Title
Galois module structure of Étale cohomology groups

Author
Dodge, Joel

Publication Date
2011

UNIVERSITY OF CALIFORNIA, SAN DIEGO

Galois Module Structure of Étale Cohomology Groups

A dissertation submitted in partial satisfaction of the
requirements for the degree
Doctor of Philosophy

in

Mathematics

by

Joel Dodge

Committee in charge:

Professor Cristian Popescu, Chair
Professor Ronald Graham
Professor Ken Intriligator
Professor Daniel Rogalski
Professor Adrian Wadsworth

2011

The dissertation of Joel Dodge is approved, and it is acceptable in quality and form for publication on microfilm:

____

____

____

____

____

Chair

University of California, San Diego

2011

This is dedicated to all of my friends and all of my family.

Whatever you can do, or dream you can, begin it. Boldness has genius, power and magic in it.

—*Goethe*

TABLE OF CONTENTS

ACKNOWLEDGEMENTS

As much as this thesis is a product of many years of hard solitary work, there is no way that I could have found myself with this finished document without the influence of many many people in my life. I will happily take a moment to thank some of them.

My life would not be anything like it is without the love and companionship of my closest friends. Alex, Sarah, Lindsay and Jessica have been my constant companions and, even from across the world, the fact that they are there gives great meaning to all the other components of my life. You are my second family, I love you guys so much and I am impatiently waiting to find out what comes next for all of us!

My family has been a great source of support throughout my life. My sister Johannah is the sweetest most thoughtful person that I know and she is always there with a kind ear when I need it. It seems almost silly to thank my mother Sharon. There's little that I do in my life on any scale that doesn't bear the mark of her influence on me. I'm blessed to have such a powerful force in my life.

I have enjoyed learning from every math professor that I have ever had but a couple of them stand above the crowd. John Loustau taught me my first abstract algebra course and helped me fall deeply in love with the subject. Beyond that, he pushed me to take all the classes that I could and to broaden my mathematical horizons as far as possible. I could not have found this path without his help and I am very grateful for it.

At UCSD, my advisor Cristian Popescu has been a continual source of knowledge and inspiration over the last four years. The path from the beginning to the end of graduate school sometimes seems longer and more hopeless than it acually is. Many times Cristian gave me the renewed enthusiasm for our beautiful subject that I needed to keep going. I have learned so much mathematics from him and I can't express how good it has been to talk and work and laugh with him.

VITA

2006	B. Sc., Mathematics, Hunter College CUNY
2006-2011	Teaching Assistant, University of California, San Diego
2009	M. Sc., Mathematics, University of California, San Diego
2011	Ph. D., Mathematics, University of California, San Diego

PUBLICATIONS

F. Blanchet-Sadri, C.D. Davis, J. Dodge, R. Mercas and M. Moorefield. *Unbordered Partial Words.* Discrete Applied Mathematics, Vol. 157, 2009.

ABSTRACT OF THE DISSERTATION

Galois Module Structure of Étale Cohomology Groups

by

Joel Dodge
Doctor of Philosophy in Mathematics

University of California San Diego, 2011

Professor Cristian Popescu, Chair

This thesis is concerned with proving a refined function field analogue of the Coates-Sinnott conjecture. The theorem we prove identifies precisely the Fitting ideal of a certain étale cohomology group. The techniques employed are directly inspired by recent work of Greither and Popescu in equivariant Iwasawa theory, both for number fields and function fields. They rest on an in-depth study of the Galois module structure of certain naturally defined 1-motives associated to a function field.

Chapter 1

Introduction

1.1 Some general notations

κ_0	A finite field of characteristic p
κ	An algebraic closure of κ_0
Z_0, Z_0'	Smooth projective curves over κ_0
Z, Z'	The base changes of Z_0 and Z_0' to curves over κ
v, w	Finite places of a global field
$\kappa(v)$	Residue field associated to a finite place v
$\mathcal{K}_0, \mathcal{K}_0', \mathcal{K}, \mathcal{K}'$	Fields of rational functions on the curves Z_0, Z_0', Z and Z'
S, S_0	Sets of closed points on the curves Z, Z_0
$\mathcal{O}_{\mathcal{K}_0,S_0}, \mathcal{O}_{\mathcal{K},S}$	Rings of regular functions on the open curves $Z_0 \setminus S_0$, $Z \setminus S$
μ_{ℓ^∞}	The group of roots of unity whose orders are a power of ℓ
$Q(R)$	Total ring of fractions of a commutative ring R
$G^{(\ell)}$	The ℓ-Sylow subgroup of a finite abelian group G
$\widehat{G}(F)$	The set of irreducible F-valued characters of the group G
χ, ψ	elements of $\widehat{G}(F)$, F will be clear from context
$R[G]$	The group ring of G with coefficients in R
$\mathbb{Z}_\ell, \mathbb{Q}_\ell$	The ring of ℓ-adic integers for ℓ-prime, respectively its field of fractions
Λ	The one variable Iwasawa algebra with coefficients in $\mathbb{Z}_\ell$; isomorphic to $\mathbb{Z}_\ell[[T]]$
$\mathrm{div}(f)$	The divisor associated to a rational function f on a curve

1.2 History

This thesis is concerned with proving a refined function field analogue of the Coates-Sinnott conjecture for number fields. We are heavily influenced by recent results of Greither and Popescu on this problem in both number fields and function fields, see [6], [7]. We begin by giving a rough account of the motivation for and formulation of the problem.

The field of Special Values of L-functions is concerned with making explicit connections between the algebraic and analytic invariants that one can associate to global fields. The prototypical example of the kind of relationship that one would like to establish is given by the classical Stickelberger theorem which we now explain.

Let $K/\mathbb{Q}$ be an abelian extension with $G = G(K/\mathbb{Q})$ and let $\mathcal{O}_K$ be the ring of integers of K. In addition, let S be a finite set of primes of $\mathbb{Q}$ containing all those primes which ramify in K. To each $\chi \in \widehat{G}(\mathbb{C})$, there is an associated S-incomplete L-function given by the Euler product

$$L_S(s, \chi) : \prod_{p \notin S} (1 - \chi(\sigma_p) p^{-s})^{-1}$$

which converges to a holomorphic function for $s \in \mathbb{C}$ with $\Re e(s) > 1$. $L_S(s, \chi)$ can be meromorphically continued to all of $\mathbb{C}$ and is actually analytic on $\mathbb{C} \setminus \{1\}$. Here σ_p denotes the Frobenius automorphism corresponding to p in $G(K/\mathbb{Q})$.

These functions can be combined in a simple way to produce an *equivariant L-function*

$$\Theta_{K/\mathbb{Q},S} : \mathbb{C} \to \mathbb{C}[G]$$

which is analytic on $\mathbb{C} \setminus \{1\}$.

Let μ_K denote the group of roots of unity in K. Observe that μ_K carries an action of the group G and so we can consider it as a $\mathbb{Z}[G]$-module. From the classical evaluation of the L-functions at $s = 0$, one can prove the following proposition.

Proposition 1.2.1 (Lemma 6.9 in [18])**.**

$$\mathrm{Ann}_{\mathbb{Z}[G]}(\mu_K) \cdot \Theta_{K/\mathbb{Q},S}(0) \subseteq \mathbb{Z}[G].$$

This already gives an interesting relationship between the $\mathbb{Z}[G]$-module structure of μ_K and the values of the L-functions at $s = 0$. Stickelberger's theorem goes much further.

Theorem 1.2.2 (Stickelberger's Theorem, Theorem 6.10 in [18])**.** *Let* $\mathrm{Cl}(\mathcal{O}_K)$ *be the class group of* $\mathcal{O}_K$*. Then*

$$\mathrm{Ann}_{\mathbb{Z}[G]}(\mu_K) \cdot \Theta_{K/\mathbb{Q},S}(0) \subseteq \mathrm{Ann}_{\mathbb{Z}[G]}(\mathrm{Cl}(\mathcal{O}_K)).$$

We can see that this theorem gives an explicit connection between the value $\Theta_{K/\mathbb{Q},S}(0)$ and the $\mathbb{Z}[G]$-module structure of μ_K and $\mathrm{Cl}(\mathcal{O}_K)$. Much of the theory of special values of L-functions is concerned with formulating and proving generalizations of this theorem.

To generalize, let K/k be an abelian extension of number fields. As above, let S be a finite set of primes of k containing those primes which ramify in K/k. Let $G = G(K/k)$ and $\chi \in \widehat{G}(\mathbb{C})$. In this context we can still define L-functions by an Euler product

$$L_S(s,\chi) = \prod_{\mathfrak{p} \notin S} (1 - \chi(\sigma_\mathfrak{p}) N\mathfrak{p}^{-s})^{-1}.$$

Again this product converges to a holomorphic function for $\Re e(s) > 1$ but can be continued to a function which is meromorphic on $\mathbb{C}$ and analytic on $\mathbb{C} \setminus \{1\}$. As above, $\sigma_\mathfrak{p}$ denotes the Frobenius automorphism associated to $\mathfrak{p}$.

We can still construct an equivariant L-function for K/k. We will denote it by

$$\Theta_{K/k,S}(s) : \mathbb{C} \to \mathbb{C}[G].$$

As in the previous case, this function is analytic on $\mathbb{C} \setminus \{1\}$ and we would like to study the values of $\Theta_{K/k,S}(s)$ at the negative integers. For reasons that will become clear, we will write these values as $\Theta_{K/k,S}(1-n)$ for $n \geq 2$. In order to find the proper analogues of μ_K and $\mathrm{Cl}(\mathcal{O}_K)$ in this context, we turn to Quillen K-theory.

Quillen K-theory refers to a sequence of functors from the category of commutative rings to the category of abelian groups. These functors, denoted K_n for $n \geq 0$, encode linear algebraic type information about the structure of a ring R. Grothendieck gave the original defintion of $K_0(R)$ as the usual group completion of the abelian semigroup of isomorphism classes of finitely generated projective R-modules. Bass gave a definition of $K_1(R)$ as the abelianization of the general linear group over R, $GL(R)$.

$$K_1(R) = GL(R)/[GL(R), GL(R)].$$

In the cases of interest to us, these definitions result in some classical objects of study in algebraic number theory.

Example 1.2.3 ([8]). *If K is a number field, then we have very concrete interpretations of both $K_0(\mathcal{O}_K)$ and $K_1(\mathcal{O}_K)$. There is a canonical isomorphism*

$$K_0(\mathcal{O}_K) \simeq \mathbb{Z} \oplus \mathrm{Cl}(\mathcal{O}_K).$$

Additionally, one can prove that

$$K_1(\mathcal{O}_K) \simeq \mathcal{O}_K^\times.$$

With this example in mind we see that Proposition 1.2.1 and Theorem 1.2.2 can be reformulated as statements about the algebraic K-theory of $\mathcal{O}_K$.

Theorem 1.2.4. *Let K be a finite abelian extension of $\mathbb{Q}$. Then*

1. $\mathrm{Ann}_{\mathbb{Z}[G]}(K_1(\mathcal{O}_K)_{\mathrm{tors}}) \cdot \Theta_{K/\mathbb{Q},S}(0) \subseteq \mathbb{Z}[G]$.
2. $\mathrm{Ann}_{\mathbb{Z}[G]}(K_1(\mathcal{O}_K)_{\mathrm{tors}}) \cdot \Theta_{K/\mathbb{Q},S}(0) \subseteq \mathrm{Ann}_{\mathbb{Z}[G]}(K_0(\mathcal{O}_K)_{\mathrm{tors}})$.

Milnor gave a defintion of $K_2(R)$, but that is as far as it went for some time. In 1972, the correct general definition of $K_n(R)$ for all $n \geq 0$ was discovered by Quillen. His definition is topological in nature but is equivalent to the previous definitions in the cases $n = 0, 1, 2$. Unfortunately, little about the groups $K_n(R)$ is obvious from first principles. Of course, as number theorists we are not interested in general commutative rings. If $R = \mathcal{O}_K$ is the ring of integers of a number field, then the groups $K_n(\mathcal{O}_K)$ are well behaved in the following sense, see the Introduction of [12].

Theorem 1.2.5 (Borel, Quillen). *Let K be a number field. The abelian groups $K_n(\mathcal{O}_K)$ are all finitely generated. The even K-groups $K_{2n}(\mathcal{O}_K)$ are all finite. The ranks of the odd K-groups $K_{2n-1}(\mathcal{O}_K)$ are given by*

$$\mathrm{rk}_{\mathbb{Z}}(K_{2n-1}(\mathcal{O}_K)) = \begin{cases} r_1 + r_2 - 1 & \text{if } n = 1, \\ r_1 + r_2 & \text{if } n \geq 1 \text{ and } n \text{ is odd}, \\ r_2 & \text{if } n \geq 2 \text{ and } n \text{ is even}, \end{cases}$$

where r_1 and r_2 are as usual.

It follows from Quillen's construction that if K/k is a Galois extension with Galois group G, then each $K_n(\mathcal{O}_K)$ is a $\mathbb{Z}[G]$-module. Using these K-groups, Coates and Sinnott proposed the correct analogue of Stickelberger's theorem in the general number field context. In order to state it, we will work ℓ-adically for each prime ℓ.

A deep theorem of Klingen-Siegel shows that $\Theta_{K/k,S}(1-n) \in \mathbb{Q}[G]$ for all $n \geq 1$, see [14]. This allows us to consider these values as elements of $\mathbb{Q}_\ell[G]$ for each ℓ. As with Stickelberger's theorem, we must first mention an integrality result on these special values. This can be proven by combining results of Soulé and Deligne-Ribet which are mentioned later.

Theorem 1.2.6 (Deligne-Ribet, [5])**.** *Let K/k be an abelian extension of number fields, let ℓ be a prime number and let $n \geq 2$ be an integer, then*

$$\mathrm{Ann}_{\mathbb{Z}_\ell[G]}(K_{2n-1}(\mathcal{O}_K)_{\mathrm{tors}} \otimes \mathbb{Z}_\ell) \cdot \Theta_{K/k,S}(1-n) \subseteq \mathbb{Z}_\ell[G].$$

With this in hand, we can state the original form of the Coates-Sinnott conjecture.

Conjecture 1.2.7 (K-theoretic Coates-Sinnott, [3])**.** *Let K/k be an abelian extension of number fields, let ℓ be a prime number and let $n \geq 2$ be an integer, then*

$$\mathrm{Ann}_{\mathbb{Z}_\ell[G]}(K_{2n-1}(\mathcal{O}_K)_{\mathrm{tors}} \otimes \mathbb{Z}_\ell) \cdot \Theta_{K/k,S}(1-n) \subseteq \mathrm{Ann}_{\mathbb{Z}_\ell[G]}(K_{2n-2}(\mathcal{O}_K) \otimes \mathbb{Z}_\ell).$$

Remark 1.2.8. *In fact, the original conjecture was a bit weaker. Coates and Sinnott were overly cautious and stated their conjecture only under the conditions that K/k was an extension of totally real fields and that n was even. It turns out that these restrictions are not necessary. See the reductions which follow Theorem 6.11 in [6].*

As alluded to above, the definition of the Quillen K-groups is unwieldy and it is difficult to attack this problem directly as stated. Fortunately, work of Soulé relates K-theory to étale cohomology and suggests an alternative approach to this problem. In [15], Soulé constructs $\mathbb{Z}_\ell[G]$-linear ℓ-adic étale Chern character maps

$$\mathrm{ch}^{(\ell)}_{n,i} : K_{2n-i}(\mathcal{O}_K) \otimes \mathbb{Z}_\ell \to H^i_{\acute{e}t}(\mathcal{O}_K[1/\ell], \mathbb{Z}_\ell(n))$$

for all $\ell > 2$ prime, $i = 1, 2$ and $n \in \mathbb{Z}_{\geq 2}$. Soulé proved that these maps were surjective and Quillen-Lichtenbaum conjectured that they were in fact isomorphisms. It is known that this follows from the Bloch-Kato conjecture, see Theorem 2.7 in [8]. The Bloch-Kato conjecture was recently proven by work of Voevodsky and Rost, see [17].

Theorem 1.2.9 (Quillen-Lichtenbaum conjecture)**.** *For all primes $\ell > 2$, $i = 1, 2$ and $n \in \mathbb{Z}_{\geq 2}$, the Chern character map $\mathrm{ch}^{(\ell)}_{n,i}$ is an isomorphism.*

To formulate the Coates-Sinnott conjecture as a statement about étale cohomology groups we first need to state an integrality result relating the $\mathbb{Z}_\ell[G]$-module structure of these groups to the denominators of the special values $\Theta_{K/k,S}(1-n)$.

Theorem 1.2.10 (Deligne-Ribet, [5]). *For K/k and S as above, ℓ prime and $n \geq 1$ we have*

$$\mathrm{Ann}_{\mathbb{Z}_\ell[G]}(H^1_{\acute{e}t}(\mathcal{O}_{K,S}, \mathbb{Z}_\ell(n))_{\mathrm{tors}}) \cdot \Theta_{K/k,S}(1-n) \subseteq \mathbb{Z}_\ell[G].$$

Now the Quillen-Lichtenbaum conjecture allows us to state a form of the Coates-Sinnott conjecture which is closer in spirit to what we will eventually prove for function fields.

Conjecture 1.2.11 (Étale Cohomological Coates-Sinnott, [12]). *Let K/k be an abelian extension of number fields with Galois group G, let S be a finite set of primes of k containing the primes which ramify in K/k, let ℓ be a prime number and let $n \geq 2$ be an integer, then*

$$\mathrm{Ann}_{\mathbb{Z}_\ell[G]}(H^1_{\acute{e}t}(\mathcal{O}_K[1/\ell], \mathbb{Z}_\ell(n))_{\mathrm{tors}}) \cdot \Theta_{K/k,S}(1-n) \subseteq \mathrm{Ann}_{\mathbb{Z}_\ell[G]}(H^2_{\acute{e}t}(\mathcal{O}_K[1/\ell], \mathbb{Z}_\ell(n))).$$

In fact, a refined version of this theorem has been proven in [6] under an additional assumption and this version will be stated in the text once we develop a little bit of commutative algebra.

As mentioned at the beginning of this introduction, we will be primarily concerned in this thesis with function fields over a finite field of characteristic p. The mathematics that will be developed in the main text is motivated heavily by the theory laid out above and the well known analogy between number fields and characteristic p function fields. This analogy will be discussed briefly in the next section.

1.3 The analogy between function fields and number fields

The analogy to which we refer begins with the observation that nearly all of the objects that we associate to a number field have natural analogues in the theory of function fields. There is a philosophy in algebraic number theory that if there is a conjecture that you would like to prove in number fields, then you should be able to formulate and prove an analogous conjecture in function fields. In many cases the geometry which is present in the function field context gives one more tools to work with.

Many of the classical conjectures in algebraic number theory have been formulated and successfully proven in the function field context.

Here we briefly recall the definition of a function field and give some indication as to the similarity between the objects on both sides of this analogy. We will also introduce some basic definions which will be used later in the text. Nothing will be proven here and a slightly more detailed discussion can be found in [9].

Let κ_0 be a finite field of characteristic p. A field $\mathcal{K}_0$ containing κ_0 is called a function field over κ_0 if $\mathcal{K}_0$ is finitely generated and has transcendence degree 1 over κ_0. If $t \in \mathcal{K}_0$ is transcendental over κ_0, then we have that $\mathcal{K}_0/\kappa_0(t)$ is a finite extension. Thinking of t now as a variable, we see that the function fields over κ_0 are nothing more than the finite extensions of the rational function field $\kappa_0(t)$, just as number fields are exactly the finite extensions of $\mathbb{Q}$.

Recall that if K is a number field, then there is a 1-to-1 correspondence between the equivalence classes of non-archimedean valuations on K and the non-zero prime ideals of its ring of integers, $\mathcal{O}_K$. In the theory of function fields, there is no canonical choice of a ring of integers. This can be seen already for the field $\kappa_0(t)$. Indeed, each non-zero prime ideal of the ring $\kappa_0[t]$ gives rise to a valuation of $\kappa_0(t)$ but there is one more! If we instead look at the ring $\kappa_0[1/t] \subseteq \kappa_0(t)$, then the valuation of $\kappa_0(t)$ which corresponds to the prime ideal $\langle 1/t \rangle$ generated by $1/t$ is not equivalent to any of the previous ones. It turns out that this accounts for all of the equivalence classes of valuations of $\kappa_0(t)$.

To compensate for the loss of a canonical choice of ring of integers, we are given a great deal of geometry. For each function field over κ_0, there is a unique isomorphism class of non-singular projective curves defined over κ_0 such that the field of rational functions on any curve in this class is isomorphic to $\mathcal{K}_0$. Any of the curves in this class is called a smooth projective model for $\mathcal{K}_0$ over κ_0.

We will sometimes need to work in a more classical geometric context. If $\mathcal{K}$ denotes a compositum of $\mathcal{K}_0$ with κ, an algebraic closure of κ_0, then there is similarly a unique isomorphism class of non-singular projective curves defined over κ such that if Z is in this class, then $\mathcal{K}$ is isomorphic to the field of rational functions on Z. As above, Z is called a smooth projective model for $\mathcal{K}$ over κ.

Fix one of the models for $\mathcal{K}_0$, say Z_0. If P is a closed point on Z_0 and $f \in \mathcal{K}_0$, then we will write $\mathrm{ord}_P(f)$ for the order of vanishing of f at P. If f is regular at P and v is the equivalence class of valuations corresponding to P, then we will write either $f(P)$

or $f(v)$ for the value of f at P in the corresponding residue field $\kappa(v)$. Similar notation will be used for the curve Z.

Instead of prime ideals of a ring, in the function field setting, there is a 1-to-1 correspondence between closed points on the curve Z_0 and equivalance classes of valuations of $\mathcal{K}_0$. In fact for each equivalence class of valuations, there is a closed point $P \in Z_0$ such that

$$\mathrm{ord}_P : \mathcal{K}_0 \to \mathbb{Z}$$

represents that class.

Although the analogy is not perfect, we see that there is a strong similarity between the valuation theory of number fields and function fields. Another important connection is that in both cases, the residue field associated to a non-archimedean valuation is finite. This allows us to define a *norm map* on the set of valuations of $\mathcal{K}_0$ by

$$N(v) = |\kappa(v)|.$$

As in the number field case, this norm map can be used to define a zeta-function and, in the presence of a Galois group, L-functions. An important difference between the number field and function field cases is that, in function fields, all of these residue fields $\kappa(v)$ have the same characteristic. It is this fact which leads to the rationality of the zeta-function of a function field as a function of q^{-s} where $q = |\kappa_0|$.

If S_0 is a finite set of closed points of Z_0, then let

$$\mathcal{O}_{\mathcal{K}_0,S_0} = \{x \in \mathcal{K}_0 \mid v(x) > 0 \text{ for } v \notin S_0\}$$

be the ring of functions which are regular away from S_0. This is just the ring of regular functions on the open curve $Z_0 \setminus S_0$. Just as in the number field case, one can prove that $\mathcal{O}_{\mathcal{K}_0,S_0}$ is a Dedekind domain. With any Dedekind domain we are entitled to talk about the class group, which in this context is called the Picard group $\mathrm{Pic}(Z_0 \setminus S_0)$. It is useful to use the more geometric language of divisors and so we will introduce some terminology.

If X is a potentially infinite set of closed points on the curve Z_0, then define $\mathrm{Div}(X)$ to be the free $\mathbb{Z}$-module on the set X, that is

$$\mathrm{Div}(X) = \bigoplus_{P \in X} \mathbb{Z} \cdot P.$$

Elements of $\mathrm{Div}(X)$ are called *divisors supported on* X. As points on Z_0 correspond to primes of a number field, we see that these divisors are the function field analogue of the fractional ideals of a number field.

We can define a degree map

$$\deg : \mathrm{Div}(X) \to \mathbb{Z}$$

by setting $\deg(P) = |\kappa(v)|$ where v is the valuation corresponding to P and extending by $\mathbb{Z}$-linearity. We will denote the kernel of this map by $\mathrm{Div}^0(X)$.

There is a divisor class map $\mathrm{div} : \mathcal{K}_0^\times \to \mathrm{Div}(Z_0)$ given by

$$\mathrm{div}(f) = \sum_{P \in Z_0} \mathrm{ord}_P(f) \cdot P$$

and it is an important fact that for any $f \in \mathcal{K}_0^\times$

$$\deg(\mathrm{div}(f)) = 0$$

i.e., $\mathrm{div}(f) \in \mathrm{Div}^0(Z_0)$ for every $f \in \mathcal{K}_0^\times$.

We now define the Picard group of Z_0.

$$\mathrm{Pic}(Z_0) = \frac{\mathrm{Div}(Z_0)}{\{\mathrm{div}(f) \mid f \in \mathcal{K}_0^\times\}}.$$

It turns out that $\mathrm{Pic}(Z_0)$ is not finite, but it has a natural subgroup which is. We have observed that $\mathrm{div}(f) \in \mathrm{Div}^0(Z_0)$ for any $f \in \mathcal{K}_0^\times$ and so it makes sense to define

$$\mathrm{Pic}^0(Z_0) = \frac{\mathrm{Div}^0(Z_0)}{\{\mathrm{div}(f) \mid f \in \mathcal{K}_0^\times\}}.$$

This group $\mathrm{Pic}^0(Z_0)$ is finite and is the correct analogue of the class group of a number field in the function field setting.

Finally, we can talk about the Picard group of an open curve sitting inside of Z_0. Let S_0 be a finite set of points on Z_0. Then we can define

$$\mathrm{Pic}(Z_0 \setminus S_0) = \frac{\mathrm{Div}(Z_0 \setminus S_0)}{\{\mathrm{div}(f) \mid f \in \mathcal{K}_0^\times \text{ and } \mathrm{ord}_P(f) = 0 \text{ for all } P \in S_0\}}.$$

This is a finite abelian group and it is not hard to see that it is isomorphic to $\mathrm{Cl}(\mathcal{O}_{\mathcal{K}_0, S_0})$.

Chapter 2

Algebraic Preliminaries

In this chapter we will introduce some of the theory of semi-local $\mathbb{Z}_\ell$-algebras, and introduce a number of functors on modules over these rings. We then discuss some of the interplay between these functors and the theory of Fitting ideals. The first Fitting ideal, from now on just referred to as *the* Fitting ideal, plays a key role in formulating refined versions of the classical conjectures on special values of L-functions. As such, the theory of Fitting ideals has become an indispensible tool in this field. Most of the material in this chapter can be found in [7].

2.1 Semi-local algebra

Let ℓ be a prime and let G be a finite abelian group. If F is a field such that $\mathrm{char}(F) \nmid |G|$ and $\chi \in \widehat{G}(F)$, then we will write $e_\chi = \frac{1}{|G|}\sum_{g\in G}\chi(g)g^{-1} \in F[G]$. Observe that if R is a ring containing the values of χ, then χ can be extended uniquely to an R-algebra homomorphism $\chi : R[G] \to R$.

Claim 2.1.1. *Suppose that F is an algebraically closed field such that* $\mathrm{char}(F) \nmid |G|$. *Then $\{e_\chi \mid \chi \in \widehat{G}(F)\}$ is a complete set of primitive orthogonal idempotents for the ring $F[G]$.*

Proof. The proof is standard. We give it for the convenience of the reader. Let $\chi, \psi \in \widehat{G}(F)$, then

$$e_\chi e_\psi = \frac{1}{|G|^2}\sum_g\sum_h \chi(g)\psi(h)(gh)^{-1} = \frac{\sum_g(\chi\psi^{-1})(g)}{|G|^2}\sum_h \psi(h)h^{-1}.$$

The orthogonality relations for characters of a finite abelian group say that

$$\sum_{g\in G}\chi(g)=\begin{cases}|G| & \text{if } \chi \text{ is the trivial character}\\ 0 & \text{if } \chi \text{ is not the trivial character}\end{cases}$$

and so we can complete the calculation above to conclude that

$$e_\chi e_\psi=\begin{cases}e_\chi & \text{if } \chi=\psi\\ 0 & \text{if } \chi\neq\psi\end{cases}$$

This shows that the e_χ are indeed orthogonal idempotents.

To see that the e_χ are primitive, we calculate that

$$h\cdot e_\chi=\frac{1}{|G|}\sum_g \chi(g)hg^{-1}=\frac{\chi(h)}{|G|}\sum_g \chi(g)g^{-1}=\chi(h)\cdot e_\chi$$

for all $h\in G$ and therefore $x\cdot e_\chi=\chi(x)\cdot e_\chi$ for all $x\in F[G]$. This shows that $F[G]\cdot e_\chi$ is a 1-dimensional F vector space with basis e_χ and therefore the e_χ are primitive.

Finally, to see that this set of idempotents is complete, we calculate that

$$\sum_\chi e_\chi=\sum_\chi \frac{1}{|G|}\sum_g \chi(g)g^{-1}=\frac{1}{|G|}\sum_g(\sum_\chi \chi(g))g^{-1}.$$

Again, the orthogonality relations for characters of a finite abelian group tell us that

$$\sum_\chi \chi(g)=\begin{cases}|G| & \text{if } g \text{ is the identity element of } G\\ 0 & \text{if } g \text{ is not the identity element of } G\end{cases}$$

Combining this with the above calculation, we get that

$$\sum_\chi e_\chi=\frac{1}{|G|}\sum_g(\sum_\chi \chi(g))g^{-1}=1$$

and hence $\{e_\chi\}$ is a complete set of idempotents.

□

Let ℓ be a prime. We will need to understand the structure of group rings $\mathbb{Z}_\ell[G]$. We will begin by considering groups whose order is co-prime to ℓ.

Proposition 2.1.2. *Suppose that G is a finite abelian group with $(|G|,\ell)=1$. For $\chi\in\widehat{G}(\overline{\mathbb{Q}_\ell})$, let $\mathbb{Z}_\ell[\chi]$ be the ring obtained by adjoining to $\mathbb{Z}_\ell$ all of the values of χ. Then*

$$\mathbb{Z}_\ell[G]\simeq\bigoplus_{\widetilde{\chi}}\mathbb{Z}_\ell[\chi],$$

where $\widetilde{\chi}$ runs over the $G(\overline{\mathbb{Q}_\ell}/\mathbb{Q}_\ell)$ equivalence classes of $\overline{\mathbb{Q}_\ell}$-valued characters of G.

Remark 2.1.3. *To clarify the equivalence relation we are imposing: If* $\chi, \psi \in \widehat{G}(\overline{\mathbb{Q}_\ell})$*, then* $\chi \sim \psi$ *if and only if there exists a* $\sigma \in G(\overline{\mathbb{Q}_\ell}/\mathbb{Q}_\ell)$ *such that* $\chi(g) = \sigma(\psi(g))$ *for all* $g \in G$*.*

Proof. Let $\widetilde{\chi}$ be an equivalence class, with $\chi \in \widetilde{\chi}$, and let

$$E_{\widetilde{\chi}} = \sum_{\varphi \in \widetilde{\chi}} e_\varphi.$$

We claim that $E_{\widetilde{\chi}} \in \mathbb{Z}_\ell[G]$. First, calculate that

$$E_{\widetilde{\chi}} = \frac{1}{|G|} \sum_{\varphi \in \widetilde{\chi}} \sum_{g \in G} \varphi(g) g^{-1} = \frac{1}{|G|} \sum_{g \in G} (\sum_{\varphi \in \widetilde{\chi}} \varphi(g)) g^{-1}.$$

Now for each $\sigma \in G(\overline{\mathbb{Q}_\ell}/\mathbb{Q}_\ell)$ we have that

$$\sigma(\sum_{\varphi \in \widetilde{\chi}} \varphi(g)) = \sum_{\varphi \in \widetilde{\chi}} \varphi^\sigma(g) = \sum_{\varphi \in \widetilde{\chi}} \varphi(g).$$

Therefore $\sum_{\varphi \in \widetilde{\chi}} \varphi(g) \in \mathbb{Z}_\ell$ and since $|G| \in \mathbb{Z}_\ell^\times$ this shows that $E_{\widetilde{\chi}} \in \mathbb{Z}_\ell[G]$.

It is easy to see that as $\widetilde{\chi}$ ranges over these equivalence classes, $\{E_{\widetilde{\chi}}\}$ gives a complete set of orthogonal idempotents for $\mathbb{Z}_\ell[G]$. I claim that the map $\chi : \mathbb{Z}_\ell[G] \to \mathbb{Z}_\ell[\chi]$, given by $g \mapsto \chi(g)$, factors through $\mathbb{Z}_\ell[G] \cdot E_{\widetilde{\chi}}$ and that it gives an isomorphism $\mathbb{Z}_\ell[G] \cdot E_{\widetilde{\chi}} \xrightarrow{\sim} \mathbb{Z}_\ell[\chi]$.

To see that the map χ factors through $\mathbb{Z}_\ell[G] \cdot E_{\widetilde{\chi}}$, let $\psi \in \widehat{G}(\overline{\mathbb{Q}_\ell})$ with $\psi \notin \widetilde{\chi}$. Then we just have to observe that for each $\varphi \in \widetilde{\psi}$, $\varphi \neq \chi$ and so

$$\chi(e_\varphi) = \sum_{g \in G} \varphi\chi^{-1}(g) = 0$$

by the orthogonality relations for characters of a finite abelian group. This shows that $\chi(E_{\widetilde{\psi}}) = 0$ for $\widetilde{\psi} \neq \widetilde{\chi}$. This implies that

$$\chi(\bigoplus_{\widetilde{\psi} \neq \widetilde{\chi}} \mathbb{Z}_\ell[G] \cdot E_{\widetilde{\psi}}) = 0$$

and therefore χ descends to a function on $\mathbb{Z}_\ell[G] \cdot E_{\widetilde{\chi}}$.

To see that χ is surjective we observe that $\mathbb{Z}_\ell[\chi]$ is, by definition, generated over $\mathbb{Z}_\ell$ by the values of χ. That is, $\mathbb{Z}_\ell[\chi]$ is the image of $\mathbb{Z}_\ell[G]$ under χ.

For injectivity, suppose that $x \in \mathbb{Z}_\ell[G]$ and that $\chi(x \cdot E_{\widetilde{\chi}}) = 0$. Then $\chi(x) = 0$ and certainly $\sigma(\chi(x)) = \chi^\sigma(x) = 0$ for all $\sigma \in G(\overline{\mathbb{Q}_\ell}/\mathbb{Q}_\ell)$. That is to say that $\varphi(x) = 0$ for all $\varphi \in \widetilde{\chi}$. Seeing as $x \cdot e_\varphi = \varphi(x) \cdot e_\varphi$ for all $\varphi \in \widehat{G}(\overline{\mathbb{Q}_\ell})$, it follows that $x \cdot E_{\widetilde{\chi}} = 0$

and hence that χ is injective. We have proven that χ is an isomorphism and therefore we have

$$\mathbb{Z}_\ell[G] \simeq \bigoplus_{\widetilde{\chi}} \mathbb{Z}_\ell[G] \cdot E_{\widetilde{\chi}} \simeq \bigoplus_{\widetilde{\chi}} \mathbb{Z}_\ell[\chi].$$

□

Corollary 2.1.4. *Let G be a finite abelian group. Write $G = G' \times \Delta$ where G' is the ℓ-Sylow subgroup of G and $(|\Delta|, \ell) = 1$. Then we have an isomorphism*

$$\mathbb{Z}_\ell[G] \simeq \bigoplus_{\widetilde{\chi}} \mathbb{Z}_\ell[\chi][G']$$

with $\widetilde{\chi}$ ranging over the $G(\overline{\mathbb{Q}_\ell}/\mathbb{Q}_\ell)$-equivalence classes of $\overline{\mathbb{Q}_\ell}$-valued characters of Δ. .

Proof. We can write $Z_\ell[G] \simeq \mathbb{Z}_\ell[\Delta][G']$ and then the previous proposition gives the result. □

Remark 2.1.5. *Let $\chi \in \widehat{G}(\overline{\mathbb{Q}_\ell})$ and define $N = |G/\ker(\chi)|$. Then $\mathbb{Z}_\ell[\chi] = \mathbb{Z}_\ell[\zeta_N]$ where ζ_N is any primitive N-th root of unity. Indeed, the image of G under χ is the same as the image of $G/\ker(\chi)$ under χ and χ gives an isomorphism of the latter group with $\langle \zeta_N \rangle \subseteq \overline{\mathbb{Q}_\ell}^\times$.*

Proposition 2.1.6. *Let G be a finite abelian ℓ-group, let $\mathcal{O}$ be the ring of integers of a finite extension of $\mathbb{Q}_\ell$, and let $\pi \in \mathcal{O}$ be a uniformizer (i.e., π generates the maximal ideal of $\mathcal{O}$). Then $\mathcal{O}[G]$ is a local ring with maximal ideal $\langle \pi, I_G \rangle$, where I_G is the ideal of $\mathcal{O}[G]$ given by $I_G = \langle g - 1 \mid g \in G \rangle$.*

Proof. The Cohen-Seidenberg theorems imply that π is contained in any maximal ideal of $\mathcal{O}[G]$ so it will suffice to show that the ring $\mathcal{O}[G]/\langle \pi \rangle$ is a local ring. Observe that $\mathcal{O}[G]/\langle \pi \rangle \simeq \mathbb{F}_q[G]$ for some q which is a power of ℓ. Write G as a product of cyclic groups, say $G \simeq C_1 \times \cdots \times C_m$ with $|C_i| = \ell^{n_i}$. Then we have an isomorphism of $\mathbb{F}_q$-algebras

$$\mathbb{F}_q[G] \simeq \mathbb{F}_q[x_1, \ldots, x_m]/\langle x_1^{\ell^{n_1}} - 1, \ldots, x_m^{\ell^{n_m}} - 1 \rangle.$$

In characteristic ℓ, we have that $x_i^{\ell^{n_i}} - 1 = (x_i - 1)^{\ell^{n_i}}$ and so we can write

$$\mathbb{F}_q[G] \simeq \mathbb{F}_q[x_1, \ldots, x_m]/\langle (x_1 - 1)^{\ell^{n_1}}, \ldots, (x_m - 1)^{\ell^{n_m}} \rangle.$$

If $\widehat{x_i}$ denotes the image of x_i in the quotient, then it is clear that the only maximal ideal of this ring is $\langle \widehat{x_1} - 1, \ldots, \widehat{x_m} - 1 \rangle$ and this shows that $\mathcal{O}[G]$ is local with maximal ideal $\langle \pi, \widehat{x_1} - 1, \ldots, \widehat{x_m} - 1 \rangle$. One sees immediately that this is the ideal $\langle \pi, I_G \rangle$. □

Definition 2.1.7. *In what follows, a commutative ring R will be called* semi-local *if there is an isomorphism $R \simeq \prod_{i=1}^{n} R_i$, with R_i local rings. (Note that this is non-standard terminology. The usual notion being that a commutative ring R is called semi-local if R has finitely many maximal ideals.)*

Corollary 2.1.8. *If G is a finite abelian group, then $\mathbb{Z}_\ell[G]$ is a semi-local ring.*

Proof. Write
$$\mathbb{Z}_\ell[G] \simeq \bigoplus_{\widetilde{\chi}} \mathbb{Z}_\ell[\chi][G']$$
as in Corollary 2.1.4. Since $\chi \in \widehat{\Delta}$ and $(|\Delta|, \ell) = 1$, the extension $\mathbb{Q}_\ell(\chi)/\mathbb{Q}_\ell$ is unramified. From Remark 2.1.5, we have that $\mathbb{Q}_\ell(\chi)$ is a cyclotomic extension of $\mathbb{Q}_\ell$, say $\mathbb{Q}_\ell(\zeta_N)$ with $\ell \nmid N$. Then Proposition 16, Chapter IV, §4 of [13] says that $\mathbb{Z}_\ell[\chi] = \mathbb{Z}_\ell[\zeta_N]$ is the full ring of integers of $\mathbb{Q}_\ell(\chi)$. The above proposition now applies to show that each $\mathbb{Z}_\ell[\chi][G']$ is a local ring. □

Lemma 2.1.9. *Let G be a finite group, let ℓ be a prime and let $x \in \mathbb{Z}_\ell[G]$. Then x is a zero-divisor if and only if $\chi(x) = 0$ for some $\chi \in \widehat{G}(\overline{\mathbb{Q}_\ell})$.*

Proof. If we think of $\mathbb{Z}_\ell[G]$ sitting inside its total ring of fractions $\mathbb{Q}_\ell[G]$, we just have to understand the zero-divisors of this larger ring. If we use Proposition 2.1.2 and invert ℓ, we get that $\mathbb{Q}_\ell[G] \simeq \bigoplus_{\widetilde{\chi}} \mathbb{Q}_\ell(\chi)$. Examining the proof of Proposition 2.1.2 shows that the isomorphism is given by $x \mapsto (\chi(x))_{\widetilde{\chi}}$. Finally, it is straightforward to check that in a product of fields, an element is a zero-divisor if and only if it is zero in at least one factor. □

2.2 Some functors

We remind the reader that a pro-finite group is a projective limit of finite groups endowed with the usual pro-finite topology. Typical examples of such groups include Galois groups with the Krull topology. If $\mathcal{G}$ is a pro-finite group which is abelian, then we define the *pro-finite group ring* associated to $\mathcal{G}$ with coefficients in $\mathbb{Z}_\ell$ as follows. Observe that if $\mathcal{H}$ and $\mathcal{H}'$ are subgroups of $\mathcal{G}$ with $\mathcal{H} \subseteq \mathcal{H}'$, then there is a natural projection map
$$\mathbb{Z}_\ell[\mathcal{G}/\mathcal{H}] \to \mathbb{Z}_\ell[\mathcal{G}/\mathcal{H}'].$$

Using these maps, we define

$$\mathbb{Z}_\ell[[\mathcal{G}]] = \varprojlim \mathbb{Z}_\ell[\mathcal{G}/\mathcal{H}]$$

where the limit is taken over all open subgroups $\mathcal{H} \subseteq \mathcal{G}$. We endow $\mathbb{Z}_\ell[\mathcal{G}/\mathcal{H}]$ with the ℓ-adic topology and $\mathbb{Z}_\ell[[\mathcal{G}]]$ is then endowed with the topology induced from the product topology on the product of the $\mathbb{Z}_\ell[\mathcal{G}/\mathcal{H}]$. In this way, $\mathbb{Z}_\ell[[\mathcal{G}]]$ becomes a compact topological ring. If $\mathcal{H}$ is one of these subgroups, then we will write $I_{\mathcal{H}}$ for the kernel of the map

$$\mathbb{Z}_\ell[[\mathcal{G}]] \to \mathbb{Z}_\ell[\mathcal{G}/\mathcal{H}].$$

Example 2.2.1. *Let $\mathcal{K}_0/\mathcal{K}'_0$ be a finite Galois extension of characteristic p function fields with Galois group G. Let κ be an algebraic closure of $\mathbb{F}_p$ and let $\mathcal{K} = \mathcal{K}_0\kappa$, $\mathcal{K}' = \mathcal{K}'_0\kappa$ be field composita of κ with $\mathcal{K}_0$, respectively $\mathcal{K}'_0$, inside some algebraic closure of $\mathcal{K}_0$. Let κ_0 be the exact field of constants of $\mathcal{K}'_0$, i.e. $\kappa_0 = \mathcal{K}'_0 \cap \kappa$. Then $\mathcal{G} = G(\mathcal{K}/\mathcal{K}'_0)$ is (non-canonically) isomorphic to $G(\mathcal{K}/\mathcal{K}') \times G(\mathcal{K}'/\mathcal{K}'_0) \simeq G(\mathcal{K}_0/\mathcal{K}_0 \cap \mathcal{K}') \times G(\mathcal{K}'/\mathcal{K}'_0)$. Of course $G(\mathcal{K}_0/\mathcal{K}_0 \cap \mathcal{K}') \subseteq G(\mathcal{K}_0/\mathcal{K}'_0)$ is finite and $\Gamma = G(\mathcal{K}'/\mathcal{K}'_0) \simeq G(\kappa/\kappa_0) \simeq \widehat{\mathbb{Z}}$. In this context we always make the canonical choice of topological generator for $G(\kappa/\kappa_0)$ to be the q-power Frobenius map $\gamma_q : x \mapsto x^q$ where $q = |\kappa_0|$. This will be the prototypical example of such a $\mathcal{G}$.*

In what follows, a module over $\mathbb{Z}_\ell[[\mathcal{G}]]$ will mean a topological $\mathbb{Z}_\ell[[\mathcal{G}]]$-module. Simple examples of such things are $\mathbb{Z}_\ell$ and $\mathbb{Q}_\ell/\mathbb{Z}_\ell$, each endowed with the trivial $\mathcal{G}$-action. The topologies here are the ℓ-adic and discrete topologies respectively. The most basic functors in the study of modules over $\mathbb{Z}_\ell[[\mathcal{G}]]$ are the invariants and co-invariants functors associated to different subgroups of $\mathcal{G}$.

Definition 2.2.2. *Let M be a finitely generated $\mathbb{Z}_\ell[[\mathcal{G}]]$-module and let $\mathcal{H} \subseteq \mathcal{G}$ be an open subgroup. Then the $\mathcal{H}$-invariants, $M^{\mathcal{H}}$, and the $\mathcal{H}$-co-invariants, $M_{\mathcal{H}}$, of M are defined by*

$$M^{\mathcal{H}} := \{x \in M \mid h \cdot x = x \text{ for all } h \in \mathcal{H}\},$$

and

$$M_{\mathcal{H}} := M/I_{\mathcal{H}} \cdot M.$$

We have that $M^{\mathcal{H}}$ is the largest submodule of M on which $\mathcal{H}$ acts trivially and $M_{\mathcal{H}}$ is the largest quotient of M on which $\mathcal{H}$ acts trivially.

Observe that with $\mathcal{H}$ as in the definition, $M^{\mathcal{H}}$ and $M_{\mathcal{H}}$ both have the structure of $\mathbb{Z}_\ell[\mathcal{G}/\mathcal{H}]$-modules. This relies on the fact that $\mathcal{G}$ is abelian. Note that if $\mathcal{H}$ is topologically cyclic with topological generator $\gamma_{\mathcal{H}}$, then there is an exact sequence

$$0 \to M^{\mathcal{H}} \to M \xrightarrow{1-\gamma_{\mathcal{H}}} M \to M_{\mathcal{H}} \to 0. \tag{2.1}$$

We record several algebraic lemmas that will be useful in the proof of the refined Coates-Sinnott conjecture.

Lemma 2.2.3. *Suppose that* $0 \to A \to B \to C \to 0$ *is a short exact sequence of finitely generated* $\mathbb{Z}_\ell[[\mathcal{G}]]$*-modules and let* $\Gamma \subseteq \mathcal{G}$ *be a subgroup which is topologically cyclic. Then there is an exact sequence of* $\mathbb{Z}_\ell[[\mathcal{G}/\Gamma]]$*-modules*

$$0 \to A^\Gamma \to B^\Gamma \to C^\Gamma \to A_\Gamma \to B_\Gamma \to C_\Gamma \to 0.$$

Proof. Let γ be a topological generator of Γ. We simply apply the snake lemma to the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle 1-\gamma} & & \downarrow{\scriptstyle 1-\gamma} & & \downarrow{\scriptstyle 1-\gamma} & & \\ 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \end{array}$$

□

Remark 2.2.4. *If* $\mathcal{G}/\Gamma$ *happens to be a finite group, then there is an isomorphism* $\mathbb{Z}_\ell[[\mathcal{G}/\Gamma]] \simeq \mathbb{Z}_\ell[\mathcal{G}/\Gamma]$*. This will be the case each time that we apply this Lemma and we will consider the resulting sequence as a* $\mathbb{Z}_\ell[\mathcal{G}/\Gamma]$*-module without further comment.*

Lemma 2.2.5. *Let* Γ *be as in the previous Lemma. Suppose that* M *is a* $Z_\ell[[\mathcal{G}]]$*-module which is finitely generated over* $\mathbb{Z}_\ell$*, then* M_Γ *is finite if and only if* M^Γ *is finite.*

Proof. Again, let γ be a topological generator for Γ. Consider the tautological exact sequence (2.1)

$$0 \to M^\Gamma \to M \xrightarrow{1-\gamma} M \to M_\Gamma \to 0.$$

Observing that $\mathbb{Q}_\ell$ is flat over $\mathbb{Z}_\ell$, we can extend scalars to get

$$0 \to M^\Gamma \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell \to M \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell \xrightarrow{1-\gamma\otimes 1} M \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell \to M_\Gamma \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell \to 0. \tag{2.2}$$

Since M is finitely generated over $\mathbb{Z}_\ell$, M^Γ is finite if and only if $M^\Gamma \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell = 0$ and similarly for M_Γ. Since the alternating sum of the $\mathbb{Q}_\ell$-vector space dimensions in sequence (2.2) is 0 we necessarily have that $\dim_{\mathbb{Q}_\ell}(M_\Gamma \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell) = \dim_{\mathbb{Q}_\ell}(M^\Gamma \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell)$ and therefore $M_\Gamma \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell = 0$ if and only if $M^\Gamma \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell = 0$.

□

We will need two notions of duality for $\mathbb{Z}_\ell[[\mathcal{G}]]$-modules. If M is such a module, then we set $M^* = \mathrm{Hom}_{\mathbb{Z}_\ell}(M, \mathbb{Z}_\ell)$. If M happens to be finite, then we further define $M^\vee = \mathrm{Hom}_{\mathbb{Z}_\ell}(M, \mathbb{Q}_\ell/\mathbb{Z}_\ell)$. M^* and $M^\vee$ are given continuous $\mathcal{G}$ actions by setting

$$(g \cdot \varphi)(m) = \varphi(g^{-1} \cdot m)$$

for $g \in \mathcal{G}$, $\varphi \in M^*$ and $\varphi \in M^\vee$ respectively.

Remark 2.2.6. *If G is a finite group and M is a $\mathbb{Z}_\ell[G]$-module, then the same formulas define two G-actions on M^* and $M^\vee$. We won't distinguish these constructions with notation but the group which is acting will always be clear from context. Observe that this is actually a special case of the previous definition. Every finite group is a pro-finite group and we've already remarked that in this case $\mathbb{Z}_\ell[G] \simeq \mathbb{Z}_\ell[[G]]$.*

Lemma 2.2.7. *Suppose that M is a finitely generated $\mathbb{Z}_\ell[[\mathcal{G}]]$-module which is $\mathbb{Z}_\ell$-free and that M_Γ is finite. Then we have an isomorphism*

$$(M_\Gamma)^\vee \simeq (M^*)_\Gamma$$

Proof. By assumption M_Γ is finite and so by Lemma 2.2.5 we have that M^Γ is finite. Since M is $\mathbb{Z}_\ell$-free, we must have that $M^\Gamma = 0$. Since $M^\Gamma = 0$, sequence (2.1) reads

$$0 \to M \xrightarrow{1-\gamma} M \to M_\Gamma \to 0.$$

Applying the functor $\mathrm{Hom}_{\mathbb{Z}_\ell}(-, \mathbb{Z}_\ell)$ produces the sequence

$$\mathrm{Hom}_{\mathbb{Z}_\ell}(M_\Gamma, \mathbb{Z}_\ell) \to \mathrm{Hom}_{\mathbb{Z}_\ell}(M, \mathbb{Z}_\ell) \xrightarrow{1-\gamma} \mathrm{Hom}_{\mathbb{Z}_\ell}(M, \mathbb{Z}_\ell) \to \mathrm{Ext}^1_{\mathbb{Z}_\ell}(M_\Gamma, \mathbb{Z}_\ell) \to \mathrm{Ext}^1_{\mathbb{Z}_\ell}(M, \mathbb{Z}_\ell).$$

Now, since M_Γ is finite, we get that $\mathrm{Hom}_{\mathbb{Z}_\ell}(M_\Gamma, \mathbb{Z}_\ell) = 0$. Also, since M is $\mathbb{Z}_\ell$-free, we get that $\mathrm{Ext}^1_{\mathbb{Z}_\ell}(M, \mathbb{Z}_\ell) = 0$. The sequence therefore reads

$$0 \to \mathrm{Hom}_{\mathbb{Z}_\ell}(M, \mathbb{Z}_\ell) \xrightarrow{1-\gamma} \mathrm{Hom}_{\mathbb{Z}_\ell}(M, \mathbb{Z}_\ell) \to \mathrm{Ext}^1_{\mathbb{Z}_\ell}(M_\Gamma, \mathbb{Z}_\ell) \to 0.$$

This says that $\mathrm{Ext}^1_{\mathbb{Z}_\ell}(M_\Gamma, \mathbb{Z}_\ell) \simeq \mathrm{Hom}_{\mathbb{Z}_\ell}(M, \mathbb{Z}_\ell)_\Gamma$ and so it will suffice to show that

$$\mathrm{Ext}^1(M_\Gamma, \mathbb{Z}_\ell) \simeq \mathrm{Hom}_{\mathbb{Z}_\ell}(M_\Gamma, \mathbb{Q}_\ell/\mathbb{Z}_\ell).$$

For this, we consider the exact sequence

$$0 \to \mathbb{Z}_\ell \to \mathbb{Q}_\ell \to \mathbb{Q}_\ell/\mathbb{Z}_\ell \to 0$$

and apply the functor $\mathrm{Hom}_{\mathbb{Z}_\ell}(M_\Gamma, -)$ to arrive at the sequence

$$\mathrm{Hom}_{\mathbb{Z}_\ell}(M_\Gamma, \mathbb{Q}_\ell) \to \mathrm{Hom}_{\mathbb{Z}_\ell}(M_\Gamma, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \to \mathrm{Ext}^1_{\mathbb{Q}_\ell}(M_\Gamma, \mathbb{Z}_\ell) \to \mathrm{Ext}^1_{\mathbb{Z}_\ell}(M_\Gamma, \mathbb{Q}_\ell).$$

Since M_Γ is finite we have that $\mathrm{Hom}_{\mathbb{Z}_\ell}(M_\Gamma, \mathbb{Q}_\ell) = 0$ and since $\mathbb{Q}_\ell$ is divisible we have that $\mathrm{Ext}^1_{\mathbb{Z}_\ell}(M_\Gamma, \mathbb{Q}_\ell) = 0$. This gives the needed isomorphism and this concludes the proof. □

When $\mathcal{G}$ is a Galois group, there is often extra structure available. Suppose that $\mathcal{L}/\mathcal{F}$ is a Galois extension of fields with $\mathcal{G} = G(\mathcal{L}/\mathcal{F})$. If we suppose that $\mu_{\ell^\infty} \subseteq \mathcal{L}$, then the ℓ-cyclotomic character, $c_\ell : \mathcal{G} \to \mathbb{Z}_\ell^\times$ is uniquely defined by $g \cdot \zeta = \zeta^{c_\ell(g)}$ for all $\zeta \in \mu_{\ell^\infty}$. In this situation, there is a family of continuous $\mathbb{Z}_\ell$-algebra automorphisms $t_n : \mathbb{Z}_\ell[[\mathcal{G}]] \to \mathbb{Z}_\ell[[\mathcal{G}]]$ uniquely characterized by $t_n(g) = c_\ell(g)^n g$ for $g \in \mathcal{G}$. It is simple to check that $(t_n)^{-1} = t_{-n}$.

If m is an integer, then we define $\mathbb{Z}_\ell(m)$ to be the abelian group $\mathbb{Z}_\ell$, endowed with a $\mathcal{G}$ action via c_ℓ^m, i.e. for $g \in \mathcal{G}$ and $x \in \mathbb{Z}_\ell(m)$, we set $g \cdot x = c_\ell(g)^m x$.

Definition 2.2.8. *If M is a $\mathbb{Z}_\ell[[\mathcal{G}]]$-module, then the m-*th Tate twist *of M is defined to be $M(m) = M \otimes_{\mathbb{Z}_\ell} \mathbb{Z}_\ell(m)$ with the diagonal $\mathcal{G}$-action. That is, if $g \in \mathcal{G}$ and $x \in M(m)$, then $g * x = c_\ell(g)^m g \cdot m$ where $g * x$ denotes the action of $\mathcal{G}$ on $M(m)$ and $g \cdot x$ denotes the original action of g on M.*

Remark 2.2.9. *Observe that $M(m)$ is nothing more than the module obtained by extending scalars from $\mathbb{Z}_\ell[[\mathcal{G}]]$ to $\mathbb{Z}_\ell[[\mathcal{G}]]$ along the automorphism t_{-m}. That is,*

$$M(m) \simeq M \otimes_{\mathbb{Z}_\ell[[\mathcal{G}]]} \mathbb{Z}_\ell[[\mathcal{G}]],$$

where we the right factor is made into a $\mathbb{Z}_\ell[[\mathcal{G}]]$-module via t_{-m}.

It is straightforward to check that $M(n)(m) = M(n+m)$ for all $n, m \in \mathbb{Z}$ and this fact will be used without further comment.

Example 2.2.10. *Keeping the notation of Example 2.2.1, suppose that $\ell \nmid \mathrm{char}(F)$. Observe that $\mu_{\ell^\infty} \subseteq \kappa$ and so the cyclotomic character is defined on Γ. Since γ_q is the q-power Frobenius we have that $c_\ell(\gamma_q) = q$. A simple calculation shows that if V is a $\mathbb{Q}_\ell$ vector space on which γ_q acts with eigenvalue λ, then γ_q acts on $V(n)$ with eigenvalue $q^n\lambda$.*

Lemma 2.2.11. *Let M be a $\mathbb{Z}_\ell[[\mathcal{G}]]$-module, let N be either $\mathbb{Z}_\ell$ or $\mathbb{Q}_\ell/\mathbb{Z}_\ell$ with trivial $\mathcal{G}$ action and let $n \in \mathbb{Z}$, then*

$$\mathrm{Hom}_{\mathbb{Z}_\ell}(M, N(n)) \simeq \mathrm{Hom}_{\mathbb{Z}_\ell}(M, N)(n) \simeq \mathrm{Hom}_{\mathbb{Z}_\ell}(M(-n), N)$$

as $\mathbb{Z}_\ell[[\mathcal{G}]]$-modules.

Proof. The three modules are actually equal as $\mathbb{Z}_\ell$-modules by definition. It is straightforward to check that the $\mathcal{G}$ actions are the same. □

Remark 2.2.12. *If we take N to be $\mathbb{Z}_\ell$ in the above Lemma, then we deduce the relation*

$$M^*(n) \simeq M(-n)^*.$$

Similarly, if $N = \mathbb{Q}_\ell/\mathbb{Z}_\ell$, then we get

$$M^\vee(n) = M(-n)^\vee.$$

2.3 Fitting ideals

Let R be a commutative ring. The R-Fitting ideal is an invariant of a finitely generated R-module that contains information about the R-module structure but that is functorial enough to enable the sorts of calculations that we need to make.

Let M be a finintely generated R-module and let $K \xrightarrow{\phi} R^n \to M \to 0$ be a presentation of M. Taking n-th exterior powers, we get a map $\wedge^n K \xrightarrow{\wedge^n \phi} \wedge^n R^n$. Once we choose a basis for R^n, there is a natural isomorphism $\det : \wedge^n R^n \simeq R$. We define the R-Fitting ideal of M to be

$$\mathrm{Fit}_R(M) = \mathrm{Image}(\det \circ \wedge^n \phi) \subseteq R.$$

In developing the theory of Fitting ideals, one proves that this definition is independent of the choices made, namely the choice of presentation of M and the choice of basis for R^n.

Fitting ideals obey the following convenient properties:

Proposition 2.3.1 (Appendix in [10])**.** *Let M, N be finitely generated R-modules, then*

1. $\mathrm{Fit}_R(M) \subseteq \mathrm{Ann}_R(M)$,

2. $\mathrm{Fit}_R(R/I) = \mathrm{Ann}_R(R/I) = I$,

3. $\mathrm{Fit}_R(M \oplus N) = \mathrm{Fit}_R(M) \cdot \mathrm{Fit}_R(N)$.

4. If $R \xrightarrow{\pi} S$ *is a morphism of rings, then* $\pi(\mathrm{Fit}_R(M)) \cdot S = \mathrm{Fit}_S(M \otimes_R S)$.

As an immediate consequence of point 4 in the above Proposition we have the following formula for the Fitting ideal of a Tate twist.

Corollary 2.3.2. *Let* M *be a* $\mathbb{Z}_\ell[[\mathcal{G}]]$*-module and let* $m \in \mathbb{Z}$*. Then*

$$\mathrm{Fit}_{\mathbb{Z}_\ell[[\mathcal{G}]]}(M(m)) = t_{-m}(\mathrm{Fit}_{\mathbb{Z}_\ell[[\mathcal{G}]]}(M)).$$

Proof. We've already observed that $M(m)$ can be viewed as an extension of scalars along the map t_{-m} and so this follows immediately from point 4 above. □

The notion of projective dimension plays an important role in many of our calculations with Fitting ideals.

Definition 2.3.3. *Let* R *be a commutative ring and let* M *be an* R*-module. Then the projective dimension of* M *over* R*,* $\mathrm{pd}_R(M)$*, is defined to be the smallest integer* n *such that there exists an exact sequence of* R*-modules*

$$0 \to P_n \to \ldots \to P_0 \to M \to 0$$

with each P_i *projective over* R*. If no such sequence exists, then we define* $\mathrm{pd}_R(M) = \infty$*.*

Remark 2.3.4. *Observe that* $\mathrm{pd}_R(M) \leq 1$ *if and only if there exist* P_0, P_1 *projective* R*-modules and an exact sequence of* R*-modules*

$$0 \to P_1 \to P_0 \to M \to 0.$$

Remark 2.3.5. *Suppose that* $R \simeq \prod_i R_i$ *is a semi-local ring and that* M *is an* R*-module. Then* M *decomposes as* $M \simeq \prod_i M_i$ *where each* M_i *is an* R_i*-module and we have that* $\mathrm{Fit}_R(M) = \prod_i \mathrm{Fit}_{R_i}(M_i)$*. This is a consequence of 3 and 4 in Proposition 2.3.1.*

In the next proposition we will need a certain $\mathbb{Z}_\ell$-algebra involution of $\mathbb{Z}_\ell[G]$. Define $\iota : \mathbb{Z}_\ell[G] \to \mathbb{Z}_\ell[G]$ by setting $\iota(g) = g^{-1}$ and extending by $\mathbb{Z}_\ell$-linearity.

Proposition 2.3.6 (Lemma 6 in [1] and Lemma 2.1 in [12])**.** *Let* G *be a finite abelian group and suppose that* M *is a finite* $\mathbb{Z}_\ell[G]$*-module with* $\mathrm{pd}_{\mathbb{Z}_\ell[G]}(M) = 1$*. We have that*

$$\mathrm{Fit}_{\mathbb{Z}_\ell[G]}(M^\vee) = \iota(\mathrm{Fit}_{\mathbb{Z}_\ell[G]}(M)).$$

Furthermore, this ideal is principal, generated by an element which is not a zero-divisor in $\mathbb{Z}_\ell[G]$*.*

Proof. Following the remark, we may assume that $\mathbb{Z}_\ell[G]$ is a local ring i.e., that G is an ℓ-group. Since finitely generated projective projective modules over a local ring are free, M has a presentation of the form

$$0 \to \mathbb{Z}_\ell[G]^n \xrightarrow{\varphi} \mathbb{Z}_\ell[G]^m \to M \to 0.$$

As M is $\mathbb{Z}_\ell[G]$-torsion, φ becomes an isomorphism upon tensoring with $\mathbb{Q}_\ell[G]$. This shows that $m = n$ and it follows from the definition of the Fitting ideal that $\mathrm{Fit}_{\mathbb{Z}_\ell[G]}(M) = \langle \det(\varphi) \rangle$. Furthermore, the fact that φ becomes an isomorphism when we extend scalars to $\mathbb{Q}_\ell[G]$, implies that $\det(\varphi)$ is not a zero-divisor in $\mathbb{Z}_\ell[G]$. This establishes the second claim.

If we apply the functor $\mathrm{Hom}_{\mathbb{Z}_\ell}(-, \mathbb{Z}_\ell)$ to the presentation, we arrive at the short exact sequence

$$0 \to \mathrm{Hom}_{\mathbb{Z}_\ell}(\mathbb{Z}_\ell[G], \mathbb{Z}_\ell)^n \xrightarrow{\varphi^*} \mathrm{Hom}_{\mathbb{Z}_\ell}(\mathbb{Z}_\ell[G], \mathbb{Z}_\ell)^n \to \mathrm{Ext}^1_{\mathbb{Z}_\ell}(M, \mathbb{Z}_\ell) \to 0. \tag{2.3}$$

The following sequence of claims will finish the proof.

1. $\mathrm{Ext}^1_{\mathbb{Z}_\ell}(M, \mathbb{Z}_\ell) \simeq M^\vee$.

   An analogous claim has been made and proven above in Lemma 2.2.7. The proof there can be adopted word for word to this situation.

2. $\mathrm{Hom}_{\mathbb{Z}_\ell}(\mathbb{Z}_\ell[G], \mathbb{Z}_\ell) \simeq \mathbb{Z}_\ell[G]$ as $\mathbb{Z}_\ell[G]$-modules.

   Define $\Psi : \mathrm{Hom}_{\mathbb{Z}_\ell}(\mathbb{Z}_\ell[G], \mathbb{Z}_\ell) \to \mathbb{Z}_\ell$ by $\varphi \mapsto \sum_{g\in G} \varphi(g)g$. If we define δ_g by $\delta_g(h) = 0$ for $h \neq g$ and $\delta_g(g) = 1$, then $\{\delta_g\}$ is a basis of $\mathrm{Hom}_{\mathbb{Z}_\ell}(\mathbb{Z}_\ell[G], \mathbb{Z}_\ell)$ which maps to the basis $\{g\}$ of $\mathbb{Z}_\ell[G]$. This shows that Ψ is an isomorphism of $\mathbb{Z}_\ell$-modules. To check that the G actions are compatible, we calculate

   $$\Psi(\sigma\varphi) = \sum_{g\in G} (\sigma\varphi)(g)g = \sum_{g\in G} \varphi(\sigma^{-1}g)g$$

   and performing the change of variables $g \mapsto \sigma g$ gives us

   $$\sum_{g\in G} \varphi(\sigma^{-1}g)g = \sum_{g\in G} \varphi(g)\sigma g = \sigma\Psi(\varphi).$$

   This shows that Ψ is an isomorphism of $\mathbb{Z}_\ell[G]$-modules.

3. If A is the matrix for φ with respect to the standard basis of $\mathbb{Z}_\ell[G]^n$ and B is the matrix for φ^* with respect to the basis of $\mathrm{Hom}_{\mathbb{Z}_\ell}(\mathbb{Z}_\ell[G]^n, \mathbb{Z}_\ell)$ induced by the above

isomorphism, then $B = \iota(A^T)$. Here A^T denotes the transpose of A and $\iota(A^T)$ is the matrix obtained by applying ι to each of the entries of A^T.

This will be proven in a lemma following this proposition.

We have already seen that $\mathrm{Fit}_{\mathbb{Z}_\ell[G]}(M) = \langle \mathrm{Det}(A) \rangle$ and combining sequence (2.3) with point 1 above shows that $\mathrm{Fit}_{\mathbb{Z}_\ell[G]}(M^\vee) = \langle \mathrm{Det}(B) \rangle$. Using point 3 we then have the following sequence of equalities

$$\mathrm{Fit}_{\mathbb{Z}_\ell[G]}(M^\vee) = \langle \mathrm{Det}(B) \rangle = \langle \mathrm{Det}(\iota(A^T)) \rangle = \langle \iota(\mathrm{Det}(A)) \rangle = \iota(\mathrm{Fit}_{\mathbb{Z}_\ell[G]}(M)).$$

□

In the proof of the above proposition, we needed the following Lemma.

Lemma 2.3.7. *With notation as in the above proposition, we have that $B = \iota(A^T)$.*

Proof. We will identify $\mathrm{Hom}_{\mathbb{Z}_\ell}(\mathbb{Z}_\ell[G]^n, \mathbb{Z}_\ell)$ with $\mathrm{Hom}_{\mathbb{Z}_\ell}(\mathbb{Z}_\ell[G], \mathbb{Z}_\ell)^n$ without further comment. The identification $\mathrm{Hom}_{\mathbb{Z}_\ell}(\mathbb{Z}_\ell[G], \mathbb{Z}_\ell) \simeq \mathbb{Z}_\ell[G]$ provides us with a $\mathbb{Z}_\ell[G]$-basis of $\mathrm{Hom}_{\mathbb{Z}_\ell}(\mathbb{Z}_\ell[G], \mathbb{Z}_\ell)^n$ which we will denote by $\{\delta_e^{(i)}\}$. The function $\delta_e^{(i)}$ is the image of the standard basis vector e_i under the isomorphism. The notation is suggestive and, writing $\delta_g^{(i)}$ for the function δ_g defined in the proof of the above proposition but in the i-th component, our notation is consistent.

Recall that $B = [b_{ij}]$ is the matrix for φ^* with respect to the basis $\{\delta_e^{(i)}\}$ and A is the matrix for φ with respect to the basis $\{e_i\}$. In order to find b_{ij} we need to have an expression for $\varphi^*(\delta_e^{(i)})(ge_j) = \delta_e^{(i)}(\varphi(ge_j))$ for each $g \in G$. We calculate that

$$(\delta_e^{(i)} \circ \varphi)(ge_j) = \delta_e^{(i)}(\sum_k g a_{jk} e_k) = \delta_e(g a_{ji}).$$

If we write $a_{ji} = \sum_\sigma n_\sigma \sigma \in \mathbb{Z}_\ell[G]$, then we have

$$\delta_e(g a_{ji}) = \delta_e(\sum_\sigma n_\sigma g \sigma) = n_{g^{-1}}$$

and this shows that

$$b_{ij} \delta_e^{(j)} = \sum_g n_{g^{-1}} \delta_g^{(j)}.$$

The relation $\delta_g = g\delta_e$ follows from our definition of the G-action on $\mathrm{Hom}_{\mathbb{Z}_\ell}(\mathbb{Z}_\ell[G], \mathbb{Z}_\ell)$ and so we finally have that

$$b_{ij} \delta_e^{(j)} = \sum_g n_{g^{-1}} \delta_g^{(j)} = (\sum_g n_{g^{-1}} g) \delta_e^{(j)} = \iota(a_{ji}) \delta_e^{(j)}.$$

This shows that $b_{ij} = \iota(a_{ji})$ and that finishes the proof. □

Remark 2.3.8. *In the following Proposition we will have need for a different action of G on the dual of a finite $\mathbb{Z}_\ell[G]$-module. If M is a finite $\mathbb{Z}_\ell[G]$-module, then define $M^\wedge = \mathrm{Hom}_{\mathbb{Z}_\ell}(M, \mathbb{Q}_\ell/\mathbb{Z}_\ell)$ with G-action defined by $g * f(m) = f(g \cdot m)$. This is a left G action because G is an abelian group.*

The following property that Fitting ideals enjoy is fundamental to the calculations that follow.

Proposition 2.3.9 (Lemma 5 in [1])**.** *Suppose that*

$$0 \to A \to B \to C \to D \to 0$$

is a short exact sequence of $\mathbb{Z}_\ell[G]$-modules which are finite and with $\mathrm{pd}_{\mathbb{Z}_\ell[G]}(B) \leq 1$ and $\mathrm{pd}_{\mathbb{Z}_\ell[G]}(C) \leq 1$. Then we have that

$$\mathrm{Fit}_{\mathbb{Z}_\ell[G]}(A^\wedge) \cdot \mathrm{Fit}_{\mathbb{Z}_\ell[G]}(C) = \mathrm{Fit}_{\mathbb{Z}_\ell[G]}(B) \cdot \mathrm{Fit}_{\mathbb{Z}_\ell[G]}(D).$$

Lemma 2.3.10. *Let M be a $\mathbb{Z}_\ell[G]$-module which is cyclic as an abelian group. Then*

$$\mathrm{Fit}_{\mathbb{Z}_\ell[G]}(M) = \mathrm{Ann}_{\mathbb{Z}_\ell[G]}(M) = \mathrm{Ann}_{\mathbb{Z}_\ell[G]}(M^\wedge) = \mathrm{Fit}_{\mathbb{Z}_\ell[G]}(M^\wedge).$$

Proof. The two outer equalities are contained in Proposition 2.3.1 so it will suffice to prove that

$$\mathrm{Ann}_{\mathbb{Z}_\ell[G]}(M) = \mathrm{Ann}_{\mathbb{Z}_\ell[G]}(M^\wedge).$$

Due to the obvious $\mathbb{Z}_\ell[G]$-module isomorphism $(M^\wedge)^\wedge \simeq M$ it will suffice to prove that $\mathrm{Ann}_{\mathbb{Z}_\ell[G]}(M) \subseteq \mathrm{Ann}_{\mathbb{Z}_\ell[G]}(M^\wedge)$. For this, suppose that $x \cdot M = 0$ for some $x \in \mathbb{Z}_\ell[G]$ and let $f \in M^\wedge$. Then $x * f(m) = f(x \cdot m) = f(0) = 0$, for all $m \in M$. This implies the containment and concludes the proof. □

Corollary 2.3.11. *Let G be a finite abelian group and let M be a $\mathbb{Z}_\ell[G]$-module. Suppose that M is finite and cyclic as an abelian group and that $\mathrm{pd}_{\mathbb{Z}_\ell[G]}(M) = 1$. Then*

$$\mathrm{Fit}_{\mathbb{Z}_\ell[G]}(M^\vee) = \iota(\mathrm{Fit}_{\mathbb{Z}_\ell[G]}(M^\wedge)).$$

Proof. This is immediate upon combining the results of Proposition 2.3.6 and Lemma 2.3.10. □

Chapter 3

Étale Cohomology Groups and L-functions

In this chapter we introduce the algebraic and analytic objects which appear in the statement of the refined Coates-Sinnott conjecture. For the algebraic side, we introduce Jacobians of curves and étale cohomology groups and present the relevant connections between the two. For the analytic side, the equivariant L-functions, both at the finite and at the infinite level, are introduced. Again, most of the material in this chapter can be found in [7].

3.1 Basic setup

From now on, $\mathcal{K}_0/\mathcal{K}_0'$ will be an abelian extension of characteristic p function fields with Galois group G. We will let κ denote an algebraic closure of $\mathbb{F}_p$ and will write $\kappa_0 = \mathcal{K}_0' \cap \kappa$ where the intersection is taken inside some algebraic closure of $\mathcal{K}_0$. κ_0 is called the field of constants of $\mathcal{K}_0'$. We specifically allow for the possibility that the field of constants of $\mathcal{K}_0$ is larger than κ_0, i.e. that $\mathcal{K}_0 \cap \kappa$ is strictly larger than κ_0.

We will often need to extend the field of constants of $\mathcal{K}_0$ and $\mathcal{K}_0'$ to κ. We will write $\mathcal{K} = \mathcal{K}_0\kappa$ and $\mathcal{K}' = \mathcal{K}_0'\kappa$ for the field composita, taken inside an algebraic closure of $\mathcal{K}_0$. By our choice of κ_0 we have that $\mathcal{K}'$ is actually isomorphic to the tensor product $\mathcal{K}_0' \otimes_{\kappa_0} \kappa$. If κ_0 is not algebraically closed inside $\mathcal{K}_0$ though, then $\mathcal{K}_0 \otimes_{\kappa_0} \kappa$ will not be a field, but rather a direct product of fields, each one isomorphic to $\mathcal{K}$. Galois theory tells us that $\mathcal{K}/\mathcal{K}'$ is an abelian extension of fields whose Galois group is isomorphic to a subgroup of $G(\mathcal{K}_0/\mathcal{K}'_0)$, namely $G(\mathcal{K}_0/\mathcal{K}_0 \cap \mathcal{K}')$.

We will write $G = G(\mathcal{K}_0/\mathcal{K}'_0)$, $\mathcal{G} = G(\mathcal{K}/\mathcal{K}'_0)$, $\Gamma = G(\mathcal{K}/\mathcal{K}_0)$ and $\Gamma' = G(\mathcal{K}'/\mathcal{K}'_0)$. Observe that we have a short exact sequence in the category of groups

$$0 \to \Gamma \to \mathcal{G} \to G \to 0.$$

We will often use this sequence to pass between $\mathbb{Z}_\ell[[\mathcal{G}]]$-modules and $\mathbb{Z}_\ell[G]$-modules. In particular, if M is a $\mathbb{Z}_\ell[[\mathcal{G}]]$-module, then M^Γ and M_Γ are both $\mathbb{Z}_\ell[G]$-modules.

We will denote by Z_0 and Z'_0 smooth projective models for $\mathcal{K}_0$ and $\mathcal{K}'_0$ over κ_0. We recall that this means that Z_0, for example, is a smooth projective curve over κ_0 whose field of rational functions is isomorphic to $\mathcal{K}_0$. Similarly, Z and Z' will be smooth projective models for $\mathcal{K}$ and $\mathcal{K}'$ over κ.

The notion of a Frobenius automorphism is ubiquitous in number theory and this thesis will be no exception. Since we are in characteristic p, there is a distinguished topological generator for $G(\kappa/\kappa_0)$. If $q = |\kappa_0|$, then this is the q-power Frobenius map γ_q defined by $\gamma_q(x) = x^q$. This can be identified with a topological generator of Γ'.

There is a 1-to-1 correspondence between equivalence classes of rank-1 discrete valuations of $\mathcal{K}$ and closed points on the curve Z. The same statement holds for equivalence classes of rank-1 discrete valuations of $\mathcal{K}_0$ and closed points of Z_0. Such equivalence classes of valuations will be referred to as *primes*. The usual notions of extensions of primes apply here along with the usual concepts of inertia, ramification and splitting.

If w is a prime of $\mathcal{K}$ which is unramified in $\mathcal{K}/\mathcal{K}'_0$, then there is an associated Frobenius automorphism in $\mathcal{G}$ and since $\mathcal{K}/\mathcal{K}'_0$ is abelian this Frobenius automorphism depends only on the restriction of w to $\mathcal{K}'_0$. If we denote this restriction by v, then we will write $\widetilde{\sigma}_v$ for the Frobenius associated to v in $\mathcal{G}$.

Similarly, if w is a prime of $\mathcal{K}_0$ which is unramified in $\mathcal{K}_0/\mathcal{K}'_0$, then there is an associated Frobenius automorphism in G which depends only on the restriction of w to $\mathcal{K}'_0$. Let v be this restriction. We will write σ_v for the Frobenius associated to v inside G.

For concreteness we recall the definition of $\sigma_v, \widetilde{\sigma}_v$. Let $\mathcal{F}$ be either $\mathcal{K}$ or $\mathcal{K}_0$ and let w be a prime of $\mathcal{F}$ lying above v. Let $\mathcal{F}_w$ be the completion of $\mathcal{F}$ at the prime w and let $\mathcal{O}_w$ be the valuation ring of $\mathcal{F}_w$. Similarly, let $\mathcal{K}'_{0,v}, \mathcal{O}_v$ be the completion and valuation ring for v. If we let $\kappa(w), \kappa(v)$ denote the respective residue fields, then the extension $\kappa(w)/\kappa(v)$ is cyclic or pro-cyclic with a distinguished (topological) generator given by the map $x \mapsto x^{|\kappa(v)|}$. Since $\mathcal{F}_w/\mathcal{K}'_{0,v}$ is unramified, this map lifts to a unique element of $G(\mathcal{F}_w/\mathcal{K}'_{0,v}) \subseteq G(\mathcal{F}/\mathcal{K}'_0)$ and this is the Frobenius automorphism σ_v or $\widetilde{\sigma}_v$ depending on whether $\mathcal{F} = \mathcal{K}$ or $\mathcal{K}_0$. We can see that $\widetilde{\sigma}_v$ and σ_v are both characterised uniquely by the condition

$$w(\widetilde{\sigma}_v(x) - x^{Nv}) > 0, \text{ for all } x \in \mathcal{O}_w$$

and

$$w(\sigma_v(x) - x^{Nv}) > 0, \text{ for all } x \in \mathcal{O}_w.$$

in their respective groups.

The fact that these formally identical properties characterise the Frobenius automorphism uniquely lets us easily check that

$$\widetilde{\sigma}_v|_{\mathcal{K}_0} = \sigma_v.$$

We will denote by $\mathcal{G}_v \simeq G(\mathcal{K}_w/\mathcal{K}'_{0,v})$, the decomposition group associated to v in the extension $\mathcal{K}/\mathcal{K}'_0$. We have that $\widetilde{\sigma}_v$ is a topological generator for $\mathcal{G}_v$. Similarly, $G_v = G(\mathcal{K}_{0,w}/\mathcal{K}'_{0,v})$ will denote the decomposition group for v in the extension $\mathcal{K}_0/\mathcal{K}'_0$ and σ_v is a generator of G_v.

Remark 3.1.1. *Let w be a prime of $\mathcal{K}$ lying over a prime v of $\mathcal{K}'_0$. If w corresponds to the closed point $P \in Z$, then since Z is defined over κ_0, we have an action of $\mathcal{G}$ on the closed points of Z. One can prove that $\mathcal{G}_v$ is the stabilizer of P under this action. Similarly, if w is a prime of $\mathcal{K}_0$ lying over v and corresponding to $P_0 \in Z_0$, then G_v is the stabilizer of P_0 under the action of G.*

3.2 Jacobians and generalized Jacobians

We keep the notation from the previous section. Let J be the Jacobian of Z. J is an abelian variety whose group of κ-rational points can be identified with the group

$$\mathrm{Pic}^0(Z) := \frac{\mathrm{Div}^0(Z)}{\{\mathrm{div}(f) \mid f \in \mathcal{K}^\times\}}.$$

We will always work with the κ-rational points rather than the scheme J and so, to save notation, the letter J will just mean the group of κ-rational points of J.

If we are given a finite nonempty set of closed points T on Z, then we want to construct the generalized Jacobian associated to T. To this end, we define the subgroup of $\mathcal{K}^\times$

$$\mathcal{K}_T^\times = \{f \in \mathcal{K}^\times \mid f(P) = 1 \text{ for all } v \in T\}.$$

The generalized Jacobian J_T is a semi-abelian variety whose group of κ-rational points can be identified with the group

$$\frac{\mathrm{Div}^0(Z \setminus T)}{\{\mathrm{div}(f) \mid f \in \mathcal{K}_T^\times\}}.$$

As above, we will simply write J_T for the group of κ-rational points of the scheme J_T.

Proposition 3.2.1. *Let* $\mathcal{K}_{(T)} = \{f \in \mathcal{K} \mid \mathrm{ord}_P(f) = 0$ *for all* $v \in T\}$.

1. *There is an isomorphism* $\mathcal{K}_{(T)}^\times / \mathcal{K}_T^\times \kappa^\times \xrightarrow{\sim} (\bigoplus_{v \in T} \kappa(v)^\times)/\kappa^\times$*, where* $\kappa^\times$ *is embedded in the sum diagonally.*

2. *There is a short exact sequence*

$$0 \to (\bigoplus_{v \in T} \kappa(v)^\times)/\kappa^\times \to J_T \to J \to 0$$

Proof. The map $J_T \to J$ is obtained from the obvious inclusion $\mathrm{Div}^0(Z \setminus T) \to \mathrm{Div}^0(Z)$ by passing to the quotient. It is surjective by the weak approximation theorem and it is easy to see that the kernel is isomorphic to $\mathcal{K}_{(T)}^\times / \mathcal{K}_T^\times \kappa^\times$. It therefore suffices to prove 1.

There is a map $\mathcal{K}_{(T)}^\times \to (\bigoplus_{v \in T} \kappa(v)^\times)/\kappa^\times$ given by evaluating $f \mapsto (f(v))_{v \in T}$ and then projecting. This is surjective by the weak approximation theorem and it obviously factors through $\mathcal{K}_T^\times \kappa^\times$. If f is in the kernel, then there is a $\lambda \in \kappa^\times$ such that $f(v) = \lambda$ for all $v \in T$. Then we have that $\lambda^{-1} f(v) = 1$ for all $v \in T$ and therefore $\lambda^{-1} f \in \mathcal{K}_T^\times$. □

The group $(\bigoplus_v \kappa(v)^\times)/\kappa^\times$ is isomorphic to the group of κ-rational points of a torus and keeping with our above convention of omitting κ from the notation we will write τ_T for this group. In the category of group-schemes, one actually has that J_T is an extension of J by τ_T.

Corollary 3.2.2. *J_T is a torsion, divisible group of finite local co-rank (meaning that for all primes ℓ, there is a $\lambda_\ell \in \mathbb{N} \cup \{0\}$ such that $J_T \otimes \mathbb{Z}_\ell \simeq (\mathbb{Q}_\ell/\mathbb{Z}_\ell)^{\lambda_\ell}$).*

Proof. Each $\kappa(v)$ is isomorphic to κ itself because κ is algebraically closed. This implies that $\bigoplus_{v \in T} \kappa(v)^\times$ is divisible and hence so is τ_T, being a quotient of a divisible group. Any abelian variety over an algebraically closed field is divisible, so J is also a divisible group. As κ is the algebraic closure of a finite field, τ_T and J are both clearly torsion.

The proposition shows that J_T is an extension of torsion divisible groups and is therefore itself torsion and divisible.

For the calculation of the λ_ℓ, we observe that $\tau_T \otimes \mathbb{Z}_\ell$ is clearly isomorphic to $(\mathbb{Q}_\ell/\mathbb{Z}_\ell)^{(|T|-1)}$. From Remark 3.3 in [7] we can deduce that $J \otimes \mathbb{Z}_\ell \simeq (\mathbb{Q}_\ell/\mathbb{Z}_\ell)^{2g_Z}$ if $\ell \neq p$ and that $J \otimes \mathbb{Z}_p \simeq (\mathbb{Q}_p/\mathbb{Z}_p)^{\alpha_p}$ for a certain $\alpha_p < g_Z$. Here g_Z denotes the genus of Z. In all cases, we have that $J_T \otimes \mathbb{Z}_\ell \simeq (\mathbb{Q}_\ell/\mathbb{Z}_\ell)^{\lambda_\ell}$ for some finite λ_ℓ. $\square$

Let A be an abelian group. Let $A[m]$ denote the kernel of the multiplication by m map $A \xrightarrow{[m]} A$. If ℓ is prime, then $A[\ell^\infty]$ will denote the union of $A[\ell^n]$ for all $n \geq 0$. Observe that $A[\ell^\infty] \simeq A \otimes \mathbb{Z}_\ell$. We note that if $n \mid m$, then $[m/n]$ restricts to a map $[m/n] : A[m] \to A[n]$.

Definition 3.2.3. *Let A be an abelian group and let ℓ be a prime number. Then the ℓ-adic realization of A, $T_\ell(A)$, is defined by*

$$T_\ell(A) = \varprojlim A[\ell^n]$$

where the transition maps are given by $[\ell] : A[\ell^n] \to A[\ell^{n-1}]$.

Corollary 3.2.4. *There is a short exact sequence of free Z-modules of finite rank*

$$0 \to T_\ell(\tau_T) \to T_\ell(J_T) \to T_\ell(J) \to 0.$$

Proof. As τ_T is divisible, the sequence from part 1 of Proposition 3.2.1 splits. This implies that, for each n, we have a short exact sequence

$$0 \to \tau_T[\ell^n] \to J_T[\ell^n] \to J[\ell^n] \to 0.$$

As taking projective limits is an exact functor on the category of finite abelian groups, we can pass to the limit with respect to the multiplication by ℓ maps to produce the desired sequence. It is easy to see that if A is a torsion divisible group of finite local co-rank, then $T_\ell(A)$ is a free $\mathbb{Z}_\ell$-module of finite rank. $\square$

Proposition 3.2.5. *Assume that T is $\mathcal{G}$-invariant and suppose that $\ell \neq p$. Then there are exact sequences in the category of $\mathbb{Z}_\ell[[\mathcal{G}]]$-modules*

$$0 \to \mathbb{Z}_\ell \to \mathrm{Div}(T) \otimes \mathbb{Z}_\ell \to \mathrm{Div}^0(T) \otimes \mathbb{Z}_\ell \to 0$$

and

$$0 \to \mathbb{Z}_\ell(1) \to \mathrm{Div}(T) \otimes \mathbb{Z}_\ell(1) \to T_\ell(\tau_T) \to 0.$$

Proof. The first exact sequence is clear from the definitions. For the second, if A is a torsion divisible group, then we clearly have that $T_\ell(A) = T_\ell(A[\ell^\infty])$. We have already observed that each $\kappa(v)^\times = \kappa^\times$. From the definition of τ_T, we then have a short exact sequence

$$0 \to \kappa^\times \to \mathrm{Div}(T) \otimes \kappa^\times \to \tau_T \to 0.$$

We always have that $\kappa^\times[\ell^\infty] = \mu_{\ell^\infty}$ and since $\ell \neq p$, we have that $\mu_{\ell^\infty} \simeq (\mathbb{Q}_\ell/\mathbb{Z}_\ell)(1)$. Taking ℓ^∞-torsion we then get an exact sequence

$$0 \to (\mathbb{Q}_\ell/\mathbb{Z}_\ell)(1) \to \mathrm{Div}(T) \otimes (\mathbb{Q}_\ell/\mathbb{Z}_\ell)(1) \to \tau_T[\ell^\infty] \to 0.$$

Observing that $T_\ell(\mathbb{Q}_\ell/\mathbb{Z}_\ell) \simeq \mathbb{Z}_\ell$ concludes the proof.

□

3.3 Some étale cohomology groups

Just as the Coates-Sinnott conjecture for number fields can be formulated as a statement about the Galois module structure of certain étale cohomology groups, we will arrive at the function field analogue of the classical Coates-Sinnott conjecture by studying cohomology groups associated to various rings of integers of the field $\mathcal{K}_0$. The machinery that we will use requires that we link these cohomology groups to the Jacobian of $\mathcal{K}$. For us, the definition of the étale cohomology groups will not be important so we will be content to present just this link.

Proposition 3.3.1 (Lemma 5.11 and Remark 5.15 in [7])**.** *Let $\widetilde{S}_0$ be a finite G-invariant set of primes of $\mathcal{K}_0$ and let S be the set primes of $\mathcal{K}$ lying over $\widetilde{S}_0$. There there are isomorphisms of $\mathbb{Z}_\ell[G]$-modules*

1. $H^2_{\acute{e}t}(\mathcal{O}_{\mathcal{K}_0,\widetilde{S}_0}, \mathbb{Z}_\ell(n)) \xrightarrow{\sim} (T_\ell(J_S)(-n)_\Gamma)^\vee$
2. $H^1_{\acute{e}t}(\mathcal{O}_{\mathcal{K}_0,\widetilde{S}_0}, \mathbb{Z}_\ell(n)) \xrightarrow{\sim} (\mathbb{Q}_\ell/\mathbb{Z}_\ell)(n)^\Gamma \simeq (\mathbb{Z}_\ell(-n)_\Gamma)^\vee$

Proof. The cited references prove everything but the last isomorphism in 2. It is easy to see that if M is a $\mathbb{Z}_\ell[[\mathcal{G}]]$-module, then $(M_\Gamma)^\vee \simeq (M^\vee)^\Gamma$. The relation $\mathbb{Z}_\ell^\vee \simeq \mathbb{Q}_\ell/\mathbb{Z}_\ell$ is easily verified and Remark 2.2.12 implies that $\mathbb{Z}_\ell(-n)^\vee \simeq (\mathbb{Z}_\ell^\vee)(n)$ Putting all this together we therefore have $\mathbb{Z}_\ell[G]$-module isomorphisms

$$(\mathbb{Z}_\ell(-n)_\Gamma)^\vee \simeq (\mathbb{Z}_\ell(-n)^\vee)^\Gamma \simeq (\mathbb{Z}_\ell^\vee(n))^\Gamma \simeq (\mathbb{Q}_\ell/\, \mathbb{Z}_\ell\, (n))^\Gamma.$$

□

3.4 The L-functions

Let $\mathcal{K}_0/\mathcal{K}_0'$ be a Galois extension of characteristic p function fields with Galois group G and let S_0, T_0 be two finite disjoint non-empty sets of primes of $\mathcal{K}_0'$ such that S_0 contains all those primes which ramify in $\mathcal{K}_0/\mathcal{K}_0'$. Let $q = |\kappa_0|$. For each prime v of $\mathcal{K}_0'$, we let d_v be the residue degree of v over κ_0, given by $|\kappa_0(v)| = q^{d_v}$. To this data we can associate the (S_0, T_0) modified equivariant L-function defined by the infinite product

$$\Theta_{\mathcal{K}_0/\mathcal{K}_0', S_0, T_0}(u) = \prod_{v \in T_0} (1 - \sigma_v^{-1} \cdot (qu)^{d_v}) \cdot \prod_{v \notin S_0} (1 - \sigma_v^{-1} u^{d_v})^{-1}. \tag{3.1}$$

This product converges in $\mathbb{Z}[G][[u]]$ to an element of $\mathbb{Z}[G][u]$, i.e. it is actually a polynomial in u. See §4.2 of [7] and Proposition 2.15 in Chapter 5 of [16]. To avoid overburdening our notation, we will suppress the extension $\mathcal{K}_0/\mathcal{K}_0'$ and simply write Θ_{S_0,T_0}.

The relation between $\Theta_{S_0,T_0}(u)$ and the L-functions attached to characters of G is given as follows. If $\chi \in \widehat{G}(\mathbb{C})$, then the (S_0, T_0) modified L-function associated to χ is defined as the convergent Euler product for $s \in \mathbb{C}$, $\Re e(s) > 1$

$$L_{S_0,T_0}(s, \chi) = \prod_{v \in T_0} (1 - \chi(\sigma_v) N v^{1-s}) \cdot \prod_{v \notin S_0} (1 - \chi(\sigma_v) N v^{-s})^{-1}.$$

For each $\chi \in \widehat{G}(\mathbb{C})$, this function can be analytically continued to all of $\mathbb{C}$. From the definitions, one can immediately prove that

$$\Theta_{S_0,T_0}(q^{-s}) = \sum_{\chi} L_{S_0,T_0}(s, \chi) e_{\chi^{-1}}. \tag{3.2}$$

Note that both of these definitions make perfect sense if T_0 is empty. In this case we will suppress T_0 from the notation and just write $\Theta_{S_0}(u)$. We warn the reader that, unlike $\Theta_{S_0,T_0}(q^{-s})$ for T_0 as above, $\Theta_{S_0}(q^{-s})$ may have a pole at $s = 1$. It is analytic on $\mathbb{C} \setminus \{1\}$ though and in particular, it is analytic at all negative integers $s = 1 - n$. If we define

$$\delta_{T_0}(s) = \prod_{v \in T_0} (1 - \sigma_v^{-1} N v^{1-s}),$$

then we have the relation $\Theta_{S_0,T_0}(q^{-s}) = \delta_{T_0}(s) \cdot \Theta_{S_0}(q^{-s})$. This factorization will turn up in later calculations.

Remark 3.4.1. *Here we would like to compare the special values of Θ_{S_0,T_0} to those which appear in the statement of the Coates-Sinnott conjecture for number fields. If K/k is an abelian extension of number fields and S is a finite set of primes of k containing all the* *primes which ramify in K/k, then the S-incomplete equivariant L-function associated to the data $(K/k, S)$ is defined by*

$$\sum_{\chi} L_S(s,\chi)e_{\chi^{-1}}.$$

The special value which appears in the Coates-Sinnott conjecture for number fields is therefore

$$\sum_{\chi} L_S(1-n,\chi)e_{\chi^{-1}}, \text{ for } n \in \mathbb{Z}_{\geq 2}$$

Using (3.2)*, we can now see that the approprate special value in the characteristic p context will be $\Theta_{S_0}(q^{n-1}) \in \mathbb{Q}_\ell[G]$, for $n \in \mathbb{Z}_{\geq 2}$.*

We fix a prime $\ell \neq p$. For the next step in our development we need to have an equivariant L-function which lives in $\mathbb{Z}_\ell[[\mathcal{G}]]$. Let γ_q denote the q-power Frobenius, identified with a topological generator for $\Gamma' = G(\mathcal{K}'/\mathcal{K}'_0)$. Since $\Theta_{S_0,T_0} \in \mathbb{Z}_\ell[G][u]$, we can evaluate Θ_{S_0,T_0} at γ_q^{-1} to get an element of $\mathbb{Z}_\ell[G \times \Gamma'] \subseteq \mathbb{Z}_\ell[[G \times \Gamma']]$. What is remarkable is that we actually get an element of $\mathbb{Z}_\ell[[\mathcal{G}]]$. What this means is made precise in the following proposition.

Proposition 3.4.2 ([7])**.** *With notations as above.*

1. *$\mathbb{Z}_\ell[[\mathcal{G}]]$ can be identified canonically with a subring of $\mathbb{Z}_\ell[[G \times \Gamma']]$.*

2. *Under this identification, we have that $\Theta_{S_0,T_0}(\gamma_q^{-1}) \in \mathbb{Z}_\ell[[\mathcal{G}]]$.*

Proof. 1. Indeed, $\mathcal{G}$ can be identified with the subgroup of $G \times \Gamma'$ consisting of those (g,σ) such that g and σ agree when restricted to $\mathcal{K}_0 \cap \kappa$. That is, there is an exact sequence

$$0 \to \mathcal{G} \to G \times \Gamma' \to G(\mathcal{K}_0 \cap \kappa/\kappa_0) \to 0$$

where the first map is given by $g \mapsto (g|_{\mathcal{K}_0}, g|_{\mathcal{K}'})$ and the second map is given by $(g,\sigma) \mapsto g|_{\mathcal{K}_0\cap\kappa_0} \cdot \sigma|^{-1}_{\mathcal{K}_0\cap\kappa_0}$. Consequently we have an injection $\mathbb{Z}_\ell[[\mathcal{G}]] \hookrightarrow \mathbb{Z}_\ell[[G \times \Gamma']]$.
2. Let $\kappa_n \subseteq \kappa$ be the unique extension of κ_0 of degree n inside κ and let $\mathcal{K}_n = \mathcal{K}_0\kappa_n$ be the compositum. We will write $G_n = G(\mathcal{K}_n/\mathcal{K}'_0)$. We will write $G^{(n)}$ for the group $G \times (\Gamma'/\Gamma'^n)$. Let $\pi_n : \mathbb{Z}_\ell[[G \times \Gamma']] \to \mathbb{Z}_\ell[G^{(n)}]$ be the natural projection map. As in part 1, we can identify G_n with a subgroup of $G^{(n)}$. We observe that $\mathbb{Z}_\ell[[\mathcal{G}]] \simeq \varprojlim \mathbb{Z}_\ell[G_n]$ and that this isomorphism is compatible with the inclusions $\mathbb{Z}_\ell[[\mathcal{G}]] \subseteq \mathbb{Z}_\ell[[G \times \Gamma']]$ and $\mathbb{Z}_\ell[G_n] \subseteq \mathbb{Z}_\ell[G^{(n)}]$. It will therefore suffice to show that $\pi_n(\Theta_{S_0,T_0}(\gamma_q^{-1})) \in \mathbb{Z}_\ell[G_n]$ for all n.

This is accomplished by a neat trick. For each n, we define a $\mathbb{Z}_\ell[G_n]$-linear map $\rho_n : \mathbb{Z}_\ell[G_n][u] \to \mathbb{Z}_\ell[G^{(n)}][u]$ by $u \mapsto \overline{\gamma}_q^{-1}u$, where $\overline{\gamma}_q$ is the image of γ_q in $G^{(n)}$. We observe that this map is continuous with respect to the u-adic topologies on both sides and so it defines a map $\rho_n : \mathbb{Z}_\ell[G_n][[u]] \to \mathbb{Z}_\ell[G^{(n)}][[u]]$. Applying ρ_n to the expression for Θ_{S_0,T_0} as an infinite product, we get

$$\rho_n(\Theta_{S_0,T_0}(u)) = \prod_{v\in T_0}(1-\sigma_v^{-1}\cdot\overline{\gamma}_q^{-d_v}(qu)^{d_v})\cdot\prod_{v\notin S_0}(1-\sigma_v^{-1}\overline{\gamma}_q^{-d_v}u^{d_v})^{-1} \in \mathbb{Z}_\ell[G^{(n)}].$$

I claim that this is actually $\Theta_{\mathcal{K}_n/\mathcal{K}_0',S_0,T_0}$, the equivariant L-function associated to the data $(\mathcal{K}_n/\mathcal{K}_0', S_0, T_0)$. This implies that $\rho_n(\Theta_{S_0,T_0}(u)) \in \mathbb{Z}_\ell[G_n][u]$, again see §4.2 of [7] and Proposition 2.15 in Chapter 5 of [16]. We assume this for the moment and finish the proof.

Evaluating $\rho_n(\Theta_{S_0,T_0}(u))$ at $u = 1$, we clearly get $\pi_n(\Theta_{S_0,T_0}(\gamma_q^{-1}))$ and since $\rho_n(\Theta_{S_0,T_0}(u)) \in \mathbb{Z}_\ell[G_n][u]$ it follows that $\pi_n(\Theta_{S_0,T_0}(\gamma_q^{-1})) \in \mathbb{Z}_\ell[G_n]$.

To see that $\rho_n(\Theta_{S_0,T_0}(u))$ is indeed just the equivariant L-function for the extension $\mathcal{K}_n/\mathcal{K}_0'$ it suffices to check that for each $v \notin S_0$, $\sigma_v\overline{\gamma}_q^{d_v}$ is in G_n and is in fact the Frobenius automorphism for v in the extension $\mathcal{K}_n/\mathcal{K}_0'$.

In the constant field extension κ_n/κ_0, the Frobenius automorphism for v corresponds exactly to the d_v-th power of the q-power Frobenius. As the Frobenius automorphism is functorial with respect to changing the top field, both σ_v and $\overline{\gamma}_q^{d_v}$ will restrict to the Frobenius automorhism for v in $G(\mathcal{K}_0 \cap \mathcal{K}'/\mathcal{K}_0')$. This shows that each of these elements is in G_n. To check that they give the respective Frobenii is straightforward once we know that they give the Frobenius in both κ_n/κ_0 and $\mathcal{K}_0/\mathcal{K}_0'$. □

We finally make the definition

$$\vartheta_{S_0,T_0}^{(\infty)} = \Theta_{S_0,T_0}(\gamma_q^{-1}) \in \mathbb{Z}_\ell[[\mathcal{G}]].$$

The importance of $\vartheta_{S_0,T_0}^{(\infty)}$ is given in the following proposition which shows that its twists know the special values of $\Theta_{S_0,T_0}(q^{-s})$ at negative integers $s = 1 - n$.

Proposition 3.4.3. *If $\pi : \mathbb{Z}_\ell[[\mathcal{G}]] \to \mathbb{Z}_\ell[G]$ is the reduction map, then*

$$\pi(t_{1-n}(\vartheta_{S_0,T_0}^{(\infty)})) = \Theta_{S_0,T_0}(q^{n-1}).$$

Proof. Thinking of $\mathcal{G} \subset G \times \Gamma'$ as in the previous proposition, the map $\mathcal{G} \hookrightarrow G \times \Gamma'$ is given by $g \mapsto (g|_{\mathcal{K}_0}, g|_{\mathcal{K}'})$. If we define $\widetilde{c_\ell} : G \times \Gamma' \to \mathbb{Z}_\ell$ by $\widetilde{c_\ell}(g,\sigma) := c_\ell(\sigma)$, then $\widetilde{c_\ell}$

extends c_ℓ to a group morphism defined on $G \times \Gamma'$ which is trivial on G. This affords us with a corresponding extension of t_{1-n} to a continuous $\mathbb{Z}_\ell$-algebra automorphism $\widetilde{t_{1-n}}$ of $\mathbb{Z}_\ell[[G \times \Gamma']]$ which is trivial on G. Also observe that the obvious reduction map $\widetilde{\pi} : \mathbb{Z}_\ell[[G \times \Gamma']] \to \mathbb{Z}_\ell[G]$ is an extension of π.

Now we calculate that

$$\pi(t_{1-n}(\vartheta^{(\infty)}_{S_0,T_0})) = \widetilde{\pi}(\widetilde{t_{1-n}}(\Theta_{S_0,T_0}(\gamma_q^{-1}))) = \widetilde{\pi}(\Theta_{S_0,T_0}(q^{n-1} \cdot \gamma_q^{-1})) = \Theta_{S_0,T_0}(q^{n-1}).$$

□

Chapter 4

Picard 1-Motives and the Proof of the Main Theorem

4.1 1-Motives

The concept of a 1-motive, introduced by Deligne in [4], has provided the foundation for recent success in proving classical conjectures on special values of L-functions. 1-motives were used by Deligne and Tate to prove the Brumer-Stark conjecture in function fields, see [16]. The notion of a 1-motive has been generalized by Greither-Popescu through the introduction of their *abstract* 1*-motives*, see [6]. With this machinery they have successfully proven the Brumer-Stark conjecture for number fields under certain hypotheses as well as many other conjectures on special values of L-functions. Here we introduce the basic language of Deligne's (geometric) 1-motives before going over the construction that will be of use to us. This material can be found in [7] or [16].

Definition 4.1.1. *Let κ be an algebraically closed field. A* 1*-motive over κ consists of the following set of data*

1. *a free $\mathbb{Z}$-module of finite rank L,*
2. *an abelian variety A and a torus τ, both defined over κ,*
3. *an extension of A by τ over κ, denoted A_τ,*
4. *a group homomorphism $d : L \to A_\tau(\kappa)$.*

To keep notation simple, we will just write A_τ, A, τ for the respective κ-valued points. Obviously, all of the information going into a 1-motive is contained in the diagram $[L \xrightarrow{d} A_\tau]$ and so we will just write $\mathcal{M} = [L \xrightarrow{d} A_\tau]$ for a 1-motive

Remark 4.1.2. *It is straightforward to see that A_τ is a divisible abelian group. Indeed, there is a short exact sequence*

$$0 \to \tau \to A_\tau \to A \to 0.$$

The group of κ-rational points of any abelian variety is divisible so both τ and A are. The divisibility of A_τ follows because extensions of divisible groups are divisible.

The usefulness of a 1-motive comes via it's ℓ-adic realizations for each prime ℓ. Let $\mathcal{M} = [L \xrightarrow{d} A_\tau]$ be a 1-motive. If $n \in \mathbb{N}$, then we would like to define the n-torsion points of $\mathcal{M}$ for all n. To do this, observe that we have the multiplication-by-n map $A_\tau \xrightarrow{[n]} A_\tau$, surjective because A_τ is a divisible group. If we let $X = A_\tau \times_{A_\tau} L$ denote the pullback of L and A_τ with respect to δ and $[n]$, then we have the following diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A_\tau[n] & \longrightarrow & X & \longrightarrow & L & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow \delta & & \\
0 & \longrightarrow & A_\tau[n] & \longrightarrow & A_\tau & \xrightarrow{[n]} & A_\tau & \longrightarrow & 0
\end{array}$$

Proposition 4.1.3 (See comments following Definition 2.5 in [7])**.** *If we define $\mathcal{M}[n] = X \otimes \mathbb{Z}/n\mathbb{Z}$, then we have a short exact sequence*

$$0 \to A_\tau[n] \to \mathcal{M}[n] \to L \otimes \mathbb{Z}/n\mathbb{Z} \to 0. \tag{4.1}$$

Proof. L is a free $\mathbb{Z}$-module so the top sequence in the above diagram splits. □

If $n \mid m$ are positive integers, then we can construct natural multiplication by m/n maps $[m/n] : \mathcal{M}[m] \to \mathcal{M}[n]$ as follows. An element of $\mathcal{M}[m]$ consists of a pair (P, λ) with $P \in A_\tau \otimes \mathbb{Z}/m\mathbb{Z}$, $\lambda \in L \otimes \mathbb{Z}/m\mathbb{Z}$ and $[m](P) = d(\lambda)$. We define

$$[m/n](P, \lambda) = ([m/n](P), \lambda).$$

Now fix a prime number ℓ. We define the ℓ-adic realization of $\mathcal{M}$ by

$$T_\ell(\mathcal{M}) = \varprojlim \mathcal{M}[\ell^n]$$

where the transition maps are given by $[\ell] : \mathcal{M}_{S,T}[\ell^n] \to \mathcal{M}_{S,T}[\ell^{n-1}]$.

The last thing that we will need to note about general 1-motives is that taking projective limits in (4.1) produces the short exact sequence of free $\mathbb{Z}$-modules of finite rank

$$0 \to T_\ell(A_\tau) \to T_\ell(\mathcal{M}) \to L \otimes \mathbb{Z}_\ell \to 0. \tag{4.2}$$

Remark 4.1.4. *We observe that these constructions are all functorial. In the next section, once we've introduced the* 1*-motives we will be working with, this will lead to natural* $\mathbb{Z}_\ell[[\mathcal{G}]]$*-module structures on all of these* ℓ*-adic realizations.*

4.2 Construction of the relevant Picard 1-motive

We keep the setup and notation from the last chapter. Suppose that we are given two finite $\mathcal{G}$-invariant sets of primes of $\mathcal{K}$, say S and T, such that $S \cap T = \emptyset$. We assume that $T \neq \emptyset$ and that S contains all the primes which are ramified in $\mathcal{K}/\mathcal{K}'$. Recalling that J_T is an extension of the abelian variety J by the torus τ_T, the machinery from the last section associates to the data $(\mathcal{K}, S, T)$ the so-called *Picard* 1*-motive*

$$\mathcal{M}_{S,T} = [\mathrm{Div}^0(S) \xrightarrow{\delta} J_T]$$

where δ is the map which sends a divisor to its class in J_T. Note that this makes sense because of the assumption that $S \cap T = \emptyset$.

In this context, the sequence (4.2) takes the form

$$0 \to T_\ell(J_T) \to T_\ell(\mathcal{M}_{S,T}) \to \mathrm{Div}^0(S) \otimes \mathbb{Z}_\ell \to 0. \tag{4.3}$$

Certain subgroups of $\mathcal{K}^\times$ play an important role in the study of $T_\ell(\mathcal{M}_{S,T})$. Recall that we have defined

$$\mathcal{K}_T^\times = \{f \in \mathcal{K}^\times \mid f(w) = 1 \text{ for all } w \in T\}.$$

We also define

$$\mathcal{K}_{S,T}^{(n)} = \{f \in \mathcal{K}_T^\times \mid \mathrm{ord}_P(f) \text{ is divisible by } n, \text{ for all } P \notin S\}.$$

Observe that $\mathcal{K}_T^{\times n} \subseteq \mathcal{K}_{S,T}^{(n)}$.

Using these two groups, we can provide the following concrete description of $\mathcal{M}_{S,T}$.

Proposition 4.2.1 (Proposition 2.9 in [7])**.** *For each n, there is an isomorphism*

$$\mathcal{M}_{S,T}[n] \xrightarrow{\sim} \mathcal{K}_{S,T}^{(n)}/\mathcal{K}_T^{\times n}.$$

Consequently, for each ℓ, we have an isomorphism

$$T_\ell(\mathcal{M}_{S,T}) \xrightarrow{\sim} \varprojlim \mathcal{K}_{S,T}^{(\ell^n)}/\mathcal{K}_T^{\times \ell^n}.$$

Remark 4.2.2. *We have already described the transition maps between the groups on the* 1*-motive side. The transition maps*

$$\mathcal{K}_{S,T}^{(\ell^{n+1})}/\mathcal{K}_T^{\times \ell^{n+1}} \to \mathcal{K}_{S,T}^{(\ell^n)}/\mathcal{K}_T^{\times \ell^n}$$

are just induced by the natural inclusion $\mathcal{K}_{S,T}^{(\ell^{n+1})} \hookrightarrow \mathcal{K}_{S,T}^{(\ell^n)}$.

Remark 4.2.3. *If S and T are $\mathcal{G}$-invariant, then $T_\ell(\mathcal{M}_{S,T})$, then these construction endow $T_\ell(\mathcal{M}_{S,T})$ with a natural $\mathbb{Z}_\ell[[\mathcal{G}]]$-module structure and the above exact sequences all exist in the category of $\mathbb{Z}_\ell[[\mathcal{G}]]$-modules.*

Greither-Popescu have recently proven a number of remarkable theorems on the $\mathbb{Z}_\ell[[\mathcal{G}]]$-module structure of $T_\ell(\mathcal{M}_{S,T})$.

Theorem 4.2.4 (Corollary 4.13 in [7])**.** *Let S, T be as above and assume further that S and T are both $\mathcal{G}$-invariant. Let S_0, T_0 be the sets of primes of $\mathcal{K}_0'$ which lie below the primes in S, T. In addition, let $H = G(\mathcal{K}/\mathcal{K}')$. Then,*

1. *$T_\ell(\mathcal{M}_{S,T})$ is $\mathbb{Z}_\ell[H]$-projective.*
2. *$\mathrm{Fit}_{\mathbb{Z}_\ell[[\mathcal{G}]]}(T_\ell(\mathcal{M}_{S,T})) = \langle \vartheta_{S_0,T_0}^{(\infty)} \rangle$.*

We will need a certain $\mathcal{G}$ equivariant duality pairing relating $\mathcal{M}_{S,T}$ to $\mathcal{M}_{T,S}$

Proposition 4.2.5 (Theorem 5.20 in [7])**.** *For each $n \in \mathbb{Z}$, there is an isomorphism of $\mathbb{Z}_\ell[[\mathcal{G}]]$-modules*

$$T_\ell(\mathcal{M}_{S,T})(n-1) \simeq T_\ell(\mathcal{M}_{T,S})(-n)^*$$

Proof. There exists a $\mathbb{Z}_\ell[[\mathcal{G}]]$-equivariant perfect pairing

$$T_\ell(\mathcal{M}_{S,T}) \times T_\ell(\mathcal{M}_{T,S}) \to \mathbb{Z}_\ell(1),$$

see the proof of Theorem 5.20 in [7]. This implies that

$$T_\ell(\mathcal{M}_{S,T}) \simeq \mathrm{Hom}_{\mathbb{Z}_\ell}(T_\ell(\mathcal{M}_{T,S}), \mathbb{Z}_\ell(1)) = T_\ell(\mathcal{M}_{T,S})^*(1).$$

Tensoring with $\mathbb{Z}_\ell(n-1)$, then gives an isomorphism

$$T_\ell(\mathcal{M}_{S,T})(n-1) \simeq T_\ell(\mathcal{M}_{T,S})^*(n)$$

and applying Lemma 2.2.11 finishes the proof. □

4.3 Statement of the refined Coates-Sinnott conjecture

First, we recall the statement of the classical Coates-Sinnott conjecture for number fields as given in the Introduction.

Conjecture 4.3.1 (Coates-Sinnott). *Let K/k be an abelian extension of number fields with Galois group G, let S be a finite set of primes of k containing the primes which ramify in K/k, let ℓ be a prime number and let $n \geq 2$ be an integer, then*

$$\mathrm{Ann}_{\mathbb{Z}_\ell[G]}(H^1_{\acute{e}t}(\mathcal{O}_K[1/\ell], \mathbb{Z}_\ell(n))_{\mathrm{tors}}) \cdot \Theta_{K/k,S}(1-n) \subseteq \mathrm{Ann}_{\mathbb{Z}_\ell[G]}(H^2_{\acute{e}t}(\mathcal{O}_K[1/\ell], \mathbb{Z}_\ell(n))).$$

Using the theory of Fitting ideals we can formulate a stronger refined version of this conjecture.

Conjecture 4.3.2 (Refined Coates-Sinnott). *Keep the same hypotheses as above and let $\widetilde{S}$ be the set of primes of K lying over the primes in S. Then,*

$$\mathrm{Ann}_{\mathbb{Z}_\ell[G]}(H^1_{\acute{e}t}(\mathcal{O}_K[1/\ell], \mathbb{Z}_\ell(n))_{\mathrm{tors}}) \cdot \Theta_{K/k,S}(1-n) \subseteq \mathrm{Fit}_{\mathbb{Z}_\ell[G]}(H^2_{\acute{e}t}(\mathcal{O}_{K,\widetilde{S}}[1/\ell], \mathbb{Z}_\ell(n))).$$

Let us see that this is a stronger conjecture. Indeed, Proposition 2.3.1 tells us that

$$\mathrm{Fit}_{\mathbb{Z}_\ell[G]}(H^2_{\acute{e}t}(\mathcal{O}_{K,\widetilde{S}}[1/\ell], \mathbb{Z}_\ell(n))) \subseteq \mathrm{Ann}_{\mathbb{Z}_\ell[G]}(H^2_{\acute{e}t}(\mathcal{O}_{K,\widetilde{S}}[1/\ell], \mathbb{Z}_\ell(n))).$$

From Remark 3.4 in [12], we have

$$\mathrm{Ann}_{\mathbb{Z}_\ell[G]}(H^2_{\acute{e}t}(\mathcal{O}_{K,\widetilde{S}}[1/\ell], \mathbb{Z}_\ell(n))) \subseteq \mathrm{Ann}_{\mathbb{Z}_\ell[G]}(H^2_{\acute{e}t}(\mathcal{O}_K[1/\ell], \mathbb{Z}_\ell(n)))$$

and so we see that this conjecture implies the previous one. In fact, for $\ell > 2$, an especially strong form of this conjecture has been proven by Greither-Popescu under one additional hypothesis.

Theorem 4.3.3 (Greither-Popescu, Theorem 6.11 in [6]). *Under the above hypotheses, let $\ell > 2$ and further suppose that the Iwasawa μ-invariant for the $\mathbb{Z}_\ell$-cyclotomic extension of K is zero (see chapter 13 of [18] for the definition). Then there exists an idempotent $\epsilon \in \mathbb{Z}_\ell[G]$, such that*

1. $\epsilon \cdot \Theta_{K/k,S}(1-n) = \Theta_{K/k,S}(1-n) \in \mathbb{Q}_\ell[G]$.

2. $\mathrm{Ann}_{\mathbb{Z}_\ell[G]}(H^1_{\acute{e}t}(\mathcal{O}_K[1/\ell], \mathbb{Z}_\ell(n))_{\mathrm{tors}}) \cdot \Theta_{K/k,S}(1-n) = \epsilon \cdot \mathrm{Fit}_{\mathbb{Z}_\ell[G]}(H^2_{\acute{e}t}(\mathcal{O}_{K,S}[1/\ell], \mathbb{Z}_\ell(n)))$.

Let $S_\infty(k)$ be the set of archimedean primes of the field k. The idempotent ϵ is defined by

$$\epsilon = \begin{cases} \prod_{v \in S_\infty(k)} \frac{1}{2}(1 + (-1)^n \sigma_v) & \text{if } k \text{ is totally real,} \\ 0 & \text{otherwise.} \end{cases}$$

In addition to their work in number fields, Greither and Popescu have proven an analogue of Conjecture 4.3.2 in the function field case. In fact, the use of Picard 1-motives in the function field case directly inspired the definition of the abstract 1-motives that were introduced in [6] and which played a key role in their work in number fields.

Theorem 4.3.4 (Greither-Popescu, Theorem 5.20 in [7])**.** *Let $\mathcal{K}_0/\mathcal{K}_0'$ be an abelian extension of characteristic p function fields, let S_0 be a finite set of primes of $\mathcal{K}_0'$ which contains all the primes which ramify in $\mathcal{K}_0/\mathcal{K}_0'$, let ℓ be a prime different from p and let $n \geq 2$ be an integer. Let $\widetilde{S}_0$ be the primes of $\mathcal{K}_0$ lying over those in S_0. Then*

$$\mathrm{Ann}_{\mathbb{Z}_\ell[G]}(H^1_{\acute{e}t}(\mathcal{O}_{\mathcal{K}_0,\widetilde{S}_0}, \mathbb{Z}_\ell(n))) \cdot \Theta_{S_0}(q^{n-1}) \subseteq \mathrm{Fit}_{\mathbb{Z}_\ell[G]}(H^2_{\acute{e}t}(\mathcal{O}_{\mathcal{K}_0,\widetilde{S}_0}, \mathbb{Z}_\ell(n))).$$

Examining theorems 4.3.3 and 4.3.4, it is natural to expect that we could prove a theorem which precisely calculates the Fitting ideal of $H^2_{\acute{e}t}(\mathcal{O}_{\mathcal{K}_0,\widetilde{S}_0}, \mathbb{Z}_\ell(n))$. The main result of this thesis is the following theorem which does exactly this and whose proof will occupy the next section.

Theorem 4.3.5. *Under the hypothese of the previous theorem, we have*

$$\mathrm{Ann}_{\mathbb{Z}_\ell[G]}(H^1_{\acute{e}t}(\mathcal{O}_{\mathcal{K}_0,\widetilde{S}_0}, \mathbb{Z}_\ell(n))) \cdot \Theta_{S_0}(q^{n-1}) = \mathrm{Fit}_{\mathbb{Z}_\ell[G]}(H^2_{\acute{e}t}(\mathcal{O}_{\mathcal{K}_0,\widetilde{S}_0}, \mathbb{Z}_\ell(n))).$$

Remark 4.3.6. *There are deep reasons why one would not expect the idempotent ϵ to play a role in the function field setting. In addition, the analogue of the μ-invariant in function fields would measure the presence of p-torsion in the ℓ-adic Tate module of the Jacobian. Since ℓ-adic Tate modules of Jacobians are free, (indeed, the ℓ-adic Tate module of any abelian variety is free) the μ-invariant vanishes automatically in function fields.*

Remark 4.3.7. *The reason that $\ell = p$ is excluded from consideration here is that the groups $H^i_{ét}(\mathcal{O}_{\mathcal{K}_0,\widetilde{S}_0}, \mathbb{Z}_p(n))$ for $i = 1, 2$ are trivial in characteristic p. There is an appropriate replacement for p-adic étale cohomology in characteristic p in the form of crystalline cohomology. The formulation and proof of a refined Coates-Sinnott conjecture at p will be forthcoming.*

4.4 Proof of the refined Coates-Sinnott conjecture

In this section, we give a proof of Theorem 4.3.5. We will keep the notation from the previous sections, but we will recall some of it here for the convenience of the reader. Let ℓ be a prime different from p and let $n \geq 2$. Let S_0, T_0 be two finite sets of primes of $\mathcal{K}'_0$ such that S_0 contains the primes which ramify in $\mathcal{K}_0/\mathcal{K}'_0$, $S \cap T = \emptyset$ and $S, T \neq \emptyset$. Let S, T denote the primes of $\mathcal{K}$ lying over the primes in S_0, T_0. Further, denote by $\widetilde{S}_0$, the primes of $\mathcal{K}_0$ lying over those in S_0.

We begin by recalling exact sequence (4.3) from section 4.2, where we have switched the roles of S and T.

$$0 \to T_\ell(J_S) \to T_\ell(\mathcal{M}_{T,S}) \to \mathrm{Div}^0(T) \otimes \mathbb{Z}_\ell \to 0$$

Tensoring this with $\mathbb{Z}_\ell(-n)$ and using the first sequence in Proposition 3.2.5 to extend the sequence by a term leads to the four term exact sequence

$$0 \to T_\ell(J_S)(-n) \to T_\ell(\mathcal{M}_{T,S})(-n) \xrightarrow{\varphi} \mathrm{Div}(T) \otimes \mathbb{Z}_\ell(-n) \to \mathbb{Z}_\ell(-n) \to 0 \tag{4.4}$$

Proposition 4.4.1. *After taking Γ co-invariants, sequence (4.4) induces a four term exact sequence of $\mathbb{Z}_\ell[G]$-modules*

$$0 \to T_\ell(J_S)(-n)_\Gamma \to T_\ell(\mathcal{M}_{T,S})(-n)_\Gamma \to (\mathrm{Div}(T) \otimes \mathbb{Z}_\ell(-n))_\Gamma \to Z_\ell(-n)_\Gamma \to 0. \tag{4.5}$$

Furthermore, all of these modules are finite.

Proof. We can split (4.4) into the two short exact sequences

$$0 \to T_\ell(J_S)(-n) \to T_\ell(\mathcal{M}_{T,S})(-n) \to \ker\varphi \to 0$$

and

$$0 \to \ker\varphi \to \mathrm{Div}(T) \otimes \mathbb{Z}_\ell(-n) \to \mathbb{Z}_\ell(-n) \to 0.$$

To check the exactness claimed in the proposition, it will suffice to check that each of these sequences stays exact when we take co-invariants. Applying Lemma 2.2.3 we, produce the sequences

$$(\ker\varphi)^{\Gamma} \to T_\ell(J_S)(-n)_\Gamma \to T_\ell(\mathcal{M}_{T,S})(-n)_\Gamma \to (\ker\varphi)_\Gamma \to 0$$

and

$$Z_\ell(-n)^{\Gamma} \to (\ker\varphi)_\Gamma \to (\mathrm{Div}(T)\otimes Z_\ell(-n))_\Gamma \to \mathbb{Z}_\ell(-n)_\Gamma \to 0.$$

We need to show that $(\ker\varphi)^{\Gamma} = 0$ and that $(\mathbb{Z}_\ell(-n))^{\Gamma} = 0$. As a free $\mathbb{Z}_\ell$-module has no non-zero finite submodules, it will suffice to show that $(\ker\varphi)^{\Gamma}$ and $(\mathbb{Z}_\ell(-n))^{\Gamma}$ are both finite. Finally, using Lemma 2.2.5 we see that this is equivalent to showing that $(\ker\varphi)_\Gamma$ and $\mathbb{Z}_\ell(-n)_\Gamma$ are both finite.

Let $\alpha \in \mathbb{N}, \gamma \in \Gamma$ be defined so that $\gamma = \gamma_q^{\alpha}$ is a topological generator for Γ. There is an obvious isomorphism of $\mathbb{Z}_\ell[[\mathcal{G}]]$-modules $\mathbb{Z}_\ell(-n) \simeq \mathbb{Z}_\ell[[\Gamma]]/\langle 1 - c_\ell(\gamma)^n\gamma\rangle$. If we pass to the quotient by taking Γ co-invariants we get

$$\mathbb{Z}_\ell(-n)_\Gamma \simeq \mathbb{Z}_\ell[[\Gamma]]/\langle 1 - c_\ell(\gamma)^n\gamma, 1-\gamma\rangle \simeq \mathbb{Z}_\ell/\langle 1 - c_\ell(\gamma)^n\rangle.$$

Since $c_\ell(\gamma)^{-n} \neq 1$ for any $n \geq 1$, this is a quotient of two free $\mathbb{Z}_\ell$-modules of rank 1 and therefore it is finite.

As $(\ker\varphi)_\Gamma \subseteq (\mathrm{Div}(T)\otimes\mathbb{Z}_\ell(-n))_\Gamma$, we can just show that the latter group is finite. Breaking up the summands in $\mathrm{Div}(T)\otimes\mathbb{Z}_\ell(-n)$ by looking at all those primes w of $\mathcal{K}$ which lie over a given prime v of $\mathcal{K}_0'$, we can write

$$\mathrm{Div}(T)\otimes\mathbb{Z}_\ell(-n) = \bigoplus_{v\in T_0}\bigoplus_{w|v}\mathbb{Z}_\ell(-n)\cdot w.$$

Fix a prime $v \in T_0$. We know that $\mathcal{G}$ acts transitively on the primes lying above v and if w_0 is a choice of a single such prime, then Remark 3.1.1 tells us that the stabilizer of w_0 in $\mathcal{G}$ is $\mathcal{G}_v$. This lets us easily show that there is an isomorphism

$$\bigoplus_{w|v}\mathbb{Z}_\ell(-n)\cdot w \simeq \mathbb{Z}_\ell[[\mathcal{G}]]\otimes_{\mathbb{Z}_\ell[[\mathcal{G}_v]]}\mathbb{Z}_\ell(-n)\cdot w_0$$

given by

$$\sum_{\overline{\sigma}\in\mathcal{G}/\mathcal{G}_v} a_{\overline{\sigma}}\cdot\sigma(w_0) \mapsto (\sum a_{\overline{\sigma}}\sigma)\otimes w_0.$$

We noted while defining $\widetilde{\sigma}_v$ that it is a topological generator for $\mathcal{G}_v$ and this easily implies that we have an isomorphism of $\mathbb{Z}_\ell[[\mathcal{G}]]$-modules

$$\mathbb{Z}_\ell[[\mathcal{G}]]\otimes_{\mathbb{Z}_\ell[[\mathcal{G}_v]]}\mathbb{Z}_\ell(-n)\cdot w_0 \simeq \mathbb{Z}_\ell[[\mathcal{G}]]/\langle 1 - c_\ell(\widetilde{\sigma}_v)^n\widetilde{\sigma}_v\rangle.$$

The coinvariants are then given by

$$(\bigoplus_{w|v} \mathbb{Z}_\ell(-n) \cdot w)_\Gamma \simeq \mathbb{Z}_\ell[[\mathcal{G}]]/\langle 1 - c_\ell(\widetilde{\sigma}_v)^n \widetilde{\sigma}_v, 1 - \gamma \rangle \simeq \mathbb{Z}_\ell[G]/\langle 1 - c_\ell(\widetilde{\sigma}_v)^n \sigma_v \rangle.$$

To prove that this is finite it suffices to show that $1 - c_\ell(\widetilde{\sigma}_v)^n \sigma_v$ is not a zero-divisor in $\mathbb{Z}_\ell[G]$ for all $n \geq 2$. By Lemma 2.1.9, this is equivalent to showing that $1 - c_\ell(\widetilde{\sigma}_v)^n \chi(\sigma_v) \neq 0$ for any $\chi \in \widehat{G}(\overline{\mathbb{Q}_\ell})$. This is clear though because $\chi(\sigma_v)$ is a root of unity but $c_\ell(\widetilde{\sigma}_v) = Nv \in \mathbb{Z}_\ell^\times$ has infinite order.

This proves exactness and along the way we've managed to prove that two of the four modules appearing in this sequence are finite. If we show that either one of the two remaining modules is finite, then the finiteness of the other will follow. Let's work with $T_\ell(J_S)(-n)_\Gamma$.

From Corollary 3.2.4, we have a short exact sequence

$$0 \to T_\ell(\tau_S) \to T_\ell(J_S) \to T_\ell(J) \to 0.$$

Twisting and taking Γ co-invariants we have the sequence

$$T_\ell(\tau_S)(-n)_\Gamma \to T_\ell(J_S)(-n)_\Gamma \to T_\ell(J)(-n)_\Gamma \to 0.$$

We will show that $T_\ell(J_S)(-n)_\Gamma$ is finite by showing that $T_\ell(\tau_S)(-n)_\Gamma$ and $T_\ell(J)(-n)_\Gamma$ are both finite.

For $T_\ell(\tau_S)(-n)$, Proposition 3.2.5 leads to a short exact sequence

$$0 \to \mathbb{Z}_\ell(1-n) \to \mathrm{Div}(S) \otimes \mathbb{Z}_\ell(1-n) \to T_\ell(\tau_S)(-n) \to 0$$

and it was proven above that the first two modules have finite co-invariants; it follows that the third must have finite co-invariants as well.

For $T_\ell(J)(-n)$ recall that the Riemann hypothesis for Z (a theorem due to Weil) says that the action of $\gamma \in \Gamma$ on $T_\ell(J) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$ has eigenvalues which are algebraic integers of absolute value $(q^\alpha)^{1/2}$ which are independent of ℓ. Using this, Example 2.2.10 tells us that the eigenvalues for the action of γ on $T_\ell(J)(-n) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$ all have absolute value $(q^\alpha)^{1/2-n}$. Since n is an integer, 1 is not an eigenvalue of this action and so $(T_\ell(J)(-n) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell)^\Gamma = 0$. Clearly $T_\ell(J)(-n)$ injects into $T_\ell(J)(-n) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$ and therefore we have that $T_\ell(J)(-n)^\Gamma = 0$. By Lemma 2.2.5 it follows that $T_\ell(J)(-n)_\Gamma$ is finite. □

We intend to apply Proposition 2.3.9 to the sequence obtained by dualizing sequence (4.5). We will check that all of the hypotheses are satisfied before dualizing.

The only hypotheses that have not already been verified are that the two modules in the middle of the sequence have projective dimension at most 1.

Proposition 4.4.2. *Both of* $(\mathrm{Div}(T) \otimes \mathbb{Z}_\ell(-n))_\Gamma$ *and* $T_\ell(\mathcal{M}_{T,S})(-n)_\Gamma$ *have projective dimension equal to 1 over* $\mathbb{Z}_\ell[G]$.

Proof. Theorem 4.2.4 implies that $T_\ell(M_{S,T})$ is $\mathbb{Z}_\ell[H]$-projective for $H = G(\mathcal{K}/\mathcal{K}')$. Propositon 1 of §3, Chapter 9 in [13] implies then that $T_\ell(M_{S,T})(n-1)^*$ is $\mathbb{Z}_\ell[H]$-projective too. Proposition 4.2.5 now implies that $T_\ell(M_{T,S})(-n)$ is also $\mathbb{Z}_\ell[H]$ projective.

Let $\mathcal{K}_\infty, \mathcal{K}'_\infty$, denote the $\mathbb{Z}_\ell$-extension of $\mathcal{K}_0$ contained in $\mathcal{K}/\mathcal{K}_0$, resepectively $\mathbb{Z}_\ell$-extension of $\mathcal{K}'_0$ contained in $\mathcal{K}'/\mathcal{K}'_0$. We will write $\Gamma'_\ell = G(\mathcal{K}'_\infty/\mathcal{K}'_0)$. Now, we can write $\Gamma \simeq \widehat{\mathbb{Z}} \simeq \prod_r \mathbb{Z}_r$, where r runs over the prime numbers. As $T_\ell(M_{T,S})$ is a free $\mathbb{Z}_\ell$-module, say of rank m, the action of Γ on $T_\ell(M_{T,S})$ factors through a map to $GL_m(\mathbb{Z}_\ell)$. It is easy to see that the prime-to-ℓ part of $GL_m(\mathbb{Z}_\ell)$ is finite and this implies that the image of $\prod_{r\neq\ell} \mathbb{Z}_r \subseteq \widehat{\mathbb{Z}}$ in $GL_m(\mathbb{Z}_\ell)$ is finite.

This shows that the action of Γ actually factors through a subextension of $\mathcal{K}/\mathcal{K}'_0$ which is finite over $\mathcal{K}_\infty$ i.e., $T_\ell(M_{T,S})$ is a module over $\mathbb{Z}_\ell[[G(\mathcal{K}_\infty(\mu_N)/\mathcal{K}'_0)]]$ for some N with $\ell \nmid |G(\mathcal{K}_\infty(\mu_N)/\mathcal{K}_\infty)|$. If we assume that $\ell \mid N$, as we now do, then we even have that $\mu_{\ell^\infty} \subseteq \mathcal{K}_\infty(\mu_N)$ and therefore that $T_\ell(M_{T,S})(-n)$ is a $\mathbb{Z}_\ell[[G(\mathcal{K}_\infty(\mu_N)/\mathcal{K}'_0)]]$-module. If we write $\Gamma_{\ell,N} = G(\mathcal{K}_\infty(\mu_N)/\mathcal{K}_0)$, then we have

$$T_\ell(M_{T,S})(-n)_\Gamma = T_\ell(M_{T,S})(-n)_{\Gamma_{\ell,N}}.$$

Recall that we have defined $H = G(\mathcal{K}/\mathcal{K}')$. Let us write $H' = G(\mathcal{K}_\infty(\mu_N)/\mathcal{K}'_\infty)$. It is easy to see that H is isomorphic to a subgroup of H' and that the quotient H'/H has order co-prime to ℓ. There is an exact sequence

$$0 \to H' \to G(\mathcal{K}_\infty(\mu_N)/\mathcal{K}'_0) \to \Gamma'_\ell \to 0$$

and the fact that Γ'_ℓ is topologically cyclic implies that this sequence is split. We can therefore write

$$\mathbb{Z}_\ell[[G(\mathcal{K}_\infty(\mu_N)/\mathcal{K}'_0)]] \simeq \mathbb{Z}_\ell[[\Gamma'_\ell]][H'].$$

We intend to apply Proposition A.2.3 from the appendix and so we observe, in the notation of that section, that $\mathbb{Z}_\ell[[\Gamma'_\ell]] \simeq \Lambda$ and we can consider $T_\ell(M_{T,S})(-n)$ as a module over $\Lambda[H']$. As $T_\ell(M_{T,S})(-n)$ is H-c.t., the fact that $\ell \nmid |H'/H|$ implies that it is H'-c.t. as well. It is clear that $T_\ell(M_{T,S})(-n)$ has no finite Λ submodules and so we conclude that there is a short exact sequence

$$0 \to P_1 \to P_0 \to T_\ell(M_{T,S})(-n) \to 0$$

where P_1, P_0 are projective over $\Lambda[H']$.

The isomorphism $G(\mathcal{K}_\infty(\mu_N)/\mathcal{K}'_0)/\Gamma_{\ell,N} \simeq G$ is clear and so Lemma 2.2.3 implies that there is an exact sequence of $\mathbb{Z}_\ell[G]$-modules

$$T_\ell(M_{T,S})(-n)^{\Gamma_{\ell,N}} \to (P_1)_{\Gamma_{\ell,N}} \to (P_0)_{\Gamma_{\ell,N}} \to T_\ell(M_{T,S})(-n)_{\Gamma_{\ell,N}} \to 0.$$

Lemma 2.2.5 now implies that $T_\ell(M_{T,S})(-n)^{\Gamma_{\ell,N}} = T_\ell(M_{T,S})(-n)^{\Gamma} = 0$ because we have already shown that $T_\ell(M_{T,S})(-n)_\Gamma$ is finite and $T_\ell(M_{T,S})(-n)$ clearly has no finite $\mathbb{Z}_\ell$-submodules.

Since the modules $(P_i)_{\Gamma_{\ell,N}}$ are clearly projective over $\mathbb{Z}_\ell[G]$, this implies that $\mathrm{pd}_{\mathbb{Z}_\ell[G]}(T_\ell(M_{T,S})(-n)_\Gamma) = 1$.

Now we deal with $(\mathrm{Div}(T) \otimes \mathbb{Z}_\ell(-n))_\Gamma$. In the previous proposition, we showed that $(\mathrm{Div}(T) \otimes \mathbb{Z}_\ell(-n))_\Gamma$ is isomorphic as a $\mathbb{Z}_\ell[G]$-module to a direct sum of modules of the form $\mathbb{Z}_\ell[G]/\langle 1 - c_\ell(\widetilde{\sigma}_v)^n \sigma_v \rangle$. It was furthermore shown that $1 - c_\ell(\widetilde{\sigma}_v)^n \sigma_v$ is not a zero–divisor in $\mathbb{Z}_\ell[G]$. This implies that we have an exact sequence of $\mathbb{Z}_\ell[G]$-modules

$$0 \to \mathbb{Z}_\ell[G] \xrightarrow{1-c_\ell(\widetilde{\sigma}_v)^n \sigma_v} \mathbb{Z}_\ell[G] \to \mathbb{Z}_\ell[G]/\langle 1 - c_\ell(\widetilde{\sigma}_v)^n \sigma_v \rangle \to 0.$$

This shows that $\mathbb{Z}_\ell[G]/\langle 1-c_\ell(\widetilde{\sigma}_v)^n \sigma_v \rangle$ has projective dimension 1 over $\mathbb{Z}_\ell[G]$ and therefore the same is true of $\mathrm{Div}(T) \otimes Z_\ell(-n)_\Gamma$. This concludes the proof. □

Applying $\mathrm{Hom}_{\mathbb{Z}_\ell}(-, \mathbb{Q}_\ell/\mathbb{Z}_\ell)$ to (4.5) we arrive at the sequence

$$0 \to (\mathbb{Z}_\ell(-n)_\Gamma)^\vee \to ((\mathrm{Div}(T) \otimes \mathbb{Z}_\ell(-n))_\Gamma)^\vee \to (T_\ell(\mathcal{M}_{T,S})(-n)_\Gamma)^\vee \to (T_\ell(J_S)(-n)_\Gamma)^\vee \to 0. \tag{4.6}$$

Observe that this sequence still satisfies the hypotheses of Proposition 2.3.9. The finiteness of the modules is clear and in the proof of Proposition 2.3.6 we proved that if $\mathrm{pd}_{\mathbb{Z}_\ell[G]}(M) = 1$, then $\mathrm{pd}_{\mathbb{Z}_\ell[G]}(M^\vee) = 1$ also.

We will adopt the convention in the remainder of this section that all Fitting ideals are taken over $\mathbb{Z}_\ell[G]$. We now apply Proposition 2.3.9 to sequence (4.6) to conclude an equality which is of fundamental importance for our proof

$$\begin{aligned} &\mathrm{Fit}(((\mathbb{Z}_\ell(-n)_\Gamma)^\vee)^\wedge) \cdot \mathrm{Fit}((T_\ell(\mathcal{M}_{T,S})(-n)_\Gamma)^\vee) \\ &\qquad = \mathrm{Fit}(((\mathrm{Div}(T) \otimes \mathbb{Z}_\ell(-n))_\Gamma)^\vee) \cdot \mathrm{Fit}((T_\ell(J_S)(-n)_\Gamma)^\vee) \end{aligned} \tag{4.7}$$

All that remains is to calculate each of these Fitting ideals in turn and then to put the pieces together.

Having already shown that

$$\mathrm{Div}(T) \otimes \mathbb{Z}_\ell(-n) \simeq \bigoplus_{v \in T_0} \mathbb{Z}_\ell[[\mathcal{G}]]/\langle 1 - c_\ell(\widetilde{\sigma}_v)^n \sigma_v \rangle,$$

it follows directly from Proposition 2.3.1 that

$$\mathrm{Fit}(\mathrm{Div}(T) \otimes \mathbb{Z}_\ell(-n))_\Gamma) = \prod_{v \in T_0} (1 - \sigma_v N v^n).$$

If we now apply Proposition 2.3.6 we see that

$$\mathrm{Fit}(((\mathrm{Div}(T) \otimes \mathbb{Z}_\ell(-n))_\Gamma)^\vee) = \iota(\prod_{v \in T_0} (1 - \sigma_v N v^n)) = \prod_{v \in T_0} (1 - \sigma_v^{-1} N v^n) = \delta_{T_0}(1-n).$$

Applying Lemma 2.2.7, and Proposition 4.2.5 we get that

$$(T_\ell(\mathcal{M}_{T,S})(-n)_\Gamma)^\vee \simeq (T_\ell(\mathcal{M}_{T,S})(-n)^*)_\Gamma \simeq T_\ell(\mathcal{M}_{S,T})(n-1)_\Gamma.$$

Combining Theorem 4.2.4 with Lemma 2.2.11 we have that

$$\mathrm{Fit}_{\mathbb{Z}_\ell[[\mathcal{G}]]}(T_\ell(\mathcal{M}_{S,T})(n-1)) = \langle t_{1-n}(\vartheta^{(\infty)}_{S_0,T_0}) \rangle.$$

Applying Proposition 3.4.3, we then have that

$$\mathrm{Fit}_{\mathbb{Z}_\ell[[\mathcal{G}]]}(T_\ell(\mathcal{M}_{S,T})(n-1)_\Gamma) = \langle \pi(t_{1-n}(\vartheta^{(\infty)}_{S_0,T_0})) \rangle = \langle \delta_{T_0}(1-n) \cdot \Theta_{S_0}(q^{n-1}) \rangle.$$

The étale cohomology groups enter the picture when we apply Proposition 3.3.1. This says that

$$(\mathbb{Z}_\ell(-n)_\Gamma)^\vee \simeq \mathrm{H}^1_{\acute{e}t}(\mathcal{O}_{\mathcal{K}_0,\widetilde{S}_0}, \mathbb{Z}_\ell(n)) \text{ and } (T_\ell(J_S)(-n)_\Gamma)^\vee \simeq \mathrm{H}^2_{\acute{e}t}(\mathcal{O}_{\mathcal{K}_0,\widetilde{S}_0}, \mathbb{Z}_\ell(n)).$$

Finally, Lemma 2.3.10 implies that

$$\mathrm{Fit}(((\mathbb{Z}_\ell(-n)_\Gamma)^\vee)^\wedge) = \mathrm{Fit}((\mathbb{Z}_\ell(-n)_\Gamma)^\vee).$$

Combining all these calculations we can write (4.7) as

$$\begin{aligned} &\mathrm{Fit}_{\mathbb{Z}_\ell[G]}(\mathrm{H}^1_{\acute{e}t}(\mathcal{O}_{\mathcal{K}_0,\widetilde{S}_0}, \mathbb{Z}_\ell(n))) \cdot \delta_{T_0}(1-n) \cdot \Theta_{S_0}(q^{n-1}) \\ &\qquad = \delta_{T_0}(1-n) \cdot \mathrm{Fit}_{\mathbb{Z}_\ell[G]}(\mathrm{H}^2_{\acute{e}t}(\mathcal{O}_{\mathcal{K}_0,\widetilde{S}_0}, \mathbb{Z}_\ell(n))). \end{aligned}$$

As $\mathrm{H}^1_{\acute{e}t}(\mathcal{O}_{\mathcal{K}_0,\widetilde{S}_0}, \mathbb{Z}_\ell(n))$ is a cyclic $\mathbb{Z}_\ell[G]$-module, Lemma 2.3.10 allows us rewrite this as

$$\mathrm{Ann}_{\mathbb{Z}_\ell[G]}(\mathrm{H}^1_{\acute{e}t}(\mathcal{O}_{\mathcal{K}_0,\widetilde{S}_0}, \mathbb{Z}_\ell(n))) \cdot \delta_{T_0}(1-n) \cdot \Theta_{S_0}(q^{n-1})$$
$$= \delta_{T_0}(1-n) \cdot \mathrm{Fit}_{\mathbb{Z}_\ell[G]}(\mathrm{H}^2_{\acute{e}t}(\mathcal{O}_{\mathcal{K}_0,\widetilde{S}_0}, \mathbb{Z}_\ell(n)))$$

We've seen that $\delta_{T_0}(1-n)$ generates the Fitting ideal of a torsion finitely generated $\mathbb{Z}_\ell[G]$-module with projective dimension 1, and so Proposition 2.3.6 implies that it is not a zero-divisor in $Z_\ell[G]$. We are therefore entitled to cancel the $\delta_{T_0}(1-n)$ term from both sides of the equation. This results in the equality

$$\mathrm{Ann}_{\mathbb{Z}_\ell[G]}(\mathrm{H}^1_{\acute{e}t}(\mathcal{O}_{\mathcal{K}_0,\widetilde{S}_0}, \mathbb{Z}_\ell(n))) \cdot \Theta_{S_0}(q^{n-1})$$
$$= \mathrm{Fit}_{\mathbb{Z}_\ell[G]}(\mathrm{H}^2_{\acute{e}t}(\mathcal{O}_{\mathcal{K}_0,\widetilde{S}_0}, \mathbb{Z}_\ell(n)))$$

and this concludes the proof.

□

Appendix A

Some Cohomological Calculations

In this Appendix, we introduce the Tate cohomology groups of a module over a finite group. The notion of cohomological triviality, due to Nakayama and Tate, is defined. We then develop a criterion for when certain étale cohomology groups are cohomologically trivial. This condition has often played a role in the calculation of Fitting ideals and this criterion should be useful in trying to remove some of the conditions in the main theorems of [6], [7].

A.1 Group cohomology

Let G be a finite group. The Tate cohomology of G is a sequence of functors $\widehat{\mathrm{H}}^i(G, -)$, for $n \in \mathbb{Z}$, from the category of $\mathbb{Z}[G]$-modules to the category of abelian groups. They satisfy the usual properties that we expect from a cohomology theory. For example, if

$$0 \to A \to B \to C \to 0$$

is a short exact sequence of $\mathbb{Z}[G]$-modules, then there is a long exact sequence in cohomology

$$\ldots \to \widehat{\mathrm{H}}^i(G, C) \to \widehat{\mathrm{H}}^{i+1}(G, A) \to \widehat{\mathrm{H}}^{i+1}(G, B) \to \widehat{\mathrm{H}}^{i+1}(G, C) \to \ldots.$$

Remark A.1.1. *For $n = -1, 0$, these cohomology groups are especially concrete. Let $N_G = \sum_{g \in G} g$ be the so-called norm element of $\mathbb{Z}[G]$. We can think of N_G as a map $N_G : M \to M$ defined by $m \mapsto N_G \cdot m$. If M is a $\mathbb{Z}[G]$-module, then*

$$\widehat{\mathrm{H}}^0(G, M) = M^G / N_G \cdot M$$

and

$$\widehat{\mathrm{H}}^{-1}(G, M) = \ker(N_G)/I_G \cdot M.$$

A full development of the theory of group cohomology can be found in chapter 4 of [2].

A.2 Cohomological triviality

We would like to have some tools in place to study the Galois-module structure of certain étale cohomology groups. We will be particularly interested in the case where the Tate cohomology groups satisfy an especially strong vanishing condition.

Definition A.2.1. *If M is a G-module, we say that M is G-*cohomologically trivial*, or G-*c.t.*, if $\widehat{H}^i(H, M) = 0$ for all $i \in \mathbb{Z}$ and for all subgroups $H \subseteq G$.*

Proposition A.2.2 (Theorem 9 in Chapter 4 of [2])**.** *Let G be a finite group. Let M be a finitely generated $\mathbb{Z}_\ell[G]$-module. Then M is G-c.t. if and only if $\mathrm{pd}_{\mathbb{Z}_\ell[G]}(M) \leq 1$.*

Let $\Lambda = \mathbb{Z}_\ell[[T]]$ be the 1-variable Iwasawa-algebra with coefficients in $\mathbb{Z}_\ell$.

Proposition A.2.3 (Proposition 2.2 in [12])**.** *Let M be a finitely generated $\Lambda[G]$-module. Then $\mathrm{pd}_{\Lambda[G]}(M) \leq 1$ if and only if the following two conditions are satisfied*

1. $\mathrm{pd}_\Lambda(M) \leq 1$,
2. *M is G-c.t.*

Proposition A.2.4 (Lemma 2.3 in [12])**.** *Let M be a Λ-module. Then $\mathrm{pd}_\Lambda(M) \leq 1$ if and only if M has no finite Λ-submodules.*

A.3 Tate cohomology of certain étale cohomology groups

Let K/k be an abelian extension of number fields or characteristic p function fields with Galois group G and let S be a finite G-invariant set of primes of K containing all the primes which ramify in K/k. Also, let ℓ be a prime different from the characteristic of k and let $n \in \mathbb{Z}$. If K/k are number fields, then we assume that n is even. We will write K_∞, k_∞ for the cyclotomic $\mathbb{Z}_\ell$-extensions of K and k i.e., $K_\infty = K(\mu_{\ell^\infty})^\Delta$ where $\Delta \subset G(K(\mu_{\ell^\infty})/K)$ is the maximal subgroup of order prime to ℓ and similarly for k_∞. We set $\Gamma_K = G(K_\infty/K)$, $\Gamma_k = G(k_\infty/k)$ and $\widetilde{\Gamma}_K = G(K(\mu_{\ell^\infty})/K)$.

We set $G' = G(K_\infty/k_\infty)$. Observe that G' is naturally identified with the group $G(K/K \cap k_\infty) \subset G$. As Γ_k is pro-cyclic, there is a splitting of the sequence

$$0 \to G' \to G(K_\infty/k) \to \Gamma_k \to 0$$

and therefore we can write $G(K_\infty/k) \simeq \Gamma_k \times G'$. This lets us identify the pro-finite group ring $\mathbb{Z}_\ell[[G(K_\infty/k)]]$ with $\mathbb{Z}_\ell[[\Gamma_k]][G'] \simeq \Lambda[G']$ and use the commutative algebra from the previous section in our study of $\mathbb{Z}_\ell[[G(K_\infty/k)]]$-modules. If we define $I_{\Gamma_K} \subset \Lambda[G']$ to be the closure of the ideal $\langle 1 - g \mid g \in \Gamma_K \rangle$, then we clearly have that $\Lambda[G']/I_{\Gamma_K} \simeq \mathbb{Z}_\ell[G]$.

Recall that $G^{(\ell)}$ has been defined to be the ℓ-Sylow subgroup of G.

Lemma A.3.1. *Let G be a finite group and let M be a finite $\mathbb{Z}_\ell[G]$-module. Then M is G-c.t. if and only if M is $G^{(\ell)}$-c.t.*

Proof. If M is G-c.t., then it is obvious that M is $G^{(\ell)}$-c.t. as every subgroup of $G^{(\ell)}$ is a subgroup of G. Conversely, if M is $G^{(\ell)}$-c.t., then we use that for any subgroup $H \subset G$, we have $\widehat{H}^i(H, M) \simeq \widehat{H}^i(H^{(\ell)}, M)$, see Corollary 3 in Chapter 4, §5 of [2]. If the latter is always zero, then so is the former. $\square$

From now on, we fix $n \geq 2$, and ℓ prime. If K, k are function fields, then we assume that $\ell \neq p$. We will be studying the Galois module structure of $H^1_{\acute{e}t}(\mathcal{O}_{K,S}, \mathbb{Z}_\ell(n))_{\text{tors}}$ and the following description of this group will be very useful.

Proposition A.3.2. *There is an isomorphism*

$$H^1_{\acute{e}t}(\mathcal{O}_{K,S}, \mathbb{Z}_\ell(n))_{\text{tors}} \simeq (\mathbb{Q}_\ell/\mathbb{Z}_\ell)(n)^{\widetilde{\Gamma}_K}.$$

Proof. For the number field case see Lemma 6.9 in [6]. For the function field case see Remark 5.15 in [7] and observe that $G(K(\mu_\infty)/K(\mu_{\ell^\infty}))$ automatically acts trivially on $(\mathbb{Q}_\ell/\mathbb{Z}_\ell)(n)$. $\square$

In what follows we will simplify our notation by setting $M = H^1_{\acute{e}t}(\mathcal{O}_{K,S}, \mathbb{Z}_\ell(n))_{\text{tors}}$.

Proposition A.3.3. $|M| = \max\{\ell^a \mid G(K(\mu_{\ell^a})/K)$ has exponent dividing $n\}$.

Proof. It is easy to see that

$$|M| = \max\{\ell^a \mid c_\ell(\sigma)^n \equiv 1 \pmod{\ell^a} \text{ for all } \sigma \in \widetilde{\Gamma}_K\}.$$

Of course, we have an injection $G(K(\mu_{\ell^a})/K) \hookrightarrow (\mathbb{Z}/\ell^a\mathbb{Z})^\times$ given by the cyclotomic character modulo ℓ^a. This allows us to say that the right hand side is the same as

$$\max\{\ell^a \mid G(K(\mu_{\ell^a})/K) \text{ has exponent dividing } n\}$$

as claimed. □

Initially, we make the following assumption

Assumption A.3.4. *If ℓ is odd, we assume that $\mu_\ell \subset K_\infty$. If $\ell = 2$, then we assume that $\mu_4 \subset K_\infty$.*

Observe that, under this assumption, the previous proposition implies that $M \neq 0$.

Proposition A.3.5. $M \simeq (\mathbb{Q}_\ell/\mathbb{Z}_\ell)(-n)^{\Gamma_K} \simeq \mathbb{Z}_\ell(-n)_{\Gamma_K}$.

Proof. Under the assumption, we have that $\widetilde{\Gamma}_K = \Gamma_K$ and so the first isomorphism follows from Proposition 3.3.1. Next, Lemma 2.2.3 applied to the short exact sequence

$$0 \to \mathbb{Z}_\ell(-n) \to \mathbb{Q}_\ell(-n) \to (\mathbb{Q}_\ell/\mathbb{Z}_\ell)(-n) \to 0$$

produces an exact sequence

$$\mathbb{Q}_\ell(-n)^{\Gamma_K} \to (\mathbb{Q}_\ell/\mathbb{Z}_\ell)(-n)^{\Gamma_K} \to \mathbb{Z}_\ell(-n)_{\Gamma_K} \to \mathbb{Q}_\ell(-n)_{\Gamma_K}.$$

The second isomorphism follows upon observing that both $\mathbb{Q}_\ell(-n)^{\Gamma_K}$ and $\mathbb{Q}_\ell(-n)_{\Gamma_K}$ are trivial. □

Theorem A.3.6. *Under Assumption A.3.4 the following are equivalent*

1. *M is G-c.t.*
2. *M is G'-c.t.*
3. *$\ell \nmid |G'|$*

Proof. Both (1) and (3) are evidently stronger conditions than (2) so it will suffice to prove that (2) implies both of (1) and (3). From now on we therefore make the assumption that M is G'-c.t.

We will prove that (2) implies (3) by considering $\mathbb{Z}_\ell(-n)$ as a module over $\Lambda[G']$. It is easy to see that $\mathbb{Z}_\ell(-n)$ has no finite Λ-submodules and so, by Propositions A.2.3 and A.2.4, we get that $\mathrm{pd}_{\Lambda[G']}(\mathbb{Z}_\ell(-n)) = 1$. If

$$0 \to P_1 \to P_0 \to \mathbb{Z}_\ell(-n) \to 0$$

is a projective resolution, then Lemma 2.2.3 produces an exact sequence of $\Lambda[G']/I_{\Gamma_K} \simeq \mathbb{Z}_\ell[G]$-modules

$$\mathbb{Z}_\ell(-n)^{\Gamma_K} \to (P_1)_{\Gamma_K} \to (P_0)_{\Gamma_K} \to M \to 0.$$

Since $\mathbb{Z}_\ell(-n)_{\Gamma_K}$ is finite, Proposition 2.2.5 implies that $\mathbb{Z}_\ell(-n)^{\Gamma_K}$ is a finite submodule of $\mathbb{Z}_\ell(-n)$. The only such submodule is trivial and so we have a projective resolution of M

$$0 \to (P_1)_{\Gamma_K} \to (P_0)_{\Gamma_K} \to M \to 0.$$

This shows that $\mathrm{pd}_{\mathbb{Z}_\ell[G]}(M) = 1$ and therefore M is G-c.t. by Proposition A.2.2. This establishes that (2) and (3) are equivalent.

To prove that (2) implies (1) we have to consider the cases when ℓ is even or odd seperately. We set $H = G(K_\infty/k_\infty(\mu_\ell)) \subset G'$ if ℓ is odd and $H = G(K_\infty/k_\infty(\mu_4))$ if $\ell = 2$. Observe that, in either case, H acts trivially on M since $c_\ell(H) = 1$.

If ℓ is odd, then since $(|G(k_\infty(\mu_\ell)/k_\infty)|, \ell) = 1$, we get that $H^{(\ell)} = G'^{(\ell)}$ and therefore $G'^{(\ell)}$ acts trivially on M. Since $M \neq 0$, the only way that $\widehat{H}^0(G'^{(\ell)}, M)$ can be zero is if $G^{(\ell)}$ is trivial i.e., if $\ell \nmid |G'|$.

If $\ell = 2$, then we no can no longer say that $(|G(k_\infty(\mu_4)/k_\infty)|, \ell) = 1$ so we need a slightly different argument. Observe though that the above argument shows that $H^{(2)}$ must be trivial. I claim that in fact, $\mu_4 \subseteq k_\infty$. If not then, since $H^{(2)} = 0$, we have a splitting $G' \simeq H \times G(k_\infty(\mu_4)/k_\infty)$. But it is easy to calculate that

$$\hat{H}^0(G(k_\infty(\mu_4)/k_\infty), M) = \begin{cases} \mu_2 & \text{if } n \text{ is odd} \\ M/2M & \text{if } n \text{ is even} \end{cases}$$

In either case we would get a contradiction with G'-cohomological triviality of M and therefore we must have $\mu_4 \subseteq k_\infty$ and $H = G'$. Since we have already observed that $H^{(2)}$ is trivial this finishes the proof. □

Having proven this theorem we can dispense with our assumption and prove the following concrete criterion for cohomological triviality.

Theorem A.3.7. *Let K/k be an arbitrary abelian extension of global fields with Galois group G.*

1. *If ℓ is odd, then $H^1_{ét}(\mathcal{O}_{K,S}, \mathbb{Z}_\ell(n))$ is G-c.t. if and only if*

$$\ell \nmid (|H^1_{ét}(\mathcal{O}_{K,S}, \mathbb{Z}_\ell(n))|, [K : K \cap k_\infty]).$$

2. *If n is odd and $\mu_4 \not\subseteq K_\infty$, then $H^1_{ét}(\mathcal{O}_{K,S}, \mathbb{Z}_2(n))$ is G-c.t. if and only if $2 \nmid |G|$. Otherwise $H^1_{ét}(\mathcal{O}_{K,S}, \mathbb{Z}_2(n))$ is G-c.t. if and only if $2 \nmid [K : K \cap k_\infty]$.*

Proof. As above, we will write $M = H^1_{ét}(\mathcal{O}_{K,S}, \mathbb{Z}_\ell(n))_{\text{tors}}$. The theorem is trivial if $M = 0$ and so we assume that this is not the case.

1. Let ℓ be an odd prime. Let $\widetilde{K} = K(\mu_\ell)$, let $\Delta = G(\widetilde{K}/K)$ and let $\widetilde{G} = G(\widetilde{K}/k)$. As we are assuming that $M \neq 0$, Proposition A.3.3 will allow us to prove that in fact there is an isomorphism of $\mathbb{Z}_\ell[G]$-modules $M \simeq H^1_{ét}(\mathcal{O}_{\widetilde{K},S}, \mathbb{Z}_\ell(n))$. Since $(|\Delta|, \ell) = 1$ we have that $G^{(\ell)} \simeq \widetilde{G}^{(\ell)}$ and so M is G-c.t. if and only if $H^1_{ét}(\mathcal{O}_{\widetilde{K},S}, \mathbb{Z}_\ell(n))$ is $\widetilde{G}$-c.t. Of course, $\widetilde{K}$ satisfies Assumption A.3.4 and so Theorem A.3.6 implies that this is so if and only if $\ell \nmid [\widetilde{K}_\infty : k_\infty]$. As $([\widetilde{K}_\infty : K_\infty], \ell) = 1$ this will hold if and only if $\ell \nmid [K_\infty : k_\infty]$. Finally, Galois theory tells us that $[K_\infty : k_\infty] = [K : K \cap k_\infty]$ and this finishes the proof in this case.

2. Let $\ell = 2$. First, suppose that $\mu_4 \not\subseteq K_\infty$ and that n is odd. Since the exponent of $K(\mu_{2^m}/K)$ is a power of 2 if $m > 1$ and n is odd, Proposition A.3.3 says that $M \simeq \mu_2$. The action of G on M is therefore trivial and we have that M is G-c.t. if and only if $2 \nmid |G|$.

If we suppose that $\mu_4 \subseteq K_\infty$ then either $\mu_4 \subseteq K$ or $\mu_4 \subseteq k_\infty$. If $\mu_4 \subseteq K$, then again K satisfies Assumption A.3.4 and therefore M is G-c.t. if and only if $2 \nmid [K : K \cap k_\infty]$. If $\mu_4 \subseteq k_\infty$, then clearly $G(K_\infty/k_\infty)$ acts trivially on M and we conclude again that M is G-c.t. if and only $2 \nmid [K : K \cap k_\infty]$.

Finally suppose that $\mu_4 \not\subseteq K_\infty$ but that n is even. If we set $G' = G(K_\infty/k_\infty) \simeq G(K/K \cap k_\infty)$, then Galois theory gives an isomorphism $G' \simeq G(K_\infty(\mu_4)/k_\infty(\mu_4))$ and so G' acts trivially on M. This gives us one direction: we've shown that if M is G-c.t., then $2 \nmid |G'| = [K : K \cap k_\infty]$. Conversely, suppose that $2 \nmid |G'|$. As in Proposition A.3.6 we therefore have that $\mathbb{Z}_\ell(n)$ has $\text{pd}_{\Lambda[G']} = 1$. Write

$$0 \to P_1 \to P_0 \to \mathbb{Z}_\ell(n) \to 0.$$

The key observation now is that Propositon A.3.5 still holds for M. This is because $\widetilde{\Gamma}_K = \Gamma_K \times \langle j \rangle$ where j is a generator for $G(K(\mu_4)/K)$. Since j has order 2, the group $\langle j \rangle$ acts trivially on $\mathbb{Z}_\ell(n)$ for n even and so $\mathbb{Z}_\ell(n)_{\widetilde{\Gamma}_K} = \mathbb{Z}_\ell(n)_{\Gamma_K}$. From here we can mimic the proof of Proposition A.3.6 to produce a sequence

$$0 \to (P_1)_{\Gamma_K} \to (P_0)_{\Gamma_K} \to \mathbb{Z}_\ell(n)_{\Gamma_K} \to 0.$$

Finally Proposition A.2.2 applies to say that M is G-c.t. $\square$

We can make some further calculations to deduce the following auxiliary theorem relating cohomological triviality of $H^1_{ét}(\mathcal{O}_{K,S}, \mathbb{Z}_\ell(n))$ to cohomological triviality of $H^2_{ét}(\mathcal{O}_{K,S}[1/\ell], \mathbb{Z}_\ell(n))$ in the number field setting.

Theorem A.3.8. *Keep the notations as above but assume that K, k are number fields. Suppose that S contains all the primes which ramify in K/k. Then $H^1_{ét}(\mathcal{O}_{K,S}, \mathbb{Z}_\ell(n))$ is G-c.t. if and only if $H^2_{ét}(\mathcal{O}_{K,S}, \mathbb{Z}_\ell(n))$ is G-c.t.*

Proof. By Corollary 2.3 in [8] we have that

$$H^1_{ét}(\mathcal{O}_{K,S}, \mathbb{Z}_\ell(n)) = H^1_{ét}(K, \mathbb{Z}_\ell(n)).$$

and by Corollary 2.11 in [8] we have that $H^1_{ét}(K, \mathbb{Z}_\ell(n))$ is G-c.t. if and only if $H^2_{ét}(K, \mathbb{Z}_\ell(n))$ is G-c.t. Finally by Remark 3.4 in [12] we have an exact sequence

$$0 \to H^2_{ét}(\mathcal{O}_{K,S}\, \mathbb{Z}_\ell\,(n)) \to H^2_{ét}(K, \mathbb{Z}_\ell(n)) \to \bigoplus_{w \notin S \cup S_\ell} H^1_{ét}(\kappa(w), \mathbb{Z}_\ell(n-1)) \to 0$$

where S_ℓ denotes the set of primes of $\mathcal{O}_K$ which lie above ℓ.

Let S_k be the set of primes of k which lie below the primes in S. Then we will show for each $v \notin S_k$ and not lying above ℓ that

$$\bigoplus_{w|v} H^1_{ét}(\kappa(w), \mathbb{Z}_\ell(n-1))$$

is G-c.t. We assume this for the moment and finish the proof. This will imply, via the long exact sequence in group cohomology, that $H^2_{ét}(\mathcal{O}_{K,S}\, \mathbb{Z}_\ell\,(n))$ is G-c.t. if and only if $H^2_{ét}(K, \mathbb{Z}_\ell(n))$ is G-c.t.

We therefore have the following sequence of implications: $H^1_{ét}(\mathcal{O}_{K,S}, \mathbb{Z}_\ell(n))$ is G-c.t. if and only if $H^1_{ét}(K, \mathbb{Z}_\ell(n))$ is G-c.t. if and only if $H^2_{ét}(K, \mathbb{Z}_\ell(n))$ is G-c.t. if and only if $H^2_{ét}(\mathcal{O}_{K,S}[1/\ell], \mathbb{Z}_\ell(n))$ is G-c.t. This concludes the argument.

Now to prove that

$$\bigoplus_{w|v} H^1_{\acute{e}t}(\kappa(w), \mathbb{Z}_\ell(n-1))$$

is G-c.t. If w_0 is a choice of one of the primes lying above v, then we can write

$$\bigoplus_{w|v} \kappa(w) = \kappa(w_0) \otimes_{\mathbb{Z}[G_v]} \mathbb{Z}[G]$$

and so we have

$$\bigoplus_{w|v} H^1_{\acute{e}t}(\kappa(w), \mathbb{Z}_\ell(n-1)) = H^1_{\acute{e}t}(\kappa(w_0), \mathbb{Z}_\ell(n-1)) \otimes_{\mathbb{Z}[G_v]} \mathbb{Z}[G].$$

By Shapiro's Lemma, Proposition 2 in Chapter 4 of [2] we have that

$$H^1_{\acute{e}t}(\kappa(w_0), \mathbb{Z}_\ell(n-1)) \otimes_{\mathbb{Z}[G_v]} \mathbb{Z}[G]$$

is G-c.t. if and only if $H^1_{\acute{e}t}(\kappa(w_0), \mathbb{Z}_\ell(n-1))$ is G_v-c.t. By Remark 3.4 in [12] again, we have that

$$H^1_{\acute{e}t}(\kappa(w_0), \mathbb{Z}_\ell(n)) \simeq (\mathbb{Q}_\ell/\mathbb{Z}_\ell)(n-1)^{G_{\kappa(w_0)}}$$

where $G_{\kappa(w_0)}$ is the absolute Galois group of the finite field $\kappa(w_0)$.

We can think of the abelian group $(\mathbb{Q}_\ell/\mathbb{Z}_\ell)(n-1)^{G_{\kappa(w_0)}}$ as the group of roots of unity of ℓ-power order in the degree $n-1$ extension of $\kappa(w_0)$. We will refer to this group as $W(n-1)$ and to the degree $n-1$ extension of $\kappa(w_0)$ as κ_{n-1}. Note that we can write $(\mathbb{Q}_\ell/\mathbb{Z}_\ell)(n-1)^{G_{\kappa(w_0)}}$ as a $G(\kappa_{n-1}/\kappa(v))$-module by writing $(\mathbb{Q}_\ell/\mathbb{Z}_\ell)(n-1)^{G_{\kappa(w_0)}} \simeq W(n-1)^{\otimes n-1}$. The G_v-module structure on $(\mathbb{Q}_\ell/\mathbb{Z}_\ell)(n-1)^{G_{\kappa(w_0)}}$ is induced from this one via restriction. Note that $G_v \subset G(\kappa_{n-1}/\kappa(v))$ because v is unramified.

Now we observe that $\kappa_{n-1}^\times$ is $G(\kappa_{n-1}/\kappa(v))$-c.t. by Hilbert's Theorem 90 and the theory of the Herbrand quotient (see Chapter 4 of [2]). Since $W(n-1)$ is the ℓ-Sylow subgroup of $\kappa_{n-1}^\times$ it too is $G(\kappa_{n-1}/\kappa(v))$-c.t. It follows from Proposition 1 in Chapter IX of [13] that $(\mathbb{Q}_\ell/\mathbb{Z}_\ell)(n-1)^{G_{\kappa(w_0)}}$ is $G(\kappa_{n-1}/\kappa(v))$-c.t. as well. Restricting to G_v we conclude that $(\mathbb{Q}_\ell/\mathbb{Z}_\ell)(n-1)^{G_{\kappa(w_0)}}$ is G_v-c.t.

□

Bibliography

[1] David Burns and Cornelius Greither. Equivariant Weierstrass preparation and values of L-functions at negative integers. *Doc. Math.*, (Extra Vol.):157–185 (electronic), 2003. Kazuya Kato's fiftieth birthday.

[2] John W. S. Cassels and Albrecht Fröhlich, editors. *Algebraic number theory.* Academic Press Inc. [Harcourt Brace Jovanovich Publishers], 1986. Reprint of the 1967 original.

[3] John Coates and Warren Sinnott. An analogue of Stickelberger's theorem for the higher K-groups. *Invent. Math.*, 24:149–161, 1974.

[4] Pierre Deligne. Théorie de Hodge. III. *Inst. Hautes Études Sci. Publ. Math.*, (44):5–77, 1974.

[5] Pierre Deligne and Kenneth A. Ribet. Values of abelian L-functions at negative integers over totally real fields. *Invent. Math.*, 59(3):227–286, 1980.

[6] Cornelius Greither and Cristian D. Popescu. An equivariant main conjecture in Iwasawa theory and applications. *http://arxiv.org/abs/1103.3069*, 2011.

[7] Cornelius Greither and Cristian D. Popescu. The Galois module structure of ℓ-adic realizations of picard 1-motives and applications. *Intl. Math. Res. Notices*, 2011.

[8] Manfred Kolster. K-theory and arithmetic. In *Contemporary developments in algebraic K-theory*, ICTP Lect. Notes, XV, pages 191–258 (electronic). Abdus Salam Int. Cent. Theoret. Phys., Trieste, 2004.

[9] Barry Mazur and Andrew Wiles. Analogies between function fields and number fields. *American Journal of Mathematics*, 105(2):pp. 507–521, 1983.

[10] Barry Mazur and Andrew Wiles. Class fields of abelian extensions of $\mathbb{Q}$. *Inventiones Mathematicae*, 76:179–+, 1984.

[11] Jürgen Neukirch. *Algebraic Number Theory.* Springer-Verlag, 1999.

[12] Cristian Popescu. On the Coates-Sinnott conjecture. *Math. Nachr.*, 282(10):1370–1390, 2009.

[13] Jean-Pierre Serre. *Local fields*, volume 67 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1979. Translated from the French by Marvin Jay Greenberg.

[14] Carl Ludwig Siegel. Über die Fourierschen Koeffizienten von Modulformen. *Nachr. Akad. Wiss. Göttingen Math.-Phys. Kl. II*, 1970:15–56, 1970.

[15] Christophe Soulé. K-théorie des anneaux d'entiers de corps de nombres et cohomologie étale. *Invent. Math.*, 55(3):251–295, 1979.

[16] John Tate. *Les conjectures de Stark sur les fonctions L d'Artin en $s = 0$*, volume 47 of *Progress in Mathematics*. Birkhäuser Boston Inc., Boston, MA, 1984. Lecture notes edited by Dominique Bernardi and Norbert Schappacher.

[17] Vladimir Voevodsky. On motivic cohomology with $\mathbb{Z}/l$-coefficients. *Annals of Math.*, 174:401–438, 2011.

[18] Lawrence Washington. *Introduction to Cyclotomic Fields*. Springer-Verlag, 1997.