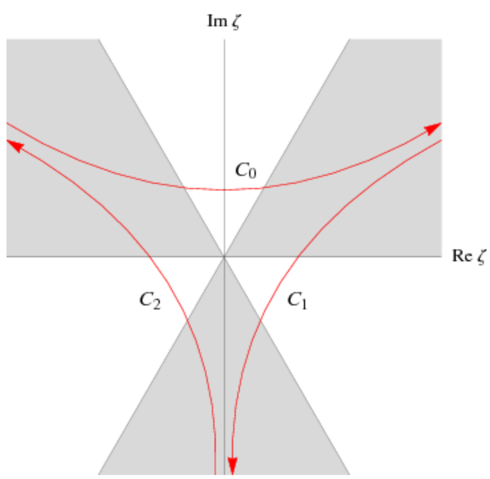

Figure 9.1: Contours of Airy integral [119]

Chapter 10

1+1D and 2+1D Examples

10.1 1+1 Dimensions

In this case the target 1-shifted symplectic stack is given by $T^*[1]P$ for a Poisson manifold P. The Poisson bivector $\pi = \pi^{ij}\xi_i\xi_j$ is a function of homological degree 2. In this context the relational symplectic groupoid [120] shows up.

Morita Theory of Poisson Geometry [22]

10.1.1 Definition (Symplectic Groupoid) *A symplectic groupoid is a Lie groupoid equipped with a multiplicative symplectic form on it's arrows X_1. Multiplicative means that on composable pairs of arrows:*

$$pr_1^*\omega + pr_2^*\omega - pr_{12}^*\omega \quad = \quad 0$$

where $pr_{1,2}$ are projections to the first and second of a pair of composable arrows and pr_{12} is projection to the composition.

The space of objects X_0 has the structure of a Poisson manifold.

10.1.2 Example (T^*G) *Trivialize the bundle $T^*G \simeq G\times\mathfrak{g}^*$. $G\times\mathfrak{g}^* \to \mathfrak{g}^*$ has the structure of an action groupoid. Together this defines T^*G as a symplectic groupoid with $X_0 = \mathfrak{g}^*$ using the Lie-Poisson structure.*

10.1.3 Example (Symplectic Double) *For a complete pair of Poisson Lie groups G and G^* we get a symplectic groupoid over G^* as* $\begin{tikzcd} G \times G^* \arrow[r, bend left, "s"] \arrow[r, bend right, "t"'] & G^* \end{tikzcd}$

This other Poisson structure is in contrast with the Drinfeld double. It is symplectic rather than being Poisson-Lie so it does not quantize to a Hopf algebra.

The phase space of the Poisson sigma model produces a relational version of this groupoid, which may or may not reduce to the above versions [120]. The distinction is due to finite dimensionality issues. So in particular, the two above examples concern the cases when the target for the AKSZ model is either $T^*[1]\mathfrak{g}^*$ or $T^*[1]G^*$

10.1.4 Definition (Morita Equivalence of Poisson) *A Morita equivalence between P_1 and P_2 is a (P_1, P_2) bimodule in the sense of providing an S:*

$$\begin{tikzcd} & S \arrow[dl, "J_1"'] \arrow[dr, "J_2"] & \\ P_1 & & P_2 \end{tikzcd}$$

such that $J_{1,2}$ are surjective submersions with fibers simply connected and symplectic orthogonals of each other.

10.1.5 Definition ($Pic(P)$) *Let Poiss be the category of integrable Poisson manifolds and the generalized morphisms are isomorphism classes of the (S, J_1, J_2) above. $Pic(P)$ is then the group of automorphisms of integrable Poisson manifold P in this category.*

10.1.6 Lemma *By taking a symplectic groupoid of an integrable Poisson manifold and modifying the source map by $\psi \in Aut(P)$ provides a map $Aut(P) \to Pic(P)$. This maps the usual automorphism group as a Poisson manifold into the automorphisms in the Morita sense.*

10.1.7 Definition ($SPic(P)$) $SPic(P) \equiv ker(f) \longrightarrow Pic(P) \xrightarrow{f} Aut(Leaf(P))$

10.1.8 Example *In the case of zero Poisson structure P we get a semidirect product $Diff(P) \ltimes SPic(P) \simeq Diff(P) \ltimes H^2(P,\mathbb{R})$. In this case $C^\infty(P)$ quantizes trivially and $Pic(C^\infty(P)) = Diff(P) \ltimes H^2(P,\mathbb{Z})$. In this context all diffeomorphisms are vacuously Poisson automorphisms.*

10.1.9 Lemma *In the case of a connected symplectic manifold, we get $Pic(M) \simeq Out(\pi_1(M))$*

Proof Let $\tilde{M}$ be the universal cover. Use $\tilde{M} \times \bar{\tilde{M}}$ with the quotient by deck transformations modified by $\phi \in Aut(\pi_1(M))$. This gives a map from $Aut(\pi_1(M))$ which actually factors through an injection from $Out(\pi_1(M))$ and in fact is an isomorphism. Just pick any basepoint you want since M was connected.

10.1.10 Example *For a punctured Riemann surface $\Sigma_{g,h}$ we see $Out(F_{2g+h-1})$*

10.1.11 Example *For $\mathfrak{g}$ a semi-simple Lie algebra of compact type, $Pic(\mathfrak{g}^*) \simeq Out(\mathfrak{g})$ [121]. As a corollary, this combines with [122] to give $Pic(G^*) \simeq Out(\mathfrak{g})$ for the Poisson-Lie group vs the KKS Lie-Poisson bracket.*

Another common target for the A-model in the study of Geometric Langlands is the Betti modul space. For this, we may take outer automorphisms of the following result:

10.1.12 Theorem ([123]) *Let G be a connected compact Lie group. Then for a punctured ($h>0$) surface $\Sigma_{g,h}$ $\pi_1(\mathcal{X}_{\Sigma_{g,h}}(G))$ is given as $\pi_1(\frac{G}{[G,G]})^{2g+h-1}$.*
For closed surfaces and $G = U(n)$, then $\pi_1(\mathcal{X}_{\Sigma_{g,0}}(G)) \simeq \mathbb{Z}^{2g}$ and trivial for $SU(n)$.

Deformation Quantization

10.1.13 Theorem (Cattaneo-Felder [124]) *The Kontsevich deformation quantization of $C^\infty(P)$ is given through evaluating a correlation function in the Poisson sigma model on a disc with 3 marked points.*

$$\begin{aligned}
(f \star g)(x) &= \int_{X(\infty)=x} f(X(1))g(X(0))e^{\frac{i}{\hbar}S[X,\eta]}dX d\eta \\
X &\in Maps(D,P) \\
\eta &\in \Omega^1(D, X^*(T^*M))
\end{aligned}$$

10.1.14 Lemma *For a map f $P_1 \to P_2$ one constructs a shifted Lagrangian correspondence as $\{(p_1, (df)^* \xi \mid_x, f(p_1), \xi)\} \subset T^*[1]P_1 \times T^*[\bar{1}]P_2$. The chain complex computing Poisson cohomology encodes point observables in the field theory by evaluating target space functions through the transgression. This construction is producing a dg bimodule for these algebras of observables (without concern for invertibility or even the Poisson structures).*

The goal is to compute $Pic((C^\infty(P)[[\hbar]], \star))$ which is showing up as the genus 0 information of the Poisson Sigma model. Using the examples of the previous section like $\Sigma_{g,h}$ and G^* provides clues to characterizing invertible bimodules for their quantization and as such defects for their respective field theories. The topological string A-model is produced through the case of P actually being symplectic and a particular choice of gauge-fixing [20]. This suggests an interpretion of these results in the Fukaya category and autoequivalences thereof.

10.2 2+1 Dimensions

BF

The target is $T^*[2]\mathfrak{g}[1] \simeq \mathfrak{g}^*[1] \bigoplus \mathfrak{g}[1]$. This is the example most tied with part II. Some Lagrangian correspondences between the $\mathfrak{d} = \mathfrak{g} \bigoplus \mathfrak{g}^*$ target and the trivial target are given by the Drinfeld Lagrangians of Poisson homogeneous spaces.

10.2.1 Proposition (Drinfeld [125]) *For a Poisson-Lie group (G, π_G), a homogeneous Poisson structure on G/H is uniquely specified by $r = \pi(eH)$. Define the Lagrangian subspace $l_r \subset \mathfrak{g} \bigoplus \mathfrak{g}^*$ given by:*

$$
\begin{aligned}
\xi \mid_{\mathfrak{h}} &= 0 \\
i_\xi r &\in x + \mathfrak{h}
\end{aligned}
$$

where $x \in \mathfrak{g}$ and $\xi \in \mathfrak{g}^*$ are the components. If l_r is an $Ad_{\mathfrak{h}}$ invariant Lie subalgebra of $\mathfrak{d}$, then this is equivalent to r defining the structure of a Poisson homogeneous space on G/H

These can be used for boundary conditions. The quantizations lead to the coideal subalgebras from chapter 4.

Split Chern Simons

The target for split Chern Simons is $\mathfrak{g}^*_{\mathbb{C}}[1]$. This is only regarded with real structure so by [126, Remark 2.2], we have a skew $r_{Iwasawa}$ defining a real Lie bialgebra. Automorphisms of this 2-shifted symplectic manifold with it's Hamiltonian function are transgressed by the general constructions of the previous chapter.

Courant Sigma Model

10.2.2 Definition (Courant Sigma Model [24]) *Let the target be $T^*[2]T^*[1]M$. Here we see the standard Courant algebroid $T^*M \bigoplus TM$ by considering degrees 0 and 1. The Hamiltonian function of degree 3 is the one coming from pairing the coordinates of the $T[1]M$ and $T^*[2]M$ fibers.*

10.2.3 Theorem ([24]) *A Dirac structure for the Courant algebroid provides boundary conditions for the BV master equation. This is done by specifying that boundary fields land in $N^*[2]K[1]$ where K is a Dirac structure for $T[1]M \bigoplus T^*[1]M$ supported on some submanifold C.*

10.2.4 Theorem (4.1 of [23]) *If two Dirac structures on $\mathbb{T}^n$ are in the same orbit of the $O(n,n,\mathbb{Z})$ action, then the quantizations of the interpreted Poisson brackets on $\mathbb{T}^n$ are in the same Morita equivalence class.*

Let the target space be $T^*[2]T^*[1]\mathbb{T}^n$ for a torus $\mathbb{T}^n$. $O(n,n,\mathbb{R})$ then acts on the translation invariant Dirac structures providing different boundary conditions. From the perspective of the sigma model we have done something strange, because we mixed up bulk dimensions. Changing Dirac structures is changing boundary conditions in 3D, but Morita equivalences for quantizing Poisson algebras would operate on a 2D bulk.

Chapter 11

3+1D Topological Insulators

Topological Insulators are phases which insulate in the bulk but support conducting states on their boundaries. They are conjectured to be described by BF theories [25, 127]. In 2+1 dimensions this is the quantum spin Hall state which can be thought of as a pair of integer quantum Hall effect states. Methods of [128, 129, 130, 131, 132, 7, 30, 133] are applicable to describe the classification. The nonabelian version of BF theory has other uses in analogy with the Moore-Read states versus the Laughlin states [134].

The feature in four dimensions is that its gauge symmetry is actually reducible [135]. This makes use of BV-BFV formalism ideal. Even more fortunately it is a topological theory of AKSZ type [20]. By general considerations of Cattaneo, Mnëv and Reshetikhin [18, 26, 136] the space of fields and the necessary symmetries, we can write down the classical action on strata of every codimension (a stratified space in the sense of [137, 138]).

In the free fermion in a crystal perspective, the goal of classification is presented as the fact that the Bloch bundle is an equivariant vector bundle and so represents a class in the appropriate choice of $KR^q_G(T^3)$ or $K^q_G(T^3)$ [139]. For related computations, see the $K(BG)$ listed in table 3.3[1]. However, this chapter will focus on the field theory instead of band structure.

11.1 Homological Degree 0 Argument

Let us first review [25] before describing how to extend with the BV formalism. This part uses only the portion of the theory in ghost degree 0. Start with an electron field coupled to both electromagnetism A and a statistical gauge field with components a_μ.

$$\begin{aligned}\mathcal{L} &= \frac{1}{2} \mid (\partial_\mu - i(A_\mu + a_\mu))\psi \mid^2 - V(\psi^\dagger \psi) \\ V &= \frac{1}{2}(\psi^\dagger \psi - v^2)^2\end{aligned}$$

This is replaced with a first order formalism by introducing a vector field ξ and using the ansatz $\psi = ve^{i\theta}$

$$\mathcal{L} = -\frac{1}{2v^2}\langle \xi, \xi \rangle_g + \langle \xi, d\theta - A - a \rangle_{ev}$$

where the g subscript indicates the term that depends on the metric and ev only uses the pairing of one-forms and vector fields.

Use the isomorphism [2]of deRham complex and divergence complex of polyvector fields to turn ξ into an $d-1$ form H. Integrate out the θ field. Because ξ is divergence free, this means that H is closed. Tracking this through gives

[1]This relation requires a localization theorem to the fixed points and completion.

[2]This uses the orientation, so remember this when folding.

$$\mathcal{L} = \frac{-1}{4v^2}\langle H, H\rangle_g - (A+a)\wedge H$$

In cases $H = dB$, and dropping the metric dependent term through a rescaling argument gives

$$\mathcal{L} = d(A+a)\wedge B$$

with B a higher abelian gauge field. If the manifold has $H^{d-1}(\Sigma) \neq 0$, then there is an additional term from a harmonic representative for $\left[H\right] \in H^{d-1}_{dR}$.

11.2 Bulk Classical 4 dimensional abelian BF theory

In this section we describe abelian BF theory for closed 4 manifolds in terms of the linear cotangent AKSZ formalism.

Let $\Sigma = S_g \times I \times S^1_{euc,time}$ be the 4 dimensional space-time and V a finite dimensional vector space regarded as an abelian Lie algebra. This choice is made because we want the spatial manifold $S_g \times I$ to be something that should be reasonable to embed in Euclidean 3-space, but rich enough to probe mapping class group information.[3] Perform the AKSZ construction with target $T^*[4-1](V[1]) = V[1]\bigoplus V^*[4-2]$ which is 3-shifted symplectic. The Hamiltonian Θ of degree 4 is 0 and therefore so is the target cohomological degree 1 vector field $Q = \{\Theta, -\}$.

The space of fields $\mathcal{F}_{bulk}$ associated to this mapping space (the coordinates on this space) is :

$$\begin{aligned}
\mathcal{F} &= \Omega^\bullet(\Sigma)\otimes V[1]\bigoplus\Omega^\bullet(\Sigma)\otimes V^*[4-2]\\
\mathcal{A} &= \sum_i A_k\\
\mathcal{B} &= \sum_i B_k
\end{aligned}$$

where A_k is a V valued k- form of cohomological/ghost degree $1-k$ and B_k is V^* valued k form of degree $(4-2)-k$. Summarize this as

With a basis of V and dual basis of V^* add another label as $A_{k,i}$ or B^i_k. By construction $\mathcal{F}$ has a (-1)- shifted symplectic structure with symplectic form in these coordinates given by:

[3] This contrasts with the difficulty of putting the fractional quantum Hall effect on higher genus Riemann surfaces because of trouble with the transverse magnetic field.

Fields	Notation from [140, 141]	form degree	ghost degree
A_0	$(-1)^{4+1}c$	0-form	ghost 1
A_1	A	1-form	ghost 0
A_2	$B^\dagger(-1)^4$	2-form	ghost -1
A_3	$(-1)^{1*0/2+4*2}\tau_1^\dagger$	3-form	ghost -2
A_4	$(-1)^{2*1/2+4*3}\tau_2^\dagger$	4-form	ghost -3
B_0	$(-1)^{2*1/2}\tau_2$	0-form	ghost 2
B_1	$(-1)^{1*0/2}\tau_1$	1-form	ghost 1
B_2	B	2-form	ghost 0
B_3	$(-1)^4 A^\dagger$	3-form	ghost -1
B_4	$c^\dagger$	4-form	ghost -2

Table 11.1: Notations for Fields

$$\begin{aligned}
\Omega &= \int_\Sigma \sum_{i,k} \delta B_k^i \wedge \delta A_{i,4-k} \\
&= \int_\Sigma \sum_i \delta B_0^i \wedge \delta A_{4i} + \delta B_1^i \wedge \delta A_{3i} + \delta B_2^i \wedge \delta A_{2i} + \delta B_3^i \wedge \delta A_{1i} + \delta B_4^i \wedge \delta A_{0i}
\end{aligned}$$

The classical action is given by:

$$\begin{aligned}
S &= \int_\Sigma \sum_{i,k} B_k^i dA_{i,3-k} \\
&= \int_\Sigma \sum_i B_0^i dA_{i,3} + B_1^i dA_{i,2} + B_2 dA_{i,1} + B_3^i dA_{i,0}
\end{aligned}$$

The cohomological vector field is the following:

$$
\begin{aligned}
Q &= \int_\Sigma d\mathbf{A}\wedge\frac{\partial}{\partial\mathbf{A}}+d\mathbf{B}\wedge\frac{\partial}{\partial\mathbf{B}}\\
&= \int_\Sigma\sum_i dA_{i,0}\wedge\frac{\partial}{\partial A_{i,1}}+dA_{i,1}\wedge\frac{\partial}{\partial A_{i,2}}+dA_{i,2}\wedge\frac{\partial}{\partial A_{i,3}}\\
&+ dB^i_0\wedge\frac{\partial}{\partial B^i_1}+dB^i_1\wedge\frac{\partial}{\partial B^i_2}+dB^i_2\wedge\frac{\partial}{\partial B^i_3}
\end{aligned}
$$

The bulk equations of motion determine that the fields are all closed forms:

$$
\begin{aligned}
dA_{i,j} &= 0\\
dB^i_j &= 0
\end{aligned}
$$

Gauge Fixed Action

For the example of the gauge fixing procedure in 2D abelian BF, see [142]. Extend the space of fields with

$$
\begin{aligned}
C_{d-m,i} &\in \Omega^{d-m}(\Sigma,V^*[d-m-3])\\
D^i_{d-m} &\in \Omega^{d-m}(\Sigma,V[d-m-2])
\end{aligned}
$$

11.2.1 Definition *The extended action is*

$$
\begin{aligned}
S &= S_{AKSZ}+S_{aux}=\int_\Sigma\sum_{m=0,1}B^i_{4-m}dA_{m-1,i}+D^i_{4-m}\wedge A_{m,i}\\
&+ \sum_{m=0}^{2}B^i_m dA_{4-m-1,i}+C_{4-m,i}\wedge B^i_m\\
&+ \sum_{m=0}^{2}C_{m,i}\wedge B^i_{4-m}\\
&+ \sum_{m=0}^{1}D^i_m\wedge C_{4-m,i}
\end{aligned}
$$

Introduce the adjoint differential which depends on the metric by

$$
Q^{gf}A_m = d^\dagger A_{m+1}
$$

and similarly for the BCD fields. Suppose an inner product is given on V so that i gives an orthonormal basis. This allows us to define a $\dagger$ on fields which swaps $A_{m,i}$ and B^i_{4-m} and similarly for C and D.

11.2.2 Definition *The gauge fixing Lagrangian is given by the relations*

$$\begin{aligned}
A_0^\dagger &= Q^{gf}(D_{4-0}) + B_{4-0} \\
A_1^\dagger &= (-1)Q^{gf}(D_{4-1}) + B_{4-1} \\
B_0^\dagger &= Q^{gf}(C_{4-0}) + A_{4-0} \\
B_1^\dagger &= (-1)Q^{gf}(C_{4-1}) + A_{4-1} \\
B_2^\dagger &= Q^{gf}(C_{4-2}) + A_{4-2} \\
C_0^\dagger &= Q^{gf}(A_0) + D_0 \\
C_1^\dagger &= Q^{gf}(A_1) + D_1 \\
D_0^\dagger &= Q^{gf}(B_0) + C_0 \\
D_1^\dagger &= Q^{gf}(B_1) + C_1 \\
D_2^\dagger &= Q^{gf}(B_2) + C_2
\end{aligned}$$

This was chosen by choosing a Ψ gauge fixing fermion of degree 1 that depends on fields and not antifields. The graph of the differential of Ψ provides the $\mathcal{L} \subset T^*[1]\mathcal{F}_{fields} = \mathcal{F}_{BV}$

11.2.3 Corollary *The restriction of S to this Lagrangian subspace reduces the action to a combination of a signature and Laplacian terms.*

Proof Using $\langle A, B\rangle_g$ for the Hodge pairing and applying the constraints given by the gauge fixing Lagrangian gives.

$$\begin{aligned}
S\mid_{\mathcal{L}} &= \langle \Phi^\bullet, \not{D}_\Sigma \Phi^\bullet\rangle_g + \langle \Phi^\bullet, \Delta_\Sigma \Phi^\bullet\rangle_g \\
&+ \langle B_0, \star D_4\rangle_g + \langle A_0, \star C_4\rangle_g
\end{aligned} \qquad \square$$

where $\Phi^\bullet$ is shorthand for all the fields $(A_0, A_1, C_0, C_1, B_0, B_1, B_2, D_0, D_1, D_2)$ leaving out antifields. $\not{D}$ is the signature Dirac operator.

The decomposition into fermions is the decomposition of $\Omega^\bullet(\Sigma)$ as spinor bundles and restricting the signature operator as the requisite Dirac operator.

11.2.4 Definition (Chirality Operator) *On $\Omega^q(\Sigma^d) \otimes_{\mathbb{R}} \mathbb{C}$ define*

$$\Gamma\omega = i^{(d+1)/2}(-1)^{q(q+1)/2} \star_\Sigma \omega$$

as an antilinear operator. In the case of concern Γ is an involution.

Proof

$$\begin{aligned}
\Gamma^2\omega &= i^{\frac{d+1}{2}}(-1)^{\frac{q(q+1)}{2}}i^{-\frac{d+1}{2}}(-1)^{\frac{(d-q)(d-q+1)}{2}}\star_M\star_M\omega \\
&= i^{\frac{d+1}{2}}(-1)^{\frac{q(q+1)}{2}}i^{-\frac{d+1}{2}}(-1)^{\frac{(d-q)(d-q+1)}{2}}(-1)^{q(d-q)}s\omega \\
&= s(-1)^{d(d+1)/2}\omega
\end{aligned}$$

□

Here $d = 4$ and $s = 1$ because it is a Riemannian 4-manifold so Γ is an involution.

11.2.5 Lemma Γ *anticommutes with the signature operator*

Proof

$$\begin{aligned}
\Gamma(d+\star d\star)\Gamma\omega &= \Gamma(d+\star d\star)(-1)^{\frac{q(q+1)}{2}}\star_M\omega \\
&= \Gamma(-1)^{\frac{q(q+1)}{2}}(d\star_M\omega+\star d\star\star\omega) \\
&= (-1)^{\frac{q(q+1)}{2}}\Gamma(d\star\omega+\star d\star\star\omega) \\
&= (-1)^{\frac{q(q+1)}{2}}((-1)^{(d-q+1)(d-q+2)/2}\star d\star\omega+(-1)^{(d-q-1)(d-q)/2}\star\star d\star\star\omega) \\
&= (-1)^{\frac{q(q+1)}{2}}(-1)^{(d-q+1)(d-q+2)/2}\star d\star\omega \\
&+ (-1)^{\frac{q(q+1)}{2}}(-1)^{(d-q-1)(d-q)/2}(-1)^{(q+1)(d-q-1)}sd(-1)^{q(d-q)}s\omega \\
&= ((-1)^{1/2(1+d)(2+d)-(1+d)q+q^2}\star d\star+(-1)^{(d+d^2+2dq-2(1+q+q^2))/2}d)\omega \\
&= -(-1)^{d(d-1)/2-qd}(\star d\star+(-1)^d d)\omega
\end{aligned}$$

□

This gives the charge conjugation symmetry that can be imposed in the decomposition of spinor bundles. In our context this complexification is thought of as $Cliff(+1)$ so the imaginary components are shifted to odd super-degree. This means that as a super vector space this is identified with the space of fields.

Analytic Torsion

This action functional is suggestive of signature and laplacian operators- the partition function for a bilinear action functional written in terms of either of these operators is known to be computable form the functional determinants or pfaffians of either of these operators, but the literature offers few clear answers in this mixed case.

11.2.6 Theorem ([143]) *For forms on Σ with coefficients in an acyclic $O(m)$ local system E the partition function can be computed explicitly in terms of Ray-Singer torsion, namely:*

$$\int_L e^{i/\hbar\hat{S}} = T_\Sigma \quad = \quad \tau_{RS}(\Sigma, E)$$

The local system is not necessarily acyclic so instead we get T_Σ as a half density on a determinant line Det $H^\bullet(\Sigma)$ instead. For the bulk problem without boundary, Σ is a closed oriented even dimensional Riemannian manifold. The Poincare Duality argument still holds and gives ± 1 when we have identified the determinant line with $\mathbb{R}$ by using an orthonormal basis for the Hodge inner product. It transforms by the determinant of an orthogonal matrix, ± 1 under change of basis, so that is quotiented out. Thus we get an identification with $+1 \in \mathbb{R}_{>0}$. [74]

Automorphisms

The operations we may perform on the space of fields that maintain or reverse it's shifted symplectic structure include:

1. $\dagger$ the shifted symplectic involution defined through $\star$ and an inner product on V
2. $A \to -A$ from the target space antisymplectomorphism
3. $B \to -B$ from the target space antisymplectomorphism
4. Both the above two from a target space symplectomorphism
5. Time reversal from $t \to -t$ extended to all differential forms.
6. Parity reversal if the spatial slice M has such an involution.

11.3 Codimension 1 Strata

Boundaries without folding

Consider the inclusion of the edge into the manifold. Because the fields are defined as sections of fiber bundles or connections, there is a canonical map π which takes bulk fields and sends them to their restrictions as elements of $\mathcal{F}_{bdry}$. This allows us to define the boundary data from the bulk with the following procedure recalled from [18, 26].

First we project the cohomological vector field to the space of boundary fields with π^*. Then we consider the one form $\tilde{\alpha}$ on the space $\mathcal{F} \mid_{bdry}$ which comes from varying the action. Let $\mathcal{F}_{bdry}$ be the space of leaves of the distribution defined by the kernel of $\delta\tilde{\alpha}$. This induces a 1-form α on $\mathcal{F}_{bdry}$. Letting ω_{bdry} be $\delta\alpha$ means that we now have a symplectic structure on boundary fields. A calculation reveals that it is invariant under the pushforward of the cohomological vector field $Q_{\partial\Sigma}$. The BFV action S is then the Hamiltonian function associated to that cohomological vector field.

$$\begin{aligned}
\delta S_{bulk} &= \int_{bulk} \delta\mathcal{B} \wedge d\mathcal{A} \pm d\mathcal{B} \wedge \delta\mathcal{A} \\
&+ \int_{bdry} \delta\mathcal{A} \wedge \mathcal{B} \\
\tilde{\alpha} &= \int_{bdry} \delta A_{0,i} \wedge B_3^i + \delta A_{1,i} \wedge B_2^i + \delta A_{2,i} \wedge B_1^i + \delta A_{3,i} \wedge B_0^i \\
\omega &= \int_{bdry} \delta\mathcal{A}_{bdry} \wedge \delta\mathcal{B}_{bdry} \\
&= \int_{bdry} \delta A_{0,i} \wedge \delta B_3^i + \delta A_{1,i} \wedge \delta B_2^i + \delta A_{2,i} \wedge \delta B_1^i + \delta A_{3,i} \wedge \delta B_0^i \\
\mathcal{A}_{bdry} &\in \Omega^\bullet(\partial\Sigma) \otimes V[1] \\
\mathcal{B}_{bdry} &\in \Omega^\bullet(\partial\Sigma) \otimes V^*[4-2] \\
S_{BFV} &= \int_{bdry} \langle \mathcal{B} \wedge d\mathcal{A} \rangle \\
&= \int_{bdry} B_0^i dA_{2,i} + B_1^i dA_{1,i} + B_2^i dA_{0,i}
\end{aligned}$$

Here we have considered a reduction modulo $\ker\tilde{\alpha}$. That was the fields which had dz's in them where the boundary is given by $z = 0$ in a local model. In this case the reduction to the space of leaves was locally easy because it was a product structure. In following with the general formalism, in the boundary the symplectic form now has degree 0 and the action has degree 1.

Codimension 1 Strata with folding

The AKSZ construction uses the fact that the manifold is oriented. This becomes relevant in the transgression map.

Because we want the folding construction, we must see the effect of orientation reversals given by reflection across some codimension 1 wall in Σ. Let us give this the coordinate transformation $or: \quad z \to -z$. For this we need to distinguish which forms have dz factors and which do not. The symplectic structure and the action transform by a factor of -1 from this transformation because there is exactly one dz in every term.

For simplicity, consider the case when cutting along $z = 0$ decomposes Σ as $N_1 \bigsqcup N_2$ (possibly with different abelian Lie algebras V and W.) By the folding trick, we can think of this as the theory with V on N_1 and the theory for W on $\bar{N}_2$ where $\bar{N}_2$ stands for the result of flipping the transverse coordinate and as a result changing the orientation. For simplicity let $N_1 = N_2 = N$. We can think of the defect as the boundary of a AKSZ theory based on target $T^*[3]((V \bigoplus W)[1])$ and source N.

$$S_{bulk} = \int_N \sum_i \mathcal{B}_l^i \wedge d\mathcal{A}_{li} - \sum_j \mathcal{B}_r^j \wedge d\mathcal{A}_{rj}$$

where l,r stand for left and right of the defect. Having reduced to the previous section, we can then immediately write down the defect action and symplectic structure

$$\begin{aligned}
\omega &= \int_{bdry} \delta A_{l,0,i} \wedge \delta B_{l,3}^i + \delta A_{l,1,i} \wedge \delta B_{l,2}^i + \delta A_{l,2,i} \wedge \delta B_{l,1}^i + \delta A_{l,3,i} \wedge \delta B_{l,0}^i \\
&- (\delta A_{r,0,j} \wedge \delta B_{r,3}^j + \delta A_{r,1,j} \wedge \delta B_{r,2}^j + \delta A_{r,2,j} \wedge \delta B_{r,1}^j + \delta A_{r,3,j} \wedge \delta B_{r,0}^j) \\
S_{BFV} &= \int_{bdry} \mathcal{B}_l^i \wedge d\mathcal{A}_{li} - \mathcal{B}_r^j \wedge d\mathcal{A}_{rj} \\
&= \int_{bdry} B_{l,0}^i dA_{l,i,2} + B_{l,1}^i dA_{l,i,1} + B_{l,2}^i dA_{l,i,0} - B_{r,0}^j dA_{r,j,2} - B_{r,1}^j dA_{r,j,1} - B_{r,2}^j dA_{r,j,0}
\end{aligned}$$

11.4 Codimension 2 Strata

This is likely the most directly visible to an experimental test. It is a one dimensional wall on the outside surface which is available to spectroscopy. The codimension 1 defect walls terminate at domain walls in the boundary anomolous 2+1 theory. Either way a codimension 2 stratum of the form 1-manifold crossed with S^1_{time}. In gluing this is also important to account for the effects of edge dislocations. [144, 145].

11.4.1 Remark Even though the the first picture in fig. 11.1 is likely to be smoothed out by wearing down of the boundary, we may use that as a very useful local picture for the calculations. The third is the one we keep in mind for the cutting out of vortices penetrating the bulk when doing the problem with superconductors for Majorana zero mode applications. This comes from the fact that remembering that vortices are disorder operators and therefore need to be thought of as cutting out and then regluing back in rather than conventional quanta of the field theory. We are not in a situation like supersymmetric Wilson-t'Hooft operators where we know that they can be put on the same footing. ♢

$$\begin{aligned}
\omega_C &= \int_C \delta A_{0,i} \wedge \delta B_2^i + \delta A_{1,i} \wedge \delta B_1^i + \delta A_{2,i} \wedge \delta B_0^i \\
S_{BFV,-2} &= \int_C B_0^i dA_{1,i} + B_1^i dA_{0,i}
\end{aligned}$$

along with another term using the restriction of fields from the other boundary.

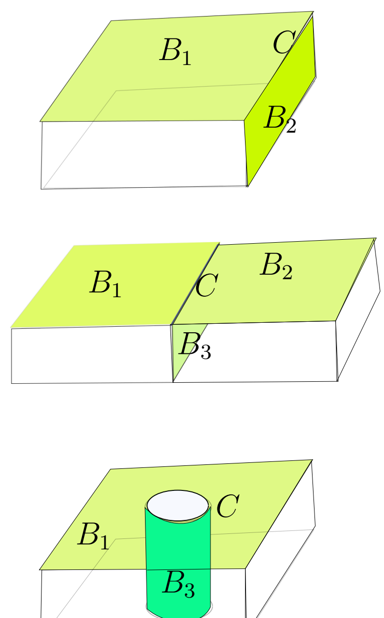

Figure 11.1: Different cases of Codimension 2 Defect

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