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Information In Coordinated System Control

A dissertation submitted in partial satisfaction of the
requirements for the degree Doctor of Philosophy

in

Engineering Sciences (Mechanical Engineering)

by

Keunmo Kang

Committee in charge:

      Professor Robert R. Bitmead, Chairperson
      Professor Philip E. Gill
      Professor Kenneth Kreutz-Delgado
      Professor Miroslav Krstić
      Professor Sonia Martínez

2008

The dissertation of Keunmo Kang is approved, and it is acceptable in quality and form for publication on microfilm and electronically:

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Chairperson

University of California, San Diego

2008

To God,

my parents, grand mother,

and

grand father in Heaven.

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Until I finish writing this thesis, I was so blessed to get so much support from many people. In this page, I would like to express my thanks to them.

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Chapter 2 includes the reprint of the following papers:

Keunmo Kang, Robert R. Bitmead - *Contraction based Model Predictive Control*, International Workshop on Assessment and Future Directions of Nonlinear Model Predictive Control, Pavia, Italy, Sep., 2008.

Keunmo Kang, Robert R. Bitmead - *Contraction based Receding Horizon Control*, submitted to Automatica, Mar., 2008.

Keunmo Kang, Robert R. Bitmead - *Online Reference Computation for Feasible Model Predictive Control*, 46th IEEE Conference on Decision and Control, New Orleans, USA, Dec., 2007.

Chapter 3 includes the reprints of the following papers:

Keunmo Kang, David D. Zhang, Robert R. Bitmead - *Disturbance Rejection Control in Coordinated Systems*, 17th IFAC World Congress, Seoul, Korea, Jul., 2008.

Keunmo Kang, Jun Yan, Robert R. Bitmead - *Cross-Estimator Design for Coordinated Systems; Constraints, Covariance, and Communication Resource Assignment*, Automatica, Vol. 44 , May., 2008, pp 1394–1401.

Keunmo Kang, Jun Yan, Robert R. Bitmead -*Communication Resources for Disturbance Rejection in Coordinated Vehicle Control*, 45th IEEE Conference on Decision and Control, San Diego, USA, Dec., 2006.

Robert R. Bitmead, Jun Yan, Keunmo Kang - *Constrained Control and Communication Resource Assignment in Coordinated Systems*, International Control Conference, Glasgow, Scotland, Aug., 2006.

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Keunmo Kang, David D. Zhang, Robert R. Bitmead - *Disturbance Rejection Control in Coordinated Systems*, 17th IFAC World Congress, Seoul, Korea, Jul., 2008.

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Robert R. Bitmead, Jun Yan, Keunmo Kang - *Constrained Control and Communication Resource Assignment in Coordinated Systems*, International Control Conference, Glasgow, Scotland, Aug., 2006.

Keunmo Kang, Jun Yan, Robert R. Bitmead - *Communication Design for Coordinated Control with a Non-Standard Information Structure*, 44th IEEE Conference on Decision and Control, Seville, Spain, Dec., 2005.

Jun Yan, Keunmo Kang, Robert R. Bitmead - *State Estimation in Coordinated Control with a Non-Standard Information Architecture*, 16th IFAC World Congress, Prague, Czech Republic, Jul., 2005.

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Journal Papers

1. Keunmo Kang, Robert R. Bitmead
   Contraction Based Receding Horizon Control
   Automatica, submitted, Mar. 2008.

2. Keunmo Kang, Jun Yan, Robert R. Bitmead
   Cross-Estimator Design for Coordinated Sytems; Constraints, Covariance, and Communication Resource Assignment
   Automatica, Vol. 44, May, 2008, pp. 1394–1401.

Conference Papers

1. Bogdan Borowy, Keunmo Kang
   Modeling of the General Atomics Maglev: System Identification Approach,
   20th International Conference on Magnetically Levitated Systems and Linear Drives, San Diego, CA, USA, Dec., 2008.

2. Keunmo Kang, Robert R. Bitmead
   Contraction Based Model Predictive Control,
   International Workshop on Assessment and Future Directions of Nonlinear Model Predictive Control, Pavia, Italy, Sep., 2008.

3. Keunmo Kang, David D. Zhang, Robert R. Bitmead
   Disturbance Rejection Control in Coordinated Systems,
   17th IFAC World Congress,
   Seoul, Korea, Jul., 2008.

4. Keunmo Kang, Robert R. Bitmead
   Online Reference Computation for Feasible Model Predictive Control,
   46th IEEE Conference on Decision and Control,
   New Orleans, LA, USA, Dec., 2007.

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FIELDS OF STUDY

Major Field: Engineering (Mechanical Engineering)

Studies in Control and Estimation.
Professors Robert R. Bitmead, Raymond de Callafon, Kenneth Kreutz-Delgado, Miroslav Krsitć, Sonia Martínez, William M. McEneaney, Robert E. Skelton.

Studies in Mathematics.
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ABSTRACT OF THE DISSERTATION

Information in Coordinated System Control

by

Keunmo Kang

Doctor of Philosophy in Engineering Sciences (Mechanical Engineering)

University of California, San Diego, 2008

Professor Robert R. Bitmead, Chairperson

In this thesis, two subjects are considered: new techniques to improve stabilizing performance and feasibility in model predictive control and disturbance rejection control in coordinated systems.

Model predictive control is powerful when a system has constraints. However, by nature, feasibility and stabilizing property of model predictive control can be lost without proper treatments. A new idea is studied for the case that a system is not well stabilized by classical model predictive control since the origin is not reachable from initial states in a limited horizon. We handle this matter by using a time-varying contractive terminal state equality constraint in model predictive control. The core condition to execute our idea is a structural property of the system such as *contractibility* or a known control lyapunov function. In addition, algorithmic approaches to guarantee feasible model predictive control are developed with several state constraint structures. Assuming that the model predictive control problem at current time is feasible, we want to know the set of terminal states or new references such that the problem at the next time instant is still feasible. Solutions are given for the linear system case using reachability analysis.

The rest of the thesis considers disturbance rejection control in coordinated systems. We employ a fixed vehicle formation problem as an working problem. The aim

is to design a controller to maintain the formation and avoid collisions in the presence of disturbance, measurement, and communication noises. Each vehicle has its own local controller that uses the state and input information from neighbors via communication. We formulate local model predictive control and estimators for one vehicle to estimate the states of the neighboring vehicles. Since coordinated systems interact via the exchange of information through communication, as the network of coordinated systems increases in the number of subsystems, natural limits on the available bandwidth of communication need to be imposed. With the gaussian assumptions on the noises and disturbance, the estimators are designed by linear matrix inequality methods, which link control objective, estimation performance, and communication limits. Even when bounds on the uncertainties are known instead of the gaussian assumptions, controllers and estimators can be formulated. Case studies are provided to demonstrate the main ideas and discuss interesting design issues.

1

Introduction

In this thesis, we study new schemes to improve feasibility and stabilizing performance of Model Predictive Control (MPC) and disturbance rejection control in coordinated systems. The main issues addressed in these subjects are:

[**New schemes in MPC**]

– MPC formulation for the situation where the system is not well stabilized by MPC with the zero-terminal-state due to an initial state far from the origin.

– Online reference computation technique for feasible MPC,

[**Disturbance rejection control in coordinated systems**]

– Local MPC design for each subsystem aiming to reflect interaction with neighboring subsystems,

– Estimation technique to predict the future actions of neighboring subsystems,

– Communication resource assignment of communication channels in coordinated systems.

In the first part, we aim to develop new algorithmic approaches to improve feasibility and stabilizing performance of MPC and provide technical conditions to realize our ideas. In the rest of the thesis, we will concentrate on design of distributed controllers and estimators for coordinated systems aiming to handle a noisy environment due to disturbance, measurement, and communication noises.

1.1   Motivation

Many systems operate under physical constraints. For instance, a car engine operates under physical limits such as maximum torque or maximum revolutions per minute. Therefore, when we design a controller such as cruise control, we should consider the physical limits. From a control design perspective, these limits are explicit constraints on inputs, or states, or both. MPC is a popular and computationally tractable control method to handle explicit input and state constraints while preserving standard design variables. A typical MPC formulation looks like a sequence of finite horizon open-loop constrained optimal control problems. For a given initial state (or state estimate), a finite horizon optimal control problem with constraints is solved, but the control solution is only applied up to the next sampling time. Then MPC takes a new measurement and repeats the same optimal control problem. This results in a so-called *receding horizon* strategy.

MPC is indeed attractive due to its capability in handling constraints. However, like any other control methods, there are some difficult design considerations. First, closed-loop stability does not come for free. Without a proper treatment in the MPC design for a given plant, closed-loop stability may not be achieved. Second, since MPC solves a constrained optimization problem at each sampling time, feasibility of the optimization problem is not guaranteed for all time. Further compounding the headache, one has to deal with uncertainties in the plant and measurement. Third, the optimization problem to be solved in the MPC may not be a convex problem. In many cases, as we shall see later, closed-loop stability arguments in the MPC-controlled systems resort to optimality of the solution obtained from the corresponding optimization problem at each time step. The solutions we can actually compute from non-convex optimization problems are not necessarily global optimizers. Therefore, foundation of the closed-loop stability by non-convex MPC may be troublesome in practice. Finally, the computational burden of the constrained optimization problem may be overly large. Since MPC solves an optimization problem at every sampling time, if the number of state and control variables is large and a long optimization horizon is considered, computation time may be a problem. This can be debilitating when a system with fast dynamics is considered.

Here we investigate the first two issues. First, we consider MPC on a general class of constrained discrete time systems. One popular way to guarantee closed-loop stability as well as feasibility by MPC is to put the zero-terminal-state-equality constraint in the finite horizon open-loop optimal control problem at each sampling time (Kwon & Pearson 1978, Keerthi & Gilbert 1988). However, if an initial state is too far from the origin, the zero terminal constraint may not be achievable in a limited horizon. We seek a remedy for this situation. Our aim is to formulate MPC so that it does not require a large optimization horizon while subject to terminal equality constraints which may vary with time. Limiting the horizon size is desirable since computation time increases as the horizon gets larger. Second, we concentrate on feasibility of MPC in linear discrete time systems by investigating two different MPC constraint structures: terminal-state-equality constraints and reference dependent state constraints. Assuming that a current MPC problem is feasible, what we look for is a set of feasible terminal states or reference trajectories that preserves the feasibility of the MPC problem at the next time instant. Limon, Alamo & Camacho (2005) investigated the feasibility of MPC with a terminal state constraint set. To guarantee feasibility, they attempted to construct a sequence of reachable sets where the system can be admissibly steered from one set to the following, ultimately reaching the terminal constraint set. The sets are computed *offline*. Our approach is to achieve the same goal *online* for the MPC with the two state constraint structures stated above. Our solution provides the MPC algorithm presented for the first issue with an actual computational tool for implementation when a linear system with polytope constraints is considered.

The rest of the thesis is devoted to disturbance rejection control in coordinated systems such as distributed power systems, networked communication nodes, a platoon of vehicles, etc. Coordinated system control has become a significant topic in control. Typically, the system size prohibits a global solution (i.e. centralized control) because the collection of the global state information and the computation of a global control law are overly demanding. Furthermore, subsystems in a coordinated system interact via exchange of their state or control information, and incorporation of the interaction into control design is the distinguishing feature that is not usually seen in the other classes of systems. Since the framework is easily understood in the context of coordinated

autonomous vehicles, we employ a coordinated vehicle system as a working problem. In recent years, many improvements in this field have been made; (Stanković, Stanojević & Šiljak 2000, Chandler, Pachter & Rasmussen 2001, Dunbar & Murray 2004, Jeanne, Leonard & Paley 2005, Dunbar & Murray 2006, Sepulchre, Paley & Leonard 2008) to list a few. The main issues are creation of a formation, communication nodes, and their relation to formation stability. However, most of their research was done in idealized environments: disturbance free, measurement noise free, and communication error free. The authors of (Richards & How 2004, Kuwata, Richards, Schouwenaars & How 2004) considered disturbances on vehicles; however, their results were in noise-free-information sharing environments. Here, our main focus is on disturbance rejection performance of local vehicles in a fixed formation with respect to vehicle disturbance, measurement, and communication noises.

We divide the problem into two categories: control and estimation. In the control part, we use a distributed control structure. In other words, each vehicle in a formation will have its own control system, which interacts with neighboring vehicles. Due to unpredictable disturbances on vehicles, collision avoidance is a critical issue and we attempt to achieve it by interaction between vehicles. Hence, the core issue in the local control design is to mathematically describe such interaction so that it can function as a control constraint. We call it a *no-collision constraint*. Since vehicles in a formation are operated in a noisy environment, no-collision constraints must include the effect of it. There exist several approaches for this. Blackmore (2006) proposed considering the possible distribution of the vehicle states using a finite number of particles to include stochastic disturbance in a no-collision constraint. In (Richards & How 2004, Kuwata, Richards, Schouwenaars & How 2004), the constraint tightening technique (Chisci, Rossiter & Zappa 2001) was used for the no-collision constraint to accommodate the bounded disturbances on vehicles. The initial set of allowable position outputs from the no-collision constraint gets smaller to handle any discrepancy between predicted positions and actual positions of the vehicles due to the disturbance. Our philosophy is similar to theirs. The difference comes from the fact that we use estimators and consider its performance measured as the estimation error covariance driven by disturbance, measurement, and communication noises to modify our no-collision constraints. For scalar linear systems,

Yan & Bitmead (2005) already developed the technique to reformulate an initial state constraint in terms of a state estimate and its corresponding estimation error covariance. Here, we want to extend their result to the multi-dimensional case. If the no-collision constraint can be converted as we hope, then MPC can be used as a means of constrained control.

In the estimation part, we focus on estimator design for one vehicle to estimate its neighbors' future behavior for use in its local control decision. We call this estimator a *cross-estimator*. This turns out to be a different estimation problem from classical ones such as the Kalman filter problem. First, since necessary state (and control input) information is transmitted from neighboring vehicles through communication channels, the information is corrupted by communication noise. If input information must be transmitted, then estimator design gets a little harder since state estimation problems usually require perfect knowledge about control inputs. Second, if the reformulated no-collision constraint is given in terms of the state estimate and its performance measure such as an error covariance, it is natural to expect different vehicle behavior with respect to estimator performance. The most exciting aspect of the cross-estimation problem is that performance of the cross-estimator is restricted by the no-collision constraints in the control problem. Finally, since the cross-estimator takes necessary state and input information from neighboring subsystems via communication, as the number of vehicles increases in the formation, limits on the available bandwidth of communication is indeed an issue, and we need a mechanism to manage this. Hence our ultimate goal in the cross-estimator design is to formulate the above issues and to incorporate them into our estimator design process.

1.2   Overview

1.2.1   Model Predictive Control: New Ideas

Consider the following discrete time-invariant system

$$x_{k+1} = f(x_k, u_k),$$

where the state and input belong to

$$x_k \in \mathbf{X} \subset \mathbb{R}^n,$$

$$u_k \in \mathbf{U} \subset \mathbb{R}^m,$$

with an initial condition x_0. Here we assume that perfect measurement of the full state is available, the origin is the equilibrium point, $0 \in \mathbf{X}$, and $0 \in \mathbf{U}$. Consider a constrained finite-horizon optimal control problem at time k with the current state $x_k \in \mathbf{X}$, the horizon N, and the terminal target state $\bar{x}_{k+N} = 0$,

$$\mathcal{P}(x_k, \bar{x}_{k+N} = 0, N):$$

$$\arg \min_{\{u_{k|k},\ldots,u_{k+N-1|k}\}} \sum_{i=0}^{N-1} h(x_{k+i|k}, u_{k+i|k}),$$

subject to

$$x_{k+i+1|k} = f(x_{k+i|k}, u_{k+i|k}),$$

$$x_{k+i|k} \in \mathbf{X},$$

$$u_{k+i|k} \in \mathbf{U},$$

$$x_{k+N|k} = \bar{x}_{k+N} = 0.$$

Here $x_{i|k}$ ($x_{k|k} = x_k$) and $u_{i|k}$ represent the states x_i and controls u_i, which are computed at time k. The cost function $h(x, u)$ is nonnegative definite, upper-semi continuous, and satisfies $h(0, 0) = 0$. Suppose that

$$\{u_{k|k}^*, u_{k+1|k}^*, \ldots, u_{k+N-2|k}^*, u_{k+N-1|k}^*\},$$

is the optimal control solution to the problem $\mathcal{P}(x_k, \bar{x}_{k+N}, N)$ [We will provide more detailed conditions for the existence of a solution to the problem $\mathcal{P}(x_k, \bar{x}_{k+N}, N)$ later.]. One widely used MPC scheme solves $\mathcal{P}(x_k, \bar{x}_{k+N} = 0, N)$ for the current state x_k and applies the very first control $u_{k|k}^*$ to the plant. Then the MPC repeats the whole process for the new measured state $x_{k+1} = x_{k+1|k}$. Assuming that the very first optimization problem with x_0 is feasible, the system is asymptotically stabilized. The core constraint to make this happen is the terminal equality constraint $x_{k+N|k} = \bar{x}_{k+N} = 0$ (Keerthi & Gilbert 1985). The problem is that, if the initial condition x_0 is too far from the origin, then the terminal equality constraint $\bar{x}_N = 0$ may not be achievable in N steps. To overcome this matter, here we consider a different strategy that consists of two stages:

– solving a finite sequence of finite horizon open-loop optimal control problems with a time-varying terminal state equality constraint $x_{k+N|k} = \bar{x}_{k+N}$ (not necessarily equal to zero) : the terminal target state changes at each time step and moves closer to the origin, then

– solving a sequence of finite horizon open-loop optimal control problems with the origin as the target terminal state to achieve closed-loop stability.

The first stage does not appear in previous MPC contexts. It functions to manipulate the system into a situation where typical zero-terminal-state MPC such as the second stage can be used. The critical issue in the first stage is how we update the terminal target state $\bar{x}_{k+N}$ while keeping $\mathcal{P}(x_k, \bar{x}_{k+N}, N)$ feasible over time. We will show that, if the system can make contractive movements in the sense of the weighted norm of the state or some energy measures such as a Control Lyapunov Function (CLF), then the first stage feasibly proceeds and shifts to the second stage in finite time. The authors of (Primbs, Nevistić & Doyle 1999, Kwon & Han 2005) discussed the idea of including a CLF in MPC and pointed out its possible performance improvement due to complementary aspects of a CLF and MPC. In this thesis, by example, we demonstrate that our scheme can also bring improvement to control performance and give flexibility to tune system behavior. In addition, our scheme can achieve faster stabilization of a system than the schemes introduced in (Primbs, Nevistić & Doyle 1999, Kwon & Han 2005).

While the above focuses on the stabilization issue, here we focus more on the feasibility of MPC. Specifically we consider two classes of state constraint structures. Consider, at time k, the MPC problem to solve

$$\arg \min_{\{u_{k|k},...,u_{k+N-1|k}\}} \sum_{i=0}^{N-1} h(x_{k+i|k}, u_{k+i|k}),$$

$$subject\ to$$

$$x_{k+i+1|k} = f(x_{k+i|k}, u_{k+i|k}),$$

$$u_{k+i|k} \in \mathbf{U},$$

$$x_{k+i|k} \in \mathbf{X}_{k+i},$$

where the state constraint is in one of the following forms:

A. A terminal state equality constraint: $x_{k+N|k} \in \mathbf{X}_{k+N} = \{\bar{x}_{k+N}\}$ (no immediate constraints $\mathbf{X}_{k+i}$ $i = 1, 2, \ldots, N-1$),

B. Reference dependent constraints with reference trajectories over N steps: $x_{k+i|k} \in \mathbf{X}_{k+i}(r_{k+i})$ (only $N-$step reference trajectories , r_{k+1}, $r_{k+2}, \ldots, r_{k+N}$, are known.).

Then the corresponding control $u^*_{k|k}$ is applied to the system and the MPC takes a new measurement to solve the above problem with newly updated $\bar{x}_{k+1+N}$ or r_{k+1+N} if available. Suppose that the above MPCs at time k are feasible. The problems we want to solve are:

A. What new terminal states $(\bar{x}_{k+N+1})$ can be chosen,

B. What new references (r_{k+1+N}) can be chosen,

so that the MPCs at time $k+1$ do not lose feasibility? The main observation about these problems is that existence of the solutions only depends on the constraints, not on the choice of the optimization objective function. That is, no matter what $h(x_{k+i|k}, u_{k+i|k})$ is chosen, what really matters is whether the combination of $\mathbf{U}$ and $\mathbf{X}_{k+i}$ results in having feasible input solutions. Therefore, the solutions to the problems are found in such a way that a choice of $\bar{x}_{k+1+N}$ or r_{k+1+N} guarantees at least one input solution for the MPCs at time $k+1$. In the linear system case, if all the constraints are given in terms of linear equalities or inequalities or both, we can provide a set of new terminal states or references that preserve the feasibility of MPC at time $k+1$. Reachability analysis and convex polytope properties are the main ingredients to attack the problems.

1.2.2   Disturbance Rejection Control in Coordinated Systems

In this thesis, our focus is limited to a fixed target vehicle formation with linear system dynamics. The task of control is to maintain the vehicles close to their target positions in the formation. If a precise model of the vehicles is given and there is no disturbance on the vehicles, then this problem might be easily solved. However, if there is disturbance acting on the vehicles, not only does the control have to steer the vehicles to their target positions, but it must also avoid possible collisions between the vehicles. Consider the second vehicle formation with 10 m target separation in Figure 1.1. If there
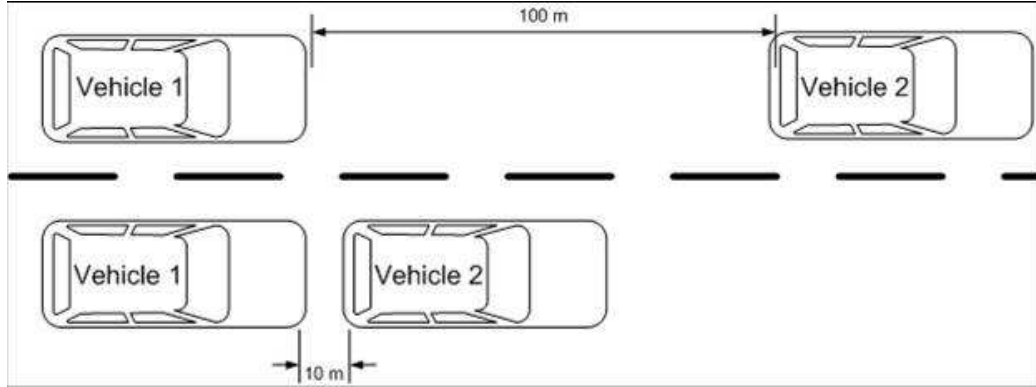
Figure 1.1: Two vehicle formation on a highway.

are disturbances acting on the vehicles, then the 10 m separation may be a dangerous choice because the vehicles' spatial variations around their target positions may be big enough to cause collision. To avoid this, the vehicles can interact via exchange of their state and input information.

While the vehicle formation with 10 m separation illustrates the motivation for local MPC at each vehicle aiming to maintain formation and avoid collision via exchange of information between the vehicles, the first formation with 100 m target separation implies that coordinated control is not always necessary. Even if there are disturbances acting on the vehicles, collision is not likely since they are too far away from each other. Therefore, before investigating the local coordinated MPC design, we first provide the criterion that clarifies the situation which requires coordinated control. Consider the following linear vehicle dynamics:

$$x_{k+1} = Ax_k + Bu_k + w_k,$$

$$y_k = Cx_k,$$

$$z_k = Dx_k + v_k,$$

where u_k, y_k, z_k and x_k represent the input vector, the position output vector, the measurement vector, and the state vector of the vehicle and of its associated disturbance process. The vehicles and their disturbance process are driven by a process noise w_k and the measurement is corrupted by a measurement noise v_k. Both noises are modeled as

white normally distributed sequences,

$$w_k \sim N(0, Q), \quad v_k \sim N(0, R).$$

Our study begins with a Linear Quadratic Gaussian (LQG) control strategy for the disturbance rejection of vehicles in a formation. Then we take the following steps:

– Calculation of the closed-loop LQG control position variance of an isolated vehicle around its nominal position under the effect of exogenous disturbances.

– Evaluation of the interaction constraint which seeks to ensure sufficiently infrequent activity of the no-collision constraint.

The last stage links formation geometry and the LQG control performance so that we can judge whether coordinated local MPC is required.

If LQG control is not sufficient to avoid collisions, MPC is used at each vehicle to maintain formation and prevent collision. The no-collision constraint basically says that the future position of one vehicle cannot overlap its neighbors' future positions. However, due to uncertainties (e.g. disturbances), calculating the exact future positions of the vehicles is not possible. Hence, proper conversion from this stochastic constraint to a deterministic one is required. If disturbance, measurement, and communication noises are gaussian, then the deterministic form of the no-collision constraint with more than $1 - \epsilon$ probability at Vehicle i is given by

$$||\hat{y}_{k+j|k}^i - \hat{y}_{i,k+j|k}^\ell|| > \alpha + \beta \sqrt{\lambda_{\max}\left(C(S_{p,j}^i + \Sigma_{i,j}^\ell)C^T\right)}, \tag{1.1}$$

where $\hat{y}_{k+j|k}^i = C\hat{x}_{k+j|k}^i$ and $\hat{y}_{i,k+j|k}^\ell = C\hat{x}_{i,k+j|k}^\ell$ are, respectively, Vehicle i's j step ahead future positions with the state error covariance $S_{p,j}^i$ and Vehicle ℓ's j step ahead predicted positions computed by Vehicle i at time k with the state error covariance $\Sigma_{i,j}^\ell$. The scalars, α and β, are the numbers associated with the size of the vehicle and the probability of collision ϵ respectively, and $|| \cdot ||$ and $\lambda_{max}(\cdot)$ represent a two norm of a vector and the maximum eigenvalue of a matrix respectively. The last term on the right hand side of the inequality (1.1) can be understood as a stand-off value by using the estimates ($\hat{y}$) in the decision making process and this is given in terms of the error covariances. For a given self-position estimate ($\hat{y}_{k|k}^i$) and cross-position predictions

$(\hat{y}^{\ell}_{i,k+j|k})$, the local MPC at Vehicle i makes a decision in a way such that $\hat{y}^{i}_{k+j|k}$ satisfies (1.1).

In order to use the no-collision constraint, the local MPC at Vehicle i takes information from two sources: one is a Kalman filter for the self-position estimate $(\hat{y}^{i}_{k|k})$, and the other is a cross-estimator to predict Vehicle ℓ's future position $(\hat{y}^{\ell}_{i,k+j|k})$. Since the Kalman filter is used for the self-state estimation, the error covariance $S^{i}_{p,j}$ in (1.1) is fixed. Then, a smaller cross-estimation error covariance $\Sigma^{\ell}_{i,j}$ will give a smaller stand off value and, therefore, will lead to less frequent activity of the no-collision constraint. Indeed, (1.1) provides the performance limit of the cross-estimator. To see this, assuming that the nominal separation between Vehicle i and Vehicle ℓ is d, consider the case that $\Sigma^{\ell}_{i,j}$ satisfies

$$d < \alpha + \beta \sqrt{\lambda_{\max}\Big(C(S^{i}_{p,j} + \Sigma^{\ell}_{i,j})C^{T} \Big)}. \tag{1.2}$$

Then, even if $\hat{y}^{i}_{k+j|k}$ and $\hat{y}^{\ell}_{i,k+j|k}$ are at their target positions in the future (i.e. $||\hat{y}^{i}_{k+j|k} - \hat{y}^{\ell}_{i,k+j|k}|| = d$), the no-collision constraint (1.1) will be active. This is not desirable since the MPC problem will encounter active constraints frequently. There are two ways to obtain a smaller $\Sigma^{\ell}_{i,j}$. One is to alter the cross-estimator gain if a better value were to exist. The other is to assign enough communication resource between the vehicles if this can solve the problem. However, in general, communication channels are band limited, which may be modeled as quantization of data transmitted over a channel. Assuming that a scalar signal has been scaled to a range of $[-0.5,\ 0.5]$, the effect of quantization to m_j bits of accuracy is to add a white, zero-mean noise of variance $\frac{1}{12}2^{-2m_j}$ (Widrow, Kollár & Liu 1996). If the total bandwidth to be assigned to a channel is limited to τ bits per sample time and each piece of information from $j = 1$ to J is assigned by m_j bits, then the limit is captured by

$$\sum_{j=1}^{J} m_j \leq \tau. \tag{1.3}$$

We formulate Linear Matrix Inequalities (LMI) to obtain the (sub) optimal estimator gains, communication resource assignment, and upper bound of the cross-estimation error covariance while satisfying the following:

R1. stable cross-estimation,

R2. the requirement from the no-collision constraint: $d \geq \alpha + \beta \sqrt{\lambda_{\max}\left(C(S_{p,j}^i + \Sigma_{i,j}^\ell)C^T\right)}$ from the observation in (1.2),

R3. the communication limit (1.3).

The main ideas presented above could be developed with the assistance of the linear gaussian assumptions on disturbance, measurement, and communication noises. Without the assumptions, it would be hard to compute estimation error covariances and, therefore, the no-collision constraint (1.1), and the LMI design approach for the cross-estimators may not work. We also provide control and estimation schemes without probabilistic models on the uncertainties: only their bounds are given. In this case, covariances are replaced by the worst case norm of the estimation error in the analysis. Then one can still construct local MPC with a similar no-collision constraint to (1.1) and design estimators to satisfy **R1** and **R2** by an approach akin to Adaptive Kalman filtering (Haykin 2001). However communication resource assignment could not be included in the analysis and, hence, we do not consider **R3** in the analysis. Rather, we will assume that bounds of communication errors are given.

1.3 Contributions

The contributions of this dissertation are summarized as follows:

(1) Development of a new MPC algorithm applicable to a system that is not effectively stabilized by classical MPC due to an initial state far from the origin. (Chapter 2)

– Identification of central conditions to run the algorithm: contractibility or existence of control Lyapunov functions,

– Performance enhancement compared to control methods that solely depend on a control Lyapunov function.

(2) Development of online reference computation algorithms for feasible MPC. (Chapter 2)

    – Algorithms based on reachability analysis and convex polytopes for the linear system case,

    – Approximation schemes to reduce computational burden of the developed algorithms.

(3) Distributed control design for disturbance rejection in coordinated systems. (Chapter 3)

    – Analysis to clarify the criterion for including cooperative control between the subsystems,

    – Conversion from a stochastic local MPC to a deterministic one.

(4) Cross-estimator design for coordinated systems. (Chapter 4)

    – Derivation of sufficient requirements on the cross-estimator performance for only sporadic activity of the constraints in the local MPC,

    – Unified approach to design the cross-estimator gain and assign communication resources.

2

Model Predictive Control: New Ideas

2.1   Introduction

Model Predictive Control (MPC) is a closed-loop control approach, which proceeds by the successive solution of open-loop finite-horizon optimal control problems. For a given system, MPC solves a finite horizon optimal control problem (with constraints) and apply the control solution to the system until the new state measurement (or state estimate) is available. Then, using the new state measurement (or state estimate), the MPC repeats the same finite horizon control problem. Therefore, MPC is often termed as Receding Horizon Control (RHC). The key idea of MPC is to apply a sequence of solutions in a receding horizon fashion, thereby converting the open-loop control to closed-loop control. The fundamental question is to understand how properties of the open-loop problem might be inherited or translated to properties of the closed-loop solution. Unfortunately, the closed-loop solution might fail to stabilize the system or the closed-loop solution might not even exist due to computational infeasibility of the open-loop constrained optimal control problem at some time instant. In this chapter, new ideas in MPC are presented:

1. Contraction Based MPC,

2. Online reference computation for feasible MPC.

1. Contraction Based MPC

We study stabilization of a constrained discrete time system, whose behavior is limited by input and state constraints. In particular, we focus on an MPC scheme for the situation where the system is not effectively stabilized by typical MPC (D.Q.Mayne, Rawlings, C.V.Rao & P.O.M.Scokaert 2000) with the zero-terminal-state or terminal set around the origin due to an initial state being too far from the origin. If the system has some structural property such as the satisfaction of *contractibility* condition, to be defined, or the existence of a Control Lyapunov Function (CLF) in some subset of the state constraint set, then one can construct feasible and stabilizing MPC solutions applicable to initial states far from the origin. Furthermore the proposed scheme gives flexibility to tune the system performance. This is not usually permitted when controller design is solely based on a CLF.

To achieve closed-loop stability, there is apparently a single central idea whereby the unapplied tail of the receding horizon solution is proven to provide a feasible solution for the next instants problem with reduced cost-to-go. This idea is traceable at least to (Keerthi & Gilbert 1988) who use it to establish conditions for both feasibility and asymptotic stability. Their approach is grounded in the specification of a finite-horizon terminal state equality constraint, which also appears in (Kwon & Pearson 1978). This has been relaxed by subsequent authors (Michalska & Mayne 1993, Chisci, Lombardi & Mosca 1996, Chen & Allgöwer 1998, Mayne 2001) from a terminal equality constraint to a terminal set constraint containing the origin. As one may already expect, the above contributions are applicable only when the origin or the terminal set around the origin is reachable from the initial state in the specified horizon, in which case the size of the domain of attraction might be quite limited. Increasing the horizon may help, but will require more computational resources. To enlarge the domain of attraction without using too much computation, (Limon, Alamo & Camacho 2005) used a sequence of reachable sets to a given initial terminal set with a control invariance property. The authors used a sequence of sets such that the system can be feasibly steered from one set to the following, finally reaching to the initial terminal set. This sequence of sets is computed offline. However, this idea may require very intensive offline computation if a given initial state is so far away from the origin that it is required to compute a sequence of overly

many sets.

Here we consider a different approach by employing a time-varying terminal state equality condition to ensure feasibility and stability with the aim of generating an MPC algorithm applicable at points, from which the origin is not reachable in small time steps. We provide conditions under which a solution to guarantee asymptotic stability is found from initial states that are not steered to the origin within the horizon. Our MPC scheme is deployed in two stages:

- solving a finite sequence of finite horizon open-loop optimal control problems with a time-varying terminal state equality constraint: the terminal target state changes at each time step and moves closer to the origin,

- solving a sequence of finite horizon open-loop optimal control problems with the origin as the target terminal state to achieve closed-loop stability.

The first stage does not appear in typical MPC contexts and it functions to manipulate the system into a situation where a typical MPC such as the second stage can be utilized. We will show that, if the system can make contractive movements in the sense of the weighted norm of the state or some energy measures such as a CLF, then the first stage operates feasibly and shifts to the second stage in finite time, and as the second stage comes into play the system enjoys closed-loop asymptotic stability. Our scheme does not require intensive offline computation or an unduly long MPC horizon.

The idea of using a contraction property of a system appears in (Polak & Yang 1993) and in (Kothare & Morari 2000) for a continuous time-invariant system. Especially, in their studies, contraction refers to a decrease in the weighted norm of the state vector. In (Polak & Yang 1993), contraction occurs at each sampling time that the finite horizon open-loop control problem is solved. But the sampling time is another decision variable in addition to the open-loop control. The interval between the sampling times is chosen as the minimum time over which contraction occurs. Hence, the horizon changes depending on how fast the system can make a movement of the desired contraction rate for the current state. In (Kothare & Morari 2000), contraction is achieved at the end of a fixed horizon. For the disturbance free case, the interval between sampling times is the horizon and the whole open-loop control solution at each sampling

time is applied to the system. While both studies use a contraction condition as a constraint on the MPC, our contraction condition is applied for the choice of the terminal target state at each sampling time. In this paper, we will show that the contraction condition can be replaced by a CLF if it known. The idea of using a CLF is also applicable to the proposed algorithms in (Polak & Yang 1993) and (Kothare & Morari 2000). We establish a system condition, contractibility, for the feasibility of the MPC problem. We show that knowledge of a CLF allows us to replace the contractibility condition.

Mayne (2001) stated that a benefit of MPC is the absence of a requirement for knowledge of a CLF. However, (Primbs, Nevistić & Doyle 1999) discussed the idea of including a CLF in the MPC formulation for an unconstrained nonlinear system and pointed out its possible performance improvement due to complementary aspects of a CLF and MPC. This idea also appears in (Kwon & Han 2005). By example, we demonstrate that our scheme can also bring improvement to control performance and give flexibility to tune system behavior.

2. Online reference computation for feasible MPC

While Contraction Based MPC focuses on the stabilization issue, here we focus more on computational feasibility of MPC by considering the following two classes of state constraint structures:

A. A terminal state equality constraint,

B. Reference dependent constraints with reference trajectories over N steps.

Suppose that the above MPC problems at time k are feasible. The problems we want to solve are:

A. What new terminal states can be chosen,

B. What new references can be chosen,

so that the MPC problems at time $k+1$ do not lose the feasibility? In the linear system case, if all the constraints are given in terms of linear equalities or inequalities or both, we can provide a set of new terminal states or references that preserve the feasibility of MPC at time $k+1$. We call the above *Online Reference Computation* problems.

The first structure is interesting in the sense that, although one of the techniques to guarantee stability is to implement either a terminal state or a terminal set with terminal cost (D.Q.Mayne, Rawlings, C.V.Rao & P.O.M.Scokaert 2000), it is valid only when the corresponding optimization problem is feasible. To guarantee feasibility, the authors of (Chen, Ballance & O'Reilly 2001, Magni, De Nicolao, Magnani & Scattolini 2001, DeDoná, Seron, Mayne & Goodwin 2002, Limon, Gomes da Silva, Alamo & Camacho 2003, Limon, Alamo & Camacho 2005) attempted to enlarge the terminal set or construct a sequence of reachable sets where the system can be admissibly steered from one set to the following, ultimately reaching the terminal set. The sets are computed *offline.* Here, we want to build an *online mechanism* to inform the system what is available for the next terminal constraint for the given current state and terminal constraint information. The first structure is also used in the Contraction Based MPC algorithms. However, they require to compute sets that are not yet efficiently obtained for a broad class of nonlinear systems. For the linear system with terminal equality constraints, the set of new feasible terminal target states can be calculated fairly efficiently.

We are also interested in the second structure since we see its potential application to a pursuit problem. Consider the Stratotanker aircraft fueling F-15K. Suppose

Figure 2.1: The USAF Stratotanker fueling F-15K of Republic of Korea

that the vehicles are unmanned (i.e. autonomous). Then the controller of each vehicle acts as the pilot. If the Stratotanker makes a dramatic movement, then the F-15K would have trouble to adjust its movement and it could lead to a failure of fueling. Therefore there exists a limit on the Stratotanker's movement depending on the dynamic properties

of the F-15K. For simplification, from the perspective of the follower, the leader's planned movements over time can be seen as a reference trajectory. Therefore we consider the system (the follower) which pursues a reference trajectory (the leader) by MPC. It is not the same as the general tracking problem which was considered by the authors of (Gilbert & Kolmanovsky 1995, Bemporad, Casavola & Mosca 1997, Bemporad 1998, Gilbert & Kolmanovsky 2001, Limon, Alvarado, Alamo & Camacho 2005, Fiacchini, Alvarado, Limon, Alamo & Camacho 2006). Rather than achieving (off-set free) tracking of a piece-wise constant reference trajectory in steady state without any constraint violations, we require the system to follow a specified reference trajectory and satisfy a specified off-set by an admissible input sequence. Thus our problem is analogous to a receding horizon variety of the Target Tube problem in (Bertsekas & Rhodes 1971), whose control objective is to keep the state in a sequence of $N-$step state constraint sets.

2.2 Basic Model Predictive Control Formulation

Prior to the main discussions of this chapter, brief review of MPC is given in this section. Consider the following discrete time-invariant system

$$x_{k+1} = f(x_k, u_k), \tag{2.1}$$

where the state and input belong to

$$x_k \in \mathbf{X} \subset \mathbb{R}^n,$$

$$u_k \in \mathbf{U} \subset \mathbb{R}^m,$$

with an initial condition x_0. Here we assume that perfect measurement of the full state is available and the origin is the equilibrium point. Consider a constrained finite-horizon optimal control problem at time k with the current state $x_k \in \mathbf{X}$, the horizon N, and

the terminal target state $\bar{x}_{k+N} \in \mathbf{X}$,

$$\mathcal{P}(x_k, \bar{x}_{k+N}, N):$$

$$\arg \min_{\{u_{k|k}, \ldots, u_{k+N-1|k}\}} \sum_{i=0}^{N-1} h(x_{k+i|k}, u_{k+i|k}),$$

subject to

$$x_{k+i+1|k} = f(x_{k+i|k}, u_{k+i|k}),$$

$$x_{k+i|k} \in \mathbf{X},$$

$$u_{k+i|k} \in \mathbf{U},$$

$$x_{k+N|k} = \bar{x}_{k+N}.$$

(2.2)

Here $x_{i|k}$ ($x_{k|k} = x_k$) and $u_{i|k}$ represent the states x_i and controls u_i, which are computed at time k. We assume the following:

A1. $\mathbf{X}$ is closed and $0 \in \mathbf{X}$,

A2. $\mathbf{U}$ is closed and $0 \in \mathbf{U}$,

A3. $f : \mathbf{X} \times \mathbf{U} \longmapsto \mathbb{R}^n$ is continuous and $0 = f(0,0)$,

A4. $h : \mathbf{X} \times \mathbf{U} \longmapsto \mathbb{R}$ is nonnegative definite, upper-semi continuous, and satisfies $h(0,0) = 0$.

Closure of $\mathbf{X}$ and $\mathbf{U}$, continuity of f and h, and positive definiteness of h in the assumptions **A1**~**4** are the conditions for the finite horizon problem $\mathcal{P}(x_k, \bar{x}_{k+N}, N)$ to have an optimal solution provided there exists an admissible sequence pair $\{(x_{k+i|k}, u_{k+i|k})\}_{i=0}^{N-1}$ (Keerthi & Gilbert 1985). If the assumptions **A1**~**4** hold, $\mathcal{P}(x_k, \bar{x}_{k+N}, N)$ has an optimal open-loop control solution sequence,

$$\{u_{k|k}^*, u_{k+1|k}^*, \ldots, u_{k+N-2|k}^*, u_{k+N-1|k}^*\},$$

and a corresponding or prediction state sequence

$$\{x_{k|k} = x_k, x_{k+1|k}^*, \ldots, x_{k+N-1|k}^*, x_{k+N|k}^* = \bar{x}_{k+N}\}.$$

For given initial state x_0, using a sequence of $\mathcal{P}(x_k, \bar{x}_{k+N}, N)$ with $\bar{x}_{k+N} = 0$, the standard MPC algorithm is stated as follows.

Offline Preparation

`choose` control horizon for the first stage (N)

Standard Model Predictive Control Law

`set` time $k = 0$,

`repeat`

> `solve` $\mathcal{P}(x_k, 0, N)$,
>
> `apply` $u^*_{k|k}$ to the system: $x_{k+1} = x^*_{k+1|k} = f(x_k, u^*_{k|k})$,
>
> `update` $x_k \leftarrow x_{k+1}$,

`end`

Remark 2.2.1. *Although the open-loop control problem is solved at each sampling time, the above MPC is actually closed-loop control since the control law uses the newly measured state every time the open-loop control problem is solved.*

Closed-loop stability of the system controlled by this Standard MPC Law is established by the following theorem.

Theorem 2.2.1. *(Keerthi & Gilbert 1988) Suppose that the following hold:*

– *Assumptions* **A1∼4**.

– *Initial* $\mathcal{P}(x_0, 0, N)$ *with* $x_0 \in \mathbf{X}$ *is feasible.*

Then the system (2.1) is asymptotically stabilized by Standard MPC Law.

Proof. For $\mathcal{P}(x_0, 0, N)$, let $J^*(x_0, 0, N)$ be the corresponding optimal cost. It can be seen as

$$J^*(x_0, 0, N) = h(x_0, u^*_{0|0}) + J^*(x^*_{1|0}, 0, N - 1).$$

By the optimality principle,

$$J^*(x^*_{1|0}, 0, N) \leq J^*(x^*_{1|0}, 0, N - 1).$$

Hence

$$J^*(x_0, 0, N) - J^*(x^*_{1|0}, 0, N) \geq h(x_0, u^*_{0|0}).$$

Successive use of this inequality leads to

$$J^*(x_0, 0, N) - J^*(x^*_{j|j-1}, 0, N) \geq \sum_{k=0}^{j-1} h(x_k, u^*_{k|k}),$$

where $j \geq 1$. Since

$$J^*(x_0, 0, N) \geq J^*(x_0, 0, N) - J^*(x^*_{j|j-1}, 0, N) \geq \sum_{k=0}^{j-1} h(x_k, u^*_{k|k})$$

holds for arbitrary j,

$$\sum_{k=0}^{\infty} h(x_k, u^*_{k|k}) < \infty \tag{2.3}$$

holds. This implies that the tails of (2.3) are zeros. Since $h(\cdot) = 0 \iff (x_k, u_k) = (0, 0)$, $x_k \longrightarrow 0$ as $k \longrightarrow \infty$. This completes the proof. $\square$

Remark 2.2.2. *Instead of using the terminal state equality constraint $x_{k+N|k} = 0$, a terminal state set constraint, $x_{k+N|k} \in \Omega \subseteq \mathbf{X}$, may be used with a proper choice of the objective function in (2.2) (Michalska & Mayne 1993, Chisci, Lombardi & Mosca 1996, Chen & Allgöwer 1998, Mayne 2001). The set Ω is a neighborhood of the origin and the purpose of MPC is to steer the state into Ω in finite time. Then a local stabilizing controller $\kappa_f(\cdot)$ is employed. In this approach, region of attraction may be expanded. This type of MPC is often referred to as dual mode MPC.*

2.3   Contraction Based Model Predictive Control

2.3.1   Problem

For the system controlled by Standard MPC Law, we saw that cost of $\mathcal{P}(x_k, 0, N)$ decreases over time and the system is asymptotically stabilized, provided that the origin is feasibly reachable from the initial state x_0 in N steps. Quite naturally the following question arises: what can we do for initial states that cannot be steered to the origin in N steps? We want to develop an MPC scheme to deal with this situation by using the same structure as (2.2), in which case $\bar{x}_{k+N}$ will not be necessarily zero. Therefore the main focus is on how we construct the sequence of terminal states $\{\bar{x}_N, \bar{x}_{1+N}, \bar{x}_{2+N}, \dots\}$ that preserve feasibility of $\mathcal{P}(x_k, \bar{x}_{k+N}, N)$ over time and finally achieve asymptotic stability.

2.3.2   Algorithm

Before stating our MPC algorithm formally, we first reveal a big picture about how it proceeds. The proposed MPC law consists of two stages. In the first stage,

finite horizon problems $\mathcal{P}(x_k, \bar{x}_{k+N}, N)$ are solved over time with a time varying least decreasing norm terminal target state $\bar{x}_{k+N}$. This manipulates the system into the situation where the second stage can take over from the first stage. In the second stage, finite horizon problems $\mathcal{P}(x_k, 0, N + N_1)$ with an extended control horizon $N_1 + N$ are solved. In fact, the second stage coincides with Standard MPC Law introduced in the previous section, which leads to asymptotic stability of the closed-loop system. We introduce notation and definitions to be used in the algorithm as well as in later discussions. For a vector $x \in \mathbb{R}^n$, $||x||_S$ is the weighted norm $\sqrt{x^T S x}$ with a positive definite matrix $S \in \mathbb{R}^{n \times n}$: when $S = I$ (an identity matrix), we use $||x||$.

Definition 2.3.1. *A ball contained in* $\mathbf{X}$ *is defined by* $\mathcal{B}(\alpha, S) \triangleq \{x \in \mathbf{X} \big| ||x||_S \leq \alpha, \ \alpha > 0\}$ *with appropriately chosen parameters* α *and* S.

Definition 2.3.2. *The subset of* $\mathbf{X}$, $\mathcal{X}(0, N_1)$ *is the set of states, from which the origin is reachable in* N_1 *steps.*

The MPC algorithm is given for an initial state $x_0 \in \mathcal{B}(\alpha, S) \subseteq \mathbf{X}$, where $\mathcal{B}(\alpha, S)$ is some known domain of attraction to be discussed later.

Offline Preparation
`choose` control horizons for the first stage (N) and the second stage $(N + N_1)$,
`choose` any feasible $\bar{x}_N \in \mathcal{B}(\alpha, S)$ based on $x_0 \in \mathcal{B}(\alpha, S)$,
`compute` $\mathcal{X}(0, N_1) \in \mathbf{X}$.

Contraction Based Model Predictive Control Law
`set` time $k = 0$,
[First Stage]
`while` $\bar{x}_{k+N} \notin \mathcal{X}(0, N_1)$

   `solve` $\mathcal{P}(x_k, \bar{x}_{k+N}, N)$,

   `compute` $\bar{x}_{k+1+N} = \arg\min_{\tilde{x}}\{||\tilde{x}||_S \big| \tilde{x} = f(\bar{x}_{k+N}, \bar{u}_{k+N}), \bar{u}_{k+N} \in \mathbf{U}\}$,

   `apply` $u_{k|k}^*$ to the system: $x_{k+1} = x_{k+1|k}^* = f(x_k, u_{k|k}^*)$,

   `update` $x_k \leftarrow x_{k+1}$ and $\bar{x}_{k+N} \leftarrow \bar{x}_{k+N+1}$,

`end`

[Second Stage]

```
repeat
        solve P(x_k, 0, N_1 + N),
        apply u*_{k|k} to the system,
        update x_k ← x_{k+1},
end
```

Remark 2.3.1. *Note that the terminal target state $\bar{x}_{k+1+N}$ can be computed offline since the choice of $\bar{x}_{k+1+N}$ is independent of the current state x_k or the predicted state $x^*_{k+1|k}$. This observation triggers further exploration of the above algorithm later in Section 2.3.5.*

At this point, it is not guaranteed that the first stage terminates in finite time and therefore that the system is stabilized by the second stage. Also the domain of attraction $\mathcal{B}(\alpha, S)$ has not been clearly specified yet. We will show that, under some conditions, the proposed MPC completes the first stage in finite time and the system enjoys asymptotic stability due to the second stage.

2.3.3 Feasibility and Stability

The machine to propel our discussion on feasibility and stability properties of the proposed MPC algorithm is contractibility, which is given bellow.

Definition 2.3.3. *The system (2.1) is contractible in the domain $\mathcal{B}(\alpha, S)$ if, for any $x \in \mathcal{B}(\alpha, S)$, there exists a feasible control $u \in \mathbf{U}$ such that $||f(x, u)||_S \leq \rho ||x||_S$, where $\rho \in [0, 1)$.*

Remark 2.3.2. *Our contractibility definition is a special case of a condition in (Kothare & Morari 2000). In their paper, the contraction occurs every j steps and in our case $j = 1$. Also contractibility appears in (Polak & Yang 1993). In their setting, for a different state, contraction can occur with different time intervals, which are also decision variables to minimize in the open-loop control problem.*

Finite time termination of the first stage is described by the following lemma.

Lemma 2.3.1. *Suppose that the following hold:*

  – *Assumptions $\mathbf{A1} \sim \mathbf{4}$.*
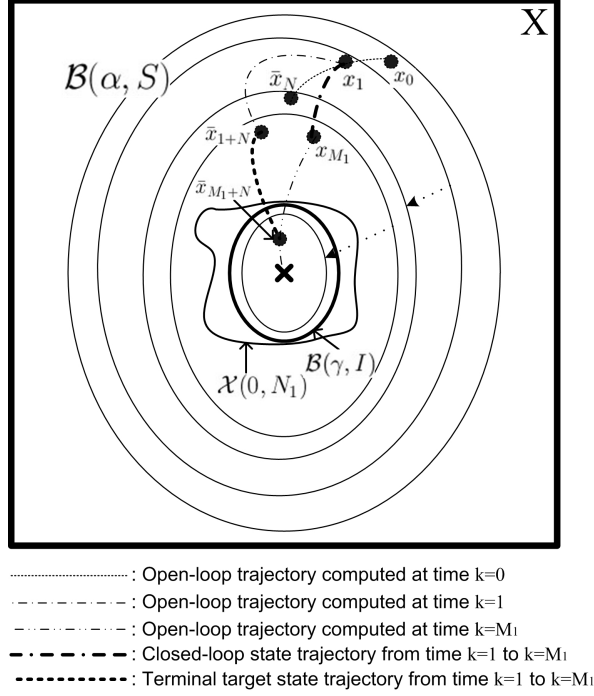
Figure 2.2: Graphical description of Lemma 2.3.1.

- *A convex domain $\mathcal{B}(\alpha, S) \subseteq \mathbf{X}$ is given in which the system is contractible.*

- *Initial $\mathcal{P}(x_0, \bar{x}_N, N)$ with $x_0, \ \bar{x}_N \in \mathcal{B}(\alpha, S)$ is feasible.*

- *The origin is an interior point of $\mathcal{X}(0, N_1)$.*

Then the first stage of the Contraction Based Model Predictive Control Law terminates in finite time.

Proof. First we show that feasibility at the current time implies feasibility all times if the terminal state constraint is chosen by

$$\bar{x}_{k+1+N} = \arg\min_{\tilde{x}}\{||\tilde{x}||_S \big| \tilde{x} = f(\bar{x}_{k+N}, \bar{u}_{k+N}), \bar{u}_{k+N} \in \mathbf{U}\}. \tag{2.4}$$

Since $\mathcal{P}(x_0, \bar{x}_N, N)$ is feasible, then due to $\mathbf{A1} \sim \mathbf{4}$, there exist an optimal input sequence and open-loop state trajectory

$$\{x_{0|0}, x^*_{1|0}, \ldots, x^*_{N-1|0}, \bar{x}_N\},$$
$$\{u^*_{0|0}, u^*_{1|0}, \ldots, u^*_{N-2|0}, u^*_{N-1|0}\}.$$

Consider the next time finite horizon problem $\mathcal{P}(x_1, \bar{x}_{1+N}, N)$ with the new current state $x_1 = x_{1|0}^*$ and the new terminal target state $\bar{x}_{1+N}$ given by (2.4). This has the following feasible input sequence and state trajectory:

$$\{u_{1|0}^*, u_{2|0}^*, \ldots, u_{N-1|0}^*, \bar{u}_N^*\},$$

$$\{x_1, x_{2|0}^*, x_{3|0}^*, \ldots, \bar{x}_N, \bar{x}_{1+N}\},$$

where $\bar{u}_N^* \in \mathbf{U}$ is the control to steer $\bar{x}_N$ to $\bar{x}_{1+N}$ that is given by (2.4). Successive use of the above argument guarantees feasibility all time $k = 0, 1, 2, \ldots$. Now it remains to show that $\bar{x}_{k+N}$ reaches $\mathcal{X}(0, N_1)$ in finite time. Since the origin is an interior point of $\mathcal{X}(0, N_1)$, there exists some positive real number $\gamma > 0$ such that $\mathcal{B}(\gamma, S) = \{x \in \mathbf{X} \| \|x\|_S \leq \gamma\} \subseteq \mathcal{X}(0, N_1)$. Since the system is contractible, $\bar{x}_{1+N}$ given by (2.4) satisfies

$$\|\bar{x}_{1+N}\|_S \leq \rho \|\bar{x}_N\|_S, \ \rho \in [0, 1).$$

By successive applications of this result,

$$\|\bar{x}_{j+N}\|_S \leq \rho^j \|\bar{x}_N\|_S$$

holds. This implies that the terminal target state will be inside $\mathcal{B}(\gamma, S)$ in M_1 steps, where M_1 is the smallest nonnegative integer j satisfying

$$j \geq \frac{\log \gamma - \log \|\bar{x}_N\|_S}{\log \rho}.$$

This completes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

Remark 2.3.3. *Keerthi & Gilbert (1988) used controllability (property C) of the system to obtain $\mathcal{X}(0, N_1)$ that has the origin as an interior point. Our condition on $\mathcal{X}(0, N_1)$ is weaker than their controllability condition.*

Remark 2.3.4. *There is no requirement that $x_{k+j|k} \in \mathcal{B}(\alpha, S), \ \forall j = 1, 2, \ldots, N - 1$. Only the current and terminal states need to be in $\mathcal{B}(\alpha, S)$.*

Once the first stage terminates, asymptotic stability of the system is achieved by the second stage as classically established in Theorem 2.2.1 by the zero terminal state constraint.

Theorem 2.3.1. *Suppose that the following hold:*

– *Assumptions* **A1∼4**.

– *A convex domain* $\mathcal{B}(\alpha, S) \subseteq \mathbf{X}$ *is given in which the system is contractible.*

– *Initial* $\mathcal{P}(x_0, \bar{x}_N, N)$ *with* $x_0, \ \bar{x}_N \in \mathcal{B}(\alpha, S)$ *is feasible.*

– *The origin is an interior point of* $\mathcal{X}(0, N_1)$.

Then Contraction Based Model Predictive Control Law asymptotically stabilizes the system.

Remark 2.3.5. *If some set* $D \subseteq \mathbf{X}$ *is a control invariant set for the system such as* $\mathcal{B}(\alpha, S)$, *using the tools of (Raković, Kerrigan, Mayne & Lygeros 2006), one might construct a larger set of states from which D is feasibly reachable in N steps. Hence, one may expand the domain of attraction.*

2.3.4 Use of Control Lyapunov Functions

The key to Contraction Based Model Predictive Control Law is to construct a sequence of finite horizon optimal control problems with contractive terminal target states in the sense of a weighted norm. The essential condition to achieve this is that the system be contractible. We now show how this might be relaxed by the knowledge of a Control Lyapunov Function.

Definition 2.3.4. *A scalar function* $\mathcal{V}(x)$ *is said to be a Control Lyapunov Function (CLF) for the system (2.1) in the domain* $\mathbf{X}^{\mathcal{V}} \subseteq \mathbf{X}$ *if the following hold:*

– *It is smooth.*

– *There exist* $\mathcal{K}_\infty$ *class functions* $W_1, \ W_2$ *such that, for all* $x \in \mathbf{X}^{\mathcal{V}}$,

$$W_1(||x||) \leq \mathcal{V}(x) \leq W_2(||x||).$$

– *There exists* $u \in \mathbf{U}$ *such that* $f(x, u) \in \mathbf{X}^{\mathcal{V}}$ *and*

$$\mathcal{V}(f(x, u)) - \mathcal{V}(x) \leq -\beta(||x||), \tag{2.5}$$

where β *is positive definite, continuous, and satisfies* $\beta(0) = 0$.

Contractibility in Definition 2.3.3 can be seen as using $x^T S x$ to be a CLF in the domain $\mathbf{X}^{\mathcal{V}} = \mathcal{B}(\alpha, S)$. This motivates the following control law.

Offline Preparation

`choose` control horizons for the first stage (N) and the second stage $(N + N_1)$,

`choose` any feasible $\bar{x}_N \in \mathbf{X}^{\mathcal{V}} \subseteq \mathbf{X}$ based on $x_0 \in \mathbf{X}^{\mathcal{V}} \subseteq \mathbf{X}$,

`compute` $\mathcal{X}(0, N_1) \in \mathbf{X}$.

Control Lyapunov Function Based Model Predictive Control Law

`set` time $k = 0$,

[First Stage]

`while` $\bar{x}_{k+N} \notin \mathcal{X}(0, N_1)$

        `solve` $\mathcal{P}(x_k, \bar{x}_{k+N}, N)$,

        `compute` $\bar{x}_{k+1+N} = \arg\min_{\tilde{x}} \{ \mathcal{V}(\tilde{x}) \big| \tilde{x} = f(\bar{x}_{k+N}, \bar{u}_{k+N}), \bar{u}_{k+N} \in \mathbf{U}, \ \tilde{x} \in \mathbf{X}^{\mathcal{V}} \}$,

        `apply` $u^*_{k|k}$ to the system: $x_{k+1} = x^*_{k+1|k} = f(x_k, u^*_{k|k})$,

        `update` $x_k \leftarrow x_{k+1}$ and $\bar{x}_{k+N} \leftarrow \bar{x}_{k+N+1}$,

`end`

[Second Stage]

`repeat`

        `solve` $\mathcal{P}(x_k, 0, N_1 + N)$,

        `apply` $u^*_{k|k}$ to the system,

        `update` $x_k \leftarrow x_{k+1}$,

`end`

Theorem 2.3.2. *Suppose that the following hold:*

  – *Assumptions* **A1~4**,

  – *There exists a CLF in the domain* $\mathbf{X}^{\mathcal{V}} \subseteq \mathbf{X}$,

  – *Initial* $\mathcal{P}(x_0, \bar{x}_N, N)$ *with* $x_0, \ \bar{x}_N \in \mathbf{X}^{\mathcal{V}} \subseteq \mathbf{X}$ *is feasible,*

  – *The origin is an interior point of* $\mathcal{X}(0, N_1)$.

Then Control Lyapunov Function Based Model Predictive Control Law asymptotically stabilizes the system.

Proof. It suffices to show that $\bar{x}_{k+N} \in \mathcal{X}(0, N_1)$ in finite time: the proof for feasibility and stability is obtained in an identical way to that of Lemma 2.3.1 and Theorem 2.2.1. Since there exists a CLF $\mathcal{V}(x)$ in the domain $\mathbf{X}^{\mathcal{V}}$, $\bar{x}_{k+1+N}$ given by

$$\bar{x}_{k+1+N} = \arg\min_{\tilde{x}}\{V(\tilde{x})\big|\tilde{x} = f(\bar{x}_{k+N}, \bar{u}_{k+N}), \bar{u}_{k+N} \in \mathbf{U}, \ \tilde{x} \in \mathbf{X}^{\mathcal{V}}\},$$

satisfies

$$\mathcal{V}(\bar{x}_{k+1+N}) \leq \mathcal{V}(\bar{x}_{k+N}) - \beta(||\bar{x}_{k+N}||),$$

where β is a positive definite function. By standard Lyapunov stability argument, this implies that $\bar{x}_{k+N} \to 0$ as $k \to \infty$. That is, given sufficiently small $\gamma > 0$ such that $\{x\big|||x|| \leq \gamma\} \subseteq \mathcal{X}(0, N_1)$, there exists an finite integer M_2 such that $||\bar{x}_{k+N}|| \leq \gamma$ from all $k \geq M_2$. This completes the proof. $\qquad\square$

2.3.5 Use of the Closed-loop State

We have presented two stabilizing MPC laws based on minimizing a positive definite function in determining the sequence of successive terminal target states. In the proposed algorithms, we see that the process to update the terminal target state could have been computed offline since it is determined solely by the positive definite function (e.g. CLF) and hence independent of the closed-loop state. Now we develop a more complicated algorithm, which can expand the achievable set of terminal target states. As we shall see, if we use the knowledge of the computed optimal state $x_{k+1|k}^*$ to determine $\bar{x}_{k+1+N}$, then the previous algorithms can be improved to accelerate convergence of the terminal target state to $\mathcal{X}(0, N_1)$. To facilitate this idea, we will rely on the $N-$step reachable set from $x_{k+1|k}^*$.

Definition 2.3.5. $\mathbb{X}_N^{fwd}(x_{k+1|k}^*)$ *is the set of states that are feasibly reachable from* $x_{k+1|k}^*$ *forward in N steps.*

The set, $\mathbb{X}_N^{fwd}(x_{k+1|k}^*)$, relies on information available at time k and so is denoted as dependent on $x_{k+1|k}^*$. The MPC algorithm is stated for an initial state $x_0 \in \mathcal{B}(\alpha, S) \subseteq \mathbf{X}$, where $\mathcal{B}(\alpha, S)$ is a domain in which the system is contractible.

Offline Preparation

`choose` control horizons for the first stage (N) and the second stage $(N + N_1)$,

`choose` any feasible $\bar{x}_N \in \mathcal{B}(\alpha, S)$ based on $x_0 \in \mathcal{B}(\alpha, S)$,

`compute` $\mathcal{X}(0, N_1) \in \mathbf{X}$.

State Dependent Contraction Based Model Predictive Control Law

`set` time $k = 0$,

[First Stage]

`while` $\bar{x}_{k+N} \notin \mathcal{X}(0, N_1)$

  `solve` $\mathcal{P}(x_k, \bar{x}_{k+N}, N)$,

  `compute` $\mathbb{X}_N^{fwd}(x_{k+1|k}^*)$,

  `compute` $\bar{x}_{k+1+N} = \arg\min_{\tilde{x}}\{||\tilde{x}||_S \big| \tilde{x} \in \mathbb{X}_N^{fwd}(x_{k+1|k}^*)\}$,

  `apply` $u_{k|k}^*$ to the system: $x_{k+1} = x_{k+1|k}^* = f(x_k, u_{k|k}^*)$,

  `update` $x_k \leftarrow x_{k+1}$ and $\bar{x}_{k+N} \leftarrow \bar{x}_{k+N+1}$,

`end`

[Second Stage]

`repeat`

  `solve` $\mathcal{P}(x_k, 0, N_1 + N)$,

  `apply` $u_{k|k}^*$ to the system,

  `update` $x_k \leftarrow x_{k+1}$,

`end`

To understand the effectiveness of this algorithm, recall the computation of $\bar{x}_{k+1+N}$ for given $\bar{x}_{k+N}$ at time k in Contraction Based Model Predictive Control Law:

$$\bar{x}_{k+1+N} = \arg\min_{\tilde{x}}\{||\tilde{x}||_S \big| \tilde{x} = f(\bar{x}_{k+N}, \bar{u}_{k+N}), \bar{u}_{k+N} \in \mathbf{U}\}.$$

This is equivalent to

$$\bar{x}_{k+1+N} = \arg\min_{\tilde{x}}\{||\tilde{x}||_S \big| \tilde{x} \in f(\bar{x}_{k+N}, \mathbf{U})\}, \tag{2.6}$$

where $f(\bar{x}_{k+N}, \mathbf{U}) \triangleq \{x | x = f(\bar{x}_{k+N}, \bar{u}_{k+N}), \forall \bar{u}_{k+N} \in \mathbf{U}\}$ is the one-step reachable set from $\bar{x}_{k+N}$. By the definition of $\mathbb{X}_N^{fwd}(x_{k+1|k}^*)$, we see that $\mathbb{X}_N^{fwd}(x_{k+1|k}^*) \supseteq f(\bar{x}_{k+N}, \mathbf{U})$.

Hence, the new construction of $\bar{x}_{k+1+N}$ by

$$\bar{x}_{k+1+N} = \arg\min_{\tilde{x}}\{||\tilde{x}||_S \big| \tilde{x} \in \mathbb{X}_N^{fwd}(x_{k+1|k}^*)\} \tag{2.7}$$

results in at least

$$||\bar{x}_{k+1+N}||_S \le \rho||\bar{x}_{k+N}||_S. \tag{2.8}$$

Therefore, if the first stage of State Dependent Contraction Based Model Predictive Control Law is feasibly executed over time, then the system may be stabilized faster by the above MPC law than Contraction Based Model Predictive Control Law.

Theorem 2.3.3. *Suppose that the following hold:*

- *Assumptions* **A1~4**.

- *A convex domain* $\mathcal{B}(\alpha, S) \subseteq \mathbf{X}$ *is given in which the system is contractible.*

- *Initial* $\mathcal{P}(x_0, \bar{x}_N, N)$ *with* $x_0, \; \bar{x}_N \in \mathcal{B}(\alpha, S)$ *is feasible.*

- *The origin is an interior point of* $\mathcal{X}(0, N_1)$.

Then the State Dependent Contraction Based Model Predictive Control Law asymptotically stabilizes the system.

Proof. Since $\bar{x}_{k+1+N}$ is chosen in the $N-$step reachable set from $x_{k+1|k}^*$, $\mathbb{X}_N^{fwd}(x_{k+1|k}^*)$, feasibility is guaranteed at each time step and this is enough to complete the proof. □

If the first stage operates feasibly then finite time termination of the first stage is immediate by the fact that the newly chosen terminal target state $\bar{x}_{k+1+N}$ achieves at least (2.8). Then the system is asymptotically stabilized by the second stage.

Remark 2.3.6. *In the above MPC law, the computational burden may be an issue for systems with high dimensions and short sampling time because* $\mathbb{X}_N^{fwd}(x_{k+1|k}^*)$ *has to be computed at each time step. This idea is tractable at least for a linear system with polytope constraints using the tools of (Kvasnica, Grieder & Baotić 2004) and Section 2.4.*

Remark 2.3.7. *Of course, a CLF $\mathcal{V}(x)$ can be used, if it exists, in the domain $\mathbf{X}^{\mathcal{V}} \subseteq \mathbf{X}$. By replacing (2.7) with*

$$\bar{x}_{k+1+N} = \arg\min_{\tilde{x}}\{\mathcal{V}(\tilde{x})\big|\tilde{x} \in \mathbb{X}_N^{fwd}(x^*_{k+1|k}) \cap \mathbf{X}^{\mathcal{V}}\}, \tag{2.9}$$

one can still construct an MPC law. This also may give faster termination time of the first stage than that of the first stage in Control Lyapunov Function Based Model Predictive Control Law. This approach deserves comparison with the control laws proposed in (Kwon & Han 2005) as well as (Freeman & Kokotović 1995) for continuous time systems. In (Kwon & Han 2005), a CLF is used to develop a state constraint. That is, for the given state $x_{k+1} = x^*_{k+1|k}$ at time $k+1$, they constrain the open-loop control to achieve

$$\mathcal{V}(x_{k+i+1|k+1}) \leq \mathcal{V}(x_{k+i|k+1}) - \beta(||x_{k+i|k+1}||), \tag{2.10}$$

for the complete horizon $i = 1, 2, \ldots, N$ or just for the next step $i = 1$. In other words, their open-loop control problem only requires the predicted states $x_{k+i+1|k+1}$ to satisfy (2.10). In our case, for the given $x^*_{k+1|k}$, we choose the terminal target state $\bar{x}_{k+N+1}$ by (2.9) to cause the greatest reduction in $\mathcal{V}(\cdot)$ and find an open-loop control solution for $x_{k+1} = x^*_{k+1|k}$ to reach $\bar{x}_{k+1+N}$. Thus, our method may lead to faster stabilization of the system than the MPC law of (Kwon & Han 2005).

2.3.6 Example: Hovercraft Control

We use a hovercraft model to demonstrate the system stabilization by our MPC algorithms: Contraction Based Model Predictive Control and State Dependent Contraction Based Model Predictive Control. These cases include taking the weighted state norm as a CLF. We use the model of HotDeC (HOvercraft Testbed for DEcentralized Control) whose details are presented in Appendix A. The vehicle model has the state vector

$$[x_{p,k}\ y_{p,k}\ \theta_{p,k}\ x_{s,k}\ y_{s,k}\ \theta_{s,k}]^T.$$

The elements $x_{p,k}$, $y_{p,k}$, $x_{s,k}$, and $y_{s,k}$ represent positions and velocities in Cartesian coordinates respectively. The variables $\theta_{p,k}$ and $\theta_{s,k}$ are angular displacement and velocity of the vehicle respectively. Each coordinate in the model is independent. Therefore, for

simplicity, we consider only the $x-$coordinate of the vehicle given by

$$\begin{bmatrix} x_{p,k+1} \\ x_{s,k+1} \end{bmatrix} = \begin{bmatrix} 1 & 0.3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_{p,k} \\ x_{s,k} \end{bmatrix} + \begin{bmatrix} 0.01 \\ 0.09 \end{bmatrix} u_k. \tag{2.11}$$

We assume that the model (2.11) is a precise description of the vehicle in the $x-$coordinate and that perfect state measurement is available. The control u_k is constrained to the set

$$\mathbf{U} = [-5, \ 5]. \tag{2.12}$$

Let us denote $[x_{p,k} \ x_{s,k}]^T$ by x_k. The goal, for a given initial state x_0, is to steer the vehicle to the zero state using the proposed MPC algorithms. At each time step k, we solve the following open-loop control problem,

$\mathcal{P}(x_k, \bar{x}_{k+N}, \bar{N}) :$

$$\arg \min_{\{u_{k|k},...,u_{k+\bar{N}-1|k}\}} \sum_{i=0}^{\bar{N}-1} x_{k+i|k}^T Q_c x_{k+i|k} + u_{k+i|k}^T R_c u_{k+i|k},$$

subject to

$$x_{k+i+1|k} = \begin{bmatrix} 1 & 0.3 \\ 0 & 1 \end{bmatrix} x_{k+i|k} + \begin{bmatrix} 0.01 \\ 0.09 \end{bmatrix} u_{k+i|k}, \tag{2.13}$$

$$u_{k+i|k} \in \mathbf{U},$$

$$x_{k+N|k} = \bar{x}_{k+N},$$

with $Q_c=I$ and $R_c=100$. For the given initial state $x_0 = [10 \ 0]^T$, two cases are considered:

Case 1 The vehicle is controlled by the Contraction Based Model Predictive Control Law (CBMPC).

Case 2 The vehicle is controlled by the State Dependent Contraction Based Model Predictive Control Law (SD-CBMPC).

For both cases, we choose $\bar{N} = N = 4$ and $\bar{N} = N + N_1 = 7$ ($N_1 = 3$) as control horizons for the first and second stages respectively. The set $\mathcal{X}(0,3)$ is shown in Figure 2.3. We choose $\bar{x}_N = [9.4 \ -1]^T$ as an initial feasible terminal target state for x_0. Note that the given initial state x_0 can never be steered to the origin within the fixed horizon $\bar{N} = 7$: the closest to the origin it can come in seven steps is $x_p = 5.82$. From the initial
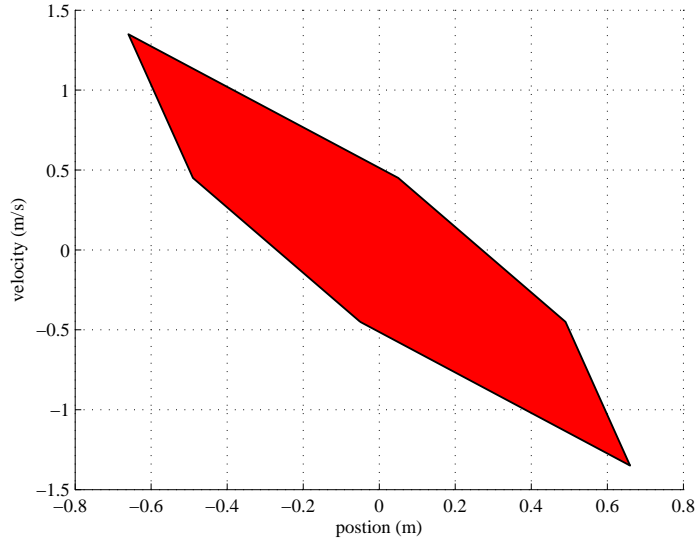
Figure 2.3: Graphical depiction of $\mathcal{X}(0,3)$.

target terminal state $\bar{x}_N$, the system is contractible in the euclidean norm sense. [It is still possible to consider the weighted norm $||x||_S$ to update $\bar{x}_{k+N}$ with an appropriately chosen S.] Once the terminal target state reaches the region shown in Figure 2.3, the MPC algorithms of both cases will switch to the second stage with zero terminal target state and the horizon $\bar{N} = 7$. Results are shown in Table 2.1 and Figure 2.3.6 and 2.5.

Table 2.1: Time comparison between the two methods.

	CBMPC	SD-CBMPC
First Stage Termination Time	31.8 sec	3.9 sec
Time for x_k inside $\{x|||x|| \leq 0.0001\}$	35.1 sec	9.3 sec

2.3.7   Example: Integrator

A simple scalar integrator is employed to show how our scheme can be beneficial to stabilization performance and tuning the system behavior. We consider a simple scalar integrator

$$x_{k+1} = x_k + u_k, \tag{2.14}$$
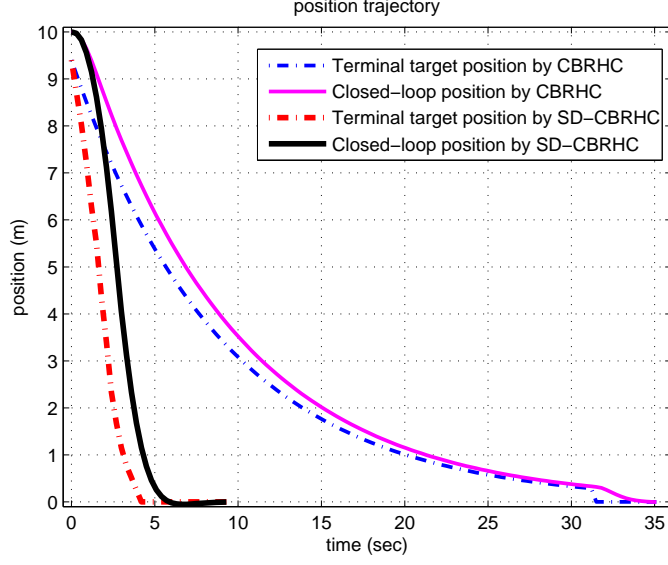
Figure 2.4: Time trajectories of the vehicle position by the two methods. The dot-dashed lines represent the terminal target position $\bar{x}_{p,k+N}$ over time.

since its behavior is quite easy and hence to appreciate the performance enhancement of our scheme. We assume 0.2 second sampling time just for visualization of the results over time. The input constraint is assumed to remain as in (2.12). The objective of control is to steer the initial state x_0 to the origin using Contraction Based Model Predictive Control Law. Consider the following open-loop control problem at time k,

$$\mathcal{P}(x_k, \bar{x}_{k+N}, \bar{N}) :$$

$$\arg \min_{\{u_{k|k}, \ldots, u_{k+\bar{N}-1|k}\}} \sum_{i=0}^{\bar{N}-1} x_{k+i|k}^T Q_c x_{k+i|k} + u_{k+i|k}^T R_c u_{k+i|k},$$

$$subject\ to \tag{2.15}$$

$$x_{k+i+1|k} = x_{k+i|k} + u_{k+i|k},$$

$$u_{k+i|k} \in \mathbf{U},$$

$$x_{k+N|k} = \bar{x}_{k+N},$$

with $Q_c = 1$. Here we use two different weightings R_c: $R_c = 0$ and $R_c = 100$. The control horizons $\bar{N} = N = 3$ and $\bar{N} = N + N_1 = 6$ ($N_1 = 3$) are chosen for the first and second stages respectively. The set $\mathcal{X}(0,3)$ is $[-15, 15]$. The initial state is given by
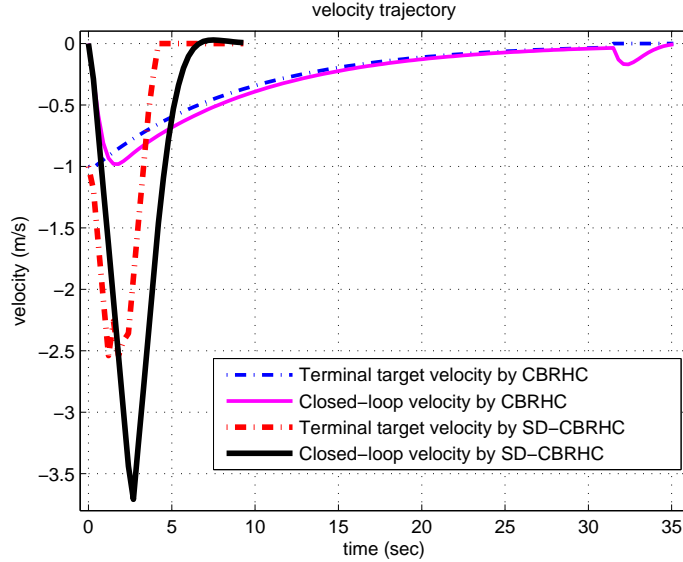
Figure 2.5: Time trajectory of the vehicle velocity. The dot-dashed lines represent the terminal target velocity $\bar{x}_{s,k+N}$ over time.

$x_0 = 50$ and the initial target terminal target state is chosen to be $\bar{x}_N = 42$. The system is contractible in the sense of the euclidean norm, which can be viewed as having a CLF $\mathcal{V}(x) = x^2$. If MPC is not considered, we might think of a nominal feedback control law $u_k = -0.1x_k$ based on $\mathcal{V}(x)$, which does not violate the control constraint. However, as shown in Figure 2.6, the state trajectory by the nominal feedback law converges to the origin much more slowly. In this particular setting, for the given $\bar{x}_{k+N}$, $u = -5$ minimizes $\mathcal{V}(\bar{x}_{k+N} + u)$ and results in the terminal target state trajectory $\{42, 37, 32, \ldots, 17\}$ until $\bar{x}_{k+N}$ enters $\mathcal{X}(0,3)$. Without control penalty R, one can expect that the MPC will allow the full control capacity to steer the state, which results in a straight-line-like trajectory as shown in Figure 2.6. The above results suggest that MPC combined with a CLF may lead to better control performance than a nominal control solely based on the known CLF.
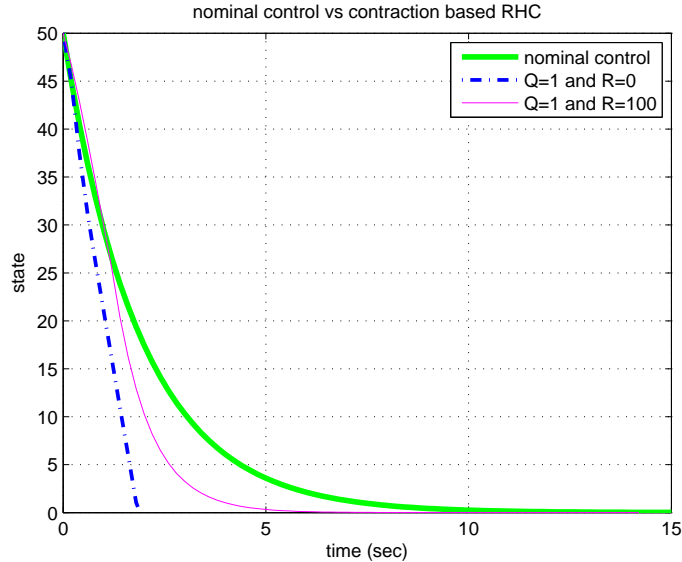
Figure 2.6: State trajectories by the nominal control and by Contraction Based Receding Horizon Control.

2.4 Online Reference Computation for Feasible Model Predictive Control

In the previous section, we discussed how we can stablize the system

$$x_{k+1} = f(x_k, u_k), \tag{2.16}$$

whose initial state is hard to be steered to the origin in a limited horizon. The core idea was constructing a sequence of feasible finite horizon optimal control problems with manipulated terminal target states. The proposed algorithms require to compute sets such as rechable sets. Here, we concentrate on computational issues associated with constructing feasible MPC

We denote the reference and the input sequences from time i to j by $r_i^j \triangleq \{r_i, r_{i+1}, \ldots, r_j\}$ and $u_i^j \triangleq \{u_i, u_{i+1}, \ldots, u_j\}$ respectively. The reference sequence r_i^j will be equivalently written as $r_i^j = \{r_i, r_{i+1}^j\} = \{r_i^{j-1}, r_j\}$. We denote the input sequence given in one-vector form by $\bar{u}_i^j \triangleq \left[u_i{}^T \ u_{i+1}{}^T \ \ldots \ u_j{}^T \right]^T$. For two vector x and y, $x > y$ and $x \geq y$ will be taken elementwise. For a given scalar s, $\bar{s}$ denotes the vector of s (i.e. $\bar{s} \triangleq s[\, 1 \ 1 \ \ldots \ 1\,]^T$) of an appropriate dimension. For the given set of ψ points,

$E = \{e^1, e^2, \ldots, e^\psi\}$, we denote by $conv(E)$ the convex hull of E. The expression $vert(F)$ represents the vertex set of convex set F and $f^\star$ represents a vertex.

2.4.1 Problems

We first state two problems using the system

$$x_{k+1} = f(x_k, u_k),$$

with perfect state measurement. Then, for simplicity, clarity, and computation, we consider a linear time invariant system.

Problem 1: Terminal state equality constraint case

Using the following finite horizon optimal control problem,

$$\mathcal{P}(x_k, \bar{x}_{k+N}, N):$$

$$\arg \min_{\{u_{k|k}, \ldots, u_{k+N-1|k}\}} \sum_{i=0}^{N-1} h(x_{k+i|k}, u_{k+i|k}),$$

subject to $\qquad\qquad$ (2.17)

$$x_{k+i+1|k} = f(x_{k+i|k}, u_{k+i|k}),$$

$$u_{k+i|k} \in \mathbf{U},$$

$$x_{k+N|k} = \bar{x}_{k+N},$$

consider the following MPC for an initial state x_0:

`set` time $k = 0$,

`repeat`

$\qquad$ `solve` $\mathcal{P}(x_k, \bar{x}_{k+N}, N)$,

$\qquad$ `apply` the optimal control $u^*_{k|k}$ to the system: $x_{k+1} = x^*_{k+1|k} = f(x_k, u^*_{k|k})$,

$\qquad$ `update` $x_k \leftarrow x_{k+1}$, $\bar{x}_{k+N} \leftarrow x_{k+1+N}$,

`end`,

which would be feasibly executed for all time provided that a feasible terminal state $\bar{x}_{i+N}$ is given for all i. With x_k and $\bar{x}_{k+N}$ given, consider $\mathcal{P}(x_k, \bar{x}_{k+N}, N)$. Assume that there exists a feasible but unknown solution at time k. Find the set, $\bar{\mathfrak{X}}_{k+N+1}(x_k, \bar{x}_{k+N})$, of subsequent terminal states $\bar{x}_{k+1+N}$ such that $\mathcal{P}(x_{k+1}, \bar{x}_{k+1+N}, N)$ is also feasible for any $\bar{x}_{k+1+N} \in \bar{\mathfrak{X}}_{k+1+N}(x_k, \bar{x}_{k+N})$.

Remark 2.4.1. *In State Dependent Contraction Based Model Predictive Control, we used current state $x_{k+1} = x_{k+1|k}$ to decide $\bar{x}_{k+N}$. However, in this problem, we attempt to obtain a set of all feasible $\bar{x}_{k+1+N}$ by only using the fact that $\mathcal{P}(x_k, \bar{x}_{k+N}, N)$ is feasible.*

In addition to the MPC with the terminal state equality constraint, we also consider an MPC problem whose state constraints are defined using reference trajectory.

Problem 2: Reference dependent state constraint case

Using the following optimal control problem

$$\mathcal{P}(x_k, r_{k+1}^{k+N}, N):$$

$$\arg \min_{\{u_{k|k}, \ldots, u_{k+N-1|k}\}} \sum_{i=0}^{N-1} h(x_{k+i|k}, u_{k+i|k}),$$

$$\textit{subject to} \tag{2.18}$$

$$x_{k+i+1|k} = f(x_{k+i|k}, u_{k+i|k}),$$

$$u_{k+i|k} \in \mathbf{U},$$

$$x_{k+i|k} \in \mathbf{X}_{k+i}(r_{k+i}),$$

consider the following MPC for an initial state x_0:

`set` time $k = 0$,

`repeat`

    `solve` $\mathcal{P}(x_k, r_{k+1}^{k+N}, N)$,

    `apply` the optimal control $u_{k|k}^*$ to the system: $x_{k+1} = x_{k+1|k}^* = f(x_k, u_{k|k}^*)$,

    `update` $x_k \leftarrow x_{k+1}$, $r_{k+1}^{k+N} \leftarrow r_{k+2}^{k+N+1}$

`end,`

where, by permissible deviation from the reference, $\mathbf{X}_{k+i}(r_{k+i})$ are defined in relation to a reference sequence $r_{k+1}^{k+N} = \{r_{k+1}, r_{k+2}^{k+N}\}$. With x_k given, suppose that there exists a feasible optimal solution $u_{k|k,*}^{k+N-1|k} \triangleq \{u_{k|k}^*, u_{k+1|k}^*, \ldots, u_{k+N-1|k}^*\}$ for $\mathcal{P}(x_k, r_{k+1}^{k+N}, N)$ and the corresponding next state x_{k+1} is available. Consider the next problem $\mathcal{P}(x_{k+1}, r_{k+2}^{k+N+1}, N)$ with extended reference sequence $r_{k+2}^{k+N+1} = \{r_{k+2}^{k+N}, r_{k+N+1}\}$, where r_{k+N+1} is the sole variable to be chosen. Find a set of r_{k+N+1} such that this problem is feasible.

In the sequel, we will consider a linear time invariant system and a polytope

for the input constraint:

$$x_{k+1} = f(x_k, u_k) = Ax_k + Bu_k,$$

$$u_{k+i-1} \in \mathbf{U} = \{u_{k+i-1} \big| \parallel u_{k+i-1} \parallel_\infty \leq c\} \tag{2.19}$$

$$= conv\{u_{k+i-1}^{\star,1}, u_{k+i-1}^{\star,2}, \ldots, u_{k+i-1}^{\star,2^m}\},$$

where $||\cdot||_\infty$ is the vector infinity norm. Note that the vertices $u_{k+i-1}^{\star,i}$ are readily obtained since $\mathbf{U}$ is symmetric about the origin. The state equality constraint in (2.17) is given by

$$\bar{x}_{k+N} = A^N x_k + \sum_{i=1}^{N} A^{N-i} Bu_{k+i-1}. \tag{2.20}$$

and, for the state constraints in (2.18), we use

$$x_{k+i} \in \mathbf{X}_{k+i}(r_{k+i}) = \{x_{k+i} \big| \parallel r_{k+i} - x_{k+i} \parallel_\infty \leq d\}. \tag{2.21}$$

What is nice about (2.19), (2.20), and (2.21) is that, if set operations associated with them result in convex sets, then they can be given in terms of a set of inequalities or a convex hull of vertices. We also note that the constraints associated with r_{k+i} and d do not have to be in the form (2.21) as long as they are in the form of linear inequalities which can be equivalently given as a convex hull of vertices. We use (2.21) for the ease of representation.

Remark 2.4.2. *Our focus here is entirely on feasibility and so, not surprisingly, the optimization objectives in (2.17) and (2.18) will not appear in our later analysis.*

2.4.2 Algorithm: Terminal state equality constraint case

In order to construct an algorithm for this problem, we first need to understand implication of the feasible problem $\mathcal{P}(x_k, \bar{x}_{k+N}, N)$ of (2.17) for given x_k and $\bar{x}_{k+N}$. Using the linear system and the symmetric input constraint in (2.19) and the state constraint (2.20), we define the following sets:

$$\mathbb{X}_1^{fwd}(x_k) = \{x | x = Ax_k + Bu_k, \forall u_k \in \mathbf{U}\}$$

$$= \{x | \Omega^f(x - Ax_k) \leq b^f\}, \tag{2.22}$$

$$\mathbb{X}_{N-j}^{bwd}(\bar{x}_{k+N}) = \{x | \bar{x}_{k+N} = A^{N-j}x + \sum_{i=j+1}^{N} A^{N-i}Bu_{k+i-1},$$

$$\forall u_{k+i-1} \in \mathbf{U}, \forall i = j+1, j+2, \ldots, N\} \tag{2.23}$$

$$= \{x | \Omega_{N-j}^{b}(\bar{x}_{k+N} - A^{N-j}x) \leq b_{N-j}^{b}\}.$$

– $\mathbb{X}_1^{fwd}(x_k)$ is the set of all x_{k+1} evaluated by all $u_k \in \mathbf{U}$ forward in time for the given x_k.

– $\mathbb{X}_{N-j}^{bwd}(\bar{x}_{k+N})$ is the set of all x_{k+j} evaluated $(N-j)-$steps backward in time for the given $\bar{x}_{k+N}$ by all $u_{k+i-1} \in \mathbf{U}$, $\forall i = j+1, j+2, \ldots, N$.

– The matrices Ω^f and b^f are obtained by writing $x - Ax_k$ in terms of feasible u_k. Likewise, Ω_{N-j}^b and b_{N-j}^b are obtained by the state evolution for all feasible u_{k+i-1}, $\forall i = j+1, j+2, \ldots, N$. If $x_k \in \mathbb{R}^1$ and $u_k \in \mathbf{U} = [-c, c]$, then

$$\mathbb{X}_1^{fwd}(x_k) = \{x | -Bc \leq x - Ax_k \leq Bc\}$$

$$= \left\{ x \middle| \begin{bmatrix} 1 \\ -1 \end{bmatrix} (x - Ax_k) \leq \begin{bmatrix} Bc \\ Bc \end{bmatrix} \right\},$$

$$\mathbb{X}_{N-j}^{bwd}(\bar{x}_{k+N}) = \left\{ x \middle| -\sum_{i=j+1}^{N} A^{N-i}Bc \leq \bar{x}_{k+N} - A^{N-j}x \leq \sum_{i=j+1}^{N} A^{N-i}Bc \right\}$$

$$= \left\{ x \middle| \begin{bmatrix} 1 \\ -1 \end{bmatrix} (\bar{x}_{k+N} - A^{N-j}x) \leq \begin{bmatrix} \sum_{i=j+1}^{N} A^{N-i}Bc \\ \sum_{i=j+1}^{N} A^{N-i}Bc \end{bmatrix} \right\}.$$

– Once the control horizon N is fixed, then Ω^f, b^f, Ω_{N-j}^b, and b_{N-j}^b are fixed.

Remark 2.4.3. *Reconsider Section 2.3 with the linear system (2.19). From $\mathbb{X}_1^{fwd}(x_k)$, we see that the set $f(\bar{x}_{k+N}, \mathbf{U})$ of (2.6) will have the same structure of $\mathbb{X}_1^{fwd}(x_k)$. Furthermore, the N-step reachable set $\mathbb{X}_N^{fwd}(x_{k+1|k}^*)$ in State Dependent Contraction Based Model Predictive Control Law will be in the form of linear inequality that is similar to $\mathbb{X}_1^{fwd}(x_k)$.*

Define $\mathbb{X}_{k+1}^{feas}(x_k, \bar{x}_{k+N}) \triangleq \mathbb{X}_1^{fwd}(x_k) \cap \mathbb{X}_{N-1}^{bwd}(\bar{x}_{k+N})$ to be the set of all x_{k+1}, which are reachable from both x_k and $\bar{x}_{k+N}$. Assuming feasibility at time k, the following statement is true.

Theorem 2.4.1. *For given x_k and $\bar{x}_{k+N}$, suppose that $\mathcal{P}(x_k, \bar{x}_{k+N}, N)$ of (2.17) is feasible. Then $\mathbb{X}_{k+1}^{feas}(x_k, \bar{x}_{k+N}) \neq \emptyset$.*

Proof. If $\mathbb{X}_{k+1}^{feas}(x_k, \bar{x}_{k+N}) = \emptyset$ then this implies no feasible solution. This contradicts the assumption. □

The algorithm is now stated and details of each step will be followed.

Algorithm for the terminal state equality constraint problem

Step 1. Compute $\mathbb{X}_{k+1}^{feas}(x_k, \bar{x}_{k+N})$.

Step 2. Compute $\bar{\mathfrak{X}}_{k+1+N}(x_k, \bar{x}_{k+N})$, the set of all $\bar{x}_{k+1+N}$ whose N-step backward sets

$$\mathbb{X}_N^{bwd}(\bar{x}_{k+1+N}) = \{x | \Omega_N^b(\bar{x}_{k+1+N} - A^N x) \le b_N^b\} \tag{2.24}$$

contains $\mathbb{X}_{k+1}^{feas}(x_k, \bar{x}_{k+N})$.

The graphical explanation is given in Figure 2.7. Since any $\bar{x}_{k+1+N} \in \bar{\mathfrak{X}}_{k+1+N}(x_k, \bar{x}_{k+N})$ is reachable from any $x_{k+1} \in \mathbb{X}_{k+1}^{feas}(x_k, \bar{x}_{k+N})$, whatever the solution at time k, if any $\bar{x}_{k+1+N} \in \bar{\mathfrak{X}}_{k+1+N}(x_k, \bar{x}_{k+N})$ is chosen then $\mathcal{P}(x_{k+1}, \bar{x}_{k+N+1}, N)$ with $x_{k+1} = x_{k+1|k}^*$ is feasible.
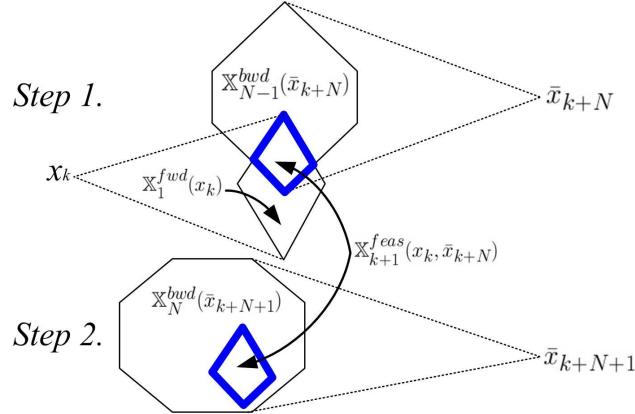
Figure 2.7: Graphical description of the algorithm.

Step 1:

Using (2.22) and (2.23), $\mathbb{X}_{k+1}^{feas}(x_k, \bar{x}_{k+N})$ is given by

$$\mathbb{X}_{k+1}^{feas}(x_k, \bar{x}_{k+N}) = \left\{ x \middle| \begin{bmatrix} \Omega^f \\ -\Omega_{N-1}^b A^{N-1} \end{bmatrix} x \le \begin{bmatrix} \Omega^f A x_k + b^f \\ -\Omega_{N-1}^b \bar{x}_{k+N} + b_{N-1}^b \end{bmatrix} \right\} \tag{2.25}$$

$$= conv\{x_{k+1}^{\star,1}, x_{k+1}^{\star,2}, \ldots, x_{k+1}^{\star,q}\}.$$

Note that, since it is a numerical procedure to obtain vertices from the set of inequalities, the number of vertices varies all the time.

Step 2:

Using the vertices of $\mathbb{X}_{k+1}^{feas}(x_k, \bar{x}_{k+N})$, we obtain $\bar{\mathfrak{X}}_{k+1+N}(x_k, \bar{x}_{k+N})$ as follows.

Theorem 2.4.2. *For given x_k and $\bar{x}_{k+N}$, suppose that $\mathcal{P}(x_k, \bar{x}_{k+N}, N)$ is feasible. Then the set of all feasible $\bar{x}_{k+1+N}$ at time $k+1$ is given by*

$$
\bar{\mathfrak{X}}_{k+1+N}(x_k, \bar{x}_{k+N}) = \left\{ x \left| \begin{bmatrix} \Omega_N^b \\ \Omega_N^b \\ \vdots \\ \Omega_N^b \end{bmatrix} x \leq \begin{bmatrix} b_N^b + \Omega_N^b A^N x_{k+1}^{\star,1} \\ b_N^b + \Omega_N^b A^N x_{k+1}^{\star,2} \\ \vdots \\ b_N^b + \Omega_N^b A^N x_{k+1}^{\star,q} \end{bmatrix} \right. \right\}, \tag{2.26}
$$

and it is not empty.

Proof. The set (2.26) is the collection of all $\bar{x}_{k+1+N}$ whose $\mathbb{X}_N^{bwd}(\bar{x}_{k+1+N})$ of (2.24) contains all the vertices of $\mathbb{X}_{k+1}^{feas}(x_k, \bar{x}_{k+N})$, which is equivalent to containing all $x_{k+1} \in \mathbb{X}_{k+1}^{feas}(x_k, \bar{x}_{k+N})$. To prove that $\bar{\mathfrak{X}}_{k+1+N}(x_k, \bar{x}_{k+N})$ is not empty, we will show its subset is not empty. If the MPC at time k is feasible, $\mathbb{X}_{k+1}^{feas}(x_k, \bar{x}_{k+N})$ exists and every element in the set can reach $\bar{x}_{k+N}$ in $(N-1)$ steps. We can construct the set $\mathbb{X}_1^{fwd}(\bar{x}_{k+N})$ which is reachable from $\bar{x}_{k+N}$ in one step. This implies that $\mathbb{X}_1^{fwd}(\bar{x}_{k+N})$ is reachable from any $x_{k+1} \in \mathbb{X}_{k+1}^{feas}(x_k, \bar{x}_{k+N})$ in N steps. Thus the set $\bar{\mathfrak{X}}_{k+1+N}(x_k, \bar{x}_{k+N}) \supseteq \mathbb{X}_1^{fwd}(\bar{x}_{k+N})$ is not empty. $\qquad\square$

2.4.3 Algorithm: Reference dependent state constraint case

To understand the problem further, the following definition is useful.

Definition 2.4.1. Feasible Input Set

For given x_k and r_{k+1}^{k+N}, define the Feasible Input Set of the MPC problem (2.18):

$$\mathbb{U}_k^{k+N-1}(x_k, r_{k+1}^{k+N}) \triangleq \{ \bar{u}_k^{k+N-1} \big| x_{j|k} \in \mathbf{X}_{k+i}(r_{k+i}) \ \forall j = k+1, \ldots, k+N \ and \ \bar{u}_k^{k+N-1} \in \mathbf{U}^N \},$$

where $\mathbf{U}^N$ denotes the N product of $\mathbf{U}$.

For given x_k and r_{k+1}^{k+N}, if there exists a feasible solution for the $N-$step optimal control problem $\mathcal{P}(x_k, r_{k+1}^{k+N}, N)$, then the $(N-1)-$step problem $\mathcal{P}(x_{k+1}, r_{k+2}^{k+N}, N-1)$ with $x_{k+1} = f(x_k, u_{k|k}^*)$ and r_{k+2}^{k+N} also has a feasible solution. That is

Lemma 2.4.1. *If $\mathbb{U}_k^{k+N-1}(x_k, r_{k+1}^{k+N}) \neq \emptyset$ then $\mathbb{U}_{k+1}^{k+N-1}(x_{k+1}, r_{k+2}^{k+N}) \neq \emptyset$.*

Consider $\mathbb{U}_{k+1}^{k+N}(x_{k+1}, r_{k+2}^{k+N+1})$ with x_{k+1} and $r_{k+2}^{k+N+1} = \{r_{k+2}^{k+N}, r_{k+N+1}\}$, where r_{k+N+1} is to be chosen. One way to solve the problem is to choose r_{k+N+1} in such a way that, by feasible $u_{k+N} \in \mathbf{U}$, $\mathbf{X}_{k+N+1}(r_{k+1+N})$ is reachable from x_{k+N} which is driven by some $\bar{u}_{k+1}^{k+N-1} \in \mathbb{U}_{k+1}^{k+N-1}(x_{k+1}, r_{k+2}^{k+N})$. Then $\mathbb{U}_{k+1}^{k+N}(x_{k+1}, r_{k+2}^{k+N+1})$ will not be empty and, therefore, $\mathcal{P}(x_{k+1}, r_{k+2}^{k+N+1}, N)$ at time $k+1$ will be feasible. For given x_k and r_{k+1}^{k+N}, consider $\mathcal{P}(x_k, r_{k+1}^{k+N}, N)$ of (2.18) with the linear system and the symmetric input constraint in (2.19) and the state constraints (2.21)

$$u_{k+i-1} \in \mathbf{U} = \{u_{k+i-1} \mid \|\, u_{k+i-1}\,\|_\infty \leq c\}$$
$$= conv\{u_{k+i-1}^{\star,1}, u_{k+i-1}^{\star,2}, \ldots, u_{k+i-1}^{\star,2m}\},$$
$$x_{k+i} \in \mathbf{X}_{k+i}(r_{k+i}) = \{x_{k+i} \mid \|\, r_{k+i} - x_{k+i}\,\|_\infty \leq d\}$$
$$= \{x_{k+i} \mid -\bar{d} + r_{k+i} \leq x_{k+i} \leq \bar{d} + r_{k+i}\}.$$

Since the system is linear, the state constraint becomes

$$-\bar{d} + r_{k+i} \leq A^i x_k + A^{i-1} B u_k + \cdots + B u_{k+i-1} \leq \bar{d} + r_{k+i}, \qquad (2.27)$$

which shows that the state constraint (2.21) can be seen as constraints on inputs. By combining all the constraints in (2.19) and (2.27), we have the feasible input set

$$\mathbb{U}_k^{k+N-1}(x_k, r_{k+1}^{k+N}) = \{\bar{u}_k^{k+N-1} \mid \mathcal{A}\bar{u}_k^{k+N-1} \leq b(x_k, r_{k+1}^{k+N})\}, \qquad (2.28)$$

where $\mathcal{A}$ and $b(x_k, r_{k+1}^{k+N})$ are given in (2.29). Once the control horizon N is fixed then $\mathcal{A}$ is fixed but $b(x_k, r_{k+1}^{k+N})$ is not since we use r_{k+2}^{k+N+1} at time $k+1$. If $\mathcal{P}(x_k, r_{k+1}^{k+N}, N)$ is feasible, $\mathbb{U}_k^{k+N-1}(x_k, r_{k+1}^{k+N})$ is not empty. It is a convex polytope since it is given as a combination of linear inequalities.

Algorithm for the reference dependent state constraint case

Step 1. For given $x_{k+1} = f(x_k, u_{k|k}^*)$, compute $\mathbb{U}_{k+1}^{k+N-1}(x_{k+1}, r_{k+2}^{k+N})$, the set of all feasible solutions $\bar{u}_{k+1}^{k+N-1}$.

Step 2. Compute $\mathbb{X}_{k+N}^{feas}(x_{k+1})$, the set of all x_{k+N} which are driven from x_{k+1} by all $\bar{u}_{k+1}^{k+N-1} \in \mathbb{U}_{k+1}^{k+N-1}(x_{k+1}, r_{k+2}^{k+N})$.

$$
\mathcal{A}=\begin{bmatrix}
B & 0 & \dots & 0 \\
AB & B & \dots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
A^{N-1}B & A^{N-2}B & \dots & B \\
-B & 0 & \dots & 0 \\
-AB & -B & \dots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
-A^{N-1}B & -A^{N-2}B & \dots & -B \\
I & 0 & \dots & 0 \\
0 & I & \dots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \dots & I \\
-I & 0 & \dots & 0 \\
0 & -I & \dots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \dots & -I
\end{bmatrix}
,\; b(x_k, r_{k+1}^{k+N})=\begin{bmatrix}
-Ax_k + \bar{d} + r_{k+1} \\
-A^2 x_k + \bar{d} + r_{k+2} \\
\vdots \\
-A^N x_k + \bar{d} + r_{k+N} \\
Ax_k + \bar{d} - r_{k+1} \\
A^2 x_k + \bar{d} - r_{k+2} \\
\vdots \\
A^N x_k + \bar{d} - r_{k+N} \\
\bar{c} \\
\bar{c} \\
\vdots \\
\bar{c} \\
\bar{c} \\
\bar{c} \\
\vdots \\
\bar{c}
\end{bmatrix}. \quad (2.29)
$$

Step 3. Compute $\mathbb{X}_{k+N+1}^{reach}$, the set of all x_{k+N+1} which are reachable from some $x_{k+N} \in \mathbb{X}_{k+N}^{feas}(x_{k+1})$ by some $u_{k+N} \in \mathbf{U}$.

Step 4. Dilate $\mathbb{X}_{k+N+1}^{reach}$ by d in every coordinate to get $\mathcal{R}_{k+N+1}$, the set of all r_{k+N+1} such that $x_{k+N+1} \in \mathbf{X}_{k+N+1}(r_{k+N+1})$ for some $x_{k+N+1} \in \mathbb{X}_{k+N+1}^{reach}$ (i.e. $\mathbb{U}_{k+1}^{k+N}(x_{k+1}, r_{k+2}^{k+N+1}) \neq \emptyset$).
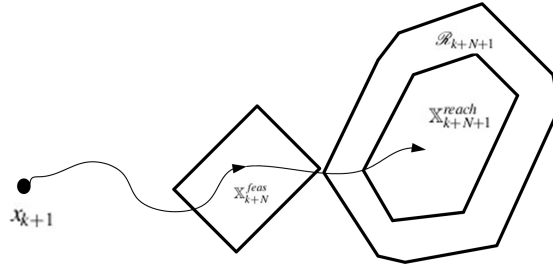

Figure 2.8: Schematic description of the algorithm.

The illustrative figure is shown in Figure 2.8. Any $x_{k+N} \in \mathbb{X}_{k+N}^{feas}(x_{k+1})$ is reachable from x_{k+1} by some $\bar{u}_{k+1}^{k+N-1} \in \mathbb{U}_{k+1}^{k+N-1}(x_{k+1}, r_{k+2}^{k+N})$ without violating any constraints. Since $\mathbb{X}_{k+N+1}^{reach}$ is reachable from some $x_{k+N} \in \mathbb{X}_{k+N}^{feas}(x_{k+1})$ by some $u_{k+N} \in \mathbf{U}$, any $x_{k+N+1} \in \mathbb{X}_{k+N+1}^{reach}$ is reachable from x_{k+1} without violating $\mathbf{X}_{k+i}(r_{k+i})$, $\forall i = 2, 3, \ldots, N$ and $\mathbf{U}$. Then we choose r_{k+N+1} whose resulting $\mathbf{X}_{k+N+1}(r_{k+N+1})$ has some $x_{k+N+1} \in \mathbb{X}_{k+N+1}^{reach}$.

Here the vertices of sets play a very important role. From *Step 2* to *Step 4*, all the set computations are indeed the Minkowski sums of two sets. In our case, since sets are convex polytopes, each vertex of the Minkowski sum is the sum of a vertex of one polytope and one of the other. That is, for linear systems with half plane constraints, the vertices of the feasible state set are given by propagation of the vertices of the feasible input set.

Step 1: The set

$$\mathbb{U}_{k+1}^{k+N-1}(x_{k+1}, r_{k+2}^{k+N}) = \{\bar{u}_{k+1}^{k+N-1} \big| \mathcal{A}' \bar{u}_{k+1}^{k+N-1} \leq b(x_{k+1}, r_{k+2}^{k+N})\},$$

is obtained by reconstructing (2.28) with $x_{k+1} = Ax_k + Bu_{k|k}^*$ and is not empty by Lemma 2.4.1. This set gives all the possible solutions that can steer x_{k+1} over $(N-1)$ steps without violating all the state and input constraints (2.19) and (2.21). It is a convex polytope in $\mathbb{R}^{(m \times [N-1])}$ which can be equivalently given as a convex hull of l vertices

$$\mathbb{U}_{k+1}^{k+N-1}(x_{k+1}, r_{k+2}^{k+N}) = conv(\{\bar{u}_{k+1}^{k+N-1,\star,1}, \bar{u}_{k+1}^{k+N-1,\star,2}, \ldots, \bar{u}_{k+1}^{k+N-1,\star,l}\}),$$

Step 2: Consider $(N-1)-$step evolution of the state for the given x_{k+1} by $\bar{u}_{k+1}^{k+N-1,\star,i} \in \mathbb{U}_{k+1}^{k+N-1}(x_{k+1}, r_{k+2}^{k+N})$:

$$x_{k+N}^i = A^{N-1}x_{k+1} + [A^{N-2}B \ A^{N-3}B \ \ldots \ B]\bar{u}_{k+1}^{k+N-1,\star,i}.$$

Then the set of all x_{k+N} which are feasibly reachable from x_{k+1} is given by

$$
\begin{aligned}
\mathbb{X}_{k+N}^{feas}(x_{k+1}) &= conv(\{x_{k+N}^1, \ x_{k+N}^2, \ldots, \ x_{k+N}^l\}), \\
vert(\mathbb{X}_{k+N}^{feas}(x_{k+1})) &= \{x_{k+N}^{\star,1}, \ x_{k+N}^{\star,2}, \ldots, \ x_{k+N}^{\star,q}\}.
\end{aligned}
\tag{2.30}
$$

One can understand $\mathbb{X}_{k+N}^{feas}(x_{k+1})$ as the Minkowski sum of the single point set $\{A^{N-1}x_{k+1}\}$ and the other associated with $\mathbb{U}_{k+1}^{k+N-1}(x_{k+1}, r_{k+2}^{k+N})$. In general, the number of vertices q in (2.30) is less than or equal to l.

Step 3: Consider one-step state evolution from some $x_{k+N}^{\star,i} \in \mathbb{X}_{k+N}^{feas}(x_{k+1})$ by some $u_{k+N}^{\star,j} \in \mathbf{U}$

$$x_{k+N+1}^{i,j} = Ax_{k+N}^{\star,i} + Bu_{k+N}^{\star,j}.$$

The set of all reachable x_{k+N+1} from some $x_{k+N} \in \mathbb{X}_{k+N}^{feas}(x_{k+1})$ is given by

$$\mathbb{X}_{k+N+1}^{reach} = conv(\{x_{k+N+1}^{i,j} | x_{k+N+1}^{i,j} = Ax_{k+N}^{\star,i} + Bu_{k+N}^{\star,j},$$

$$\forall x_{k+N}^{\star,i} \in \mathbb{X}_{k+N}^{feas}(x_{k+1}),\ \forall u_{k+N}^{\star,j} \in \mathbf{U}\}), \tag{2.31}$$

$$vert(\mathbb{X}_{k+N+1}^{reach}) = \{x_{k+N+1}^{\star,1},\ x_{k+N+1}^{\star,2}, \ldots, x_{k+N+1}^{\star,w}\}.$$

Step 4: Consider the state constraint

$$\mathbf{X}_{k+N+1}(r_{k+N+1}) = \{x_{k+N+1}| \parallel r_{k+N+1} - x_{k+N+1} \parallel_\infty \le d\}.$$

The inequality in $\mathbf{X}_{k+N+1}(r_{k+N+1})$ is equivalent to

$$x_{k+N+1} - \bar{d} \le r_{k+N+1} \le x_{k+N+1} + \bar{d}, \tag{2.32}$$

which implies that $\mathcal{R}_{k+N+1}$ is the collection of all r_{k+N+1} satisfying (2.32) for some $x_{k+N+1} \in \mathbb{X}_{k+N+1}^{reach}$. In addition, the expression (2.32) implies that $\mathcal{R}_{k+N+1}$ is obtained by dilating $\mathbb{X}_{k+N+1}^{reach}$ by $\pm d$ in every coordinate. Consider the set

$$D = \{v| \parallel v \parallel_\infty \le d\}$$

$$= conv(\{v^{\star,1}, v^{\star,2}, \ldots, v^{\star,2^n}\}),$$

as the dilation of the single point set $\{0\}$ by $\pm d$. Its vertices are readily obtained since it is symmetric about the origin. Now, for some $x_{k+N+1}^{\star,i} \in \mathbb{X}_{k+N+1}^{reach}$ we can define a feasible r_{k+N+1} satisfying (2.32) using the vertices of D

$$r_{k+N+1}^{i,j} \triangleq x_{k+N+1}^{\star,i} + v^{\star,j}. \tag{2.33}$$

Then, using (2.33), $\mathcal{R}_{k+N+1}$ is given in the following theorem.

Theorem 2.4.3. *For given x_k and r_{k+1}^{k+N}, suppose that $\mathcal{P}(x_k, r_{k+1}^{k+N}, N)$ is feasible. Then the set, $\mathcal{R}_{k+N+1}$, of all feasible r_{k+N+1} is given by*

$$\mathcal{R}_{k+N+1} = conv(\{r_{k+N+1}^{i,j} | r_{k+N+1}^{i,j} = x_{k+N+1}^{\star,i} + v^{\star,j},$$

$$\forall x_{k+N+1}^{\star,i} \in \mathbb{X}_{k+N+1}^{reach},\ \forall v^{\star,j} \in D\}). \tag{2.34}$$

Note that, if we do not use the vector-infinity-norm state constraints, then D may have a different form.

Remark 2.4.4. *To solve the problem, we assume to have an available optimal solution and its resulting x_{k+1}. Instead, one could be tempted to consider the set of feasible x_{k+1} using all the feasible u_k in the solutions of (2.28), obtain $\mathbb{X}_{k+N}^{feas}$ which is now reachable from any feasible x_{k+1}, and finally $\mathbb{X}_{k+N+1}^{reach}$ and $\mathcal{R}_{k+N+1}$ as in (2.31) and (2.34). However, such $\mathbb{X}_{k+N}^{feas}$ that is reachable from any feasible x_{k+1} is not guaranteed to be non-empty. Hence it is not a reliable idea to consider all feasible x_{k+1}.*

2.4.4 Computational Cost and Approximation

In general, the complexity of convex set computation is measured by critical input parameters such as the number of inequalities or vertices and the dimension of the inequality solutions (Fukuda 2004). It is commonly agreed to consider the worst case computational size to measure the complexity. The worst computational size of the convex hull algorithm for d points in $\mathbb{R}^n$ is bounded above $O(d^n)$ ($O(d^{n/2})$ if $d \geq 4$ (Chazelle 1993)). Therefore the costs for the proposed algorithms may not be cheap. If we have a fast system with large dimensions and control horizon, the algorithms might not be viable. In that case, provided that cheap but guaranteed feasible approximations exist, we should trade off between computational cost and full description of the sets $(\bar{\mathfrak{X}}_{k+1+N}(x_k, \bar{x}_{k+N}), \mathcal{R}_{k+N+1})$. For $\bar{\mathfrak{X}}_{k+1+N}(x_k, \bar{x}_{k+N})$, one could use the optimal solution to consider only one $x_{k+1} = x_{k+1|k}^*$ (i.e. $\mathbb{X}_{k+1}^{feas}(x_k, \bar{x}_{k+N}) = \{x_{k+1}\}$). Then $\bar{\mathfrak{X}}_{k+1+N}(x_k, \bar{x}_{k+N})$ is simply the $N-$step reachable set from x_{k+1} in which case we avoid the computations (2.25) and most of (2.26). The complexity is just as much as evaluating (2.26) for only one vertex. Note that this is exactly the same situation as computing the N-step reachable set in State Dependent Contraction Based Model Predictive Control of Section 2.3.5. For $\mathcal{R}_{k+N+1}$, we observe that the set computations from *Step 2* to *Step 4* are the Minkowski sum of multiple sets. That is, using

$$
\begin{aligned}
\mathcal{R}_{k+N+1}^p = \{ & r_{k+N+1}^{i,j,h} | r_{k+N+1}^{i,j,h} = A^N x_{k+1} + [A^{N-1}B \ A^{N-2}B \ \ldots AB \ ] \bar{u}_{k+1}^{k+N-1,*,i} \\
& + B u_{k+N}^{*,j} + v^{*,h}, \ \forall v^{*,h} \in D, \forall \bar{u}_{k+1}^{k+N-1,*,i} \in \mathbb{U}_{k+1}^{k+N-1}(x_{k+1}, r_{k+2}^{k+N}), \ \forall u_{k+N}^{*,j} \in \mathbf{U} \},
\end{aligned}
\tag{2.35}
$$

we get the convex hull $conv(\mathcal{R}^p_{k+N+1})=\mathcal{R}_{k+N+1}$. Since $\mathcal{R}^P_{k+N+1}$ will have a large number of points, $conv(\mathcal{R}^P_{k+N+1})$ will not be a cheap computation. We propose the following approximation

$$\mathcal{R}^{approx}_{k+N+1} = conv(min\{\mathcal{R}^P_{k+N+1}\} \cup max\{\mathcal{R}^P_{k+N+1}\}), \tag{2.36}$$

where $min\{\mathcal{R}^P_{k+N+1}\}$ $(max\{\mathcal{R}^P_{k+N+1}\})$ denotes the set of the vectors in $\mathcal{R}^P_{k+N+1}$ that have at least one minimum (maximum) value in some coordinate. The illustrative picture in $\mathbb{R}^2$ is given in Figure 2.9. To complete (2.35) and (2.36), only one convex hull com-

P : set of points $conv$ (P) $conv$ $(min\{P\} \cup max\{P\})$
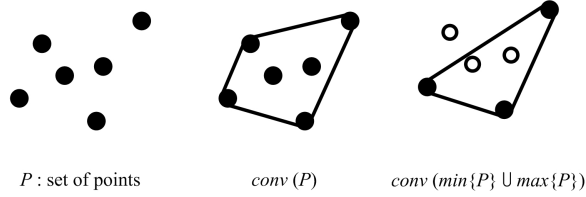
Figure 2.9: Graphical description of the approximation for $\mathcal{R}^p_{k+N+1}$. The transparent circles are the points excluded by the approximation.

putation with reduced number of points is needed. It is easy to see $\mathcal{R}^{approx}_{k+N+1} \subset \mathcal{R}_{k+N+1}$. Therefore, $\mathcal{R}^{approx}_{k+N+1}$ is relatively conservative.

2.4.5 Example

The following is the example for the reference dependent state constraint problem. Consider the linear system $x_{k+1} = Ax_k + Bu_k$ with the following matrices

$$A = \begin{bmatrix} 0.9 & 0.5 \\ 0 & 0.9 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

The initial condition $x_k = [x^1_k \quad x^2_k]^T = [0 \quad 0]^T$. Consider the $5-$step MPC with the following constrained optimal control problem:

$$\arg \min_{\{u_{k|k},...,u_{k+N-1|k}\}} \sum_{i=0}^{4} x^T_{k+i+1|k} Q x_{k+i+1|k} + u^T_{k+i|k} R u_{k+i|k},$$

$$subject\ to$$

$$x_{k+i+1|k} = Ax_{k+i|k} + Bu_{k+i|k},$$

$$x_{k+i+1|k} \in \mathbf{X}_{k+i+1}(r_{k+i+1}) = \{x \mid \| r_{k+i+1} - x \|_\infty \le 0.5\}, \tag{2.37}$$

$$u_{k+i|k} \in \mathbf{U} = \{u \mid \| u \|_\infty \le 1\},$$

where

$$Q = I_2, \qquad R = I_1,$$

$$\begin{bmatrix} r_{k+1} & r_{k+2} & \dots & r_{k+5} \end{bmatrix} = \begin{bmatrix} 0 & 0.6 & 1.4 & 2.5 & 3 \\ 1.2 & 2.0 & 2.8 & 3.5 & 4.0 \end{bmatrix}.$$

After x_{k+1} is computed by $u^*_{k|k}$, the set $\mathbb{X}^{feas}_{k+5}(x_{k+1})$ is obtained as shown in Figure 2.10. The reference sequence r^{k+5}_{k+1} and resulting state x_{k+1} are shown as $*$ and o respectively. The red square region represents the state constraint $\mathbf{X}_{k+5}(r_{k+5})$ (The other state constraints were not depicted in the figure.) and the green polytope in the lower right corner is $\mathbb{X}^{feas}_{k+5}(x_{k+1})$. Then $\mathbb{X}^{reach}_{k+6}$ is generated by evaluating all the vertices in $\mathbb{X}^{feas}_{k+5}(x_{k+1})$ and
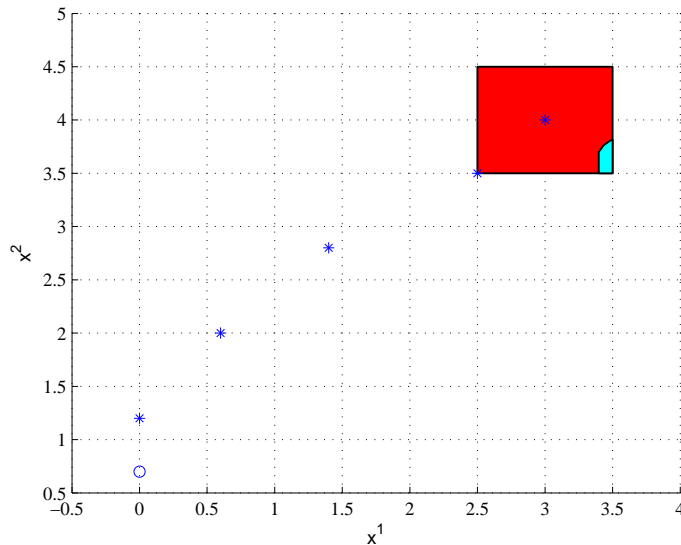
Figure 2.10: Graphical description from Step 1 to 2 of the algorithm.

$\mathcal{R}_{k+6}$ are obtained by dilating $\mathbb{X}^{reach}_{k+6}$ with ± 0.5 in x^1 and x^2 coordinates. The graphical representation is given in Figure 2.11. The small green polytope represents $\mathbb{X}^{reach}_{k+6}$ and the big red region in addition to $\mathbb{X}^{reach}_{k+6}$ is $\mathcal{R}_{k+6}$.

2.5   Conclusion

First, it has been shown that, using a system's structural property of contractibility, one may construct MPC for the case where the initial states are not able to
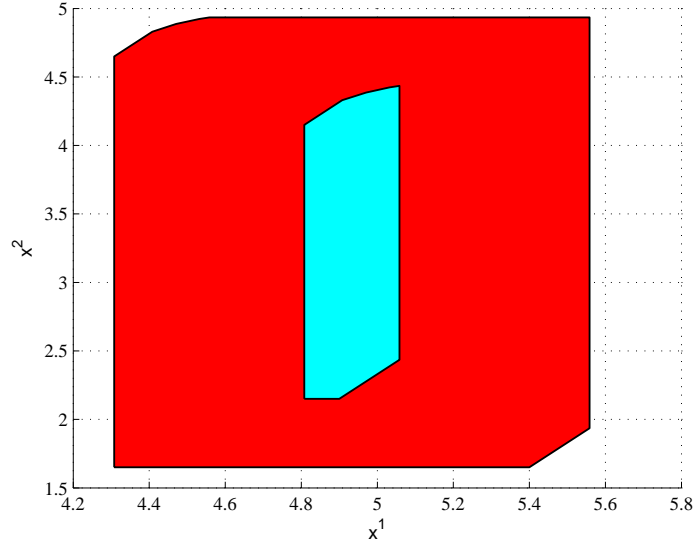
Figure 2.11: Graphical description from Step 3 to 4 of the algorithm.

be feasibly steered to the origin in reasonable time by classical MPC schemes. Furthermore, the proposed strategy may lead to improvement in the stabilizing performance and give freedom to tune the controlled behavior of the system compared to the case that the control law is solely based on a CLF. Second, we have introduced the online reference computation problem and provided algorithms for the linear system with linear equality and inequality constraints. The algorithms we suggested give the set of terminal states or references to guarantee feasibility.

Further investigation is necessary for the system with disturbance for robustness and the computability of the proposed MPC algorithms. Especially, in terms of the computability, the proposed MPC algorithms are tractable for linear systems with polytope constraints. However, required set calculations might not be achieved efficiently and accurately for some constrained nonlinear systems. Therefore, more analysis on the computability is desired.

This chapter is in part a reprint of the materials as it appears in,

Keunmo Kang, Robert R. Bitmead - *Contraction Based Model Predictive Control*, International Workshop on Assessment and Future Directions of Nonlinear Model Predictive Control, Pavia, Italy, Sep., 2008.

Keunmo Kang, Robert R. Bitmead - *Contraction Based Receding Horizon Control*, submitted to Automatica, 2008.

Keunmo Kang, Robert R. Bitmead -*Online Reference Computation for Feasible Model Predictive Control*, 46th IEEE Conference on Decision and Control, New Orleans, LA, USA, Dec., 2007.

The dissertation author was the primary author and the co-author, professor Bitmead, directed and supervised the research.

3

Coordinated System Control

3.1 Introduction

In this chapter, we seek to formulate control schemes for coordinated systems. Since the coordinated system is understood well in the coordinated vehicle context, we employ coordinated vehicles as our working problems. However, the developed ideas in the coordinated vehicle control and estimation may be inherited to the other coordinated systems. Here we only consider a fixed vehicle formation with noisy environments such as disturbance on vehicles, measurement noise, and communication error if any communication is used.

The vehicle formation control problem was considered in (Kang, Xi & Sparks 2000) using the reference projection method. Artificial forces between each vehicle and virtual leaders were used in (Leonard & Fiorelli 2001, Xi & Abed 2005). Model Predictive Control (MPC) was considered in (Dunbar & Murray 2002, Dunbar & Murray 2004, Dunbar & Murray 2006). Their investigation was based on a deterministic system and focused on creating the specified formation from the vehicles' initial locations and then on the stability of the vehicles in the formation. Hence, in the stationary stage, the vehicles achieve their desired formation asymptotically. Our concern is different from theirs; we consider steady state disturbance rejection with respect to vehicle disturbances, measurement noise, and control design parameters. Main features of this chapter are:

– Analysis to understand the need for coordinated control in a fixed vehicle forma-

tion,

– Formulation of local MPC for each vehicle as a method of coordinated control,

– Communication resources in the coordinated vehicle control.

In order to understand why coordinated control is necessary, we first consider a disturbance rejection control problem on each vehicle in the formation and relate the covariance of neighboring vehicles positions to no-collision between vehicles. This can be done by calculating the closed-loop Linear Quadratic Gaussian (LQG) control position variance of an isolated vehicle around its nominal position under the effect of exogenous disturbances. Our approach gives a quite useful perspective on when the coordinated control scheme may help to avoid collision.

When LQG control of each vehicle does not handle collision avoidance well enough in the formation, one should seek ways to prevent possible collisions between vehicles: one vehicle's future position cannot overlap neighbors' future positions. Due to the uncertainties such as disturbance, predicting exact future position outputs is not possible and therefore control design for collision avoidance requires a careful treatment about this. If the variances of predicted vehicle position errors are given, then uncertainty regions of the vehicles can be obtained, which contain the actual vehicle locations with a certain probability. The uncertainty regions are centered at the position estimates ($\hat{y}$). Using these regions, the no-collision constraint is re-formulated for each vehicle. Since implementing explicit constraints such as the no-collision constraint is not tractable in an infinite horizon control formulation, we attempt to solve a series of constrained finite horizon control problems in a receding horizon fashion. This is exactly why MPC is an attractive control approach to deal with coordinated vehicles.

The no-collision constraint in one vehicle requires knowledge about future positions of the other vehicles. So communication is necessary and, in our development, it defines the coordinated control. Each vehicle receives the state and input information from the neighboring vehicles and uses it to predict their future positions. We call the device for this a *cross-estimator*. The covariance of the cross-estimator is affected by inter-vehicle communications. Sufficient communication resources might reduce communication errors and lead to better estimation and control performance. In this chapter,

our discussion will be limited to this fundamental perspective. More details will be given in the next chapter along with the cross-estimator design.

The above overlook is given with gaussian assumptions on the uncertainties. We will extend our results in the coordinated vehicle control to the same vehicle formation scenario without probabilistic models on the uncertainties. In this case, the only information about the uncertainties is their bounds or a representative time profile so that we can model the overall behavior of them with fictitious white noise.

3.2 Coordinated Vehicle Problem

The fundamental task of vehicles that we consider is to follow specified trajectories while attempting to keep a certain formation and avoid collisions. We also use the following assumptions The assumptions are easily removable, but at the expense of clarity.

A1. The dynamics of each vehicle are linear, time-invariant, and identical.

A2. We consider only stationary values of covariances, without introducing the transient calculations, which are easy to accommodate.

A3. The spectral properties of the disturbances acting on each vehicle are identical and independent.

A4. For the theoretical development, the disturbances are zero-mean and gaussian.

A5. The nominal reference trajectory of each vehicle is known to all others in advance and the target formation geometry is fixed.

A6. Some limited communication is permissible: this leads to a decentralized or distributed disturbance rejection control problem.

Since the target trajectory is known ahead of time and the vehicle dynamics are linear, we may re-center the local control tasks of formation maintenance to keeping each vehicle to a neighborhood of its own origin.

Our approach to coordination is to pose the problem as disturbance rejection – perhaps best captured by imagining the interaction between aerial acrobatic team aircraft

under the influence of heavy gusts. Without this stochastic aspect to the coordination problem, a no-communication, open-loop control strategy might suffice. However, we see it as important to capture the environmental effects on the formation via posing a disturbance rejection problem.

Now we consider the following dynamical model for the vehicles and assume the models are perfect and identical:

$$
\begin{aligned}
x_{k+1} &= Ax_k + Bu_k + w_k, \\
y_k &= Cx_k, \\
z_k &= Dx_k + v_k,
\end{aligned}
\tag{3.1}
$$

where u_k, y_k, z_k and x_k represent the input vector, the position output vector, the measurement vector, and the state vector of the vehicle and of its associated disturbance process. The vehicles and their disturbance process are driven by a process noise w_k and the measurement is corrupted by a measurement noise v_k. Both noises are modeled as white normally distributed sequences,

$$
w_k \sim N(0, Q), \quad v_k \sim N(0, R).
\tag{3.2}
$$

Remark 3.2.1. *We identify here that we model the exogenous disturbances as gaussian, although we seek to guarantee non-collision between vehicles. This requires assuming a density of bounded support for the noises, which we comment upon later.*

In the absence of plant uncertainty, exogenous disturbances, and measurement noise, this control problem becomes a tracking problem, which might be solved by open-loop or closed-loop methods without the need to introduce communication between vehicles. Since we assume perfect modeling and consider only stationary formation-keeping in the face of stationary disturbances, our analysis reduces to the consideration of each vehicle's controlled motion around its local zero point. We first consider the position variability of an isolated vehicle under LQG control with the disturbance operating. This will provide a useful benchmark for the need to introduce constrained local control and communications.

3.3   Steady State LQG Control of an Isolated Vehicle

The feedback control problem of formation keeping is tied to disturbance rejection in our synthesis. Ignoring, for the moment, interaction between vehicles, one may formulate a local disturbance rejection control problem for an isolated vehicle around the origin. If the formation geometry is such that the permissible deviations of the vehicle from their nominal origins are so small as to preclude collision, then there is no need for constrained control and inter-vehicle communication. This observation establishes the link between target formation geometry, disturbance rejection, constrained control, and communication requirements.

We pose an LQG disturbance rejection control problem for each individual vehicle in isolation. The LQG problem captures, via selection of penalty matrices Q_c and R_c, the derivable cost function associated with solo vehicle control. Later, we will introduce interaction between the vehicles via a no-collision constraint imposed over and above this nominal LQG control. The distraction from the LQG control problem by the constrained problem rests in the choice of a control penalty, by which normal unconstrained operation seeks to use a modest amount of control as captured by the penalty matrix R_c, which is not infinitesimal. Under active no-collision constrained operation, the LQG penalty is subservient to the necessary control to maintain feasible operation.

An initial analysis will be to study the dependence of the stationary spatial deviation of the vehicles from their nominal zero positions as a function of the LQG control penalty R_c. The LQG control law is given by solving the following minimization problem, with the stochastic disturbance operating,

$$\min_{u_k} \left[\lim_{N \to \infty} \frac{1}{N} E(\sum_{k=0}^{N-1} x_k^T Q_c x_k + u_k^T R_c u_k) \right].$$

The corresponding control Discrete-time Algebraic Riccati Equation (DARE) and control signal are given by

$$P = A^T P A - A^T P B (B^T P B + R_c)^{-1} B^T P A + Q_c,$$

$$u_k = -K_c \hat{x}_{k|k}, \tag{3.3}$$

$$K_c = (B^T P B + R_c)^{-1} B^T P A,$$

where $\hat{x}_{k|k}$ is the filtered state estimate of the vehicle. We use the Kalman filter for state estimation. The filtering problem is obtained by solving the following estimation DARE

$$S_p = AS_pA^T - AS_pD^T(DS_pD^T + R)^{-1}DS_pA^T + Q, \tag{3.4}$$

where

$$S_p = \lim_{k\to\infty} E[(x_k - \hat{x}_{k|k-1})(x_k - \hat{x}_{k|k-1})^T|Z_{k-1}].$$

Here Z_{k-1} is the measurement sequence $\{z_0, z_1, \ldots, z_{k-1}\}$ up to time $k - 1$. We have the corresponding filtering gain

$$K_f = S_pD^T(DS_pD^T + R)^{-1}.$$

By the separation principle, the optimal control law from (3.3) is optimal for our stochastic linear system. We assume that the weightings Q_c, R_c, Q, and R are chosen so that K_f and K_c are stabilizing filter and control gains respectively. The closed loop system and state estimator error equation are given by

$$x_{k+1} = (A - BK_c)x_k + BK_c\tilde{x}_{k|k} + w_k,$$
$$\tilde{x}_{k+1|k+1} = (A - K_fDA)\tilde{x}_{k|k} + (I - K_fD)w_k - K_fv_k, \tag{3.5}$$

where $\tilde{x}_{k|k} \triangleq x_k - \hat{x}_{k|k}$. Define

$$S \triangleq \lim_{k\to\infty} E[x_kx_k^T|Z_k],$$

$$S_c \triangleq \lim_{k\to\infty} E[x_k\tilde{x}_{k|k}^T|Z_k],$$

$$S_f \triangleq \lim_{k\to\infty} E[\tilde{x}_{k|k}\tilde{x}_{k|k}^T|Z_k].$$

Then, using (3.5), we have the following standard Lyapunov equation for these variables

$$
\begin{bmatrix} S & S_c \\ S_c^T & S_f \end{bmatrix} = \begin{bmatrix} A - BK_c & BK_c \\ 0 & A - K_fDA \end{bmatrix} \begin{bmatrix} S & S_c \\ S_c^T & S_f \end{bmatrix} \begin{bmatrix} A - BK_c & BK_c \\ 0 & A - K_fDA \end{bmatrix}^T
$$
$$
+ \begin{bmatrix} Q & Q(I - K_fD)^T \\ (I - K_fD)Q & (I - K_fD)Q(I - K_fD)^T + K_fRK_f^T \end{bmatrix}. \tag{3.6}
$$

This equation describes the stationary closed loop covariance of the system (vehicle+ disturbance) state, the spatial position components of which contain the information about the vehicle's deviation from nominal position. Thus we are interested in analyzing these components of S as a function of R_c. Note that the dependence of S on R_c in (3.6) is manifested via K_c and the control DARE.

3.3.1 Analysis of Controlled System Outputs

To discuss dependence of S on the choice of R_c, we introduce the LQG control performance results of Maciejowski (Maciejowski 1985).

Theorem 3.3.1. *(Maciejowski 1985) Consider the given system (3.1) under LQG control. If $det(CB) \neq 0$, $Q_c = C^T C$, $R_c = 0$, and the system is minimum-phase, then $S = S_p$.*

The minimum phase property is required. Otherwise the closed loop system will lose internal stability. This theorem establishes that minimum variance position control ($Q_c = C^T C$, $R_c = 0$) for a minimum phase system yields a closed-loop state covariance identical to the Kalman predictor state estimation error covariance. For the vehicle control problem, this has the immediate interpretation that if we do not penalize the control and focus entirely on maintaining the position, then the achieved position deviation will be, subject to assumptions, identical to the variance of the one-step-ahead position estimate. When this result is brought into the vehicle coordination problem, as we shall see, it results in separating the class of control problem into those where collisions are precluded even without communications and those where inter-vehicle communication does not lead to effective collision avoidance with reasonable control energy.

If we accept that it is desirable not normally to allow cost-free control, so that $R_c > 0$, then we shall see that the deviation of the vehicles under LQG control will exceed their prediction error deviation. First we establish that, with $R_c > 0$, we must have $S \geq S_p$.

Theorem 3.3.2. *Suppose the following conditions hold.*

- *A has full rank and $[A, B]$ is stabilizable.*

- *$R_c > 0$, $Q_c \geq 0$, and $[A, Q_c^{1/2}]$ is observable.*

Then $S \geq S_p$.

Proof. First we will show that $S - S_f$ is symmetric positive definite. From (3.6)

$$
\begin{aligned}
S =& (A - BK_c)S(A - BK_c)^T + BK_cS_c(A - BK_c)^T \\
& + (A - BK_c)S_c^T K_c^T B^T + BK_cS_f K_c^T B^T + Q.
\end{aligned}
$$

$$(3.7)$$

Note that $\tilde{x}_{k|k}$ is zero mean white noise process and independent of the estimate $\hat{x}_{k|k}$. Thus

$$
\begin{aligned}
S_c &= \lim_{k \to \infty} E[x_k \tilde{x}_{k|k}^T | Z_k] \\
&= \lim_{k \to \infty} E[(\hat{x}_{k|k} + \tilde{x}_{k|k}) \tilde{x}_{k|k}^T | Z_k] \\
&= \lim_{k \to \infty} E[\tilde{x}_{k|k} \tilde{x}_{k|k}^T | Z_k] \\
&= S_f.
\end{aligned}
$$

Hence (3.7) becomes

$$
\begin{aligned}
S =&(A - BK_c)S(A - BK_c)^T + BK_cS_f(A - BK_c)^T \\
&+ (A - BK_c)S_fK_c^T B^T + BK_cS_fK_c^T B^T + Q.
\end{aligned}
$$

By subtracting S_f from the left and right hand sides, it is easy to show that

$$
\begin{aligned}
S - S_f &= (A - BK_c)(S - S_f)(A - BK_c)^T + AS_fA^T + Q - S_f \\
&= (A - BK_c)(S - S_f)(A - BK_c)^T + S_p - S_f.
\end{aligned}
\tag{3.8}
$$

The last equality is obtained from the Kalman filtering and prediction error covariance relation. This is the Lyapunov equation for a closed-loop LQG-controlled system. de Souza, Gevers & Goodwin (1986) showed that, if $[F, G]$ is stabilizable, $[A, Q_c^{1/2}]$ is observable, $R_c > 0$, and $Q_c \geq 0$, then the control DARE in (3.3) has a symmetric positive definite solution P and $A - BK_c$ has all eigenvalues inside the unit circle. The solution of (3.8) is given by

$$
S - S_f = \sum_{k=0}^{\infty}(A - BK_c)^k(S_p - S_f)(A - BK_c)^{kT},
$$

from which it is immediate that

$$
S - S_f \geq S_p - S_f.
$$

Hence $S \geq S_p$. $\qquad\square$

The property $S \geq S_p$ implies that, if we penalize the control, the LQG-controlled position has greater variability than that of the position prediction error. In the vehicle formation context, two situations can result. In one, collision is precluded

by sufficient separation between the vehicles even though $S \geq S_p$. In this case, communication and local constrained control are not needed to prevent collision. In the other case, target separation does not rule out possible collision because the two vehicles' controlled spatial uncertainty regions intercept. Then a collision might be prevented by constrained control and sufficient inter-vehicle communication.

We make additional observations about the stationary spatial uncertainty regions in this vehicle coordination context.

– Define the actual vehicle position by $\bar{y}_k = Cx_k \in \mathbb{R}^2$. Then, with this model, $\bar{y}_k$ is distributed as an $N(0, CSC^T)$ random 2-vector. A controlled spatial uncertainty set is defined as follows

$$\mathbb{S} = \{\bar{y}_k \big| \bar{y}_k^T (CSC^T)^{-1} \bar{y}_k \leq \kappa\}. \tag{3.9}$$

Here κ is related to the probability that $\bar{y}_k$ stays close to its origin (or target position) when the vehicle is affected by the disturbance and is under LQG control. Once we assign the desired probability then κ can be obtained by the chi-square distribution of the two degree of freedom. Larger κ means larger probability. For the implementation of a non-collision constraint, we need to assume disturbance and measurement noise with distribution of compact support. Then, for more appropriate value of κ, $\bar{y}_k$ will be guaranteed to lie within this uncertainty set. We note that this set is centered at the vehicle's nominal position.

– Define the vehicle position prediction error measurement by $\tilde{y}_k = C(x_k - \hat{x}_{k|k-1}) \in \mathbb{R}^2$. Then $\tilde{y}_k$ has the distribution $N(0, CS_pC^T)$. We define a prediction error uncertainty set associated with S_p

$$\mathbb{S}_p = \{\tilde{y}_k \big| \tilde{y}_k^T (CS_pC^T)^{-1} \tilde{y}_k \leq \kappa\}. \tag{3.10}$$

The interpretation of this set is that, with the same probability we assigned for (3.9), the vehicle's controlled real position is close to the one-step-ahead prediction of its position. Thus this set is centered at the predicted position $C\hat{x}_{k+1|k}$. We note that this latter region lies entirely with the regions defined by (3.9).

– When $R_c = 0$ and the vehicle is minimum-phase, the sets $\mathbb{S}$ and $\mathbb{S}_p$ are coincident.

$R_c > 0$
$d_1 < d_2$

✘ : nominal positions of vehicles
--➤ : communication channels
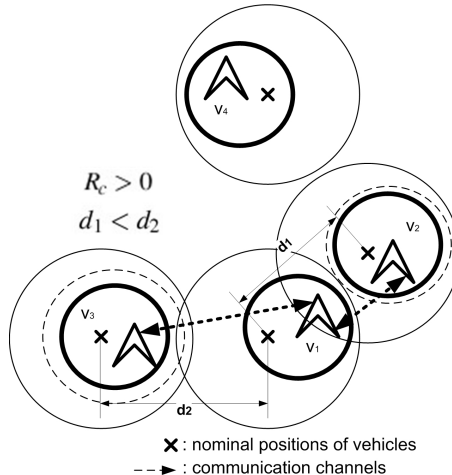
Figure 3.1: A formation of four vehicles as viewed from the perspective of V_1.

– If the control is penalized ($R_c > 0$), the vehicle's prediction-error uncertainty region, given by (3.10), lies entirely within its controlled uncertainty region, given by (3.9). This is the importance of Theorem 3.3.2. The smaller $\mathbb{S}_p$ region jiggles inside the bigger $\mathbb{S}$ region.

In the next section, we will link the developed ideas about the LQG-controlled isolated vehicle of this section to the requirement of local constrained control via communication in the vehicle formation.

3.4 Links to Local Constrained Control via Communication

The four-vehicle formation is given in Figure 3.1 representing the case that the conditions in Theorem 3.3.2 hold. We denote vehicle i by V_i ($i = 1, 2, 3, 4$). The largest circles with solid, thin lines indicate the boundary of the set $\mathbb{S}$ in (3.9). Now we call the region inside the circle *controlled uncertainty region*. The smallest solid, heavy lines represent the boundary of the set $\mathbb{S}_p$ in (3.10). We give the name *prediction-error uncertainty region* to the area inside the circle. The dashed circles refer only to V_1 and to its estimates of predicted positions of V_2 and V_3. These estimators, to be discussed in the next chapter, are called *cross-estimators* and rely on communicated

information from the other vehicles to V_1. The radii of these circles will depend on the cross-estimate covariances. Although the uncertainty regions are circles for simplicity, they will generally be ellipsoids depending on the dynamical model that one obtains. In the following subsections, our main concern will be limited to V_1 only. The others' cases can be analyzed in the same manner.

3.4.1   No Communication Advantage

Consider V_1 and V_4. Their nominal separation in the formation provides full separation of their own controlled uncertainty regions. Hence collision cannot occur. In general, for any given vehicle formation, if the controlled uncertainty regions do not overlap in the formation, then they are guaranteed not to collide with each other. Hence, in this case, we no longer require any cooperative control via communication between V_1 and V_4 to avoid collision.

Consider the case that the conditions in Theorem 3.3.1 hold. Then the controlled uncertainty regions of all the vehicles are identical to their prediction-error uncertainty regions. As a special case, one can consider the case that the prediction-error uncertainty regions overlap. The vehicles may be present anywhere within the regions at the next time instant and this is unpredictable. If the prediction-error uncertainty regions overlap, the eventual collision is guaranteed with LQG control whatever the communication between the vehicles. If their prediction-error uncertainty regions do not overlap because of formation design, then no communication is necessary to avoid collision. This leads us to the surprising conclusion that, for coordinating vehicles operating under minimum variance control, the communication of information between vehicles is immaterial to their collision prospects.

3.4.2   Communication Advantage

Consider V_1, V_2, and V_3. The LQG controlled uncertainty regions of V_2 and V_3 overlap that of V_1 but their prediction-error uncertainty regions do not. If the LQG control of V_1 were replaced by a constrained control, say MPC, and V_1 were provided with an estimate of the positions of V_2 and V_3, then collision could be avoided as long as V_1's prediction error circle (centered at $\hat{x}_{k+1|k}$) can be kept outside the estimate-error

circles of V_2 and V_3. The radii of these dashed circles in Figure 3.1 depend on covariances of cross-estimates for V_2 and V_3. [Details on the cross estimator design, covariances, and communication resource assignment will be fully discussed in the next chapter.]

In the case that vehicles' prediction-error uncertainty regions overlap, of course one may formulate a constrained control problem to separate the circles by control action and so to preclude collisions. Unfortunately, if the overlapping circle is the result of specification of the target positions, then this constrained control will lead to continual (or at least very frequent) activity of the no-collision constraint in the constrained solution. This is undesirable from the perspective of average control energy and strong instability. Hence the advantage of communication is very limited.

3.5 Coordinated System Control by Constrained Model Predictive Control

In this section, we formulate local constrained MPC to avoid collision between vehicles in a given formation. Denote the joint state of the ith vehicle and of its disturbance at time k by x_k^i. Denote; the control signal computed at time k as $u_{k|k}^i$, the actual position by y_k^i, and the measurement by z_k^i. We assume that the vehicle and its locally acting disturbance are jointly described by

$$
\begin{align}
x_{k+1}^i &= A x_k^i + B u_{k|k}^i + w_k^i, \tag{3.11}\\
y_k^i &= C x_k^i, \tag{3.12}\\
z_k^i &= D x_k^i + v_k^i. \tag{3.13}
\end{align}
$$

We take w_k^i and v_k^i to be independent, zero-mean, white noise sequences. We capture the non-collision of Vehicle i with Vehicle ℓ through a minimal separation requirement

$$
||y_{k+j}^i - y_{k+j}^\ell||_M > \alpha, \tag{3.14}
$$

where positive definite matrix $M > 0$ defines an ellipse with center y_{k+j}^ℓ and semi-axes given by the eigenvectors of M. This condition says that, from Vehicle i's perspective, Vehicle i's position must be outside the ellipsoid given by (3.14) centered at y_{k+j}^ℓ.

3.5.1 Local Probabilistic MPC Problem for Vehicle i

At time k and from current state, x_k^i, the MPC problem is given by

$$\arg\min_{\{u_{k|k}^i,\dots u_{k+N-1|k}^i\}} \sum_{j=1}^{N} x_{k+j}^{i^T} Q_c x_{k+j}^i + u_{k+j-1|k}^{i^T} R_c u_{k+j-1|k}^i,$$

$$\text{subject to: } x_{k+j|k}^i = A x_{k+j-1|k}^i + B u_{k+j-1|k}^i + w_{k+j-1}^i,$$

$$y_{k+j|k}^i = C x_{k+j|k}, \tag{3.15}$$

$$x_{k+j}^i \in \mathbf{X}_k,$$

$$u_{k+j-1|k}^i \in \mathbf{U}_k,$$

$$Pr\left(||y_{k+j|k}^i - y_{k+j|k}^\ell||_M < \alpha\right) < \epsilon,$$

where $x_{k+j|k}^i$ $(y_{k+j|k}^i)$ is the j-step-ahead state (output) computed at time k and $\mathbf{X}_k$ $(\mathbf{U}_k)$ represents the local state (input) constraint set. Currently, the objective function and the probabilistic no-collision constraint are expressed in terms of the actual vehicle states and positions at the future time $k+j$. Due to unknown disturbance $(w_{k+j-1}^i, w_{k+j-1}^j)$, we cannot solve the above MPC directly. Our objective is to recast (3.15) in terms of local predictions, so that the MPC problem might be converted into a deterministic problem soluble with local information.

3.5.2 Deterministic Restatement of the MPC

The local MPC control problem is couched in terms of the prediction of the local Vehicle i's dynamic performance over horizon length N based on Kalman state predictions, $\hat{x}_{k+j|k}^i$, with corresponding covariance matrices $S_{p,k+j|k}^i$ computed via standard covariance recursions. In this framework above with the inclusion of a gaussian assumption on the noises and initial state estimate, the future state is random and gaussian with,

$$x_{k+j}^i \sim N(\hat{x}_{k+j|k}^i, S_{p,k+j|k}^i). \tag{3.16}$$

Suppose for the moment that Vehicle i also has an unbiased estimate of the state of Vehicle ℓ distributed similarly,

$$x_{k+j}^\ell \sim N(\hat{x}_{i,k+j|k}^\ell, \Sigma_{i,k+j|k}^\ell). \tag{3.17}$$

Note that this distribution describes the likelihood of Vehicle ℓ's state given the information received up to time k at Vehicle i with local information including the cross-estimate. The mean of this distribution, $\hat{x}^\ell_{i,k+j|k}$, is Vehicle i's *cross-estimate* of Vehicle ℓ's state. The corresponding cross-estimate of Vehicle ℓ's position is $\hat{y}^\ell_{i,k+j|k} = C\hat{x}^\ell_{i,k+j|k}$ with covariance $C\Sigma^\ell_{i,k+j|k}C^T$. Define the two j-step-ahead prediction errors

$$\begin{aligned}
\tilde{y}^i_{k+j|k} &= y^i_{k+j} - \hat{y}^i_{k+j|k} \sim N(0, CS^i_{p,k+j|k}C^T), \\
\tilde{y}^\ell_{i,k+j|k} &= y^\ell_{k+j} - \hat{y}^\ell_{i,k+j|k} \sim N(0, C\Sigma^\ell_{i,k+j|k}C^T).
\end{aligned} \tag{3.18}$$

Using (3.18), the no-collision constraint (3.14) becomes

$$||\hat{y}^i_{k+j|k} - \hat{y}^\ell_{i,k+j|k} + \tilde{y}^i_{k+j|k} - \tilde{y}^\ell_{i,k+j|k}||_M > \alpha. \tag{3.19}$$

The triangle inequality ensures the inequality (3.19) is guaranteed by satisfaction of

$$||\hat{y}^i_{k+j|k} - \hat{y}^\ell_{i,k+j|k}||_M > \alpha + ||\tilde{y}^i_{k+j|k} - \tilde{y}^\ell_{i,k+j|k}||_M. \tag{3.20}$$

From (3.16-3.17) plus the earlier assumption of independence of noises, the following holds:

$$\tilde{y}^i_{k+j|k} - \tilde{y}^\ell_{i,k+j|k} \sim N\left(0, C\left(S^i_{p,k+j|k} + \Sigma^\ell_{i,k+j|k}\right)C^T\right). \tag{3.21}$$

Then the probabilistic statement of no-collision is converted into a deterministic one by the following lemma.

Lemma 3.5.1. *Assume that all estimation errors are gaussian with covariances given above. Denote the dimension of the position vector by d and denote the cumulative distribution function of the χ^2 density with d degrees of freedom by $\Psi_d(\cdot)$. Consider the probabilistic no-collision constraint in (3.15) with weighting matrix $M > 0$, and define the value β to be any value satisfying $\Psi_d(\beta^2) \geq 1 - \epsilon$. Then, with $P^\ell_{i,j} = C\left(S^i_{p,k+j|k} + \Sigma^\ell_{i,k+j|k}\right)C^T$, satisfaction of the probabilistic no-collision constraint is implied by*

$$||\hat{y}^i_{k+j|k} - \hat{y}^\ell_{i,k+j|k}||_M > \alpha + \beta\sqrt{\lambda_{\max}\left(P^{\ell\,\frac{1}{2}}_{i,j} M P^{\ell\,\frac{1}{2}}_{i,j}\right)}. \tag{3.22}$$

Proof. From (3.21) we have that $\tilde{y}^i_{k+j|k} - \tilde{y}^\ell_{i,k+j|k} = P^{\ell\,\frac{1}{2}}_{i,j}z_j$, where z_j is a d-dimensional random vector with $N(0, I_d)$ and $Pr(z_j^T z_j < \gamma^2) = \Psi_d(\gamma^2)$. Thus, with the defined β, we

have $Pr\left(z^T z > \beta^2\right) \le \epsilon$. Then $Pr\left(\bar{\lambda} z^T z > \bar{\lambda}\beta^2\right) \le \epsilon$ holds with $\bar{\lambda} = \lambda_{\max}\left(P_{i,j}^{\ell\,\frac{1}{2}} M P_{i,j}^{\ell\,\frac{1}{2}}\right)$. Since

$$||\tilde{y}_{k+j|k}^i - \tilde{y}_{i,k+j|k}^\ell||_M^2 = \left(\tilde{y}_{k+j|k}^i - \tilde{y}_{i,k+j|k}^\ell\right)^T M\left(\tilde{y}_{k+j|k}^i - \tilde{y}_{i,k+j|k}^\ell\right)$$
$$= z^T P_{i,j}^{\ell\,\frac{1}{2}} M P_{i,j}^{\ell\,\frac{1}{2}} z \le \bar{\lambda} z^T z,$$

we have $Pr\left(||\tilde{y}_{k+j|k}^i - \tilde{y}_{i,k+j|k}^\ell||_M^2 \ge \bar{\lambda}\beta^2\right) \le \epsilon$. This and (3.22) imply that (3.20) and (3.19) hold with probability at least $1 - \epsilon$, and therefore, the satisfaction of the probabilistic no-collision constraint in (3.15). $\square$

A similar result for a scalar case appears in (Yan & Bitmead 2005). Now it remains to express the state constraint in (3.15) in terms of $\hat{x}_{k+j|k}^i$, as from (3.16) the true plant state has been modeled as a gaussian random vector and, accordingly, cannot be guaranteed to satisfy the state constraint at any time in the future. However, if we adopt a similar probabilistic formulation of Lemma 3.5.1, then we may replace the state constraint by altering the feasible set $\mathbf{X}_k$ as follows

$$\hat{x}_{k+j|k}^i \in \bar{\mathbf{X}}_k = \mathbf{X}_k - \tilde{\mathbf{X}}(\delta, S_{p,k+j|k}^i), \tag{3.23}$$

indicating that these state constraints are tightened in accordance with the probability of satisfaction, δ, and the Kalman prediction error covariance, $S_{p,k+j|k}^i$. If the densities involved have compact support, then one might guarantee almost sure satisfaction of these constraints via gaussian overbounds. Finally the MPC problem (3.15) becomes deterministic as following:

$$\arg \min_{\{u_{k|k}^i, \dots u_{k+N-1|k}^i\}} \sum_{j=1}^N \hat{x}_{k+j|k}^{i^T} Q_c \hat{x}_{k+j|k}^i + u_{k+j-1|k}^{i^T} R_c u_{k+j-1|k}^i,$$
$$\text{subject to: } \hat{x}_{k+j|k}^i = A\hat{x}_{k+j-1|k}^i + Bu_{k+j-1|k}^i,$$
$$\hat{y}_{k+j|k}^i = C\hat{x}_{k+j|k}^i, \tag{3.24}$$
$$\hat{x}_{i,k+j|k}^\ell \in \bar{\mathbf{X}}_k,$$
$$u_{k+j-1|k}^i \in \mathbf{U}_k,$$
$$||\hat{y}_{k+j|k}^i - \hat{y}_{i,k+j|k}^\ell||_M > \alpha + \beta\sqrt{\lambda_{\max}\left(P_{i,j}^{\ell\,\frac{1}{2}} M P_{i,j}^{\ell\,\frac{1}{2}}\right)}.$$

Remark 3.5.1. *The proposed MPC (3.24) cannot be solved by standard quadratic programming due to non-convex nature of the no-collision constraint. We use a nonlinear*
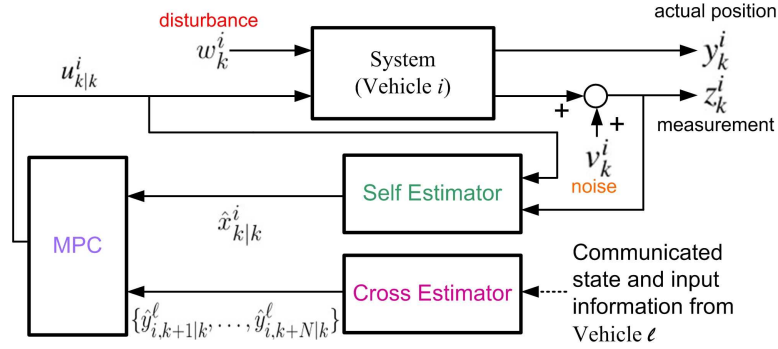
Figure 3.2: The block diagram at Vehicle i.

optimization solver such as SNOPT (Gill, Murray & Saunders 2006) to compute a sub-optimal solution.

Remark 3.5.2. *The no-collision constraint provides a AHA! moment for the communication requirement in control. If the communication quality is poor (i.e. noisy communication), then we can expect to obtain larger $\Sigma^\ell_{i,k+j|k}$. This leads to more frequent activities of the no-collision constraint. This justifies assigning enough communication resource on communication channels if available. We will further investigate this issue in the next chapter.*

Remark 3.5.3. *The right-hand-side of the no-collision constraint may be fixed with $P^\ell_{i,1}{}^{\frac{1}{2}}$ since vehicle i is assured receiving updated measurements from Vehicle ℓ before being required to assert the constraint.*

3.6 Local Constrained MPC with Bounded Noise and Disturbance

We developed the local constrained MPC formulation to maintain a vehicle formation and avoid collisions under the gaussian assumption on disturbance and noise processes. However, gaussian process is unbounded and, therefore, it is not possible to guarantee %100 collision avoidance. In practice, physical disturbance, measurement, and communication noises are generally bounded and hard to capture their probabilistic characteristics. Hence it is worth investigating the coordinated control without prob-

abilistic assumptions on the disturbance and noise processes. Especially, we consider the case that the area where the vehicles operate is known to have a specific class of disturbance so that we can sample the disturbance data for modeling.

The focus of this section is on the formulation of a disturbance rejection controller capable of preventing possible collisions. It starts from fitting a disturbance description by data record alone into an approximating state-space model. In this way, we hope to provide an approach suited to the rejection of realistic disturbances such as wind gusts on vehicles; deterministic disturbance cases were investigated in (Francis & Wonham 1976). Due to possible collisions between the vehicles, the controller must be able to avoid the collisions. Therefore, we will formulate a deterministic no-collision constraint similar to the one in (3.24).

3.6.1 Local MPC Formulation

Consider a vehicle formation scenario in the previous section. Consider Vehicle i whose dynamic equation is given by

$$x_{k+1}^{i,v} = A_v x_k^{i,v} + B_v u_{k|k}^i + G_v q_k^{i,real}, \tag{3.25}$$

where $x_k^{i,v}$ is the state vector and $u_{k|k}^i$ denotes the MPC-input vector u_k^i computed at time k. We assume that the dynamics of each vehicle are linear, time-invariant, identical, and precisely given by A_v, B_v, and G_v. The real disturbance $q_k^{i,real}$ can be empirically measured from nature as time-domain-data sets before the estimator and control design processes and it is assumed that $q_k^{i,real}$ is the typical disturbance class acting on Vehicle i. We seek to describe $q_k^{i,real}$ by a linear model

$$\begin{aligned} x_{k+1}^{i,d} &= A_d^i x_k^{i,d} + G_d^i w_k^i, \\ q_k^i &= C_d^i x_k^{i,d}, \end{aligned} \tag{3.26}$$

with the disturbance state $x_k^{i,d}$, the output q_k^i (same dimension as $q_k^{i,real}$), and fictitious white noise input w_k^i. Since we will never obtain a perfect model for $q_k^{i,real}$, we only aim to obtain a reasonable model, where the acceptability of a model is determined by its ability to yield a good prediction of $q_k^{i,real}$. Then we have the following equations:

$$x_{k+1}^i = A^i x_k^i + B u_{k|k}^i + G^i w_k^i, \tag{3.27}$$

where

$$x_k^i = \begin{bmatrix} x_k^{i,v} \\ x_k^{i,d} \end{bmatrix}, A^i = \begin{bmatrix} A_v & G_v C_d^i \\ 0 & A_d^i \end{bmatrix}, B = \begin{bmatrix} B_v \\ 0 \end{bmatrix}, G^i = \begin{bmatrix} 0 \\ G_d^i \end{bmatrix},$$

$$position\text{-}output: \ y_k^i = C x_k^{i,v},$$

$$measurement: \ z_k^i = D x_k^{i,v} + v_k^i.$$

The measurement z_k^i is corrupted by bounded random noise v_k^i. Here we assume that the choice of A_d^i, G_d^i, and C_d^i satisfies the requirements. Recall the no-collision constraint (3.14)

$$||y_{k+j}^i - y_{k+j}^\ell|| > \alpha, \tag{3.28}$$

with $M = I$. The issue is how we could convert this constraint without any probabilistic assumptions on disturbance and noise processes. Indeed it is not possible to obtain deterministic form of the no-collision constraint in terms of covariances due to the absence of probabilistic models for disturbances and noises. However, since we have data profiles of disturbances and noises are bounded, we may be able to compute the worst estimation errors of the state estimators. Consider

$$|\tilde{y}_{k+j|k}^i|_p \triangleq \max_k ||\tilde{y}_{k+j|k}||, \quad |\tilde{y}_{i,k+j|k}^\ell|_p \triangleq \max_k ||\tilde{y}_{i,k+j|k}^\ell||, \tag{3.29}$$

the worst two norms of the j-step self- and cross-prediction errors over time. Then using the relationship $y = \hat{y} + \tilde{y}$ and the triangle inequality, (3.28) is converted to

$$||\hat{y}_{k+j+1|k}^i - \hat{y}_{i,k+j+1|k}^\ell|| > |\tilde{y}_{k+j|k}^i|_p + |\tilde{y}_{i,k+j|k}^\ell|_p + \alpha. \tag{3.30}$$

The constraint (3.30) means that the control for Vehicle i must be chosen in such a way that the distance between $\hat{y}_{k+j|k}^i$ and $\hat{y}_{i,k+j|k}^\ell$ is greater than the sum of the worst position error bounds. This can be understood as no overlapping of two circles around $\hat{y}_{k+j|k}^i$ and $\hat{y}_{i,k+j|k}^\ell$ as shown in Figure 3.3. The deterministic MPC for Vehicle i is

$$\arg \min_{\{u_{k|k}^i, \dots, u_{k+N-1|k}^i\}} \sum_{j=0}^{N-1} (\hat{y}_{k+j+1|k}^i - r^i)^T Q_c (\hat{y}_{k+j+1|k}^i - r^i) + u_{k+j|k}^{1\,T} R_c u_{k+j|k}^1,$$

$$subject \ to: \ \hat{x}_{k+j+1|k}^1 = A^i \hat{x}_{k+j|k}^i + B u_{k+j|k}^i,$$

$$\hat{y}_{k+j+1|k}^i = C \hat{x}_{k+j+1|k}^i,$$

$$u_{k+j|k}^i \in \mathbf{U},$$

$$||\hat{y}_{k+j+1|k}^i - \hat{y}_{i,k+j+1|k}^j|| > |\tilde{y}_{k+j|k}^i|_p + |\tilde{y}_{i,k+j|k}^\ell|_p + \alpha.$$
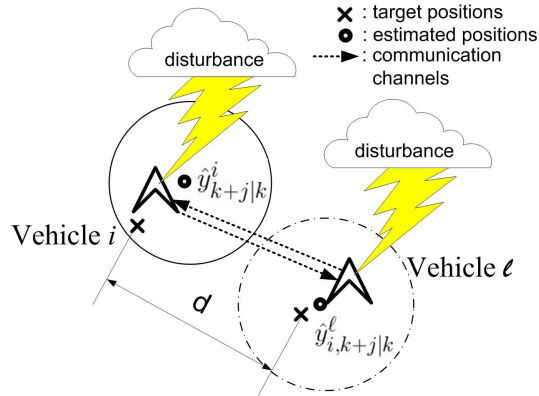
Figure 3.3: The graphical description of the control at Vehicle i. The actual vehicle positions are inside the solid and dashed circles.

Remark 3.6.1. *As discussed earlier, one may use $|\tilde{y}^i_{k+1|k}|_p + |\tilde{y}^\ell_{i,k+1|k}|_p$ instead of $|\tilde{y}^i_{k+j|k}|_p + |\tilde{y}^\ell_{i,k+j|k}|_p$.*

Remark 3.6.2. *If there exists a state constraint $\mathbf{X}_k$, then it should be modified to reflect estimation errors in a similar way to (3.23). In this case, the constraint tightening technique (Richards & How 2004) may be used.*

Remark 3.6.3. *Since the proposed MPC is also a non-convex problem due to the no-collision constraint, we use a nonlinear optimization solver such as SNOPT (Gill, Murray & Saunders 2006).*

3.7 Conclusion

We have shown that the requirement of the local constrained control via communication in the vehicle formation can be understood by analyzing the regions of the LQG-controlled closed-loop state covariance and the prediction error covariance of an isolated vehicle position associated with formation geometry. In the case that the local constrained control is required, we can formulate a local constrained MPC to reject disturbance and avoid collisions. Since the no-collision constraint takes information from the cross-estimator, in the next chapter, we will focus on the cross-estimator design and its related issues such as communication resource.

This chapter is in part a reprint of the materials as it appears in,

Keunmo Kang, David D. Zhang, Robert R. Bitmead - *Disturbance Rejection Control in Coordinated Systems*, 17th IFAC World Congress, Seoul, Korea, Jul., 2008.

Keunmo Kang, Jun Yan, Robert R. Bitmead - *Cross-Estimator Design for Coordinated Sytems; Constraints, Covariance, and Communication Resource Assignment*, Automatica, Vol. 41, May. 2008, pp. 1394–1401.

Keunmo Kang, Jun Yan, Robert R. Bitmead - *Communication Resources for Disturbance Rejection in Coordinated Vehicle Control* , 45th IEEE Conference on Decision and Control, San Diego, CA, USA, Dec., 2006.

Robert R. Bitmead, Jun Yan, Keunmo Kang - *Constrained Control and Communication Resource Assignment in Coordinated Systems*, International Control Conference, Glasgow, Scotland, Aug., 2006.

The dissertation author was the primary or co-author in these publications and professor Bitmead directed and supervised the research.

4

Cross-Estimator for Coordinated Systems: Constraints, Covariance, and Communication Resource Assignment

4.1 Introduction

In this chapter, we concentrate on the technique for one vehicle to estimate neighbors' future behavior so that each vehicle can use this for its control decision process. The main features of this chapter are:

– Deriving, from this deterministic constrained control problem in the previous chapter, sufficient requirements on the covariance of predictions of other vehicles state for only sporadic activity of the constraints in the feedback control,

– Tying together the covariance requirement on the cross-estimator design and the communication resource assignment using a Linear Matrix Inequality (LMI) approach.

Control approaches to coordination in (Dunbar & Murray 2004, Kuwata, Richards, Schouwenaars & How 2004, Richards & How 2004, Jeanne, Leonard & Paley 2005,

Dunbar & Murray 2006, Sepulchre, Paley & Leonard 2008) were proposed in a full-information-sharing environment. Here we adopt an approach which permits the inclusion of limited information sharing by introducing disturbances and the associated uncertainty of systems states. Of central concern is the covariance of the predicted state of a cross-estimator, which estimates the state of neighbors from communicated information consisting of some limited data concerning the neighbor's self-estimated state and computed control inputs.

The cross-estimation error covariance is important in the sense that the overall control performance is related to the prediction error covariances. This is captured by no-collision constraints. No-collision corresponds to non-intersection of the regions of uncertainty of the two systems which is quantified by an underlying estimation error covariance matrix. The cross-estimate error covariance calculation accommodates the following effects: limited communication bandwidth, delay, random packet dropout, disturbance acting on neighboring vehicles, and neighboring vehicles' state estimate errors. Especially, as the network of coordinated systems increases in the number of subsystems, natural limits on the available communication bandwidth need to be imposed. In this chapter, we shall show how cross-estimator design and communication resource management can be treated in one set of Linear Matrix Inequalities (LMI). It is still possible to accommodate random packet loss or delay in our formulation.

In addition, we develop a cross-estimator design technique for a coordinated system with a different controller structure. We consider the case that a subsystem has a fixed feedback control law that uses its self- and cross-state estimates of its neighboring subsystems. In this case, the goal is to design a cross-estimator that minimizes the variance of the cross-estimation error. We can still provide LMI formulation to achieve our goal.

In the previous chapter, we also discussed how we may control a coordinated system if the only available information about the disturbance and the noise is their bounds, but not probabilistic descriptions for them. We have seen that interaction constraints such as the no-collision constraint contain the worst case estimation error magnitudes. This impacts the estimator design process and we will show how we can design self- and cross-estimators under this situation.

4.2 Cross-Estimator and Information Architecture

4.2.1 Control Requirements of Estimation

We recall the coordinated system control problem in the previous chapter with the same assumptions:

A1. The dynamics of each vehicle are linear, time-invariant and identical.

A2. We consider only stationary values of covariances, without introducing the transient calculations, which are easy to accommodate.

A3. The spectral properties of the disturbances acting on each vehicle are identical and independent.

A4. For the theoretical development, the disturbances are zero-mean and gaussian.

A5. The nominal reference trajectory of each vehicle is known to all others in advance and the target formation geometry is fixed.

A6. Some limited communication is permissible: this leads to a decentralized or distributed disturbance rejection control problem.

Denote the joint state of the ith vehicle and of its disturbance at time k by x_k^i. Denote the control signal as u_k^i, the actual position by y_k^i, and the measurement signal by z_k^i. The vehicle and its locally acting disturbance are jointly described by

$$x_{k+1}^i \quad = \quad Ax_k^i + Bu_{k|k}^i + w_k^i, \tag{4.1}$$

$$y_k^i \quad = \quad Cx_k^i, \tag{4.2}$$

$$z_k^i \quad = \quad Dx_k^i + v_k^i, \tag{4.3}$$

with $x_k^i \in \mathbb{R}^\xi$ and $u_{k|k}^i \in \mathbb{R}^\psi$. We take w_k^i and v_k^i to be independent, zero-mean, white noise sequences. Then the deterministic MPC for formation keeping and collision avoidance is constructed by solving the following problem at each time step k with $\hat{x}_{k|k}^i$ and

$$\{\hat{y}_{i,k+1|k}^{\ell},\ \hat{y}_{i,k+2|k}^{\ell}, \ldots, \hat{y}_{i,k+N|k}^{\ell}\}:$$

$$\arg \min_{\{u_{k|k}^i, \ldots u_{k+N-1|k}^i\}} \sum_{j=1}^{N} \hat{x}_{k+j|k}^{i}{}^{T} Q_c \hat{x}_{k+j|k}^{i} + u_{k+j-1|k}^{i}{}^{T} R_c u_{k+j-1|k}^{i}$$

$$\textit{subject to:}\ \ \hat{x}_{k+j|k}^{i} = A\hat{x}_{k+j-1|k}^{i} + Bu_{k+j-1|k}^{i},$$

$$\hat{y}_{k+j|k}^{i} = C\hat{x}_{k+j|k}^{i},$$

$$\hat{x}_{k+j|k}^{i} \in \bar{\mathbf{X}}_k,$$ (4.4)

$$u_{k+j-1|k}^{i} \in \mathbf{U}_k,$$

$$||\hat{y}_{k+j|k}^{i} - \hat{y}_{i,k+j|k}^{\ell}||_M \ge \alpha + \beta\sqrt{\lambda_{\max}\left(P_{i,j}^{\ell}{}^{\frac{1}{2}} M P_{i,j}^{\ell}{}^{\frac{1}{2}}\right)},$$

where $P_{i,j}^{\ell} = C\left(S_{p,k+j|k}^{i} + \Sigma_{i,k+j|k}^{\ell}\right)C^T$. Henceforth, since we are assuming that our problem is stationary, we take $\Sigma_{i,j}^{\ell}$ not to be a function of time k. Where it clarifies matters, we use the $k+j|k$ subscript even though we are considering the stationary case. The cross-estimates $\hat{x}_{i,k+j|k}^{\ell}$ play a central role in the no-collision constraint, and their stationary position covariances $C\Sigma_{i,j}^{\ell}C^T$ appear in the standoff added to the constraint margins. We now focus on the connection between the control objective and the estimation tasks. Suppose that $y_{\star,k}^{i}$ is the target position of Vehicle i at time k. Since the target formation geometry is fixed, $||y_{\star,k}^{i} - y_{\star,k}^{\ell}||_M$ will be constant and is denoted by $||y_{\star}^{i} - y_{\star}^{\ell}||_M$. We observe that, under a stationarity assumption, it is desirable to ensure that the constraints are not always active, but only become active as needed and in response to the disturbances. This provides the key to formulating the control requirements of estimation via an inequality on state/position cross-estimation, $P < W$, where P is the matrix of covariances of the state cross-estimators and W is a design parameter reflecting the vehicle dynamics, the disturbance environment, the tightness of the no-collision constraints, and the information exchange between vehicles.

Definition 4.2.1. *We say that the (i, ℓ) no-collision constraint is **usually inactive**, if the target positions satisfy, $||y_{\star}^{i} - y_{\star}^{\ell}||_M > \alpha + \beta\sqrt{\lambda_{\max}\left(P_{i,j}^{\ell}{}^{\frac{1}{2}} M P_{i,j}^{\ell}{}^{\frac{1}{2}}\right)}$, for $j = 1, \ldots, N$.*

Note that having a constraint usually inactive implies that, in nominal operation, the constraint does not activate other than as an exception to deal with relatively infrequent disturbance-induced movement of a neighbor.

Lemma 4.2.1. *The (i, ℓ) no-collision constraint is usually inactive provided the covariance satisfies*

$$C\Sigma_{i,k+j|k}^{\ell}C^T \leq \frac{(||y_{\star}^i - y_{\star}^{\ell}||_M - \alpha)^2}{\beta^2}M^{-1} - CS_{p,k+j|k}^i C^T. \tag{4.5}$$

This lemma describes an important connection between the control problem of managing a fleet of vehicles and the concomitant estimation performance to achieve this. Shortly, we shall formulate this further in terms of communications resource management.

4.2.2   Modeling the Communication Channel

The cross-estimation of neighbors' states demands a non-standard information architecture, because the control signal of Vehicle ℓ is not perfectly known to Vehicle i. This information, like the state information, must be transmitted over an imperfect communication channel.

Communication modeling should capture a number of effects, the relative importance of which will depend on the application; limited bandwidth or equivalently maximal total bit-rate, dropped packets and unreliability of the network, and network delays in the arrival of data. The principal feature that we seek to develop is the requirement for the communications to scale properly with increasing numbers of interacting systems. As this number increases, so too does the demand on communications resources and some methodology needs to be presented to assist in the orderly solution of this problem. Here we link the assignment of communications resources to the achieved covariance of the cross-estimators.

In underwater operations, communications channels are typically very low bit-rate. Likewise in situations requiring stealth or limited environmental impact, there might be a desire to limit communications rates seriously. Our development will focus on the assignment of bit-rates in situations such as this and will rely on modeling the effect of bit-rate on the cross-estimator covariances, via a convexified optimization problem. For systems with deterministic communication delay, it is straightforward to incorporate delay effects directly into the prediction error covariance models. When one introduces stochastically varying delays or packet drops, then one can still compute the covariance in real time and, with tools such as those of (Sinopoli, Schenato, Franceschetti, Poolla,

Jordan & Sastry 2004), stationary statistics of these resulting covariances. The methodology of using cross-estimator covariance calculation appears to capture most of these phenomena well. While our approach is to analyze the stationary case, the techniques would seem to apply equally to real-time evaluation of constraints.

We model the quantization error of data transmitted over such a channel as additive white noise. Assuming that a scalar signal has been scaled properly to a range of $[-0.5, 0.5]$, the effect of quantization to m_j bits of accuracy is to add a white, zero-mean noise of variance $\frac{1}{12}2^{-2m_j}$ (Widrow, Kollár & Liu 1996). From a linear estimation perspective, this has the effect of adding measurement noise. If the total bandwidth to be assigned to a channel is limited to, say, τ bits per sample time and each piece of information from $j = 1$ to J $(= \xi + N \times \psi)$ is assigned by $\{m_j\}$ bits, then the total bandwidth limit is captured by the inequality

$$\sum_{j=1}^{J} m_j \leq \tau. \tag{4.6}$$

4.2.3   Dealing with the Control Signal

Recall the linear equations describing the ℓ-th dynamical system's evolution and measurement z_k^ℓ as in (4.1-4.3). On the basis of known control signal $\{u_{k-1|k-1}^\ell\}$ and measured output $\{z_k^\ell\}$, Vehicle ℓ uses a Kalman filter to estimate $\hat{x}_{k|k}^\ell$ and, from this to solve the MPC. This Kalman estimate has associated covariance $\Sigma_{k|k}^\ell$. Vehicle ℓ might transmit this own-state estimate to Vehicle i over the limited-bit-rate channels and acquire an increase in measurement noise covariance to $\Sigma_{i,k|k}^\ell = \Sigma_{k|k}^\ell + R_x$. Here R_x is the covariance matrix associated with the bit-rate limitations and Vehicle i might use an observer to update its estimate of x^ℓ; these are to be discussed shortly. However, to compute further predicted cross state estimates Vehicle ℓ's control signal is needed. This introduces new issues to the problem of cross state estimation. Since MPC is used at each vehicle, at time k prior to the computation of the new control, only the previous control sequence is available for transmission over the limited-bit-rate channels to Vehicle i for use in its constrained MPC computation. The time-k control calculation at Vehicle i requires prediction of Vehicle ℓ's position up to time $k + N$, which in turn requires control values up to $u_{k+N-1|k}^\ell$ – one short of the full sequence available for transmission.

Likewise, because of obvious timing issues, the control sequence used is one sample out-of-date compared with the current value. How ought this be accommodated in this framework? At time k, Vehicle i receives the following information from Vehicle ℓ:

$$
\begin{bmatrix}
\check{x}^\ell_{i,k|k} \\
\hat{u}^\ell_{i,k-1|k} \\
\hat{u}^\ell_{i,k|k} \\
\vdots \\
\hat{u}^\ell_{i,k+N-2|k}
\end{bmatrix}
=
\begin{bmatrix}
\hat{x}^\ell_{k|k} \\
u^\ell_{k-1|k-1} \\
u^\ell_{k|k-1} \\
\vdots \\
u^\ell_{k+N-2|k-1}
\end{bmatrix}
+
\begin{bmatrix}
\nu^{i,\ell}_{x,k} \\
\nu^{i,\ell}_{u_1,k} \\
\nu^{i,\ell}_{u_2,k} \\
\vdots \\
\nu^{i,\ell}_{u_N,k}
\end{bmatrix}.
\tag{4.7}
$$

where $\nu^{i,\ell}_{\cdot,k}$ is the quantization noise corresponding to the bit-rate assignment for that information.

Remark 4.2.1. *It is possible to transmit the measurement z^ℓ_k to Vehicle i instead of $\hat{x}^\ell_{k|k}$. In this case, instead of $\check{x}^\ell_{i,k|k}$, Vehicle i will receive partial state measurement corrupted by Vehicle ℓ's measurement noise (v^ℓ_k) and the communication error.*

By adopting the model for quantization effects in (Widrow, Kollár & Liu 1996), we can model $Cov(\nu^{i,\ell})$ as follows

$$
\begin{aligned}
R_x &\triangleq Cov(\nu^{i,\ell}_x) = \frac{1}{12}\mathrm{diag}\{2^{-2m^{i,\ell}_{x1}}, 2^{-2m^{i,\ell}_{x2}}, \cdots, 2^{-2m^{i,\ell}_{x\xi}}\}, \\
R_{u_j} &\triangleq Cov(\nu^{i,\ell}_{u_j}) = \frac{1}{12}\mathrm{diag}\{2^{-2m^{i,\ell}_{u_j1}}, 2^{-2m^{i,\ell}_{u_j2}}, \cdots, 2^{-2m^{i,\ell}_{u_j\psi}}\},
\end{aligned}
\tag{4.8}
$$

where $m^{i,\ell}_{\cdot,r}$ is the r th element of $m^{i,\ell}_{\cdot}$. A filter is used to incorporate the transmitted state information into the current cross-estimate of Vehicle ℓ's state,

$$
\hat{x}^\ell_{i,k|k} = \left(I-K^\ell_i\right)A\hat{x}^\ell_{i,k-1|k-1} + \left(I-K^\ell_i\right)B\hat{u}^\ell_{i,k-1|k} + K^\ell_i\check{x}^\ell_{i,k|k}.
\tag{4.9}
$$

The following predictor can then be applied for the propagation of the state cross-predictions.

$$
\hat{x}^\ell_{i,k+j+1|k} = A\hat{x}^\ell_{i,k+j|k} + B\hat{u}^\ell_{i,k+j|k}, \quad \forall j = 1, 2, \cdots, N-1,
\tag{4.10}
$$

$$
\hat{x}^\ell_{i,k+N|k} = A\hat{x}^\ell_{i,k+N-1|k} + B\hat{u}^\ell_{i,k+N-2|k}.
\tag{4.11}
$$

That is, the final (missing) control element, $\hat{u}^\ell_{i,k+N-1|k}$, is taken to be identical to its predecessor, $\hat{u}^\ell_{i,k+N-2|k}$. Note that (4.9-4.11) could be standard Kalman filter/prediction

equations if the covariance matrices R_x and R_{u_j} were known. In our framework, the goal is to design the cross-estimator gain and the bit-rate variables so that (4.5) can be satisfied.

This is done via the calculation of covariances of $x^\ell_{k+j} - \hat{x}^\ell_{i,k+j|k}$ that involves errors due to the discrepancy of control signals and the communication bandwidth assigned. Specifically because of the timing issues of coordinated MPC, we are constructing state cross-prediction based on applying noisy versions of the previous time sample's control sequence $\{u^\ell_{k+j|k-1}\}$ in place of the actual control $\{u^\ell_{k+j|k}\}$ and we need to limit the variability of the MPC control solutions. In modeling terms it may be captured by writing

$$\begin{bmatrix} u^\ell_{k|k} \\ \vdots \\ u^\ell_{k+N-2|k} \\ u^\ell_{k+N-1|k} \end{bmatrix} = \begin{bmatrix} u^\ell_{k|k-1} \\ \vdots \\ u^\ell_{k+N-2|k-1} \\ u^\ell_{k+N-2|k-1} \end{bmatrix} + \begin{bmatrix} \eta^\ell_{1,k} \\ \vdots \\ \eta^\ell_{N-1,k} \\ \eta^\ell_{N,k} \end{bmatrix}, \tag{4.12}$$

for some noise process $\{\eta^\ell_{j,k}\}$ with appropriately chosen covariance matrix

$$H \triangleq \mathrm{diag}\{Cov(\eta^\ell_{1,k}), Cov(\eta^\ell_{2,k}), \cdots, Cov(\eta^\ell_{N,k})\}$$

$$\triangleq \mathrm{diag}\{H_1, H_2, \cdots, H_N\}.$$

Remark 4.2.2. *The implementation of such as restriction on the variation of successive control solutions would be effected through the selection of $\mathbf{U}_k$ in the MPC problem (Dunbar & Murray 2004, Dunbar & Murray 2006).*

Remark 4.2.3. *In some applications, it might not be reasonable to employ the above control signal information structure. In the worst case, one could model the control signal (restricted by $\mathbf{U}_k$ that contains zero) by a white, zero-mean noise sequence with an approximated covariance matrix based on $\mathbf{U}_k$.*

The cross-estimator is formed by combining the recursions from (4.10-4.12) across the horizon of interest. This yields a linear recursion for cross-predictions based on current estimates and transmitted data. The covariance is calculated by standard means, to be detailed in the next section, and then absorbed into the constraint management of a deterministic problem. The bandwidth limitation lies in the restriction that (4.6) places on the assigned covariances of quantization noises $\nu^{i,\ell}$.

4.3 Cross-Estimator Design and Bandwidth Assignment via LMI

In Section 4.2, we specified a set of linear filters for the propagation of cross-predictions from transmitted data. The covariance of the estimates can then be calculated via estimate errors produced from the state and control transmission equation (4.7), the control delay error captured by (4.12), the recursion (4.9), the computation of cross-predictions via (4.10-4.11), and the true system (4.1). The driving noises for these covariance calculations are the local process noises $\{w_k^i, w_k^\ell\}$ and measurement noises $\{v_k^i, v_k^\ell\}$ from (4.1) and (4.3) affecting own-state estimation, the control variation noise η_k^ℓ from (4.12), and the communications quantization noise $\nu^{i,\ell}$ from (4.7). For the moment, the sole design variable of the cross-estimator is the estimator gain matrix K_i^ℓ. Combining (4.1) and the first element of (4.7),

$$\tilde{x}_{i,k|k}^\ell = Ax_{k-1}^\ell + Bu_{k-1|k-1}^\ell + w_{k-1}^\ell + \tilde{x}_{k|k}^\ell + \nu_{x,k}^{i,\ell},$$

where $\tilde{x}_{k|k}^\ell$ denotes the own-state Kalman filter error signal. This may be substituted into (4.9) to yield an expression for the filtered cross-state error:

$$\tilde{x}_{i,k|k}^\ell = (I-K_i^\ell)A\tilde{x}_{i,k-1|k-1}^\ell - (I-K_i^\ell)B\nu_{u_1,k}^{i,\ell} + (I-K_i^\ell)w_{k-1}^\ell - K_i^\ell\tilde{x}_{k|k}^\ell - K_i^\ell\nu_{x,k}^{i,\ell}. \qquad (4.13)$$

This equation (4.13) describes the evolution of the filtered cross-state estimate error signal. If we consider fixed gain matrix K_i^ℓ then, provided the matrix $(I - K_i^\ell)A$ is stable, this error has a fixed covariance, denoted $\Sigma_{i,k|k}^\ell$. Using the error equation (4.13) we can establish the following second moment inequality.

Lemma 4.3.1. *Suppose that the following inequality holds with $P > 0$.*

$$P - (I-K_i^\ell)APA^T(I-K_i^\ell)^T - (I-K_i^\ell)(BR_{u_1}B^T+Q_o)(I-K_i^\ell)^T - K_i^\ell(S_f^\ell + R_x)K_i^{\ell T} > 0, \quad (4.14)$$

with S_f^ℓ the stationary Kalman filter covariance at Vehicle ℓ and $Q_o \triangleq Cov(w_k^\ell)$. Then the filtering cross-estimator is stable and has a limiting covariance $\lim_{k\to\infty}\left[\Sigma_{i,k|k}^\ell\right] < P$.

The subsequent cross-predictor covariances $\Sigma_{i,k+j|k}^\ell$ are computable in similar

fashion from (4.1), (4.10), (4.11), and (4.12):

$$\tilde{x}^\ell_{i,k+j|k}=A^j\tilde{x}^\ell_{i,k|k}+\sum_{m=1}^{j}A^{j-m}B(\sum_{n=1}^{m}\eta^\ell_{n,k+m-n})-\sum_{m=2}^{j+1}A^{j+1-m}B\nu^{i,\ell}_{u_m,k}+\sum_{m=0}^{j-1}A^m w^\ell_{k+(N-1-m)},$$

(4.15)

whence

$$\Sigma^\ell_{i,k+j|k}\triangleq \mathcal{A}_j\begin{bmatrix}\Sigma^\ell_{i,k|k} & & & \\ & R_{u_2} & & \\ & & \ddots & \\ & & & R_{u_{j+1}}\end{bmatrix}\mathcal{A}_j^T+\mathcal{Q}_j,$$

(4.16)

where

$$\mathcal{A}_j \triangleq [A^j, A^{j-1}B,\ldots, B],$$

$$\mathcal{Q}_j \triangleq \sum_{m=1}^{j}A^{j-m}B(\sum_{n=1}^{m}H_n)B^T A^{j-m^T} + \sum_{m=0}^{j-1}A^m Q_o A^{m^T}.$$

(4.17)

The expression H_n denotes $Cov(\eta_{n,k})$ as in (4.12). Also note that in this iteration, when $j = N$, $R_{u_{N+1}}$ is equal to R_{u_N} due to the particular choice of control sequence update (4.12). Recall that $\Sigma^\ell_{i,k+j|k}$ should be bounded from above as in (4.5) in Lemma 4.2.1. Substituting (4.16) into (4.5) yields a new inequality imposing bounds on $\Sigma^\ell_{i,k|k}$ and R_{u_j}s:

$$\mathcal{C}_j\begin{bmatrix}\Sigma^\ell_{i,k|k} & & & \\ & R_{u_2} & & \\ & & \ddots & \\ & & & R_{u_{j+1}}\end{bmatrix}\mathcal{C}_j^T \leq \mathcal{W}_j,$$

(4.18)

where $\mathcal{C}_j \triangleq C\mathcal{A}_j$ and $\mathcal{W}_j \triangleq \frac{(||y_\star^i-y_\star^\ell||_M-\alpha)^2}{\beta^2}M^{-1} - CS^i_{p,k+j|k}C^T- C\mathcal{Q}_jC^T$. By taking the Schur complement of (4.18),

$$\begin{bmatrix}\mathcal{W}_j & CA^j & CA^{j-1}B & \ldots & CB \\ A^{j^T}C^T & Y & & & \\ B^T A^{j-1^T}C^T & & R^{-1}_{u_2} & & \\ \vdots & & & \ddots & \\ B^T C^T & & & & R^{-1}_{u_{j+1}}\end{bmatrix} \geq 0,$$

(4.19)

where $P = Y^{-1}$. Clearly, if $\mathcal{W}_j$ is negative definite, then (4.19) is infeasible. In this case, we can restore the feasibility (positive definiteness of $\mathcal{W}_j$) by increasing the distance

between the target positions, by reducing Vehicle i's self-prediction covariance, or by decreasing the accumulative noise covariance $\mathcal{Q}_j$ via decreasing H_n (i.e. limiting the changes between the control sequences).

4.3.1 Three Cross-Estimator Requirements

We have three specific requirements of the cross-estimator:

R1. The cross-estimator should be stable.

R2. The error covariance of the cross-estimator should satisfy the control performance requirement (4.5) or (4.19) of achieving an overbound on the prediction error covariance associated with the target positions of the vehicles.

R3. The total bit-rate applied to all communicated information per sample should satisfy the upperbound (4.6).

Condition **R3** introduces the channel bit-rates as additional design variables to guarantee conditions **R1** and **R2**, which are nonlinear and, indeed, non-convex in the bit-rates. They can, however, be satisfied by a convexified LMI formulation to be discussed later.

4.3.2 Incorporating Communications Limits

In the LMI formulation of the covariance constraints, the communications noises $(\nu^{i,\ell})$ are represented by their covariances, as is done by R_x and R_{u_j} in (4.14) and (4.19). Here we attempt to make the bit-rate assignment to particular channels also a part of the design problem, provided the inclusion of the bandwidth constraint (4.6) is amenable. Using the inequality, $x \geq 1 + \ln x$ for $x > 0$, the inverse expressions of (4.8) are

$$
\begin{aligned}
R_x^{-1} &\geq \begin{bmatrix} 1+\ln 12+2(\ln 12)m_{x,1}^{i,\ell} & & & \\ & 1+\ln 12+2(\ln 12)m_{x,2}^{i,\ell} & & \\ & & \ddots & \\ & & & 1+\ln 12+2(\ln 12)m_{x,\xi}^{i,\ell} \end{bmatrix} \triangleq \bar{R}_x^{-1}, \\
R_{u_j}^{-1} &\geq \begin{bmatrix} 1+\ln 12+2(\ln 12)m_{u_j,1}^{i,\ell} & & & \\ & 1+\ln 12+2(\ln 12)m_{u_j,2}^{i,\ell} & & \\ & & \ddots & \\ & & & 1+\ln 12+2(\ln 12)m_{u_j,\psi}^{i,\ell} \end{bmatrix} \triangleq \bar{R}_{u_j}^{-1}.
\end{aligned}
\tag{4.20}
$$

We immediately have a set of linear inequalities in bit-rate variables $m_x^{i,\ell}$ and $m_u^{i,\ell}$. Note that $\bar{R}_x, \bar{R}_{u_j}$ serve as upper bounds on the original R_x and R_{u_j}.

4.3.3 Cross-Estimator LMIs

We are now ready to draw the strings together from control-specified estimator performance (4.5), cross-estimator stability and covariance calculation (4.14), and total bit-rate limit (4.6) to yield a single set of LMIs, whose solution, if it exists, would yield; bit-rates $m_x^{i,\ell}, m_{u_1}^{i,\ell}, \cdots, m_{u_N}^{i,\ell}$, cross-estimator gains K_i^ℓ, and limiting covariances P.

Theorem 4.3.1. *Any solution $\{P = Y^{-1}, L = YK_i^\ell, m_x^{i,\ell}, m_{u_1}^{i,\ell}, \cdots, m_{u_N}^{i,\ell}\}$ to the following set of LMIs,*

$$\begin{bmatrix} Y & (Y-L)A & (Y-L) & L & L & (Y-L)B \\ A^T(Y-L)^T & Y & 0 & 0 & 0 & 0 \\ (Y-L)^T & 0 & Q_o^{-1} & 0 & 0 & 0 \\ L^T & 0 & 0 & S_f^{\ell^{-1}} & 0 & 0 \\ L^T & 0 & 0 & 0 & \bar{R}_x^{-1} & 0 \\ B^T(Y-L)^T & 0 & 0 & 0 & 0 & \bar{R}_{u_1}^{-1} \end{bmatrix} > 0, \tag{4.21}$$

$$\sum_{k=1}^{\xi} m_{x,k}^{i,\ell} + \sum_{j=1}^{N} \sum_{k=1}^{\psi} m_{u_j,k}^{i,\ell} < \tau, \tag{4.22}$$

$$\begin{bmatrix} \mathcal{W}_j & CA^j & CA^{j-1}B & \dots & CB \\ A^{Tj}C^T & Y & & & \\ B^T A^{j-1^T}C^T & & \bar{R}_{u_2}^{-1} & & \\ \vdots & & & \ddots & \\ B^T C^T & & & & \bar{R}_{u_{j+1}}^{-1} \end{bmatrix} > 0, \tag{4.23}$$

where τ is the total bit-rate, $\bar{R}_x^{-1}$ and $\bar{R}_{u_j}^{-1}$ are given by (4.20), and $\mathcal{W}_j = \frac{(\|y_\star^i - y_\star^\ell\|_M - \alpha)^2}{\beta^2} \times M^{-1} - CS_{p,k+j|k}^i C^T - CQ_j C^T$, yields a solution for (4.5), (4.14), and (4.6).

Proof. The Schur complements of (4.14) lead to same the expression as (4.21) but with R_x^{-1} and $R_{u_1}^{-1}$ instead of $\bar{R}_x^{-1}$ and $\bar{R}_{u_1}^{-1}$. Due to the relationship (4.20), the satisfaction of (4.21) implies (4.14). If (4.23) is satisfied then (4.19) is satisfied. The scalar LMI (4.22) is a restatement of (4.6). □

The LMIs in Theorem 4.3.1 may be infeasible if we consider a fixed limited total available bandwidth τ in (4.22) and a fixed bound on the covariance P in (4.23) at the same time. This can be easily amended by taking the total bandwidth τ as a variable to be minimized subject to (4.21)-(4.23). Due to the approximation (4.20), the solution of the LMIs is a feasible solution but not the global minimum if applied to the vehicles.

Once the solution of the above LMIs is obtained, one may construct a new Kalman filter as a cross-estimator as if the transmitted information $(\check{x}_{i,k|k}^{\ell},\ \hat{u}_{i,k-1|k}^{\ell},\ \hat{u}_{i,k|k}^{\ell},\ \cdots,\ \hat{u}_{i,k+N-2|k}^{\ell})$ of (4.7) is the measured signal for Vehicle ℓ corrupted by quantization errors $\nu_x^{i,\ell}$ and $\nu_u^{i,\ell}$. In this case, using the bit-rate assignment solution from the above LMIs, one can compute R_x and R_u of (4.8) and use them as a measurement noise error covariance. Since $\bar{R}_x$ and $\bar{R}_u$ overbound R_x and R_u, the resulting state estimation error covariance from the newly constructed Kalman filter will underbound P from the LMIs in Theorem 4.3.1. In the case that K_i^{ℓ} is used in the cross-estimator, actually achieved state estimation error covariance is also smaller than P. To see this, consider the solution P, K_i^{ℓ}, $\bar{R}_x$, and $\bar{R}_u$ from Theorem 4.3.1. They satisfy the inequality

$$P-(I-K_i^{\ell})APA^T(I-K_i^{\ell})^T-(I-K_i^{\ell})(B\bar{R}_{u_1}B^T+Q_o)(I-K_i^{\ell})^T-K_i^{\ell}(S_f^{\ell}+\bar{R}_x)K_i^{\ell T}>0.$$

This implies that there exist a positive definite matrix Γ such that

$$P-\Gamma=(I-K_i^{\ell})APA^T(I-K_i^{\ell})^T+(I-K_i^{\ell})(B\bar{R}_{u_1}B^T+Q_o)(I-K_i^{\ell})^T+K_i^{\ell}(S_f^{\ell}+\bar{R}_x)K_i^{\ell T} \quad (4.24)$$

holds. The achieved estimation error covariance is given by solving

$$S_{f,i}^{\ell}=(I-K_i^{\ell})AS_{f,i}^{\ell}A^T(I-K_i^{\ell})^T+(I-K_i^{\ell})(BR_{u_1}B^T+Q_o)(I-K_i^{\ell})^T+K_i^{\ell}(S_f^{\ell}+R_x)K_i^{\ell T}, \quad (4.25)$$

where $S_{f,i}^{\ell}$ is the actual cross-estimation error covariance of Vehicle ℓ at Vehicle i. By subtracting (4.25) from (4.24), we obtain

$$P-S_{f,i}^{\ell}=(I-K_i^{\ell})A(P-S_{f,i}^{\ell})A^T(I-K_i^{\ell})^T+(I-K_i^{\ell})(B(\bar{R}_{u_1}-R_{u_1})B^T)(I-K_i^{\ell})^T$$
$$+K_i^{\ell}(\bar{R}_x-R_x)K_i^{\ell T}+\Gamma.$$

Since $\bar{R}_{u_1}>R_{u_1}$, $\bar{R}_x>R_x$, and $\Gamma>0$, we have $P>S_{f,i}^{\ell}$.
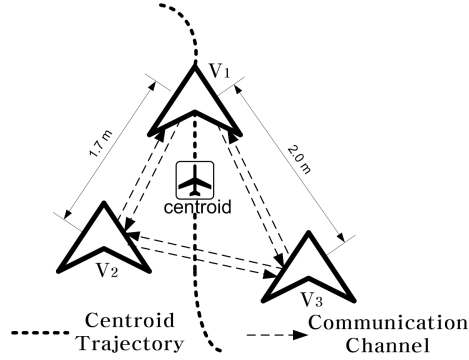
Figure 4.1: A formation of three vehicles.

4.4 Example

We consider a three vehicle formation as shown in Figure 4.1. Vehicles track a known $2D$ trajectory while keeping a certain formation. The vehicle model of HoTDeC in Appendix A is used. The model of each vehicle is assumed to be identical, with the state vector $x_k = [x_{p,k}\ y_{p,k}\ \theta_{p,k}\ x_{s,k}\ y_{s,k}\ \theta_{s_k}]^T$, where $(x_{p,k}, y_{p,k})$ and $(x_{s,k}, y_{s,k})$ represent positions and velocities in cartesian coordinates, and $\theta_{p,k}$ and $\theta_{s,k}$ are angular displacement and velocity. Full state measurement is assumed available. The vehicle produces actuation for x, y, and θ coordinates (i.e. $u_k \in \mathbb{R}^3$). The ambient disturbance processes are modeled by simple white noises with normal distributions $N(0, Cov(w_k))$ impinging directly onto the states of the vehicles where $Cov(w_k) = \text{diag}\{1, 1, 1, 10, 10, 10\}$. Each vehicle has self-measurements and communications with the other two. Each local controller is assumed to be a three-step MPC with 99% probability of no-collision with the others. As shown in the figure, the nominal separations between V_1 and V_2, V_3 are $s_{12} = 1.7$m and $s_{13} = 2.0$m. Our goal is to derive cross-estimator gains and a bandwidth assignment scheme with minimal total bit-rates for V_1 only. The problem is to minimize τ subject to two sets of (4.21) and (4.23), one for V_2 and one for V_3, and a combined version of (4.22) which sums up all the bit-rates. The formulation can be expanded to include others similarly. The additional noises on the control sequences $\{\eta_k\}$ in (4.12) are zero-mean, white gaussian with $Cov(\eta_k) = \text{diag}\{5, 5, 5, 5, 5, 5, 10, 10, 10\}$. The computation results showing bitrate assignments from LMI and corresponding integer approximation are shown in Table 4.1. A total of 120 bits per sample is assigned after

Table 4.1: Bitrate assignment at Vehicle 1. The bitrate assignment for the other state variables and control signals are omitted since they are all 1.

	Assigned bit-rates for states			
	x_p	y_p	x_s	y_s
V2	19.64 (20)	19.64 (20)	18.68 (19)	18.68 (19)
V3	4.20 (5)	4.20 (5)	4.19 (5)	4.19 (5)

rounding up the bandwidth values from the LMI. Since the effect of the inputs on the vehicle motion covariances is relatively small (due to the model), very small bit rates for the controls are enough to capture the neighbors' motion. The numerical results show another benefit of considering communication resource assignment. That is, one can observe what is more important and what is not to predict neighbors' behaviors by looking at bandwidth assignment results. For V_1, capturing neighbors' angular state information and control sequence information is not as important as state information of x and y coordinates to predict neighbors' positions. Our computation shows that a small difference in nominal separation of vehicles (say $s_{12} = 1.69 \ m$) may lead to dramatic increase in the required quality of information (more bits). This means a slight decrease in the bound $\mathcal{W}_j$ over prediction error covariance demands many additional bits. Very small bit-rate assignments may be rounded to zero, as all other assignments may be rounded up or down, and a computation (4.25) of the resultant exact cross-estimate covariance is made to validate the specific integer choice.

4.5 Cross-Estimator Design for Coordinated Systems with Fixed Feedback Control

As a special case, we consider a coordinated system whose subsystems use fixed control laws that use self- and cross-estimates. For simplicity, we assume a coordinated system with two subsystems ($i = 1, 2$):

$$Dynamic \ Equation : x_{k+1}^i = Ax_k^i + Bu_k^i + w_k^i,$$

$$Output : y_k^i = Cx_k^i,$$

$$Measurement : z_k^1 = Dx_k^i + v_k^i,$$

with the disturbance $w_k^i \sim (0, Q_i)$ and the measurement noise $v_k^i \sim (0, R_i)$. Their control laws are assumed to be

$$u_k^1 = M_{11}\hat{x}_{k|k}^1 + M_{12}\hat{x}_{1,k|k}^2 + l_1,$$

$$u_k^2 = M_{21}\hat{x}_{2,k|k}^1 + M_{22}\hat{x}_{k|k}^2 + l_2,$$

where the control gains M_{ij} and the additive constants l_i are known to both subsystems. The self-estimates $\hat{x}_{k|k}^1$ and $\hat{x}_{k|k}^2$ are computed by standard Kalman filters. The cross-estimators are given by

$$Cross - Estimator\ 1\ at\ 2:\ \hat{x}_{2,k+1|k}^1 = A\hat{x}_{2,k|k}^1 + B\bar{u}_k^1,$$

$$\hat{x}_{2,k+1|k+1}^1 = x_{2,k+1|k}^1 + K_2^1(z_{2,k}^1 - D\hat{x}_{2,k+1|k}^1),$$

$$Cross - Estimator\ 2\ at\ 1:\ \hat{x}_{1,k+1|k}^2 = A\hat{x}_{1,k|k}^2 + B\bar{u}_k^2,$$

$$\hat{x}_{1,k+1|k+1}^2 = x_{1,k+1|k}^2 + K_1^2(z_{1,k}^2 - D\hat{x}_{1,k+1|k}^2),$$

where

$$z_{2,k}^1 = Dx_k^1 + v_{2,k}^1, \quad \bar{u}_k^1 = M_{11}\hat{x}_{2,k|k}^1 + M_{12}\hat{x}_{k|k}^2 + l_1,$$

$$z_{1,k}^2 = Dx_k^2 + v_{1,k}^2, \quad \bar{u}_k^2 = M_{21}\hat{x}_{k|k}^1 + M_{22}\hat{x}_{1,k|k}^2 + l_2.$$

We model $v_{2,k}^1$ and $v_{1,k}^2$ as the noise caused by measurement and communication noises with $v_{j,k}^i \sim (0, R_j^i)$. Since the subsystem i does not have the direct knowledge of u_k^j, we attempt to capture u_k^i by $\bar{u}_k^i$. As usual define $\tilde{x} = x - \hat{x}$, then we obtain

$$\tilde{X}_{k+1} = (I - \bar{K}\bar{D})\bar{A}\tilde{X}_k + (I - \bar{K}\bar{D})\bar{B}w_k - \bar{K}v_{k+1},$$

where

$$\tilde{X}_k = \begin{bmatrix} \tilde{x}_{k|k}^1 \\ \tilde{x}_{2,k|k}^1 \\ \tilde{x}_{1,k|k}^2 \\ \tilde{x}_{k|k}^2 \end{bmatrix}, \quad w_k = \begin{bmatrix} w_k^1 \\ w_k^2 \end{bmatrix}, \quad v_{k+1} = \begin{bmatrix} v_{1,k+1}^1 \\ v_{2,k+1}^1 \\ v_{1,k+1}^2 \\ v_{2,k+1}^2 \end{bmatrix},$$

$$\bar{A} = \begin{bmatrix} A & 0 & 0 & 0 \\ -BM_{11} & (A+BM_{11}) & -BM_{12} & BM_{12} \\ BM_{21} & -BM_{21} & (A+BM_{22}) & -BM_{22} \\ 0 & 0 & 0 & A \end{bmatrix},$$

$$
\bar{B} = \begin{bmatrix} I & 0 \\ I & 0 \\ 0 & I \\ 0 & I \end{bmatrix}, \quad
\bar{K} = \begin{bmatrix} K^1 & 0 & 0 & 0 \\ 0 & K_2^1 & 0 & 0 \\ 0 & 0 & K_1^2 & 0 \\ 0 & 0 & 0 & K^2 \end{bmatrix}, \quad
\bar{D} = \begin{bmatrix} D & 0 & 0 & 0 \\ 0 & D & 0 & 0 \\ 0 & 0 & D & 0 \\ 0 & 0 & 0 & D \end{bmatrix}.
$$

Consider the steady state covariance $P = Cov(\tilde{X}_k)$ that satisfies

$$
P = (I - \bar{K}\bar{D})\bar{A}P\bar{A}^T(I - \bar{K}\bar{D})^T + (I - \bar{K}\bar{D})\bar{B}Q\bar{B}^T(I - \bar{K}\bar{D})^T + \bar{K}R\bar{K}^T,
$$

where

$$
Q = \begin{bmatrix} Q_1 & 0 \\ 0 & Q_2 \end{bmatrix}, \quad
R = \begin{bmatrix} R_1 & 0 & 0 & 0 \\ 0 & R_2^1 & 0 & 0 \\ 0 & 0 & R_1^2 & 0 \\ 0 & 0 & 0 & R_2 \end{bmatrix}.
$$

First of all, the estimators must be stable. Using the same idea of Lemma 4.3.1, if we find an upper bound of positive definite P for

$$
-P + (I - \bar{K}\bar{D})\bar{A}P\bar{A}^T(I - \bar{K}\bar{D})^T + (I - \bar{K}\bar{D})\bar{B}Q\bar{B}^T(I - \bar{K}\bar{D})^T + \bar{K}R\bar{K}^T < 0, \qquad (4.26)
$$

then the estimators are stable. Taking the Schur complement of (4.26) yields

$$
\begin{bmatrix}
-Y & (Y - L\bar{D})\bar{A} & (Y - L\bar{D})\bar{B}Q^{\frac{1}{2}} & LR^{\frac{1}{2}} \\
\bar{A}^T(Y - \bar{D}^T L^T) & -Y & 0 & 0 \\
Q^{\frac{1}{2}}\bar{B}^T(Y - \bar{D}^T L^T) & 0 & -I & 0 \\
R^{\frac{1}{2}}L^T & 0 & 0 & -I
\end{bmatrix} < 0, \qquad (4.27)
$$

where $P = Y^{-1}$ and $\bar{K} = PL$. From the perspective of formulating and solving non-classical information architecture control problems, however, an incipient problem arises through the inability to explore a block diagonal structure on the computed solution $\bar{M}$ from (4.27) without also imposing such a structure on Y and P. Evidently from the structure of $\bar{A}$ a block diagonal P is not typically of interest. Indeed it is the cross-covariance between terms such as $\tilde{x}_{1,k|k}^1$ and $\tilde{x}_{2,k|k}^1$ that captures the information architecture. Without a structural condition on $\bar{K}$, the solution of minimizing $\mathbf{tr}(Y^{-1})$ subject to (4.27) would yield the classical, fully-shared-measurement Kalman filtering solution. To explore the development of an LMI approach to finding feasible solutions to the non-standard information architecture problem, we employ a result of (de Oliveira, J.Bernussou & J.C.Geromel 1999).

Lemma 4.5.1. *(de Oliveira, J.Bernussou & J.C.Geromel 1999) The following statements are equivalent.*

(i) There exists a symmetric matrix $P > 0$ such that

$$A^T P A - P < 0.$$

(ii) There exist a symmetric matrix P and a matrix G such that

$$\begin{bmatrix} -P & A^T G^T \\ GA & -G - G^T + P \end{bmatrix} \leq 0.$$

Now we use Lemma 4.5.1 to establish the following theorem.

Theorem 4.5.1. *Matrices G, Y, and L satisfying*

$$\begin{bmatrix} -G-G^T+Y & (G-L\bar{D})\bar{A} & (G-L\bar{D})\bar{B} & LR^{\frac{1}{2}} \\ \bar{A}^T(G^T-\bar{D}^T L^T) & -Y & 0 & 0 \\ \bar{B}^T(G^T-\bar{D}^T L^T) & 0 & -I & 0 \\ R^{\frac{1}{2}}L^T & 0 & 0 & -I \end{bmatrix} \leq 0, \qquad (4.28)$$

yield $P = Y^{-1}$ and $\bar{K} = G^{-1}L$ which satisfy (4.26). Conversely, P and $\bar{K}$ satisfying (4.26) provide $Y = G = P^{-1}$, $L = P^{-1}\bar{K}$ which satisfy (4.28).

Corollary 4.5.1. *If G and L are constrained to be block diagonal matrices in (4.28), then $P = Y^{-1}$ and $\bar{K} = G^{-1}L$ are also feasible in (4.26) with $\bar{M}$ block diagonal.*

Corollary 4.5.2. *If matrices G, Y, and L, with G and L block diagonal conformably with $\tilde{X}_k$, can be found satisfying (4.28) then the state estimators, with the gains given by the block diagonal elements of $\bar{K} = G^{-1}L$, are stable and their covariances are bounded above by the corresponding diagonal blocks of P.*

Since our aim is to seek a solution of (4.28), which minimizes the variance of the estimation error, we introduce a new variable W such that

$$W > Y^{-1}, \qquad (4.29)$$

and then minimize the $\mathbf{tr}(W)$. The Schur compliment of (4.29) is

$$\begin{bmatrix} -W & I \\ I & -Y \end{bmatrix} < 0.$$

This yields the following convex LMI optimization problem to provide a solution for the observer gains for coordinated control with non-standard information structure.

$$\min_{G,L,W,Y} \quad \mathbf{tr}(W)$$

subject to:

$$
\begin{bmatrix}
-G-G^T+Y & (G-L\bar{C})\bar{A} & (G-L\bar{C})\bar{B} & LR^{\frac{1}{2}} \\
\bar{A}^T(G-L\bar{C})^T & -Y & 0 & 0 \\
\bar{B}^T(G-L\bar{C})^T & 0 & -I & 0 \\
R^{\frac{1}{2}}L^T & 0 & 0 & -I
\end{bmatrix} \leq 0, \tag{4.30}
$$

$$
\begin{bmatrix}
-W & I \\
I & -Y
\end{bmatrix} < 0,
$$

where G and L are block diagonal, and Y and W are symmetric.

Note that we use Kalman filters for $\hat{x}^i_{k|k}$. Hence, once we get solutions G, Y and L from (4.30), the gains K^1 and K^2 will be replaced by their initial Kalman filtering gains.

4.6 Self- and Cross-Estimator Design with Bounded Noise and Disturbance

As seen in the previous sections, if probabilistic properties of the noise and disturbance are known, then cross-estimator design can be given in a set of LMIs. If we do not have any probabilistic models on the noise and disturbance, cross-estimator design takes an different approach from what we have seen in the previous sections. Here we attempt to design state estimators for coordinated systems under this circumstance.

We consider the control problem presented in Section 3.6. Recall the vehicle dynamics of Vehicle i

$$x^{i,v}_{k+1} = A_v x^{i,v}_k + B_v u^i_k + G_v q^{i,real}_k, \tag{4.31}$$

where $x^{i,v}_k$ is the state vector and $q^{i,real}_k$ is the actual disturbance acting on the vehicle. We seek to describe $q^{i,real}_k$ by a linear model

$$
\begin{aligned}
x^{i,d}_{k+1} &= A^i_d x^{i,d}_k + G^i_d w^i_k, \\
q^i_k &= C^i_d x^{i,d}_k,
\end{aligned}
\tag{4.32}
$$

with the disturbance state $x_k^{i,d}$, the output q_k^i (same dimension as $q_k^{i,real}$), and fictitious white noise input w_k^i. This model is required to

- allow the construction of an observer-based predictor for $q_k^{i,real}$,

- preserve detectability (preferably observability) of the vehicle model

$$x_{k+1}^i = A^i x_k^i + B u_{k|k}^i + G^i w_k^i, \tag{4.33}$$

where

$$x_k^i = \begin{bmatrix} x_k^{i,v} \\ x_k^{i,d} \end{bmatrix}, A^i = \begin{bmatrix} A_v & G_v C_d^i \\ 0 & A_d^i \end{bmatrix}, B = \begin{bmatrix} B_v \\ 0 \end{bmatrix}, G^i = \begin{bmatrix} 0 \\ G_d^i \end{bmatrix},$$

$$position\text{-}output: \ y_k^i = C x_k^{i,v},$$

$$measurement: \ z_k^i = D x_k^{i,v} + v_k^i.$$

The measurement noise v_k is random and bounded. Here we assume that the choice of A_d^i, G_d^i, and C_d^i satisfies the requirements. We consider the estimators at Vehicle i: the design process for the other vehicles will be identical. The self-estimator of Vehicle i is given by

$$\hat{x}_{k|k}^i = (I - \mathcal{K}^i \bar{D}) A^i \hat{x}_{k-1|k-1}^i + (I - \mathcal{K}^i \bar{D}) B u_{k-1|k-1}^i + \mathcal{K}^i z_k^i, \tag{4.34}$$

where $\bar{D} = \begin{bmatrix} D & 0 \end{bmatrix}$ and $\mathcal{K}^i$ is the self-estimator gain. Since $(A^i, \bar{D})$ is detectable there exists $\mathcal{K}^i$ such that $(I - \mathcal{K}^i \bar{D}) A^i$ is stable. For cross-estimation, Vehicle i receives the *full-state* information from Vehicle ℓ via a communication channel:

$$z_{i,k}^\ell = \begin{bmatrix} \hat{x}_{k|k}^{\ell,v} + v_{i,k}^{\ell,v} \\ \hat{x}_{k|k}^{\ell,d} + v_{i,k}^{\ell,d} \end{bmatrix}.$$

The vectors $\nu_{i,k}^{\ell,v}$ and $v_{i,k}^{\ell,d}$ capture the bounded random communication noises such as quantization errors, packet dropout, and delay. Then the cross-estimator for Vehicle ℓ is

$$\hat{x}_{i,k+1|k}^\ell = (A^\ell - K_i^\ell) \hat{x}_{i,k|k-1}^\ell + B u_{i,k|k-1}^\ell + K_i^\ell z_{i,k}^\ell, \tag{4.35}$$

where K_i^ℓ is the cross-estimator gain. There exists a stabilizing K_i^ℓ by the same argument as for the self-estimator case. For $j-$step ahead estimation, the input signal $u_{i,k+j-1|k-1}^\ell$ is also transmitted from Vehicle ℓ via the communication channel:

$$u_{i,k+j-1|k-1}^\ell = u_{k+j-1|k-1}^\ell + \nu_{i,k}^{\ell,j-1}, \ j = 1, 2, \ldots, N-1.$$

We also consider $v_{i,k}^{\ell,j-1}$ as bounded random communication noise. Due to the timing issue, the control sequence at time $k-1$ is transmitted The final control element is taken to be identical to $u_{k+N-2|k-1}^{\ell}$ since $u_{k+N-1|k-1}^{\ell}$ does not exist. The error by using the previous control is captured by

$$
\begin{aligned}
\eta_k^{\ell,j-1} &= u_{k+j-1|k}^{\ell} - u_{k+j-1|k-1}^{\ell}, \ j = 1, \ldots, N-1, \\
\eta_k^{\ell,N-1} &= u_{k+N-1|k}^{\ell} - u_{k+N-2|k-1}^{\ell}.
\end{aligned}
\tag{4.36}
$$

The variability $(\eta_k^{\ell,j-1})$ is assumed to be bounded by the control constraint $\mathbf{U}$.

4.6.1 Control Requirement of Estimation

From Section 3.6, the no-collision constraint of Vehicle i to avoid collision with Vehicle ℓ is given by

$$
||\hat{y}_{k+j|k}^i - \hat{y}_{i,k+j|k}^{\ell}|| \geq |\tilde{y}_{k+j|k}^i|_p + |\tilde{y}_{i,k+j|k}^{\ell}|_p + \alpha,
\tag{4.37}
$$

where $|\tilde{y}_{k+j|k}^i|_p \triangleq \max_k ||\tilde{y}_{k+j|k}||$, $|\tilde{y}_{i,k+j|k}^{\ell}|_p \triangleq \max_k ||\tilde{y}_{i,k+j|k}^{\ell}||$, and α is the constant that corresponds to the size of the vehicles. Since the right-hand-side of (4.37) can be fixed by $|\tilde{y}_{k+1|k}^i|_p + |\tilde{y}_{i,k+1|k}^{\ell}|_p + \alpha$, for simplicity, we only consider this choice of the no-collision constraint. The communication and estimation objective is to ensure that constraints are not always active. We want to have as small $|\tilde{y}_{k+1|k}^i|_p + |\tilde{y}_{i,k+1|k}^{\ell}|_p$ as possible so that the following condition is met:

$$
|\tilde{y}_{k+1|k}^i|_p + |\tilde{y}_{i,k+1|k}^{\ell}|_p + \alpha < d,
\tag{4.38}
$$

where d is the nominal distance between the vehicles in the formation. This is the minimum requirement for the estimators that is similar to (4.5) of the gaussian noise and disturbance. Otherwise the no-collision constraint will be active even when $\hat{y}_{k+j|k}^i$ and $\hat{y}_{i,k+j|k}^{\ell}$ are at the target positions. This provides the key guideline for the estimator design.

The condition (4.38) also suggests that the Kalman filters are not the best choice for the self-estimators. It is because that the Kalman filter is the optimal filter in terms of achieving the minimum estimation error variance: this does not imply minimization of $\max_k |\tilde{y}_{k+1|k}^i|_p$. Therefore, the self-estimator design should also be reconsidered along with the cross-estimator design.

4.6.2 Estimator Gain Tuning and Performance

The gain tuning begins with calculating $\tilde{y}^i_{k+1|k}$ and $\tilde{y}^\ell_{i,k+1|k}$ as shown in the following theorems.

Theorem 4.6.1. *Suppose that Vehicle i has the model (4.33) and a self-estimator as in (4.34). If $u^i_{k|k}$ is applied, then the input to output map $q^{i,real}_k$ and v^i_k to $\tilde{y}^i_{k+1|k}$ is given by*

$$
\begin{bmatrix} \tilde{x}^{i,v}_{k+1|k} \\ \hat{x}^{i,d}_{k+1|k} \end{bmatrix} = \begin{bmatrix} A_v - K^{i,1}D & -G_v C^i_d \\ K^{i,2}D & A^i_d \end{bmatrix} \begin{bmatrix} \tilde{x}^{i,v}_{k|k-1} \\ \hat{x}^{i,d}_{k|k-1} \end{bmatrix} + \begin{bmatrix} G_v \\ 0 \end{bmatrix} q^{i,real}_k + \begin{bmatrix} -K^{i,1} \\ K^{i,2} \end{bmatrix} v^i_k,
$$

$$
\tilde{y}^i_{k|k-1} = \begin{bmatrix} C & 0 \end{bmatrix} \begin{bmatrix} \tilde{x}^{i,v}_{k|k-1} \\ \hat{x}^{i,d}_{k|k-1} \end{bmatrix},
$$

(4.39)

where $K^i = A^i \mathcal{K}^i = [K^{i,1^T}\ K^{i,2^T}]^T$.

Theorem 4.6.2. *Suppose that Vehicle ℓ has a model as in (4.33) and its cross-estimator at Vehicle i as in (4.35). Then the input to output map from $q^{\ell,real}_k$, $\eta^{\ell,0}_k$, $\nu^{\ell,0}_{i,k}$, $\nu^{\ell,v}_{i,k}$, $\nu^{\ell,d}_{i,k}$, and v^ℓ_{k+1} to $\tilde{y}^\ell_{i,k+1|k}$ is given by (4.40) with*

$$
K^\ell_i = \begin{bmatrix} K^{\ell,11}_i & K^{\ell,12}_i \\ K^{\ell,21}_i & K^{\ell,22}_i \end{bmatrix}, \quad \mathcal{K}^\ell = \begin{bmatrix} \mathcal{K}^{\ell,1} \\ \mathcal{K}^{\ell,2} \end{bmatrix}.
$$

Denote $|\tilde{y}_{k+1|k}|_p$ evaluated for a single input e_k by $|\tilde{y}_{k+1|k}|^e_p$. Then the worst error bounds are given by

$$
|\tilde{y}^i_{k+1|k}|_p = |\tilde{y}^i_{k+1|k}|^{q^{i,real}}_p + |\tilde{y}^i_{k+1|k}|^{v^i}_p,
$$

$$
|\tilde{y}^\ell_{i,k+1|k}|_p = |\tilde{y}^\ell_{i,k+1|k}|^{q^{\ell,real}}_p + |\tilde{y}^\ell_{i,k+1|k}|^{\eta^{\ell,0}}_p + |\tilde{y}^\ell_{i,k+1|k}|^{\nu^{\ell,0}_i}_p
$$

$$
+ |\tilde{y}^\ell_{i,k+1|k}|^{\nu^{\ell,v}_i}_p + |\tilde{y}^\ell_{i,k+1|k}|^{\nu^{\ell,d}_i}_p + |\tilde{y}^\ell_{i,k+1|k}|^{v^\ell}_p.
$$

To compute $|\tilde{y}^i_{k+1|k}|_p$ and $|\tilde{y}^\ell_{i,k+1|k}|_p$, one may use peak-induced system norms ($\star$-norm) (Bu, Sznaier & Holmes 1996) of (4.39) and (4.40) for each input. This would be useful when the only knowledge about input is its bound. However since $q^{i,real}_k$ and $q^{\ell,real}_k$ are available as data records, $|\tilde{y}^i_{k+1|k}|^{q^{i,real}}_p$ and $|\tilde{y}^\ell_{i,k+1|k}|^{q^{\ell,real}}_p$ can be obtained by numerical simulation for given K^i and K^ℓ_i. This usually achieves less conservative $|\tilde{y}^i_{k+1|k}|^{q^{i,real}}_p$ and $|\tilde{y}^\ell_{i,k+1|k}|^{q^{\ell,real}}_p$ than computing them via peak-induced system norms. Since the rest of the inputs are bounded random sequences, for fixed K^i and K^ℓ_i, the remainder of $|\tilde{y}^i_{k+1|k}|_p$ and $|\tilde{y}^\ell_{i,k+1|k}|_p$ is calculated via peak-induced system norms.

$$
\begin{bmatrix}
\tilde{x}_{i,k+1|k}^{\ell,v} \\
\hat{x}_{i,k+1|k}^{\ell,d} \\
\tilde{x}_{k+1|k+1}^{\ell,v} \\
\hat{x}_{k+1|k+1}^{\ell,d}
\end{bmatrix}
=
\begin{bmatrix}
A_v - K_i^{\ell,11} & K_i^{\ell,12} - G_v C_d^\ell & K_i^{\ell,11} & -K_i^{\ell,12} \\
K_i^{\ell,21} & A_d^\ell - K_i^{\ell,22} & -K_i^{\ell,21} & K_i^{\ell,22} \\
0 & 0 & (I - \mathcal{K}^{\ell,1}\bar{D})A_v & -(I - \mathcal{K}^{\ell,1}\bar{D})G_v C_d^\ell \\
0 & 0 & \mathcal{K}^{\ell,2}\bar{D}A_v & A_d^\ell - \mathcal{K}^{\ell,2}\bar{D}G_v C_d^\ell
\end{bmatrix}
\begin{bmatrix}
\tilde{x}_{i,k|k-1}^{\ell,v} \\
\hat{x}_{i,k|k-1}^{\ell,d} \\
\tilde{x}_{k|k}^{\ell,v} \\
\hat{x}_{k|k}^{\ell,d}
\end{bmatrix}
$$

$$
+
\begin{bmatrix}
G_v \\
0 \\
(I - L^{\ell,1}\bar{D})G_v \\
L^{\ell,2}\bar{D}G_v
\end{bmatrix}
q_k^{\ell,real}
+
\begin{bmatrix}
-B_v \\
0 \\
0 \\
0
\end{bmatrix}
\eta_k^{\ell,0}
+
\begin{bmatrix}
-B_v \\
0 \\
0 \\
0
\end{bmatrix}
\nu_{i,k}^{\ell,0}
+
\begin{bmatrix}
-K_i^{\ell,11} \\
K_i^{\ell,21} \\
0 \\
0
\end{bmatrix}
\nu_{i,k}^{\ell,v}
$$

$$
+
\begin{bmatrix}
-K_i^{\ell,12} \\
K_i^{\ell,22} \\
0 \\
0
\end{bmatrix}
\nu_{i,k}^{\ell,d}
+
\begin{bmatrix}
0 \\
0 \\
-\mathcal{K}^{\ell,1} \\
\mathcal{K}^{\ell,2}
\end{bmatrix}
v_{k+1},
$$

$$
\tilde{y}_{i,k|k-1}^\ell =
\begin{bmatrix}
C & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\tilde{x}_{i,k|k-1}^{\ell,v} \\
\hat{x}_{i,k|k-1}^{\ell,d} \\
\tilde{x}_{k|k}^{\ell,v} \\
\hat{x}_{k|k}^{\ell,d}
\end{bmatrix},
$$

$$(4.40)$$

Our aim is to design the estimators that achieve small $|\tilde{y}_{k+1|k}^i|_p$ and $|\tilde{y}_{i,k+1|k}^\ell|_p$. We may tune K^i and K_i^ℓ directly by evaluating the difference in $|\tilde{y}_{k+1|k}^i|_p$ and $|\tilde{y}_{i,k+1|k}^\ell|_p$ with respect to variation in the elements of K^i and K_i^ℓ. However, if their dimension is high, this approach would not be tractable. Furthermore it is hard to tune K^i and K_i^ℓ only over the elements which are stabilizing. Alternatively since we can approximate the covariance matrix (R) of measurement and communication noises (denote R for self- and cross-estimators by R^i and R_i^ℓ respectively), for fixed R^i and R_i^ℓ, one may use the Discrete time Algebraic Riccati Equation (DARE) to obtain stabilizing K^i and K_i^ℓ. In this case, one should pick Q^i and Q_i^ℓ as the covariance matrices of the fictitious noise w_k in (4.33) for self- and cross-estimators. If we limit Q^i and Q_i^ℓ to be symmetric positive semi-definite, then it would be manageable to tune Q^i and Q_i^ℓ by evaluating the difference in $|\tilde{y}_{k+1|k}^i|_p$ and $|\tilde{y}_{i,k+1|k}^\ell|_p$ with respect to variation in the elements of Q^i and Q_i^ℓ.

Gain Tuning and Performance Determination of Self- (Cross-) estimator

For fixed R^i (R_i^ℓ),

① *make an initial guess of Q^i (Q_i^ℓ),*

② *solve the DARE to obtain K^i (K_i^ℓ),*

③ *evaluate $|\tilde{y}_{k+1|k}^i|_p$ $(|\tilde{y}_{i,k+1|k}^\ell|_p)$,*

④ *adjust elements of Q^i (Q_i^ℓ) and take the same procedure above (②∼③),*

⑤ *evaluate the difference in $|\tilde{y}_{k+1|k}^i|_p$ $(|\tilde{y}_{i,k+1|k}^\ell|_p)$ to determine the direction of steepest descent. Repeat (④∼⑤) until the descent is sufficiently small,*

⑥ *check if the final K^i and K_i^ℓ satisfy (4.38). If so, move on to the control design. Otherwise use better measurement and communication systems or d must be increased.*

Remark 4.6.1. *In Adaptive Kalman Filtering (Haykin 2001), gains are tuned on-line using sampled error covariance information. Here $|\tilde{y}_{k+1|k}^i|_p$ and $|\tilde{y}_{i,k+1|k}^\ell|_p$ information is used to adjust elements in Q^i and Q_i^ℓ and, hence, tuning is performed off-line.*

Remark 4.6.2. *To facilitate the deign process we proposed, one can use `fminsearch` in `matlab` that does not use analytic gradients.*

4.7 Conclusion

We developed cross-estimation techniques for coordinated systems. The emphasis has been to link the constraint specification of control strategy, cross-estimation performance, and the inherent requirements of inter-subsystem communication. While conceptually straightforward, the surprising aspect is that the treatment using covariance calculations permits the simultaneous specification of a single set of LMIs whose solution yields communication resource assignment and cross-estimator gains. When each subsystem has a known fixed feedback control law, we can still formulate a set of LMIs to design the cross-estimators. In addition, when statistical knowledge about uncertainties is absent, we take a quite different approach from solving LMIs to design self- and cross-estimators.

This chapter is in part a reprint of the materials as it appears in,

Keunmo Kang, David D. Zhang, Robert R. Bitmead - *Disturbance Rejection Control in Coordinated Systems*, 17th IFAC World Congress, Seoul, Korea, Jul., 2008.

Keunmo Kang, Jun Yan, Robert R. Bitmead - *Cross-Estimator for Coordinated Systems: Constraints, Covariance, and Communication Resource Assignment*, Automatica, May, 2008. pp. 1394–1401

Robert R. Bitmead, Jun Yan, Keunmo Kang - *Constrained Control and Communication Resource Assignment in Coordinated Systems*, International Control Conference, Glasgow, Scotland, Aug., 2006.

Keunmo Kang, Jun Yan, Robert R. Bitmead -*Communication Design for Coordinated Control with a Non-Standard Information Structure*, 44th IEEE Conference on Decision and Control, Seville, Spain, Dec., 2005.

Jun Yan, Keunmo Kang, Robert R. Bitmead -*State Estimation in Coordinated Control with a Non-Standard Information Architecture*, 16th IFAC World Congress, Prague, Czech Republic, Jul., 2005.

The dissertation author was the primary or co-author in these publications and professor Bitmead directed and supervised the research.

5

Simulation Studies: Hovercraft Formation Control

5.1 Introduction

In this chapter, we demonstrate our main ideas presented in Chapter 3 and 4. We use the dynamical model of the HotDec hovercraft, which is described in Appendix A. The vehicle levitates on a flat ground and hence moves in the $2D$ space. The dynamics are linear. We consider the case in which only bounds on the noise are known and a specific disturbance class is acting. Therefore, the control and estimator design process is based on Sections 3.6 and 4.6.

5.2 Scenario

We consider the two-vehicle-formation control problem depicted in Figure 5.1. Both vehicles are initially on their target positions. The offset d is 0.6 m. To effectively reveal the control performance in the presence of disturbances, we consider the following scenario:

- The gusts on Vehicles 1 and 2 blow from the positive $\mathbb{X}$ and the negative $\mathbb{X}$ directions respectively with a slight time delay,

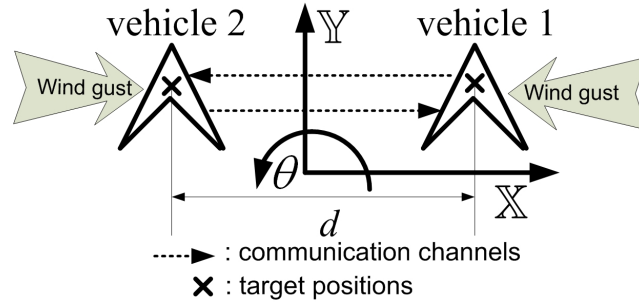- The vehicles' target positions are fixed for all time. That is, their reference trajec-

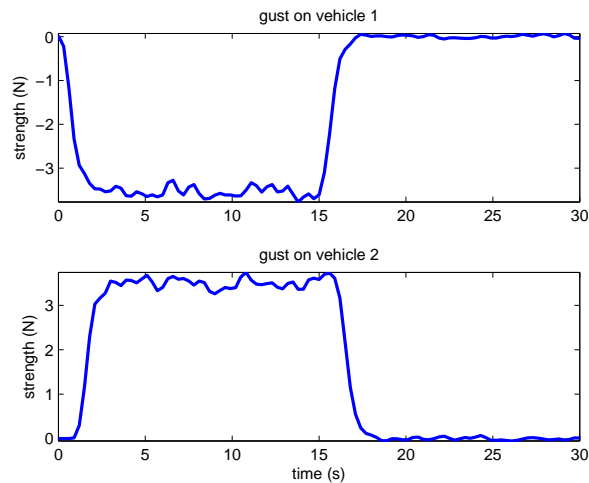Figure 5.1: A formation of two vehicles and the coordinates.

Figure 5.2: Sampled wind gust in thrust (N).

tories are constant.

One can easily implement more realistic wind gust scenarios and consider time-varying reference trajectories. Typical and representative wind gusts are shown in Figure 5.2. The control objective is to steer the vehicles to their target positions while avoiding collision in the presence of the disturbance.

The vehicles are represented by

$$x_{k+1}^{i,v} = A_v x_k^{i,v} + B_v u_k^i + G_v q_k^{i,real},$$

with the state vector

$$x_k^{i,v} = \begin{bmatrix} x_{p,k}^i & y_{p,k}^i & \theta_{p,k}^i & x_{s,k}^i & y_{s,k}^i & \theta_{s,k}^i \end{bmatrix}^T,$$

and the actual disturbance $q_k^{i,real}$. The elements x_p^i, y_p^i, x_s^i, and y_s^i respectively represent the positions and velocities in Cartesian coordinates. The variables θ_p^i and θ_s^i are the angular displacements and velocities of the vehicles respectively. Three control inputs control $\mathbb{X}$, $\mathbb{Y}$, and angular motions of the vehicle, and have the limits defined by

$$\mathbf{U}^i \triangleq \{u \in \mathbb{R}^3 \big| \; ||u||_\infty \leq 5\}, \; i = 1, 2, \tag{5.1}$$

where $|| \cdot ||_\infty$ denotes the vector infinity norm. We model both disturbances in Figure 5.2 as the outputs of linear systems (4.32):

$$x_{k+1}^{i,d} = A_d^i x_k^{i,d} + G_d^i w_k^i,$$
$$q_k^i = C_d^i x_k^{i,d},$$

with the disturbance state $x_k^{i,d} \in \mathbb{R}^4$, the output q_k^i, and fictitious white noise input w_k^i. Then the combined vehicle models for Vehicles 1 and 2 are constructed as (4.33):

$$\begin{bmatrix} x_{k+1}^{i,v} \\ x_{k+1}^{i,d} \end{bmatrix} = \begin{bmatrix} A_v & G_v C_d^i \\ 0 & A_d^i \end{bmatrix} \begin{bmatrix} x_k^{i,v} \\ x_k^{i,d} \end{bmatrix} + \begin{bmatrix} B_v \\ 0 \end{bmatrix} u_{k|k}^i + \begin{bmatrix} 0 \\ G_d^i \end{bmatrix} w_k^i.$$

Measurement noise is assumed to be uniformly distributed for numerical simulation and bounded by

$$||v_k^i||_\infty \leq 0.01.$$

5.3   Control without Communication

We first demonstrate the results of Section 3.3. We have shown that vehicles may be able to avoid collision even without using communications if the formation geometry is such that the deviations of the vehicles from their target positions are small enough to preclude collision. The results were developed by using the Kalman filter and LQG control. Here the vehicles have input constraints and, therefore, they use MPC. Furthermore, the measurement and communication noises are bounded without any probabilistic knowledge. However, we still expect that the overall MPC-controlled behavior of the vehicles would be quite similar to that in Section 3.3.

Each vehicle uses MPC to solve the following problem at each time step:

$$arg \min_{\{u^i_{k|k},...,u^i_{k+N-1|k}\}} \sum_{j=0}^{N-1} (\hat{y}^i_{k+j+1|k} - r^i)^T Q_c (\hat{y}^i_{k+j+1|k} - r^i) + u^{i^T}_{k+j|k} R_c u^i_{k+j|k},$$

$$subject\ to: \quad \hat{x}^i_{k+j+1|k} = A^i \hat{x}^i_{k+j|k} + Bu^i_{k+j|k},$$

$$\hat{y}^i_{k+j+1|k} = C\hat{x}^i_{k+j+1|k},$$

$$u^i_{k+j|k} \in \mathbf{U}^i,$$

where r^i is the vehicle's reference trajectory. Since there is no communication between the vehicles, there is only a self-estimator at each vehicle. In this case, we look for self-estimator gains that minimize $|\tilde{y}^i_{k+1|k}|_p$. This can be done in a similar fashion to the gain tuning process presented in Section 4.6.2:

① select a measurement noise (v^i_k) covariance R^i,

② select a disturbance driving noise (w^i_k) covariance Q^i ,

③ solve the DARE to obtain the estimator gain K^i,

④ optimize K^i over Q^i to minimize $|\tilde{y}^i_{k+1|k}|_p$.

Simulation results are shown in Figure 5.3. The horizon N is 5 and the other control parameters are described in Figure 5.3. When we do not penalize the control input at all, the vehicles stay at their target positions very closely. Even when we do not allow control-cost-free control, as seen in Figure 5.3 (b), the vehicles seem to avoid collisions. Although the trajectories in Figure 5.3 (c) do not overlap, due to the size of the vehicles (0.175 m radius), the vehicles collide. Hence the vehicles may collide when the control is given a relatively stronger penalty R_c. If it is desired that the vehicles avoid cost-free control, each vehicle needs to use both MPC with the no-collision constraint to avoid collision and cross-estimators to compute the future states of its neighbor.

5.4 Control with Communication

5.4.1 Self- and Cross-Estimators

We describe the design process of the cross-estimator for Vehicle 1. The counterparts including noise descriptions for Vehicle 2 are parallel. We use the formulation
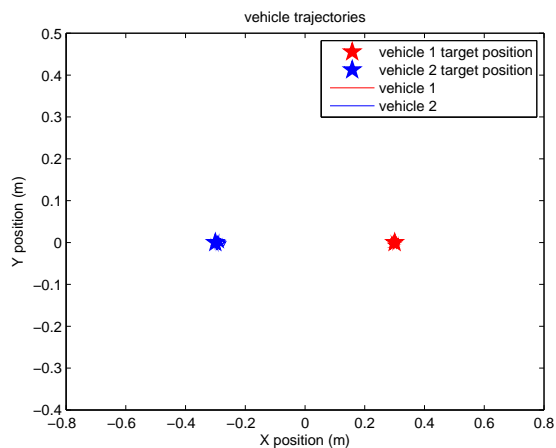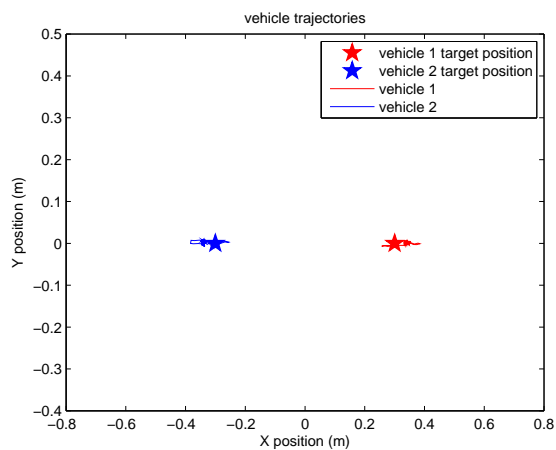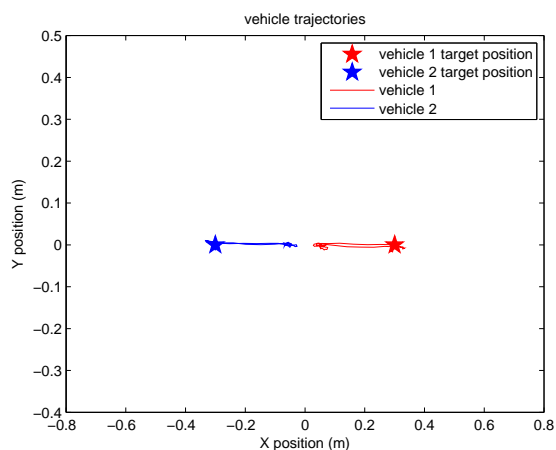
(a) $d = 0.6m$, $Q_c > 0$, $R_c = 0$

(b) $d = 0.6m$, $Q_c = 100 \times I$, and $R_c = I$

(c) $d = 0.6m$, $Q_c = 25 \times I$, and $R_c = I$

Figure 5.3: Vehicle trajectories without communication

developed in Section 4.6. At each sampling time, each vehicle transmits its self-vehicle-state estimate, the self-disturbance-state estimate, and the one step out-of-date control sequence to its neighbor via a communication channel. Communication errors are assumed to be uniformly distributed for the numerical simulation with the following bounds:

$$control\ signal:\ ||\nu_{1,k}^{2,j-1}||_\infty \leq 0.001,\ j = 1, 2, \ldots, N$$

$$vehicle\ state:\ ||\nu_{1,k}^{2,v}||_\infty \leq 0.001, \tag{5.2}$$

$$disturbance\ state:\ ||\nu_{1,k}^{2,d}||_\infty \leq 0.001.$$

Since the transmitted control information is one step out-of-date, one should consider the error $\eta_k^{2,j}$ between this previous control and unknown current control as in (4.36). From the description of the input constraint (5.1), $\eta_k^{2,j}$ of (4.36) is assumed to have the following limit

$$||\eta_k^{2,j}||_\infty \leq 10,\ j = 0, 1, 2, \ldots, N - 1. \tag{5.3}$$

The self- and cross-estimators satisfying

$$|\tilde{y}_{k+1|k}^1|_p + |\tilde{y}_{1,k+1|k}^2|_p < d - 0.35 = 0.25 \tag{5.4}$$

are designed. Here we include the effect of vehicles' radii $(2 \times 0.175\ m)$. After the tuning process, the achieved worst error bounds are

$$|\tilde{y}_{k+1|k}^1|_p = 0.038,\quad |\tilde{y}_{1,k+1|k}^2|_p = 0.167, \tag{5.5}$$

which satisfy the condition (5.4).

5.4.2 Local Model Predictive Control

The local MPC at Vehicle i is

$$arg \min_{\{u_{k|k}^i, \ldots, u_{k+N-1|k}^i\}} \sum_{j=0}^{N-1} (\hat{y}_{k+j+1|k}^i - r^i)^T Q_c (\hat{y}_{k+j+1|k}^i - r^i) + u_{k+j|k}^{i^T} R_c u_{k+j|k}^i,$$

$$subject\ to:\quad \hat{x}_{k+j+1|k}^i = A^i \hat{x}_{k+j|k}^i + B u_{k+j|k}^i,$$

$$\hat{y}_{k+j+1|k}^i = C \hat{x}_{k+j+1|k}^i, \tag{5.6}$$

$$u_{k+j|k}^i \in \mathbf{U}^i,$$

$$||\hat{y}_{k+j+1|k}^i - \hat{y}_{i,k+j+1|k}^\ell|| > |\tilde{y}_{k+1|k}^i|_p + |\tilde{y}_{i,k+1|k}^\ell|_p + 0.35.$$

Simulations are performed for the following parameters:

$$N = 5, \quad Q_c = 100 \times I, \quad R_c = I, \tag{5.7}$$

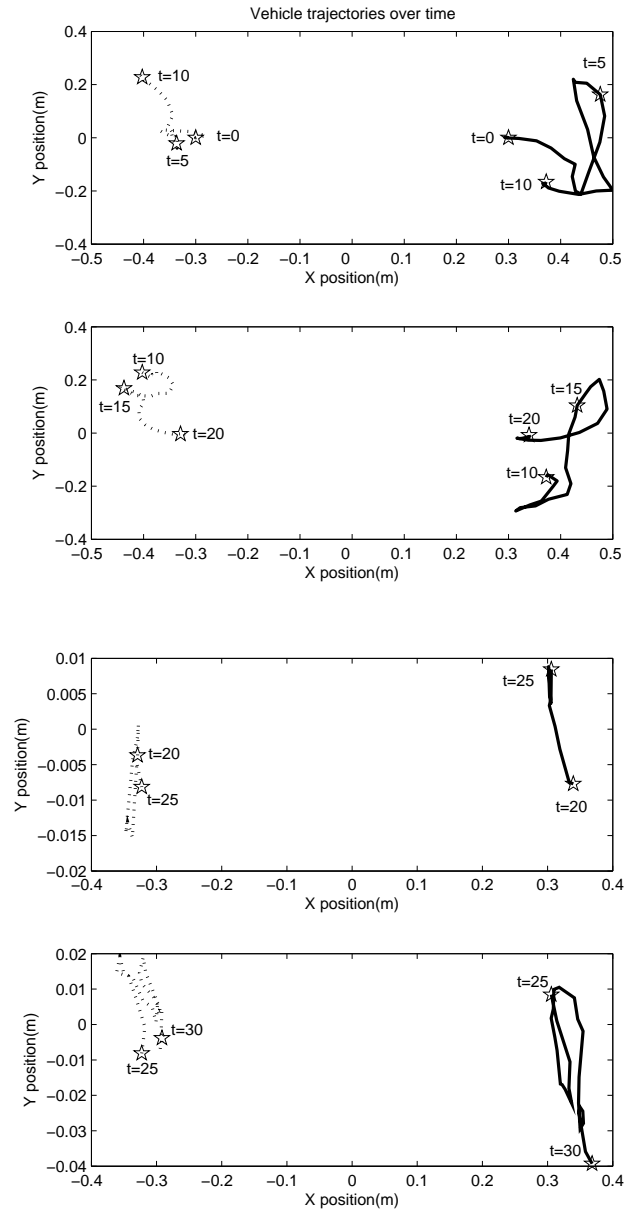with appropriate dimensions of identity matrices I. Entire vehicle trajectories with mea-

Figure 5.4: Time trajectories of Vehicle 1 (solid lines) and Vehicle 2 (dotted lines) with star-time (t sec) marks.

surement and communication noises are depicted in Figure 5.4. Due to the choice of Q_c and R_c in (5.7), the vehicles' overall positions are fairly close to their target positions [If larger R_c is used, for fixed Q_c, the vehicles will make more dramatic movements to avoid collisions. This will be further discussed in the next subsection.]. From Figure 5.4 and Figure 5.3 (b), it is interesting to see the behavioral difference between the control with communication and the control without communication. Without communication, the local MPCs only attempt to steer the vehicles to their target positions against the wind gusts. When communication is used (i.e. no-collision constraints in effect), each vehicle estimates the future positions of its neighbor and, at some steps, the vehicles confront active constraints as shown in Figure 5.5. (In Figure 5.5, the solid lines are the squared-expected distance between the vehicles for the computed input sequences. Contact between the solid line and the dashed line $[(\parallel \tilde{y}^1_{k+1|k} \parallel_p + \parallel \tilde{y}^2_{1,k+1|k} \parallel_p +0.35)^2]$ indicates constraint activity at that time.) This appears to increase each vehicle's movement in the $\mathbb{Y}$ direction.

In terms of activity of no-collision constraints, since there are no measurement and communication noises acting in Figure 5.7, all active constraints are solely caused by the disturbance $q^{1,real}_k$ and $q^{2,real}_k$. On the other hand, when the measurement and communication noises (5.2) come into play, then the active constraints are caused by the combination of the disturbance and the noises. In particular, in the cross-prediction process, communicated inputs are corrupted by $\nu^{2,j-1}_{1,k}$ and the errors are accumulated as the prediction step j increases. As a result, more active constraints are observed as shown in Figure 5.5. This leads to more conservative control overall, i.e. as shown in Figure 5.6, the more uncertainty the vehicles have, the more stand-off they need to avoid collision. In both cases, active constraints are not observed at the one-step prediction stage especially when the wind gusts are active ($0 \sim 20 \ sec$) due to the integral nature of the vehicle dynamics: extensive first-control-sequence to avoid constraint violations at the last prediction stage.
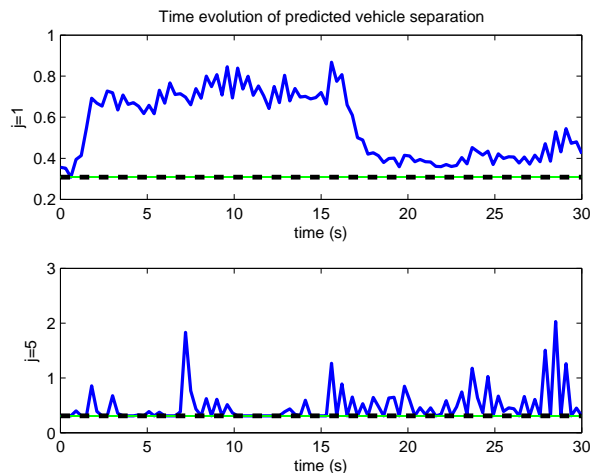
Figure 5.5: Activity of one- and five-step ahead no-collision constraints at Vehicle 1 when measurement and communication noises are acting.
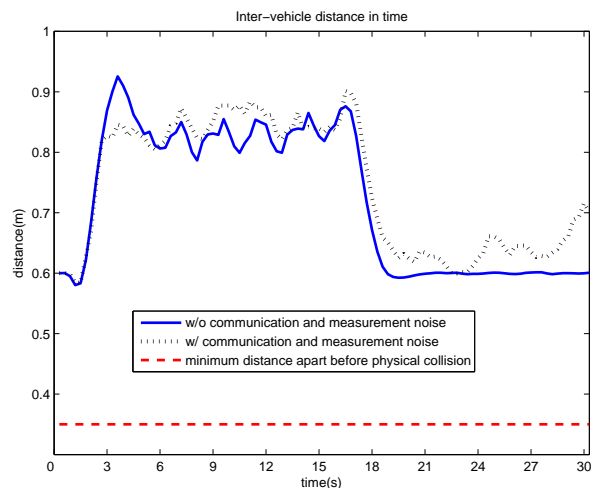

Figure 5.6: Actual vehicle distance with and without measurement and communication noises. If the dotted or the solid line is below the dashed line, then collision has occurred.

5.5   Additional Simulation Results

From the simulation results in the previous subsection, the following questions arise naturally: How would the vehicles behave if the control were more heavily penalized? What happens if their nominal separation is larger? Is two-way communication
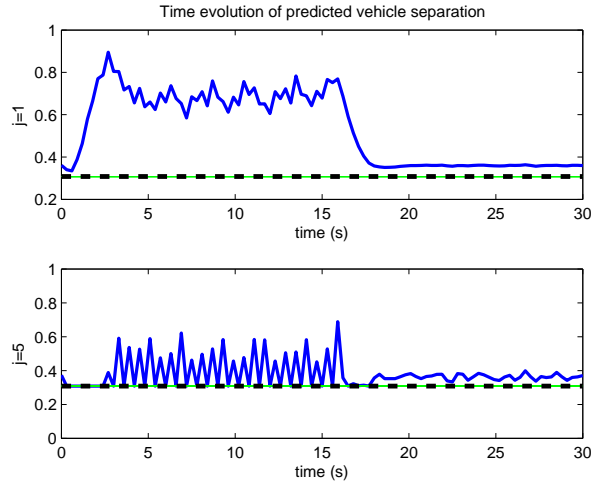
Figure 5.7: Activity of one- and five-step ahead no-collision constraints at vehicle 1 without measurement and communication noises.

always better than one-way communication? Rather than a theoretical development, we provide simulation results for these questions. Every simulation is done with active communication noises (5.2).

5.5.1 Control Parameters and Formation Geometry

Figure 5.8(a) shows the vehicle trajectories using the same parameters as (5.7). As shown in Figure 5.8(b), when we penalize the control more, the vehicles move further away from the target positions and make more pronounced maneuvers. Although the parameters ($Q_c = 25 \times I$, and $R_c = I$) do not result in any collisions, they are not desirable choices for maintaining the formation. Furthermore, if more vehicles are in the formation, then collision avoidance might be unmanageable. Both cases show very different constraint activities in Figure 5.9(a) and 5.9(b). The parameter combination of $Q_c = 25 \times I$ and $R_c = I$ causes limited control efforts: bigger predicted vehicle separations in the figures are due to bigger control efforts.

When the nominal separation is larger, then regardless of the control parameter choices, we obtain better formation keeping performance as shown in Figure 5.8(c) and 5.8(d). Less frequent activity of the no-collision constraints due to larger separation led

(a) $d = 0.6m$, $Q_c = 100 \times I$, and $R_c = I$

(b) $d = 0.6m$, $Q_c = 25 \times I$, and $R_c = I$

(c) $d = 1.2m$, $Q_c = 100 \times I$, and $R_c = I$

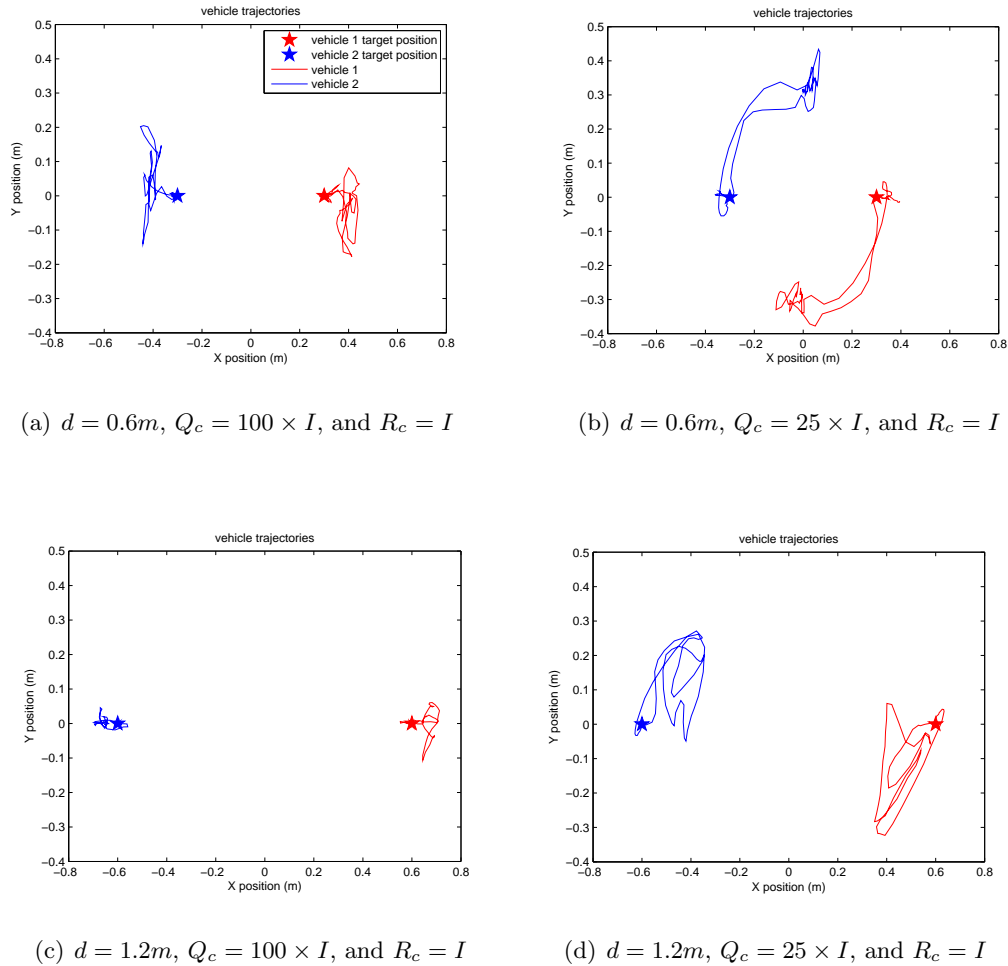(d) $d = 1.2m$, $Q_c = 25 \times I$, and $R_c = I$

Figure 5.8: Time trajectories of the vehicles by two-way communication

to less dramatic movements by the vehicles. The active no-collision constraints in Figure 5.9(c) and 5.9(d) are less frequent than those with the 0.6 m separation.

5.5.2 One-Way Communication

Now we turn off the communication from Vehicle 2 to Vehicle 1. This leads to leader-follower-like coordination. Vehicle 1 does not have the no-collision constraint and it only aims to stay close to its target position. The results are shown in Figure 5.10 and 5.11. The movement of Vehicle 1 is very similar to those of Figure 5.3. Compared to the two-way communication case, Vehicle 2 shows greater spatial variability around its target position in Figure 5.10 and more frequent active no-collision constraints in Figure 5.11. In the leader-follower vehicle coordination contexts, effects of disturbance and noise

(a) $d = 0.6m$, $Q_c = 100 \times I$, and $R_c = I$

(b) $d = 0.6m$, $Q_c = 25 \times I$, and $R_c = I$

(c) $d = 1.2m$, $Q_c = 100 \times I$, and $R_c = I$

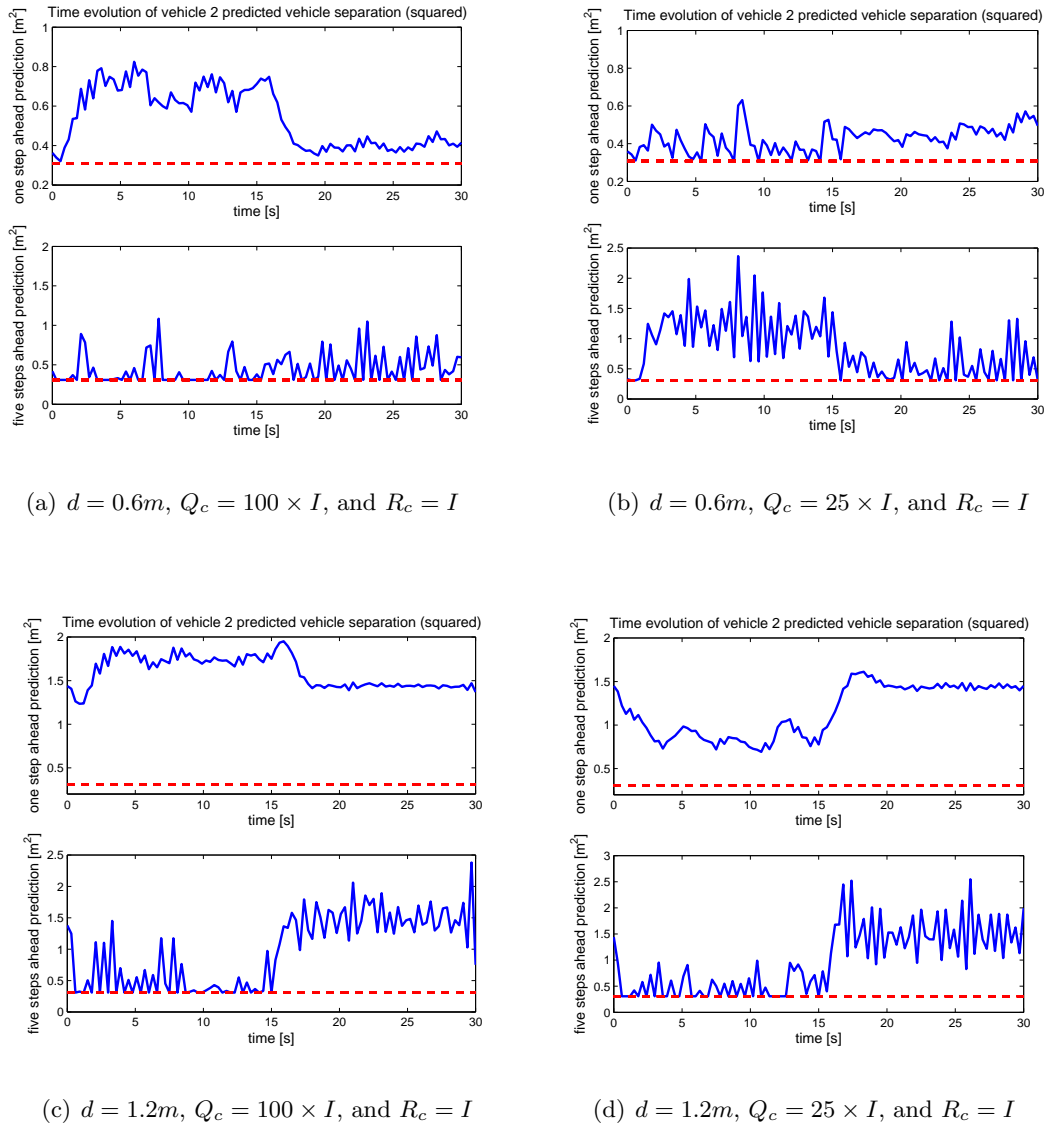(d) $d = 1.2m$, $Q_c = 25 \times I$, and $R_c = I$

Figure 5.9: Constraint activity of Vehicle 2 by two-way communication presented in the same way as Figure 5.5 and 5.7.

on the leader are conveyed to its immediate follower, and the effects are amplified by addition of each successive vehicle's disturbance and noise. This phenomenon is known as *string instability* (Swaroop & Hedrick 1996) and the one-way communication in our vehicle coordination scenario shows features quite similar to it. For any combination of control parameters and formation geometry, more active constraints are observed in the one-way communication case (Figure 5.11) than in the two-way communication case (Figure 5.9). Since it is desirable to encounter less active no-collision constraints, the

(a) $d = 0.6m$, $Q_c = 100 \times I$, and $R_c = I$

(b) $d = 0.6m$, $Q_c = 25 \times I$, and $R_c = I$

(c) $d = 1.2m$, $Q_c = 100 \times I$, and $R_c = I$

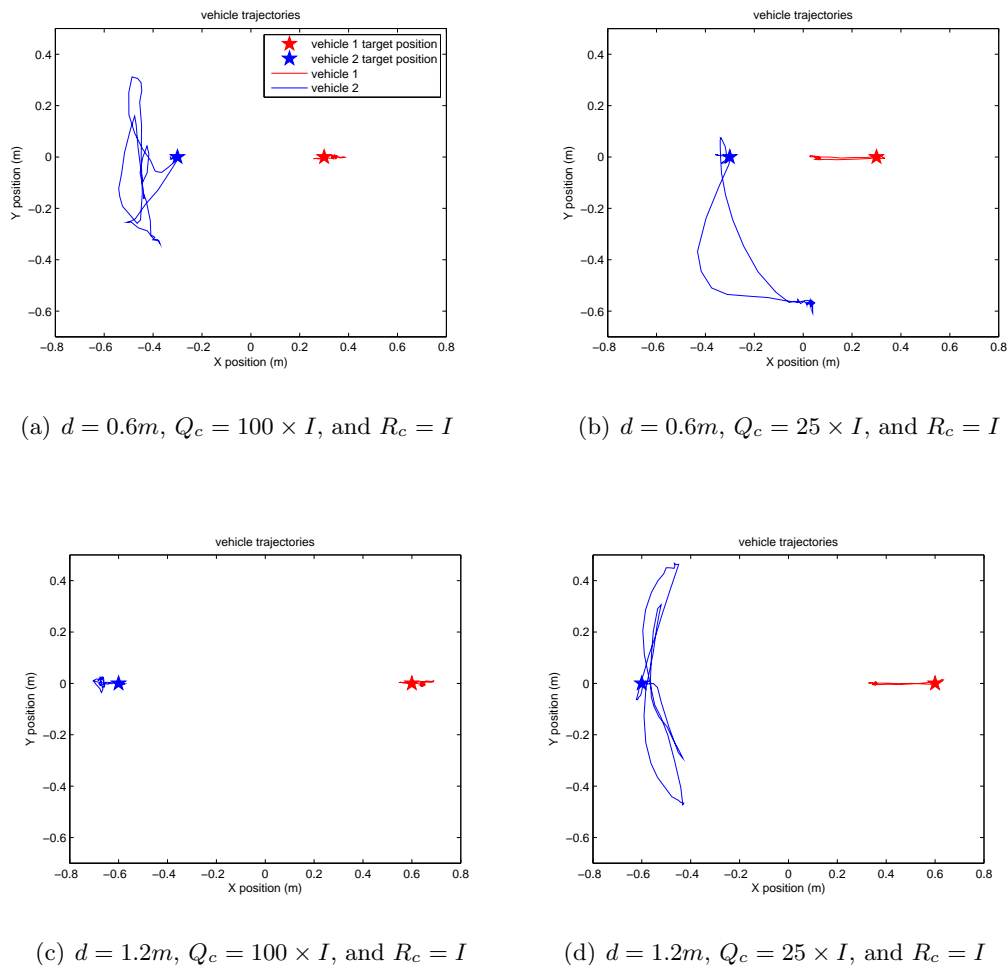(d) $d = 1.2m$, $Q_c = 25 \times I$, and $R_c = I$

Figure 5.10: Time trajectories of the vehicles when Vehicle 1 does not receive information from Vehicle 2.

two-way communication appears to be a sounder choice for our vehicle formation control scenario.

5.6 Conclusion

The disturbance rejection control in coordinated systems was demonstrated using a local constrained MPC and cross-estimators. The developed control and estimation techniques achieve our goal of formation keeping and collision avoidance fairly well. However, this chapter reveals a few areas requiring further research such as theoretical development of the MPC-controlled performance with respect to control parameters, ve-
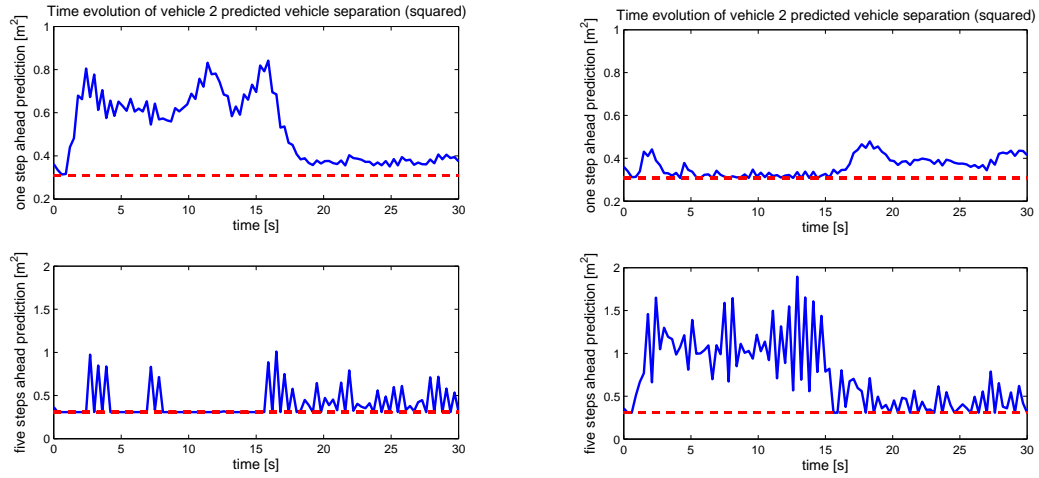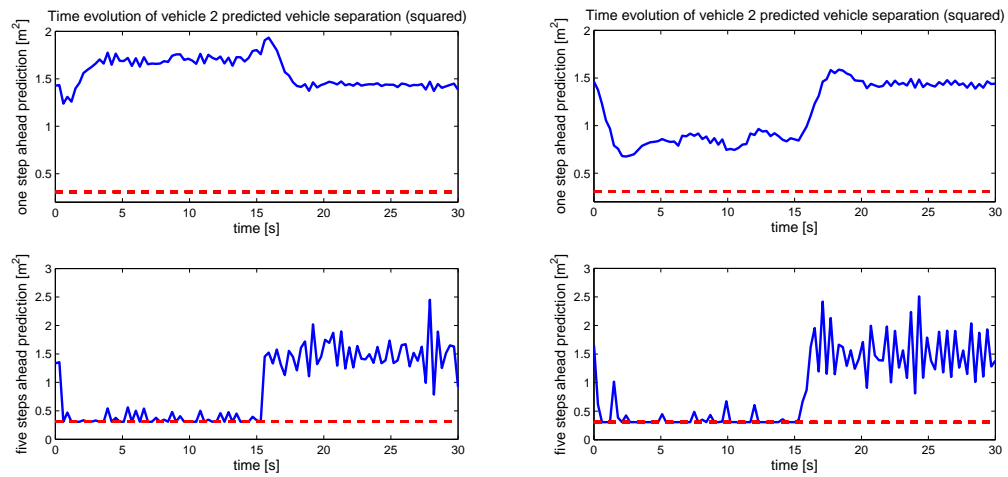
(a) $d = 0.6m$, $Q_c = 100 \times I$, and $R_c = I$

(b) $d = 0.6m$, $Q_c = 25 \times I$, and $R_c = I$

(c) $d = 1.2m$, $Q_c = 100 \times I$, and $R_c = I$

(d) $d = 1.2m$, $Q_c = 25 \times I$, and $R_c = I$

Figure 5.11: Constraint activity of Vehicle 2 when Vehicle 1 does not receive information from Vehicle 2.

hicle formation geometry, and communications.

This chapter is in part a reprint of the materials as it appears in,

Keunmo Kang, David D. Zhang, Robert R. Bitmead - *Disturbance Rejection Control in Coordinated Systems*, 17th IFAC World Congress, Seoul, Korea, Jul., 2008.

The dissertation author was the primary author in this publication and the co-author, professor Bitmead, directed and supervised the research.

6

Conclusion

6.1 Conclusions

In this dissertation, we investigated

– New techniques to improve stabilizing performance and to guarantee feasibility of MPC (Chapter 2),

– Disturbance Rejection Control in Coordinated Systems (Chapter 3 and 4).

The main conclusions are the following:

– In Chapter 2, Contraction Based MPC was developed for a general class of discrete time invariant systems with constraints. The algorithm is useful for a constrained system whose initial states are far from the origin. A system property such as contractibility or existence of CLF is a core condition to realize the algorithm. As a by-product, our MPC scheme may bring faster system stabilization than control methods solely associated with the CLF being used for feedback control.

– In Chapter 2, in addition to the above, algorithmic methods are developed to generate feasible MPC with two state constraint structures: terminal state equality constraint and reference dependent constraints. When a linear system is considered and constraints are in the forms of convex polytopes, then the algorithms can be formulated by using convex set and reachable set properties.

– In Chapter 3, disturbance rejection control schemes were studied using the fixed vehicle formation scenario. The role of control is to maintain the formation without collision between the vehicles in the presence of disturbance. One surprising discovery is that, to achieve our aims, coordinated control via interaction between the vehicles is not always necessary. With linear and gaussian assumptions, we provided the analytical tool to show when coordinated control is required especially for collision avoidance. If coordinated control is necessary, then MPC may be used to accommodate no-collision between the vehicles as a constraint. A method to incorporate disturbance and noise effects into the MPC design was developed.

– In Chapter 4, cross-estimator design was studied as a tool for one subsystem to predict neighboring subsystems' future behavior. Under linear and gaussian assumptions, a set of LMIs was formulated and it is capable of accommodating the stabilizing filter requirement, the communication limit, and the performance requirement from the no-collision constraint in the feedback control. If bounded noises and a specific disturbance class are considered, then we take a different approach that is akin to Adaptive Kalman Filtering.

6.2 Future Work

The following issues need further investigations or will help in advancing the presented results:

– In Chapter 2, Contraction Based MPC needs more work on the actual set computations for the nonlinear systems. Recall the terminal target state computation:

$$\bar{x}_{k+1+N} = \arg\min_{\tilde{x}}\{||\tilde{x}||_S \big| \tilde{x} = f(\bar{x}_{k+N}, \bar{u}_{k+N}), \bar{u}_{k+N} \in \mathbf{U}\}.$$

This involves an one-step reachable set using the plant $f(x, u)$ and the set $\mathbf{U}$. The difficulty is that reachable sets computed for a nonlinear system usually lose nice set properties such as convexity even when $\mathbf{U}$ is convex. Furthermore, reachable sets would have very different characteristic depending on what nonlinear system is considered. The algorithms for feasible MPC also require more investigations

for the nonlinear system case for the same difficulty that Contraction Based MPC has. Inclusion of disturbance in this chapter's results is also a great problem.

– In Chapter 3, the local MPC design attempted to maintain the formation by using the quadratic cost function to minimize the distance between the vehicle's target position and its expected future position. Although a treatment to guarantee closed-loop stability (i.e. formation keeping) such as a terminal constraint is desired, further studies are required associated with some issues: incorporating the uncertainties, effect on the other constraints and the other vehicles, and feasibility. In addition, the current MPC formulation does not have a mechanism to guarantee feasibility all time. This issue mainly comes from the difference between the state estimate $\hat{x}_{k+1|k+1}$ and the one step ahead prediction $\hat{x}_{k+1|k}$: if $\hat{x}_{k+1|k+1} = \hat{x}_{k+1|k}$, then one might have been able to have feasibility by a similar argument used in Section 2.2. In this case, one possible remedy is to include this effect by tightening the constraints that was introduced in (Kuwata, Richards, Schouwenaars & How 2004, Richards & How 2004). However, this approach may results in much more conservative control inputs.

– In Chapter 4, when bounded noises and a class of disturbance are considered, the current cross-estimator deign seems unable to include the communication resource assignment as done in the gaussian case. Further investigations are needed to see what one can do for this.

– In Chapter 5, interesting issues arose in the controlled vehicle behavior by MPC with respect to control parameters, formation geometry, communication (one way vs. two way), etc. The biggest headache to analyze such issues theoretically is that MPC does not produce a fixed control law and the control input is never predictable since it solves a numerical optimization problem at each sampling time. This makes the analysis harder even for the steady state analysis under the linear gaussian assumption.

– The ideas in Chapter 3, 4, and 5 were developed for the coordinated vehicles. As we mentioned, we used the coordinated vehicle problem since the framework is easy to understand. It would be interesting to investigate whether the developed results

can be applied to other coordinated systems such as distributed power systems, networked communication nodes, etc.

Appendix A

Hovercraft Model: HOvercraft Testbed for DEcentralized Control

Here we present the dynamical model of the hovercraft that is mainly used for example problems in this thesis. The hovercraft we consider is known as HOvercraft Testbed for DEcentralized Control (HoTDeC) of the University of Illinoise, Urbana-Champagne. We provide the main features of the model based on (Rubel 2004, Stubbs, Vladimerou, Fulford, Strick, Di & Dullerud 2005).

The hovercraft has the three degrees of freedom; $\mathbb{X}$ and $\mathbb{Y}$ in the typical Cartesian coordinate, and the rotational direction θ as shown in Figure A.1. The vehicle's equation of motion is given by

$$
\begin{bmatrix} \dot{x}_p \\ \dot{y}_p \\ \dot{\theta}_p \\ \dot{x}_s \\ \dot{y}_s \\ \dot{\theta}_s \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -\frac{b_t}{m} & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{b_t}{m} & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{b_r}{J} \end{bmatrix} \begin{bmatrix} x_p \\ y_p \\ \theta_p \\ x_s \\ y_s \\ \theta_s \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{1}{m} & 0 & 0 \\ 0 & \frac{1}{m} & 0 \\ 0 & 0 & \frac{1}{J} \end{bmatrix} \begin{bmatrix} F_x \\ F_y \\ T \end{bmatrix}, \qquad (A.1)
$$

whose state variables and parameters are given in Table A.1. We discretize the system
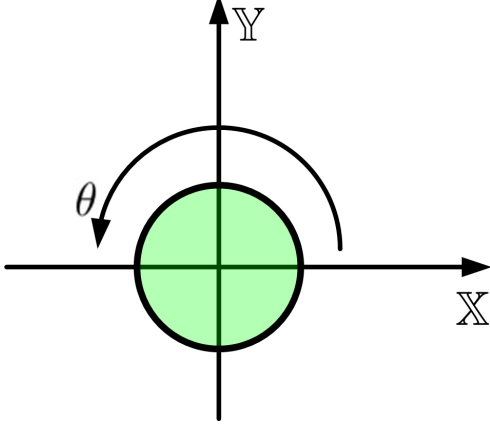
Figure A.1: Coordinates considered in the modeling of the hovercraft (green).

Table A.1: State and input variables and physical parameters of the HoTDeC hovercraft.

State and input variables	Definition
x_p	position in $\mathbb{X}$ coordinate
y_p	position in $\mathbb{Y}$ coordinate
θ	angular position
F_x	thrust in $\mathbb{X}$ coordinate
F_y	thrust in $\mathbb{Y}$ coordinate
T	angular thrust

Physical parameters	Definition	Value
m	mass of the vehicle	$3.44\ kg$
J	moment of inertia	$0.061\ kg \times m^2$
b_t	translational viscous friction constant	$3.50 \times e^{-3}\ \frac{N \times s}{m}$
b_r	rotational viscous friction constant	$2.63 \times e^{-4}\ \frac{N \times m \times s}{rad}$
	diameter of the vehicle	$0.35\ m$

equation (A.1) with $0.3\ sec$ sampling time:

$$
\begin{bmatrix} x_{p,k+1} \\ y_{p,k+1} \\ \theta_{p,k+1} \\ x_{s,k+1} \\ y_{s,k+1} \\ \theta_{s,k+1} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0.30 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0.30 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0.30 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_{p,k} \\ y_{p,k} \\ \theta_{p,k} \\ x_{s,k} \\ y_{s,k} \\ \theta_{s,k} \end{bmatrix} + \begin{bmatrix} 0.01 & 0 & 0 \\ 0 & 0.01 & 0 \\ 0 & 0 & 0.74 \\ 0.09 & 0 & 0 \\ 0 & 0.09 & 0 \\ 0 & 0 & 4.93 \end{bmatrix} \begin{bmatrix} F_{x,k} \\ F_{y,k} \\ T_k \end{bmatrix}, \quad \text{(A.2)}
$$

where $x_{p,k}$, $y_{p,k}$, $x_{s,k}$ and $y_{s,k}$ represent positions and velocities in $\mathbb{X}$ and $\mathbb{Y}$ coordinates respectively, $\theta_{p,k}$ and $\theta_{s,k}$ are angular displacement and velocity respectively, and

$[F_{x,k} \ F_{y,k} \ T_k]^T$ is the discrete time counterpart of $[F_x \ F_y \ T]^T$.

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