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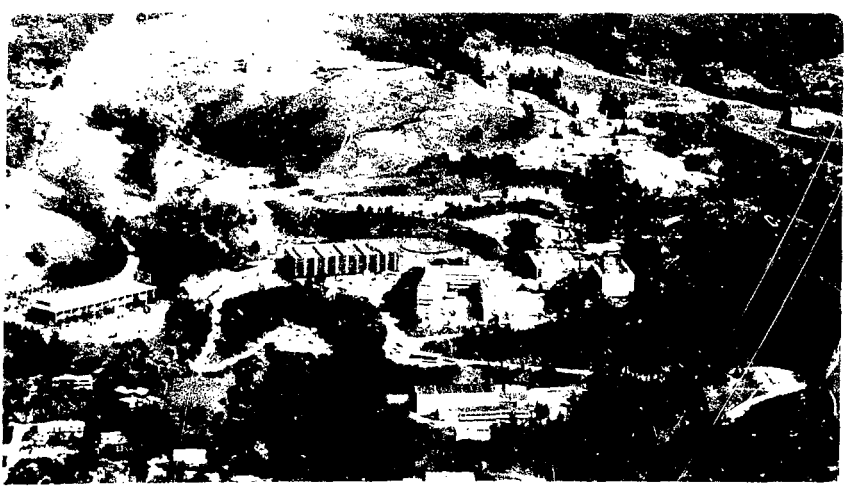
THE VARIATION OF THE MOYER MODEL PARAMETER, H_o, WITH PRIMARY PROTON ENERGY

Liu Kuei-Lin, Graham R. Stevenson,
Ralph H. Thomas, and Simon V. Thomas

August 1982

MASTER



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THE VARIATION OF THE MOYER MODEL PARAMETER, H_0, WITH PRIMARY PROTON ENERGY

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"Experience Joined With Common Sense,
to Mortals is a Providence"

From,
"The Spleen"
Matthew Green
1696-1737

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ABSTRACT

Experimental values of the Moyer Model Parameter H_o were summarized and presented as a function of proton energy, E_p. The variation of $H_o(E_p)$ with E_p was studied by regression analysis. Regression Analysis of the data under log-log transformation gave a best value for the exponent m of 0.77 ± 0.26, but a t-test did not reject m = 1 (p ± 20 percent). Since m = 1 was not excluded, and a Fisher's F-test did not exclude linearity, a linear regression analysis was performed. A line passing through the origin was not rejected (Student's t-test, p = 30 percent) and has the equation: $H_o(E_p) = (1.61 \pm 0.19) \times 10^{-13}$ Sv·m^2/GeV to be compared with a value of $(1.65 \pm 0.21) \times 10^{-13}$ Sv·m^2/GeV published by Stevenson et al. (St 82).

1. INTRODUCTION

High-energy proton accelerators must be surrounded by substantial shielding to protect, for example: operators; scientists; hospital staff and members of the general public (Pa 73). If the accelerator is above ground this shielding generally consists of concrete or steel structures and the cost of providing shielding can be a significant fraction of the total cost of the facility. Some larger accelerators have been constructed underground in order to reduce shielding costs but even then excavation costs are high and the depth below ground level must be optimized.

Detailed shield design using sophisticated Monte-Carlo radiation transport calculations can be expensive and is used only for the final design stages of a large proton accelerator. Semi-empirical calculations are often employed in the preliminary design phase.

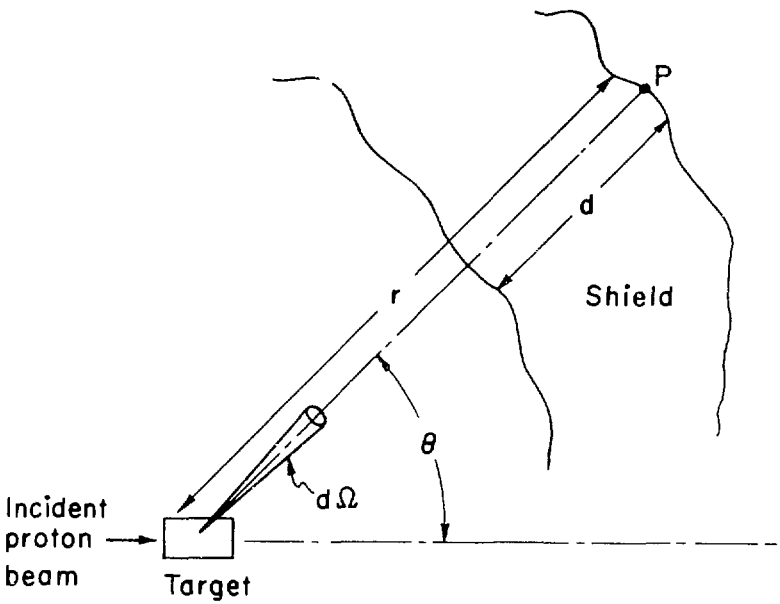
Moyer was one of the first to use such a technique in the design of the shielding for the Bevatron (Mo 62). The basic equation used by Moyer to calculate the radiation intensity, H, outside a shield around a target bombarded by protons is:

$$H = (1/r^2)\, H_0 \exp(-B\theta) \exp(-d/\lambda) \qquad (1)$$

where the symbols r, θ and d are explained in Figure 1. The angular distribution parameter, B, and the attenuation length, λ, are well determined both by theoretical and experimental means (Le 72, Pa 73, St 82).

The parameter H_0, which is the subject of this paper, may be determined empirically. (It does, in fact, have a complex relation to several parameters related to particle production in the target and their transport through the shield.)

Recently Stevenson et al., have summarized experimental determinations of H_0 at several different proton accelerators (St 82).



XBL 733-2478

Fig. 1

2. EXPERIMENTAL DATA

Table 1 summarizes the experimental determinations of the Moyer Parameter, H_o, discussed by Stevenson et al., (St 82) and, in addition, includes the value reported by Routti and Thomas (Ro 69), as corrected by Stevenson et al. These data are plotted in Figures 2 and 3. Table 1 also indicates the original sources of these data. The value of H_o reported are clearly independent, having largely been obtained at different accelerators by different experimenters and by different experimental techniques. The group of data obtained at the CERN PS in 1966 (Gi 68) were obtained by separate experiments.

Table 1. Summary of published values of moyer model parameters $H_o(E_p)$ (from St 82).

Primary Proton Energy, $[E_p]$ (GeV)	Moyer Parameter, $[H_o(E_p)]$ ($Sv\ m^2$)		Source
7.4	1.4	10-12	Sh69, St69
7.4	2.1	10-12	Sh69, St69
10.0	0.96	10-12	Ho 66
13.7	2.5	10-12	Gi 68
13.7	3.1	10-12	Gi 68
21.0	1.6	10-12	Ho 79
23.0	3.5	10-12	Ma 79
25.5	3.3	10-12	Gi 68
25.5	5.0	10-12	Gi 68
25.5	6.6	10-12	Ro69, St82
30.0	3.4	10-12	Aw 70

3. ENERGY VARIATION OF H_o

It is important to understand the variation of H_o with proton energy, both so that the experimental determinations of H_o at various proton energies may be combined to permit accurate interpolation and perhaps, more importantly, to allow extrapolation to higher energies. Such a need arose, for example in the design of shielding for the 50 Gev Beijing Proton Synchrotron (Ch 80, Li 79).

Since the principle use of the Moyer Model is in the calculation of tranverse shielding, we are interested in the global production of neutrons at large angles to the interaction target, as determined outside substantial shielding. At energies below 1 Gev there is evidence that the global production of neutrons is roughly proportional to neutron energy (for a summary see Pat 73). If an exponential variation of the form:

$$H_o = cE_p^m \qquad (2)$$

is assumed a value of $m = 1$ sets an upper limit to the variation of neutron production with proton energy and this is therefore a conservative assumption for extrapolating the experimental determinations of H_o to higher energies.

There has been some speculation in the literature as to the value of the coefficient m. Lindenbaum pointed out that the production of shower particles varied as $E^{0.25}$ and suggested a value of $m = 0.50$ for fast nucleons, intermediate between that for shower particles and low energy neutrons (Li 61, Pa 73). The data obtained from Monte-Carlo calculations of the Hadron Cascades generated in matter by high-energy protons suggest a value of $m = 0.75$ (Fe 72).

Until recently there were insufficient experimental data to empirically investigate the relationship between H_o and E_p but the experimental data of Table 1 now make this possible. The experimental data are shown plotted on log-log graph paper in Figure 2 and on linear paper on Figure 3. The regression analysis of these data is the subject of this note.

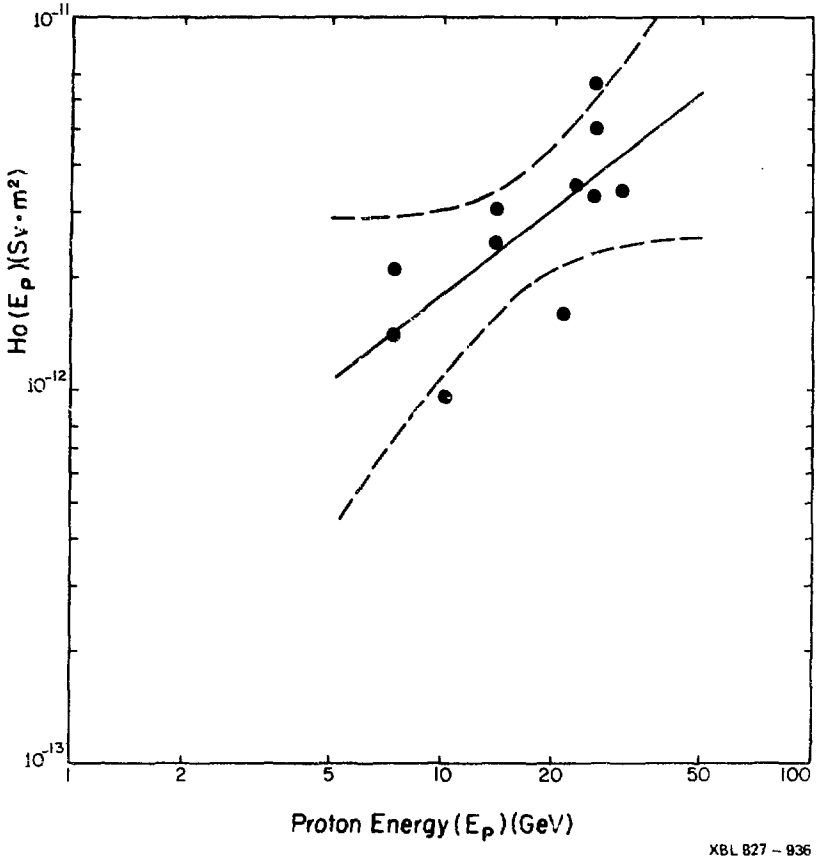

10^{-11}
10^{-12}
10^{-13}
$H_0(E_p)(Sv \cdot m^2)$
1
2
5
10
20
50
100
Proton Energy (E_p)(GeV)
XBL 827 – 936

Fig. 2

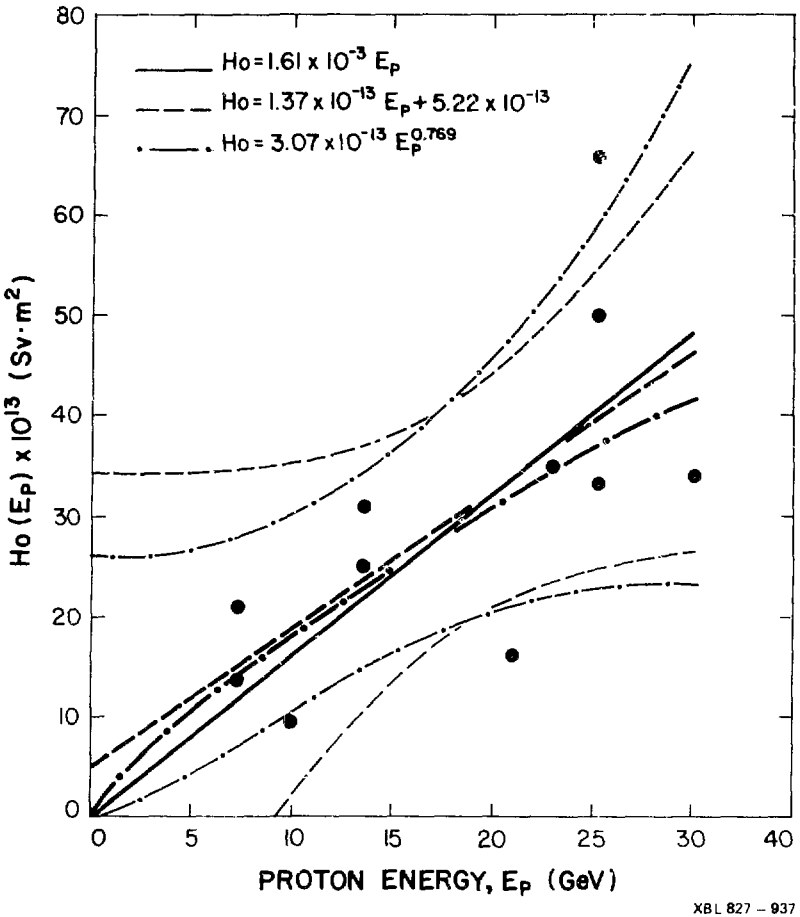

Ho = 1.61 x 10^-3 Ep
Ho = 1.37 x 10^-13 Ep + 5.22 x 10^-13
Ho = 3.07 x 10^-13 Ep^0.769
Ho (Ep) x 10^13 (Sv·m^2)
PROTON ENERGY, Ep (GeV)
0
10
20
30
40
50
60
70
80
0
5
10
15
20
25
30
40
XBL 827 – 937

Fig. 3

4. DATA ANALYSIS

The number of data points severely limit the analysis and our purpose here is to show that the assumption of linearity between the random variables $H_o(E_p)$ and E_p is not excluded by the experimental data. The progression of statistical tests used was:

(a) Fisher's F-test of the hypothesis of linearity of the data under log-log transformation.
(b) Regression analysis to determine the best value of the coefficient m.
(c) Student's t-Test of log-log transformed data for m = 1.
(d) Regression analysis under the assumption $H_o(E_p) = a + bE_p$.
(e) Student's t-Test for a = o.
(f) Linear regression analysis with fit forced through the origin: $H_o(E_p) = b' E_p$.
(g) Analysis of variance techniques to calculate 95 percent confidence bands to regression lines.

4.1 REGRESSION ANALYSIS OF LOG-LOG TRANSFORMED DATA

A Fisher's F-test (Fi 70, Sn 80) of the log-log transformed data does not reject linearity and a regression analysis of the data gives:

$$H_o(E_p) = 3.07\ E_p^{0.769}$$

with

$$S_{\hat{m}} = \pm 0.257$$

A Student's t-test does not reject m = 1.0 (p=20 percent).

For details see the analysis in Appendix 1.

4.2 LINEAR REGRESSION ANALYSIS

Since m = 1 is not excluded, and a Fisher's F-Test does not exclude linearity a linear regression analysis assuming $H_o(E_p)$ was performed giving:

$$H_o(E_p) = 5.22 \times 10^{-13} + (1.37 \times 10^{-13})Ep$$

The estimated variance of the intercept, $S_{\hat{a}}$, is:

$$S_{\hat{a}} = \pm 9.86 \times 10^{-13}$$

Details of the regression analysis are given in Appendix 2. A Student's t-test does not reject a = o (p = 30 percent).

4.3 LINEAR REGRESSION ANALYSIS FORCED THROUGH THE ORIGIN

Finally, a regression analysis with the line forced through the origin:

$$H_o(E_p) = b'E_p$$

gives:

$$\hat{b}' = 1.608 \times 10^{-13}$$

$$S_{\hat{b}'} = \pm 0.47 \times 10^{-13}$$

For details of the calculation see Appendix 3.

Figure 3 shows the experimental data, the three lines obtained by regression analysis and the 95 percent confidence band to the lines obtained by log-log and linear regression analysis.

5. CONCLUSIONS

The experimental data may be fitted by straight lines either in linear or log-log transformation. The best value of the coefficient, m, is 0.77 ± 0.26, but linearity is not excluded by a Student's t-test (p = 20 percent). A straight line forced through the origin has a slope (1.61 ± 0.19) x 10^{-13} Sv·m^2/GeV, to be compared with a value of (1.65 ± 0.21) x 10^{-13} Sv·m^2/GeV given by Stevenson et al. (St 82).*

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*In coherent SI units: the slope of the straight line forced through the origin is (1.00 ± 0.12) x 10^{-3} Sv·m^2/J, to be compared with a value of (1.03 ± 0.13) x 10^{-3} Sv·m^2/J given by Stevenson et al. (St 82).

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APPENDICES

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APPENDIX 2. Linear Regression analysis and hypothesis testing 21

APPENDIX 3. Regression analysis with line forced through the origin. . . 27

APPENDIX 1. REGRESSION ANALYSIS AND HYPOTHESIS TESTING USING A LOG-LOG TRANSFORMATION

In this appendix the data of Table A1-1 are first tested for the hypothesis of linearity using a Fisher's F-Test; the data are then subjected to regression analysis to determine the equation of the line fitted to the log-log transformed data; finally, the hypothesis that the estimated slope of the regression line is consistent with the value m = 1 is tested using a Student's t-test.

A1.1. LOG-LOG TRANSFORMATION

As discussed in the main body of the test there are good reasons for first investigating the regression of $\log_{10} H_o(E_p)$ on $\log_{10} E_p$ of the data shown in the scatter-diagram of Fig. 2. In this appendix, for simplicity, the variables $\log_{10} E_p$ and $\log_{10} 10^{13}\ (H_o(E_p))$ will be designated by x and y respectively.

A1.2. NUMERICAL DATA

Table A1-1. Numerical data for determination of regression line and for test of linearity log-log transformed data.

i	j	E_o (GeV)	$H_o(E_p)$ ($S_v \cdot m^2$)	x_{ij}	y_{ij}	$\bar{y}_i$	n_i
1	1	7.4	14×10^{-13}	0.8692	1.1461	1.2342	2
	2	7.4	21		1.3222		
2	1	10.0	9.6	1.0000	0.9823	0.9823	1
3	1	13.7	25	1.1367	1.3979	1.4447	2
	2	13.7	31		1.4914		
4	1	21.0	16	1.3222	1.2041	1.2041	1
5	1	23.0	35	1.3617	1.5441	1.5441	1
6	1	25.5	33	1.4065	1.5185	1.6790	3
	2	25.5	50		1.6990		
	3	25.5	66		1.8195		
7	1	30.0	34	1.4771	1.5315	1.5315	1

$x = \log_{10} E_p$

$y = \log_{10} 10^{13} \cdot H_o(E_p)$

A1.3. SUMMATIONS

The basic summations of the data of Table A1-1 required for regression analysis and hypothesis testing are:

$\Sigma x = \sum_i \sum_j x_{ij} = 13.3923$ $\quad$ $\Sigma y = \sum_i \sum_j y_{ij} = 15.6566$

$\Sigma x^2 = \sum_i \sum_j (x_{ij})^2 = 16.81418195$ $\quad$ $\Sigma y^2 = \sum_i \sum_j (y_{ij})^2 = 22.88768108$

$\Sigma xy = \sum_i \sum_j (X_{ij}Y_{ij}) = 14.45322097$ $\quad$ $N = \sum_i n_i = 11$

$\bar{x} = 1.21748188$ $\quad$ $\bar{y} = 1.423327273$

A1.4. CALCULATIONS

A1.4.1. Regression Line Parameters

The estimated values of constant, $\hat{k}$, and slope, $\hat{m}$, (Eq. 2) are given by:

$$\hat{m} = \frac{\Sigma xy - \frac{1}{N}\Sigma x \Sigma y}{\Sigma x^2 - \frac{1}{N}(\Sigma x)^2} = 0.7688886239 \qquad \text{(A1-1)}$$

$$\hat{k} = \frac{\Sigma y - \hat{m}\Sigma x}{N} = 0.4872193527$$

A1.4.2. Error Variance and Standard Error on $\hat{m}$ are given by:

$$S^2_{y|x} = \frac{1}{N-2}[\Sigma y^2 - \hat{k}\Sigma y - \hat{m}\Sigma xy] = 0.0335698066 \qquad \text{(A1-2)}$$

$$S^2_{\hat{m}} = \frac{S^2_{y|x}}{x^2 - N(\bar{x})^2} = 0.065913594 \qquad \text{(A1-3)}$$

$S_{\hat{m}} = 0.2567$

A1.4.3. Analysis of Variance (Partitioning Sums of Squares)

$$\text{Total sum of squares} = \Sigma y^2 - \frac{1}{N}(\Sigma y)^2 = 0.60322$$

$$\text{Regression sum of squares} = \hat{m}^2[\Sigma x^2 - \frac{1}{N}(\Sigma x)^2] = 0.30136 \qquad \text{(A1-4)}$$

$$\text{Residual sum of squares} = \text{Total SS} - \text{Regression SS} = 0.30186$$

$$\text{Within group sum of squares} = \Sigma y^2 - \sum_i \frac{(\sum_j y_{ij})^2}{n_i} = 0.06578$$

$$\text{About regression SS} = \text{Residual SS} - \text{Within SS} = 0.23608$$

A.5. TEST FOR LINEARITY

The experimental data of Table A1-1 have more than one determination of the y-values ($H_o(E_p)$) at energies of 7.4, 13.7 and 25.5 GeV and the assumption of linearity of the log-log transformed data may be tested.

Table A1-1 presents the data grouped into 7 compartments and the summations necessary for the test of linearity are given in Section A1.3. Table A1-2 summarises the analysis of variance data. Linearity may be tested using a Fisher's F-test. The null hypothesis, HYP(0), is:

HYP(0): $E(y|x) = k + mx$ or regression is linear
HYP(1): not linear

Table A1-2. Analysis of variance.

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square
Regression	0.30136	1	0.30136
About Regression	0.23608	k-2 = 5	0.04722
Within Group	0.06578	N-k = 4	0.01644
Residual	0.30186	N-2 = 9	0.03354
Total	0.60322	N-1 = 10	

The Test-Statistic is given by:

$$F = \frac{\text{About Regression Mean Square}}{\text{Within Group Mean Square}}$$

$$= \frac{0.04722}{0.01644} \qquad \text{(from Table A1-2)}$$

$$= 2.90$$

The critical region with a level of significance, $\alpha = 0.05$, $N = 11$ and $k = 7$ is $F > F_{0.95,(5,4)} = 6.26$, and thus the data do not support rejection of the hypothesis that the relation between x and y, ($\log_{10} H_o(\bar{E}_p)$ and $\log_{10} E_p$), is linear.

A1.6. EQUATION OF TRANSFORMED LINE

We have:

$$y = \hat{k} + \hat{m}x \qquad \text{(A1-10)}$$

Remembering that

$$y = \log_{10} 10^{13} H_o(E_p)$$

$$x = \log_{10} E_p \qquad \text{(A1-11)}$$

Then:

$$H_o(E_p) = (10^{-13} \cdot 10^{\hat{k}}) E_p^{\hat{m}} \qquad \text{(A1-12)}$$

Substituting the values for $\hat{k}$ and m we obtain:

$$H_o(E_p) = (3.07 \times 10^{-13}) E_\mu^{0.769} \qquad \text{(A1-13)}$$

This line is drawn on Fig. 2. The confidence band for this regression line is calculated in the following section.

A1.7. CALCULATION OF CONFIDENCE BAND FOR REGRESSION LINE

The boundaries for a $(1 - \alpha)$ confidence level band may be calculated from:

Upper limit: $\hat{y}_\ell + c\, S_{\hat{y}_\ell}$ (A1-14a)

Lower limit: $\hat{y}_\ell + c\, S_{\hat{y}_\ell}$ (A1-14b)

Where $c\, S_{\hat{y}_\ell}$ is given by:

$$c^2 S^2_{\hat{y}_\ell} = 2F_{2,N-2}\, S^2_{y|x} \left[\frac{1}{N} + \frac{(x_\ell - \bar{x})^2}{\Sigma (x_i - \bar{x})^2} \right] \qquad \text{(A1-15)}$$

Substituting:

$$F_{0.95(2,9)} = 4.26 \quad \text{for} \quad (\alpha = 0.05)$$

$$S^2_{y|x} = 0.03357$$

$$N = 11$$

$$\Sigma(x_i - \bar{x}) = 0.5093$$

$$\bar{x} = 1.217482$$

We obtain:

$$c\, S_{\hat{y}_\ell} = \sqrt{0.02598 + 0.561120(x_\ell - 1.21748)^2} \qquad \text{(A1.16)}$$

Table A1-3 summarizes the values of lower and upper bounds calculated using Eqs. (A1-14), (A1-15) and (A1-16).

Table A1-3. Calculated values of upper and lower bounds for 95 percent confidence band.

E_p (GeV)	$H_o(E_p)$ (Lower Limit) $Sv \cdot m^2 (x10^{13})$	$H_o(E_p)$ (Upper Limit) $Sv \cdot m^2 (x10^{13})$
1	----	25.9
2	1.03	26.5
5	4.02	27.8
10	10.6	30.5
15	16.9	35.9
20	20.6	45.7
30	23.5	75.0
50	25.0	154

A1.8. TEST OF HYPOTHESIS m = 1.0

Having established that the experimental data under log-log transformation are consistent with the hypothesis of linearity, and having calculated the best value of slope, $\hat{m}$, as 0.77 ± 0.26 we now proceed to test the hypothesis that this estimated value of $\hat{m}$ differs from the upper limit value of 1.0 due only to random variations.

Hypothesis:

HYP(0): $m_o = 1$
HYP(1): $m_o < 1$

Test statistic:

$$T = \frac{\hat{m} - m_o}{S_{\hat{m}}} \qquad \text{(A1-17)}$$

Substituting the values $\hat{m} = 0.769$ and $S_{\hat{m}} = 0.257$ from Section A1-4:

$$T = \frac{0.769 - 1.0}{0.257} = -0.90 \quad .$$

The critical region at a level of significance of 5 percent, df = 9 is $T < -2.26$. Since the calculated value of T is not less than -2.26, the null hypothesis is not rejected. The experimental data are not inconsistent with the value of b = 1.0. The probability of observing a value of T less than -0.90 (one tail test) is 20 percent.

APPENDIX 2. LINEAR REGRESSION ANALYSIS AND HYPOTHESIS TESTING

Having shown in Appendix 1 that linearity ($m = 1.0$) is not excluded linear regression can be carried out. In this appendix the data are tested for linearity, subjected to regression analysis [$H_o(E_p) = \hat{a} + \hat{b}E_p$] and the hypothesis $\hat{a} = 0$ tested. For convenience, throughout this appendix the variables $H_o(E_p)$ and E_p are denoted by y and x respectively.

A2.1. NUMERICAL DATA

Table A2-1 summarizes the experimental data drawn up into 7 groups (as was done in Table A1-1) for the original measurements given in Table 1.

Table A2-1. Numerical data for determination of regression line and for hypothesis testing.

Group	Energy x (GeV)	Moyer Parameter y ($Sv\ m^2$)	n_i $\sum_j x_{ij}$	$\sum_j x_{ij}^2$	$\sum_j y_{ij}$	$\sum_j (y_{ij})^2$	$\sum x_{ij}y_{ij}$	n_i
1	7.4 7.4	14×10^{-13} 21×10^{-13}	14.8	109.52	35	637	259	2
2	10.0	9.6×10^{-13}	1	100	9.6	92.16	96	1
3	13.7 13.7	25×10^{-13} 31×10^{-13}	27.4	375.38	56	1586	767.2	2
4	21	16×10^{-13}	21	441	16	256	336	1
5	23	35×10^{-13}	23	529	35	1225	805	1
6	25.5 25.5 25.5	33×10^{-13} 50×10^{-13} 66×10^{-13}	76.5	1950.75	149	7945	3799.5	3
7	30	34×10^{-13}	30	900	34	1156	1020	1

A2.2. SUMMATIONS

The summation data required for regression analysis and hypothesis testing are:

$\Sigma x = \sum_i \sum_j x_{ij} = 202.7$

$\Sigma y = \sum_i \sum_j y_{ij} = 334.60 \times 10^{-13}$

$\Sigma x^2 = \sum_i \sum_j x_{ij}^2 = 4405.65$

$\Sigma y^2 = \sum_i \sum_j y_{ij}^2 = 12897.16 \times 10^{-26}$

$\Sigma xy = \sum_i \sum_j x_{ij} y_{ij} = 7082.7 \times 10^{-13}$

$N = \sum_i n_i = 11$

$\bar{x} = 18.4272723$

$\bar{y} = 30.418182.$

A2.3. CALCULATIONS

A2.3.1. Regression Line Parameters

Substituting the appropriate sums into the regression line equations (see A1.4.1) we obtain:

$\hat{a} = 5.215990555 \times 10^{-13}$

$\hat{b} = 1.367657148 \times 10^{-13}$

A2.3.2. Error Variance and Estimated Variance of $\hat{a}$ and $\hat{b}$

$$S^2_{y|x} = \frac{1}{N-2}\left[\Sigma y^2 - \hat{a}\Sigma y - \hat{b}\Sigma xy\right] = 162.7982531 \times 10^{-26} \qquad \text{(A1-2)}$$

$\therefore S_{y|x} = 1.275924 \times 10^{-12}$

$$S^2_{\hat{b}} = \frac{S^2_{y|x}}{\Sigma x^2 - N(\bar{x})^2} = 2.42822799 \times 10^{-27}$$

$S_{\hat{b}} = 4.927700071 \times 10^{-14}$

The estimated variance of $\hat{a}$, $S_{\hat{a}}$, is given by:

$$S^2_{\hat{a}} = S^2_{y|x}\left[\frac{1}{N} + \frac{\bar{x}^2}{\Sigma(x_{ij} - \bar{x})^2}\right] = 9.725365962 \times 10^{-25} \qquad \text{(A2-1)}$$

$$S_{\hat{a}} = 9.86172701 \times 10^{-13}$$

A2.3.2. Analysis of Variance (Partitioning Sums of Squares)

The formulae for calculating the various sums of squares are given in Section A1.4.3.

Total sums of squares = 2.71924×10^{-23}
Regression sum of squares = 1.25405×10^{-23}
Residual sum of squares = 1.46518×10^{-23}
Within group sum of squares = 0.58717×10^{-23}
About regression sum of squares = 0.87801×10^{-23}

A2.4. TEST FOR LINEARITY

Table A2-2. Analysis of Variance.

Source	Sum of Squares	Degrees of Freedom	Mean Square
Regression	1.2540×10^{-23}	1	1.2540×10^{-23}
About Regression	0.8790×10^{-23}	k-2 = 5	0.1756×10^{-23}
Within Group	0.5872×10^{-23}	N-k = 4	0.1468×10^{-23}
Residual	1.4652×10^{-23}	N-2 = 9	0.1628×10^{-23}
Total	2.7192×10^{-23}	N-1 = 10	

The null hypothesis, HYP(0), is that:

HYP(0): $E(y|x) = a + bx$

HYP(1): Not linear

The test statistic is given by:

$$F = \frac{\text{ABOUT REGRESSION MEAN SQUARE}}{\text{WITHIN GROUP MEAN SQUARE}}$$

$$= \frac{0.1756 \times 10^{-23}}{0.1468 \times 10^{-23}}$$

$$= 1.20$$

The critical region with a level of significance $\alpha = 0.05$, $N = 11$, $k = 7$ is $F < F_{0.95(5,4)} = 6.26$.

Since $F < 6.26$ the data do not support the rejection of the hypothesis of linearity.

A2.5. CALCULATION OF CONFIDENCE BAND FOR REGRESSION LINE

A2.5.1. Calculation of 95 Percent Confidence Band

The boundaries for the $(1 - \alpha)$ confidence level band are:

$$\text{Upper Bound} = y_\ell + CS_{y_\ell}$$

$$\text{Lower Bound} = y_\ell - CS_{y_\ell}$$

where:

$$S^2_{\hat{y}_\ell} = S^2_{y|x}\left[\frac{1}{N} + \frac{(x_\ell - \bar{x})^2}{\Sigma(x_i - \bar{x})^2}\right] \quad ,$$

$$c^2 = 2\,F_{2,N-2}$$

Substituting:

$$S^2_{y|x} = 1.62798 \times 10^{-24}$$

$$N = 11$$

$$\Sigma(x_i - \bar{x})^2 = 610.441818$$

$$F_{2,9} = 4.26 \quad (\alpha = 0.05)$$

$$S^2_y = 1.479982\times10^{-25} + 2.428219655\times10^{-27}\,(x_\ell - 18.427272)^2$$

$$c^2S^2_y = 1.2609446\times10^{-24} + 2.068843146\times10^{-26}\,(x_\ell - 18.427272)^2 \qquad \text{(A2-2)}$$

Table A2-3 summarizes values of the upper and lower limits of the 95 percent confidence band as a function of x. These values are plotted on Fig. 3.

Table A2-3. Calculated values of upper and lower bounds for 95 percent confidence band.

E_0 (GeV) x	$H_0 \times 10^{13}$ ($Sv \cdot m^2$) y	$cS_y \times 10^{13}$	Upper Limit $y + cS_{y_\ell}$ $Sv\cdot m^2$ ($\times 10^{13}$)	Lower Limit $y - cS_{y_\ell}$ $Sv\cdot m^2$ ($\times 10^{13}$)
0	5.22	28.79	34.00	-23.57
5	12.07	22.34	34.39	-10.29
10	18.92	16.52	35.42	2.37
15	25.77	12.26	37.99	13.47
20	32.62	11.46	44.03	21.11
25	39.47	14.68	54.09	24.73
30	46.32	20.08	66.33	26.17

A2.6. TEST FOR $a = 0$

The working hypothesis is that the straight line passes through the origin (i.e., $a = 0$).

Student's t-test gives:

$$T = \frac{\hat{a} - a_0}{S_{\hat{a}}} \qquad \text{(A2-2)}$$

Substituting the values:

$a_0 = 0$ (null hypothesis)
$\hat{a} = 5.216 \times 10^{-13}$
$S_{\hat{a}} = 9.862 \times 10^{-13}$

into Eq. (A2-2) we obtain:

$T = 0.529$

We reject the null hypothesis if

$|T| > t_{\alpha,N-2}$ (9 degrees of freedom) = 2.262

$0.529 \not> 2.262$ and thus the null hypothesis is not excluded ($p \cong 0.30$) and the experimental data are consistent with the hypothesis that they may be represented by a straight line passing through the origin.

APPENDIX 3. REGRESSION ANALYSIS WITH LINE FORCED THROUGH THE ORIGIN

In Appendix 2 it was shown that the experimental data are compatible with the hypothesis that the relationship between $H_o(E_p)$ and E_p is of the form:

$$H_o(E_p) = b'E_p \qquad \text{(A3-1)}$$

In this appendix, as in Appendix 2, the variables $H_o(E_p)$ and E_p will be denoted by y and x.

For a line forced through the origin:

$$b' = \frac{\Sigma xy}{\Sigma x^2} \qquad \text{(A3-2)}$$

and substituting the values for the summations from Appendix 2:

$$b = 1.60764 \times 10^{-13}$$

The error variance is:

$$S^2_{y\cdot x} = \frac{y^2 - b'^2 x^2}{N - 1} \qquad \text{(A3-3)}$$

which yields upon substitution:

$$S^2_{y\cdot x} = 1.5107295 \times 10^{-24}$$

The estimated variance of b' is given by:

$$S^2_{b'} = \frac{S^2_{y\cdot x}}{\Sigma x^2} \qquad \text{(A3-4)}$$

Substitution yields:

$$S^2_{\hat{b}'} = \frac{1.5107245 \times 10^{-24}}{4405.65}$$

$$= 3.429073 \times 10^{-28}$$

$$S_{\hat{b}'} = 1.852 \times 10^{-14}$$

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