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Publication Date

2016-07-26

Peer reviewed

DEFINABLE REGULARITY LEMMAS FOR NIP HYPERGRAPHS

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ABSTRACT. We present a systematic study of the regularity phenomena for NIP hypergraphs and connections to the theory of (locally) generically stable measures, providing a model-theoretic hypergraph version of the results from [21]. Besides, we revise the two extremal cases of regularity for stable and distal hypergraphs, improving and generalizing the results from [6] and [23]. Finally, we consider a related question of the existence of large (approximately) homogeneous definable subsets of NIP hypergraphs and provide some positive results and counterexamples.

1. INTRODUCTION

Szemerédi’s regularity lemma is a fundamental result in (hyper-)graph combinatorics with numerous applications in extremal combinatorics, number theory and computer science (see [19] for a survey). We recall it in a simplest form. By a graph $G = (V, E)$ we mean a set G with a symmetric subset $E \subseteq V^2$. For $A, B \subseteq V$ we denote by $E(A, B)$ the set of edges between A and B , i.e. $E(A, B) = E \cap (A \times B)$.

Fact 1.1 (Szemerédi regularity lemma). *Let $G = (V, E)$ be a finite graph and $\varepsilon > 0$. There is a partition $V = V_1 \cup \dots \cup V_M$ into disjoint sets for some $M < M(\varepsilon)$, where the constant $M(\varepsilon)$ depends on ε only, real numbers $\delta_{ij}, i, j \in [M]$, and an exceptional set of pairs $\Sigma \subseteq [M] \times [M]$ such that*

$$\sum_{(i,j) \in \Sigma} |V_i||V_j| \leq \varepsilon |V|^2$$

and for each $(i, j) \in [M] \times [M] \setminus \Sigma$ we have

$$||E(A, B)| - \delta_{ij}|A||B|| < \varepsilon |V_i||V_j|$$

for all $A \subseteq V_i, B \subseteq V_j$.

The bounds on the size of such a partition, however, are known to be extremely bad: Gowers had demonstrated that $M(\varepsilon)$ grows as an exponential tower of height polynomial in $(\frac{1}{\varepsilon})$ (see e.g. [26]).

Several recent results demonstrate that better bounds and stronger regularity can be obtained for certain restricted families of hypergraphs. For example, in [9, 10] it is shown that when the edge relation is semialgebraic, of bounded description complexity, then the size of the partition can be bounded by a polynomial in terms of $\frac{1}{\varepsilon}$, all good pairs are actually homogeneous, and the sets in the partition can be chosen to be semialgebraic, of bounded complexity. Similar polynomial bounds were

A.C. was supported by the NSF Research Grant DMS-1600796 and by an Alfred P. Sloan Fellowship.

S.S. was supported by the NSF Research Grant DMS-1500671.

obtained by Tao [39] for algebraic hypergraphs of bounded description complexity in large finite fields and by Lovász, Szegedy [21] for graphs of bounded VC-dimension.

These results can be naturally viewed as results about hypergraphs with the edge relation definable, in the sense of first-order logic, in certain tame structures, and the restrictions on the complexity of the edge relation in all of the results above are surprisingly well aligned with generalized stability and classification in model theory. For example, as demonstrated in [6], the results in [9, 10] can be generalized to graphs definable in arbitrary *distal* structures (see Section 4.2), and that moreover this strong form of regularity characterizes distality. Here “semialgebraic graphs” corresponds to the special case of “graphs definable in the field of reals”, but the result also applies to graphs definable in the p -adics, for example. Similarly, the result in [39] can be viewed as a result about graphs definable in *pseudofinite fields*, and admits a natural model theoretic proof and generalizations [12, 14, 28]. Another very important example is given by the regularity lemma for *stable graphs* [23] (model-theoretic stability is the notion of tameness at the core of Shelah’s classification [32], see Section 4.1). Similarly, the results in [21] can be interpreted as results about graphs definable in *NIP structures* (see below).

Another point of view on the hypergraph regularity phenomenon is through the prism of probability theory. Namely, the existence of a regular partition can be viewed as a finitary version of the existence of the conditional expectation. There are several proofs of the hypergraph regularity lemma in the literature making this precise by reducing working with a family of finite graphs to working with some kind of an analytic “limit object” equipped with a probability measure (see [8, 20, 38]).

Similarly, regularity for restricted families of graphs can be viewed as the study of (finitely additive) probability measures on certain restricted families of Boolean algebras. Such measures in the model-theoretic setting of Boolean algebras of definable sets were introduced by Keisler [17], and recently the study of Keisler measures has attracted a lot of attention, especially the study of *generically stable measures* in *NIP structures* [15, 16, 37]. The class of NIP structures was introduced by Shelah in his work on the classification program [32]. It contains all stable and σ -minimal structures, along with other important algebraic examples, and we refer to [1, 35] for an introduction to the area (see also Section 3.3 for the definition and some examples). The study of Keisler measures in NIP structures can be viewed as a model theoretic counterpart of the Vapnik-Chervonenkis theory [41], and generically stable measure are those Keisler measures that satisfy a form of the VC-theorem for all uniformly definable families (see Section 3.3).

The connection between the study of generically stable measures in model theory and regularity lemmas for definable hypergraphs was pointed out in the distal case in [6], and the aim of this article is to systematically develop these connections for the general (local) NIP setting.

In Section 2 we give a decomposition result for products of finitely additive probability measures that are well-approximated by counting measures (which we call *fap measures*, see Section 2.4), with and without the assumption of finite VC-dimension. Namely, assume we are given some sets V_1, \dots, V_k equipped with Boolean algebras $\mathcal{B}_1, \dots, \mathcal{B}_k$ of subsets and measures μ_1, \dots, μ_k on them. Let $R \subseteq V_1 \times \dots \times V_k$ be an edge relation such that all of its fibers are measurable. It then follows from the fap assumption that there is a Boolean algebra \mathcal{B} of subsets of $V_1 \times \dots \times V_k$ extending the product Boolean algebra $\mathcal{B}_1 \otimes \dots \otimes \mathcal{B}_k$ and such that $R \in \mathcal{B}$, and

such that \mathcal{B} can be equipped with a natural product measure μ satisfying a Fubini condition (Section 2.4). Moreover, relatively to μ , the set R can be approximated by a union of boxes (i.e. sets of the form $A_1 \times \dots \times A_k$ with $A_i \in \mathcal{B}_i$) up measure ε , and in the finite VC-dimension case the number of boxes needed is polynomial in $\frac{1}{\varepsilon}$ (Theorem 2.18). On the one hand, this can be viewed as a version of the results for graphons from [21] in a setting better suited for the model-theoretic applications, and generalized to hypergraphs. On the other hand, this result can also be viewed as developing elements of the *local* theory of generically stable measures, and refining some of the results in [16] for such measures. In our setting, instead of working with Borel measures on the space of types, we use the theory of integration for finitely additive measures (sometimes called the theory of charges [5]), which we believe provides a more streamlined account, and we give some details for the sake of exposition. Note that we are only assuming bounded VC-dimension on R -definable sets, and our definition of a fap measure is weaker than the definition of fim measures in [16] (see Remark 3.7), so we have to redefine the product of fap measures.

In Section 3 we apply these results to obtain a *definable* regularity lemma for hypergraphs of bounded VC-dimension, in particular for hypergraphs definable in an NIP structure, uniformly over all generically stable measures. In Section 4 we discuss regularity in two extreme opposite special cases of the NIP hypergraphs. Namely, we revise and improve the aforementioned stable [23, 24] and distal [6] regularity lemmas in our setting. The (global) model-theoretic implications of these results can be summarized as follows.

Theorem 1.2. (1) (Corollary 3.8) *Let \mathcal{M} be an NIP structure. For every definable relation $E(x_1, \dots, x_n)$ there is some $c = c(E)$ such that: for any $\varepsilon > 0$ and any generically stable Keisler measures μ_i on $M^{|x_i|}$ there are partitions $M^{|x_i|} = \bigcup_{j < K} A_{i,j}$ and a set $\Sigma \subseteq \{1, \dots, K\}^n$ such that:*

- (a) $K \leq \left(\frac{1}{\varepsilon}\right)^c$.
 - (b) $\mu\left(\bigcup_{(i_1, \dots, i_n) \in \Sigma} A_{1, i_1} \times \dots \times A_{n, i_n}\right) \leq 1 - \varepsilon$, where $\mu = \mu_1 \otimes \dots \otimes \mu_n$,
 - (c) for all $(i_1, \dots, i_n) \notin \Sigma$ and definable $A'_1 \subseteq A_{1, i_1}, \dots, A'_n \subseteq A_{n, i_n}$ either $d_E(A'_1, \dots, A'_n) < \varepsilon$ or $d_E(A'_1, \dots, A'_n) > 1 - \varepsilon$, where $d_E(A'_1, \dots, A'_n) = \frac{\mu(E \cap A'_1 \times \dots \times A'_n)}{\mu(A'_1 \times \dots \times A'_n)}$ denotes the edge density.
 - (d) each $A_{i,j}$ is defined by an instance of an E -formula depending only on E and ε .
- (2) (Corollary 4.13) *Assume in addition that \mathcal{M} is stable. Then:*
- (a) we can take the μ_i 's to be arbitrary Keisler measures (i.e., no need to assume generic stability),
 - (b) we may assume that $\Sigma = \emptyset$, i.e. all tuples in the partition are ε -regular.
- (3) (Theorem 4.15) *Assume in addition that \mathcal{M} is distal. Then we have instead:*
- (a) for all $(i_1, \dots, i_n) \notin \Sigma$, either $(A_{1, i_1} \times \dots \times A_{n, i_n}) \cap E = \emptyset$ or $A_{1, i_1} \times \dots \times A_{n, i_n} \subseteq E$,
 - (b) if the relation E is defined by an instance of a formula θ , then we can take each $A_{i,j}$ to be defined by an instance of a formula $\psi_i(x_i, z_i)$ which only depends on θ (and not on ε).

Finally, in Section 5, we consider a related question of the existence of large (approximately) homogeneous definable subsets of definable NIP hypergraphs (i.e.,

the measure theoretic versions of the results of Erdős, Hajnal and Rödl, see e.g. [11]). As a corollary of the regularity lemma, we show that for every d and $\alpha, \varepsilon > 0$ there is some $\delta = \delta(d, \alpha, \varepsilon) > 0$ such that the following holds. Let a hypergraph $R \subseteq V_1 \times \dots \times V_k$ of VC-dimension at most d be given, and let μ_i be measures on V_i which are all fap on R . Assume that the density of R on $V_1 \times \dots \times V_k$ (relatively to the product measure) is at least α . Then it is possible to find R -definable sets $A_i \subseteq V_i$ such that $\mu_i(A_i) \geq \delta$ and such that the density of R on $A_1 \times \dots \times A_k$ is at least $1 - \varepsilon$ (Theorem 5.1). The situation is quite different in the non-partitioned case. Namely, when $V = V_1 = \dots = V_k$, $\mu = \mu_1 = \dots = \mu_k$ and R is a symmetric relation, we would like to find a definable subset A of V of positive measure, such that the density of R on A is ε -close to 0 or 1 (the result above applied to this situation would typically produce *disjoint* sets A_1, \dots, A_k). A classical theorem of Rödl (see Fact 5.4) implies that this is indeed possible for pseudofinite counting measures, with all internal sets added to the language. We provide an example of a definable graph in the p -adics which does not admit uniformly definable sets of positive measure with this property, relatively to the additive Haar measure (Section 5.2.1) (hence demonstrating that an analogue of Rödl's theorem doesn't hold for fap measures in general).

2. DECOMPOSING PRODUCT MEASURES

In this section, we present some general results on decomposing products of finitely additive probability measures that can be locally approximated by frequency measures.

2.1. Notation. We will use the following notation:

- For $k \in \mathbb{N}$ we will denote by $[k]$ the set $\{1, \dots, k\}$.
- For an integer k and $I \subseteq [k]$ we will denote by I^c the complement $I^c = [k] \setminus I$. For $i \in [k]$ instead of $\{i\}^c$ we write i^c .
- For sets V_1, \dots, V_k and $I \subseteq [k]$ we denote by V_I the product $V_I = \prod_{i \in I} V_i$.
- Let $R \subseteq V_1 \times \dots \times V_k$ and $I \subseteq [k]$. Viewing R as a subset of $V_I \times V_{I^c}$, for $b \in V_{I^c}$ we denote by R_b the fiber

$$R_b = \{a \in V_I : (a, b) \in R\}.$$

Definition 2.1. Let V_1, \dots, V_k be sets, $R \subseteq V_1 \times \dots \times V_k$ and $I \subseteq [k]$. We say that a subset $X \subseteq V_I$ is *R -definable over a set $D \subseteq V_{I^c}$* if it is a finite Boolean combination of sets of the form R_b with $b \in D$, and say that X is *R -definable* if it is R -definable over V_{I^c} .

Definition 2.2. Let V_1, \dots, V_k be sets and $A \subseteq V_1 \times \dots \times V_k$.

For a set $R \subseteq V_1 \times \dots \times V_k$ we say that A is *R_{\otimes} -definable* if A can be written as a finite union of sets of the form $X_1 \times \dots \times X_k$, such that each $X_i \subseteq V_i$ is R -definable.

In addition for a tuple $\vec{D} = (D_1, \dots, D_k)$ with $D_i \subseteq V_{i^c}$ we say that A is *R_{\otimes} -definable over \vec{D}* if every X_i above is R -definable over D_i . For such a tuple \vec{D} we use notation $\|\vec{D}\| = \max\{|D_i| : i \in [k]\}$.

We recall the notion of VC-dimension (see e.g. [25, Chapter 10]). Let V be a set, finite or infinite, and let \mathcal{F} be a family of subsets of V . Given $A \subseteq V$, we say that it is *shattered* by \mathcal{F} if for every $A' \subseteq A$ there is some $S \in \mathcal{F}$ such that $A \cap S = A'$. The *VC-dimension of \mathcal{F}* , that we will denote by $VC(\mathcal{F})$, is the smallest integer d such that no subset of V of size $d + 1$ is shattered by \mathcal{F} . For a

set $B \subseteq V$, let $\mathcal{F} \cap B = \{A \cap B : A \in \mathcal{F}\}$. The *shatter function* of \mathcal{F} is defined as $\pi_{\mathcal{F}}(n) = \max\{|\mathcal{F} \cap B| : B \subseteq V, |B| = n\}$.

Fact 2.3 (Sauer-Shelah lemma). *If $VC(\mathcal{F}) \leq d$ then for $n \geq d$ we have $\pi_{\mathcal{F}}(n) \leq \sum_{i \leq d} \binom{n}{i} = O(n^d)$.*

Definition 2.4. For sets V_1, \dots, V_k and a set $R \subseteq V_1 \times \dots \times V_k$ we say that R has *VC-dimension at most d* if for every $I \subseteq [k]$ the family $\{R_a : a \in V_{I^c}\}$ is a family with VC-dimension at most d .

The next fact follows from the Sauer-Shelah Lemma.

Fact 2.5. *For every $d \in \mathbb{N}$ there is a constant C_d such that for any relation $R \subseteq V \times W$ of VC-dimension at most d and any finite $D \subseteq V$, the number of subsets of W that are R -definable over D is at most $C_d |D|^d$.*

2.2. Basics on Boolean algebras and measures. Recall that for a set V , a *field on V* is a Boolean algebra of subsets of V .

For sets V_1, \dots, V_k and fields \mathcal{B}_i on V_i , $i \in [k]$, as usual, we denote by $\mathcal{B}_1 \otimes \dots \otimes \mathcal{B}_k$ the field on $V_1 \times \dots \times V_k$ generated by the sets $X_1 \times \dots \times X_k$ with $X_i \in \mathcal{B}_i$. It is not hard to see that every set in $\mathcal{B}_1 \otimes \dots \otimes \mathcal{B}_k$ is a disjoint union of sets of the form $X_1 \times \dots \times X_k$ with $X_i \in \mathcal{B}_i$. Given $I = \{i_1, \dots, i_n\} \subseteq [k]$, we let $\mathcal{B}_I := \bigotimes_{i \in I} \mathcal{B}_i = \mathcal{B}_{i_1} \otimes \dots \otimes \mathcal{B}_{i_n}$.

2.2.1. Finitely additive probability measures.

Definition 2.6. Let V be a set and \mathcal{B} be a field on V . In this paper, a *measure on \mathcal{B}* is a finitely additive probability measure on \mathcal{B} , i.e. a function $\mu : \mathcal{B} \rightarrow \mathbb{R}$ such that $\mu(\emptyset) = 0$, $\mu(V) = 1$ and $\mu(A \cup B) = \mu(A) + \mu(B) - \mu(A \cap B)$ for all $A, B \in \mathcal{B}$.

Let V_1, \dots, V_k be sets and \mathcal{B}_i be fields on V_i , $i \in [k]$. Assume we have a measure μ_i on \mathcal{B}_i for each $i \in [k]$. It is not hard to see that there is a unique measure μ on $\mathcal{B}_1 \otimes \dots \otimes \mathcal{B}_k$ with $\mu(A_1 \times \dots \times A_k) = \prod_{i=1}^k \mu_i(A_i)$ for all $A_i \in \mathcal{B}_i$, $i \in [k]$. We will denote this measure μ by $\mu_1 \times \dots \times \mu_k$.

2.2.2. Integration with respect to finitely additive measures. We will need some basic facts about integration relatively to finitely additive measures (and refer to [5] for a detailed account).

As usual for a set V and a subset $X \subseteq V$ we will denote by $\mathbf{1}_X$ the indicator function of X on V .

We fix a set V and a field \mathcal{B} on V .

We say that a function $f : V \rightarrow \mathbb{R}$ is *\mathcal{B} -simple* if there are $X_1, \dots, X_n \in \mathcal{B}$ and $r_1, \dots, r_n \in \mathbb{R}$ with $f = \sum_{i=1}^n r_i \mathbf{1}_{X_i}$. Obviously the set of all \mathcal{B} -simple functions forms an \mathbb{R} -algebra.

For a measure μ on \mathcal{B} and a \mathcal{B} -simple function $f = \sum_{i=1}^n r_i \mathbf{1}_{X_i}$ we define

$$\int_V f d\mu = \sum_{i=1}^n r_i \mu(X_i).$$

It is easy to see that the above integral does not depend on a representation of f as a simple function. If a subset $A \subseteq V$ is in \mathcal{B} then we also define

$$\int_A f d\mu = \int_V \mathbf{1}_A f d\mu = \sum_{i=1}^n r_i \mu(A \cap X_i).$$

Remark 2.7. Clearly for $A \in \mathcal{B}$ we have $\mu(A) = \int_V \mathbf{1}_A d\mu$.

We say that a function $f: V \rightarrow \mathbb{R}$ is \mathcal{B} -integrable, or just *integrable*, if it is in the closure of the set of \mathcal{B} -simple functions with respect to the L_∞ -norm, i.e. for all $\varepsilon > 0$ there is a \mathcal{B} -simple function g with $|f(x) - g(x)| < \varepsilon$ for all $x \in V$. The following claim is obvious.

Claim 2.8. *A function $f: V \rightarrow \mathbb{R}$ is \mathcal{B} -integrable if and only if for any $\varepsilon > 0$ there are $Y_1, \dots, Y_n \in \mathcal{B}$ covering V such that for any $i \in [n]$ and any $c, c' \in Y_i$ we have $|f(c) - f(c')| < \varepsilon$.*

If f is \mathcal{B} -integrable and μ is a measure on \mathcal{B} then the integral of f with respect to μ is defined as

$$\int_V f d\mu = \lim_{n \rightarrow \infty} \int_V g_n d\mu,$$

where $(g_n)_{n \in \mathbb{N}}$ is a sequence of \mathcal{B} -simple functions convergent to f . It is easy to see that this integral does not depend on the choice of a convergent sequence. Also for a \mathcal{B} -integrable function f and a set $A \in \mathcal{B}$ we define

$$\int_A f d\mu = \int_V \mathbf{1}_A f d\mu.$$

2.3. On ε -nets. Let V be a set, \mathcal{B} a field on V and μ a measure on \mathcal{B} . Let \mathcal{F} be a family of subsets of V with $\mathcal{F} \subseteq \mathcal{B}$. As usual, for $\varepsilon > 0$ we say that a subset $T \subseteq V$ is an ε -net for \mathcal{F} if for every $F \in \mathcal{F}$ we have $\mu(F) \geq \varepsilon \implies F \cap T \neq \emptyset$.

The following is a well-known consequence of the classical VC-theorem (see [18, 41] and also [21]).

Fact 2.9. *Let V be a set, \mathcal{B} a field on V and μ a measure on \mathcal{B} with a finite support (i.e. there exists a finite set $A \in \mathcal{B}$ with $\mu(A) = 1$). If $\mathcal{F} \subseteq \mathcal{B}$ is a VC-family with VC-dimension at most d then for any $\varepsilon > 0$ it admits an ε -net T with $|T| \leq 8d \frac{1}{\varepsilon} \log \frac{1}{\varepsilon}$.*

2.4. Fap measures and their products. Let V, W be sets with fields $\mathcal{B}_V, \mathcal{B}_W$ on them. Let \mathcal{F} be a family of subsets of V in \mathcal{B}_V .

Definition 2.10. Let μ be a measure on \mathcal{B}_V . We say that μ is *fap* (“finitely approximated”) on \mathcal{F} if for every $\varepsilon > 0$ there are $p_1, \dots, p_n \in V$ (possibly with repetitions) with

$$|\mu(F) - \text{Av}(p_1, \dots, p_n; F)| < \varepsilon \text{ for every } F \in \mathcal{F},$$

where $\text{Av}(p_1, \dots, p_n; F) = \frac{1}{n} |\{i \in [n]: p_i \in F\}|$. We say that p_1, \dots, p_n is an ε -approximation of μ on \mathcal{F} .

Definition 2.11. Let $R \subseteq V \times W$ be such that $R_b \in \mathcal{B}_V$ for all $b \in W$. Let μ be a measure on \mathcal{B}_V . We say that μ is *fap on R* if it is fap on \mathcal{F}_m for all $m \in \mathbb{N}$, where \mathcal{F}_m is the family of all subsets of V given by the Boolean combinations of at most m sets of the form $R_b, b \in W$.

Remark 2.12. (1) In particular, if μ is fap on R , then it is fap on the family $\mathcal{R}^\Delta = \{R_b \Delta R_{b'} : b, b' \in W\}$.

(2) Note that μ being fap on \mathcal{R}^Δ does not imply that μ is fap on $\mathcal{R}_V = \{R_b : b \in W\}$. For example, $V = \mathbb{R}$, let \mathcal{B}_V be the field generated by all intervals in V , and let \mathcal{R}_V be the family of all intervals unbounded from above. Let μ be the

0 – 1 measure on \mathcal{B}_V such that the measure of a set is 1 if and only if it is unbounded from above. Then all sets in \mathcal{R}^Δ have measure 0, so we can take the empty set as an ε -approximation for μ on \mathcal{R}^Δ , for any $\varepsilon > 0$. But there are no finite ε -approximations for μ on \mathcal{R} , for any $\varepsilon < 1$, as any finite set can be avoided by some unbounded interval of measure 1.

Similarly, if \mathcal{R} is the family of all intervals bounded from above, then μ is trivially fap on \mathcal{R} . However, it is not fap on the family of all complements of the sets in \mathcal{R} .

See Example 3.11 for more examples of fap measures.

For any set $A \in \mathcal{B}_V$, consider the function $h_{R,A} : W \rightarrow \mathbb{R}$ given by $h_{R,A}(b) = \mu(R_b \cap A)$.

Claim 2.13. *Assume that μ is fap on R (or just on \mathcal{R}^Δ) and that $R_a \in \mathcal{B}_W$ for all $a \in V$. Then for any set $A \in \mathcal{B}_V$, the function $h_{R,A}$ is \mathcal{B}_W -integrable.*

Proof. Let $\varepsilon > 0$. By assumption we can choose $p_1, \dots, p_n \in V$ such that

$$|\mu(R_b \Delta R_{b'}) - \text{Av}(p_1, \dots, p_n; R_b \Delta R_{b'})| < \varepsilon$$

for every $b, b' \in W$.

For $I \subseteq [n]$ let $C_I \subseteq W$ be the set $C_I = \{b \in W : p_i \in R_b \Leftrightarrow i \in I\} \in \mathcal{B}_W$. Clearly the sets $C_I, I \subseteq [n]$, cover W and for every $I \subseteq [n]$ and $b, b' \in C_I$ we have $\mu(R_b \Delta R_{b'}) < \varepsilon$. Hence, for any $b, b' \in C_I$ we have

$$|h_{R,A}(b) - h_{R,A}(b')| \leq \mu(A \cap (R_b \Delta R_{b'})) \leq \mu(R_b \Delta R_{b'}) < \varepsilon.$$

By Claim 2.8 the function $h_{R,A}$ is \mathcal{B}_W -integrable. \square

Let now V, W, Z be sets and $R \subseteq V \times W \times Z$. Assume that $\mathcal{R}_V = \{R_{(b,c)} : (b,c) \in W \times Z\} \subseteq \mathcal{B}_V$ and $\mathcal{R}_W = \{R_{(a,c)} : (a,c) \in V \times Z\} \subseteq \mathcal{B}_W$. Let μ be a measure on \mathcal{B}_V which is fap on $R \subseteq V \times (W \times Z)$, and ν a measure on \mathcal{B}_W . Note that by assumption and Claim 2.13, if E is an arbitrary R -definable subset of $V \times W$ and $A \in \mathcal{B}_V$, then the function $h_{E,A}$ is \mathcal{B}_W -integrable. And $h_{E,A}(b) = \int_A \mathbf{1}_E(x,b) d\mu$. Hence the double integral

$$\omega_E(A, B) = \int_B \left(\int_A \mathbf{1}_E(x,y) d\mu \right) d\nu.$$

is well defined for any $A \in \mathcal{B}_V, B \in \mathcal{B}_W$.

Let now $\mathcal{B}_{V \times W}$ be the field on $V \times W$ generated by $\mathcal{B}_V \otimes \mathcal{B}_W$ and $\{R_c : c \in Z\}$, in particular it contains all R -definable sets. Then we have the following.

Proposition 2.14. (1) *There is a unique measure ω on $\mathcal{B}_{V \times W}$ whose restriction to $\mathcal{B}_V \otimes \mathcal{B}_W$ is $\mu \times \nu$ and such that $\omega(E \cap (A \times B)) = w_E(A, B)$ for every R -definable $E \subseteq V \times W, A \in \mathcal{B}_V, B \in \mathcal{B}_W$. We denote this measure by $\mu \times \nu$.*
(2) *If in addition ν is fap on R , then $\mu \times \nu$ is also fap on R and $\mu \times \nu(E) = \nu \times \mu(E)$ for all R -definable sets.*

Proof. (1) It is easy to see that every set Y in $\mathcal{B}_{V \times W}$ is a finite disjoint union of sets of the form $E_i \cap (A_i \times B_i)$ where E_i is an atom of the Boolean algebra of all R -definable subsets of $V \times W$ and $A_i \in \mathcal{B}_V, B_i \in \mathcal{B}_W$. We define $\omega(Y) = \sum \omega_{E_i}(A_i, B_i)$. It is easy to check that ω is well-defined (for all $A' \in \mathcal{B}_V, B' \in \mathcal{B}_W$ and R -definable $E' \subseteq V \times W$, if $(A \times B) \cap E = (A' \times B') \cap E'$, then $w_E(A, B) = w_{E'}(A', B')$) and is a finitely additive probability measure on $\mathcal{B}_{V \times W}$ satisfying the requirements. Uniqueness is straightforward from the definition of ω .

(2) It is enough to show that $\mu \times \nu$ is fap on any R -definable relation $E \subseteq (V \times W) \times Z$. Fix an arbitrary $\varepsilon > 0$. Let us take $p_1, \dots, p_n \in V$ such that $\mu(E_{b,c}) \approx^\varepsilon \text{Av}(p_1, \dots, p_n; E_{b,c})$ for all $(b, c) \in W \times Z$, and $q_1, \dots, q_m \in W$ such that $\nu(E_{a,c}) \approx^\varepsilon \text{Av}(q_1, \dots, q_m, E_{a,c})$ for all $(a, c) \in V \times Z$.

We claim that the set $\{(p_i, q_j) : 1 \leq i < n, 1 \leq j < m\}$ gives a 2ε -approximation for $\mu \times \nu(E_c)$, for any $c \in Z$. Namely, using linearity of integration, we have

$$\begin{aligned} \mu \times \nu(E_c) &= \int_W \left(\int_V \mathbf{1}_{E_c}(v, w) d\mu \right) d\nu \approx^\varepsilon \\ &\int_W \left(\frac{1}{n} \sum_{i=1}^n \mathbf{1}_{E_{w,c}}(p_i) \right) d\nu = \frac{1}{n} \sum_{i=1}^n \left(\int_W \mathbf{1}_{E_{w,c}}(p_i) d\nu \right) = \\ &\frac{1}{n} \sum_{i=1}^n \left(\int_W \mathbf{1}_{E_{p_i,c}}(w) d\nu \right) \approx^\varepsilon \frac{1}{n} \sum_{i=1}^n \left(\frac{1}{m} \sum_{j=1}^m \mathbf{1}_{E_{p_i,c}}(q_j) \right) = \\ &= \frac{1}{nm} \sum_{1 \leq i \leq n, 1 \leq j \leq m} \mathbf{1}_{E_c}(p_i, q_j), \end{aligned}$$

so $\mu \times \nu(E_c) \approx^{2\varepsilon} \text{Av}(\{(p_i, q_j) : 1 \leq i \leq n, 1 \leq j \leq m\}; E_c)$.

The fact that $\mu \times \nu(E_c) = \nu \times \mu(E_c)$ follows as, by the above, for any $\varepsilon > 0$ we have

$$\begin{aligned} \mu \times \nu(E_c) &\approx^{2\varepsilon} \frac{1}{n} \sum_{i=1}^n \left(\frac{1}{m} \sum_{j=1}^m \mathbf{1}_{E_{p_i,c}}(q_j) \right) = \\ &\frac{1}{m} \sum_{j=1}^m \left(\frac{1}{n} \sum_{i=1}^n \mathbf{1}_{E_{q_j,c}}(p_i) \right) \approx^\varepsilon \frac{1}{m} \sum_{j=1}^m \left(\int_V \mathbf{1}_{E_{q_j,c}}(v) d\mu \right) = \\ &\int_V \left(\frac{1}{m} \sum_{j=1}^m \mathbf{1}_{E_{v,c}}(q_j) \right) d\mu \approx^\varepsilon \int_V \left(\int_W \mathbf{1}_{E_c}(v, w) d\nu \right) d\mu = \\ &\nu \times \mu(E_c), \end{aligned}$$

hence $\mu \times \nu(E_c) \approx^{4\varepsilon} \nu \times \mu(E_c)$ for arbitrary $\varepsilon > 0$. \square

It is not hard to see that a product of fap measures satisfies a weak Fubini's property.

Lemma 2.15. *Let V, W be sets, μ a measure on \mathcal{B}_V which is fap on R , ν a measure on \mathcal{B}_W . For $\varepsilon > 0$ if $\mu_V(R_a) < \varepsilon$ for all $a \in W$ then $(\mu_V \times \nu_W)(R) < \varepsilon$.*

We extend products of fap measures to an arbitrary number of sets.

Definition 2.16. Let V_1, \dots, V_k be sets, $R \subseteq V_1 \times \dots \times V_k$ and assume that for each $i \in [k]$ we have a field \mathcal{B}_i on V_i and a measure μ_i on \mathcal{B}_i which is fap on R (viewed as a binary relation on $V_i \times V_i^c$). Then, by induction on k , we define a measure $\mu_1 \times \dots \times \mu_k = (\mu_1 \times \dots \times \mu_{k-1}) \times \mu_k$ on $\mathcal{B}_{V_1} \times \dots \times \mathcal{B}_{V_k}$.

2.5. Approximations by rectangular sets.

Proposition 2.17. *Let $V, W, R \subseteq V \times W$ be sets, μ a measure on V which is fap on R . Then for any $\varepsilon > 0$ there are R -definable subsets $X_1, \dots, X_m \subseteq W$ partitioning W such that for every $i \in [m]$ and any $a, a' \in X_i$ we have $\mu_V(R_a \Delta R_{a'}) < \varepsilon$.*

In addition, if the family $\mathcal{R} = \{R_a : a \in W\}$ has VC-dimension at most d then we can choose $D \subseteq V$ of size at most $320d(\frac{1}{\varepsilon})^2$ such that every X_i is R -definable over D .

Proof. Let $\mathcal{R}^\Delta = \{R_a \Delta R_{a'} : a, a' \in W\}$. Since μ is fap on R , there are $p_1, \dots, p_n \in V$ with $|\mu(F) - \text{Av}(p_1, \dots, p_n; F)| < \varepsilon$ for any $F \in \mathcal{R}^\Delta$.

For each $I \cap [n]$ let $X_i = \{a \in W : p_i \in R_a \Leftrightarrow i \in I\}$. It is easy to see that the sets $X_I, I \subseteq [n]$ partition W , every X_i is R -definable and for every $I \subseteq [n]$ and $a, a' \in X_I$ we have $\mu(R_a \Delta R_{a'}) < \varepsilon$.

Assume in addition that \mathcal{R} is a VC-family with VC-dimension at most d . As above we choose $p_1, \dots, p_n \in V$ with

$$|\mu(F) - \text{Av}(p_1, \dots, p_n; F)| < \varepsilon/2$$

for any $F \in \mathcal{R}^\Delta$.

Let w be a measure on \mathcal{B}_V given by $w(X) = \text{Av}(p_1, \dots, p_n; X)$. Since \mathcal{R} has VC-dimension at most d , the family \mathcal{R}^Δ had dimension at most $10d$ (see [21, Lemma 4.5]), and by Fact 2.9 we can choose $\varepsilon/2$ -net D for \mathcal{R}^Δ and w with $|D| \leq 80d \frac{2}{\varepsilon} \log \frac{2}{\varepsilon}$. Clearly

$$80d \frac{2}{\varepsilon} \log \frac{2}{\varepsilon} \leq 80d \left(\frac{2}{\varepsilon}\right)^2 = 320d \left(\frac{1}{\varepsilon}\right)^2.$$

For each $I \cap D$ let $X_I = \{a \in W : R_a \cap D = I\}$. It is easy to see that the sets $X_I, I \subseteq D$, partition W and every X_i is R -definable over D . Let $I \subseteq D$ and $a, a' \in X_I$. Then $R_a \cap D = R_{a'} \cap D$, hence $w(R_a \Delta R_{a'}) \leq \varepsilon/2$, and $\mu_V(R_a \Delta R_{a'}) < \varepsilon$. \square

Theorem 2.18. *Let V_1, \dots, V_k and $R \subseteq V_1 \times \dots \times V_k$ be sets, and μ_1, \dots, μ_k measures on V_1, \dots, V_k , respectively, which are all fap on R . Then for every $\varepsilon > 0$ there is an R_\otimes -definable $A \subseteq V_1 \times \dots \times V_k$ with*

$$(\mu_1 \times \dots \times \mu_k)(R \Delta A) < \varepsilon.$$

In addition, if R has VC-dimension at most d (see Definition 2.4) then we can choose A to be R_\otimes -definable over some \vec{D} with $\|\vec{D}\| \leq C_{k,d} \left(\frac{1}{\varepsilon}\right)^{2(k-1)d}$, where $C_{k,d}$ is a constant that depends on k and d only.

Proof. We proceed by induction on k .

The case $k = 2$. Let V_1, V_2 and $R \subseteq V_1 \times V_2$ be given. Using proposition 2.17 we can find R -definable sets X_1, \dots, X_m partitioning V_2 such that for every $i \in [m]$ and any $a, a' \in X_i$ we have $\mu_1(R_a \Delta R_{a'}) < \varepsilon$.

For each $i \in [m]$ we pick $a_i \in X_i$ and let $A = \bigcup_{i \in [m]} R_{a_i} \times X_i$. Obviously A is R_\otimes -definable. It is not hard to see that for every $a \in W$ we have $\mu_1(R_a \Delta A) < \varepsilon$, hence, by Lemma 2.15, $(\mu_1 \times \mu_2)(R \Delta A) < \varepsilon$.

Assume in addition that R has VC-dimension at most d . Then by Proposition 2.17, we can assume that for some $D_2 \subseteq V_2$ with $|D_2| \leq 320d \left(\frac{1}{\varepsilon}\right)^2$ every X_i is R -definable over D_2 . Let $D_1 = \{a_1, \dots, a_m\}$, and $\vec{D} = (D_1, D_2)$. Obviously A is R_\otimes -definable over \vec{D} . By Fact 2.5, $m < C_d |D_2|^d$, hence $|D_1| \leq C_d (320d)^d \left(\frac{1}{\varepsilon}\right)^{2d}$. And we can take $C_{2,d} = C_d (320d)^d$.

Inductive step $k + 1$. Let V_1, \dots, V_{k+1} and $R \subseteq V_1 \times \dots \times V_{k+1}$ be given.

Viewing $V_1 \times \cdots \times V_{k+1}$ as $V_{[k]} \times V_{k+1}$ and using the case of $k = 2$ we obtain R -definable X_1, \dots, X_m partitioning V_{k+1} and points $a_i \in X_i$, $i \in [m]$, such that for the set $A' = \bigcup_{i \in [m]} R_{a_i} \times X_i$ we have $(\mu_1 \times \cdots \times \mu_{k+1})(R\Delta A') < \varepsilon/2$.

For each $i \in [m]$ let $R^i = R_{a_i}$. It is an R -definable subset of $V_1 \times \cdots \times V_k$. It is easy to see that each R^i has VC-dimension at most d . Applying induction hypothesis to each R^i we obtain R_{\otimes}^i -definable sets $A_i \subseteq V_1 \times \cdots \times V_k$ such that $(\mu_1 \times \cdots \times \mu_k)(R^i \Delta A_i) < \varepsilon/2$. Let $A = \bigcup_{i \in [m]} A_i \times X_i$. It is an R_{\otimes} -definable set and using Lemma 2.15, it is not hard to see that $(\mu_1 \times \cdots \times \mu_{k+1})(A' \Delta A) < \varepsilon/2$, hence $(\mu_1 \times \cdots \times \mu_{k+1})(R\Delta A) < \varepsilon$, as required.

Assume in addition that R has VC-dimension at most d . As in the case $k = 2$ we can assume that every X_i is R -definable over $D_{k+1} \subseteq V_1, \dots, V_k$ with $|D_{k+1}| \leq 320d(\frac{2}{\varepsilon})^2$ and also assume that

$$m \leq C_d |D_{k+1}|^d \leq C_d \left[320d \left(\frac{2}{\varepsilon} \right)^2 \right]^d = C_d (1280d)^d \left(\frac{1}{\varepsilon} \right)^{2d}.$$

Applying induction hypotheses we can assume that each A_i above is R_{\otimes}^i -definable over $\vec{D}^i = (D_1^i, \dots, D_k^i)$ with $\|\vec{D}^i\| \leq C_{k,d} (\frac{2}{\varepsilon})^{2(k-1)d}$, where $D_j^i \subseteq \prod_{l \in [k] \setminus \{j\}} V_l$.

For each $i \in [m]$ and $j \in [k]$ let $\bar{D}_j^i = \{(c, a_i) : c \in D_j^i\}$, $D_j = \bigcup_{i \in [m]} \bar{D}_j^i$, and $\vec{D} = (D_1, \dots, D_{k+1})$.

It is not hard to see that A above is R -definable over \vec{D} and

$$\begin{aligned} \|\vec{D}\| &\leq m C_{k,d} \left(\frac{2}{\varepsilon} \right)^{2(k-1)d} \leq C_d (1280d)^d \left(\frac{1}{\varepsilon} \right)^{2d} 2^{2(k-1)d} \left(\frac{1}{\varepsilon} \right)^{2(k-1)d} = \\ &= C_{k+1,d} \left(\frac{1}{\varepsilon} \right)^{2kd} \end{aligned}$$

□

3. DEFINABLE REGULARITY LEMMA FOR HYPERGRAPHS OF BOUNDED VC DIMENSION

In this section we apply the product measure decomposition results from Section 2 to regularity of definable hypergraphs. Our goal is to prove a stronger version of Fact 1.1 for hypergraphs of bounded VC-dimension.

3.1. Regularity Lemmas for Hypergraphs. A k -hypergraph $G = (V_1, \dots, V_k; E)$ consists of sets V_1, \dots, V_k and a subset $E \subseteq V_1 \times \cdots \times V_k$. We don't assume that the sets V_i , $i \in [k]$ are pairwise distinct.

A k -uniform hypergraph $G = (V; E)$ is a set V with a subset $E \subseteq V^k$. Of course every k -uniform hypergraph $G = (V; E)$ can be also viewed as a k -hypergraph $(V, \dots, V; E)$ that we will denote by \tilde{G} .

For a k -hypergraph $G = (V_1, \dots, V_k; E)$ and $A_1 \subseteq V_1, \dots, A_k \subseteq V_k$ we will denote by $E(A_1, \dots, A_k)$ the set $E(A_1, \dots, A_k) = E \cap A_1 \times \cdots \times A_k$.

Let $G = (V_1, \dots, V_k; E)$ be a k -hypergraph. By a *rectangular partition* of G we mean a k -tuple $\vec{\mathcal{P}} = (\mathcal{P}_1, \dots, \mathcal{P}_k)$ where each \mathcal{P}_i is a finite partition of V_i . For a rectangular partition $\vec{\mathcal{P}} = (\mathcal{P}_1, \dots, \mathcal{P}_k)$ we define $\|\vec{\mathcal{P}}\| = \max\{|\mathcal{P}_i| : i \in [k]\}$, and for a set $X \subseteq V_1 \times \cdots \times V_k$ we write $X \in \vec{\mathcal{P}}$ if $X = X_1 \times \cdots \times X_k$ for some $X_i \in \mathcal{P}_i$, $i \in [k]$. We will also write $\Sigma \cap \vec{\mathcal{P}}$ to indicate that Σ consists of subsets $X \subseteq V_1 \times \cdots \times V_k$ with $X \in \vec{\mathcal{P}}$.

For a set $A \subseteq V_1 \times \cdots \times V_k$ and a rectangular partition $\vec{\mathcal{P}} = (\mathcal{P}_1, \dots, \mathcal{P}_k)$ we say that A is compatible with $\vec{\mathcal{P}}$ if for any $X \in \vec{\mathcal{P}}$ either $X \subseteq A$ or $X \cap A = \emptyset$, in other words A is a finite union of sets $X \in \vec{\mathcal{P}}$.

A k -hypergraph $G = (V_1, \dots, V_k; E)$ has *VC-dimension at most d* if E has VC-dimension at most d in the sense of Definition 2.4. A k -uniform hypergraph $G = (V; E)$ has VC-dimension at most d if the corresponding k -hypergraph \tilde{G} is NIP with VC-dimension at most d .

Let V_1, \dots, V_k and $E \subseteq V_1 \times \cdots \times V_k$ be sets, and let \mathcal{B}_i be a field on V_i , $i = 1, \dots, k$. We will consider the k -hypergraph $G = (V_1, \dots, V_k; E)$, and let $\vec{\mathcal{P}} = (\mathcal{P}_1, \dots, \mathcal{P}_k)$ be a rectangular partition of G . We say that $\vec{\mathcal{P}}$ is *definable* if each \mathcal{P}_i consists of sets in \mathcal{B}_i . We say that $\vec{\mathcal{P}}$ is *E -definable* if each \mathcal{P}_i consists of E -definable sets. For a tuple $\vec{D} = (D_1, \dots, D_k)$ as in Definition 2.2 we say that $\vec{\mathcal{P}}$ is *E -definable over \vec{D}* if for each $i \in [k]$ every $X \in \mathcal{P}_i$ is E -definable over E .

Definition 3.1. Let V_1, \dots, V_k and $E \subseteq V_1 \times \cdots \times V_k$ be given, and let μ_1, \dots, μ_k be measures on V_1, \dots, V_k which are fap on E . Let $\mu = \mu_1 \times \cdots \times \mu_k$.

Given $\varepsilon > 0$, we say that a definable rectangular partition $\vec{\mathcal{P}}$ of $V_1 \times \cdots \times V_k$ is *ε -regular with 0-1-densities* if there is $\Sigma \subseteq \vec{\mathcal{P}}$ such that

$$\sum_{X \in \Sigma} \mu(X) \leq \varepsilon,$$

and for every $X_1 \times \cdots \times X_k \in \vec{\mathcal{P}} \setminus \Sigma$ either

$$\mu(Y_1 \times \cdots \times Y_k) - \mu(E(Y_1, \dots, Y_k)) < \varepsilon \mu(X_1 \times \cdots \times X_k)$$

for all sets $Y_i \in \mathcal{B}_i$, $i = 1, \dots, k$;

or

$$\mu(E(Y_1, \dots, Y_k)) < \varepsilon \mu(X_1 \times \cdots \times X_k)$$

for all sets $Y_i \in \mathcal{B}_i$, $i = 1, \dots, k$.

The next proposition demonstrates how existence of an approximation by rectangular sets for the product measure can be used to obtain a regular partition.

Proposition 3.2. Let V_1, \dots, V_k and $E \subseteq V_1 \times \cdots \times V_k$ be given, and let μ_1, \dots, μ_k be measures on V_1, \dots, V_k which are all fap on E . Let $\mu = \mu_1 \times \cdots \times \mu_k$.

Let $\vec{\mathcal{P}}$ be a definable rectangular partition of $V_1 \times \cdots \times V_k$. If there is $A \subseteq V_1 \times \cdots \times V_k$, an E_{\otimes} -definable set compatible with $\vec{\mathcal{P}}$ with $\mu(A \Delta E) < \varepsilon^2$, then $\vec{\mathcal{P}}$ is ε -regular with 0-1-densities.

Proof. Let

$$\Sigma = \{X \in \vec{\mathcal{P}} : \mu(X \cap (A \Delta E)) \geq \varepsilon \mu(X)\}.$$

Since $\mu(A \Delta E) < \varepsilon^2$ and μ is finitely additive we obtain that

$$\sum_{X \in \Sigma} \mu(X) \leq \varepsilon.$$

Let $X = X_1 \times \cdots \times X_k \in \vec{\mathcal{P}} \setminus \Sigma$. We have

$$\mu(X \cap (A \Delta E)) < \varepsilon \mu(X).$$

Since A is compatible with $\vec{\mathcal{P}}$ either $X \subseteq A$ or $X \cap A = \emptyset$.

Assume first $X \subseteq A$. Let $Y_i \subseteq X_i$ be from $\mathcal{B}_i, i = 1, \dots, k$, and let $Y = Y_1 \times \dots \times Y_k$. Since $Y \subseteq X$, by monotonicity of μ we have

$$\mu(Y \cap (A\Delta E)) < \varepsilon\mu(X).$$

As $Y \subseteq A$ we have $Y \cap (A\Delta E) = Y \setminus E(Y_1, \dots, Y_k)$. Since $E(Y_1, \dots, Y_k) \subseteq Y$ we also have

$$\mu(Y \setminus E(Y_1, \dots, Y_k)) = \mu(Y) - \mu(E(Y_1, \dots, Y_k)),$$

hence

$$\mu(Y_1 \times \dots \times Y_k) - \mu(E(Y_1, \dots, Y_k)) \leq \varepsilon\mu(X_1 \times \dots \times X_k).$$

If $X \cap A = \emptyset$ similar arguments show that

$$\mu(E(Y_1, \dots, Y_k)) < \varepsilon\mu(X_1, \dots, X_k).$$

for all $Y_i \subseteq X_i$ from $\mathcal{B}_i, i = 1, \dots, k$. \square

Combining this observation with the results of Section 2, we obtain a regularity lemma for NIP hypergraphs.

Theorem 3.3. *Let V_1, \dots, V_k and $E \subseteq V_1 \times \dots \times V_k$ be given, and let μ_1, \dots, μ_k be measures on V_1, \dots, V_k which are all fap on E . Let $\mu = \mu_1 \times \dots \times \mu_k$.*

For any $\varepsilon > 0$ there is an E -definable ε -regular partition $\vec{\mathcal{P}}$ with 0-1-densities.

In addition, if E is NIP with VC dimension at most d we can choose $\vec{\mathcal{P}}$ with $\|\vec{\mathcal{P}}\| \leq C_d(C_{k,d})^d \left(\frac{1}{\varepsilon}\right)^{2(k-1)d^2}$, where C_d and $C_{k,d}$ are constants from Fact 2.5 and Theorem 2.18.

Proof. Using Theorem 2.18 there is E_{\otimes} -definable A with $\mu(A\Delta E) < \varepsilon^2$. Say $A = \cup_{j \in [m]} A_1^j \times \dots \times A_k^j$ where each $A_i^j \subseteq V_i$ is E -definable.

For each $i \in [k]$ let \mathcal{P}_i be the set of all atoms in the Boolean algebra generated by A_i^1, \dots, A_i^m . Obviously each \mathcal{P}_i consists of E -definable sets partitioning V_i , and A is compatible with $\vec{\mathcal{P}} = (\mathcal{P}_1, \dots, \mathcal{P}_k)$. By Proposition 3.2 $\vec{\mathcal{P}}$ is ε -regular with 0-1-densities.

Assume in addition that E is NIP with VC-dimension at most d . Then using Theorem 2.18 we can assume that A is E_{\otimes} -definable over $\vec{D} = (D_1, \dots, D_k)$ with $|D_i| \leq C_{k,d} \left(\frac{1}{\varepsilon}\right)^{2(k-1)d}$ for $i \in [k]$. For each $i \in [k]$ let \mathcal{P}_i be the set of all atoms in the Boolean algebra generated by E -definable over D_i subsets of V_i . Obviously each \mathcal{P}_i consists of E -definable subsets partitioning V_i and A is compatible with $\vec{\mathcal{P}} = (\mathcal{P}_1, \dots, \mathcal{P}_k)$. Also, by Fact 2.5,

$$|\mathcal{P}_i| \leq C_d |D_i|^d \leq C_d \left(C_{k,d} \left(\frac{1}{\varepsilon}\right)^{2(k-1)d}\right)^d = C_d(C_{k,d})^d \left(\frac{1}{\varepsilon}\right)^{2(k-1)d^2}$$

\square

Remark 3.4. In the case when \mathcal{M} is finite the above theorem without the NIP part is trivial, since we can take \mathcal{P}_i to be the set of all atoms in the Boolean algebra of all E -definable subsets of V_i .

We also have an analogous theorem for k -uniform hypergraphs. We state it only in the NIP case.

Theorem 3.5. *Let V_1, \dots, V_k and $E \subseteq V_1 \times \dots \times V_k$ be given, and let μ_1, \dots, μ_k be measures on V_1, \dots, V_k which are all fap on E . Let $\mu = \mu_1 \times \dots \times \mu_k$.*

For any $\varepsilon > 0$ there is an E -definable partition \mathcal{P} of V such that $\vec{\mathcal{P}} = (\mathcal{P}, \dots, \mathcal{P})$ is an ε -regular partition of the k -hypergraph $(V, \dots, V; E)$ with 0-1-densities, and $|\mathcal{P}| \leq C_d(kC_{k,d})^d \left(\frac{1}{\varepsilon}\right)^{2(k-1)d^2}$.

Proof. Using Theorem 2.18 there is $A \subseteq V^k$ which is E_\otimes -definable over some $\vec{D} = (D_1, \dots, D_k)$, and such that $\mu(A\Delta E) < \varepsilon^2$ and each D_i is a subset of V^{k-1} with $|D_i| \leq C_{k,d} \left(\frac{1}{\varepsilon}\right)^{2(k-1)d}$. Let $D = \bigcup_{i \in [k]} D_i$. We take \mathcal{P} to be the set of all atoms in the Boolean algebra of all E -definable over D subsets of V . By Fact 2.5,

$$|\mathcal{P}| \leq C_d |D|^d \leq C_d \left(kC_{k,d} \left(\frac{1}{\varepsilon}\right)^{2(k-1)d}\right)^d = C_d(kC_{k,d})^d \left(\frac{1}{\varepsilon}\right)^{2(k-1)d^2}$$

□

Now we give some examples where Theorems 3.3 and 3.5 apply.

3.2. The finite case. Let $G = (V_1, \dots, V_k; E)$ be a finite k -hypergraph. For each $i \in [k]$ let μ_i be the counting measure on V_i , i.e. $\mu_i(X) = \frac{|X|}{|V_i|}$ and μ be the counting measure on $V_1 \times \dots \times V_k$. Then all μ_i and μ are fap measures with $\mu = \mu_1 \times \dots \times \mu_k$. Hence all the results of the previous section can be applied to finite k -hypergraphs with respect to counting measures.

Corollary 3.6. *Let $G = (V; E)$ be a k -uniform hypergraph. Assume E has VC-dimension at most d , as a relation on V^k .*

There is a partition $V = V_1 \dot{\cup} \dots \dot{\cup} V_M$ for some $M \leq C_d(C_{k,d})^d \left(\frac{1}{\varepsilon}\right)^{2(k-1)d^2}$, numbers $\delta_{\vec{i}} \in \{0, 1\}$ for $\vec{i} \in [M]^k$, and an exceptional set $\Sigma \subseteq [M]^k$ such that

$$\sum_{(i_1, \dots, i_k) \in \Sigma} |V_{i_1}| \cdots |V_{i_k}| \leq \varepsilon |V|^k$$

and for each $\vec{i} = (i_1, \dots, i_k) \in [M]^k \setminus \Sigma$ we have

$$| |E(A_1, \dots, A_k)| - \delta_{\vec{i}} |A_1| \cdots |A_k| | < \varepsilon |V_{i_1}| \cdots |V_{i_k}|$$

for all $A_1 \subseteq V_{i_1}, \dots, A_k \subseteq V_{i_k}$.

3.3. Hypergraphs definable in NIP structures. Now we discuss the model theoretic setting, which is the main motivating example for this article. For a detailed account of this setting, we refer to the introduction in [6] and to [37].

Let \mathcal{M} be a first-order structure. Recall that a *Keisler measure* on M^n is a finitely additive probability measure on the Boolean algebra of all definable subsets of M^n . Given a formula $\phi(x)$ with parameters from M and a Keisler measure μ on $M^{|x|}$, we will write $\mu(\phi(x))$ to denote $\mu(\phi(M^{|x|}))$. Let us fix a definable relation $E(x_1, \dots, x_k)$, let $V_i = M^{|x_i|}$ and let \mathcal{B}_i be the Boolean algebra of all definable subsets of $M^{|x_i|}$. Let μ_i be a Keisler measure on $M^{|x_i|}$, equivalently a measure on \mathcal{B}_i .

Recall that a structure \mathcal{M} is an *NIP structure* if for every formula $\phi(x, y)$ the family of all ϕ -definable sets $\mathcal{F}_\phi = \{\phi(M, a) : a \in M^{|y|}\}$ has finite VC-dimension. In particular, if \mathcal{M} is NIP, then any definable relation $E(x_1, \dots, x_k) \subseteq M^{|x_1|} \times \dots \times M^{|x_k|}$ has finite VC dimension (in the sense of Definition 2.4). Recall that, in an NIP structure \mathcal{M} , a Keisler measure μ on $M^{|x|}$ is *generically stable* if it is fap on all definable relations $\phi(x, y) \subseteq M^{|x|} \times M^{|y|}$, in particular on E .

Remark 3.7. There are several equivalent characterizations of generically stable measures in NIP structures. Our definition of fap only requires the *existence* of an ε -approximation for every ε . A stronger notion of a *fim measure* is given in [16] requiring that in fact for every ε , there is sufficiently large n such that *almost all* n -tuples (in the sense of the product measure $\mu^{(n)}$) give an ε -approximation. While fap is equivalent to fim under the NIP assumption (by the results in [16]), it is not so clear if the equivalence holds in general.

Now, the semidirect product $\mu = \mu_1 \times \cdots \times \mu_k$ corresponds to the non-forking product $\mu_1 \otimes \cdots \otimes \mu_k$. Hence Theorem 3.3 translates into the following.

Corollary 3.8. *Let \mathcal{M} be NIP. For every definable relation $E(x_1, \dots, x_n)$ there is some $c = c(E)$ such that: for any $\varepsilon > 0$ and any generically stable Keisler measures μ_i on $M^{|x_i|}$ there are partitions $M^{|x_i|} = \bigcup_{j < K} A_{i,j}$ and a set $\Sigma \subseteq \{1, \dots, K\}^n$ such that:*

- (1) $K \leq (\frac{1}{\varepsilon})^c$.
- (2) $\mu \left(\bigcup_{(i_1, \dots, i_n) \in \Sigma} A_{1,i_1} \times \cdots \times A_{n,i_n} \right) \leq 1 - \varepsilon$, where $\mu = \mu_1 \otimes \cdots \otimes \mu_n$,
- (3) for all $(i_1, \dots, i_n) \notin \Sigma$ and all definable $A'_1 \subseteq A_{1,i_1}, \dots, A'_n \subseteq A_{n,i_n}$, either $d_E(A'_1, \dots, A'_n) < \varepsilon$ or $d_E(A'_1, \dots, A'_n) > 1 - \varepsilon$, where $d_E(A'_1, \dots, A'_n) = \frac{\mu(E \cap A'_1 \times \cdots \times A'_n)}{\mu(A'_1 \times \cdots \times A'_n)}$ denotes the edge density.
- (4) each $A_{i,j}$ is defined by an instance of an E -formula depending only on E and ε .

Theorem 3.3 is more general however as both NIP and fap are only assumed locally for R , and can be applied outside of the context of NIP structures.

Example 3.9. Let \mathcal{M} be a pseudo-finite field, viewed as a structure in the ring language (e.g. an ultraproduct of finite fields modulo some non-principal ultrafilter). Then the ultralimit of the counting measures gives a measure on the definable sets in \mathcal{M} . This measure is fap on all quantifier-free definable relations (by Lemma 4.3, as it is well-known that all quantifier-free formulas in \mathcal{M} are stable), but not fap for general definable relations (e.g. because the random graph is definable). Still, Theorem 3.3 can be applied to any quantifier-free definable relation in this situation.

We list some specific structures and Keisler measures for which Corollary 3.8 applies to all definable relations (again, see introduction in [6] for more details).

Example 3.10. Examples of NIP structures:

- (1) Abelian groups and modules (see e.g. [40]),
- (2) $(\mathbb{C}, +, \times, 0, 1)$ (see e.g. [40]),
- (3) Differentially closed fields (see e.g. [40]),
- (4) free groups (in the pure group language $(\cdot, {}^{-1}, 0)$, see [31]),
- (5) Planar graphs (in the language with a single binary relation corresponding to the edges, see [29]).
- (6) (Weakly) o -minimal structures, e.g. $M = (\mathbb{R}, +, \times, e^x)$ (see [6]).
- (7) Presburger arithmetic, i.e. the ordered group of integers (see [6]).
- (8) p -minimal structures with Skolem functions. E.g. $(\mathbb{Q}_p, +, \times)$ for each prime p is distal (see [6]).
- (9) The (valued differential) field of transseries ([2, 4]).

- (10) Algebraically closed valued fields (see e.g. [35])

Example 3.11. Examples of generically stable Keisler measures (see e.g. the introduction in [6]):

- (1) Any Keisler measure concentrated on a finite set (as it is clearly fap).
- (2) Let λ_n be the Lebesgue measure on the unite cube $[0, 1]^n$ in \mathbb{R}^n . Let \mathcal{M} be an o-minimal structure expanding the field of real numbers. If $X \subseteq \mathbb{R}^n$ is definable in \mathcal{M} , then, by o-minimal cell decomposition, $X \cap [0, 1]^n$ is Lebesgue measurable, hence λ_n induces a Keisler measure on M^n .
- (3) Similarly to (2), for every prime p a (normalized) Haar measure on a compact ball in \mathbb{Q}_p induces a smooth Keisler measure on \mathbb{Q}_p^n .

4. STABLE AND DISTAL CASES

Next we consider two extreme opposite special cases of NIP hypergraphs: stable and distal ones. Stable theories are at the cornerstone of Shelah's classification theory [32], and we refer to e.g. [27, 40] for a general exposition of stability. Examples (1) – (5) in Example 3.10 are stable. Distal theories were introduced more recently in [34] aiming to capture “purely unstable” structures in NIP theories. Examples (6) – (9) in Example 3.10 are distal. Example (10) gives a combination of these two cases: it has a stable part (the algebraically closed residue field) and distal part (the value group), and the theory developed in [13] demonstrates that the whole structure can be analyzed in terms of these two parts. There are certain generalizations of this decomposition principle for arbitrary NIP theories [33, 36].

4.1. Stable case. Regularity lemma for stable graphs was proved in [23] for counting measures. Later, [24] provides a proof for general measures. However, the proof in [24] does not give any bounds on the size of the partition. In this section we combine these two approaches and prove a regularity lemma for stable hypergraphs relatively to arbitrary measures, bounding the size of the partition by a polynomial in $\frac{1}{\varepsilon}$.

We work in the same setting as in Section 2. Let the sets V_1, \dots, V_k and $R \subseteq V_1 \times \dots \times V_k$ be given, let \mathcal{B}_i be a field on V_i , and let μ_i be a measure on \mathcal{B}_i . Assume moreover that for every $i \in [k]$, $R_b \in \mathcal{B}_i$ for all $b \in V_i^c$.

- Definition 4.1.** (1) A binary relation $R(x, y) \subseteq V \times W$ is *d-stable* if there is no tree of parameters $(b_\eta : \eta \in 2^{<d})$ in W such that for any $\eta \in 2^d$ there is some $a_\eta \in V$ such that $a_\eta \in R_{b_\nu} \iff \nu \frown 1 \leq \eta$ (where \leq is the tree order).
- (2) A relation $R \subseteq V_1 \times \dots \times V_k$ is *d-stable* if for every $I \subseteq [k]$ the family R viewed as a binary relation on $V_I \times V_{I^c}$ is *d-stable*.
- (3) A relation R is *stable* if it is *d-stable* for some d .

Remark 4.2. Alternatively, stability of a relation can be defined in terms of the so called *order property*. Namely, $R \subseteq V \times W$ has the *d-order property* if there are some elements a_i in V and b_i in W , $i = 1, \dots, d$, such that $a_i \in R_{b_j} \iff i \leq j$ for all $1 \leq i, j \leq d$. It is a standard fact in basic stability theory that R is stable (in the sense of Definition 4.1) if and only if it does not have the *d-order property* for some d .

Lemma 4.3. *Let R be a stable relation. Then any measure μ_i on \mathcal{B}_i is fap on R .*

Proof. Let E be an arbitrary R -definable relation.

Claim 1. For any $\varepsilon > 0$ there is some $m = m(\varepsilon, E)$ and some 0-1 measures $\delta_1, \dots, \delta_m$ on \mathcal{B}_i (possibly with repetitions) such that $\mu_i(R_c) \approx^\varepsilon \frac{1}{m} \sum_{j=1}^m \delta_j(R_c)$ for all $c \in V_i^c$.

Proof. As R is stable, it follows that E has finite VC-dimension. Then the claim follows from the VC-theorem applied on the *compact* space of 0-1 measures on \mathcal{B}_i . See [15, Lemma 4.8] for the details.

Claim 2. Every 0 – 1 measure δ on \mathcal{B}_i is fap on E .

Proof. This is a straightforward consequence of the explicit form of the definability of types in local stability. See e.g. the proof of [27, Lemma 2.2]: identifying our measure δ restricted to E with a complete E -type, an ε -approximation of δ on E is given by the c_1, \dots, c_m constructed in that proof, for any m large enough so that $\frac{N}{m} < \varepsilon$.

Now, let $\varepsilon > 0$ be arbitrary, and let $\delta_1, \dots, \delta_m$ be as given by Claim 1. By Claim 2, let A_j be a multiset in V_i giving an ε -approximation for δ_j . It is straightforward to verify that $A = \bigcup_{j=1}^m A_j$ is a 2ε -approximation for μ_i . \square

In view of this lemma, for $I = \{i_1, \dots, i_n\} \subseteq [k]$ we have a semi-direct product measure $\mu_I = \mu_{i_1} \times \dots \times \mu_{i_n}$ on $\mathcal{B}_I = \mathcal{B}_{i_1} \times \dots \times \mathcal{B}_{i_n}$ (see Definition 2.16) which is fap on R (Proposition 2.14).

Definition 4.4. A set $A \in \mathcal{B}_I$ is ε -good if for any $b \in V_{I^c}$, either $\mu_I(A \cap R_b) < \varepsilon \mu_I(A)$ or $\mu_I(A \cap R_b) > (1 - \varepsilon) \mu_I(A)$.

Remark 4.5. Notice that if a set is ε -good then it has measure greater than 0.

Lemma 4.6. Assume that μ_{I^c} is fap on R . For any $\varepsilon > 0$, consider the set

$$A = \{a \in V_I : \mu_{I^c}(R_a) < \varepsilon\}.$$

Then there is an R -definable set $A' \supseteq A$ such that $\mu_{I^c}(R_a) < 2\varepsilon$ for all $a \in A'$.

Proof. Let $b_1, \dots, b_n \in V_{I^c}$ be such that $\mu_{I^c}(R_a) \approx^{\frac{\varepsilon}{2}} \text{Av}(b_1, \dots, b_n; R_a)$ for all $a \in V_I$. Let $\mathcal{J} = \{J \subseteq [n] : \frac{|J|}{n} < \frac{3}{2}\varepsilon\}$, and let $A' = \bigcup_{J \in \mathcal{J}} \left(\bigcap_{j \in J} R_{b_j} \right)$. It is easy to check that A' satisfies the requirements. \square

Lemma 4.7. Fix some $I \subseteq [k]$ and some $J \subseteq [k] \setminus I$. Let $B \in \mathcal{B}_J$ be an ε -good set, and let $A \in \mathcal{B}_I$ and $c \in V_{[k] \setminus (I \cup J)}$ be arbitrary, such that both A and B are of positive measure. Then (by Definition 4.4) A is a disjoint union of the sets

$$A_{B,c}^0 = \{a \in A : \mu_J(R_{a,c} \cap B) < \varepsilon \mu_J(B)\}$$

and

$$A_{B,c}^1 = \{a \in A : \mu_J(R_{a,c} \cap B) > (1 - \varepsilon) \mu_J(B)\}.$$

Assume that $\varepsilon < \frac{1}{4}$. Then $A_{B,c}^0, A_{B,c}^1 \in \mathcal{B}_I$.

Proof. Indeed, let μ'_I be the restriction of μ_I to A and let μ'_I be the restriction of μ_I to B . As R is stable, by Lemma 4.3 both μ'_I, μ'_{I^c} are fap on R . Hence, by Lemma 4.6 applied to μ'_I, μ'_{I^c} we can find some R -definable $A'_0 \supseteq A_{B,c}^0, A'_1 \supseteq A_{B,c}^1$ such that $\mu'_{I^c}(R_{a,c}) < 2\varepsilon$ for all $a \in A'_0$ and $\mu'_{I^c}(R_{a,c}) > (1 - 2\varepsilon)$ for all $a \in A'_1$. As $\varepsilon < \frac{1}{4}$, it follows that in fact $A_{B,c}^0 = A'_0 \cap A, A_{B,c}^1 = A'_1 \cap A$. \square

In particular, it makes sense to speak of the μ_I -measure of $A_{B,c}^0, A_{B,c}^1$.

Definition 4.8. Let $0 < \varepsilon < \frac{1}{4}$ be arbitrary, and let $I \subseteq [k]$. We say that a set $A \in \mathcal{B}_I$ is ε -excellent if it is ε -good and for every $J \subseteq [k] \setminus I$, every ε -good $B \in \mathcal{B}_J$ and every $c \in V_{[k] \setminus (I \cup J)}$, either $\mu_I(A_{B,c}^0) < \varepsilon \mu_I(A)$ or $\mu_I(A_{B,c}^1) < \varepsilon \mu_I(A)$ (in the notation from Lemma 4.7).

The following lemma is a generalization of [23, Claim 5.4], with an additional observation that the proof can be performed “definably”.

Lemma 4.9. *Let $R \subseteq V_1 \times \dots \times V_k$ be d -stable and let $0 < \varepsilon < \frac{1}{2^d}$ be arbitrary. Assume that $A \in \mathcal{B}_n$ and $\mu_n(A) > 0$. Then there is an ε -excellent R -definable set $A' \in \mathcal{B}_n$ with $\mu_n(A') \geq \varepsilon^d \mu_n(A)$.*

Proof. We will need the following claim.

Claim. Assume that $0 < \varepsilon < \frac{1}{4}$ and $A \in \mathcal{B}_n$ is not ε -excellent. Then there are disjoint $A^0, A^1 \subseteq A$ with $A_i \in \mathcal{B}_n$ and $\mu(A_i) \geq \varepsilon \mu(A)$ for $i \in \{0, 1\}$, and such that for any finite $S^0 \subseteq A^0, S^1 \subseteq A^1$ with $|S^0| + |S^1| \leq \frac{1}{\varepsilon}$ there is some $c \in V_{n^c}$ such that $a \in R_c$ for all $a \in S^1$ and $a \notin R_c$ for all $a \in S^0$.

Proof. If A is not ε -good, there is some $c \in V_{n^c}$ such that $\mu_n(A \cap R_c) \geq \varepsilon \mu_n(A)$ and $\mu_n(A \cap (R_c)^c) \geq \varepsilon \mu_n(A)$. We let $A^1 = A \cap R_c$ and $A^0 = A \cap (R_c)^c$.

If A is ε -good, as it is not ε -excellent, there are some $J \subseteq [k] \setminus \{n\}$, some set $B \in \mathcal{B}_J$ which is ε -good, and some $c' \in V_{[k] \setminus (n \cup J)}$ such that A is a disjoint union of the sets $A^0 := A_{B,c'}^0, A^1 := A_{B,c'}^1$ (in the notation from Lemma 4.7) and $\mu_n(A^t) \geq \varepsilon \mu_n(A)$ for both $t \in \{0, 1\}$. Now given S^0, S^1 as in the claim, we have $\mu_J(B \cap R_{a,c'}) \leq \varepsilon \mu_J(B)$ for all $a \in S^0$ and $\mu_J(B \cap (R_{a,c'})^c) \leq \varepsilon \mu_J(B)$ for all $a \in S^1$. Let

$$B' = B \cap \left(\bigcup_{b \in S^0} R_{a,c'} \cup \bigcup_{b \in S^1} (R_{a,c'})^c \right).$$

As $|S^0| + |S^1| < \frac{1}{\varepsilon}$, it follows that $\mu_J(B') \leq \frac{1}{\varepsilon} \varepsilon \mu_J(B) < \mu_J(B)$. In particular there is some $b' \in B \setminus B'$, and taking $c = b' \frown c'$ satisfies the claim.

Assume now that the conclusion of the lemma fails. By induction we choose sets $(A_\eta : \eta \in 2^{\leq d})$ in \mathcal{B}_n such that $A_\emptyset = A$ and given $\eta \in 2^{< d}$, we take $A_{\eta \frown 0} := (A_\eta)^0, A_{\eta \frown 1} := (A_\eta)^1$ as given by the claim applied to A_η . For every $\eta \in 2^d$, pick some $a_\eta \in A_\eta$ (possible as $\mu_n(A_\eta) \geq \varepsilon^d \mu_n(A) > 0$). For every $\nu \in 2^{< d}$ there is some $c_\nu \in V_{n^c}$ such that $a_\eta \in R_{c_\nu}$ if and only if $\nu \frown 1 \sqsubseteq \eta$ – which gives contradiction to the d -stability of R . Namely we can take c given by the claim for $S^0 = \{a_\eta : \eta \in 2^d, \nu \frown 0 \sqsubseteq \eta\}$ and $S^1 = \{a_\eta : \eta \in 2^d, \nu \frown 1 \sqsubseteq \eta\}$ (note that $|S^0| + |S^1| \leq 2^d < \frac{1}{\varepsilon}$ by assumption). \square

Lemma 4.10. *Let $R \subseteq V_1 \times \dots \times V_k$ be d -stable, and let $0 < \varepsilon < \frac{1}{2^d}$ be arbitrary. For any $n \in [k]$, there is a partition of V_n into ε -excellent sets from \mathcal{B}_n , and the size of the partition can be bounded by a polynomial of degree $d+1$ in $\frac{1}{\varepsilon}$.*

Proof. Repeatedly applying Lemma 4.9, we let A_{m+1} be an $\frac{\varepsilon}{2}$ -excellent subset of $B_m := V_n \setminus (\bigcup_{1 \leq i \leq m} A_i)$ with $\mu_n(A_{m+1}) \geq (\frac{\varepsilon}{2})^d \mu_n(B_m)$. Then $\mu_n(B_{m+1}) \leq \mu_n(B_m) - (\frac{\varepsilon}{2})^d \mu_n(B_m) \leq (1 - (\frac{\varepsilon}{2})^d) \mu_n(B_m)$, hence $\mu_n(B_m) \leq (1 - (\frac{\varepsilon}{2})^d)^{m-1}$ for all m . Thus $\mu_n(B_m) \leq \frac{\varepsilon}{2} \mu_n(A_1)$ after $m = \frac{\log(\frac{\varepsilon}{2})^{d+1}}{\log(1 - (\frac{\varepsilon}{2})^d)}$ steps. Letting $A'_1 = A_1 \cup B_m$, it is easy to check that A'_1 is an ε -excellent set, and A'_1, A_2, \dots, A_m is a partition of V_n .

Finally, for the size of the partition we have an estimate

$$m = -\frac{(d+1)\log 2}{\log(1 - (\frac{\varepsilon}{2})^d)} \log(\frac{1}{\varepsilon}) \leq -\frac{c}{\ln(1 - (\frac{\varepsilon}{2})^d)} \ln(\frac{1}{\varepsilon})$$

for some constant $c \in \mathbb{N}$ depending just on d . And as $-\ln(1-x) \geq x$ for all x , this gives

$$m \leq c \left(\frac{\varepsilon}{2}\right)^d \ln\left(\frac{1}{\varepsilon}\right) \leq c' \left(\frac{1}{\varepsilon}\right)^{d+1}$$

for some $c' = c'(d) \in \mathbb{N}$. □

Finally we can use the partition in Lemma 4.10 to obtain a regular partition for $R \subseteq V_1 \times \dots \times V_k$.

Lemma 4.11. *If $A \subseteq V_n$ is ε -excellent and $B \subseteq V_{[n-1]}$ is ε -good then $B \times A$ is 2ε -good.*

Proof. Let $c \in V_{[n]^c}$ be arbitrary. As B is ε -good and A is ε -excellent, by Definition 4.8 we have $A = A_{B,c}^0 \cup A_{B,c}^1$ and either $\mu_n(A_{B,c}^0) < \varepsilon\mu_n(A)$ or $\mu_n(A_{B,c}^1) < \varepsilon\mu_n(A)$. Assume we are in the first case. Then, using the definition of $\mu_{[n]}$ and Lemma 2.15, we have

$$\begin{aligned} \mu_{[n]}((B \times A) \cap R_c) &= \int_A (\mu_{[n-1]}(R_{a,c} \cap B)) d\mu_n \geq \\ &\int_{A_{B,c}^1} (\mu_{[n-1]}(R_{a,c} \cap B)) d\mu_n \geq \int_{A_{B,c}^1} (1-\varepsilon)\mu_{[n-1]}(B) d\mu_n \geq \\ &(1-\varepsilon)^2\mu_n(A)\mu_{[n-1]}(B) > (1-2\varepsilon)\mu_{[n]}(A \times B). \end{aligned}$$

Similarly, in the second case we obtain that $\mu_{[n]}((B \times A) \cap R_c) \leq 2\varepsilon\mu_{[n]}(A \times B)$. □

Theorem 4.12. *Let $R \subseteq V_1 \times \dots \times V_k$ be d -stable, and let $0 < \varepsilon < \frac{1}{2^d}$ be arbitrary. Then there is an R -definable ε -regular partition $\vec{\mathcal{P}}$ of $V_1 \times \dots \times V_k$ with 0-1-densities (see Definition 3.1) without any bad k -tuples in the partition (i.e. $\Sigma = \emptyset$) and such that the size of the partition $\|\vec{\mathcal{P}}\|$ is bounded by a polynomial of degree $d+1$ in $\frac{1}{\varepsilon}$.*

Proof. For each $n \leq k$, let \mathcal{P}_n be a partition of V_n into $\frac{\varepsilon}{2^{k+1}}$ -excellent sets as given by Lemma 4.10, and let $\mathcal{P} := \{X_1 \times \dots \times X_k : X_n \in \mathcal{P}_n\}$. We claim that \mathcal{P} is ε -regular with $\Sigma = \emptyset$. Indeed, let $X = X_1 \times \dots \times X_k \in \mathcal{P}$ be arbitrary, and let $Y = Y_1 \times \dots \times Y_k$, where $Y_n \subseteq X_n, Y_n \in \mathcal{B}_n$ are arbitrary. Let $X' := X_1 \times \dots \times X_{k-1}, Y' := Y_1 \times \dots \times Y_{k-1}$. Applying Lemma 4.11 k times, the set X' is $\frac{\varepsilon}{2}$ -good, and X_k is $\frac{\varepsilon}{2}$ -excellent. Then, by Definition 4.8, X_k is a disjoint union of the sets $(X_k)_{X'}^0, (X_k)_{X'}^1 \in \mathcal{B}_k$ and $\mu_k((X_k)_{X'}^t) < \frac{\varepsilon}{2}\mu_k(X_k)$ for one of $t \in \{0, 1\}$. Let $Y_k^0 := (X_k)_{X'}^0 \cap Y_k$ and $Y_k^1 := (X_k)_{X'}^1 \cap Y_k$. We have

$$\mu_{[k]}(R \cap Y) = \int_{Y_k} \mu_{[k-1]}(R_c \cap Y') d\mu_k(c).$$

As Y_k is a disjoint union of Y_k^0, Y_k^1 and $\mu(Y_k^t) \leq \frac{\varepsilon}{2}\mu_k(X_k)$ for some $t \in \{0, 1\}$, we have

$$\begin{aligned} |\mu_{[k]}(R \cap Y) - \int_{Y_k^t} \mu_{[k-1]}(R_c \cap Y') d\mu_k(c)| &\leq \\ \frac{\varepsilon}{2}\mu_k(X_k)\mu_{[k-1]}(Y_1 \times \dots \times Y_{k-1}) &\leq \frac{\varepsilon}{2}\mu_{[k]}(X_1 \times \dots \times X_k) \end{aligned}$$

for some $t \in \{0, 1\}$.

Assume that $t = 0$. Then for all $c \in Y_k^0$ we have $\mu_{[k-1]}(R_c \cap X') < \frac{\varepsilon}{2} \mu_{[k-1]}(X')$. Hence

$$\int_{Y_k^0} \mu_{[k-1]}(R_c \cap Y') d\mu_k(c) \leq \mu(Y_k^0) \frac{\varepsilon}{2} \mu_{[k-1]}(X') \leq \frac{\varepsilon}{2} \mu_{[k]}(X_1 \times \dots \times X_k),$$

and so $\mu_{[k]}(R \cap Y) \leq \varepsilon \mu_{[k]}(X)$.

If $t = 1$, applying the same argument to R^c we obtain $\mu_{[k]}(R^c \cap Y) \leq \varepsilon \mu_{[k]}(X)$, hence $|\mu_{[k]}(R^c \cap Y) - \mu_{[k]}(Y)| \leq \varepsilon \mu_{[k]}(X)$. \square

Similarly to Corollary 3.8, Theorem 4.12 gives the following in the definable case. Recall that a structure \mathcal{M} is stable if every relation definable in it is stable.

Corollary 4.13. *Let M be a stable structure. For every definable $E(x_1, \dots, x_n)$ there is some $c = c(E)$ such that: for any $\varepsilon > 0$ and any Keisler measures μ_i on $M^{|x_i|}$ there are partitions $M^{|x_i|} = \bigcup_{j < K} A_{i,j}$ satisfying*

- (1) $K \leq \left(\frac{1}{\varepsilon}\right)^c$.
- (2) for all $(i_1, \dots, i_n) \in \{1, \dots, K\}^n$ and definable $A'_1 \subseteq A_{1,i_1}, \dots, A'_n \subseteq A_{n,i_n}$ either $d_E(A'_1, \dots, A'_n) < \varepsilon$ or $d_E(A'_1, \dots, A'_n) > 1 - \varepsilon$.
- (3) Each $A_{i,j}$ is defined by an instance of an E -formula depending only on E and ε ,

$$\text{where } d_E(A'_1, \dots, A'_n) = \frac{\mu(E \cap A'_1 \times \dots \times A'_n)}{\mu(A'_1 \times \dots \times A'_n)} \text{ and } \mu = \mu_1 \otimes \dots \otimes \mu_n.$$

4.2. Distal case. The class of distal theories is defined and studied in [34], with the aim to isolate the class of purely unstable NIP theories (as opposed to the class of stable theories, see also [35]). For completeness of the exposition, we recall the distal regularity lemma established in [6], pointing out a stronger form of definability for the regular partition than the one stated there. First we recall the definition of distality (and refer to the introduction in [6] for more details).

Definition 4.14. [6] An NIP structure \mathcal{M} is *distal* if and only if for every definable family $\{\phi(x, b) : b \in M^d\}$ of subsets of $M^{|x|}$ there is a definable family $\{\psi(x, c) : c \in M^{kd}\}$ such that for every $a \in M$ and every finite set $B \subset M^d$ there is some $c \in B^k$ such that $a \in \psi(x, c)$ and for every $a' \in \psi(x, c)$ we have $a' \in \phi(x, b) \Leftrightarrow a \in \phi(x, b)$, for all $b \in B$.

Theorem 4.15. *Let \mathcal{M} be distal. For every definable $E(x_1, \dots, x_n)$, defined by an instance of some formula $\theta(x_1, \dots, x_n; z)$, there is some $c = c(\theta)$ such that: for any $\varepsilon > 0$ and any generically stable Keisler measures μ_i on $M^{|x_i|}$ there are partitions $M^{|x_i|} = \bigcup_{j < K} A_{i,j}$ and a set $\Sigma \subseteq \{1, \dots, K\}^n$ such that*

- (1) $K \leq \left(\frac{1}{\varepsilon}\right)^c$.
- (2) $\mu\left(\bigcup_{(i_1, \dots, i_n) \in \Sigma} A_{1,i_1} \times \dots \times A_{n,i_n}\right) \leq \varepsilon$, where $\mu = \mu_1 \otimes \dots \otimes \mu_n$.
- (3) for all $(i_1, \dots, i_n) \notin \Sigma$, either $(A_{1,i_1} \times \dots \times A_{n,i_n}) \cap E = \emptyset$ or $A_{1,i_1} \times \dots \times A_{n,i_n} \subseteq E$.
- (4) Each $A_{i,j}$ is defined by an instance of a formula $\psi_i(x_i, z_i)$ which only depends on θ (and **not on** $\varepsilon!$).

Proof. This is proved in [6, Section 5.2], except for the fact that in (4) the formulas $\psi_i(x_i, z_i)$ can be chosen independently of ε — and we explain how to modify the proof there to obtain it. Namely, the proof of [6, Proposition 5.3] shows that, under the assumptions of the lemma, for each $i = 1, \dots, n$ we can find a finite set

of formulas Δ_i and a constant $c \in \mathbb{N}$ depending *only on* θ (in view of [6, Corollary 4.6]), a finite set of parameters A_N depending on θ and ε with $|A_N| \leq (\frac{1}{\varepsilon})^c$, and partitions $\mathcal{P}_i = \{A_{i,j} : j < K\}$ of $M^{|x_i|}$ satisfying the conclusion of the lemma, except for the bold font part, such that each $A_{i,j}$ is Δ_i -definable over A_N .

Let \mathcal{Q}_i be a partition of $M^{|x_i|}$ into the sets of realizations of complete Δ_i -types over A_N . By distality of \mathcal{M} , let Δ'_i be a finite set of formulas such that for every $\phi \in \Delta_i$ it contains a formula ψ as in Definition 4.14. Let $\psi_i(x_i, z_i)$ be a conjunction of all formulas in Δ'_i . Then for every $a \in M^{|x_i|}$ there is a single instance $\psi_i(x_i, e)$ such that its parameters e are all from A_N and such that $\psi_i(x_i, e)$ isolates the complete Δ_i -type of a over A_N . Using this, we can choose a partition \mathcal{Q}'_i of $M^{|x_i|}$ which refines \mathcal{Q}_i (and so also refines \mathcal{P}_i) and such that every set in \mathcal{Q}'_i is defined by an instance of $\phi_i(x_i, z_i)$ over A_N . Then the size of \mathcal{Q}'_i is bounded by $|A_N|^{|z_i|} \leq (\frac{1}{\varepsilon})^{c'}$ where $c' = c|z_i|$ only depends on θ . Hence $\mathcal{Q}'_i, i = 1, \dots, n$ give the desired partition. \square

5. DEFINABLE VARIANTS OF THE ERDŐS-HAJNAL AND RÖDL THEOREMS

In this section, we are concerned with the question of finding a “large” “approximately homogeneous” definable subset of a definable hypergraph. “Large” here refers to positive measure, relatively to a fap measure, and “approximately homogeneous” means that the edge density on the set is close to 0 or 1 (see below for precise definitions). We consider two very different situations — (k -partite) k -hypergraphs and k -uniform hypergraphs (in the sense of Section 3.1).

5.1. Partitioned hypergraphs. First we consider the “partite” situation. We are working in the same setting as in Section 3.1.

Theorem 5.1. *Let $E \subseteq V_1 \times \dots \times V_k$ be a k -hypergraph of VC-dimension at most d . Then for every $\alpha, \varepsilon > 0$ there is some $\delta = \delta(d, \alpha, \varepsilon) > 0$ such that the following holds.*

Let \mathcal{B}_i be a field on V_i , and let μ_i be a measure on \mathcal{B}_i which is fap on E , for $i = 1, \dots, k$. Let $\mu = \mu_1 \times \dots \times \mu_k$. Assume that $\mu(E) \geq \alpha$. Then there are some E -definable sets $A_i \subseteq V_i$ such that $\mu(A_i) > \delta$ for all $i = 1, \dots, k$ and $d_E(A_1, \dots, A_k) > 1 - \varepsilon$.

As usual, $d_E(A_1, \dots, A_k) = \frac{\mu(E \cap (A_1 \times \dots \times A_k))}{\mu_1(A_1) \dots \mu_k(A_k)}$ denotes the E -density.

Proof. This follows from the regularity lemma for NIP hypergraphs (Theorem 3.3).

Let $\varepsilon' = \frac{\min\{\alpha, \varepsilon\}}{4} > 0$. Applying Theorem 3.3 with respect to ε' , we get that there are some constants c_1, c_2 depending just on E and E -definable partitions $V_i = \bigcup_{j=1, \dots, n} A_{i,j}$ for each $i = 1, \dots, k$ with $n \leq c_1(\frac{1}{\varepsilon'})^{c_2}$ such that if $\Sigma \subseteq [n]^k$ is the set of all bad k -tuples then we have $\sum_{(j_1, \dots, j_k) \in \Sigma} \mu_1(A_{1,j_1}) \dots \mu_k(A_{k,j_k}) < \varepsilon'$. Here a k -tuple of sets $(A_{1,j_1}, \dots, A_{k,j_k})$ is bad if it is not good, at it is good if for any $A'_{i,j_i} \subseteq A_{i,j_i}$ with $\mu_i(A'_{i,j_i}) > \varepsilon' \mu_i(A_{i,j_i})$ for $i = 1, \dots, k$, we have that either $d_E(A'_{1,j_1}, \dots, A'_{k,j_k}) < \varepsilon'$ or $d_E(A'_{1,j_1}, \dots, A'_{k,j_k}) > 1 - \varepsilon'$.

Let $\delta := \frac{\varepsilon'}{c_1(\frac{1}{\varepsilon'})^{kc_2}} > 0$, it only depends on α, ε, E . To prove the theorem, it is enough to find a good k -tuple $(A_{1,j_1}, \dots, A_{k,j_k})$ such that $\mu_i(A_{i,j_i}) > \delta$ for $i = 1, \dots, k$ and $d_E(A_{1,j_1}, \dots, A_{k,j_k}) > \varepsilon'$ (as then necessarily $d_E(A_{1,j_1}, \dots, A_{k,j_k}) > 1 - \varepsilon' > 1 - \varepsilon$).

Assume that this fails. Then we have:

$$\begin{aligned}
\mu(E) &= \sum_{(j_1, \dots, j_k) \in [n]^k} \mu(E \cap (A_{1, j_1} \times \dots \times A_{k, j_k})) \leq \\
&\quad \sum_{(j_1, \dots, j_k) \in \Sigma} \mu(A_{1, j_1} \times \dots \times A_{k, j_k}) + \\
&\quad \sum_{(j_1, \dots, j_k) \notin \Sigma, \mu(A_{1, j_1} \times \dots \times A_{k, j_k}) > \delta} \mu(A_{1, j_1} \times \dots \times A_{k, j_k}) \varepsilon' + \\
&\quad \sum_{(j_1, \dots, j_k) \notin \Sigma, \mu(A_{1, j_1} \times \dots \times A_{k, j_k}) \leq \delta} \delta d_E(A_{1, j_1}, \dots, A_{k, j_k}) \leq \\
&\quad \varepsilon' + \varepsilon' + n^k \delta \leq 2\varepsilon' + \delta c_1 \left(\frac{1}{\varepsilon'}\right)^{kc_2},
\end{aligned}$$

which by the choice of δ is at most $3\varepsilon'$. But this contradicts the assumption that $\mu(E) \geq \alpha \geq 4\varepsilon'$. \square

Remark 5.2. In the special case when μ is an ultraproduct of counting measures concentrated on finite sets, this gives a density version of the well-known lemma of Erdős and Hajnal, see e.g. [11, Lemma 2.1]

In particular, the result holds when E is a definable relation in an NIP structure (see Section 3.3), giving uniform definability of the sets A_i in terms of E, α, ε .

In the case when E is definable in a distal structure we have the following strengthening proved in [6, Corollary 4.6].

Fact 5.3. *Let \mathcal{M} be a distal structure and $\theta(x_1, \dots, x_k, y)$ a formula. Given $\alpha > 0$ there is $\delta > 0$ such that: for any relation $E(x_1, \dots, x_k)$ defined by an instance of θ and any generically stable measures μ_i on $M^{|x_i|}$, if $\mu(R) \geq \alpha$ (where $\mu = \mu_1 \otimes \dots \otimes \mu_k$), then there are definable sets $A_i \subseteq M^{|x_i|}$ with $\mu_i(A_i) \geq \delta$ for all $i = 1, \dots, k$ and $\prod_{i=1}^k A_i \subseteq R$. Moreover, each A_i can be defined by an instance of a formula $\psi_i(x_i, z_i)$ that depends only on θ and α .*

5.2. Non-partitioned case. In the non-partite case, however, it is much harder to find a large homogeneous subset (i.e. a clique or an anti-clique), as it is well-known in combinatorics, and we give some examples in the definable setting illustrating it.

The following is a classical result of Rödl.

Fact 5.4. ([30], see also [11, Theorem 1.1]) *For each $\varepsilon \in (0, \frac{1}{2})$ and finite graph H there is some $\delta = \delta(H, \varepsilon) > 0$ such that every H -free graph on n vertices contains an induced subgraph on at least δn vertices with edge density either at most ε or at least $1 - \varepsilon$.*

We consider a generalization of this property to fap measures.

Definition 5.5. Let \mathcal{M} be a structure and let \mathfrak{M} be a class of Keisler measures. Let \mathcal{E} be a collection of definable (symmetric) (hyper-)graphs in (some powers of) \mathcal{M} .

- (1) We will say that \mathcal{E} satisfies the *Rödl property* with respect to \mathfrak{M} if for every $E \subseteq (M^n)^k$ in \mathcal{E} and every $\varepsilon > 0$ there is some $\delta > 0$ such that for every $\mu \in \mathfrak{M}$, a Keisler measure on M^n which is fap on E , there is some definable $A \subseteq M^n$ such that $\mu(A) \geq \delta$ and the $\mu^{(k)}$ -density of E on A is either $< \varepsilon$ or $> 1 - \varepsilon$.

- (2) If in addition such an A can be defined by an instance of some formula that depends only on E , and not on ε , then we say that \mathcal{E} satisfies the *uniform Rödl property* with respect to \mathfrak{M} .
- (3) We will say that \mathcal{E} satisfies the *strong Rödl property* with respect to \mathfrak{M} if in (1) we can find a definable E -homogeneous subset of positive μ -measure.

Fact 5.4 implies that if \mathcal{E} is a family of pseudofinite hypergraphs of bounded VC-dimension, then it satisfies the Rödl property with respect to the class \mathcal{M} of pseudofinite counting measures, in the language of set theory. We give some examples showing that there is little hope in generalizing this to arbitrary generically stable measures.

Example 5.6. The strong Rödl property does not hold for graphs definable in the field of reals, with respect to the Lebesgue measure. To see this, consider the relation $E \subseteq \mathbb{R}^2 \times \mathbb{R}^2$ defined by $(a, b)E(a', b') \iff |a - a'| < |b - b'|$, and let μ be the generically stable measure on \mathbb{R}^2 given by restricting the Lebesgue measure on $[0, 1]^2$ to the definable sets. We claim that there is no definable E -homogeneous subset of \mathbb{R}^2 of positive measure. Indeed, any such set $A \subseteq [0, 1]^2$ would have to contain an E -homogeneous square, and it is easy to see that this is impossible by the definition of E (one can check, however, that the uniform Rödl property is satisfied as for any $\varepsilon > 0$ we can choose a sufficiently thin vertical stripe of positive measure such that the E -density on it is ε -close to 1).

It may be tempting to use the NIP regularity lemma as in the partitioned case (Theorem 5.1) to establish the Rödl property (applying it for a symmetric relation $R \subseteq V_1 \times V_2$ with $V = V_1 = V_2$, $\mu = \mu_1 = \mu_2$). However, it doesn't work. The reason is that, given an ε -regular partition A_1, \dots, A_n of V , it is perfectly possible that all of the pairs on the diagonal (A_i, A_i) , $1 \leq i \leq n$ are bad simultaneously. Namely, if Σ is the collection of all bad pairs, we have that $\sum_{(i,j) \in \Sigma} \mu(A_i)\mu(A_j) < \varepsilon$. On the other hand, if let's say $(A_i : 1 \leq i \leq n)$ is an equipartition, we have $\sum_{1 \leq i \leq n} \mu(A_i)^2 \leq n \frac{1}{n^2} \leq \frac{1}{n}$, which can be smaller than ε when n is sufficiently large. In fact, this observation suggests an idea of a counter-example to the uniform Rödl property, which we present in the next subsection.

5.2.1. A counterexample to the uniform Rödl property. We are working in the field of 2-adics \mathbb{Q}_2 , in the Macintyre language. Let μ be the Haar measure on \mathbb{Q}_2 normalized on the compact ball \mathbb{Z}_p (restricted to definable sets). Then \mathbb{Q}_2 is a distal structure, and μ is generically stable measure (w.g. see the introduction in [6]). We think of elements in \mathbb{Q}_2 as branches of a binary tree and define $E \subseteq M^2$ by saying that $E(x, y)$ holds if and only if $v(x - y)$ is odd (i.e. if the branches x and y split at an odd level). This is a symmetric relation definable in the Macintyre's language. We estimate the $\mu^{(2)}$ -density of E on certain definable sets.

Lemma 5.7. *Assume that A is a ball, then the density $d_E(A)$ is either $\frac{1}{3}$ or $\frac{2}{3}$ (depending on the radius of the ball).*

Proof. We have $d_E(A) = \frac{\mu^{(2)}(E \cap A^2)}{\mu(A)^2}$ and $\mu^{(2)}(E \cap A^2) = \int_A \mu(E_a \cap A) d\mu$.

We think of the elements of \mathbb{Q}_2 as infinite binary sequences, and let's say $A = \{\tau_0 \frown \tau : \tau \in 2^\omega\}$ for some $\tau_0 \in 2^{<\omega}$. For each $n \in \omega$, consider the partition $A = \bigcup_{\sigma \in 2^n} A_\sigma$, where $A_\sigma = \{\tau_0 \frown \sigma \frown \tau : \tau \in 2^\omega\}$.

In the following calculations, "on step n " we only look at the edges that go between different parts in the partition $\{A_\sigma : \sigma \in 2^n\}$.

By the definition of E (and since for $\sigma \in 2^n$, $\mu(A_\sigma) = \frac{1}{2^n}\mu(A)$), the $\mu^{(2)}$ -measure of E -edges on A added on each step n such that $|\tau_0| + n$ is even, is at least $r_n := 2^{n-1} \cdot 2 \cdot (\frac{1}{2^n}\mu(A))^2 = \frac{1}{2^n}\mu(A)^2$.

On the other hand, the number of edges omitted on each step n such that $|\tau_0| + n$ is odd, is at least $s_n := 2^{n-1} \cdot 2 \cdot (\frac{1}{2^n}\mu(A))^2 = \frac{1}{2^n}\mu(A)^2$.

Thus we have the following estimates.

(1) If $|\tau_0|$ is odd, then

$$\sum_{n \geq 1 \text{ odd}} r_n \leq d_E(A)\mu(A)^2 \leq \mu(A)^2 - \left(\sum_{n \geq 1 \text{ even}} s_n \right).$$

We have:

$$\begin{aligned} \sum_{n \geq 1 \text{ odd}} r_n &= \sum_{n \geq 1 \text{ odd}} \frac{1}{2^n}\mu(A)^2 = \mu(A)^2 \sum_{m \geq 0} \frac{1}{2^{2m+1}} = \\ &= \mu(A)^2 \sum_{m \geq 0} \frac{1}{2} \frac{1}{4^m} = \mu(A)^2 \frac{1/2}{1 - 1/4} = \frac{2}{3}\mu(A)^2, \\ \sum_{n \geq 1 \text{ even}} s_n &= \sum_{n \geq 1 \text{ even}} \frac{1}{2^n}\mu(A)^2 = \mu(A)^2 \sum_{m \geq 1} \frac{1}{2^{2m}} = \\ &= \mu(A)^2 \left(\sum_{m \geq 0} \frac{1}{4^m} - 1 \right) = \left(\frac{1}{1 - 1/4} - 1 \right) \mu(A)^2 = \frac{1}{3}\mu(A)^2. \end{aligned}$$

(2) If $|\tau_0|$ is even, then

$$\sum_{n \geq 1 \text{ even}} r_n \leq d_E(A)\mu(A)^2 \leq \mu(A)^2 - \left(\sum_{n \geq 1 \text{ odd}} s_n \right),$$

and a similar computation shows that $d_E(A) = \frac{1}{3}$. □

Lemma 5.8. *Fix a formula $\phi(x, y)$. Then there is some $\gamma \in (0, 1)$ such that: for any parameter b , if $\mu(\phi(x, b)) > 0$, then $\phi(x, b)$ contains some ball B with $\mu(B) \geq \gamma\mu(\phi(x, b))$.*

Proof. If $\mu(\phi(x, b)) > 0$, then $\phi(x, b)$ has to be infinite. As demonstrated in the original paper of Macintyre [22, Theorem 2], every infinite definable subset of M in the p -adics has non-empty interior. In particular it must contain some open ball of positive Haar measure.

However, to prove the claim we need a slightly more careful analysis. We recall a couple of facts about the p -adic cell decomposition (see e.g. [3, Section 7]). Let $\phi(x, y)$ be fixed. Then there is some $N \in \mathbb{N}$, definable functions f_i, g_i and elements $\lambda_i \in M$ for $i \leq N$ such that for every $b \in M_y$, the set $\phi(M, b)$ is a union of at most N cells of the form

$$U_i(b) = \{x \in M : v(f_i(b)) \leq v(x - c_i(b)) < v(g_i(b)) \wedge P_{n_i}(\lambda_i(x - c_i(b)))\}.$$

Besides, we have the following fact.

Fact 5.9. *(see e.g. [3, Lemma 7.4]) Suppose $n > 1$, and let $x, y, a \in K$ be such that $v(y - x) > 2v(n) + v(y - a)$. Then $x - a$ and $y - a$ are in the same coset of P_n .*

Assume now that $\mu(\phi(x, b)) = \delta > 0$. As μ concentrates on the valuation ring V , we have $\mu(\phi(M, b) \cap V) > \delta > 0$. Then there is at least one cell $U(b) \subseteq \phi(M, b)$ with $\mu(U(b)) > \frac{\delta}{N}$.

We claim that there is some element $a \in U(b)$ with $v(a - c(b)) \leq v(f(b)) + n$. First, as $\Gamma = \mathbb{Z}$, there must be some $\beta \in \Gamma$ such that $v(f(b)) \leq \beta \leq v(f(b)) + n$ and $v(\lambda) + \beta = n\alpha$ for some $\alpha \in \Gamma$. Let $e \in M$ be arbitrary with $v(e) = \alpha$, and let $a = \frac{e^n}{\lambda} + c(b)$. Then:

(1) $\lambda(a - c(b)) = e^n$ (in particular $P_n(\lambda(a - c(b)))$ holds),

(2) $v(a - c(b)) = v(\frac{e^n}{\lambda}) = v(e^n) - v(\lambda) = nv(e) - v(\lambda) = n\alpha - v(\lambda) = \beta$.

Now either $\beta < v(g(b))$, in which case $a \in U(b)$, or $\beta \geq v(g(b))$, in which case any element in $U(b)$ satisfies the claim.

Now we consider the ball $B = B_{\geq m}(a)$ for $m = 2v(n) + v(f(b)) + n$. We claim that $B \subseteq U(b)$. Indeed, for any $x \in B$ we have $v(a - x) > 2v(n) + v(a - c(b))$, hence by Fact 5.9, $x - c(b)$ and $a - c(b)$ are in the same coset of P_n , and of course $v(x - c(n)) = v(a - c(n))$, so $x \in U(b)$.

Finally, we have $\mu(B) \geq \frac{1}{2^{2v(n)+n}} \mu(B_{\geq v(f(b))}(c(b)))$ and, as $U(b) \subseteq B_{\geq v(f(b))}(c(b))$, $\mu(U(b)) \leq \mu(B_{\geq v(f(b))}(c(b)))$. Hence

$$\mu(B) \geq \frac{1}{2^{2v(n)+n}} \mu(U(b)) \geq \left(\frac{1}{2^{2v(n)+n}} \cdot \frac{1}{N} \right) \mu(\phi(M, b)).$$

Note that the coefficient only depends of $\phi(x, y)$, and not on the choice of the parameter b . □

We show that the uniform Rödl property fails for E . Assume towards contradiction that we can find some $\phi(x, y)$ such that for every $\varepsilon > 0$ there is some set $A \subseteq M$ definable by an instance of $\phi(x, y)$ and satisfying $\mu(A) > 0$ and $d_E(A) \in [0, \varepsilon) \cup (1 - \varepsilon, 1]$. Let's say $d_E(A) > 1 - \varepsilon$ (if $d_E(A) < \varepsilon$, we work with the complement of E instead). Let $\gamma > 0$ be as given by Lemma 5.8 for $\phi(x, y)$, and let's take $\varepsilon \ll \gamma$.

Now A contains some ball B with $\mu(B) = \delta \mu(A)$ for some $0 < \gamma < \delta \leq 1$, and we estimate the number of edges on A using Lemma 5.7.

$$\mu^{(2)}(E \cap A^2) = \mu^{(2)}(E \cap B^2) + \mu^{(2)}(E \cap (A \setminus B)^2) + 2\mu^{(2)}(E(A \setminus B, B)) \leq$$

$$\frac{2}{3}\mu(B)^2 + \mu(A \setminus B)^2 + 2\mu(A \setminus B)\mu(B) =$$

$$\frac{2}{3}\delta^2\mu(A)^2 + (1 - \delta)^2\mu(A)^2 + 2(1 - \delta)\delta\mu(A)^2 =$$

$$\left(\frac{2}{3}\delta^2 + 1 - 2\delta + \delta^2 + 2\delta - 2\delta^2 \right) \mu(A)^2 =$$

$$\left(1 - \frac{1}{3}\delta^2 \right) \mu(A)^2 \leq \left(1 - \frac{1}{3}\gamma^2 \right) \mu(A)^2.$$

But as we have assumed $\varepsilon \ll \gamma \in (0, 1)$, this contradicts the assumption that $d_E(A) > 1 - \varepsilon$.

5.2.2. *Uniform Rödl property fails for semialgebraic hypergraphs.* It is well-known that Fact 5.4 fails for hypergraphs (see the example at the very end of [30]). We observe that the uniform Rödl property fails already in the case of 3-hypergraphs in the *semialgebraic* setting.

For this, let $E(x_1, x_2, x_3) \subseteq \mathbb{R}^3$ be the relation given by $(x_1 < x_2 < x_3) \wedge (x_1 + x_3 - 2x_2 \geq 0)$, it is definable in the field of reals (it is considered in [7, Section 3.1]). We claim that it doesn't satisfy the uniform Rödl property relatively to the class of measures concentrated on finite sets. If we assume that it holds, then by α -minimality for every $\varepsilon > 0$ there is some $\delta > 0$ such that for any finite set $A \subseteq \mathbb{R}$ there is some interval $B \subseteq \mathbb{R}$ such that for $C = A \cap B$ we have $d_E(C) > 1 - \varepsilon$ or $d_E(C) < \varepsilon$. We observe that in fact the E -density tends to be $\frac{1}{2}$. Let arbitrary $\varepsilon < \frac{1}{2}$ and $\delta > 0$ be fixed. Let us take $A = \{1, 2, 3, \dots, N\}$ for some $N \in \mathbb{N}$ large enough (such that δN is also large), and let $C \subseteq A, C = \{p_1, \dots, p_n\}$ be an arbitrary interval of integers in A , $p_1 < \dots < p_n$, $|C| \geq \delta N$.

Assume that $E(p_i, p_j, p_k)$ doesn't hold for some $p_1 < i < j < k < p_n$. Let us define $q_i := p_n - p_{n-i+1} + p_1$. Then we have $p_1 < q_i < q_j < q_k < p_n$, and q_i, q_j, q_k are all in C since C is an interval. Moreover it's easy to see that $E(q_i, q_j, q_k)$ holds. This establishes a bijection between edges and non-edges in C , showing that the density on C is arbitrary close to $1/2$ for N large enough.

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