

UCLA

UCLA Electronic Theses and Dissertations

Title

Intermittency of the Malliavin Derivatives and Regularity of the Densities for a Stochastic Heat Equation

Permalink

<https://escholarship.org/uc/item/9zc7t81k>

Author

Mahboubi, Pejman

Publication Date

2012

Peer reviewed|Thesis/dissertation

UNIVERSITY OF CALIFORNIA
Los Angeles

**Intermittency of the Malliavin Derivatives and
Regularity of the Densities for a Stochastic Heat
Equation**

A dissertation submitted in partial satisfaction
of the requirements for the degree
Doctor of Philosophy in Mathematics

by

Pejman Mahboubi

2012

ABSTRACT OF THE DISSERTATION

Intermittency of the Malliavin Derivatives and Regularity of the Densities for a Stochastic Heat Equation

by

Pejman Mahboubi

Doctor of Philosophy in Mathematics

University of California, Los Angeles, 2012

Professor Thomas Liggett, Chair

In recent decades, as a result of mathematicians' endeavor to come up with more realistic models for complex phenomena, the acceptance of a stochastic model seemed inevitable. One class of these models are Stochastic Partial Differential Equations (SPDEs).

The solution to a SPDE, considered as a Wiener functional, can be analyzed by means of Malliavin calculus. Malliavin calculus, which is a calculus on the Wiener space, is becoming a standard method for investigating the existence of the density of random variables.

In this thesis, we study nonlinear SPDEs of the form $\partial_t u(t, x) = \mathcal{L}u(t, x) + \sigma(u(t, x))\dot{w}$ with a periodic boundary condition on a torus, where $\mathcal{L}$ is the generator of a Lévy process on the torus. We used the technique of Malliavin calculus to show that when σ is smooth, under a mild condition on $\mathcal{L}$, the law of the solution has a density with respect to Lebesgue measure for all $t > 0$ and $x \in \mathbf{T}$. It turns out that the density of $u(t, x)$ has an upper bound that is independent of x . We also prove that the Malliavin derivatives grow in time with an exponential rate. This result, in certain cases, extends to the weak intermittency of the random field of the Malliavin derivatives.

The dissertation of Pejman Mahboubi is approved.

Jean Turner

Sebastian Roch

Marek Biskup

Davar Khoshnevisan

Thomas Liggett, Committee Chair

University of California, Los Angeles

2012

To: Mr. Kamran Mortezaei Farid and Mr. Morad Arabi, who taught me calculus.

TABLE OF CONTENTS

1	Introduction and main results	1
2	Preliminaries and background	10
2.1	Lévy processes on a torus	10
2.2	A Stochastic Partial Differential Equation	17
2.3	Elements of Malliavin's calculus	25
2.3.1	The Wiener chaos	25
2.3.2	The derivative operator	26
2.3.3	The divergence operator	27
3	Smoothness of the Solution	33
3.1	The Malliavin derivatives of the solution	33
3.1.1	The first derivative	34
3.1.2	Preliminaries and notations	39
3.1.3	The k th derivatives of the v_n 's and the smoothness of the solution	45
3.1.4	Proof of the Theorem 3.1.2	49
4	Analysis of the Malliavin Matrix	51
4.1	Existence of a density	51
4.2	Upper bound for the density	57
5	Intermittency and Malliavin Derivatives	60
6	New directions and further developments	65
6.0.1	The Malliavin calculus on $\mathbf{R}$	65

6.0.2	Estimates on the densities	66
6.0.3	Existence of the joint densities	66
References		67

ACKNOWLEDGMENTS

This thesis would have been impossible without the guidance of my advisor, Prof. Davar Khoshnevisan. I am deeply grateful to him for introducing me to the field, his constant encouragement as well as many fruitful and insightful discussions.

My thanks go to Profs. Marek Biskup, Thomas Liggett, Sebastian Roch and Jean Turner for accepting to be on my Ph.D committee. I am particularly thankful to Prof. Liggett for his continued support of my plan.

I am thankful to my friends in the UCLA Mathematics Department and the University of Utah, including Miljan Brakočević, Alex Chen, Mohammud Foondun, Nicos Georgiou, Mathew Joseph, Lê Thái Hoàng, Liang Zhang, whose company I have enjoyed over the past years. Mathew deserves a special thanks for many discussions, mathematical and non-mathematical.

Lastly and most importantly, I wish to thank my family, especially my brother Payam and his wife Amanda whose love and encouragement have helped me complete graduate school.

As of May 22, 2012, only one year has passed since the attack by the Iranian government on the Bahai Institute for Higher Education (BIHE). Thirty-nine homes were raided, and many instructors and others who were worked in BIHE were arrested. Laboratories and classrooms were forced to close, and computers and other materials were confiscated. By October, seven educators that were associated with the BIHE were sentenced to four and five year terms in prison and many others have been given similar long prison terms. Their only crime was efforts to educate those whose rights to education have been denied by the government since 1979. As a former student of BIHE, I would like to dedicate this work to my educators and other members who served in BIHE, especially to: Mr. Morad Arabi, who taught me calculus, and Mr. Kamran Mortezaei Farid, who is now serving his five year prison sentence for his contribution in BIHE.

VITA

1973	Born, Qaemshahr, Iran.
2005	B.S. (Mathematics), University of Utah.
2005–2008	Teaching Assistant, Mathematics Department, UCLA,
2008-2009	Research Assistant, Mathematics Department, University of Utah.
2009-2010	Teaching Assistant, Mathematics Department, UCLA.
2010-2012	Research Assistant, Mathematics Department, University of Utah.
2012	Teaching Assistant, Mathematics Department, UCLA.

PUBLICATIONS

A parabolic Anderson equation, smoothness and chaos, submitted.

CHAPTER 1

Introduction and main results

Let $\{\dot{w}(t, x)\}_{t \geq 0, x \in [0, 2\pi]}$ denote space-time white noise on the torus $\mathbf{T}$, and let $\sigma : \mathbf{R} \rightarrow \mathbf{R}$ be a nice function. For every $T \in [0, \infty]$, we define $E_T := [0, T] \times \mathbf{T}$, and let E denote $E_\infty := \bigcup_{T > 0} E_T$. We study the parabolic stochastic partial differential equation [SPDE]

$$\left\{ \begin{array}{ll} \partial_t u(t, x) = \mathcal{L}u(t, x) + \sigma(u(t, x))\dot{w} & (t, x) \in E, \\ u(t, 0) = u(t, 2\pi) & t \geq 0, \\ u(0, x) = u_0(x) & x \in \mathbf{T}, \end{array} \right. \quad (1.1)$$

where $\mathcal{L}$ is the $L^2(\mathbf{T})$ -generator of a Lévy process $X := \{X_t\}_{t \geq 0}$, and acts only on the variable x , and u_0 is a bounded, measurable real function on $\mathbf{T}$. We denote by $C_b^\infty(\mathbf{R})$ the space of all smooth functions on $\mathbf{R}$ with bounded derivatives of all orders. Note that we *do not* require σ to be bounded, however, the bound on $|\sigma'|$ requires σ to be Lipschitz. Let $\Phi : \mathbf{Z} \rightarrow \mathbf{C}$ denote the characteristic exponent of X normalized so that $\mathbb{E} \exp(inX_t) = \exp(-t\Phi(n))$ for all $n \in \mathbf{Z}$ and $t > 0$. In other words, Φ is the Fourier multiplier of $\mathcal{L}$ and $\hat{\mathcal{L}}(n) = -\Phi(-n)$ holds for all $n \in \mathbf{Z}$; see section 2.1 for details.

We show that (1.1) has a well-defined and unique solution and let u denote this solution. The idea for the existence and uniqueness of the solutions to (1.1) come from [24] and [23]. A linearized version of (1.1) on $\mathbf{R}$, with vanishing initial data, in which the noise is additive; i.e.,

$$\left\{ \begin{array}{ll} \partial_t u(t, x) = \mathcal{L}u(t, x) + \dot{w}, \\ u(0, x) = 0, \end{array} \right. \quad (1.2)$$

is studied by Foondun et al. in [24]. They have shown a one-to-one correspondence between the existence of a unique random field solution to (1.2) and the existence of the local times

for the symmetrized underlying Lévy process $\bar{Y}$, where

$$\bar{Y}_t = Y_t - Y'_t \quad \forall t \geq 0, \quad (1.3)$$

and $Y' = \{Y'_t\}_{t \geq 0}$ is an independent copy of Y . Their result is the following:

Theorem 1.0.1 (Foondun-Khoshnevisan). *The stochastic heat equation (1.2) has random field solutions if and only if the symmetric Lévy process Y has local times.*

In [23], the authors consider a multiplicative white noise and study the existence and uniqueness of the mild solution to the equation

$$\begin{cases} \partial_t u = \mathcal{L}u + \sigma(u)\dot{w} & t \geq 0, x \in \mathbf{R}, \\ u(x, 0) = u_0(x) & x \in \mathbf{R}, \end{cases} \quad (1.4)$$

with a nonnegative initial data u_0 . In this paper, Foondun and Khoshnevisan combine the existence result of [24] with a result of Hawkes (see Theorem 2.1.1 below) to show that (1.4) has a strong solution, whenever $v(\beta) < \infty$, for some $\beta > 0$ where

$$v(\beta) := \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{d\xi}{\beta + 2\operatorname{Re} \varphi(\xi)},$$

where φ denotes the characteristic exponent of Y . Therefore, it is natural to consider a solution to equation (1.1) under a similar hypothesis. We define

$$\Upsilon(\beta) := \frac{1}{4\pi} \sum_{-\infty}^{\infty} \frac{1}{\beta + 2\operatorname{Re} \Phi(n)}.$$

Theorem 2.2.7, Lemma 2.2.4 and Lemma 2.2.5 deal with the existence and uniqueness of the solution to Eq. (1.1). In Hypothesis **H1** below, we will discuss briefly how the existence of a mild solution imposes a restriction on the underlying Lévy process X and the corresponding gauge function $\Upsilon(\beta)$.

Let $\{f(t, x)\}_{t \geq 0, x \in \mathbf{T}}$ be a predictable random field. For each $\beta > 0$ and $p \geq 2$ define a family of seminorms via

$$\|f\|_{\beta, p} := \left\{ \sup_{(t, x) \in \mathbf{R}_+ \times \mathbf{T}} e^{-\beta t} \mathbf{E}(|f(t, x)|^p) \right\}^{1/p}, \quad (1.5)$$

and let $\mathcal{D}_{\beta,p} = \{f = f(t, x) : \|f\|_{\beta,p} < \infty\}$. Let $L^{k,p}$ be the collection of all random fields $f \in \mathcal{D}_{\beta,p}$ such that $f(t, x) \in \mathbf{D}^{k,p}$ for all $t \geq 0$ and $x \in \mathbf{T}$. Define $\Gamma_{t,x}^k f := \|D^k f(t, x)\|_{L^2(T^k)}$ and let

$$\mathcal{D}_{k,\beta,p} := \{f \in L^{k,p} : \|\Gamma^k f\|_{\beta,p} < \infty\}. \quad (1.6)$$

Here $\mathbf{D}^{k,p}$ and the D^k operator denote a Malliavin Sobolev space and the Malliavin derivative operator of k th order respectively; for precise definitions see Chapter 2.3.

Fix $p \geq 1$, we will show that, for any k , there is $\beta = \beta(k)$ so that $\beta(k) \geq \beta(k-1)$ and $u \in \mathcal{D}_{k,\beta,p}$. This property of u , when interpreted as a rate of growth for the Malliavin derivatives with time, translates into the existence of an upper bound for the Liapounov exponents for the field $D^k u$. We will show that, when u_0 is sufficiently large, under some conditions on σ , the Malliavin derivatives are intermittent. This property of the random field describes the pronounced spatial structure of the field; for the precise definition and more details we refer to Chapter 5. When σ is linear, this result holds for the derivatives of all orders. For the general σ we will prove the intermittency for the Malliavin derivative of the first and second order. To be more precise, let us define the upper p th-moment Liapounov exponent $\bar{\gamma}^k(p)$ of $D^k u(t, x)$ by

$$\bar{\gamma}^k(p) = \limsup_{t \rightarrow \infty} \frac{1}{t} \ln(\mathbb{E} \|D^k u(t, x)\|_{H^{\otimes k}}^p) \quad \text{for all } p \in (0, \infty), \quad (1.7)$$

where $H = L^2((0, \infty) \times \mathbf{T})$. We say that $D^k u$ is *weakly intermittent* if

$$\bar{\gamma}^k(2) > 0 \quad \text{and} \quad \bar{\gamma}^k(p) < \infty \quad \text{for all } p > 2. \quad (1.8)$$

If we interpret the Malliavin derivatives as derivatives with respect to ω , then intermittency implies an unusually big derivative with respect to path ω . In this context, the following two theorems show the sensitivity of $u(t, x)$ to the change of paths.

Theorem 1.0.1. *Let $u = u(t, x)$ be the solution of (1.1). If $\sigma \in C_b^\infty(\mathbf{R})$, then $u \in \mathbf{D}^{k,p}$ for all $k \geq 1$ and $p \geq 1$, and*

$$\bar{\gamma}^k(p) < \infty. \quad (1.9)$$

For the following theorem, we assume that $u_0 \geq 0$ is sufficiently large.

Theorem 1.0.2. 1. When $\sigma(x) = \lambda x$, then (1.8) holds for all $k \geq 1$.

2. Furthermore, assume $q_0 := \liminf_{x \neq 0} |\sigma(x)/x| > 0$. Then

(a) If σ is differentiable with a bounded derivative, then (1.8) holds for $k = 1$.

(b) If σ is twice differentiable with bounded derivatives and $\inf_x |\sigma'(x)| > 0$, then (1.8) holds for $k = 1, 2$.

Chapter 5 of this thesis is devoted to the proof of the existence of the Liapounov exponents and the intermittency of the Malliavin derivatives $D^k u$. These results can be interpreted as an indication of a “chaotic” behavior of the system defined by (1.1). We also establish sufficient conditions for the existence of a smooth density for $u(t, x)$; see Theorem 1.0.2.

By replacing σ with $\lambda\sigma$, where $\lambda > 0$, we compare the dependencies of the upper and lower bounds of the Liapounov exponents on λ . We note that the upper and lower bounds have the same forms of dependency on λ . More precisely, we have

$$\Upsilon^{-1} \left(\frac{1}{c_0^2 \lambda^2} \right) \leq \bar{\gamma}^1(p) \leq \frac{p}{2} \Upsilon^{-1} \left(\frac{1}{2\lambda^2 |\sup \sigma'|^2 (1 + z_p^2)} \right),$$

where Υ^{-1} is the pseudo inverse of Υ , and defined by

$$\Upsilon^{-1}(\theta) := \inf\{\beta > 0 : \Upsilon(\beta) < \theta\},$$

$c_0 \in (0, q_0)$ and z_p is the optimal constant in Burkholder-Davis-Gundy inequality; see [23] and the references therein for more details on the optimal constants z_p .

The most common application of the Malliavin calculus is the investigation of the existence of the densities. The following result is in this direction.

Theorem 1.0.2. Let u be the mild solution to the equation (1.1), where $\sigma \in C_b^\infty(\mathbf{R})$ and suppose that there is a $\kappa > 0$ such that $\inf_x \sigma(x) \geq \kappa > 0$. Assume that there exist finite constants $c, C \geq 0$ and $1 < \alpha \leq \beta \leq 2$, such that

$$c|n|^\alpha \leq \operatorname{Re} \Phi(n) \leq C|n|^\beta, \tag{1.10}$$

for all $n \geq 1$. If $\alpha \geq 2\beta/(\beta + 1)$, then $u(t, x)$ has a smooth density $p_t(x)$ at every $t > 0$ and $x \in \mathbf{T}$. This holds, in particular when

$$c n^{\frac{4}{3}+\epsilon} \leq \operatorname{Re} \Phi(n) \quad \forall n \geq 1,$$

where $0 < \epsilon < \frac{2}{3}$.

We would like to remark that when (1.1) is linear, and $\alpha, \beta \leq 1$, a solution does *not* exist. This observation might explain why in the nonlinear case of (1.1) we considered $\alpha > 1$; see Theorem 1.0.2.

Some variants of this result can be found in the literature. For example the case of $\mathcal{L} = \Delta$, Laplacian, which is the well-known Stochastic Heat Equation (SHE), is treated in [4]. In that article, the authors considered the Neumann boundary condition and showed that, if σ is infinitely differentiable with bounded derivatives, then the law of any vector $(u(t_1, x), \dots, u(t_n, x))$ has a smooth and strictly positive density with respect to Lebesgue measure on the set $\{\sigma > 0\}^d$. In [50] the authors studied (1.1), for $\mathcal{L} = \Delta$ on the interval $[0, 1]$, with Dirichlet boundary conditions. They showed that if σ is Lipschitz, then the following holds true.

Theorem 1.0.3. *Let $(t, x) \in (0, \infty) \times (0, 1)$. The law of $u(t, x)$ is absolutely continuous with respect to Lebesgue measure if there exists $x_0 \in [0, 1]$ such that $\sigma(u_0(x_0)) \neq 0$.*

The regularity of the density, under the condition $\sigma(u_0(x_0)) \neq 0$ for some x_0 , is an open problem [15, page 99]. For the same equation the smoothness of the density was proved by Muller and Nualart [43]. They assumed that σ is infinitely differentiable with bounded derivatives. We also name [53, 33] as examples of Malliavin calculus for the SHEs on $\mathbf{R}^d$ with colored noise. In [33], Hu, Nualart and Song considered the solution to (1.1) in which $\dot{w}(t, x)$ denotes a colored noise with covariance function $\mathbb{E} \dot{w}(t, x) \dot{w}(s, y) = t \wedge q(x, y)$, for a γ_0 -Hölder continuous function q that satisfies $|q(x_1, x_2)| \leq C(1 + |x_1|^\beta + |x_2|^\beta)$, for some $\beta \in [0, 2)$. They further assumed that there is some $\gamma > -1$ such that for each $t \geq 0$,

$$\sup_{x \in \mathbf{R}^d} \int_{\mathbf{R}^{2d}} p_t(x, z_1) p_t(x, z_2) q(z_2, z_2) dz_2 dz_2 \geq Ct^\gamma.$$

They proved the following result.

Theorem 1.0.4. *Suppose there is $x_0 \in \mathbf{R}$ such that $u_0(x_0) \neq 0$, and $q(x_0, x_0) \neq 0$, where u_0 is a bounded Hölder continuous function. The following holds true: If σ is infinitely differentiable with bounded derivatives of all orders, then for any $t > 0$ and $x \in \mathbf{R}^d$, the probability law of $u(t, x)$ has a smooth density with respect to Lebesgue measure.*

To the best of our knowledge, the case of the SHE with multiplicative white noise on $\mathbf{R}$ is not studied yet. See section 6.0.1 for more details.

Proposition 4.2.1 below, allows us to find a bound for density in Theorem 1.0.2. To state the result we define the following quantities. Let $C = 2C_\alpha/c^{1/\alpha}$, where c is defined in (1.10) (also defined as C_1 in Hypothesis **H2** below), and $C_\alpha = \int_0^\infty \frac{dx}{1+x^\alpha} = \pi/\alpha \csc(\pi/\alpha)$. Define ν , d , and b by

$$\nu = \frac{2\alpha - 1}{\alpha - 1}, \quad b > \frac{2\text{Lip}_\sigma^2}{\pi} \vee \left(\frac{2\text{Lip}_\sigma^2 C}{\pi} \right)^{\frac{\alpha}{\alpha-1}}, \quad d = \frac{1 - \nu}{4(b\nu^\nu)^{1/(\nu-1)}}. \quad (1.11)$$

Note that $d < 0$, as $\nu > 2$.

Corollary 1.0.5. *Let $p_{t,y}(x)$ denote the density of $u(t, y)$ and let b and ν be defined as above. There are $K > 0$ and $\beta_0 > 0$ such that*

$$p_{t,y}(x) \leq K \exp(\beta_0 t + \Lambda(t, x)), \quad (1.12)$$

where $\Lambda(t, x) \leq 0$ uniformly in $t > 0$ and $x \in \mathbf{R}$, and is defined by

$$\Lambda(t, x) = \begin{cases} d t^{-1/(\nu-1)} (\ln(|x|/m))^{\nu/(\nu-1)} & |x| \geq m, \\ 0 & |x| \leq m, \end{cases} \quad (1.13)$$

and $m = 2 \sup_{x \in \mathbf{T}} |u_0(x)| + 2|\sigma(0)/\text{Lip}_\sigma|$.

Remark 1.0.6. *Since $\nu > 1$, we have $\Lambda(t, x) < 0$. We can also define $b > \frac{\text{Lip}_\sigma^2}{(1-\epsilon)\pi} \vee \left(\frac{\text{Lip}_\sigma^2 C}{\epsilon\pi} \right)^{\frac{\alpha}{\alpha-1}}$, for any $\epsilon \in (0, 1)$.*

The proof of this corollary is in Chapter 4. The global nature of this estimate makes it different from similar results in [17, 18, 20, 19]. In [17, 18], Dalang, Khoshnevisan and

Nualart study the stochastic heat equation with additive and multiplicative white noise respectively. They find a Gaussian upper bound, which works only for t and y in compact subsets of $\mathbf{R}_+ \times (0, 1)$. This is in contrast with our result which holds for all $t > 0$ and $y \in [0, 1]$. In [20], Dalang and Sanz-Solé investigate the hitting probability of a stochastic wave equation with colored noise.

The Kardar-Parisi-Zhang (KPZ) equation [35, 5], $\partial_t h = -(\partial_x h)^2 + \partial_x^2 h + \dot{w}$, which is a standard model for random interface growth, is related to (1.1). KPZ is ill-posed. However, if we apply the Hopf-Cole nonlinear transformation [7], $u(t, x) := \exp\{-h(t, x)\}$, then u solves the well-posed SHE $\partial_t u = \partial_x^2 u + u\dot{w}$. This is a special case of (1.1), in which $\mathcal{L} = \partial_x^2$ and $\sigma(x) = x$. The Hopf-Cole transformation suggests that we can define the solution h to KPZ via the well-defined random field u by

$$h(t, x) := -\log u(t, x). \quad (1.14)$$

For more details we refer to [7, 28]. If we start with $u_0(x) > 0$ for all $x \in \mathbf{T}$, then $u(t, x) > 0$ for all $(t, x) \in E$ by the Mueller's comparison theorem; see [15, Theorem 5.1] or [42, section 3]. Intuitively, when u is small, $u(t, x)\dot{w}$ is small. Therefore, when $u \ll 1$, the effect of noise become negligible and the equation behaves like the nonrandom heat equation.

One important feature of the KPZ equation is its scaling limit behavior and universality [1, 3, 22, 25]. A growth model which has a long time behavior similar to that of KPZ is in KPZ universality class. Physicists employed the renormalization group method and computed the dynamic scaling exponent $z = 3/2$; see [26, 5, 35]. This means that, under the rescaling $h_\epsilon(t, x) = \epsilon^{1/2}h(\epsilon^{-z}t, \epsilon^{-1}x)$, we have a nontrivial limit, as $\epsilon \downarrow 0$. For a brief introduction to KPZ, we refer to the unpublished survey [14].

Corollary 1.0.5 implies that a similar bound for the density of the KPZ equation might hold true. Although, our result is not directly applicable to $\sigma(x) = x$, it certainly covers the case of $\sigma(x) = x + \epsilon$, for all $\epsilon > 0$. This solution converges to the Hopf-Cole solution of KPZ as $\epsilon \downarrow 0$.

To elaborate more, let $u^\epsilon(t, y)$ and $p_{t,y}^\epsilon(x)$ denote the solution and density of the perturbed

equation respectively. We also let $\sup_{x \in \mathbf{T}} |u_0(x)|$ be a small positive number. If $\epsilon > 0$ is sufficiently small, then we have $m < 1$, and by Remark 1.0.6, $b = 1$ is admissible. In particular, we find an upper bound for $p_{t,y}^\epsilon(x)$, which is independent of ϵ ,

$$p_{t,y}^\epsilon(x) \leq K \exp \left[\beta_0 t - c \frac{(\ln |x|)^{3/2} \mathbf{1}_{|x| \geq 1}}{\sqrt{t}} \right], \quad (1.15)$$

where $c = (12\sqrt{3})^{-1}$. As $\epsilon \downarrow 0$, we expect $P(u^\epsilon(t, x) \leq -e^{-x}) \downarrow 0$, by the Mueller's comparison theorem. Therefore,

$$\begin{aligned} \frac{d}{dx} P(-\log |u^\epsilon(t, x)| \leq x) &= \frac{d}{dx} P(u^\epsilon(t, x) \geq e^{-x}) + \frac{d}{dx} P(u^\epsilon(t, x) \leq e^{-x}) \\ &\approx -\frac{d}{dx} P(u^\epsilon(t, x) \leq e^{-x}) \leq K \exp \left[\beta_0 t - c \left(\frac{|x|^{3/2}}{\sqrt{t}} + x \right) \mathbf{1}_{|x| \geq 1} \right]. \end{aligned}$$

Since the right-hand-side is independent of ϵ , one might be able to take the limit as $\epsilon \downarrow 0$, and show that the density $\bar{p}_{t,y}(x)$ of KPZ satisfies

$$\bar{p}_{t,y}(x) \leq K \exp \left[\beta_0 t - c \left(\frac{|x|^{3/2}}{\sqrt{t}} + x \right) \mathbf{1}_{|x| \geq 1} \right],$$

for any bounded $u_0(x)$.

The technique of Malliavin calculus is normally implemented in two steps:

Step 1 is to prove that the solution is smooth; i.e., the existence of the Malliavin derivatives of all orders, and

Step 2 is the proof of the nondegeneracy of the Malliavin matrix; i.e., the study of the corresponding Malliavin matrix and existence of the negative moments.

In “Step 1” we offer a new method, which, in contrast to the other works [4, 50], does not rely on the approximations that use the detailed features of the transition probabilities of the Lévy process. This feature of our proof has enabled us to prove the Malliavin differentiability of the solution for all Lévy processes for which the existence of the mild solution is proved. To emphasize, we mention that, in this step we only require that Hypothesis **H1** holds.

In “Step 2” we followed carefully [15, pages 97-98], and could find an “ ϵ -room” to extend the results from Brownian motion to a large group of Lévy processes, characterized by the rate of the growth of their Lévy exponents.

The rest of of this thesis is organized as follows. In chapter 2 we collect some results about Lévy processes that are relevant to our study. We also discuss briefly the Walsh method of integration. This discussion includes the result about the existence and uniqueness of the solution to Eq (1.1). In Chapter 2, we also reviewed some elements of Malliavin calculus as economically as possible. In Chapter 3 we show that the Malliavin derivative of u of all order exists, i.e., the solution to (1.1) is smooth. In Chapter 4 we give a proof for Theorem 1.0.2 and its corollary, Corollary 1.0.5. Chapter 5 start with a short introduction to intermittency. The proof of Theorem 1.0.2 is in this chapter. Finally, Chapter 6 is about the continuation of this project.

CHAPTER 2

Preliminaries and background

2.1 Lévy processes on a torus

In this section we review some results about Lévy processes. This material will be used in the sequel. Let $\{Y_t\}$ be a Lévy process on $\mathbf{R}$. This means that

1. $Y_0 = 0$ a.s.
2. Y has independent and stationary increments.
3. It is stochastically continuous; i.e., for all $\epsilon > 0$ and for all $s \geq 0$,

$$\lim_{t \rightarrow s} P(|Y_t - Y_s| > \epsilon) = 0.$$

4. There is $\Omega_0 \in \mathcal{F}$ with $P(\Omega) = 1$ such that, for every $\omega \in \Omega$, $Y_t(\omega)$ is right-continuous in $t \geq 0$ and has left limits in $t > 0$.

Let φ denote the characteristic exponent of Y ; i.e,

$$E e^{i\lambda(Y_{s+t} - Y_s)} = e^{-t\varphi(\lambda)}, \quad s, t \geq 0, \lambda \in \mathbf{R}. \quad (2.1)$$

The existence of the characteristic exponents for the rational numbers is a result of the stationary and independent increments (property 2.) of the Lévy process. It extends to the real numbers by the *càdlàg* property (property 4) of the paths.

As we discussed before — see the paragraph before (1.3)— we assume the following:

H 1. Let $\bar{Y}_t = Y_t - Y'_t$, where Y' is an independent copy of Y . $\bar{Y}_t$ has local times.

Combining Theorem 1.0.1 with the following theorem from [32] and the fact that $\bar{Y}_t$ is a Lévy process with characteristic exponent $2\operatorname{Re} \varphi$ imply that $\Upsilon(\beta) < \infty$, for all $\beta > 0$.

Theorem 2.1.1 (Hawkes [32]). *Let X be a Lévy process having exponent φ . Then a local time exists if and only if*

$$\operatorname{Re} \left(\frac{1}{1 + \varphi} \right) \in L^1(\mathbf{R}). \quad (2.2)$$

Lemma 8.1 in [24] tells us that under hypothesis **H1** process Y_t has transition densities $\{p_t(x, y)\}$ such that $\int_{\mathbf{T}} p_t(x, y)^2 dy < \infty$ for all $x \in \mathbf{T}$. More precisely,

Theorem 2.1.2 (Foondun-Khoshnevisan-Nualart). *If (1.2) has a random-field solution, then the process Y has a jointly measurable transition density $\{p_t(x)\}_{t>0, x \in \mathbf{R}}$ that satisfies the following: For all $\eta > 0$ there exists a constant $C := C_\eta \in (0, \infty)$ such that for all $t > 0$,*

$$\int_0^t \|p_s\|_{L^2(\mathbf{R})}^2 ds \leq C e^{\eta t}. \quad (2.3)$$

Let $\mathbf{T} := [0, 2\pi)$. Define a process X_t on $\mathbf{T}$, via Y_t , by

$$X_t := Y_t - 2n\pi \quad \text{when} \quad 2n\pi \leq Y_t < 2(n+1)\pi. \quad (2.4)$$

Let $\{q_t(x, \cdot)\}_{x \in \mathbf{T}}$ denote the transition probability densities for the process X . A simple calculation shows that the transition densities of X are given by

$$q_t(x, y) = \sum_{n=-\infty}^{\infty} p_t(x, y + 2n\pi) \quad \forall x, y \in \mathbf{T}. \quad (2.5)$$

Let us introduce a function $\Phi : \mathbf{Z} \rightarrow \mathbf{C}$ by

$$\Phi(n) = \varphi(n) \quad n \in \mathbf{Z}. \quad (2.6)$$

As is shown below, Φ is the characteristic exponent of the process X . It is clear from the definition of Φ that $\Upsilon(\beta) < \infty$ for all $\beta > 0$, when Hypothesis **H2** below holds; i.e., **H2** implies **H1**. The function Υ continues to have a crucial role in “Step 1” above. The convergence of all Picard iterations relies on the fact that $\Upsilon(\beta) \rightarrow 0$ as $\beta \rightarrow \infty$.

The L^2 generator of X is defined via the concept of the Fourier Multiplier. To explain this, we start with the definition of the Fourier integrals. For every $g \in L^2(\mathbf{T})$ we have

$$g(x) = \sum_{n=-\infty}^{\infty} \hat{g}(n) e^{-inx},$$

where

$$\hat{g}(n) = \frac{1}{2\pi} \int_{\mathbf{T}} e^{inx} g(x) dx.$$

Here the convergence of the series holds in $L^2(\mathbf{T})$. Since $q_t(x, y)$ is a function of $y - x$ for each $t \geq 0$, we occasionally abuse notation and write $q_t(y - x)$ instead of $q_t(x, y)$.

Lemma 2.1.3. *Under Hypothesis **H1**, $q_t(x, \cdot) \in L^2(\mathbf{T})$ for all $x \in \mathbf{T}$ and $t > 0$. Furthermore,*

$$\hat{q}_t(x, n) = \frac{1}{2\pi} e^{inx} e^{-t\Phi(n)}, \quad \|q_t(\cdot)\|_{L^2(\mathbf{T})}^2 = \frac{1}{4\pi^2} \sum_{n=-\infty}^{\infty} e^{-2t\operatorname{Re} \Phi(n)}, \quad (2.7)$$

and

$$q_t(x, y) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} e^{inx} e^{-t\Phi(n)} e^{-iny}. \quad (2.8)$$

Proof. In order to show that $q_t(x, \cdot) \in L^2([0, 2\pi))$ we need only to show that its Fourier coefficients are in $\ell^2(\mathbf{Z})$. We can write $\hat{q}_t$ in terms of Φ as follows:

$$\hat{q}_t(x, n) = \frac{1}{2\pi} \int_{\mathbf{T}} e^{iny} q_t(y - x) dy = \frac{1}{2\pi} e^{inx} \int_{-\infty}^{\infty} e^{inz} p_t(z) dz = \frac{1}{2\pi} e^{inx} e^{-t\Phi(n)}. \quad (2.9)$$

Therefore, Φ is the characteristic exponent of the Lévy process X_t . To prove the second formula, we need only to show that the sum in (2.7) converges, because then this equation would be the Parseval identity. An application of Fubini and **H1** imply that

$$\int_0^{\infty} \sum_{n=-\infty}^{\infty} e^{-(\beta t + 2t\operatorname{Re} \Phi(n))} dt = 4\pi^2 \Upsilon(\beta) < \infty. \quad (2.10)$$

Therefore, by the continuity of the integrand,

$$\sum_{n=-\infty}^{\infty} e^{-(\beta t + 2t\operatorname{Re} \Phi(n))} < \infty \quad \forall t > 0.$$

Therefore $\sum_{n=-\infty}^{\infty} e^{-2t\operatorname{Re} \Phi(n)} < \infty$ for all $t > 0$. Finally, (2.8) is a consequence of the inversion formula. □

The transition densities q_t induce a semigroup T_t on $L^2(\mathbf{T})$ defined by

$$T_t f(x) = \mathbb{E}^x f(X_t) := \int_{\mathbf{T}} f(y) q_t(x, y) dy. \quad (2.11)$$

Lemma 2.1.4. *The semigroup operator defined in (2.11) is a convolution operator and*

$$T_t f(x) = \sum_{n=-\infty}^{\infty} e^{-inx} e^{-t\Phi(-n)} \hat{f}(n). \quad (2.12)$$

Proof. Since

$$T_t f(x) = \int_{\mathbf{T}} f(y) \frac{1}{2\pi} \left(\sum_{n=-\infty}^{\infty} e^{inx} e^{-t\Phi(n)} e^{-iny} \right) dy,$$

an application of Fubini gives us the result. $\square$

Let $\mathcal{L}$ be the generator of X_t in L^2 sense. This means

$$\mathcal{L}f(x) = \lim_{t \rightarrow 0^+} \frac{T_t f(x) - f(x)}{t} \text{ in } L^2(\mathbf{T}),$$

whenever the limit exists. It is natural to define

$$\text{Dom}[\mathcal{L}] := \left\{ \varphi \in L^2(\mathbf{T}) : \mathcal{L}(\varphi) := \lim_{t \rightarrow 0^+} \frac{T_t \varphi - \varphi}{t} \text{ exists in } L^2(\mathbf{T}) \right\}.$$

Next, we characterize $\text{Dom}[\mathcal{L}]$ in terms of the characteristic exponent.

Proposition 2.1.5. *We have*

$$\text{Dom}[\mathcal{L}] = \left\{ f \in L^2(\mathbf{T}) : \sum_{n=-\infty}^{\infty} |\Phi(n)|^2 |\hat{f}(n)|^2 < \infty \right\}.$$

Proof. From the definition and the continuity of the Fourier transform,

$$\widehat{\mathcal{L}f}(n) = \lim_{t \rightarrow 0^+} \frac{\widehat{T_t f}(n) - \hat{f}(n)}{t} = \widehat{\mathcal{L}f}(n) = \hat{f}(n) \lim_{t \rightarrow 0^+} \frac{e^{-t\Phi(-n)} - 1}{t} = -\Phi(-n) \hat{f}(n)$$

Then $\Phi(n) \hat{f}(n) \in \ell^2(\mathbf{Z})$. Since $\Phi(-n) = \overline{\Phi(n)}$, this is equivalent to what we wanted to prove. $\square$

Therefore $\mathcal{L}$ can be viewed as a convolution operator with Fourier multiplier $\hat{\mathcal{L}}(n) = -\Phi(-n)$. We state the result as follows.

Lemma 2.1.6. *The $L^2(\mathbf{T})$ generator $\mathcal{L}$ of T_t can be written as*

$$\mathcal{L}u_0(x) = - \sum_{n=-\infty}^{\infty} e^{inx} \Phi(-n) \hat{u}_0(n) \quad x \in [0, 2\pi), \quad (2.13)$$

for all $u_0 \in L^2(\mathbf{T})$.

We borrow the following lemma from [23]; it plays a key role in the proof of the existence of Malliavin derivatives.

Lemma 2.1.7. *For all $\beta > 0$,*

$$\sup_{t>0} e^{-\beta t} \int_0^t \|q_s\|_{L^2(T)}^2 ds \leq \int_0^\infty e^{-\beta s} \|q_s\|_{L^2(T)}^2 ds = \Upsilon(\beta). \quad (2.14)$$

Proof. Since $e^{-\beta t} \leq e^{-\beta s}$ for all $s \leq t$, then we have the inequality. The equality follows from (2.7) and (2.9),

$$\begin{aligned} \int_0^\infty e^{-\beta s} \|q_s\|_{L^2(T)}^2 ds &= \int_0^\infty e^{-\beta s} \sum_{n=1}^\infty |\hat{q}_s(x, n)|^2 ds \\ &= \sum_{n=-\infty}^\infty \int_0^\infty e^{-\beta s} |\hat{q}_s(x, n)|^2 ds = \Upsilon(\beta). \end{aligned}$$

This finishes the proof. □

We refer the reader to [2, page 172] for further details. The following results are used in “Step 2” of our proof, that is the existence of negative moments.

Lemma 2.1.8. *Let $1 < \alpha \leq 2$. There is $C \in (0, \infty)$ such that*

$$\lim_{\lambda \rightarrow 0} \lambda^{\frac{1}{\alpha}} \sum_{n=1}^\infty e^{-n^\alpha \lambda} = C. \quad (2.15)$$

A proof for the special case of $\alpha = 2$ is given in [15, pages 34-35]. We extend the result to $1 < \alpha \leq 2$ by modifying the same idea. We start with the following lemma.

Lemma 2.1.9. *There is $c < \infty$ depending only on α such that*

$$n^\alpha - x^\alpha \leq cx^{\alpha-1}.$$

for every $x \in [n-1, n]$ and for all $n \geq 1$.

Proof. Since $n^\alpha - x^\alpha \leq n^\alpha - (n-1)^\alpha$, by an application of the mean value theorem,

$$n^\alpha - x^\alpha \leq \alpha n^{\alpha-1}.$$

Since $\lim_{n \rightarrow \infty} \frac{n^{\alpha-1}}{(n-1)^{\alpha-1}} = 1$, there is N such that $n^{\alpha-1} \leq 2(n-1)^{\alpha-1}$ for all $n \geq N$. But for every $n \leq N$, there is C_n such that

$$n^{\alpha-1} \leq C_n(n-1)^{\alpha-1}.$$

Then, if we let $C = \max\{2, C_1, \dots, C_N\}$, we get

$$n^{\alpha-1} \leq C(n-1)^{\alpha-1} \quad n \geq 1.$$

Therefore, the inequality $(n-1)^{\alpha-1} \leq x^{\alpha-1}$ completes the proof. $\square$

Proof of Lemma 2.1.8. Let $\int_0^\infty e^{-x^\alpha} dx = C$. Then $\int_0^\infty e^{\lambda x^\alpha} dx = \frac{C}{\lambda^{\frac{1}{\alpha}}}$. Let

$$T = \int_2^\infty e^{-x^\alpha \lambda} dx - \sum_{n=3}^\infty e^{-\lambda n^\alpha} \geq 0. \quad (2.16)$$

Since, $1 - e^{-\theta} \leq 1 \wedge \theta$,

$$T = \sum_{n=3}^\infty \int_{n-1}^n e^{-x^\alpha \lambda} (1 - e^{-\lambda(n^\alpha - x^\alpha)}) dx \leq \sum_{n=3}^\infty \int_{n-1}^n e^{-x^\alpha \lambda} (1 \wedge \lambda(n^\alpha - x^\alpha)) dx.$$

Then, by Lemma 2.1.9 followed by a change of variable,

$$T \leq \int_0^\infty e^{-x^\alpha \lambda} (1 \wedge \lambda c x^{\alpha-1}) dx = \frac{1}{\lambda^{\frac{1}{\alpha}}} \int_0^\infty e^{-y^\alpha} (1 \wedge \lambda^{\frac{1}{\alpha}} y) dy.$$

Therefore, $0 \leq \lambda^{\frac{1}{\alpha}} T \leq \int_0^\infty e^{-y^\alpha} (1 \wedge \lambda^{\frac{1}{\alpha}} y) dy$. Since by the dominated convergence theorem

$$\lim_{\lambda \rightarrow 0} \int_0^\infty e^{-y^\alpha} (1 \wedge \lambda^{1-\frac{1}{\alpha}} y) dy = 0,$$

then by the squeeze theorem,

$$\lim_{\lambda \rightarrow 0} \lambda^{\frac{1}{\alpha}} T = 0. \quad (2.17)$$

Since $\lim_{\lambda \rightarrow 0} \lambda^{\frac{1}{\alpha}} \int_2^\infty e^{-y^\alpha \lambda} dy = C > 0$, (2.17) implies that

$$\lim_{\lambda \rightarrow 0} \lambda^{\frac{1}{\alpha}} \sum_{n=1}^\infty e^{-n^\alpha \lambda} = C.$$

$\square$

Next we introduce the second hypothesis.

H 2. *There are $1 < \alpha < \beta \leq 2$ and $0 < C_1 < C_2$ such that*

$$C_1|n|^\alpha \leq \operatorname{Re} \Phi(n) \leq C_2|n|^\beta \quad \forall n \geq 1. \quad (2.18)$$

Corollary 2.1.10. *Let $\Phi(n)$ denotes the Lévy exponent of a Lévy process with transition probability $q = q_t(x)$. If Φ satisfies Hypothesis **H2**, then, for $t \in [0, T]$, there are constants $0 < A_1 < A_2$ depending on T , such that*

1. *For all $t > 0$,*

$$A_1 t^{-\frac{1}{\beta}} \leq \|q_t\|_{L^2(\mathbf{T})}^2 \leq A_2 t^{-\frac{1}{\alpha}}; \quad (2.19)$$

2. *For every $\delta \in (0, T)$,*

$$A_1 \delta^{1-\frac{1}{\beta}} \leq \int_0^\delta \|q_t\|_{L^2(\mathbf{T})}^2 dt \leq A_2 \delta^{1-\frac{1}{\alpha}}. \quad (2.20)$$

Proof. We prove only the first part; the second part follows from the first part by integration.

It follows from (2.18) and (2.7) that

$$\sum_{n=-\infty}^{\infty} e^{-2tC_2|n|^\beta} \leq 4\pi^2 \|q_t\|_{L^2(\mathbf{T})}^2 \leq \sum_{n=-\infty}^{\infty} e^{-2tC_1|n|^\alpha}. \quad (2.21)$$

The first inequality in (2.21) implies that

$$t^{1/\beta} \|q_t\|_{L^2(\mathbf{T})}^2 \geq \frac{1}{(2C_2)^{1/\beta} 4\pi^2} (2tC_2)^{1/\beta} \sum_{n=-\infty}^{\infty} e^{-2tC_2|n|^\beta}. \quad (2.22)$$

By (2.15) the right-hand-side of (2.22) converges to a number $B_1 > 0$. Therefore there is an $\epsilon_1 > 0$ such that

$$t^{1/\beta} \|q_t\|_{L^2(\mathbf{T})}^2 \geq B_1/2 \quad \forall t \in (0, \epsilon_1).$$

To extend the inequality to $t \in (0, T)$, we note that by (2.15), $q_t \neq 0$ and is continuous for $t > 0$. Therefore, there is $A_1 > 0$ such that

$$t^{1/\beta} \|q_t\|_{L^2(\mathbf{T})}^2 \geq A_1 \quad \forall t \in (0, T). \quad (2.23)$$

Similarly, the second inequality in (2.21) implies that there is A_2 such that

$$t^{1/\beta} \|q_t\|_{L^2(\mathbf{T})}^2 \leq A_2 \quad \forall t \in (0, T). \quad (2.24)$$

Inequalities (2.23) and (2.24) imply (2.19). $\square$

2.2 A Stochastic Partial Differential Equation

Equation (1.1) is formal; we interpret it, in the Walsh sense, as the solution to the integral equation

$$u(t, x) = v(t, x) + \int_{\mathbf{T}} \int_0^t \sigma(u(s, y)) q_{t-s}(y - x) w(ds dy), \quad (2.25)$$

where $v(t, x) = T_t u(x)$, and the integral on the right-hand-side is with respect to white noise.

The white noise also defines a filtration $\{\mathcal{F}_t\}_{t \geq 0}$ via

$$\mathcal{F}_t = \sigma(\dot{w}([0, s] \times A), 0 \leq s \leq t, \text{ and } A \in \mathcal{B}(\mathbf{T})),$$

where $\mathcal{B}(\mathbf{T})$ denotes the Borel σ -algebra on $\mathbf{T}$ equipped with Lebesgue measure, normalized to have mass 1. In (2.25) a solution u that satisfies (2.25) is called a *mild solution* to (1.1), if

$$\sup_{(t, x) \in [0, T] \times \mathbf{T}} \mathbb{E}(|u(t, x)|^2) < \infty \quad \text{for all } T < \infty;$$

see [23, page 4]. We assume $\mathcal{F}_t$ satisfies the *usual condition* for all $t \geq 0$. This means that $\mathcal{F}_t$ is right continuous and contains all the null sets [51, page 22].

To better understand what this equation represents, consider the following noninteracting particle system in a random environment. Particles are initially distributed on $\mathbf{T}$ according to the initial density $u_0(x)$. At time $t = 0$ particles start a continuous time random walk on the torus. The motion of the particles is governed by the generator of the Lévy process. At each time-point $(t, x) \in E$, particles either multiply or die at a rate proportional to the amount of the noise at (x, t) . This branching mechanism is responsible for the term $\sigma(u(t, x))\dot{w}$ in (1.1). In this model the diffusive effect of the operator $\mathcal{L}$ competes with the white noise. While the diffusion is trying to flatten the solution, the noise roughens it up.

White noise $\dot{w}$ is a continuous analogue of a sequence of i.i.d. Gaussian random variables. Roughly speaking, it is a Gaussian process with covariance function,

$$\mathbb{E} \dot{w}(s, x) \dot{w}(t, y) = \delta(t - s) \delta(x - y)$$

More precisely, let $\mathcal{B}(E)$ denote the Borel σ -algebra on E , and let $|\cdot|$ denote the product Lebesgue measure on $\mathcal{B}(E)$. White noise is a centered Gaussian process indexed by the

elements in $\mathcal{B}(E)$, whose covariance function $C : \mathcal{B}(E) \times \mathcal{B}(E) \rightarrow \mathbf{R}$ is defined by

$$C(A, B) = |A \cap B|. \quad (2.26)$$

Although $\dot{w}(A \cup B) = \dot{w}(A) + \dot{w}(B)$ a.s. for all disjoint A and B , white noise is not a σ -additive set function and fails to define a pathwise signed measure. However, for all disjoint sets $A_1, A_2, \dots \in \mathcal{B}(E)$,

$$\mathbf{P} \left\{ \dot{w} \left(\bigcup_{n=1}^{\infty} A_n \right) = \sum_{i=1}^{\infty} \dot{w}(A_n) \right\} = 1, \quad (2.27)$$

where the infinite sum converges in $L^2(\mathbf{P})$. To define an Itô type integral against $\dot{w}$, for every $A \in \mathcal{B}(E)$ we set

$$\int \mathbf{1}_A(t, x) w(dt, dx) = \dot{w}(A).$$

By linearity of integration, we can define $\int f(t, x) w(dt, dx)$ for every f of form

$$f(t, x) = \sum_{i=1}^n c_i \mathbf{1}_{A_i}(t, x),$$

where $A_i \in \mathcal{B}(E)$ for $i = 1, \dots, n$. Then the Itô isometry,

$$\|f\|_{L^2(E)} = \left\| \int f(t, x) w(dt, dx) \right\|_{L^2(\Omega)},$$

allows us to extend this definition to all functions in $L^2(E)$. It remains to define the integral for the random integrands, which will be discussed briefly next. Let $A = [0, t] \times B$, where $B \subset \mathcal{B}(\mathbf{T})$. For sets of this form we define $w_t(B) = \dot{w}(A)$. Then following lemma holds.

Lemma 2.2.1. $\{w_t(B)\}_{t \geq 0, B \in \mathcal{B}(\mathbf{T})}$ is a “martingale measure” in the sense that:

1. For all $A \in \mathcal{B}(\mathbf{T})$, $w_0(B) = 0$ a.s.;
2. If $t > 0$ when w_t is a sigma-finite, $L^2(\mathbf{P})$ -valued signed measure; and
3. For all $B \in \mathcal{B}(\mathbf{T})$, $\{w_t(B)\}_{t \geq 0}$ is a mean-zero martingale.

Proof. See [15] page 15. □

Definition 1. A function $f : \mathbf{R}_+ \times \mathbf{T} \times \Omega \rightarrow \mathbf{R}$ is *elementary*, if for some X and A ,

$$f(t, x, \omega) = X(\omega) \mathbf{1}_{(a,b]}(t) \mathbf{1}_A(x), \quad (2.28)$$

where X is bounded and $\mathcal{F}_a$ measurable random variable, and $A \in \mathcal{B}(\mathbf{T})$. Finite [nonrandom] linear combination of elementary functions are called *simple functions*. Let $\mathcal{S}$ denote the class of all simple functions.

If f is an elementary function, then we define the stochastic-integral process of f as

$$\int_0^t \int_B f(s, x, \omega) w(ds, dx) = X[w_{t \wedge b}(B \cap A) - w_{t \wedge a}(B \cap A)], \quad (2.29)$$

for every $B \in \mathcal{B}(\mathbf{T})$. This definition extends to $\mathcal{S}$ by linearity. The sigma algebra $\mathcal{P}$ generated by $\mathcal{S}$ is called the *predictable σ -algebra*. We restrict the time variable t in a finite interval $[0, T]$ and let $\mathcal{P}_w$ denote the collection of predictable functions f such that

$$\mathbb{E} \left[\int_0^T \int_{\mathbf{T}} |f(s, x)|^2 ds dx \right] < \infty,$$

where the index w in $\mathcal{P}_w$ emphasizes the dependency on the white noise. Let $A, B \in \mathcal{B}(\mathbf{T})$. By Lemma 2.2.1, $\{w_t(A)\}_{t \geq 0}$ and $\{w_t(B)\}_{t \geq 0}$ are martingales, and we have

$$\langle w(A), w(B) \rangle_t = t|A \cap B|, \quad (2.30)$$

where $\langle \cdot, \cdot \rangle_t$ denotes the *covariance process* of the two martingales [36, page 205]. One way to check (2.30) is to observe that $w_t(A)/\sqrt{|A|}$ is a standard Brownian motion for all $A \in \mathcal{B}(\mathbf{T})$. (2.30) is best appreciated in view of the Burkholder inequality and (2.34), as it leads to the following statement.

Theorem 2.2.2. Let $f \in \mathcal{P}_w$, and define $(f \cdot w) := \int_0^\bullet \int_{\mathbf{T}} f(s, x) w(ds, dx)$. For all $t \in (0, T]$ and $A, B \in \mathcal{B}(\mathbf{T})$,

$$\langle (f \cdot w), (f \cdot w) \rangle_t = \int_0^t \int_{\mathbf{T}} |f(s, x)|^2 ds dx. \quad (2.31)$$

Furthermore, we have the following Itô type isometry:

$$\mathbb{E} [(f \cdot w)]^2 = \mathbb{E} \left[\int_{\mathbf{T}} \int_0^t |f(s, x)|^2 ds dx \right]. \quad (2.32)$$

Proof. A proof for a general martingale measure can be found in [15, page 21]. $\square$

Let $\{f(t, x)\}_{t \geq 0, x \in \mathbf{T}}$ be a predictable random field. For any $T > 0$, $\beta > 0$ and $p \geq 2$ define

$$\|f\|_{\beta, p, T} := \left\{ \sup_{(t, x) \in [0, T] \times \mathbf{T}} e^{-\beta t} \mathbf{E}(|f(t, x)|^p) \right\}^{1/p}. \quad (2.33)$$

To analyze this family of p -norms we will need the following inequality. It is an $L^p(\mathbf{P})$ version of (2.32), which can be proved by the Itô formula and Doob's inequality.

Theorem 2.2.3. *[Burkholder-Davis-Gundy Inequality[10]] Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbf{P})$ be a filtered probability space. Let $p > 0$. There exist two universal constants C_p and c_p , depending only on p , such that for every $(\mathcal{F}_t)_{t \geq 0}$ continuous local martingale M , with $M_0 = 0$, and any stopping time ρ , we have*

$$c_p \mathbf{E}(\langle M \rangle_\rho^{1/2})^p \leq \mathbf{E}(\sup_{s \leq \rho} |M_s|)^p \leq C_p \mathbf{E}(\langle M \rangle_\rho^{1/2})^p, \quad (2.34)$$

where the optimal constant C_p , when $\rho = t$, and $p \geq 2$ is given by

$$z_p := \text{optimal } C_p = \sup \left\{ \frac{\|N_t\|_{L^p(\mathbf{P})}}{\|\langle N, N \rangle^2\|_{L^{p/2}(\mathbf{P})}} : N \in \mathfrak{M}_p \right\}, \quad (2.35)$$

where $0/0 = 0$, and $\mathfrak{M}_p$ denotes the collection of all continuous $L^p(\mathbf{P})$ martingales.

Most of above discussion is borrowed from [15]. For more details on the theory and examples of the stochastic partial differential equations we also refer to [13]. We aim next to prove the existence and uniqueness of the solution to (1.1). It is well-known that the stochastic heat equation (1.1), in which $\mathcal{L} = \Delta$, has solution $\{u(t, x)\}_{t \geq 0, x \in \mathbf{R}}$ that is jointly continuous. The solution is unique up to modification. This result can be found, for example in Walsh[55, page 312], see also [16]. This special case arises also in the study of stochastic Burger equation[30], the parabolic Anderson Model [12] and KPZ equation [35].

In order to analyze the mild solution (2.25), we define an operator $\mathcal{A}$ by

$$(\mathcal{A}f)(t, x) = \int_0^{2\pi} \int_0^t q_{t-s}(x, y) \sigma(f(s, y)) w(ds dy) \quad \forall x \in [0, 2\pi), t \geq 0, \quad (2.36)$$

provided that the stochastic integral exists in the sense of Walsh [55].

Lemma 2.2.4. *If f is predictable and $\|f\|_{\beta,p} < \infty$ for a real $\beta > 0$ and $p \geq 1$, then*

$$\|\mathcal{A}f\|_{\beta,p} \leq z_p (\sigma(0) + \text{Lip}_\sigma \|f\|_{\beta,p}) \sqrt{\Upsilon(2\beta/p)}, \quad (2.37)$$

where Υ and q_t correspond to the Lévy process X .

Proof. The integrand in $\mathcal{A}f(t, x)$ depends on t . Therefore, $\mathcal{A}f(t, x)$ is not a martingale. However, if we choose and fix $r > 0$, then the operator

$$A_r f(t, x) := \int_0^t \int_{\mathbf{T}} q_{r-s}(x, y) \sigma(f(s, y)) w(ds, dy)$$

is a martingale for all $t \leq r$, with quadratic process,

$$\langle A_r f(\bullet, x) \rangle_t = \int_0^t \int_{\mathbf{T}} q_{t-s}^2(x, y) |\sigma(f(s, y))|^2 dy ds.$$

If we apply the Burkholder inequality to $\mathcal{A}_r f(t, x)$, for $t \leq r$,

$$\|\mathcal{A}_r f(t, x)\|_{L^p(\mathbf{P})}^p \leq z_p^p \mathbb{E} \left| \int_0^t \int_{\mathbf{T}} q_{r-s}^2(x, y) |\sigma(f(s, y))|^2 dy ds \right|^{p/2}.$$

Now let $t = r$ to get

$$\|\mathcal{A}f(t, x)\|_{L^p(\mathbf{P})}^2 \leq z_p^2 \left\| \int_0^t \int_{\mathbf{T}} q_{r-s}^2(x, y) |\sigma(f(s, y))|^2 dy ds \right\|_{L^{p/2}(\mathbf{P})}.$$

Minkowski's inequality allows us to switch the norm $[L^{p/2}(\Omega)]$ with the integral $[dy ds]$,

$$\|\mathcal{A}f(t, x)\|_{L^p(\mathbf{P})}^2 \leq z_p^2 \int_0^t \int_{\mathbf{T}} q_{t-s}^2(x, y) \|\sigma(f(s, y))\|_{L^p(\mathbf{P})}^2 dy ds.$$

Since $\sigma(x)$ is Lipschitz, $|\sigma(x)| \leq c_0 + c_1|x|$, where $c_0 = |\sigma(0)|$, and $c_1 = \text{Lip}_\sigma$. This combined with inequality

$$(a + b)^2 \leq (1 + \epsilon^{-1})a^2 + (1 + \epsilon)b^2 \quad a, b \in \mathbf{R}, \epsilon > 0,$$

yields,

$$|\sigma(f(s, y))|^2 \leq (1 + \epsilon^{-1})c_0^2 + (1 + \epsilon)c_0^2|f(s, y)|^2.$$

Therefore, by the triangle inequality,

$$\|\sigma(f(s, y))\|_{L^p(\mathbf{P})}^2 \leq (1 + \epsilon^{-1})c_0^2 + (1 + \epsilon)c_0^2\|f(s, y)\|_{L^p(\mathbf{P})}^2.$$

Then, by replacing $|\sigma|^2$ in (4.1) by this upper bound, and using the fact that $\|f(s, y)\|_{L^2(\mathbf{P})}^2 \leq e^{2\beta s/p} \|f\|_{p,\beta}^2$ we will arrive at

$$\begin{aligned}
\|\mathcal{A}f(t, x)\|_{L^p(\mathbf{P})}^2 &\leq z_p^2(1 + \epsilon^{-1})c_0^2 \int_0^t \|q_{t-s}\|_{L^2(\mathbf{T})}^2 ds \\
&\quad + z_p^2(1 + \epsilon)c_1^2 \|f\|_{p,\beta}^2 \int_0^t \|q_{t-s}\|_{L^2(\mathbf{T})}^2 e^{\frac{2\beta s}{p}} ds \\
&= z_p^2(1 + \epsilon^{-1})c_0^2 \int_0^t \|q_s\|_{L^2(\mathbf{T})}^2 ds \\
&\quad + z_p^2(1 + \epsilon)c_1^2 \|f\|_{p,\beta}^2 e^{2\beta t/p} \int_0^t \|q_s\|_{L^2(\mathbf{T})}^2 e^{-2\beta s/p} ds.
\end{aligned} \tag{2.38}$$

We multiply the above inequality by $e^{-2\beta t/p}$ and recall that by (2.1.7),

$$e^{-2\beta t/p} \int_0^t \|q_s\|_{L^2(\mathbf{T})}^2 ds \leq \int_0^t \|q_s\|_{L^2(\mathbf{T})}^2 e^{-2\beta s/p} ds \leq \Upsilon(2\beta/p),$$

to conclude that

$$e^{-2\beta t/p} \|\mathcal{A}f(t, x)\|_{L^p(\mathbf{P})}^2 \leq z_p^2(1 + \epsilon^{-1})c_0^2 \Upsilon(2\beta/p) + z_p^2(1 + \epsilon)c_1^2 \|f\|_{\beta,p}^2 \Upsilon(2\beta/p).$$

From the definition of $\|\cdot\|_{\beta,p}$, we have

$$\|\mathcal{A}f\|_{\beta,p}^2 \leq [(1 + \epsilon^{-1})c_0^2 + (1 + \epsilon)c_1^2 \|f\|_{\beta,p}^2] z_p^2 \Upsilon(2\beta/p).$$

Finally, we choose

$$\epsilon = \begin{cases} \frac{|c_0|}{c_1 \|f\|_{\beta,p}} & \text{if } c_1 \|f\|_{\beta,p} > 0, \\ 0 & \text{if } c_0 = 0, \\ \infty & \text{if } \|f\|_{\beta,p} = 0, \end{cases} \tag{2.39}$$

and the proof is complete. $\square$

Lemma 2.2.5. *Let $p \geq 2$. For every $\beta > 0$, and all predictable random fields f and g that satisfy $\|f\|_{\beta,p} + \|g\|_{\beta,p} < \infty$,*

$$\|\mathcal{A}f - \mathcal{A}g\|_{\beta,p} \leq z_p \text{Lip}_\sigma \sqrt{\Upsilon(2\beta/p)} \|f - g\|_{\beta,p}. \tag{2.40}$$

Remark 2.2.6. *This result obviously implies the uniqueness of the solution to Eq. (1.1).*

Proof. As we did in the proof of Lemma 2.2.4, we can apply the Burkholder inequality. Then the Lipschitz property of σ yields,

$$\mathbb{E}|\mathcal{A}f(t, x) - \mathcal{A}g(t, x)|^p \leq \text{Lip}_\sigma^p z_p^p \mathbb{E} \left| \int_0^t \int_{\mathbf{T}} q_{t-s}^2(x, y) |f(s, y) - g(s, y)|^2 dy ds \right|^{p/2}.$$

Then after raising the both sides to the power of $2/p$ we apply the Minkowski's inequality on the right-hand-side to conclude that

$$\begin{aligned} \|\mathcal{A}f(t, x) - \mathcal{A}g(t, x)\|_{L^p(\mathbf{P})}^2 &\leq (\text{Lip}_\sigma z_p)^2 \int_0^t \int_{\mathbf{T}} q_{t-s}(x, y) \|f(s, y) - g(s, y)\|_{L^p(\mathbf{P})}^2 dy ds \\ &\leq (\text{Lip} z_p)^2 \|f - g\|_{\beta, p}^2 \int_0^{2\pi} \int_0^t e^{\frac{2\beta s}{p}} q_{t-s}^2(x, y) ds dy. \end{aligned}$$

Then we get

$$\|\mathcal{A}f(t, x) - \mathcal{A}g(t, x)\|_{L^p(\mathbf{P})}^2 \leq (\text{Lip}_\sigma z_p)^2 \|f - g\|_{\beta, p}^2 e^{2\beta t/p} \Upsilon(2\beta/p).$$

This finishes the proof. $\square$

Now, we can prove the following existence theorem. We omit the proof of the L^p continuity of u as it will not be used in the sequel.

Theorem 2.2.7. *Under the hypothesis **H1**, (1.1) has a solution u that is unique up to a modification. The solution is finite in $\|\cdot\|_{\beta, p}$ norm, for some $\beta > 0$, and all $p \geq 2$. Furthermore, when u_0 is continuous, u is continuous in $L^p(\mathbf{P})$ for all $p > 0$.*

Proof. It is easy to check that, if we substitute $t = 0$ in the mild solution given by (2.25), then $u(0, x) = u_0(x)$, and also, if the solution exists, then $u(t, 2\pi) = u(t, 0)$. Notice that if v is defined by

$$v(t, x) = T_t u_0(x) \quad (t, x) \in E, \tag{2.41}$$

then v satisfies $\partial_t v(t, x) = \mathcal{L}v(t, x)$ weakly. Furthermore, the periodic condition $v(t, 0) = v(t, 2\pi)$ on $\mathbf{T}$ and the initial condition $v(0, x) = u_0(x)$ are satisfied. That is v is the Green's

function for the operator $\partial_t - \mathcal{L}$. We consider the following Picard iteration. Define $v_0(t, x) = T_t u_0(x)$, and for $n \geq 1$ set

$$v_{n+1}(t, x) = v_0(t, x) + \int_0^t \int_{\mathbf{T}} q_{t-r}(x, z) \sigma(v_n(r, z)) w(dr dz). \quad (2.42)$$

Then the existence of the solution boils down to the convergence of v_n . We first show, by induction, that $\|\mathcal{A}v_n\|_{\beta, p} < \infty$ for all n . Since u_0 is bounded, then $\|v_0\|_{\beta, p} < \infty$ and then, by Lemma 2.2.4,

$$\|\mathcal{A}v_0\|_{\beta, p} \leq z_p(|\sigma(0)| + \text{Lip}_\sigma \|v_0\|_{\beta, p}) \sqrt{\Upsilon(2\beta/p)} < \infty.$$

Similarly, if $\|v_n\|_{\beta, p} < \infty$, then $\|\mathcal{A}v_n\|_{\beta, p} < \infty$. Then the triangle inequality on (2.42), would give us $\|v_{n+1}\|_{\beta, p} < \infty$, and hence $\|\mathcal{A}v_{n+1}\|_{\beta, p} < \infty$. Next we find a bound on $\mathcal{A}v_n$ that is uniform in n . If we let $a_n := \|\mathcal{A}v_n\|$, then, by (2.37) and the triangle inequality,

$$a_{n+1} = \alpha + \beta a_n, \quad (2.43)$$

where $\alpha = z_p \sqrt{\Upsilon(2\beta/p)} (|\sigma(0)| + \text{Lip}_\sigma \|v_0\|_{\beta, p})$ and $\beta = z_p \sqrt{\Upsilon(2\beta/p)} \text{Lip}_\sigma$. Iterating (2.43) yields

$$a_{n+1} \leq \alpha(1 + \beta + \cdots + \beta^n) + \beta^{n+1} a_0. \quad (2.44)$$

Since $\lim_{\beta \uparrow \infty} \Upsilon(2\beta/p) = 0$, then we can choose β sufficiently large to have $\beta < 1$. Therefore,

$$\sup_{n \geq 1} a_n \leq \frac{\alpha}{1 - \beta}.$$

Then $\sup_n \|\mathcal{A}v_n\|_{\beta, p} < \infty$, and so is $\sup_{n \geq 1} \|v_n\|_{\beta, p} < \infty$. This is because $T_t u_0$ is bounded uniformly by $\sup_{x \in \mathbf{T}} |u_0(x)|$, and

$$\sup_{n \geq 1} \|v_n\|_{\beta, p} \leq \sup_{x \in \mathbf{T}} |u_0(x)| + \frac{\alpha}{1 - \beta} < \infty. \quad (2.45)$$

Therefore, by Lemma (2.2.5) for all $n \geq 1$ we have

$$\|v_{n+1} - v_n\|_{\beta, p} = \|\mathcal{A}v_n - \mathcal{A}v_{n-1}\|_{\beta, p} \leq z_p \text{Lip}_\sigma \sqrt{\Upsilon(2\beta/p)} \|v_n - v_{n-1}\|_{\beta, p}.$$

This proves that $\{v_n\}_{n=1}^\infty$ is Cauchy in $\|\cdot\|_{\beta, p}$ norm, and so is convergent to some predictable random field u with

$$\|u\|_{\beta, p} < \infty, \quad \|v_n\|_{\beta, p} < \infty \quad \forall p \geq 1, \beta > 0, n \geq 1. \quad (2.46)$$

This also shows that

$$\lim_{n \rightarrow \infty} \|v_n(t, x) - u(t, x)\|_p = 0 \quad \forall (t, x) \in \mathbf{R}_+ \times [0, 2\pi]. \quad (2.47)$$

This and Remark 2.2.6 together prove more than what we promised to show. $\square$

2.3 Elements of Malliavin's calculus

The Malliavin calculus is an infinite-dimensional differential calculus on the Wiener space. It is an appropriate method for investigating the regularity of the law of functionals on the Wiener space. Such functionals include the solutions to stochastic (partial) differential equations. The integration by parts formula for an infinite dimensional space, against the Gaussian measure is central in this calculus. Paul Malliavin[54] initially invented this method to produce an alternative proof of Hörmanders condition[38, 31].

Most of this section is borrowed from [47] and [53]. Let $C_p^\infty(\mathbf{R}^n)$ denote the space of the smooth real-valued functions f on $\mathbf{R}^n$, such that f and all its partial derivatives have at most polynomial growth.

2.3.1 The Wiener chaos

For every $h \in H := L^2(E_T)$ let $w(h)$ denote the Wiener integral

$$w(h) = \int_0^T \int_{\mathbf{T}} h(t, x) w(dt, dx). \quad (2.48)$$

We call $W = \{w(h)\}_{h \in H}$ a Gaussian process on H and we let $\mathcal{G} \subset \mathcal{F}$ denote the σ -algebra generated by W .

For $n \geq 0$, let H_n be the *Hermite polynomial* of degree n . These are functions defined by

$$H_n(x) = \frac{(-1)^n}{n!} e^{\frac{x^2}{2}} \frac{d^n}{dx^n} (e^{-\frac{x^2}{2}}) \quad n \geq 1.$$

We let $H_0 = 1$. For each $n \geq 1$, let $\mathcal{H}_n$ denote the closed linear subspace of $L^2(\Omega, \mathcal{F}, \mathbf{P})$ generated by the random variables $\{H_n(w(h)), h \in H, \|h\|_H = 1\}$, and $\mathcal{H}_0$ is the set of

constants. The space H_n is called the *Wiener chaos* of order n . $\mathcal{H}_n$ are orthogonal with respect to P , and the space $L^2(\Omega, \mathcal{G}, P)$ can be decomposed into the infinite orthogonal sum of $\mathcal{H}_n$ [47, Theorem 1.1.1].

2.3.2 The derivative operator

Let $\mathcal{S}$ denote the class of smooth random variables. A random variable F belongs to $\mathcal{S}$, if there is $n \geq 1$ and a function $f : \mathbf{R}^n \rightarrow \mathbf{R}$ such that $F = f(w(h_1), \dots, w(h_n))$, where $f \in C_p^\infty(\mathbf{R}^n)$ and $w(h_i)$ is defined by (2.48) for $1 \leq i \leq n$. We initially define the Malliavin derivative operator $D : \mathcal{S} \rightarrow L^2(\Omega; H) \approx L^2(\Omega \times E_T)$ as the following: If $F \in \mathcal{S}$ is of the form above, then we define

$$D_{t,x}F = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(w(h_1), \dots, w(h_n)) h_i(t, x). \quad (2.49)$$

The following result is an *integration by parts* formula in its simplest form. As mentioned before, It plays an important role in the theory of Malliavin calculus.

Lemma 2.3.1. *If $F \in \mathcal{S}$ and $h \in H$, then*

$$E\langle DF, h \rangle_H = EFw(h). \quad (2.50)$$

Proof. The proof follows from the definition. See [47, Lemma 1.2.1] for details. $\square$

A consequence of Lemma 2.3.1 is the fact that the derivative operator D is closable. The closure of the $\mathcal{S}$ under the closed graph norm is called $\mathbf{D}^{1,2}$. To be able to investigate the smoothness of a random variable $X \in L^2(\Omega, \mathcal{G}, P)$, we need to define the Malliavin derivatives of higher orders. Let $F \in \mathcal{S}$. For $p \geq 1$ and $j \in \mathbf{N}$, the Malliavin derivative $D^j : \mathcal{S} \rightarrow L^p(\Omega; H^{\otimes j})$ of order j is defined by

$$D^j F = \sum \frac{\partial^j f}{\partial x_{i_1} \dots \partial x_{i_j}}(w(h_1), \dots, w(h_n)) h_{i_1} \otimes \dots \otimes h_{i_j}, \quad (2.51)$$

where the sum is over all j -tuples $(i_1, \dots, i_j) \in \{1, \dots, n\}^j$. The operator $D^j : \mathcal{S} \rightarrow L^p(\Omega; H^{\otimes j}) \approx L^p(\Omega \times E_T^j)$ is closable for all $p \geq 1$ and $n \in \mathbf{N}$. The domain of the closed

operator is called $\mathbf{D}^{j,p}$. More precisely, $\mathbf{D}^{k,p}$ is the closure of $\mathcal{S}$ under the $\|\cdot\|_{k,p}$ norm which is defined by

$$\|F\|_{k,p} = \left(\mathbb{E}|F|^p + \sum_{j=1}^k \mathbb{E}\|D^j F\|_{H^{\otimes j}}^p \right)^{1/p}. \quad (2.52)$$

Example 2.3.1 (Standard Wiener Space[21]). In some cases, the Malliavin derivative coincides with the Frechet derivative. Let Ω be the standard Wiener space of all continuous functions on $[0, 1]$ starting from zero. Let $\mathcal{H}$ be the space of all continuous functions ω such that $\omega(t) = \int_0^t g(s)ds$, where $g \in L^2([0, 1])$. The space of all such paths is called the *Cameron-Martin space*. The Cameron-Martin space and $L^2([0, 1])$ are isometric via $\langle \omega, \lambda \rangle_{\mathcal{H}} := \langle g, h \rangle_{L^2([0, 1])}$, where $\lambda(t) = \int_0^t h(s)ds$. We define the directional derivative of $F : \Omega \rightarrow \mathbf{R}$ at the point $\omega \in \Omega$ in direction of $\gamma \in \mathcal{H}$ by

$$D_{\gamma}F(\omega) := \lim_{\epsilon \rightarrow 0} \frac{F(\omega + \epsilon\gamma) - F(\omega)}{\epsilon},$$

where the limit is in $L^2(P)$. We say F is differentiable, if there exists $\psi(s, \omega) \in L^2(P \times \lambda)$ such that

$$D_{\gamma}F(\omega) = \int_0^t g(s)\psi(s, \omega)ds,$$

and we set $D_t F(\omega) = \psi(t, \omega)$. We call $D_{\bullet}F \in L^2(P, \lambda)$ the Malliavin derivative of F . Now assume $f \in L^2([0, 1])$ and let $F = \int_0^1 f(s)dB_s$, then

$$\frac{F(\omega + \epsilon\gamma) - F(\omega)}{\epsilon} = \int_0^T f(s)g(s)ds.$$

This implies that $D_t F = f(t)$. For example, since $B(s) = \int_0^s \mathbf{1}_{[0, s]}(s)dB_r$, $s \leq 1$, then

$$D_t B_s = \mathbf{1}_{[0, s]}(t).$$

For a different approach on the definition of the integration by parts formula, based on the Cameron-Martin space, and the Girsanov theorem see [6, Chapter 8].

2.3.3 The divergence operator

We start with an elementary result [53, page 2].

Proposition 2.3.2. *Let F be an $\mathbf{R}$ -valued random variable. Assume there is a function $H \in L^1(\Omega)$ such that*

$$\mathbb{E}\varphi'(F) = \mathbb{E}[\varphi(F)H], \quad (2.53)$$

for all bounded and differentiable function φ , whose first derivative is bounded. Then the probability law of F has density $p(x)$ with respect to Lebesgue measure on $\mathbf{R}$. Moreover,

$$p(x) = \mathbb{E}[\mathbf{1}_{x \leq F} H].$$

Proof (non-rigorous). Loosely speaking, $p(x) = \mathbb{E}\delta(F - x)$. If we let $\varphi(x) = \mathbf{1}_{[0, \infty)}(x)$, then

$$p(x) := \mathbb{E}(\delta_0(F - x)) = \mathbb{E}\varphi'(F - x) = \mathbb{E}[\mathbf{1}_{[0, \infty)}(F - x)H].$$

One can make this argument rigorous by approximating the delta function by smooth functions. $\square$

In this section, our goal is to characterize a large class of functions in $L^2(\Omega, \mathcal{G})$ for which the random variable H in (2.53) is defined; as an example of such results see Theorem 2.3.3.

If $F := p(X_1, \dots, X_m)$, where $p : \mathbf{R}^m \rightarrow \mathbf{R}$ is a polynomial, $\varphi : \mathbf{R}^m \rightarrow \mathbf{R}^m$ is compactly supported smooth functions, and $\{X_i\}_{i=1}^m$ are i.i.d. Gaussian random variables, then

$$\mathbb{E}\langle \nabla p(X), \varphi(X) \rangle_{\mathbf{R}^m} = \mathbb{E}[p(X)(\delta_m \varphi)(X)], \quad (2.54)$$

where $X = (X_1, \dots, X_m)$ and $(\delta_m \varphi)(x_1, \dots, x_m) = \sum_{i=1}^m x_i \varphi(x_1, \dots, x_m) - \frac{\partial \varphi}{\partial x_i}(x_1, \dots, x_m)$. After a simple computation we have,

$$\langle \nabla(\varphi op)(X), \nabla p(X) \rangle_{\mathbf{R}^m} = \varphi'(p(X)) |\nabla p(X)|_{\mathbf{R}^m}^2,$$

where $|\cdot|_{\mathbf{R}^m}$ denotes the Euclidean norm. Therefore, under the nondegeneracy condition $|\nabla p(X)|_{\mathbf{R}^m} \neq 0$ almost surely with respect to the m dimensional Gaussian measure, we have

$$\varphi'(p(X)) = \langle \nabla(\varphi op)(X), \frac{\nabla p(X)}{|\nabla p(X)|_{\mathbf{R}^m}^2} \rangle_{\mathbf{R}^m}$$

Therefore, if we take expectation, by (2.54), we arrive at the integration by part formula

$$\mathbb{E}\varphi'(p(X)) = \mathbb{E}[(\varphi op)(X) \delta_m(\frac{\nabla p(X)}{|\nabla p(X)|_{\mathbf{R}^m}^2})]. \quad (2.55)$$

We can extend the integration by parts formula (2.55) to the functions in $\mathbf{D}_m^\infty := \bigcap_{p \geq 2, k \geq 1} \mathbf{D}_m^{k,p}$, where the space $\mathbf{D}_m^{k,p}$ is the finite dimensional counterpart of $\mathbf{D}^{k,p}$; see [53, Chapter 2].

But the above analysis is not efficient for the investigation of the existence of the density, when $p(X)$ is replaced by the solution of a SPDE such as (1.1). This is because these solutions are functions of *infinitely* many increments that are Gaussian and independent. We define the infinite dimensional version δ of δ_m abstractly via duality. We saw an example of such duality in Lemma 2.3.1, where D is restricted to $\mathcal{S}$. Generally, we can define an adjoint operator for $D : \mathbf{D}^{1,2} \rightarrow L^2(\Omega; H)$ through the duality relation (2.50). More precisely we have the following definition.

Definition 2. An unbounded operator $\delta : L^2(\Omega; H) \rightarrow L^2(\Omega)$ is called the divergence operator if:

1. The domain of δ , denoted by $\text{Dom}\delta$, is the set of all $u \in L^2(\Omega; H)$ such

$$|\mathbb{E}\langle DF, u \rangle_H| \leq c\|F\|_2, \quad (2.56)$$

for all $F \in D^{1,2}$, where c is constant depending only on u .

2. If $u \in \text{Dom}\delta$, then $\delta(u)$ is an element of $L^2(\Omega)$ characterized by the

$$\mathbb{E}\langle DF, u \rangle_H = \mathbb{E}[F\delta(u)],$$

for all $F \in D^{1,2}$.

Now, we state the infinite dimensional counterpart of Proposition 2.3.2. The statement of the theorem and its proof are from [15, page 83-84].

Theorem 2.3.3. *Let F be a random variable in the space $\mathbf{D}^{1,2}$. Suppose that $DF/\|DF\|_H^2$ belongs to the domain of the operator δ . Then the law of F has a continuous and bounded density given by*

$$p(x) = \mathbb{E} \left[\mathbf{1}_{F > x} \delta \left(\frac{DF}{\|DF\|_H^2} \right) \right]. \quad (2.57)$$

Proof. Let ψ be a nonnegative smooth function with compact support, and set $\varphi(y) = \int_{-\infty}^y \psi(z)dz$. A chain rule formula for the Malliavin derivatives implies that $\varphi(F) \in \mathbf{D}^{1,2}$, and $\langle D(\varphi(F)), DF \rangle_H = \psi(F)\|DF\|_H^2$. Therefore, by duality,

$$\mathbb{E}[\psi(F)] = \mathbb{E} \left[\left\langle D(\varphi(F)), \frac{DF}{\|DF\|_H^2} \right\rangle_H \right] = \mathbb{E} \left[\varphi(F) \delta \left(\frac{DF}{\|DF\|_H^2} \right) \right].$$

By an approximation argument, the equation above holds for $\psi(y) = \mathbf{1}_{[a,b]}(y)$, where $a < b$.

We apply Fubini's theorem to get

$$\mathbb{P}(a < F < b) = \mathbb{E} \left[\left(\int_{-\infty}^F \psi(x)dx \right) \delta \left(\frac{DF}{\|DF\|_H^2} \right) \right] = \int_a^b \mathbb{E} \left[\mathbf{1}_{F > x} \delta \left(\frac{DF}{\|DF\|_H^2} \right) \right] dx,$$

which implies the desired result. $\square$

As we will see in the next two chapters, the solution to the SPDE (1.1) satisfies the hypothesis of Theorem 2.3.3. In this thesis, we always apply δ to an adapted processes. A stochastic process $u = \{u(t, x), t \geq 0, 0 \leq x \leq 2\pi\}$ is called *adapted* if $u(t, x)$ is $\mathcal{F}_t$ measurable for any $(t \geq 0, x) \in E_T$. Fix a (finite or infinite) time interval $[0, t]$, and denote by $L_a^2([0, t] \times \Omega)$ the set of all square integrable and adapted processes. The divergence operator is an extension of the Itô integral in the following sense:

Proposition 2.3.4. $L_a^2([0, t] \times \Omega) \subset \text{Dom} \delta$, and the operator δ restricted to $L_a^2([0, t] \times \Omega)$ coincides with the Itô integral; that is,

$$\delta(u) = \int_0^t \int_{\mathbf{T}} u(s, x) w(ds, dx). \quad (2.58)$$

Proof. A proof can be found in [47, Proposition 1.3.11]. $\square$

For the rest of this section, we state those theorems which will be used in Chapter 3.

Proposition 2.3.5. Suppose $F \in L^2(\Omega)$, and let J_n denote the projection to the n th Wiener chaos. Then, $F \in \mathbf{D}^{k,2}$ if and only if

$$\sum_{n=1}^{\infty} n^k \|J_n F\|_{L^2(\Omega)}^2 < \infty. \quad (2.59)$$

Proof. See [47, Proposition 1.2.2], and the paragraph after the proof of Proposition 1.2.2. $\square$

Proposition 2.3.6. *Let $\{F_n, n \geq 1\}$ be a sequence of random variables that converge to F in $L^p(\Omega)$ for some $p > 1$. Suppose that,*

$$\sup_{n \geq 1} \|F_n\|_{k,p} < \infty, \quad \text{for some } k \geq 1. \quad (2.60)$$

Then F belongs to $\mathbf{D}^{k,p}$, and the sequence of derivatives $\{D^k F_n, n \geq 1\}$ converges to $D^k F$ in the weak topology of $L^p(\Omega; H)$.

Proof. A proof for this proposition can be found in [15, page 78] or [47, Lemma 1.2.3]. $\square$

Remark 2.3.4. The space $\mathbf{D}^{1,2}(L^2(T))$, denoted by $L^{1,2}$, coincides with the class of processes $u \in L^2(T \times \Omega)$ such that $u(t) \in \mathbf{D}^{1,2}$ for almost all $t \in T$, and there exists a measurable version of the two-parameter process $D_s u_t$ verifying $E \int_T \int_T (D_s u_t)^2 \mu(ds) \mu(dt) < \infty$. The space $L^{1,2}$ is included in $\text{Dom} \delta$.

To apply the Malliavin calculus to our problem, we usually need to compute the Malliavin derivatives of the integrals. In this regard the following proposition [47, Proposition 1.3.8] is useful.

Proposition 2.3.7. *Suppose that $u \in L^{1,2}$. Furthermore assume that the following two conditions are satisfied:*

1. *For almost all $(s, y) \in E$ the process $\{D_{s,y} u(r, z), (r, z) \in E\}$ is Skorohod integrable;*
2. *There is a version of the process*

$$\left\{ \int_{\mathbf{T}} \int_0^T D_{s,y} u(r, z) W(dr, dz), (s, y) \in E \right\},$$

which is in $L^2(\Omega \times E)$.

Then $\delta(u) \in \mathbf{D}^{1,2}$ and we have

$$D_{s,y}(\delta(u)) = u(s, y) + \int_{\mathbf{T}} \int_0^T D_{s,y} u(r, z) W(dr, dz). \quad (2.61)$$

If $F = (F^1, \dots, F^n)$ is a random vector with $F^i \in \mathbf{D}^{1,1}$, then we define its Malliavin matrix γ_F to be

$$\gamma_F = (\langle F^i, F^j \rangle)_{1 \leq i, j \leq m}.$$

Before we state the main theorem we introduce the following definition from [15, page 86].

Definition 1. We say that a random vector $F = (F^1, \dots, F^n)$ is non degenerate if it satisfies the following conditions:

1. $F^i \in \mathbf{D}^\infty$ for all $i = 1, \dots, m$;
2. The matrix γ_F satisfies $E[(\det \gamma_F)^{-p}] < \infty$ for all $p \geq 2$.

The following is a key result and can be found in [15, page 86].

Theorem 2.3.8. *If $F = (F^1, \dots, F^n)$ is a non degenerate random vector, then the law of F possesses an infinitely-differentiable density.*

The hypothesis of Theorem 2.3.8 can be relaxed significantly, if we only demand the existence of a density; see [9]. While Proposition 2.3.7 allows us to prove an integral is in $\mathbf{D}^{1,2}$ it falls short of telling us whether or not it belongs to $\mathbf{D}^{1,p}$ for $p > 2$. The following proposition, which is a result of Meyer's inequality (see [47] page 72) states the required conditions for going from $p = 2$ to $p > 2$, [47, Proposition 1.5.5].

Proposition 2.3.9. *Let F be a random variable in $\mathbf{D}^{k,\alpha}$, where $\alpha > 1$. suppose that $D^i F$ belongs to $L^p(\Omega, H^{\otimes i})$ for $i = 0, 1, \dots, k$, and for some $p > \alpha$. Then $F \in \mathbf{D}^{k,p}$.*

Remark 2.3.10. *We will frequently use two families of semi-norms, indexed by two parameters. One is defined in (2.33), which always comes with parameter β , and the other one is the norm $\|\cdot\|_{k,p}$, on space $\mathbf{D}^{k,p}$, which is indexed by integers such as k, m, n etc., and p . For the sake of clarity, in the sequel, when these two norms are both used, we will use a new notation defined by (3.16).*

CHAPTER 3

Smoothness of the Solution

3.1 The Malliavin derivatives of the solution

In this section we assume that $\sigma \in C_b^\infty(\mathbf{R})$ and the underlying Lévy process satisfies **H1**.

Remark 3.1.1. *We occasionally use the symbol “ $\lesssim$ ” in our proofs. By $X \lesssim Y$ we mean there is a positive C such that $|X| \leq CY$. We might also subscript this by a parameter to denote dependence on this parameter.*

The main result in this section is the following theorem¹. Before we state the theorem, we introduce a notation. Let $\alpha = ((s_1, y_1), \dots, (s_k, y_k))$ be a string of k pairs, where $(s_i, y_i) \in E_T$. Such α is an element of E_T^k . Let $(\hat{s}, \hat{y})$ denote the pair with the largest first coordinate, i.e., $\hat{s} = s_1 \vee \dots \vee s_k$. To each $\alpha \in E_T^k$, we assign an element $\hat{\alpha} \in E_T^{k-1}$ obtained by eliminating the $(\hat{s}, \hat{y})$ from the string of the pairs that define α . We refer the reader to the section 3.1.2 for more details on this notation.

Theorem 3.1.2. *If u is the solution to the Eq. (2.25) then $u(t, x) \in \mathbf{D}^\infty$ for almost all $(t, x) \in [0, T] \times [0, 2\pi]$. Furthermore*

$$D_\alpha^k u(t, x) = q_{t-\hat{s}}(x, \hat{y}) D_{\hat{\alpha}}^{k-1} \sigma(u(\hat{s}, \hat{y})) + \int_0^t \int_0^{2\pi} q_{t-r}(x, z) D_\alpha^k \sigma(u(r, z)) w(dr, dz), \quad (3.1)$$

where $\alpha \in E_T^k$.

We have a sequence of random functions v_n , defined through the Picard iteration (2.42), which converge to u in $\|\cdot\|_{\beta, p}$, for β sufficiently large, and consequently in $L^p(\Omega)$, for all

¹Most of the material in this chapter is from [37].

$(t, x) \in E_T$ and $p \geq 1$. We would like to use Theorem 2.3.6 to show that $u \in \mathbf{D}^{k,p}$ for all $p \geq 1$ and $k = 1, 2, \dots$. Therefore the first step is to show that v_n 's are in $\mathbf{D}^{k,p}$ for all k and p . Then we need to show that $D^k v_n$'s are convergent weakly for all $k \geq 1$; i.e., it is sufficient to have

$$\sup_n \|v_n(t, x)\|_{k,p} < \infty,$$

where the $\|\cdot\|_{k,p}$ is the norm of $\mathbf{D}^{k,p}$ space. To this end we need a quantitative bound on the growth of Malliavin derivatives as well. We carry out this task by induction. The case $n = 1$; i.e., $v_n(t, x) \in \mathbf{D}^{1,p}$ is the subject of subsection 3.1.1. The second subsection is devoted to introducing a few notations and some technical lemmas that will allow us to go from the first derivative to the higher-order derivatives in the upcoming subsection. The third subsection deals with the k 'th derivatives of v_n 's and the short final subsection concludes the this section with proving main result of this section; i.e., Theorem 3.1.2.

3.1.1 The first derivative

The first Malliavin derivative is the only derivative that has $\sigma(x)$ in its formulation. Here, as me mentioned before, we require σ to be Lipschitz continuous.

Proposition 3.1.3. *If the v_n 's are defined by (2.42), then $v_n \in \mathbf{D}^{1,p}$ for all $n \geq 0$, and:*

1. $Dv_0 = 0$ and

$$\begin{aligned} D_{s,y} v_{n+1}(t, x) = & q_{t-s}(x, y) \sigma(v_n(s, y)) \\ & + \int_0^t \int_{\mathbf{T}} q_{t-r}(x, z) D_{s,y} \sigma(v_n(r, z)) w(dr, dz). \end{aligned} \quad (3.2)$$

2. For all $n \geq 0$, and $T \in [0, \infty)$,

$$\|\Gamma^1 v_{n+1}\|_{\beta,p,T}^2 \leq C_p \text{Lip}_{\sigma'} \Upsilon(2\beta/p) (1 + \|\Gamma^1 v_n\|_{\beta,p,T}^2), \quad (3.3)$$

where $\Gamma_{t,x}^1 f = \|Df(t, x)\|_H$, and

$$\text{Lip}_{\sigma'} = 2 \max \left\{ \sup_{x \in \mathbf{T}} |\sigma(x)|^2, \sup_{x \in \mathbf{T}} |\sigma'(x)|^2 \right\}.$$

Proof. We need to mention that $D_{s,y}\sigma(v_n(r,z)) = 0$ when $r < s$; i.e., the integral vanishes on the subinterval $[0, s]$, [47, Corollary 1.21].

We proceed by applying induction on n . We will find that the proofs of these three conclusions go hand in hand, i.e., we use (3.3) for $n = n_0$ to show that v_{n_0+1} is in $\mathbf{D}^{1,2}$, with a derivative which satisfies (3.2). Then we use (3.2), to show (3.3) holds for $n = n_0 + 1$.

For $n = 0$ all these three conclusions hold vacuously. Next assume (3.3) holds for v_n , then $v_{n+1} \in \mathbf{D}^{1,2}$ and (3.2) holds by Proposition 2.3.7. Here we will not go through the details to show this, as we will do this later for the derivatives of higher order [see subsection 3.1.2 below]. Next, we show that (3.3) holds for v_{n+1} . Since

$$\begin{aligned} D_{s,y}v_{n+1}(t,x) &= q_{t-s}(x,y)\sigma(v_n(s,y)) \\ &\quad + \int_0^t \int_{\mathbf{T}} q_{t-r}(x,z)D_{s,y}\sigma(v_n(r,z))w(dr,dz), \end{aligned} \tag{3.4}$$

then by the triangle inequality for the H norm

$$\begin{aligned} \Gamma_{t,x}^1 v_{n+1} &\leq \text{Lip}_\sigma \left(\int_0^t \int_{\mathbf{T}} q_{t-s}^2(x,y)|v_n(s,y)|^2 dy ds \right)^{1/2} \\ &\quad + \left(\int_0^t \int_{\mathbf{T}} \left| \int_0^t \int_{\mathbf{T}} q_{t-r}(x,z)D_{s,y}\sigma(v_n(r,z))w(dr,dz) \right|^2 dy ds \right)^{1/2}. \end{aligned}$$

Take the $L^p(\Omega)$ norm from the both sides. Since, by the chain rule [47, Proposition 1.2.3]

$$D_{s,y}\sigma(v_n(r,z)) = \sigma'(v_n(r,z))D_{s,y}v_n(r,z),$$

and by the Burkholder's inequality for the Hilbert space-valued martingales,

$$\begin{aligned} &\mathbb{E} \left\| \int_0^t \int_{\mathbf{T}} q_{t-r}(x,z)D\sigma(v_n(r,z))w(dr,dz) \right\|_H^p \\ &\lesssim_p \mathbb{E} \left(\int_0^t \int_{\mathbf{T}} q_{t-r}^2(x,z) \|D\sigma(v_n(r,z))\|_H^2 dz dr \right)^{p/2} \\ &\leq \sup_{x \in \mathbf{T}} |\sigma'(x)|^p \mathbb{E} \left(\int_0^t \int_{\mathbf{T}} q_{t-r}^2(x,z) \|Dv_n(r,z)\|_H^2 dz dr \right)^{p/2}, \end{aligned}$$

we obtain

$$\begin{aligned} & \|\Gamma_{t,x}^1 v_{n+1}\|_{L^p(\Omega)}^2 \lesssim_p \text{Lip}_\sigma^2 \int_0^t \int_{\mathbf{T}} q_{t-s}^2(x, y) \|v_n(s, y)\|_{L^p}^2 dy ds \\ & + \sup_{x \in \mathbf{T}} |\sigma'(x)|^2 \left\{ \mathbb{E} \left(\int_0^t \int_{\mathbf{T}} q_{t-r}^2(x, z) |\Gamma_{r,z}^1 v_n|^2 dz dr \right)^{p/2} \right\}^{2/p}. \end{aligned}$$

Now, apply Minkowski's inequality $[dP \times dz dr]$ to the last term to switch the expectation and the double integral, as follows

$$\begin{aligned} & \|\Gamma_{t,x}^1 v_{n+1}\|_{L^p(\Omega)}^2 \\ & \lesssim_{p,\sigma,\sigma'} \left(\|u\|_{\beta,p}^2 \int_0^t \int_{\mathbf{T}} q_{t-s}^2(x, y) e^{2\beta r/p} dy ds \right. \\ & \quad \left. + \int_0^t \int_{\mathbf{T}} q_{t-r}^2(x, z) \|\Gamma_{r,z}^1 v_n\|_{L^p(\Omega)}^2 dz dr \right) \\ & \lesssim_{p,\sigma,\sigma'} (\|u\|_{\beta,p}^2 + \|\Gamma^1 v_n\|_{\beta,p,T}^2) \int_0^t \int_{\mathbf{T}} q_{t-r}^2(x, z) e^{2\beta r/p} dz dr, \end{aligned}$$

where we used the trivial inequality

$$\|\Gamma_{r,z}^1 v_n\|_{L^p(\Omega)}^2 \leq e^{2\beta r/p} \|\Gamma^1 v_n\|_{\beta,p,T}^2 \quad (r, z) \in [0, T] \times \mathbf{T}. \quad (3.5)$$

Therefore if we change the variable $t - r \rightarrow r$ in the last integral, and multiply the inequality by $e^{-2\beta t/p}$ we arrive at

$$\begin{aligned} & e^{-2\beta t/p} \|\Gamma_{t,x}^1 v_{n+1}\|_{L^p(\Omega)}^2 \\ & \lesssim_p \text{Lip}_{\sigma'} (\|u\|_{\beta,p}^2 + \|\Gamma^1 v_n\|_{\beta,p}^2) \int_0^t \|q_r\|_{L^2(\mathbf{T})}^2 e^{-2\beta r/p} dr \\ & \lesssim_p c_{\sigma,\sigma'} \Upsilon(2\beta/p) (1 + \|\Gamma^1 v_n\|_{\beta,p,T}^2), \end{aligned}$$

where the last inequality follows from Lemma 2.1.7 and $c_{\sigma,\sigma'} = \text{Lip}_\sigma^2 + \sup_x |\sigma(x)|^2 + \|u\|_{\beta,p}$.

By optimizing this expression over all $(t, x) \in E_T$ we arrive at

$$\|\Gamma^1 v_{n+1}\|_{\beta,p,T}^2 \leq C_{p,\sigma,\sigma'} \Upsilon(2\beta/p) (1 + \|\Gamma^1 v_n\|_{\beta,p,T}^2). \quad (3.6)$$

This proves that $\|Dv_{n+1}\|_H \in L^p(\Omega)$, and so $v_{n+1}(t, x) \in \mathbf{D}^{1,p}$, for all $(t, x) \in E_T$ by Proposition 2.3.9. $\square$

Proposition 3.1.4. *If $u = u(t, x)$ denote the solution to (2.25), then $u(t, x) \in \mathbf{D}^{1,p}$ and Du satisfies (3.1); i.e.,*

$$D_{s,y}u(t, x) = q_{t-s}(x, y)u(s, y) + \int_0^t \int_{\mathbf{T}} q_{t-r}(x, z)D_{s,y}\sigma(u(r, z))w(dr, dz). \quad (3.7)$$

Proof. The fact that $u \in \mathbf{D}^{1,2}$ follows easily from the bound (3.6), because by iterating this bound we get

$$\|\Gamma^1 v_n\|_{\beta,p}^2 \leq \alpha + \alpha^2 + \cdots + \alpha^n,$$

where $\alpha := C_p \text{Lip}_{\sigma'} \Upsilon(2\beta/p)$. Since $\lim_{\beta \rightarrow \infty} \Upsilon(\beta) = 0$, we can choose $\beta > 0$ sufficiently large so that $\alpha < 1$ and consequently we get

$$\sup_n \|\Gamma^1 v_n\|_{\beta,p,T}^2 \leq \frac{\alpha}{1-\alpha} < \infty. \quad (3.8)$$

Then $u(t, x) \in \mathbf{D}^{1,2}$, by Proposition 2.3.6. Then Proposition 2.3.9 proves that $u(t, x) \in \mathbf{D}^{1,p}$ for all $p \geq 1$. Since the right-hand-side of (3.8) is independent of T , it holds for $T = \infty$. Because of this, in the rest of the paper, we only work with the norm $\|\cdot\|_{\beta,p}$.

In order to derive (3.7), it suffices to show that

$$\|\Gamma^1(v_n - u)\|_{\beta,p} \rightarrow 0, \quad \text{as } n \rightarrow \infty, \quad (3.9)$$

where

$$\Gamma_{t,x}^1 u := \|Du(t, x)\|_H.$$

By the triangle inequality, applied first to the H norm and then to the $L^p(\Omega)$ norm, and squaring both sides, we have

$$\begin{aligned} \frac{1}{2} \|\Gamma^1(v_{n+1} - u)\|_{L^p(\Omega)}^2 &\leq \{E\|q_{t-\bullet}(x, *)[\sigma(v_n(\bullet, *)) - \sigma(u(\bullet, *))]\|_H^p\}^{2/p} \\ &+ \left\{E\left\|\int_0^t \int_{\mathbf{T}} q_{t-r}(x, z)D[\sigma(v_n(r, z)) - \sigma(u(r, z))]w(dr, dz)\right\|_H^p\right\}^{2/p}. \end{aligned}$$

Since σ is Lipschitz, by applying Minkowski's inequality to the first term and Burkholder's inequality followed by Minkowski's inequality to the second term, we get

$$\begin{aligned} C\|\Gamma_{t,x}^1(v_{n+1} - u)\|_{L^p(\Omega)}^2 &\leq \int_0^t \int_{\mathbf{T}} q_{t-s}^2(x, y) \|v_n(s, y) - u(s, y)\|_{L^p(\Omega)}^2 dy ds \\ &+ \int_0^t \int_{\mathbf{T}} q_{t-r}^2(x, z) \|\Gamma_{r,z}^1(\sigma(v_n) - \sigma(u))\|_{L^p(\Omega)}^2 dz dr. \end{aligned}$$

Then, by (3.5) and Lemma 2.1.7 we obtain

$$\begin{aligned} C\|\Gamma_{t,x}^1(v_{n+1} - u)\|_{L^p(\Omega)}^2 &\leq \|v_n - u\|_{\beta,p}^2 e^{2\beta t/p} \Upsilon(2\beta/p) \\ &\quad + \|\Gamma^1(\sigma(v_n) - \sigma(u))\|_{\beta,p}^2 e^{2\beta t/p} \Upsilon(2\beta/p), \end{aligned}$$

where C depends on $\text{Lip}_{\sigma'}$ and C_p , where C_p is the constant in Burkholder's inequality. After we optimize on $t \in [0, T]$ and $x \in [0, 2\pi]$, we have

$$\|\Gamma^1(v_{n+1} - u)\|_{\beta,p}^2 \leq C_{\beta,p,\sigma'} \left(\|v_n - u\|_{\beta,p}^2 + \|\Gamma^1(\sigma(v_n) - \sigma(u))\|_{\beta,p}^2 \right), \quad (3.10)$$

where $C_{\beta,p,\sigma'} \rightarrow 0$ as $\beta \rightarrow \infty$. Since

$$\begin{aligned} D[\sigma(v_n(r, z)) - \sigma(u(r, z))] &= \sigma'(v_n(r, z))[Dv_n(r, z) - Du(r, z)] \\ &\quad + [\sigma'(v_n(r, z)) - \sigma'(u(r, z))]Du(r, z), \end{aligned}$$

then by applying the triangle inequality, and considering the boundedness and the Lipschitz property of σ' we obtain the following:

$$C\Gamma_{r,z}^1(\sigma(v_n) - \sigma(u)) \leq \Gamma_{r,z}^1(v_n - u) + |v_n(r, z) - u(r, z)|\Gamma_{r,z}^1 u.$$

Therefore,

$$C\|\Gamma_{r,z}^1(\sigma(v_n) - \sigma(u))\|_{\beta,p} \leq \|\Gamma^1(v_n - u)\|_{\beta,p} + \|(v_n - u)\Gamma^1 u\|_{\beta,p}. \quad (3.11)$$

By optimizing over $(t, x) \in E_T$ and substituting in (3.10) we obtain

$$\begin{aligned} \|\Gamma^1(v_{n+1} - u)\|_{\beta,p}^2 &\leq C_{\beta,p,\sigma'} (\|v_n - u\|_{\beta,p}^2 + \|(v_n - u)\Gamma^1 u\|_{\beta,p}^2 \\ &\quad + \|\Gamma^1(v_n - u)\|_{\beta,p}^2). \end{aligned} \quad (3.12)$$

Consider the first two terms in the parenthesis in (3.12). From Theorem 2.2.7 we know that $\|v_n - u\|_{\beta,p}^2 \rightarrow 0$ as $n \rightarrow \infty$, while the second term in (3.12) vanishes as $n \rightarrow \infty$, for example, by the Cauchy-Schwarz inequality. Therefore (3.12) can be written as

$$\|\Gamma^1(v_{n+1} - u)\|_{\beta,p}^2 \leq C_{\beta,p,\sigma'} (\lambda_n + \|\Gamma^1(v_n - u)\|_{\beta,p}^2), \quad (3.13)$$

where $\lambda_n \rightarrow 0$ as $n \rightarrow \infty$. Choose β sufficiently large such that $C_{\beta,p,\sigma'} < 1$. Then (3.13) implies that

$$\lim_{n \rightarrow \infty} \|\Gamma^1(v_{n+1} - u)\|_{\beta,p}^2 = 0.$$

This finishes the proof. $\square$

To state and prove the result for derivatives of higher order, we need to introduce some notations and prove some preparatory lemmas that will be stated next.

3.1.2 Preliminaries and notations

We know that the m th derivative of $v_n(t, x)$, if it exists, belongs to the space $L^2(E_T^{m+1} \times \Omega)$. Recall that $L^{1,2} = \mathbf{D}^{1,2}(E_T^{m+1} \times \Omega)$. Let $\alpha = \alpha_m$ denote an element in E_T^m . We can write

$$\alpha = ((s_1, y_1), \dots, (s_m, y_m)).$$

Let $\hat{s} := \max\{s_1, \dots, s_m\}$. If $i \in \{1, \dots, m\}$ is so that $s_i = \hat{s}$, then we let $\hat{y}$ denote y_i and $\hat{\alpha}_m := ((s_1, y_1), \dots, (s_{i-1}, y_{i-1}), (s_{i+1}, y_{i+1}), \dots, (s_m, y_m))$. Note that $\hat{\alpha}_m \in E_T^{m-1}$.

If we think of $\alpha = \alpha_m$ as a set of m pairs $\alpha = \{(s_1, y_1), \dots, (s_m, y_m)\}$, instead of an ordered m -tuple, then the partitions of α are defined. Let $\mathcal{P}^m :=$ the set of all partitions of $\alpha \in E_T^m$.

If $\deg = \{\deg_1, \dots, \deg_l\} \in \mathcal{P}^m$, then let $|\deg_j|$ denote the cardinality of $\deg_j$, where $j = 1, \dots, l$. Clearly $|\deg_1| + \dots + |\deg_l| = m$. If $\deg = \{\deg_1, \dots, \deg_l\} \in \mathcal{P}^m$, then $D_{\deg_j}^{|\deg_j|} F$ makes sense for $j = 1, \dots, l$. For example if $\deg_1 = \{(s_1, y_1), (s_3, y_3)\}$, then $D_{\deg_1}^{|\deg_1|} F = D_{(s_1, y_1)(s_3, y_3)}^2 F$. Furthermore, if $\deg = \{\deg_1, \dots, \deg_l\} \in \mathcal{P}^m$, then we introduce the new notation $D^{\deg} F$, and define it by

$$D_{\alpha}^{\deg} F := D_{\deg_1}^{|\deg_1|} F \times \dots \times D_{\deg_l}^{|\deg_l|} F. \quad (3.14)$$

Notice that

$$\|D^{\rho} F\|_{H^{\otimes m}} = \prod_{i=1}^l \|D^{\deg_i} F\|_{H^{\otimes |\deg_i|}}. \quad (3.15)$$

Fix $l \leq m$ and let $\mathcal{P}_l^m$ denote the set of all $\deg \in \mathcal{P}^m$ such that $|\deg| = l$. We have

$$\mathcal{P}^m = \cup_{l=1}^m \mathcal{P}_l^m.$$

We let $\Gamma_{x,y}^{\deg} v$ denote the H^m norm of $D^\rho v(t, x)$; i.e.,

$$\Gamma_{t,x}^{\deg} v = \|D^{\deg} v(t, x)\|_{H^{\otimes m}}. \quad (3.16)$$

If $\deg$ denotes the only member of $\mathcal{P}_1^m$, i.e., $\deg = \{(s_1, y_1), \dots, (s_k, y_k)\}$, then write $\Gamma_{t,x}^m v$ instead of $\Gamma_{t,x}^{\deg} v$.

The following lemma allows us to approximate $\|\sigma(v_n(t, x))\|_{k,p}$, where $\|\cdot\|_{k,p}$ denotes the norm on $\mathbf{D}^{k,p}$.

Lemma 3.1.5. *Assume σ is smooth and bounded together with all its derivatives. If $F \in \cap_{p \in [1, \infty)} \mathbf{D}^{m,p}$, then so is $\sigma(F)$. Furthermore, for $\alpha \in E_T^m$,*

$$D_\alpha^m \sigma(F) = \sum_{j=1}^m \sigma^{(j)}(F) \sum_{\deg \in \mathcal{P}_j^m} D_\alpha^{\deg} F, \quad (3.17)$$

where $\mathcal{P}_j^m$ is the set of all partitions of α , comprised of j components $\deg_1, \dots, \deg_j$, and $\sigma^{(j)}$ denotes the j th derivative of σ .

Proof. We can easily prove this for the smooth functionals by induction, and then extend the result to $F \in \cap_{p \in [1, \infty)} \mathbf{D}^{m,p}$ by approximation by the smooth functionals. $\square$

Lemma 3.1.6. *Let $\alpha \in E_T^m$, and $\deg = \{\deg_1, \dots, \deg_l\} \in \mathcal{P}^m$ denote a partition of α . Choose and fix $\beta > 0$. Let $U(t, z)$ and $V(t, z)$ belong to $\cap_{p \geq 1} \mathbf{D}^{m,p}$ for almost all r, z and $\|\Gamma^{|\deg_j|} V\|_{\beta,p} < \infty$ for all $p \geq 1$. If σ is bounded and smooth with bounded derivatives of all orders, then*

$$\|\Gamma^m \sigma(V)\|_{\beta,p} \lesssim \|\Gamma^m V\|_{\beta,p} + \sum_{j=2}^m \sum_{\deg_1 \dots \deg_j \in \mathcal{P}^m} \Pi_{i=1}^j \|\Gamma^{|\deg_j|} V\|_{\beta,jp}. \quad (3.18)$$

Furthermore, if $\|\Gamma^{|\deg_j|} U\|_{\beta,p} < \infty$ for all $p \geq 1$, then we have

$$\|\Gamma^m(\sigma(V) - \sigma(U))\|_{\beta,p} \lesssim \|\Gamma^m(V - U)\|_{\beta,p} + \sum_{j=2}^m \sum_{\deg_1 \dots \deg_j \in \mathcal{P}^m} \Pi_{i=1}^j \|\Gamma^{|\deg_j|}(V - U)\|_{\beta,jp}. \quad (3.19)$$

Proof. According to (3.17) we have

$$\Gamma_{r,z}^m \sigma(V) \leq C \sum_{j=1}^m \sum_{\deg_1 \dots \deg_j \in \mathcal{P}^m} \Pi_{i=1}^j \|D^{|\deg_i|} V(r, z)\|_{H^{\otimes |\deg_i|}},$$

where $C = \sup_x \{\sigma(x), \sigma'(x), \dots, \sigma^m(x)\}$. Then

$$\|\Gamma_{r,z}^m \sigma(V)\|_{L^p(\Omega)} \lesssim \|\Gamma_{r,z}^m V\|_{L^p(\Omega)} + \sum_{j=2}^m \sum_{\deg_1 \dots \deg_j \in \mathcal{P}^m} \|\Pi_{i=1}^j \Gamma_{r,z}^{|\deg_i|} V\|_{L^p(\Omega)}.$$

Therefore, by the generalized Hölder inequality,

$$\|\Gamma_{r,z}^m \sigma(V)\|_{L^p(\Omega)} \lesssim \|\Gamma_{r,z}^m V\|_{L^p(\Omega)} + \sum_{j=2}^m \sum_{\deg_1 \dots \deg_j \in \mathcal{P}^m} \Pi_{i=1}^j \|\Gamma_{r,z}^{|\deg_i|} V\|_{L^{jp}(\Omega)}.$$

Multiplying both sides by $e^{\beta r/p}$ we get

$$\begin{aligned} e^{\beta r/p} \|\Gamma_{r,z}^m \sigma(V)\|_{L^p(\Omega)} &\lesssim e^{\beta r/p} \|\Gamma_{r,z}^m V\|_{L^p(\Omega)} + \sum_{j=2}^m \sum_{\deg_1 \dots \deg_j \in \mathcal{P}^m} \Pi_{i=1}^j e^{\beta r/jp} \|\Gamma_{r,z}^{|\deg_i|} V\|_{L^{jp}(\Omega)} \\ &\lesssim \|\Gamma^m V\|_{\beta,p} + \sum_{j=2}^m \sum_{\deg_1 \dots \deg_j \in \mathcal{P}^m} \Pi_{i=1}^j \|\Gamma^{|\deg_j|} V\|_{\beta,jp}. \end{aligned}$$

Therefore,

$$\|\Gamma^m \sigma(V)\|_{\beta,p} \leq C \left(\|\Gamma^m V\|_{\beta,p} + \sum_{j=2}^m \sum_{\deg_1 \dots \deg_j \in \mathcal{P}^m} \Pi_{i=1}^j \|\Gamma^{|\deg_j|} V\|_{\beta,jp} \right). \quad (3.20)$$

The proof of the second statement is similar, if we observe that,

$$D(\sigma(V) - \sigma(U)) = \sum_{j=1}^m \sum_{\deg \in \mathcal{P}_j^m} \sigma^j(V)(D^{\deg_j}(V - U)) + [\sigma^j(V) - \sigma^j(U)] D^{\deg_j} U.$$

In this case, the constant C in (3.20), is replaced by $C' = C \vee \text{Lip}_{\sigma'} \vee \dots \vee \text{Lip}_{\sigma^{(m)}}$. □

The following lemma must be known, but we could not find a reference for it. It explains the method that we use to prove a random variable is in $\mathbf{D}^{k+1,p}$, when we know it is in $\mathbf{D}^{k,p}$.

Lemma 3.1.7. *Let $F \in \mathbf{D}^{k,p}$ satisfy $D_\alpha^k F \in \mathbf{D}^{1,p}$ for almost all $\alpha \in E_T$. If $\mathbb{E} \|DD^k F\|_{H^{\otimes k+1}} < \infty$, then $F \in \mathbf{D}^{k+1,p}$.*

Proof. To make the notation simpler, we prove the lemma only for $k =$ and $p = 2$. In this case we have $F \in \mathbf{D}^{1,2}$ and $DF \in \mathbf{D}^{1,2}(L^2(T))$. By Proposition 2.3.5 we need only to show that

$$\sum_{n=1}^{\infty} n(n-1) \|J_n F\|_{L^2(\Omega)}^2 < \infty. \quad (3.21)$$

Since $DF \in \mathbf{D}^{1,2}(L^2(T))$, then

$$\sum_{n=1}^{\infty} n \|J_n DF\|_{L^2(\Omega \times T)}^2 < \infty.$$

Since $F \in \mathbf{D}^{1,2}$, then $\langle DJ_n F, h \rangle_{L^2(T)} = J_{n-1}(\langle DF, h \rangle_{L^2(T)})$, then

$$\|J_n DF\|_{L^2(\Omega \times T)}^2 = \mathbb{E} \|J_n DF\|_{L^2(T)}^2 = \mathbb{E} \|DJ_{n+1} F\|_{L^2(T)}^2 = (n+1) \|J_n F\|_2^2.$$

For the proof of the last equality we refer the reader to [48], Proposition 1.12. $\square$

Lemma 3.1.8. *Let $V(t, z) \in \cap_{p>1} D^{m+1,p}$ for almost all r, z and let $\|\Gamma^{|\deg_j|} V\|_{\beta,p} < \infty$ for all $p > 1$, where $\deg = \{\deg_1, \dots, \deg_l\} \in \mathcal{P}^{m+1}$. For $\alpha \in E_T^m$ let*

$$f_\alpha(r, z) = q_{t-r}(x, z) D_\alpha^m \sigma(V(r, z)). \quad (3.22)$$

Then $f_\alpha \in L^{1,2}$.

Proof. We need to verify that the three conditions mentioned in Remark 2.3.4 hold for f_α .

1. By Lemma 3.1.6, $\|\Gamma^m \sigma(V)\|_{\beta,2} < \infty$. Then

$$\begin{aligned} \int_{E_T^m} \mathbb{E} \|f_\alpha\|_{L^2(E_T \times \Omega)}^2 d\alpha &= \mathbb{E} \int_0^t \int_{\mathbf{T}} q_{t-r}^2(x, z) \int_{E_T^m} |D_\alpha^m \sigma(V(r, z))|^2 d\alpha dz dr \\ &= \int_0^t \int_{\mathbf{T}} q_{t-r}^2(x, z) \mathbb{E} [\Gamma_{r,z}^m \sigma(V)]^2 dz dr \\ &\leq \int_0^t \int_{\mathbf{T}} q_{t-r}^2(x, z) e^{\beta r} \|\Gamma^m \sigma(V)\|_{\beta,2}^2 dz dr < \infty. \end{aligned}$$

This also means that $\|f_\alpha\|_{L^2(E_T \times \Omega)}^2 < \infty$ for almost all $\alpha \in E_T$.

2. $f_\alpha(r, z) \in \mathbf{D}^{1,2}$ because $V(t, x) \in D^{m+1,2}$.

3. Since $\|\Gamma^{m+1}\sigma(V)\|_{\beta,2} < \infty$,

$$\begin{aligned}
& \int_{E_T^m} \mathbb{E} \|Df_\alpha\|_{L^2(E_T^2 \times \Omega)}^2 d\alpha \\
&= \mathbb{E} \int_0^t \int_{\mathbf{T}} q_{t-r}^2(x, z) \int_{E_T^{m+1}} |D_{s,y} D_\alpha^m \sigma(V(r, z))|^2 d\lambda dz dr \\
&= \int_0^t \int_{\mathbf{T}} q_{t-r}^2(x, z) \mathbb{E} [\Gamma_{r,z}^{m+1} \sigma(V)]^2 dz dr \\
&\leq \int_0^t \int_{\mathbf{T}} q_{t-r}^2(x, z) e^{\beta r} \|\Gamma^{m+1}\sigma(V)\|_{\beta,2}^2 dz dr < \infty,
\end{aligned} \tag{3.23}$$

where $d\lambda = d\alpha dy ds$. The last result also shows that $Df_\alpha \in L^2(E_T^2 \times \Omega)$ for almost all $\alpha \in E_T$.

Therefore $f_\alpha \in L^{1,2}$ for almost all α . □

Lemma 3.1.9. *If V and f_α are defined as in Lemma 3.1.8 and satisfy the same conditions, then $D_{s,y}f_\alpha \in \text{Dom} \delta$.*

Proof. Applying Fubini's theorem to (3.23) yields,

$$\|D_{s,y}f_\alpha\|_{L^2(E_T \times \Omega)} < \infty \quad \text{for almost all } ((s, y), \alpha) \in E_T^{m+1}.$$

Since $D_{s,y}f_\alpha$ is adapted and belongs to $L^2(E_T \times \Omega)$ for almost all (s, y) and α , then the Itô integral of $D_{s,y}f_\alpha$ is defined and coincides with $\delta(f_\alpha)$. □

Lemma 3.1.10. *If V and f_α are as defined in Lemma 3.1.8, and satisfy the same conditions, then for each $\alpha \in E_T^m$*

$$\int_0^t \int_{\mathbf{T}} Df_\alpha(r, z) w(dr, dz) \in L^2(E_T \times \Omega). \tag{3.24}$$

Proof. This follows from Burkholder's inequality:

$$\begin{aligned}
& \mathbb{E} \int_0^t \int_{\mathbf{T}} \left| \int_0^t \int_{\mathbf{T}} Df_\alpha(r, z) w(dr, dz) \right|^2 dy ds \\
& \leq \mathbb{E} \int_0^t \int_{\mathbf{T}} q_{t-r}^2(x, z) \|DD_\alpha^m \sigma(v_N(r, z))\|_H^2 dz dr.
\end{aligned}$$

To show that the last expectation is finite for almost all $\alpha \in E_T^m$, we take integral with respect α , and then Fubini's theorem implies that

$$\begin{aligned} & \int_{E_T^m} \mathbb{E} \int_0^t \int_{\mathbf{T}} q_{t-r}^2(x, z) \|DD_\alpha^m \sigma(v_N(r, z))\|_H^2 dz dr d\alpha \\ & \leq \mathbb{E} \int_0^t \int_{\mathbf{T}} q_{t-r}^2(x, z) \|D^{m+1} \sigma(v_N(r, z))\|_{H^{\otimes m+1}}^2 dz dr \\ & \leq \int_0^t \int_{\mathbf{T}} q_{t-r}^2(x, z) e^{\beta r} \|\Gamma^{m+1} \sigma(v_N)\|_{\beta, 2}^2 dz dr < \infty. \end{aligned}$$

Among other things, this proves that the integrand is finite for almost all α . $\square$

Lemma 3.1.11. *Let V and f_α be as defined in Lemma 3.1.8, and satisfy the same conditions.*

Define

$$F_1(\alpha) = \int_0^t \int_{\mathbf{T}} f_\alpha(r, z) w(dr, dz). \quad (3.25)$$

Then:

1. $F_1(\alpha) \in \mathbf{D}^{1,2}$ for almost all $\alpha \in E_T^m$;
2. DF_1 is given by

$$D_{s,y} F_1(\alpha) = f_\alpha(s, y) + \int_0^t \int_{\mathbf{T}} D_{s,y} f_\alpha(r, z) w(dr, dz); \quad (3.26)$$

3. We have

$$\mathbb{E} (\|DF_1\|_{H^{\otimes m+1}}^p) < \infty. \quad (3.27)$$

Proof. After proving Lemma 3.1.8, 3.1.9 and 3.1.10, we know that $F_1(\alpha)$ satisfies the assumptions of Proposition 2.3.7. Therefore, it is an immediate consequence of Proposition 2.3.7 that $F_1(\alpha) \in \mathbf{D}^{1,2}$ and (3.26) holds. We finally prove (3.27) as follows. By the Burkholder's inequality,

$$\begin{aligned} \mathbb{E} (\|F_1\|_{H^{\otimes m+1}}^p) &= \mathbb{E} \left\| \int_0^t \int_{\mathbf{T}} q_{t-r}(x, z) D^{m+1} \sigma(V(r, z)) w(dr, dz) \right\|_{H^{\otimes m+1}}^p \\ &\leq C_p \mathbb{E} \left(\int_0^t \int_{\mathbf{T}} q_{t-r}^2(x, z) \|D^{m+1} \sigma(V(r, z))\|_{H^{\otimes m+1}}^2 dz dr \right)^{p/2}. \end{aligned}$$

By Minkowski's inequality we have

$$\{\mathbb{E}(\|F_1\|_{H^{\otimes m+1}}^p)\}^{2/p} \leq C_p^{2/p} \int_0^t \int_{\mathbf{T}} q_{t-r}^2(x, z) \{\mathbb{E}|\Gamma_{r,z}^{m+1}\sigma(V)|^p\}^{2/p} dz dr.$$

Therefore,

$$\{\mathbb{E}(\|F_1\|_{H^{\otimes m+1}}^p)\}^{2/p} \leq C_p^{2/p} e^{2\beta t/p} \|\Gamma^{m+1}\sigma(V)\|_{\beta,p}^2 \int_0^t \|q_\tau\|_{L^2(T)}^2 e^{\frac{-2\beta\tau}{p}} d\tau. \quad (3.28)$$

After rearranging and choosing a new constant, we arrive at

$$\{e^{-\beta t} \mathbb{E}\|F_1\|_{H^{\otimes m+1}}^p\}^{1/p} \leq C_p \|\Gamma^{m+1}\sigma(V)\|_{\beta,p} \sqrt{\Upsilon(2\beta/p)}. \quad (3.29)$$

This ends the proof. $\square$

Remark 3.1.12. *If we define a random variable $\tilde{F}_1 := \|F_1\|_{H^{\otimes m+1}}$, then the (3.29) can be written as*

$$\|\tilde{F}_1\|_{\beta,p} \leq C_p \|\Gamma^{m+1}\sigma(V)\|_{\beta,p} \sqrt{\Upsilon(2\beta/p)}. \quad (3.30)$$

3.1.3 The k th derivatives of the v_n 's and the smoothness of the solution

Proposition 3.1.13. *Let v_n be defined by (2.42) for $n = 0, 1, \dots$, where $\sigma \in C_b^\infty(\mathbf{R})$ and $q_t(x)$ satisfies hypothesis **H1**. Then $v_n \in \mathbf{D}^{k,p}$ for $k = 1, 2, \dots$ and $p \geq 2$. Furthermore if $\alpha \in E_T^k$, then:*

1. $D^k v_0 = 0$ and

$$\begin{aligned} D_\alpha^k v_{n+1}(t, x) &= q_{t-\hat{s}}(\hat{y}, x) D_{\hat{\alpha}}^{k-1} \sigma(v_n(\hat{s}, \hat{y})) \\ &\quad + \int_0^t \int_{\mathbf{T}} q_{t-r}(x, z) D_\alpha^k \sigma(v_n(r, z)) w(dr, dz); \end{aligned} \quad (3.31)$$

2. For some $C > 0$ which only depends on p :

$$\|\Gamma^{m+1} v_{n+1}\|_{\beta,p}^2 \leq C \Upsilon(2\beta/p) (1 + \|\Gamma^{m+1} v_n\|_{\beta,p}^2); \quad (3.32)$$

3. If $\deg = \{\deg_1, \dots, \deg_l\}$, then

$$\|\Gamma^{\deg} v_{n+1}\|_{\beta,p} \leq \prod_{j=1}^l \|\Gamma^{|\deg_j|} v_{n+1}\|_{l\beta,lp} < \infty. \quad (3.33)$$

Proof. We proceed by applying induction on n and k . When $k = 1$ and $\alpha = (s, y) = \hat{\alpha}$, we have shown in Proposition 3.1.3 that all above claims hold. Next, by assuming that the claims hold for all $n \geq 0$ and $k = 1, \dots, m$, we will prove that they also hold for $m + 1$ and all $n \geq 0$. Since $D^{m+1}v_0 = 0$, (3.32) and (3.33) hold for $n = 0$. Suppose the claims hold for $n = 0, \dots, N$. To prove the claims for $N + 1$, notice that by Lemma 3.1.6,

$$\|\Gamma^{m+1}\sigma(v_N)\|_{\beta,p} < \infty.$$

Then by Lemma 3.1.11, $\int_0^t \int_{\mathbf{T}} q_{t-r}(x, z) D_{\alpha}^k \sigma(v_n(r, z)) w(dr, dz)$ belongs to $\mathbf{D}^{1,2}$. If we let $\gamma = ((s, y), \alpha) \in E_T^{m+1}$, then after relabeling γ , we have

$$\gamma = ((s_1, y_1), \dots, (s_{m+1}, y_{m+1})).$$

We let $\hat{s} = \max\{s_1, \dots, s_{m+1}\}$, and define $\hat{\gamma}$ and $\hat{y}$ accordingly. Then by (3.26),

$$\begin{aligned} D_{\gamma}^{m+1}v_{N+1}(t, x) &= q_{t-\hat{s}}(x, \hat{y}) D_{\hat{\gamma}}^m v_N(\hat{s}, \hat{y}) \\ &+ \int_0^t \int_{\mathbf{T}} q_{t-r}(x, z) D_{\gamma}^{m+1} \sigma(v_N(r, z)) w(dr, dz), \end{aligned} \quad (3.34)$$

where we applied the fact that $D_{s,y} f_{\alpha}(r, z) = 0$ if $s > r$. By the triangle inequality for the $H^{\otimes m+1}$ norm,

$$\begin{aligned} \Gamma_{t,x}^{m+1}v_{N+1} &\leq \left(\sum_{j=1}^{m+1} \underbrace{\int_0^t \int_{\mathbf{T}} \dots \int_0^t \int_{\mathbf{T}}}_{m+1 \text{ times}} q_{t-s_j}^2(x, y_j) \left| D_{\gamma_j}^m \sigma(v_N(s_j, y_j)) \right|^2 \mathbf{1}_{\hat{s}=s_j}(\gamma) d\alpha_j dy_j ds_j \right)^{1/2} \\ &+ \left\| \int_0^t \int_{\mathbf{T}} q_{t-r}^2(x, z) D^{m+1} \sigma(v_N(r, z)) w(dr, dz) \right\|_{H^{\otimes m+1}}, \end{aligned}$$

where $\gamma_j = ((s_1, y_1), \dots, (s_{j-1}, y_{j-1}), (s_{j+1}, y_{j+1}), \dots, (s_{m+1}, y_{m+1}))$. Notice that $\hat{\gamma} = \gamma_j$ when $\hat{s} = s_j$. All the integrals inside the sum are equal, and by omitting the indicator function $\mathbf{1}_{\hat{s}=s_j}(\alpha)$ we arrive at

$$\begin{aligned} \Gamma_{t,x}^{m+1}v_{N+1} &\leq \left((m+1) \int_0^t \int_{\mathbf{T}} q_{t-s_1}^2(x, y_1) \|D^m \sigma(v_N(s_1, y_1))\|_{H^{\otimes m}}^2 dy_1 ds_1 \right)^{1/2} \\ &+ \left\| \int_0^t \int_{\mathbf{T}} q_{t-r}^2(x, z) D^{m+1} \sigma(v_N(r, z)) w(dr, dz) \right\|_{H^{\otimes m+1}}. \end{aligned}$$

Then, by the triangle inequality for the $L^p(\Omega)$ norm, followed by Burkholder's inequality applied to the second integral on the right-hand-side,

$$\begin{aligned} & \|\Gamma_{t,x}^{m+1} v_{N+1}\|_{L^p(\Omega)} \\ & \leq (m+1)^{1/2} \left\{ \mathbb{E} \left(\int_0^t \int_{\mathbf{T}} q_{t-s_1}^2(x, y_1) |\Gamma_{s_1, y_1}^m \sigma(v_N)|^2 dy_1 ds_1 \right)^{p/2} \right\}^{1/p} \\ & + C_p \left\{ \mathbb{E} \left(\int_0^t \int_{\mathbf{T}} q_{t-r}^2(x, z) |\Gamma_{r,z}^{m+1} \sigma(v_N)|^2 dz dr \right)^{p/2} \right\}^{1/p}. \end{aligned}$$

If we square both sides of the last inequality and then apply Minkowski's inequality to the both integrals on the right-hand-side, then we obtain

$$\begin{aligned} A_p \|\Gamma_{t,x}^{m+1} v_{N+1}\|_{L^p(\Omega)}^2 & \leq \int_0^t \int_{\mathbf{T}} q_{t-s_1}^2(x, y_1) \|\Gamma_{s_1, y_1}^m \sigma(v_N)\|_{L^p(\Omega)}^2 dy_1 ds_1 \\ & + \int_0^t \int_{\mathbf{T}} q_{t-r}^2(x, z) \|\Gamma_{r,z}^{m+1} \sigma(v_N)\|_{L^p(\Omega)}^2 dz dr, \end{aligned}$$

where $A_p = \frac{1}{2(m+1)\sqrt{2C_p^2}}$. By (3.5) we have

$$\begin{aligned} A_p \|\Gamma_{t,x}^{m+1} v_{N+1}\|_{L^p(\Omega)}^2 & \leq \left(\|\Gamma^m \sigma(v_N)\|_{\beta,p}^2 + \|\Gamma^{m+1} \sigma(v_N)\|_{\beta,p}^2 \right) \\ & \times \int_0^t \int_{\mathbf{T}} q_{t-r}^2(x, z) e^{2\beta r/p} dz dr. \end{aligned}$$

By optimizing on all $t > 0$, for some constant $B_p > 0$ which only depends on p , we have

$$\|\Gamma^{m+1} v_{N+1}\|_{\beta,p}^2 \leq B_p \left(\|\Gamma^m \sigma(v_N)\|_{\beta,p}^2 + \|\Gamma^{m+1} \sigma(v_N)\|_{\beta,p}^2 \right) \Upsilon(2\beta/p).$$

Therefore, (3.18), and the induction hypothesis (3.33) for $n = N$, imply that $\|\Gamma^m \sigma(v_N)\|_{\beta,p}^2 < \infty$. Therefore, by choosing a constant $C > 0$ sufficiently large, we obtain

$$\|\Gamma^{m+1} v_{N+1}\|_{\beta,p}^2 \leq C \left(1 + \|\Gamma^{m+1} v_N\|_{\beta,p}^2 \right) \Upsilon(2\beta/p).$$

This proves (3.32) for $k = m+1$ and all $n \geq 0$, in the sense that $v_{N+1}(t, x) \in \mathbf{D}^{m+1,p}$, for all $p \geq 2$.

The proof is not complete yet, as we need to study the case that $\deg \neq \alpha_{m+1}$. Let $\deg = \{\deg_1, \dots, \deg_l\}$. Since by definition

$$D^{\deg} v_{N+1}(t, x) = D_{\deg_1}^{|\deg_1|} v_{N+1}(t, x) \cdots D_{\deg_l}^{|\deg_l|} v_{N+1}(t, x),$$

then

$$\begin{aligned} \mathbb{E} \|D^{|\deg|} v_{N+1}(t, x)\|_{H^{\otimes m+1}}^p \\ = \mathbb{E} (\|D^{|\deg_1|} v_{N+1}(t, x)\|_{H^{\otimes |\deg_1|}}^p \cdots \|D^{|\deg_l|} v_{N+1}(t, x)\|_{H^{\otimes |\deg_l|}}^p), \end{aligned}$$

where, $l \geq 2$, and $|\deg_1| + \cdots + |\deg_l| = m + 1$. Then by the generalized Hölder's inequality,

$$\begin{aligned} \{\mathbb{E} \|D^{|\deg|} v_{N+1}(t, x)\|_{H^{\otimes m+1}}^p\}^l &\leq \mathbb{E} \|D^{|\deg_1|} v_{N+1}(t, x)\|_{H^{\otimes |\deg_1|}}^{lp} \times \cdots \\ &\quad \times \mathbb{E} \|D^{|\deg_l|} v_{N+1}(t, x)\|_{H^{\otimes |\deg_l|}}^{lp}. \end{aligned}$$

Equivalently,

$$\begin{aligned} e^{\frac{-\beta t}{l^p}} \{\mathbb{E} \|D^{|\deg|} v_{N+1}(t, x)\|_{H^{\otimes m+1}}^p\}^{1/p} \\ \leq \left\{ e^{-\beta t} \mathbb{E} |\Gamma_{t,x}^{|\deg_1|} v_{N+1}|^{lp} \right\}^{\frac{1}{l^p}} \times \cdots \times \left\{ e^{-\beta t} \mathbb{E} |\Gamma_{t,x}^{|\deg_l|} v_{N+1}|^{lp} \right\}^{\frac{1}{l^p}}. \end{aligned}$$

We optimize, first the right-hand-side and then the left-hand-side of the latter inequality over all $t > 0$ and $x \in \mathbf{T}$ in order to find that

$$\|\Gamma^{\deg} v_{N+1}\|_{\frac{\beta}{l}, p} \leq \|\Gamma^{|\deg_1|} v_{N+1}\|_{\beta, lp} \cdots \|\Gamma^{|\deg_l|} v_{N+1}\|_{\beta, lp}.$$

If we replace β by $l\beta$, then we have

$$\begin{aligned} \|\Gamma^{\deg} v_{N+1}\|_{\beta, p} &\leq \|\Gamma^{|\deg_1|} v_{N+1}\|_{l\beta, lp} \cdots \|\Gamma^{|\deg_l|} v_{N+1}\|_{l\beta, lp} \\ &= \prod_{j=1}^l \|\Gamma^{|\deg_j|} v_{N+1}\|_{l\beta, lp} < \infty. \end{aligned} \tag{3.35}$$

Therefore, $v_n \in \mathbf{D}^{m+1, p}$ for all n . This finishes the proof. $\square$

Remark 3.1.14. *As in Proposition 3.1.4, we can iterate (3.32), and choose $\beta > 0$ sufficiently large to obtain*

$$\sup_n \mathbb{E} \|D^m v_n(t, x)\|_{H^{\otimes m}}^p < \infty.$$

This in turn implies that

$$\sup_n \|v_n(t, x)\|_{m, p} < \infty. \tag{3.36}$$

3.1.4 Proof of the Theorem 3.1.2

We prove Theorem 3.1.2 by applying induction on the order of the derivative k . In Proposition 3.1.4 we showed that $u \in \mathbf{D}^{1,p}$ and its derivative Du satisfies (3.1) for $k = 1$.

Assume now that $u \in \mathbf{D}^{k,p} < \infty$ for all $k \leq m-1$, $p \geq 1$ and the k th derivative $D^k u$ satisfies (3.1) for $k = 1, \dots, m-1$. This together with (3.36) imply that $u(t, x) \in \mathbf{D}^{m,p}$. Next we show that (3.1) also holds for $k = m$. This proof is basically repeating what we did for the proof of (3.32), and therefore we avoid going through the details. Define

$$\begin{aligned} c_n^2(t, x) &= \frac{1}{2} \|\Gamma_{t,x}^m(v_{n+1} - u)\|_{L^p(\Omega)}^2, \\ b_n^2(t, x) &= \left\{ \mathbb{E} \left\| q_{t-\bullet}(x, *) D_{\diamond}^{m-1}(\sigma(v_n(\bullet, *)) - \sigma(u(\bullet, *))) \right\|_{H^{\otimes m}}^p \right\}^{2/p}, \\ a_n^2(t, x) &= \left\{ \mathbb{E} \left\| \int_0^t \int_{\mathbf{T}} q_{t-r}(x, z) D^m(\sigma(v_n(r, z)) - \sigma(u(r, z))) w(dr, dz) \right\|_{H^{\otimes m}}^p \right\}^{2/p}. \end{aligned}$$

Our goal is to show that $\lim_{n \rightarrow \infty} \sup_{0 < t \leq T} \sup_{x \in \mathbf{T}} c_n^2(t, x) = 0$. By the triangle inequality,

$$c_n^2(t, x) \leq b_n^2(t, x) + a_n^2(t, x),$$

A similar argument as the proof of Proposition 3.1.13 leads to the following bound on b_n :

$$e^{-2\beta t/p} b_n(t, x)^2 \leq m \|\Gamma^{m-1}(\sigma(v_n) - \sigma(u))\|_{\beta,p}^2 \Upsilon(2\beta/p).$$

Finding an upper bound for $a_n(t, x)$ is similar to what we have done for F_1 , which led to (3.29). For a_n we have

$$e^{-2\beta t/p} a_n^2(t, x) \leq C_p \|\Gamma^m(\sigma(v_n) - \sigma(u))\|_{\beta,p}^2 \Upsilon(2\beta/p).$$

Another application of (3.19), together with the induction hypothesis shows that

$$e^{-2\beta t/p} a_n^2(t, x) \leq C_p (\lambda_n + \|\Gamma^m(v_n - u)\|_{\beta,p}^2) \Upsilon(2\beta/p),$$

where λ_n is independent of t and x , and $\lambda_n \rightarrow 0$ as $n \rightarrow \infty$. Therefore

$$e^{-2\beta t/p} c_n^2(t, x) \leq K_{p,m} (\theta_n + \|\Gamma^m(v_n - u)\|_{\beta,p}^2) \Upsilon(2\beta/p),$$

where $K_{p,m} = \max\{C_p, m\}$ and θ_n is independent of t and x and $\theta_n \rightarrow 0$ as $n \rightarrow \infty$. Therefore, by choosing β sufficiently large so that $K_{p,m}\Upsilon(2\beta/p) < 1$, we have

$$\|\Gamma^m(v_{n+1} - u)\|_{\beta,p}^2 \leq C_{k,m,\beta}(\theta_n + \|\Gamma^m(v_n - u)\|_{\beta,p}^2).$$

The latter inequality implies that $\|\Gamma^m(v_n - u)\|_{\beta,p}^2 \rightarrow 0$ as $n \rightarrow \infty$ which is equivalent to what we wanted to prove.

Remark 3.1.15. *The value of β transfers through the induction steps; i.e., its value in the m th step must be at least as large as its value in $(m - 1)$ th step.*

CHAPTER 4

Analysis of the Malliavin Matrix

In this chapter, we study the $L^p(\Omega)$ -integrability of the inverse of the Malliavin matrix. Here is the first place where we use the second assumption, Hypothesis **H2**, of this paper, which asserts that there are $1 < \alpha < \beta \leq 2$ and $0 < C_1 < C_2$, such that

$$C_1|n|^\alpha \leq \operatorname{Re} \Phi(n) \leq C_2|n|^\beta.$$

When (1.1) is linear, we know that if $\beta \leq 1$, then a solution does not exist. In this section, we want to show that for every $(t, x) \in E_T$, and $p \geq 2$,

$$\mathbb{E}(\|Du(t, x)\|^{-p}) < \infty. \quad (4.1)$$

This will finish the proof for the existence of the density. We will also find a bound for the density $p_t(x)$.

4.1 Existence of a density

We start with the following lemma.

Lemma 4.1.1. *Let u be the solution to Eq. (1.1). Let $p \geq 1$.*

1. *If we define*

$$V(t) = \sup_{x \in [0, 2\pi]} \mathbb{E} \left(\int_0^t \int_{\mathbf{T}} |D_{s,y}u(t, x)|^2 dy ds \right)^{p/2},$$

then

$$V(t) \leq C_{T,p} t^{(\alpha-1)p/2\alpha} \quad \forall t \in [0, T].$$

2. If we fix $t \in [0, T]$ and for any $\delta \in (0, t)$ define

$$W(\delta) = \sup_{x \in [0, 2\pi]} \mathbb{E} \left(\int_{t-\delta}^t \int_{\mathbf{T}} |D_{s,y} u(t, x)|^2 dy ds \right)^{p/2},$$

then

$$W(\delta) \leq C_{T,p} \delta^{(\alpha-1)p/2\alpha}.$$

We prove only the first part in detail, as the second part can be proved similarly. We will only mention the minor changes. But before starting the proof, we state the well-known Bellman-Gronwall's inequality, which will be used in our proof.

Lemma 4.1.2 (Bellman-Gronwall's Lemma). *Let $\lambda(t)$ be a nonnegative piecewise continuous function of time t and $C \geq 0$. If the function $y(t)$ satisfies the inequality*

$$y(t) \leq \lambda(t) + C \int_0^t y(s) ds,$$

then

$$y(t) \leq \lambda(t) + C \int_0^t \lambda(s) e^{C(t-s)} ds.$$

Proof. See[34, Lemma A.6.1.]. □

Proof of Lemma 4.1.1. We start with the expansion of $D_{s,y} u(t, x)$. Because, by (3.7),

$$D_{s,y} u(t, x) = q_{t-s}(x, y) \sigma(u(s, y)) + \int_0^t \int_{\mathbf{T}} q_{t-r}(x, z) D_{s,y} \sigma(u(r, z)) w(dr, dz),$$

it follows that

$$\begin{aligned} \|Du(t, x)\|_H &\leq \text{Lip}_\sigma \left\{ \int_0^t \int_{\mathbf{T}} q_{t-s}^2(x, y) dy \right\}^{1/2} \\ &\quad + \left\| \int_0^t \int_{\mathbf{T}} q_{t-r}(x, z) D\sigma(u(r, z)) w(dr, dz) \right\|_H. \end{aligned} \tag{4.2}$$

Then, by (2.20),

$$\begin{aligned} \|Du(t, x)\|_H &\leq C_\alpha t^{\frac{\alpha-1}{2\alpha}} \\ &\quad + \left\| \int_0^t \int_{\mathbf{T}} q_{t-r}(x, z) D\sigma(u(r, z)) w(dr, dz) \right\|_H. \end{aligned} \tag{4.3}$$

By raising to the power of p and taking expectation,

$$\mathbb{E} \|Du(t, x)\|_H^p \lesssim_{p, \alpha} t^{\frac{p(\alpha-1)}{2\alpha}} + \mathbb{E} \left\| \int_0^t \int_{\mathbf{T}} q_{t-r}^2(x, z) D\sigma(u(r, z)) w(dr, dz) \right\|_H^p.$$

Next, the Burkholder's inequality yields,

$$\mathbb{E} \|Du(t, x)\|_H^p \lesssim_{p, \alpha} t^{\frac{p(\alpha-1)}{2\alpha}} + \mathbb{E} \left| \int_0^t \int_{\mathbf{T}} q_{t-r}^2(x, z) \|D\sigma(u(r, z))\|_H^2 dz dr \right|^{p/2}.$$

Then

$$\mathbb{E} \|Du(t, x)\|_H^p \lesssim_{p, \alpha, \sigma, \sigma'} t^{\frac{p(\alpha-1)}{2\alpha}} + \mathbb{E} \left| \int_0^t \int_{\mathbf{T}} q_{t-r}^2(x, z) \|Du(r, z)\|_H^2 dz dr \right|^{p/2}.$$

Next, by observing that

$$q_{t-r}^2(x, z) \|D(u(r, z))\|_H^2 = q_{t-r}^{\frac{2p-4}{p}}(x, z) \left(q_{t-r}^{\frac{4}{p}}(x, z) \|D(u(r, z))\|_H^2 \right),$$

we may apply the Hölder inequality to obtain

$$\begin{aligned} \int_0^t \int_{\mathbf{T}} q_{t-r}^2(x, z) \|D(u(r, z))\|_H^2 dz dr &\leq \left(\int_0^t \int_{\mathbf{T}} q_{t-r}^2(x, z) dz dr \right)^{(p-2)/p} \times \\ &\quad \left(\int_0^t \int_{\mathbf{T}} q_{t-r}^2(x, z) \|D(u(r, z))\|_H^p dz dr \right)^{2/p}. \end{aligned}$$

This yields

$$\begin{aligned} \left(\int_0^t \int_{\mathbf{T}} q_{t-r}^2(x, z) \|D(u(r, z))\|_H^2 dz dr \right)^{p/2} &\leq \left(\int_0^t \int_{\mathbf{T}} q_{t-r}^2(x, z) dz dr \right)^{(p-2)/2} \times \\ &\quad \left(\int_0^t \int_{\mathbf{T}} q_{t-r}^2(x, z) \|Du(r, z)\|_H^p dz dr \right). \end{aligned}$$

Another application of (2.20) yields

$$\begin{aligned} \left(\int_0^t \int_{\mathbf{T}} q_{t-r}^2(x, z) \|D(u(r, z))\|_H^2 dz dr \right)^{p/2} &\leq C t^{\frac{(\alpha-1)(p-2)}{2\alpha}} \\ &\quad \left(\int_0^t \int_{\mathbf{T}} q_{t-r}^2(x, z) \|Du(r, z)\|_H^p dz dr \right). \end{aligned}$$

Therefore,

$$\begin{aligned} \mathbb{E} \|Du(t, x)\|_H^p &\lesssim_{p, \alpha, \sigma, \sigma'} t^{\frac{p(\alpha-1)}{2\alpha}} + \\ &\quad t^{\frac{(\alpha-1)(p-2)}{2\alpha}} \int_0^t \int_{\mathbf{T}} q_{t-r}^2(x, z) \mathbb{E} \|Du(r, z)\|_H^p dz dr. \end{aligned}$$

Then, for a new constant C , we have

$$\begin{aligned} & \frac{1}{C} \mathbb{E} \|Du(t, x)\|_H^p \\ & \leq t^{\frac{(\alpha-1)p}{2\alpha}} + t^{\frac{(p-2)(\alpha-1)}{2\alpha}} \int_0^t \sup_{z \in [0, 2\pi]} \mathbb{E} \|Du(r, z)\|_H^p (t-r)^{-1/\alpha} dr. \end{aligned}$$

We write everything in terms of $V(t)$,

$$V(t) \leq C \left(t^{\frac{(\alpha-1)p}{2\alpha}} + t^{\frac{(p-2)(\alpha-1)}{2\alpha}} \int_0^t V(r) (t-r)^{-1/\alpha} dr \right).$$

Next, apply Hölder's inequality to the integral on the right in order to find that

$$\int_0^t V(r) (t-r)^{-1/\alpha} dr \leq \left(\int_0^t V(r)^{p_1} dr \right)^{1/p_1} \left(\int_0^t (t-r)^{-q_1/\alpha} dr \right)^{1/q_1},$$

where $p_1 = (\alpha + 1)/2$ and $q_1 = (\alpha + 1)/(\alpha - 1)$. Because $q_1/\alpha < 1$,

$$\left(\int_0^t (t-r)^{-\frac{q_1}{\alpha}} dr \right)^{\frac{1}{q_1}} < \infty.$$

Therefore there is C such that

$$V(t) \leq C \left(t^{\frac{(\alpha-1)p}{2\alpha}} + t^{\frac{(p-2)(\alpha-1)}{2\alpha}} \left(\int_0^t V(r)^{p_1} dr \right)^{1/p_1} \right).$$

Consequently for some $C > 0$,

$$V(t)^{p_1} \leq C \left(t^{\frac{(\alpha-1)pp_1}{2\alpha}} + t^{\frac{p_1(p-2)(\alpha-1)}{2\alpha}} \int_0^t V(r)^{p_1} dr \right).$$

Again, since $0 \leq t \leq T$, then we can choose C such that

$$V(t)^{p_1} \leq C \left(t^{\frac{(\alpha-1)pp_1}{2\alpha}} + \int_0^t V(r)^{p_1} dr \right).$$

Then by the Gronwall's lemma we have

$$V(t) \leq Ct^{\frac{(\alpha-1)p}{2\alpha}}.$$

This finishes the proof of (1). Next we prove the second claim. If we proceed as what we have done for part 1, we will get

$$\begin{aligned} \|Du(t, x)\|_{H^*} & \leq \int_{t-\delta}^t \int_{\mathbf{T}} q_{t-s}(x, y) \sigma(u(s, y)) dy ds \\ & \quad + \left\| \int_0^t \int_{\mathbf{T}} q_{t-r}(x, z) D\sigma(u(r, z)) w(dr, dz) \right\|_{H^*}, \end{aligned}$$

where H^* denotes $L^2([t - \delta] \times \mathbf{T})$. We change the variable $t - s \rightarrow r$ on the first integral on the right and apply (2.20) to find an upper bound for it. Then we continue similar to our previous proof with H replaced by H^* to arrive at

$$W(\delta) \leq C \left(\delta^{\frac{(\alpha-1)p}{2\alpha}} + \delta^{\frac{(p-2)(\alpha-1)}{2\alpha}} \int_0^\delta W(r)(\delta - r)^{-1/\alpha} dr \right).$$

Then an application of Gronwall's lemma finishes the proof. $\square$

The following corollary is an estimate on the Malliavin's derivative of the solution of the equation (1.1).

Corollary 4.1.3. *Let u be the solution to the equation (1.1), and Φ the Lévy exponent corresponding the differential operator $\mathcal{L}$. Define*

$$I_\delta = \int_{t-\delta}^t \int_{\mathbf{T}} \left| \int_0^t \int_{\mathbf{T}} q_{t-r}(x, z) D_{sy} \sigma(u(r, z)) w(dr, dz) \right|^2 dy ds, \quad (4.4)$$

where $q = q_t(x)$ is the transition density corresponding to $\mathcal{L}$. If the Φ satisfies Hypothesis **H** 2, then

$$\mathbb{E}(|I_\delta|^p) \leq C \delta^{2p(\alpha-1)/\alpha},$$

Proof. By the Burkholder's inequality

$$\begin{aligned} \mathbb{E}(|I_\delta|^p) &= \mathbb{E} \left(\left| \int_{t-\delta}^t \int_{\mathbf{T}} \left| \int_s^t \int_{\mathbf{T}} q_{t-r}(x, z) D_{s,y} \sigma(u(r, z)) w(dr, dz) \right|^2 dy ds \right|^p \right) \\ &\leq c_p \text{Lip}_\sigma^{2p} \mathbb{E} \left| \int_{t-\delta}^t \int_{\mathbf{T}} q_{t-r}^2(x, z) \left(\int_{t-\delta}^r \int_{\mathbf{T}} |D_{sy} u(r, z)|^2 dy ds \right) dz dr \right|^p. \end{aligned}$$

Raising to the power $1/p$ and applying Minkowski's inequality gives us

$$\begin{aligned} \{\mathbb{E}(|I_\delta|^p)\}^{1/p} &\leq c_p^{1/p} \text{Lip}_\sigma^2 \left\{ \mathbb{E} \left| \int_{t-\delta}^t \int_{\mathbf{T}} q_{t-r}^2(x, z) \left(\int_{t-\delta}^r \int_{\mathbf{T}} |D_{s,y} u(r, z)|^2 dy ds \right) dz dr \right|^p \right\}^{1/p} \\ &\leq c_p^{1/p} \text{Lip}_\sigma^2 \left(\int_{t-\delta}^t \int_{\mathbf{T}} q_r^2(z, x) dz dr \right) \sup_{(r,z) \in [0, \delta] \times [0, 2\pi]} \left\{ \mathbb{E} \left| \int_{t-\delta}^t \int_{\mathbf{T}} |D_{s,y} u(r, z)|^2 dy ds \right|^p \right\}^{1/p}. \end{aligned}$$

Then, by (2.20), and Lemma 4.1.1,

$$\begin{aligned} \mathbb{E}(|I_\delta|^p) &\leq C \left(\int_0^\delta \int_{\mathbf{T}} q_r^2(z, x) dz dr \right)^p \\ &\quad \times \sup_{(r, z) \in [0, \delta] \times [0, 2\pi]} \mathbb{E} \left| \int_{t-\delta}^t \int_{\mathbf{T}} |D_{s,y} u(r, z)|^2 dy ds \right|^p \\ &\leq C \delta^{(\alpha-1)p/\alpha} \delta^{(\alpha-1)p/\alpha}. \end{aligned}$$

The proof is complete. $\square$

Finally we quote from [15, page 97] a lemma which allows us to put together the results of Lemma 4.1.1 and Corollary 4.1.3 and prove the existence of the negative moments (4.1).

Lemma 4.1.4. *Let F be nonnegative random variable. Then property (4.1) holds for all $p \geq 2$ if and only if for every $q \in [2, \infty)$ there exists $\epsilon_0 = \epsilon_0(q) > 0$, such that*

$$\mathbb{P}(\|Du(t, x)\|_H^2 < \epsilon) < C\epsilon^q,$$

for all $\epsilon < \epsilon_0$.

Proof of Theorem 1.0.2. We need only to show that (4.1) holds for every $(t, x) \in E_T$ and all $p \geq 2$. Let $q = q_t(x)$ be the transition density corresponding to Φ and $\mathcal{L}$. Since

$$q_{t-s}(y, x) \sigma(u(s, y)) = D_{sy} u(t, x) - \int_0^t \int_{\mathbf{T}} q_{t-r}(x, z) D_{sy} \sigma(u(r, z)) w(dr, dz),$$

considering the fact that $\sigma \geq \kappa > 0$, then

$$|D_{sy} u(t, x)|^2 \geq \frac{\kappa^2}{2} q_{t-s}^2(y, x) - \left| \int_0^t \int_{\mathbf{T}} q_{t-r}(x, z) D_{sy} \sigma(u(r, z)) w(dr, dz) \right|^2.$$

Therefore

$$\begin{aligned} \|Du(t, x)\|_H^2 &\geq \int_{t-\delta}^t \int_{\mathbf{T}} |D_{sy} u(t, x)|^2 dy ds \\ &\geq \frac{\kappa^2}{2} \int_{t-\delta}^t \int_{\mathbf{T}} q_{t-s}^2(y, x) dy ds - \int_{t-\delta}^t \int_{\mathbf{T}} \left| \int_0^t \int_{\mathbf{T}} q_{t-r}(x, z) D_{sy} \sigma(u(r, z)) w(dr, dz) \right|^2 dy ds. \end{aligned}$$

If we let $\tau = t - s$ in the first integral on the right, then we get

$$\|Du(t, x)\|_H^2 \geq J_\delta - I_\delta \quad \forall \delta \in (0, t),$$

where I_δ is defined in (4.4) and

$$J_\delta := \frac{\kappa^2}{2} \int_0^\delta \|q_u\|_{L^2(\mathbf{T})} du.$$

If we choose $\delta > 0$ such that $J_\delta - \epsilon > 0$, then by the Chebyshev's inequality

$$\mathbb{P}(\|D_{sy}u(t, x)\|_H^2 < \epsilon) \leq \mathbb{P}(J_\delta - I_\delta < \epsilon) \leq \frac{\mathbb{E}|I_\delta|^p}{(J_\delta - \epsilon)^p}.$$

Then, by Corollary 4.1.3 and (2.20) we have

$$\mathbb{P}(\|Du(t, x)\|_H^2 < \epsilon) \leq \frac{C_1 \delta^{2(\alpha-1)p/\alpha}}{(\frac{C}{2} \delta^{(\beta-1)/\beta} - \epsilon)^p}.$$

Take $\delta = (4\epsilon/C)^{\frac{\beta}{\beta-1}}$ to get

$$\mathbb{P}(\|Du(t, x)\|_H^2 < \epsilon) \leq C_{\alpha,p} \epsilon^{\theta p}, \quad (4.5)$$

where $\theta = \frac{2\beta(\alpha-1)}{\alpha(\beta-1)} - 1 > 0$. Combine this with Lemma 4.1.4 to complete the proof of the existence of the a density. $\square$

4.2 Upper bound for the density

Next, we prove the last claim of Theorem 1.0.2. We first mention the following result which is a consequence of Eq. (2.57). Then come a few lemmas, which pave the road for applying that result.

Proposition 4.2.1. *Let q, α, λ be three positive real numbers such that $q^{-1} + \alpha^{-1} + \lambda^{-1} = 1$. Let F be a random variable in the space $\mathbf{D}^{2,\alpha}$, such that $\mathbb{E}\|DF\|_H^{-2\lambda} < \infty$. Then the density $p(x)$ of F can be estimated as follows.*

$$p(x) \leq c_{q,\alpha,\beta} (\mathbb{P}(|F| > |x|))^{1/q} \times (\mathbb{E}\|DF\|_H^{-1} + \|D^2F\|_{L^\alpha(\Omega, H \otimes H)} \|\|DF\|_H^{-2}\|_\lambda) \quad (4.6)$$

Proof. See [15] page 85. $\square$

Lemma 4.2.2. *We have*

$$l\Upsilon(2\beta/l) \leq \frac{1}{8\pi} \left[\frac{l^2}{\beta} + C \frac{l^{2-\frac{1}{\alpha}}}{\beta^{1-\frac{1}{\alpha}}} \right]. \quad (4.7)$$

where $C = 2C_\alpha/c^{1/\alpha}$, $c = C_1$ is defined in Hypothesis **H2**, and $C_\alpha = \int_0^\infty \frac{dx}{1+x^\alpha} = \pi/\alpha \csc(\pi/\alpha)$; see [29].

Proof. Starting with the definition of $\Upsilon(\beta)$, we have,

$$8\pi\Upsilon(2\beta) = \frac{1}{\beta} + 2 \sum_{n=1}^{\infty} \frac{1}{\beta + cn^{\alpha}} \leq \frac{1}{\beta} + 2 \int_0^{\infty} \frac{dx}{\beta + cx^{\alpha}} = \frac{1}{\beta} + \frac{2C}{\beta^{1-\frac{1}{\alpha}}}.$$

This will give us the result. $\square$

Lemma 4.2.3. *Let C be as in Lemma 4.2.2, and define,*

$$\nu = \frac{2\alpha - 1}{\alpha - 1}, \quad b > \frac{2\text{Lip}_{\sigma}^2}{\pi} \vee \left(\frac{2\text{Lip}_{\sigma}^2 C}{\pi} \right)^{\frac{\alpha}{\alpha-1}}. \quad (4.8)$$

If $\beta > bl^{\nu}$, then

$$z_l^2 \text{Lip}_{\sigma}^2 \Upsilon(2\beta/l) < 1/2. \quad (4.9)$$

Proof. By combining (4.7) and the fact that $z_l \leq 2l^{1/2}$ (see [11]), we only need to show that

$$\frac{l^2}{\beta} + C \frac{l^{2-\frac{1}{\alpha}}}{\beta^{1-\frac{1}{\alpha}}} < \frac{\pi}{\text{Lip}_{\sigma}^2}.$$

It is sufficient to show that each summand on the left is smaller than $\frac{1}{2} \frac{\pi}{\text{Lip}_{\sigma}^2}$. The first summand is l^2/bl^{ν} . Since $\nu \geq 2$, we only need to have $b > 2\text{Lip}_{\sigma}^2/\pi$, which holds true. A simple computation shows that, if $b > \left(\frac{2\text{Lip}_{\sigma}^2 C}{\pi} \right)^{\frac{\alpha}{\alpha-1}}$, and ν as given above, the second summand is also smaller than $\frac{1}{2} \frac{\pi}{\text{Lip}_{\sigma}^2}$. $\square$

Lemma 4.2.4. *Let ν and b be as in (4.8). For $|x| \geq m$ we have*

$$\mathbb{P}(|u(t, y)| > |x|) \leq \exp(\Lambda(t, x)), \quad (4.10)$$

where

$$\Lambda(t, x) = d t^{-1/(\nu-1)} (\ln(|x|/m))^{\nu/(\nu-1)}, \quad (4.11)$$

and $d = \frac{1-\nu}{(b\nu)^{1/(\nu-1)}} < 0$.

Proof. From (2.45) we know that

$$\mathbb{E}|u(t, y)|^l \leq \left(\frac{k + z_l |\sigma(0)| \sqrt{\Upsilon(2\beta/l)}}{1 - z_l \text{Lip}_{\sigma} \sqrt{\Upsilon(2\beta/l)}} \right)^l e^{\beta t},$$

where $k = \sup_{x \in \mathbf{T}} u_0(x)$. Then, from (4.9), we conclude that

$$\mathbb{E}|u(t, y)|^l \leq m^l e^{\beta t}, \quad (4.12)$$

where $m = 2k + 2|\sigma(0)/\text{Lip}_\sigma|$, and $k = \sup_{x \in \mathbf{T}} u_0(x)$. Since by Chebyshev's inequality, $\mathbb{P}(|u(t, y)| > |x|) \leq \mathbb{E}|u(t, y)|^l / |x|^l$, for every $l \geq 1$, then

$$\mathbb{P}(|u(t, y)| \geq |x|) \leq \exp[tbl^\nu - l \ln(|x|/m)] \quad \forall l \geq 1. \quad (4.13)$$

Therefore,

$$\mathbb{P}(|u(t, y)| \geq |x|) \leq \inf_{l \geq 1} \exp[tbl^\nu - l \ln(|x|/m)]. \quad (4.14)$$

We can solve this optimization problem as follows: let $A(l) := tbl^\nu - l \ln(|x|/m)$. Then $A'(l_{\min}) = 0$, if and only if $l_{\min} = \left(\frac{\ln(|x|/m)}{tb\nu}\right)^{1/(\nu-1)}$. We have where the last inequality follows from the fact that $\nu > 2$. $\square$

Proof of Corollary 1.0.5. In (4.6) let $F = u(t, y)$, $q = \lambda = 4$ and $\alpha = 2$, to get the bound,

$$(\mathbb{P}(|F| > |x|))^{1/4} \times (\mathbb{E}\|DF\|_H^{-1} + \|D^2F\|_{L^2(\Omega, H \otimes H)} \|\|DF\|_H^{-2}\|_4). \quad (4.15)$$

Jensen's inequality followed by (4.5) implies that $\mathbb{E}\|DF\|_H^{-1} \leq (\mathbb{E}\|DF\|_H^{-2})^{1/2} < \infty$, uniformly on $x \in \mathbf{T}$ and $t > 0$. With the same argument, we have $\|\|DF\|_H^{-2}\|_4 < \infty$. Choose $C_1 \in (0, \infty)$ such that

$$\mathbb{E}\|DF\|_H^{-1} \vee \|\|DF\|_H^{-2}\|_4 \leq C_1.$$

From (3.32) we conclude, when $\beta > 0$ is sufficiently large,

$$\sup_{(t, x) \in (0, \infty) \times \mathbf{T}} e^{-\beta t} \mathbb{E}\|D^2u(t, y)\|_{H \otimes H}^2 \leq \frac{C\Upsilon(\beta)}{1 - C\Upsilon(\beta)}.$$

Therefore, there are a constants $\beta, C > 0$, such that $\|D^2F\|_{L^2(\Omega, H \otimes H)} < Ce^{\beta t}$. Therefore, by choosing $\beta_0 = \beta$, the parenthesis on the right is bounded by $C(1 + Ce^{\beta_0 t/2})$. Now, apply Lemma 4.2.4 to finish. $\square$

CHAPTER 5

Intermittency and Malliavin Derivatives

In this chapter we investigate the intermittency of the random field $D^k u(t, x)$ for $k = 1, 2, \dots$, which is an asymptotic property of these random fields. Roughly speaking, an intermittent random field is distinguished by its sharp peaks. The following three examples will illustrate the concept.

Example 5.0.1. consider a triangular array $\{X_i^n, n \geq 1, i = 1, \dots, n\}$ of Bernoulli *i.i.d.* random variables with $P(X_i^n = 1 + a^2) = P(X_i^n = 0) = 1/2$. Let $Y_n = \prod_{i=1}^n X_i^n$. Then the function

$$\gamma(p) := \frac{1}{n} \log EY_n^p = p \log(1 + a^2) - \log 2 \quad (5.1)$$

satisfies $0 < \gamma(1) < \frac{\gamma(2)}{2} < \dots$. We also have $\gamma(p) < \infty$ for all $p \geq 1$.

If we let $a^2 = 2$, then the random field Y_n is zero with a large probability, and takes a large value with a small probability. Nevertheless, an application of the Borel-Cantelli lemma shows that $P(Y_n = 0 \text{ i.o.}) = 0$. This means that the large peaks will eventually occur for almost all paths. The next example is from the theory of stochastic ordinary differential equations.

Example 5.0.2 (Exponential Martingale). Consider the stochastic differential equation

$$dX_t = X_t dB_t, \quad X_0 = 1. \quad (5.2)$$

We know that $X_t = \exp(B_t - t/2)$ is the solution to (5.2). We also know that $X_t \rightarrow 0$ a.s. as $t \rightarrow \infty$, and $EX_t^p = \exp(\frac{p(p-1)t}{2})$. We can check that the map $p \mapsto \gamma(p)/p$ defined by (5.1) is finite, positive and strictly increasing.

Our next example from [40] is more sophisticated.

Example 5.0.3 (Anderson parabolic problem). Consider the following parabolic anderson equation on $\mathbf{Z}^d$.

$$\left\{ \begin{array}{ll} \partial_t u = \kappa \Delta u + \xi(x)u & t \geq 0, x \in \mathbf{Z}^d, \\ u(0, x) = 1 & x \in \mathbf{Z}^d, \end{array} \right. \quad (5.3)$$

where $\Delta\psi(x) := \sum_{|x'-x|=1} [\psi(x') - \psi(x)]$; i.e., $\kappa\Delta$ is a generator of the homogeneous random symmetrical walk x_t on $\mathbf{Z}^d$ with continuous time and the rate of jumps κ in all directions $x \rightarrow x'$, $|x - x'| = 1$. Then $\bar{\gamma}(p) = p^2/2$, and (5.5) below holds. If $u(t, x)$ is the density of particles, the intermittency of the solution fields indicates a highly nonuniform, i.e., as t goes to ∞ , there are times at which most of the mass is concentrated at on the peaks.

To define the intermittency, we choose and fix some $x_0 \in \mathbf{R}$. Define the upper p th-moment Liapounov exponent $\bar{\gamma}(p)$ of u as

$$\bar{\gamma}(p) = \limsup_{t \rightarrow \infty} \frac{1}{t} \log \mathbb{E}|u(t, x_0)|^p \quad p \in [1, \infty). \quad (5.4)$$

When $u(t, x)$ is ergodic this limit is independent of x_0 . If $u(t, x)$ is the solution to (1.1), then $\bar{\gamma}(p)$ is independent of x_0 if $u_0(x)$ is a constant. We say u is *intermittent* if regardless of value of x_0 ,

$$0 < \gamma(1) < \frac{\gamma(2)}{2} < \dots < \infty. \quad (5.5)$$

This implies a progressive increase of the moments. For example the second moment increases faster than the square the second moment. To quote from [27], “When a random field is intermittent, asymptotically as $t \rightarrow \infty$, the main contribution to each moment function is carried by higher and higher and more and more widely spaced “overshoots” (“peaks”) of the random field.” In this thesis, we consider a variant of the above intermittency; i.e., the *weak intermittency*. This means that we only require that

$$\bar{\gamma}(2) > 0, \quad \text{and} \quad \bar{\gamma}(p) < \infty, \quad \forall p \in [2, \infty). \quad (5.6)$$

In some situations, the weak intermittency may imply the full intermittency; i.e., (5.6) may imply (5.5). Since $\bar{\gamma}$ is convex and $\bar{\gamma}(0) = 0$, then the map $p \rightarrow \bar{\gamma}(p)/p$ is nondecreasing. It is also easy to check that the map $p \rightarrow \bar{\gamma}(p)/p$ is strictly increasing if $2\bar{\gamma}(1) < \bar{\gamma}(2)$. We also

observe that convexity implies that if $\bar{\gamma}(1) = 0$, then the weak intermittency implies the full intermittency.

There is a big body of works on this intermittency. We refer to [12, 40, 41] for more details. In [12], the authors device a probabilistic method for proving the intermittency. This method employs the Feynman-Kac formula to formulate (1.1). Here, we borrow the idea of our analytical approach from [23], in which the authors introduced this technique to prove that the solution $u(t, x)$ to (1.1) is intermittent.

Proof of Theorem 1.0.2. When $\sigma(x) = \lambda x$, by (3.1), we have

$$D_\alpha^k u(t, x) = \lambda q_{t-\hat{s}}(x, \hat{y}) D_\alpha^{k-1} u(\hat{s}, \hat{y}) + \lambda \int_{\hat{s}}^t \int_0^{2\pi} q_{t-r}(x, z) D_\alpha^k u(r, z) w(dr, dz).$$

Therefore,

$$\begin{aligned} \mathbb{E} \int_{E^k} |D_\alpha^k u(t, x)|^2 d\alpha &= \lambda^2 \int_{E^k} q_{t-\hat{s}}^2(x, \hat{y}) \mathbb{E} |D_\alpha^{k-1} u(\hat{s}, \hat{y})|^2 d\alpha \\ &\quad + \mathbb{E} \int_{E^k} \lambda^2 \left| \int_{\hat{s}}^t \int_0^{2\pi} q_{t-r}(x, z) D_\alpha^k u(r, z) w(dr, dz) \right|^2 d\alpha. \end{aligned}$$

This yields

$$\mathbb{E} \|D^k u(t, x)\|_{H^{\otimes k}}^2 \geq \lambda^2 \|q_{t-\bullet}^2(x, *) \mathbb{E} D_\diamond^{k-1} u(\bullet, *)\|_{H^{\otimes k}}^2.$$

To show that $\int_0^\infty e^{-\beta t} \mathbb{E} \|D^k u(t, x)\|_{H^{\otimes k}}^2 dt = \infty$, it is sufficient to show that

$$\int_{\mathbf{T}^k} d\xi \int_0^\infty dt e^{-\beta t} \int_0^t ds_1 \cdots \int_0^t ds_k (q_{t-\hat{s}}^2(x, \hat{y}) \mathbb{E} |D_\alpha^{k-1} u(\hat{s}, \hat{y})|^2) = \infty,$$

where $d\xi = dy_1 \cdots dy_k$. Consequently, it is sufficient to show that

$$\int_0^\infty e^{-\beta t} \int_0^t q_{t-s}^2(x, y) \|\mathbb{E} D^{k-1} u(s, y)\|_{H^{\otimes k-1}}^2 ds dt = \infty,$$

for almost all $y \in \mathbf{T}^k$. Therefore, after a change of the variables $[t - s_i \rightarrow \tau]$, we should show that

$$\left(\int_0^\infty e^{-\beta \tau} q_\tau^2(x, y) d\tau \right) \int_0^\infty e^{-\beta s} \mathbb{E} \|D^{k-1} u(s, y)\|_{H^{\otimes k-1}}^2 ds = \infty.$$

We conclude that $\int_0^\infty e^{-\beta s} \mathbb{E} \|D^{k-1}u(s, y)\|_{H^{\otimes k-1}}^2 ds = \infty$ implies that

$$\int_0^\infty e^{-\beta s} \mathbb{E} \|D^k u(s, y)\|_{H^{\otimes k}}^2 ds = \infty.$$

Therefore, if we prove the claim for $k = 1$, then, by an application of induction on k , we can finish the proof. Next, we prove the case $k = 1$ for general σ . By squaring (3.7) we obtain

$$\mathbb{E} \|Du(t, x)\|_H^2 \geq \int_0^t \int_0^{2\pi} p_{t-s}^2(x, y) \mathbb{E} |\sigma(u(s, y))|^2 dy ds.$$

If we assume $c := \liminf_{|x| \rightarrow \infty} |\sigma(x)/x| > 0$, then for every $c_0 \in (0, c)$, there is A such that $|\sigma(x)| \geq c_0|x|$ for $|x| > A$. Therefore

$$\mathbb{E} |\sigma(u(s, y))|^2 \geq c_0^2 \mathbb{E} (|u(s, y)|^2; A) \geq c_0^2 \mathbb{E} |u(s, y)|^2 - c_0^2 A^2. \quad (5.7)$$

Therefore,

$$\begin{aligned} & \int_0^\infty e^{-\beta t} \mathbb{E} \|Du(t, x)\|_H^2 dt \\ & \geq c_0^2 \int_0^\infty e^{-\beta t} \int_0^t \int_{\mathbf{T}} q_{t-s}^2(x, y) \mathbb{E} |u(s, y)|^2 dy ds dt \\ & \quad - c_0^2 A^2 \int_0^\infty e^{-\beta t} \int_0^t \int_{\mathbf{T}} q_{t-s}^2(x, y) dy ds dt. \end{aligned}$$

We know that the last integral is less than or equal to $\Upsilon(\beta)/\beta \in [0, \infty)$. An application of Fubini's lemma followed by a change of variable $t - s \rightarrow \tau$ yields

$$\begin{aligned} & \int_0^\infty e^{-\beta t} \int_0^t \int_{\mathbf{T}} q_{t-s}^2(x, y) \mathbb{E} |u(s, y)|^2 dy ds dt \\ & = \int_{\mathbf{T}} \left(\int_0^\infty \mathbb{E} |u(s, y)|^2 e^{-\beta s} ds \right) \left(\int_0^\infty e^{-\beta \tau} q_\tau^2(x, y) d\tau \right) dy. \end{aligned}$$

From [23, proof of Theorem 2.10] we know that $\int_0^\infty \mathbb{E} |u(s, y)|^2 e^{-\beta s} ds = \infty$, when the initial profile u_0 is sufficiently large. Here we summarize their proof. From (5.7) we conclude that

$$F_\beta(x) - (\mathcal{H} F_\beta)(x) \geq G_\beta(x) + \frac{-c_0^2 A^2}{\beta} \Upsilon(\beta), \quad (5.8)$$

where

$$\begin{aligned} F_\beta(x) &= \int_0^\infty e^{-\beta t} \mathbb{E} |u(t, x)|^2 dt, \\ G_\beta(x) &= \int_0^\infty e^{-\beta t} \mathbb{E} |u_0(x + X_t)|^2 dt, \\ \mathcal{H}(x) &= \int_0^\infty e^{-\beta t} c_0^2 q_t^2(x) dt, \end{aligned}$$

where $\mathcal{H}f(x) := H_\beta * f(x)$. Therefore, by linearity and positivity of $\mathcal{H}$,

$$\mathcal{H}^n F_\beta(x) - (\mathcal{H}^{n+1} F_\beta)(x) \geq \mathcal{H}^n G_\beta(x) + \mathcal{H}^n \frac{-c_0^2 A^2}{\beta} \Upsilon(\beta). \quad (5.9)$$

Let us assume that $u_0 \geq k$. Then $\mathcal{H}G_\beta(x) \geq \frac{c_0^2 k^2}{\beta} \Upsilon(\beta)$. Therefore,

$$\mathcal{H}^n F_\beta(x) - (\mathcal{H}^{n+1} F_\beta)(x) \geq \frac{k^2 - A^2 c_0^2 \Upsilon(\beta)}{\beta} |c_0^2 \Upsilon(\beta)|^n. \quad (5.10)$$

Summing this inequality from $n = 0$ to $n = \infty$ yields

$$F_\beta(x) \geq \frac{k^2 - A^2 c_0^2 \Upsilon(\beta)}{\beta} \sum_{n=0}^{\infty} |c_0^2 \Upsilon(\beta)|^n = \infty,$$

given $k^2 \geq A^2 c_0^2 \Upsilon(\beta)$, and $\Upsilon(\beta) \geq c_0^{-2}$. Such $\beta > 0$ always exists, because X is a Lévy process on a compact set and so is recurrent (see, for example, [39, page 144]); i.e., $\Upsilon(0) = \infty$; see Port-Stone [52] and Bertoin [8, page 33]. Therefore $\int_0^\infty e^{-\beta t} \mathbb{E} \|Du(t, x)\|_H^2 dt = \infty$ for all $x \in \mathbf{T}$. This implies that $\bar{\gamma}^2(\beta) \geq \alpha > 0$ for any $\alpha < \beta$. To see this, assume to the contrary $\bar{\gamma}^1(2) < \alpha$. Then

$$\limsup_{t \rightarrow \infty} \frac{\log \mathbb{E} \|Du(t, x)\|_H^2}{t} < \alpha.$$

This means that $\mathbb{E} \|Du(t, x)\|_H^2 \lesssim e^{\alpha t}$, and contradicts the fact that

$$\int_0^\infty e^{-\beta t} \mathbb{E} \|Du(t, x)\|_H^2 dt = \infty.$$

Similarly, if we assume that $\sigma'(x) \geq c > 0$, then

$$\begin{aligned} \mathbb{E} \|D^2 u(t, x)\|_{H^{\otimes 2}}^2 &\geq \int_{E_t^2} p_{t-\hat{s}}^2(x, \hat{y}) \mathbb{E} |D_{\hat{\alpha}} \sigma(u(\hat{s}, \hat{y}))|^2 d\alpha \\ &\geq 2c^2 \int_0^t \int_0^{2\pi} \int_0^s \int_0^{2\pi} p_{t-s}^2(x, y) \mathbb{E} |D_{r,v} u(s, y)|^2 dv dr dy ds, \end{aligned}$$

Therefore,

$$\begin{aligned} \int_0^\infty e^{-\beta t} \mathbb{E} \|D^2 u(t, x)\|_{H^{\otimes 2}}^2 dt &\geq \int_{E_t^2} p_{t-\hat{s}}^2(x, \hat{y}) \mathbb{E} |D_{\hat{\alpha}} \sigma(u(\hat{s}, \hat{y}))|^2 d\alpha \\ &\geq 2c^2 \int_0^\infty e^{-\beta t} \int_0^t \int_0^{2\pi} \int_0^s \int_0^{2\pi} p_{t-s}^2(x, y) \mathbb{E} |D_{r,v} u(s, y)|^2 dv dr dy ds dt \\ &\geq 2c^2 \int_0^{2\pi} \int_0^\infty \mathbb{E} \|Du(s, y)\|_H^2 e^{-\beta s} ds \int_0^\infty e^{-\beta t} p_t^2(x, y) dt dy, \end{aligned}$$

which is infinity, by the first part. □

CHAPTER 6

New directions and further developments

6.0.1 The Malliavin calculus on $\mathbf{R}$

Consider the parabolic Anderson model on $\mathbf{R}$; i.e.,

$$\left\{ \begin{array}{ll} \partial_t u(t, x) = \mathcal{L}u(t, x) + \sigma(u(t, x))\dot{w} & (t, x) \in (0, \infty) \times \mathbf{R}, \\ u(0, x) = 1 & x \in \mathbf{R}, \end{array} \right. \quad (6.1)$$

The existence and uniqueness of the solution to this equation is illustrated in [23]. Intuitively, the solution should admit a density at all $(t, x) \in \mathbf{R}_+ \times \mathbf{R}$. But the lack of the integrability condition

$$\mathbb{E} \int_0^t \int_{\mathbf{R}} |u(s, x)|^2 dx ds < \infty,$$

causes a challenge. This is because Proposition 2.3.4 does not hold and the stochastic integral in the formulation of the mild solution no longer coincides with the divergence operator. This means that the commutative relation between the Malliavin derivative and the divergence operator will not hold. There are two ways of overcoming this challenge: We can approximate the solution on $\mathbf{R}$ by the solution to the truncated equation (truncated on the intervals $[-n, n]$, and then keep track of their Malliavin derivatives. The other possible approach is through the method offered by the recent paper [33]. The authors were able to use a Feynman-Kac representation for the SHE driven by a non homogeneous Gaussian noise, which allowed them to write an explicit formula for the Malliavin derivatives. There is a possibility that their method could be adapted to the white noise case.

6.0.2 Estimates on the densities

Finding an upper and lower gaussian bound for the density of the solution to (1.1) is another direction for further research. A new method for obtaining such estimate is developed by Nourdin and Peccati [44, 45], which is rooted in the interaction between the Stein's method and the Malliavin Calculus. In this context, Nualart and Quer-Sardanyons in [49] have applied a result by Nourdin and Viens [46] to find such gaussian bounds.

6.0.3 Existence of the joint densities

There is also an interest in the existence and regularity of the joint density. Bally and Pardoux proved such result [4] for (1.1), when $\mathcal{L} = \Delta$. The case for general $\mathcal{L}$ is yet to be proved. The main challenge of this problem is the analysis of the Malliavin matrix and the prove of the nondegeneracy.

REFERENCES

- [1] Gideon Amir, Ivan Corwin, and Jeremy Quastel. Probability distribution of the free energy of the continuum directed random polymer in $1 + 1$ dimensions. *Comm. Pure Appl. Math.*, 64(4):466–537, 2011.
- [2] David Applebaum. *Lévy processes and stochastic calculus*, volume 116 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, second edition, 2009.
- [3] M. Balázs, J. Quastel, and T. Seppäläinen. Fluctuation exponent of the KPZ/stochastic Burgers equation. *J. Amer. Math. Soc.*, 24(3):683–708, 2011.
- [4] Vlad Bally and Etienne Pardoux. Malliavin calculus for white noise driven parabolic SPDEs. *Potential Anal.*, 9(1):27–64, 1998.
- [5] Albert-László Barabási and H. Eugene Stanley. *Fractal concepts in surface growth*. Cambridge University Press, Cambridge, 1995.
- [6] Richard F. Bass. *Stochastic processes*, volume 33 of *Cambridge Series in Statistical and Probabilistic Mathematics*. Cambridge University Press, Cambridge, 2011.
- [7] Lorenzo Bertini and Giambattista Giacomin. Stochastic Burgers and KPZ equations from particle systems. *Comm. Math. Phys.*, 183(3):571–607, 1997.
- [8] Jean Bertoin. *Lévy processes*, volume 121 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1996.
- [9] Nicolas Bouleau and Francis Hirsch. *Dirichlet forms and analysis on Wiener space*, volume 14 of *de Gruyter Studies in Mathematics*. Walter de Gruyter & Co., Berlin, 1991.
- [10] D. L. Burkholder. Martingale inequalities. In *Martingales (Report Meeting, Oberwolfach, 1970)*, pages 1–8. Lecture Notes in Math., Vol. 190. Springer, Berlin, 1971.
- [11] Eric Carlen and Paul Krée. L^p estimates on iterated stochastic integrals. *Ann. Probab.*, 19(1):354–368, 1991.
- [12] René A. Carmona and S. A. Molchanov. Parabolic Anderson problem and intermittency. *Mem. Amer. Math. Soc.*, 108(518):viii+125, 1994.
- [13] Pao-Liu Chow. *Stochastic partial differential equations*. Chapman & Hall/CRC Applied Mathematics and Nonlinear Science Series. Chapman & Hall/CRC, Boca Raton, FL, 2007.
- [14] Ivan Corwin. The kardar-parisi-zhang equation and universality class, preprint available at [http : //arxiv.org/abs/1106.1596](http://arxiv.org/abs/1106.1596), 2011.

- [15] Robert Dalang, Davar Khoshnevisan, Carl Mueller, David Nualart, and Yimin Xiao. *A minicourse on stochastic partial differential equations*, volume 1962 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2009. Held at the University of Utah, Salt Lake City, UT, May 8–19, 2006, Edited by Khoshnevisan and Firas Rassoul-Agha.
- [16] Robert C. Dalang. Extending the martingale measure stochastic integral with applications to spatially homogeneous s.p.d.e.’s. *Electron. J. Probab.*, 4:no. 6, 29 pp. (electronic), 1999.
- [17] Robert C. Dalang, Davar Khoshnevisan, and Eulalia Nualart. Hitting probabilities for systems of non-linear stochastic heat equations with additive noise. *ALEA Lat. Am. J. Probab. Math. Stat.*, 3:231–271, 2007.
- [18] Robert C. Dalang, Davar Khoshnevisan, and Eulalia Nualart. Hitting probabilities for systems for non-linear stochastic heat equations with multiplicative noise. *Probab. Theory Related Fields*, 144(3-4):371–427, 2009.
- [19] Robert C. Dalang, C. Mueller, and L. Zambotti. Hitting properties of parabolic s.p.d.e.’s with reflection. *Ann. Probab.*, 34(4):1423–1450, 2006.
- [20] Robert C. Dalang and Marta Sanz-Solé. Criteria for hitting probabilities with applications to systems of stochastic wave equations. *Bernoulli*, 16(4):1343–1368, 2010.
- [21] Giulia Di Nunno, Bernt Øksendal, and Frank Proske. *Malliavin calculus for Lévy processes with applications to finance*. Universitext. Springer-Verlag, Berlin, 2009.
- [22] Patrik L. Ferrari and Herbert Spohn. Scaling limit for the space-time covariance of the stationary totally asymmetric simple exclusion process. *Comm. Math. Phys.*, 265(1):1–44, 2006.
- [23] Mohammud Foondun and Davar Khoshnevisan. Intermittence and nonlinear parabolic stochastic partial differential equations. *Electron. J. Probab.*, 14:no. 21, 548–568, 2009.
- [24] Mohammud Foondun, Davar Khoshnevisan, and Eulalia Nualart. A local-time correspondence for stochastic partial differential equations. *Trans. Amer. Math. Soc.*, 363(5):2481–2515, 2011.
- [25] Dieter Forster, David R. Nelson, and Michael J. Stephen. Large-distance and long-time properties of a randomly stirred fluid. *Phys. Rev. A (3)*, 16(2):732–749, 1977.
- [26] Dieter Forster, David R. Nelson, and Michael J. Stephen. Large-distance and long-time properties of a randomly stirred fluid. *Phys. Rev. A (3)*, 16(2):732–749, 1977.
- [27] J. Gärtner and S. A. Molchanov. Parabolic problems for the Anderson model. I. Intermittency and related topics. *Comm. Math. Phys.*, 132(3):613–655, 1990.
- [28] Jürgen Gärtner. Convergence towards Burgers’ equation and propagation of chaos for weakly asymmetric exclusion processes. *Stochastic Process. Appl.*, 27(2):233–260, 1988.

- [29] I. S. Gradshteyn and I. M. Ryzhik. *Table of integrals, series, and products*. Academic Press Inc., San Diego, CA, fifth edition, 1996. CD-ROM version 1.0 for PC, MAC, and UNIX computers.
- [30] István Gyöngy and David Nualart. On the stochastic Burgers' equation in the real line. *Ann. Probab.*, 27(2):782–802, 1999.
- [31] Martin Hairer. On Malliavin's proof of Hörmander's theorem. *Bull. Sci. Math.*, 135(6-7):650–666, 2011.
- [32] J. Hawkes. Local times as stationary processes. In *From local times to global geometry, control and physics (Coventry, 1984/85)*, volume 150 of *Pitman Res. Notes Math. Ser.*, pages 111–120. Longman Sci. Tech., Harlow, 1986.
- [33] Yaozhong Hu, David Nualart, and Xiaoming Song. A nonlinear stochastic heat equation: Hlder continuity and smoothness of the density of the solution, preprint available at [http : //arxiv.org/pdf/1110.4855v1.pdf](http://arxiv.org/pdf/1110.4855v1.pdf), 2011.
- [34] Petros Ioannou and Barış Fidan. *Adaptive control tutorial*, volume 11 of *Advances in Design and Control*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2006.
- [35] Mehran Kardar, Giorgio Parisi, and Yi-Cheng Zhang. Dynamic scaling of growing interfaces. *Phys. Rev. Lett.*, 56:889–892, Mar 1986.
- [36] Thomas M. Liggett. *Continuous time Markov processes*, volume 113 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2010. An introduction.
- [37] Pejman Mahboubi. Regularity of the density for a stochastic heat equation, preprint available at [http : //arxiv.org/pdf/1109.2833v2.pdf](http://arxiv.org/pdf/1109.2833v2.pdf), 2011.
- [38] Paul Malliavin. Stochastic calculus of variation and hypoelliptic operators. In *Proceedings of the International Symposium on Stochastic Differential Equations (Res. Inst. Math. Sci., Kyoto Univ., Kyoto, 1976)*, pages 195–263, New York, 1978. Wiley.
- [39] Michael B. Marcus and Jay Rosen. *Markov processes, Gaussian processes, and local times*, volume 100 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2006.
- [40] S. Molchanov. Lectures on random media. In *Lectures on probability theory (Saint-Flour, 1992)*, volume 1581 of *Lecture Notes in Math.*, pages 242–411. Springer, Berlin, 1994.
- [41] Stanislav A. Molchanov. Ideas in the theory of random media. *Acta Appl. Math.*, 22(2-3):139–282, 1991.

- [42] Carl Mueller. On the support of solutions to the heat equation with noise. *Stochastics Stochastics Rep.*, 37(4):225–245, 1991.
- [43] Carl Mueller and David Nualart. Regularity of the density for the stochastic heat equation. *Electron. J. Probab.*, 13:no. 74, 2248–2258, 2008.
- [44] Ivan Nourdin and Giovanni Peccati. Stein’s method on Wiener chaos. *Probab. Theory Related Fields*, 145(1-2):75–118, 2009.
- [45] Ivan Nourdin, Giovanni Peccati, and Anthony Réveillac. Multivariate normal approximation using Stein’s method and Malliavin calculus. *Ann. Inst. Henri Poincaré Probab. Stat.*, 46(1):45–58, 2010.
- [46] Ivan Nourdin and Frederi G. Viens. Density formula and concentration inequalities with Malliavin calculus. *Electron. J. Probab.*, 14:no. 78, 2287–2309, 2009.
- [47] David Nualart. *The Malliavin calculus and related topics*. Probability and its Applications (New York). Springer-Verlag, Berlin, second edition, 2006.
- [48] David Nualart. *Malliavin calculus and its applications*, volume 110 of *CBMS Regional Conference Series in Mathematics*. Published for the Conference Board of the Mathematical Sciences, Washington, DC, 2009.
- [49] David Nualart and Lluís Quer-Sardanyons. Gaussian density estimates for solutions to quasi-linear stochastic partial differential equations. *Stochastic Process. Appl.*, 119(11):3914–3938, 2009.
- [50] Étienne Pardoux and Tu Sheng Zhang. Absolute continuity of the law of the solution of a parabolic SPDE. *J. Funct. Anal.*, 112(2):447–458, 1993.
- [51] S. Peszat and J. Zabczyk. *Stochastic partial differential equations with Lévy noise*, volume 113 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 2007. An evolution equation approach.
- [52] Sidney C. Port and Charles J. Stone. Infinitely divisible processes and their potential theory. *Ann. Inst. Fourier (Grenoble)*, 21(2):157–275; *ibid.* 21 (1971), no. 4, 179–265, 1971.
- [53] Marta Sanz-Solé. *Malliavin calculus*. Fundamental Sciences. EPFL Press, Lausanne, 2005. With applications to stochastic partial differential equations.
- [54] Daniel W. Stroock, Marc Yor, Jean-Pierre Kahane, Richard Gundy, Leonard Gross, and Michèle Vergne. Remembering Paul Malliavin. *Notices Amer. Math. Soc.*, 58(4):568–573, 2011.
- [55] John B. Walsh. An introduction to stochastic partial differential equations. In *École d’été de probabilités de Saint-Flour, XIV—1984*, volume 1180 of *Lecture Notes in Math.*, pages 265–439. Springer, Berlin, 1986.