UNIVERSITY OF CALIFORNIA

Lawrence Radiation Laboratory
Berkeley, California

AEC Contract No. W-7405-eng-48

TRANSVERSE COHERENT RESISTIVE INSTABILITIES OF AZIMUTHALLY BUNCHED BEAMS IN PARTICLE ACCELERATORS

Ernest D. Courant and Andrew M. Sessler

April 1, 1966

UCRL-16751

TRANSVERSE COHERENT RESISTIVE INSTABILITIES OF AZIMUTHALLY BUNCHED BEAMS IN PARTICLE ACCELERATORS*

Ernest D. Courant

Brookhaven National Laboratory
Associated Universities, Inc.
Upton, New York

and

Andrew M. Sessler

Lawrence Radiation Laboratory
University of California
Berkeley, California

April 1, 1966

ABSTRACT

The transverse electromagnetic coupling of bunches of particles with each other is investigated theoretically, and shown to incorporate the possibility (due to the effect of nonperfectly conducting vacuum chamber walls) of coherent instability even when the longitudinal distance between bunches is much larger than the transverse dimensions of the vacuum tank. The modes of oscillation in which the bunches move rigidly are investigated; criteria for stability, and expressions for the small amplitude growth rates under unstable conditions are presented. The case of a single bunch is considered in detail and demonstrated to be stable (even in the absence of Landau damping) provided ν lies between an integer and the next higher half-integer, where ν is the number of transverse free betatron oscillations

* Research supported by the U. S. Atomic Energy Commission.

occurring in one revolution; for many bunches which are sensibly different in intensity (a criterion for this is presented), all modes are stable provided ν satisfies the same restriction. For equally spaced bunches of equal numbers of particles, approximately half the modes are unstable without Landau damping. Numerical examples are presented covering some intermediate situations.

I. INTRODUCTION

The possible instability of coherent transverse oscillations of an azimuthally uniform beam of particles circulating in a metallic vacuum chamber has been studied by Laslett, Neil, and Sessler[1] (LNS), who showed that under certain circumstances the finite resistivity of the vacuum walls could cause growing oscillations. In most accelerators, the rf acceleration mechanism generates azimuthal non-uniformity of particle density, and consequently the work of LNS is not applicable to the analysis of transverse instabilities of the beam. In this work we treat a complementary idealization to that of LNS--namely, a beam consisting of a number of bunches which are assumed to have no coherent motion of the internal degrees of freedom.

We have not, in this paper, studied coherent modes within a bunch. We expect that in the absence of Landau damping some of these modes will be unstable, but we also expect that the synchrotron motion will introduce considerable Landau damping and that--in practice--these modes will not impose a restraint upon beam intensity.

The physical concepts which form the basis of resistive instabilities have been expounded in LNS; there is no need to repeat the discussion here. However, the physics for bunches of particles is, perhaps, somewhat more transparent than that for a uniform beam, and consequently we present it in Section II. Section III contains the body of the analysis, culminating in a dispersion relation involving the

solution of a set of homogeneous equations. The consequences of the dispersion relation are explored in Section IV, first for a single bunch, secondly for bunches which have different numbers of particles, thirdly for equally spaced bunches of equal numbers, fourthly (numerically) for intermediate cases, and finally for unequal bunch spacings. An Appendix is devoted to analysis of a function--the Bunch Function--which plays a fundamental role in the theory.

The reader interested only in results may turn directly to Section IV; readers not interested in mathematics but wanting to "understand" the phenomena may find Sections II and IV adequate.

A report on part of this work was presented at the Particle Accelerator Conference in March 1965;[2] a preliminary report and abstract of this work appears in the Summary Report of the SIAC Summer Study on Instabilities in Stored Particle Beams.[3]

II. PHYSICAL CONSIDERATIONS

In this section we limit our attention to the case of a single bunch having no internal degrees of freedom. The analysis could readily be extended to include many bunches, and also to include spreads in particle revolution frequency (and hence Landau damping), but the resulting analysis would then become more cumbersome than that employed in Section III where the completely general problem is considered.

The simplified problem of this section has already been treated in the literature;[4,5] we repeat the discussion because (i) it is so relevant to an appreciation of the contents of this paper, (ii) it is

much more transparent than previous discussions (Ref. 1) or the analysis of Section III, and (iii) it is rather brief.

The physical basis of the instability is that in a resistive vacuum tank, fields due to a particle decay only very slowly in time after the particle has left. The decay can be so slow that when a bunch returns after one (or more) revolutions it is subject to its own residual field which--depending upon its phase relative to the wake field--can lead to damped or undamped tranverse motion. We need, as a first ingredient, the solution to the electromagnetic problem and this has been given by a number of authors.[5,6,7,8] From Ref. 8 we know that a particle of charge Ne passing the point $z = 0$ at time $t = 0$ while traveling with speed βc down a straight pipe of circular cross section and radius b and oscillating transversely with displacement $\xi \exp(+i\omega t)$ will exert a force on a particle of charge e having speed βc and passing the point z at time t given by

$$F = \frac{4 e^2 N \xi \beta^2 e^{+i\omega z/\beta c}}{(\pi \mathcal{K})^{\frac{1}{2}} b^3 |z - \beta c t|^{\frac{1}{2}}} , \quad \text{for} \quad z < \beta c t \tag{2.1}$$

where $\mathcal{K} = 4 \pi \beta \sigma / c$ and σ is the conductivity of the pipe walls. For $z > \beta c t$, the force is negligible in comparison with that of Eq. (2.1).

We can, with this force, immediately write an equation for the transverse displacement y of the bunch, namely:

$$\gamma m_0 \frac{d^2 y}{dt^2} = F + e\beta \left.\frac{\partial H_x}{\partial y}\right|_{y=0} y , \tag{2.2}$$

where γm_0 is the mass of, and F is the force on, one particle of rest mass m_0 in the bunch. In Eq. (2.2) we have neglected any local fields of a bunch upon itself; these fields are generally less important than the wake field and, in any case, of such a sign as to cause damping. The last term in Eq. (2.2) is the force due to the external field which determines the transverse oscillation frequency $\nu_0\omega_0$ of the unperturbed bunch, in terms of which Eq. (2.2) may be written as

$$\frac{d^2y}{dt^2} + \nu_0^2 \omega_0^2 y = \frac{F}{\gamma m_0} , \tag{2.3}$$

with the particle circulation frequency $\omega_0 \equiv \beta c/R$. The force F must be evaluated as a sum over contributions from all previous turns, ($z = -2\pi Rn$), and assuming that y varies harmonically (as it does), we see that Eq. (2.3) becomes

$$\left[\frac{d^2}{dt^2} + \nu_0^2 \omega_0^2\right] \xi e^{+i\nu\omega_0 t} = \frac{4e^2 N\beta^2 \xi\, e^{+i\nu\omega_0 t}}{(\pi R)^{\frac{1}{2}}\, b^3\, R^{\frac{1}{2}}\, \gamma m_0} \sum_{n=1}^{\infty} \frac{e^{-i\nu 2\pi n}}{(2\pi n)^{\frac{1}{2}}} , \tag{2.4}$$

where we have replaced ω with $\nu\omega_0$. The sum is conveniently expressed in terms of a function--the Bunch Function--and by Eq. (A9) of the Appendix, Eq. (2.4) yields

$$\nu_0^2 - \nu^2 = \frac{4Ne^2\beta^2\, G(2\pi,\nu)}{(\pi R)^{\frac{1}{2}}\, b^3\, \gamma m_0\, R^{\frac{1}{2}}\, \omega_0^2\, 2\pi^{\frac{1}{2}}} , \tag{2.5}$$

with solution

$$\nu \simeq \nu_0 \left[1 - \frac{Ne^2 \beta^2 G(2\pi,\nu)}{\pi (\Pi)^{\frac{1}{2}} b^3 \gamma m_0 R^{\frac{1}{2}} \omega_0^2 \nu_0^2} \right], \qquad (2.6)$$

where the positive sign is required to be consistent with the force assumed in Eq. (2.1). Instability occurs for $\mathrm{Im}\ \nu < 0$ and thus is confined to those regions in which $\mathrm{Im}\ G(2\pi,\nu) > 0$. It is shown in the Appendix [discussion following Eq. (A5)] that $\mathrm{Im}\ G(2\pi,\nu) > 0$ when $I - \frac{1}{2} < \nu < I$ where I is any integer. [This result is consistent with that derived with only the first term in G; i.e., the residual field from only the last revolution.]

The physical basis of the instability is thus clear; more bunches will simply cause mathematical complications, whereas allowing frequency spread of the particles in the bunch will give possible stability from Landau damping in the range of instability disclosed by the present analysis. In the absence of Landau damping, Eq. (2.6) gives a growth time τ, for $I - \frac{1}{2} < \nu < I$:

$$\tau = \frac{\pi \gamma \nu_0}{N} \left(\frac{b^3}{R r_0 c} \right) \left(\frac{4\pi\sigma R}{\beta c} \right)^{\frac{1}{2}} \frac{1}{|\mathrm{Im}\ G(2\pi,\nu_0)|}, \qquad (2.7)$$

where $r_0 = e^2/m_0 c^2$ is the classical particle radius.

III. DERIVATION OF THE DISPERSION RELATION

We proceed directly, now, to the analysis of the general M-bunch problem, including the dispersion of particle frequencies and hence Landau damping. We first consider the electromagnetic problem, then particle dynamics.

A. Fields

We obtain the requisite field expressions by employing the results of LNS, whose treatment is confined to a continuous beam, of azimuthally constant density and dimensions, oscillating coherently in such a mode that its transverse electric dipole moment per unit length is of the form

$$P(\theta,t) \equiv \int y\, \rho(r,\theta,z,t)\,dr\,dz = P_n\, e^{i(n\theta-\omega t)} , \qquad (3.1)$$

where ρ is the charge density of the beam per unit volume. We employ cylindrical coordinates r, θ, z ; y is the direction of transverse oscillations, and we have ignored effects associated with the major radius of the beam. From LNS Eq. (2.25), the average force per unit charge acting on the beam is

$$\left(\frac{F_y}{e}\right) = P_n \left[U + W\left(\frac{1}{\omega}\right)^{1/2} \right] e^{i(n\theta-\omega t)} , \qquad (3.2)$$

where U and W depend on the geometry of the beam and the vacuum chamber. For a circular beam (radius a) in a circular vacuum chamber (radius b) they obtain, approximately,

$$U = -\frac{2}{\gamma^2}\left(\frac{1}{a^2} - \frac{1}{b^2}\right)$$

$$W = \frac{4c\beta^2}{b^3}\,(4\pi\sigma)^{-1/2} , \qquad (3.3)$$

where σ is the conductivity of the wall material, expressed in Gaussian units (dimension T^{-1}) and βc is the velocity of the particles in the beam. The expressions of Eq. (3.3) are valid if $\sigma \gg \omega$, $\sigma \gg c^2/d^2\omega$ (d = thickness of vacuum chamber wall $\gg$ skin depth), $R/n \gg b$ (wave length of oscillation $\gg$ transverse dimension of chamber). For other geometries the expressions for U and W are different, but subject to the above conditions, they still possess the following characteristics: (a) U and W are independent of ω and of the mode number n; (b) U contains the factor $1/\gamma^2$, W does not; (c) U is sensitive to the beam dimensions, W is not; (d) W is proportional to $\sigma^{-1/2}$.

The resistive (W) term in Eq. (3.2) arises from the skin effect in the chamber wall. The derivation of this effect shows that the sign of the square root must be chosen, regardless of the sign of ω, in such a way that $(i/\omega)^{1/2}$ has a positive real part, corresponding to an attenuated wave in the metal.

For a non-uniform beam with arbitrary time dependence, we may write $P(\theta,t)$ as a periodic function of θ and a Fourier integral in t:

$$P(\theta,t) = \int_{-\infty}^{\infty} Q(\theta,\omega)\, e^{-i\omega t}\, d\omega \,. \tag{3.4}$$

By Eq. (3.2) the Fourier transform of F is then

$$\left[U + \left(\frac{i}{\omega}\right)^{1/2} W \right] Q(\theta,\omega) \,. \tag{3.5}$$

Inverting the Fourier transform and noting that U and W are independent of ω, we obtain

$$\frac{F_y}{e}(\theta,t) = U\,P(\theta,t) + \frac{W}{2\pi^{\frac{1}{2}}} \int_{-\infty}^{t} \frac{P(\theta,t')}{(t-t')^{\frac{1}{2}}}\,dt' . \qquad (3.6)$$

To find the fields associated with bunches of arbitrary shape, we use the somewhat indirect (but transparent) method of first finding the field due to a single particle at the position of another single particle, and then superimposing the results. Consider, therefore, a single particle--the rth particle--circulating with angular velocity ω_0 and oscillating transversely with angular frequency $\nu\omega_0$ and amplitude ξ (we assume that all particles have the same angular velocity ω_0).

The dipole moment per unit length due to this particle is

$$P(\theta,t) = \frac{e}{R}\,\xi_r\, e^{i(\phi_r + \nu\omega_0 t)}\,\delta_p(\theta - \theta_r - \omega_0 t) , \qquad (3.7)$$

where e is the charge of the particle, δ_p is the periodic delta function, ϕ_r is the transverse phase, and θ_r is the azimuthal location of the particle at $t = 0$. Substituting Eq. (3.7) in Eq. (3.6), we find

$$\frac{F}{e}(\theta,t) = U\,P(\theta,t) + \frac{W\,e\,\xi_r}{2\pi R\,\omega_0^{\frac{1}{2}}}\, e^{i(\phi_r + \nu\omega_0 t)}\, G(\alpha,\nu) , \qquad (3.8)$$

where $\alpha = \theta - \theta_r - \omega_0 t$, and we have introduced the function $G(\alpha,\nu)$.

The "Bunch Function" $G(\alpha,\nu)$ is defined as

$$G(\alpha,\nu) = 2\pi^{\frac{1}{2}} \sum_{k=0}^{\infty} \frac{e^{-i\nu(\alpha+2\pi k)}}{(\alpha + 2\pi k)^{\frac{1}{2}}} \qquad (3.9)$$

for $0 < \alpha \leq 2\pi$, and is defined to be periodic in α with period 2π for other values of α [equivalent to starting the summation over k, in Eq. (3.9), with the smallest integer greater than $-\alpha/2\pi$]. The Appendix is devoted to a study of the properties of this function; it contains alternative representations, approximate formulas, numerical values, and some general theorems which will be employed subsequently.

B. Particle Dynamics

From Eq. (3.8) the force per unit charge on a particle moving with velocity βc, due to the oscillation and longitudinal motion of the rth particle, is:

$$\left(\frac{F}{e}\right) = U\,P(\theta,t) + \frac{W}{\omega_0^{\frac{1}{2}}} \frac{e\xi_r}{2\pi R}\, G(\alpha,\nu)\, \exp[(\phi_r + \nu\omega_0 t)i] \,, \qquad (3.10)$$

where P is given by Eq. (3.7), and $\alpha = \theta_r + \omega_0 t - \theta$. Consider the motion of a particle--the sth particle, subject to the force of Eq. (3.10) (evaluated at $\theta = \omega_0 t + \theta_s$) as well as the restoring force of the external focusing field. Its equation of transverse motion is

$$m_0\gamma(\ddot{y}_s + \omega_0^2\, \nu_s^2\, y_s) = e\,U\,P(\theta_s + \omega_0 t, t) + \frac{We^2\xi_r}{2\pi R\omega_0^{\frac{1}{2}}} \exp[i(\phi_r + \nu\omega_0 t)]\; G(\theta_r - \theta_s, \nu) \,. \qquad (3.11)$$

We study the normal modes of oscillation of an arbitrary collection of particles by assuming they all oscillate coherently, with transverse angular frequency $\nu\omega_0$. Thus the motion of the $\underline{s}$th particle is described by:

$$m_0\gamma\omega_0^2(\nu_s^2 - \nu^2)\xi_s \exp[i(\phi_s + \nu\omega_0 t)] = e\,U\,P(\theta_s + \omega_0 t, t)$$

$$+ \frac{We^2}{2\pi R\omega_0^{\frac{1}{2}}} \sum_r \xi_r \exp[i(\phi_r + \nu\omega_0 t)]\, G(\theta_r - \theta_s, \nu) , \tag{3.12}$$

where $\nu_s\omega_0$ is the frequency of free oscillation of the $\underline{s}$th particle.

To proceed further, we assume that the particles are bunched tightly into M bunches, each of length L , the $\underline{m}$th having N_m particles. The particles have various amplitudes of oscillation ξ , phases ϕ , azimuthal location θ , and betatron frequencies $\nu_s\omega_0$. We describe this situation with a distribution function ψ , taken of the form

$$\psi(\theta, \xi, \phi, \nu_s) = N_m \left(\frac{2\pi R}{L}\right) D(\xi, \phi)\, f(\nu_s) \tag{3.13}$$

for θ in the range $(L/2\pi R)$, and zero elsewhere. The functions D and f are normalized to unity. The dipole moment of a bunch, Q_m , is given by

$$Q_m = e \int \psi(\theta, \xi, \phi, \nu_s)\, \xi\, e^{i\phi}\, d\xi\, d\phi\, d\theta , \tag{3.14}$$

whereas the dipole moment per unit length $P(\theta, t) = Q_m/L$.

We obtain an equation for Q_n by multiplying Eq. (3.12) by $e\psi$, dividing by $(\nu_s^2 - \nu^2)$, and then integrating ξ, ϕ, θ, and ν_s over the nth bunch. We also replace the summation over r by summation over bunches and integration within the bunches:

$$\sum_{r\ (\text{particles})} \longrightarrow \sum_{m} \left(\frac{2\pi R}{L}\right) \int d\theta_m \ , \tag{3.15}$$

to obtain

$$m_0 \gamma \omega_0^2 Q_n = \int \frac{f(\nu_s)d\nu_s}{(\nu_s^2 - \nu^2)} \left\{ \frac{e^2 U N_n Q_n}{L} + \sum_{m} \left(\frac{2\pi R}{L}\right)^2 \frac{We^2 N_n Q_m}{2\pi R \omega_0^{\frac{1}{2}}} \int_0^{L/2\pi R} \int d\theta_m \, d\theta_n \, G(\theta_m - \theta_n, \nu) \right\} . \tag{3.16}$$

In the summation over m we must treat the nth bunch specially; for all other bunches the bunch function may be treated as a constant and removed from the integral. Within the nth bunch we use Eqs. (A12) and (A13) to obtain

$$m_0 \gamma \omega_0^2 Q_n = \int \frac{f(\nu_s)d\nu_s}{(\nu_s^2 - \nu^2)} \left\{ \frac{e^2 U N_n Q_n}{L} + \frac{We^2 N_n}{2\pi R \omega_0^{\frac{1}{2}}} \left[\sum_{m \neq n} Q_m G(\theta_m - \theta_n, \nu) + Q_n \left(\frac{2\pi R}{L}\right)^2 \int_0^{L/2\pi R} d\theta_m \int_0^{Q_m} d\theta_n \, 2\left(\frac{\pi}{\theta_m - \theta_n}\right)^{\frac{1}{2}} \exp[-i(\theta_m - \theta_n)] + Q_n \, G(2\pi, \nu) \right] \right\} . \tag{3.17}$$

Letting

$$\lambda = \frac{m_0 \gamma \omega_0^2}{\int \frac{f(\nu_s)d\nu_s}{\nu_s^2 - \nu^2}} , \tag{3.18}$$

and expanding the exponential in the integral of Eq. (3.17), valid for $\nu L/2\pi R \ll 1$, we obtain

$$\lambda Q_n = \frac{e^2 U N_n Q_n}{L} + \frac{W e^2 N_n}{2\pi R \omega_0^{\frac{1}{2}}} \left\{ \sum_{m \neq n} Q_m G(\theta_m - \theta_n, \nu) + Q_n \left[G(2\pi,\nu) + \frac{8\pi^{\frac{1}{2}}}{3} \left[\left(\frac{2\pi R}{L} \right)^{\frac{1}{2}} - \frac{i\nu}{5} \left(\frac{L}{2\pi R} \right)^{\frac{1}{2}} \right] \right] \right\} . \tag{3.19}$$

(Higher order terms in $\nu L/2\pi R$ can easily be generated, if needed.) Finally, we may write Eq. (3.19) in the compact form

$$(N_n U' - \lambda)Q_n + N_n W' \sum_m Q_m G_{mn} = 0 , \tag{3.20}$$

where

$$G_{mn} = G(\theta_m - \theta_n, \nu) , \tag{3.21}$$

$$G_{nn} = G(2\pi,\nu) - \frac{8\pi^{\frac{1}{2}}}{15} \left(\frac{L}{2\pi R} \right)^{\frac{1}{2}} \nu i \tag{3.22}$$

and

$$W' = \frac{e^2 W}{2\pi R \omega_0^{\frac{1}{2}}} , \tag{3.23}$$

$$U' = \frac{e^2 U}{L} + \frac{8\pi^{\frac{1}{2}}}{3} \left(\frac{2\pi R}{L} \right)^{\frac{1}{2}} W' . \tag{3.24}$$

In the next section we shall discuss the solution of Eqs. (3.20); the equations are valid for the coherent motion of short bunches.

IV. CONSEQUENCES OF THE DISPERSION RELATION

We will, in this section, study the set of homogeneous linear equations [Eq. (3.20)] for the dipole moments Q_n. These equations are of the form of a standard eigenvalue problem: The eigenvalue λ must be determined in such a way that the determinant of the coefficients of the Q_n vanishes. Then, from Eq. (3.18), one solves for ν which gives immediately--by Eq. (3.7)--the time development of the coherent motion. Clearly the motion is unstable if the imaginary part of ν is negative, stable if the imaginary part is positive.

The case of bunches with no spread in betatron frequencies, and hence no Landau damping, is simplest to consider. From Eq. (3.18), with ν_0 the common betatron tune,

$$\nu^2 = \nu_0^2 - \frac{\lambda}{m_0 \gamma \omega_0^2} , \tag{4.1}$$

and hence

$$\nu \approx \nu_0 - \frac{\lambda}{2m_0 \gamma \nu_0 \omega_0^2} , \tag{4.2}$$

since ν must have the sign of ν_0 . Thus the motion is unstable if and only if $\mathrm{Im}\,\lambda > 0$.

With Landau damping included, the motion is always stable if the $\mathrm{Im}\,\lambda < 0$; with $\mathrm{Im}\,\lambda > 0$ the motion can still be stable, with the stability depending upon the $\mathrm{Re}\,\lambda$ and the distribution function $f(\nu_s)$. This point is discussed at some length in LNS, and all the analysis given there is applicable here. The new feature, of this paper, is the expression for λ in terms of the properties of the accelerator and the nature of the particle beam. We shall concentrate upon this aspect of the problem, treating a number of different cases.

A. One Bunch

For one bunch of N particles Eq. (3.20) becomes

$$\lambda = N[U' + W' G_{nn}] . \tag{4.3}$$

Inserting Eq. (4.3) into Eq. (4.1)--corresponding to no Landau damping--and using Eqs. (3.22), (3.23), and (3.24) yields:

$$\nu_0^2 - \nu^2 = \frac{N}{m_0 \gamma \omega_0^2} \left\{ \frac{e^2 U}{L} + \frac{e^2 W}{2\pi R \omega_0^{\frac{1}{2}}} \left[\frac{8\pi^{\frac{1}{2}}}{3} \left(\frac{2\pi R}{L} \right)^{\frac{1}{2}} + G(2\pi, \nu) - \frac{8\pi^{\frac{1}{2}}}{15} \left(\frac{L}{2\pi R} \right)^{\frac{1}{2}} \nu i \right] \right\} .$$

If we drop the terms which are purely real--as they won't affect the stability analysis (to lowest order)--and employ Eq. (3.3), we have

$$\nu_0^2 - \nu^2 = \frac{4 N e^2 \beta^2 \left[G(2\pi, \nu) - \frac{8\pi^{\frac{1}{2}}}{15} \left(\frac{L}{2\pi R} \right)^{\frac{1}{2}} \nu i \right]}{(\pi(\quad))^{\frac{1}{2}} b^3 \gamma m_0 R^{\frac{1}{2}} \omega_0^2 2\pi^{\frac{1}{2}}} , \tag{4.4}$$

where $\zeta' = 4\pi\beta\sigma/c$. Compare this result with Eq. (2.5), which was derived employing wake fields. It agrees with the simple analysis except for the addition of the local-field term [which had its source in $G(\theta,\nu)$ for $0 < \theta << 2\pi$]. For a short bunch the local field is negligible compared to the residual field from previous turns, and the analysis of Section II is valid: The motion is stable if and only if ν lies above an integer; namely $I < \nu < I + \frac{1}{2}$, for any integer I . [Derivation of this result and further discussion may be found in Section II, following Eq. (2.6).]

In the more general case, where local fields are important, one can employ Eq. (4.4). If Landau damping is to be considered also, then one must resort to Eqs. (4.3) and (3.18).

It is interesting to consider the case of a very large accelerator --that is, a particle moving down a long straight resistive pipe. Is it stable or unstable with respect to transverse oscillations? To study this case, we take the limit of Eq. (4.4) as $R \to \infty$. Introducing in place of ν , the distance, λ_β, that the particle travels during one transverse oscillation period [duration $(w\omega_0)^{-1}$] , we observe that $\nu = R/\lambda_\beta \to \infty$. Consequently the local-field term in Eq. (4.4) dominates[9] $G(2\pi,\nu)$ --in agreement with Eq. (A9) of the Appendix which shows that $G(2\pi,\nu)$ consists only of contributions from previous turns. The remaining term yields $\mathrm{Im}\ \nu > 0$, and hence the motion is stable.[10] We may readily pursue the problem further and compute the damping rate, which is a factor of $\exp[-(\mathrm{Im}\ \nu)\nu_0^{-1}]$ in each transverse oscillation

period. From Eq. (4.4), the damping factor per period, f, is:

$$f = \exp\left[\frac{-8N}{15\pi\gamma}\left(\frac{r_0\lambda_\beta}{b^3}\right)\left(\frac{L}{2\pi}\right)^{\frac{1}{2}}\right], \tag{4.5}$$

where $r_0 = e^2/m_0c^2$ is the classical particle radius, the bunch of length L has N particles and travels down a resistive tube of radius b while oscillating with transverse wave length λ_β. The quantity ρ^{-1} is a skin depth, and Eq. (4.5) is valid for $\lambda_\beta \gg L \gg b \gg \rho^{-1}$.

B. Many Nonequal Bunches

If the number of particles, N_n, in the various bunches are unequal, then the set of equations for the Q_n [Eq. (3.20)] has non-degenerate eigenvalues in the limit that $W' \to 0$. In this case, and for small W', the eigenvalues, $\lambda_{(n)}$, are given to first order in W' only by the diagonal terms of the matrix:

$$\lambda_{(n)} = N_n[U' + W' G_{nn}], \; n = 1, \cdots M. \tag{4.6}$$

The M eigenvalues of Eq. (4.6) are the same as one would obtain for M independent bunches. Just as for one bunch, for many bunches we are assured of stability if $\mathrm{Im}\,\lambda < 0$; that can be accomplished by choosing $I < \nu < I + \frac{1}{2}$, for any integer I.

The result obtained is easily understood since for bunches of unequal number N_n, the natural frequency of each bunch is different from that of any other bunch. Thus most of the influence of one bunch

on another averages out to a large extent (to be precise, it is removed from first order), and hence the bunch motion is dominated by the influence of one bunch upon itself. The natural frequencies of the bunches are almost equal, however, since the frequency difference is due only to the effect of image terms. Quantitatively, the bunches will act independently when the interbunch contribution to the coherent frequency is small compared with the difference in bunch frequency: For all m and n, $|N_n W' G_{mn}| \ll (N_n - N_m)|U'|$. Since W' involves the resistivity and U' does not, U' is often much larger than W' and this condition is satisfied with only modest differences in the bunch numbers. In the extreme relativistic limit, however, U' vanishes since the electric and magnetic images tend to cancel. Dielectric loading and other similar devices can be used to keep $U' \gg W'$, as has been discussed in the literature;[11] for a smooth vacuum tank the criterion for independent bunch motion is, from Eqs. (3.23), (3.24), (3.3), and (A9) (taking $a = b/2$),

$$\frac{\Delta N}{N} \gg \frac{(\beta\gamma)^2 M^{\frac{1}{2}}}{4\pi} \left| \frac{L}{b(R/b)^{\frac{1}{2}}} \right| , \tag{4.7}$$

where N is the number of particles in one of the M bunches--each of length L --and ΔN is the required difference in number between bunches. More generally, the requirement on ΔN for independent bunch motion is:

$$\frac{\Delta N}{N} \gg (2M)^{\frac{1}{2}} \left| \frac{W'}{U'} \right| . \tag{4.8}$$

In the case of independent bunch motion, and when ν is below an integer, the motion is unstable except for Landau damping. The extensive discussion of LNS may now be applied, with $\lambda_{(n)}$ given by Eq. (4.6): For $|U'| \gg |W'G_{nn}|$ the threshold particle intensity for an instability is approximately proportional to U' and almost independent of W'. From Eq. (3.24) it is seen that the threshold intensity depends upon the tightness of bunching [$U' \propto L^{-1}$], whereas from Eq. (3.23) it is seen that the growth rate (when above threshold) is independent of the degree of bunching.

C. Equally Spaced Bunches of Equal Intensity

In some circumstances--usually for beams of extremely relativistic particles--the inequalities for independent bunch motion are strongly violated. It is then possible that a different approximation becomes valid; namely, that all the bunches are sensibly equal in intensity. The case of equally spaced bunches of equal intensity is one for which the solution of Eq. (3.20) is immediate.

Taking $N_n = N$, and $\theta_m = 2\pi\, m/N$, we observe that Eq. (3.20) can be written in the form

$$[NU' + NW'G_{nn} - \lambda]Q_n + NW' \sum_{m \neq n} G_{mn} Q_m = 0 , \qquad (4.9)$$

where $G_{mn} = G\left(\frac{2\pi}{M}(m-n), \nu\right)$ and G_{nn} is [from Eq. (3.22)] independent of n. Relabelling the sum, we obtain

$$[NU' + NW'G_{nn} - \lambda]Q_n + NW' \sum_{r=1}^{M-1} G\left(\frac{2\pi r}{M}, \nu\right) Q_{n+r} = 0 , \tag{4.10}$$

in which all the coefficients are independent of n. The matrix is cyclic and the solution well-known. In particular, let

$$a_{(m)} = e^{-2\pi mi/M} \tag{4.11}$$

be the mth of the M roots of unity. Then clearly an mth solution of the set of equations is

$$Q_{(m)n} = a_{(m)}^{n} , \tag{4.12}$$

with associated eigenvalue:

$$\lambda_{(m)} = NU' + NW'G_{nn} + NW' \sum_{r=1}^{M-1} G\left(\frac{2\pi r}{M}, \nu\right) a_{(m)}^{r} . \tag{4.13}$$

This may be written, from Eq. (3.22), in the form:

$$\lambda_{(m)} = NU' - NW' \frac{8\pi^{\frac{1}{2}}}{15} \left(\frac{L}{2\pi R}\right)^{\frac{1}{2}} \nu i + NW' \sum_{r=1}^{M} G\left(\frac{2\pi r}{M}, \nu\right) e^{-2\pi mri/M} . \tag{4.14}$$

By Eq. (A18) of the Appendix, the M eigenvalues are

$$\lambda_{(m)} = NU' + NW' \left\{ M^{\frac{1}{2}} G\left(2\pi, \frac{m+\nu}{M}\right) - \frac{8\pi^{\frac{1}{2}}}{15} \left(\frac{L}{2\pi R}\right) \nu i \right\} . \tag{4.15}$$

If we ignore the self-field term then, by Eq. (A5), the imaginary part is positive when $(\nu + m)/M$ lies between an integer and the next lower half-integer, and negative in the other half-interval. Therefore, if M is even, half the eigenvalues have positive and half have negative imaginary parts; if M is odd, one more has a positive imaginary part than a negative one (or vice versa). The only case where there is no eigenvalue with a positive imaginary part occurs when there is only one bunch and ν lies in the proper range.

The self-field term is stabilizing, of course, and could improve the situation, but it appears unlikely that machine parameters would be such as to have this term important. Also, finally, Landau damping can make some (or all) of the modes with $\mathrm{Im}\,\lambda > 0$ stable.

D. Numerical Calculations

A computer program has been prepared which obtains the eigenvalues and eigenvectors of Eqs. (3.20) [with the second term in Eq. (3.22) omitted] for given values of the ratio W'/U' and given distributions of bunch populations N_n, and for uniform spacing of the bunches. As is expected, it is found that if $W'/U \ll \Delta N/N$ and $W'/U' \gg \Delta N/N$, respectively, the results behave as described in Secs. IV.B and IV.C.

In the intermediate case, for $M = 12$ bunches and $\nu = 8.85$ (corresponding to the Brookhaven AGS), the real parts of the normalized

eigenvalues are plotted as functions of W'/U' in Fig. 1, for the case that the relative bunch populations vary in steps of 10^{-3} from 1.000 to 0.989. The largest value of W'/U' for which all modes have positive imaginary parts is 1.5×10^{-4}, which just about corresponds to replacing the inequality in Eq. (4.8) with an equality. For W'/U' four times as large, or larger, six modes have negative imaginary parts, as in the limit where all bunches are equally populated.

Note that, for W'/U' greater than the "threshold" value (1.5×10^{-4} in this case), the real part of the highest eigenvalue increases rapidly with W'/U' while the lower ones change much less. Examination of the corresponding eigenvectors discloses that this mode is a "collective" mode in which all bunches participate in the motion, with relative phases corresponding to that integral wave number which lies closest to ν (in this case, 9). In all the other modes some of the bunches participate far less than others, especially for relatively small W'/U'. For example, with $W'/U' = 5 \times 10^{-4}$, the amplitudes of oscillations of the various bunches vary from 1.0 to 0.58 in the "collective" mode, from 1.0 to 0.068 in the next highest mode, and from 1.0 to 1.3×10^{-5} in the mode whose eigenvalue has the smallest real part.

A more detailed study of these regularities lies beyond the scope of the present paper and will have to be left to future investigations.

E. Unequal Spacing

For the case where the bunches are not equally spaced, we have not succeeded in deriving any general theorem about the behavior of the solutions. Numerical studies show that with just two equally populated bunches there is always one stable and one unstable mode, no matter how close the two bunches are; when two of many bunches have the same population there is always at least one stable and one unstable mode. When there are just two bunches this property can be shown to be equivalent to the statement that

$$| \operatorname{Im} G(2\pi,\nu)| \; < \; |\operatorname{Im} [G(\theta,\nu)\, G(2\pi - \theta, \nu)]^{\frac{1}{2}} | \, ,$$

a relation which can be inferred for small θ from the approximations (A12) and (A13), but which we have not yet demonstrated for all θ.

This result indicates that, with bunches of finite length, there will always be unstable modes corresponding to relative motion within a bunch. We believe, however, that these modes will, in practice, be stabilized by Landau damping, as stated in the Introduction.

ACKNOWLEDGMENTS

Much of this work was performed during the Stanford Storage Ring Summer Study (1965) during which the authors enjoyed the hospitality of the Stanford Linear Accelerator Center. We would like to thank our hosts, and acknowledge many illuminating discussions with the various other participants. We also profited from conversations, with colleagues from many lands, held during the conference on storage rings in Novosibirsk, U.S.S.R. (March, 1965).

APPENDIX

In this Appendix we analyze the Bunch Function $G(\theta,\nu)$, which is defined by:

$$G(\theta,\nu) = 2\pi^{\frac{1}{2}} \sum_{k=0}^{\infty} \frac{e^{-i\nu(\theta+2\pi k)}}{(\theta + 2\pi k)^{\frac{1}{2}}} , \qquad (A1)$$

where k ranges over nonnegative integers, ν is nonintegral, and $0 < \theta < 2\pi$. Outside this range of θ, $G(\theta,\nu)$ is defined by the periodic continuation of Eq. (A1). There is evidently no difficulty in passing to the limit $\theta \to 2\pi$, and we define $G(\theta,\nu)$ by Eq. (A1) also for $\theta = 2\pi$.

1. Alternative representations.

Because of the general formula for the Gaussian integral

$$\int_0^\infty e^{-\alpha y} \frac{dy}{\sqrt{y}} = \left(\frac{\pi}{\alpha}\right)^{\frac{1}{2}} , \qquad (\mathrm{Re}\ \alpha \geq 0) , \qquad (A2)$$

we may rewrite Eq. (A1) in the form

$$G(\theta,\nu) = 2 \sum_{k=0}^{\infty} \int_0^\infty e^{-(y+i\nu)(\theta+2\pi k)}\, dy/y^{\frac{1}{2}} . \qquad (A3)$$

Interchanging summation and integration, we have a convenient integral representation:

$$G(\theta,\nu) = 2 \int_0^\infty \frac{e^{-(y+i\nu)\theta}}{1 - e^{-2\pi(y+i\nu)}} \frac{dy}{y^{\frac{1}{2}}} , \quad 0 < \theta \leq 2\pi . \qquad (A4)$$

The sign of the imaginary part of the Bunch Function plays a crucial role in stability analysis. From Eq. (A4) it is clear that, for $\theta = 2\pi$,

$$\text{Im } G(2\pi,\nu) = -2 \sin 2\pi\nu \int_0^\infty \frac{dy\, e^{-2\pi y}}{y^{\frac{1}{2}} \left|1 - e^{-2\pi(y+i\nu)}\right|^2} , \quad (A5)$$

and from the positive definite character of the integrand, the Im $G(2\pi,\nu)$ is negative for $I < \nu < I + \frac{1}{2}$ and positive for $I - \frac{1}{2} < \nu < I$ for any integer I . Since G is, by definition, periodic, it may be expanded in a Fourier series

$$G(\theta,\nu) = \sum_{n=-\infty}^{\infty} g_n \, e^{-in\theta} ,$$

with

$$g_n = \frac{1}{2\pi} \int_0^{2\pi} G(\theta,\nu)\, e^{in\theta} \, d\theta . \quad (A6)$$

Using Eq. (A4) we find

$$g_n = \frac{1}{\pi} \int_0^\infty \frac{dy}{[y + i(\nu - n)]y^{\frac{1}{2}}}$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{dz}{z^2 + i(\nu - n)}$$

(Equation A7 continued)

$$= \begin{cases} [1/(n-\nu)]^{\frac{1}{2}} & n > \nu \\ [-1/(\nu-n)]^{\frac{1}{2}} & n < \nu \end{cases} , \qquad (A7)$$

with the sign of the root chosen so as to make the real part of g_n positive in all cases.

Therefore an alternate form for G is

$$G(\theta,\nu) = \sum_{n=-\infty}^{\infty} \left(\frac{i}{n-\nu}\right)^{\frac{1}{2}} e^{-in\theta} \qquad (A8)$$

with the sign of $(i/n-\nu)^{\frac{1}{2}}$ chosen so that its real part is positive.

2. Summation formulas and approximations.

For computational purposes it is convenient to compute the sum (A1) over a finite number of terms and to estimate the remainder. We need only carry out the procedure leading to Eq. (A4) with the sum from $k = M$ to ∞ converted to an integral:

$$G(\theta,\nu) = 2\pi^{\frac{1}{2}} \sum_{m=0}^{M-1} \frac{e^{-i\nu(\theta+2\pi m)}}{[\theta+2\pi m]^{\frac{1}{2}}} + 2\int_0^{\infty} \frac{dy\, e^{-(\theta+2\pi M)(y+i\nu)}}{y^{\frac{1}{2}}\,[1-e^{-2\pi(y+i\nu)}]} . \qquad (A9)$$

For large M, the integral in Eq. (A9) becomes very small and may be approximated by an asymptotic formula. Writing

$$\frac{1}{1 - e^{-2\pi(y+i\nu)}} = \frac{1}{[1 - e^{-2\pi i\nu}]} \sum_{\ell=0}^{\infty} \left\{ \frac{e^{-2\pi i\nu}(e^{-2\pi y} - 1)}{[1 - e^{-2\pi i\nu}]} \right\}^{\ell} , \tag{A10}$$

we can easily generate such a formula. In particular, the first two terms yield

$$2 \int_0^{\infty} \frac{dy\, e^{-(\theta+2\pi M)(y+i\nu)}}{y^{\frac{1}{2}} [1 - e^{-2\pi(y+i\nu)}]} \approx \frac{2\pi^{\frac{1}{2}}\, e^{-(\theta+2\pi M)i\nu}}{(1 - e^{-2\pi i\nu})(\theta + 2\pi M)^{\frac{1}{2}}}$$

$$\times \left[1 - \frac{\pi\, e^{-2\pi i\nu}}{(1 - e^{-2\pi i\nu})(\theta + 2\pi M)} \right] . \tag{A11}$$

From Eq. (A9) we can readily obtain limiting values of the Bunch Function. Thus for $0 < \theta \leq 2\pi$,

$$G(\theta,\nu) = 2\pi^{\frac{1}{2}} \frac{e^{-i\nu\theta}}{\theta^{\frac{1}{2}}} + 2\pi^{\frac{1}{2}} \sum_{m=1}^{\infty} \frac{e^{-i\nu(\theta+2\pi m)}}{[\theta + 2\pi m]^{\frac{1}{2}}} ,$$

and for $\theta \ll 2\pi$ we can neglect θ in the sum to obtain

$$G(\theta,\nu) \approx 2(\pi/\theta)^{\frac{1}{2}}\, e^{-i\nu\theta} + G(2\pi,\nu), \quad \text{for } \theta \ll 2\pi . \tag{A12}$$

For $\theta < 0$, $G(-|\theta|,\nu) = G(2\pi - |\theta|,\nu)$, and since $G(\theta,\nu)$ varies slowly for $\theta \to 2\pi$, we have:

$$G(-|\theta|,\nu) \approx G(2\pi,\nu) , \qquad \theta << 2\pi . \tag{A13}$$

Numerical values[12] of $G(\theta,\nu)$, obtained employing Eq. (A9), are displayed in Figs. 2 and 3. Values of the Bunch Function outside the range displayed can be obtained from the relation

$$G(\theta + 2\pi m,\nu + n) = e^{-in\theta} G(\theta,\nu) , \tag{A14}$$

valid for all integers m and n , which follows immediately from the definition of $G(\theta,\nu)$ [see Eq. (A8)].

3. Addition theorem.

We wish to evaluate the sum S defined as

$$S = \sum_{r=1}^{M} e^{-2\pi mri/M} \, G\left(\frac{2\pi r}{M} , \nu\right) , \tag{A15}$$

where $m,r,$ and M are integers. Employing the representation of Eq. (A4), and interchanging the finite summation and the integration yields

$$S = 2 \int_0^{\infty} \frac{dy}{y^2 \left[1 - e^{-2\pi(y+i\nu)}\right]} \sum_{r=1}^{M} e^{-2\pi[(m+\nu)i+y]r/M} . \tag{A16}$$

The summation is immediate and yields

$$S = 2 \int_0^\infty \frac{dy\, e^{-2\pi[(m+\nu)i+y]/M} \left[1 - e^{-2\pi[(m+\nu)i+y]}\right]}{y^{\frac{1}{2}}\left[1 - e^{-2\pi(y+i\nu)}\right]\left[1 - e^{-2\pi[(m+\nu)i+y]/M}\right]} . \tag{A17}$$

Since $\exp(-2\pi m) = 1$, we have--after replacing y/M with $\bar{y}$--

$$S = 2M^{\frac{1}{2}} \int_0^\infty \frac{d\bar{y}\, e^{-2\pi[\bar{y}+(m+\nu)i/M]}}{\bar{y}^{\frac{1}{2}}\left[1 - e^{-2\pi[\bar{y}+(m+\nu)i/M]}\right]} , \tag{A18}$$

which, on comparison with Eq. (A4), yields $S = M^{\frac{1}{2}}\, G(2\pi, \frac{m+\nu}{M})$.

FOOTNOTES AND REFERENCES

1. L. J. Laslett, V. K. Neil, and A. M. Sessler, Rev. Sci. Instr. 36, 436 (1965).

2. E. D. Courant, Proceedings of the Particle Accelerator Conference, Washington, D. C., March 10-12, 1965, IEEE Trans. Nucl. Sci. NS-12 [3], p. 550.

3. E. D. Courant and A. M. Sessler, in Storage Ring Summer Study, 1965, on Instabilities in Stored Particle Beams, a Summary Report, SLAC-49, Aug. 1965, p. 36.

4. A. M. Sessler, Instabilities of Relativistic Particle Beams, Lawrence Radiation Laboratory Report UCRL-16440, Oct. 1965 (unpublished). Submitted for publication to the Proceedings of the International Conference on High Energy Accelerators, Frascati, 1966.

5. N. S. Dekonski,) and A. N. Skrinski, Coherent Instability of Bunches of Charged Particles, Institute of Nuclear Physics Report, Novosibirsk, USSR, 1965 (unpublished).

6. K. W. Robinson, in SLAC-49, Aug. 1965 (see Ref. 3), p. 32.

7. L. J. Laslett and A. M. Sessler, in SLAC-49, Aug. 1965 (see Ref. 3), p. 23.

8. P. L. Morton, V. K. Neil, and A. M. Sessler, Wake Fields of a Pulse of Charge Moving in a Highly Conducting Pipe of Circular Cross Section, Lawrence Radiation Laboratory Report UCRL-16740, Feb. 1966 (unpublished). To be published in J. Appl. Phys.

9. In the limiting process we must observe the condition $\nu L/2\pi R \ll 1$ required to obtain Eq. (3.19). This condition is satisfied if $L \ll 2\pi\lambda_\beta$.

10. The result for rectilinear motion has been obtained in a somewhat roundabout manner; the reader may welcome the following more straightforward argument. For a single particle in a straight pipe the Bunch Function, as defined by Eq. (3.9), becomes modified in an obvious way; namely, the periodic delta function of Eq. (3.7) is replaced with an ordinary delta function with the result that [see Eq. (A8)]

$$G(\theta,\nu) \rightarrow G_{s.p.}(\theta,\nu) \equiv \int_{-\infty}^{\infty} dk \left(\frac{1}{k-\nu}\right)^{\frac{1}{2}} e^{-ik\theta} .$$

The Straight Pipe Bunch Function may readily be evaluated by contour integration with the result

$$G_{sp}(\theta,\nu) = \begin{cases} 2\left(\frac{\pi}{\theta}\right)^{\frac{1}{2}} e^{-i\nu\theta} , & \theta > 0 \\ 0 , & \theta < 0 . \end{cases}$$

This is seen to be exactly the same as the $m = 0$ term in Eq. (A9); dynamical analysis will consequently lead to a result analogous to Eq. (4.4), but with the term $G(2\pi,\nu)$ absent.

The argument just given is not, however, immune to criticism: In the integration over k there is a range where k is near ν and

one of the criteria for valid field expressions, namely $\omega \gg c^2/d^2\sigma$, is not satisfied. One can answer the criticism by replacing the field expressions of Eq. (3.3) with more generally valid expressions, given in Ref. 8, and then evaluating $G_{sp}(\theta,\nu)$. This is a very tedious calculation--which has not been performed--but, because the range of invalidity of Eq. (3.3) is exceedingly narrow one expects only very small corrections to $G_{sp}(\theta,\nu)$.

Note that the derivation given in the body of the paper is not subject to criticism, since for any large (but not fantastically large) R , the sum employed in the definition of $G(\theta,\nu)$ completely avoids contributions from the small region where the skin depth exceeds the vacuum chamber wall thickness.

11. A. M. Sessler, in SIAC-49, Aug. 1965 (see Ref. 3), p. 8.

12. M. Allen, M. Lee, and J. Rees, in SIAC-49, Aug. 1965 (see Ref. 3), p. 49. We wish to thank these authors for supplying us with the numerical results presented here.

FIGURE CAPTIONS

Fig. 1. Real parts of the eigenvalues of the matrix defined by Eq. (3.20) as a function of W'/U' for $M = 12$ bunches, $\nu = 8.85$, and bunch populations ranging from 1.00 to 0.989 in steps of 0.001. The dots are cases in which the imaginary part of the eigenvalues are positive; crosses correspond to negative imaginary parts.

Fig. 2. Values of the real part of the Bunch Function $G(\theta,\nu)$ for $0 < \theta < 2\pi$ and $\nu = 0.1, 0.9\ (0.2)$. The function is defined by Eq. (A1). (See Fig. 2a and Fig. 2b)

Fig. 3. Values of the imaginary part of the Bunch Function $G(\theta,\nu)$ for $0 < \theta < 2\pi$ and $\nu = 0.1, 0.9\ (0.2)$. The function is defined by Eq. (A1). (See Fig. 3a and Fig. 3b)

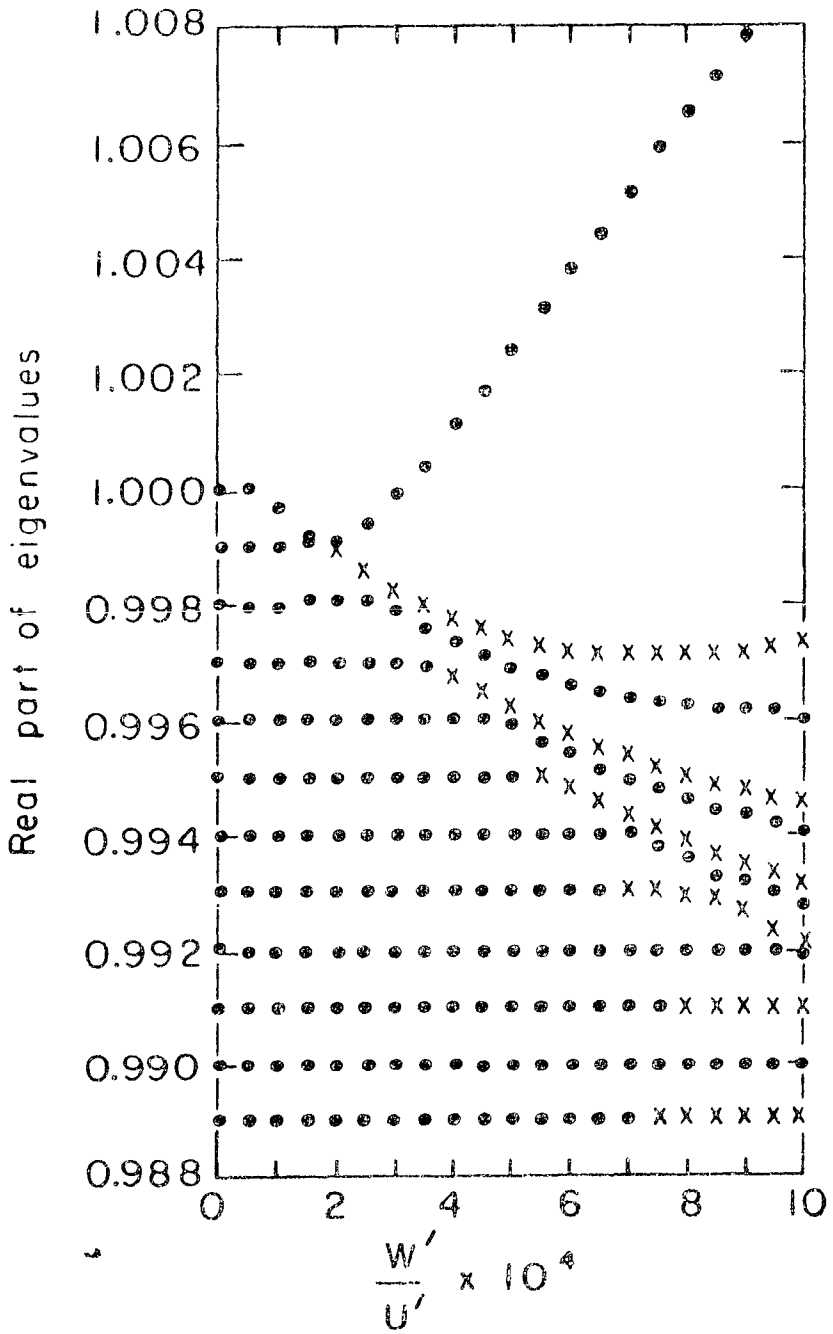
1.008
1.006
1.004
1.002
1.000
0.998
0.996
0.994
0.992
0.990
0.988
Real part of eigenvalues
0
2
4
6
8
10
W'/U' x 10⁴
MUB-10862

Fig. 1

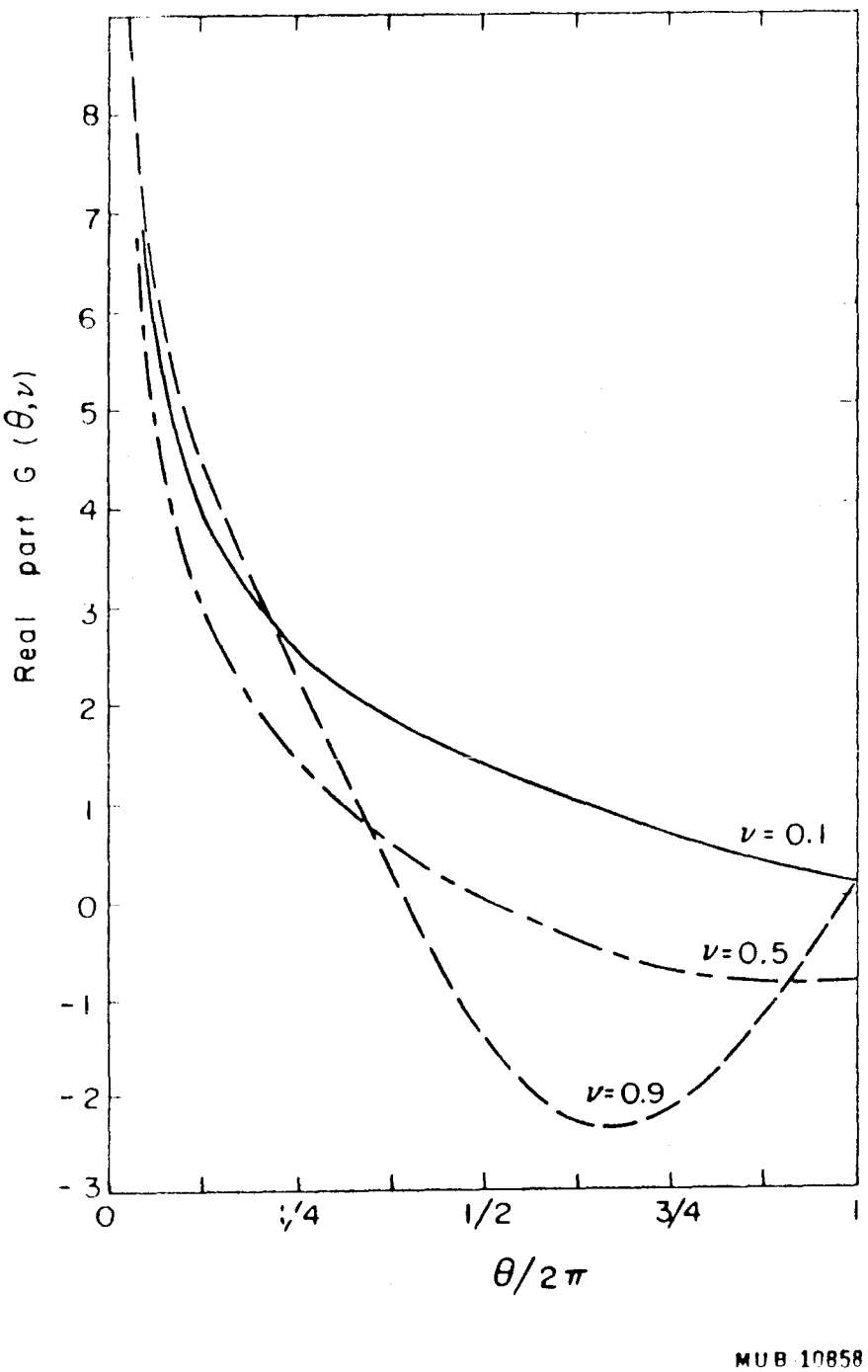
Real part G (θ,ν)
8
7
6
5
4
3
2
1
0
-1
-2
-3
0
1/4
1/2
3/4
1
θ/2π
ν = 0.1
ν=0.5
ν=0.9
MUB 10858

Fig. 2a

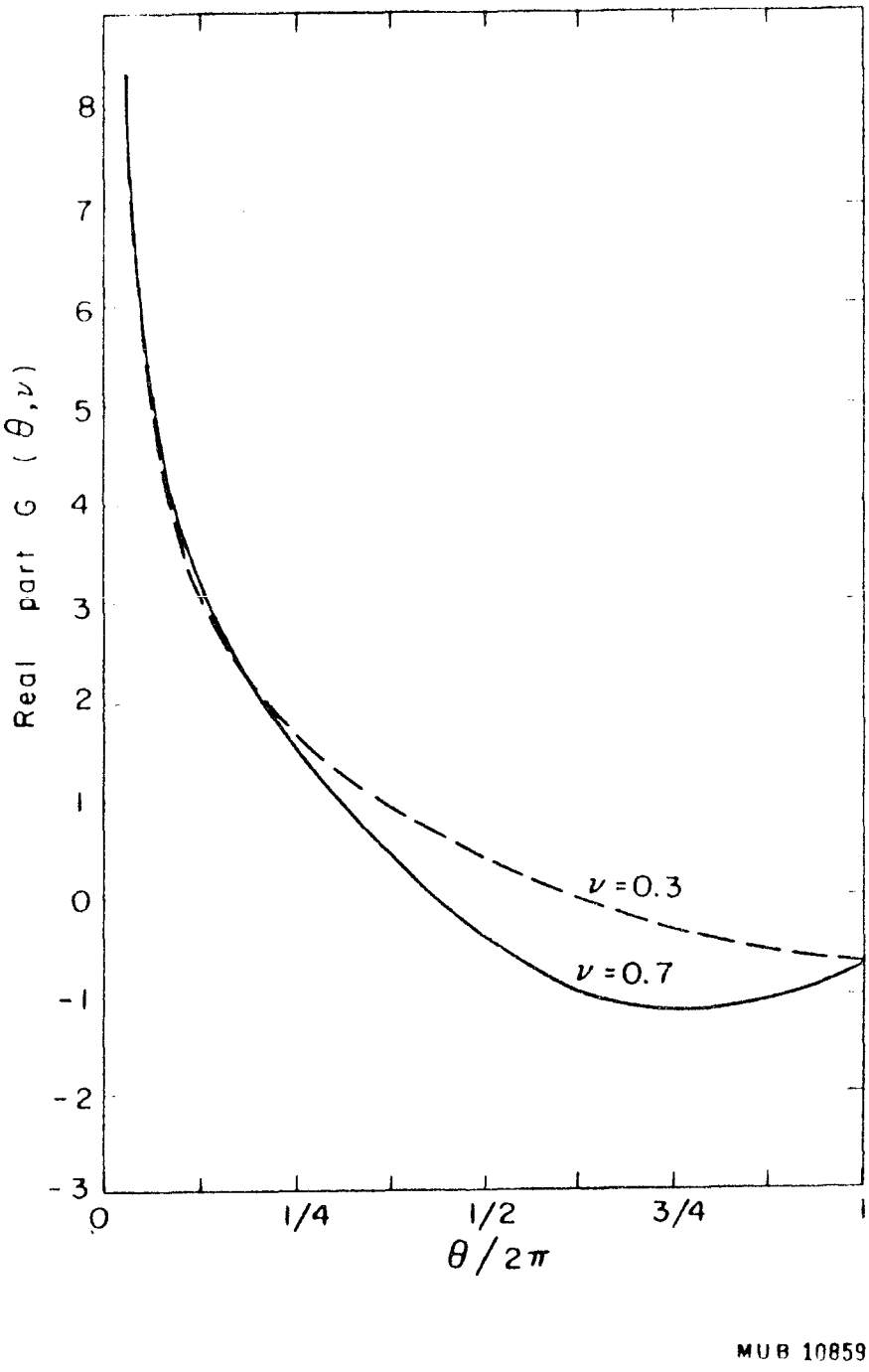

Fig. 2b

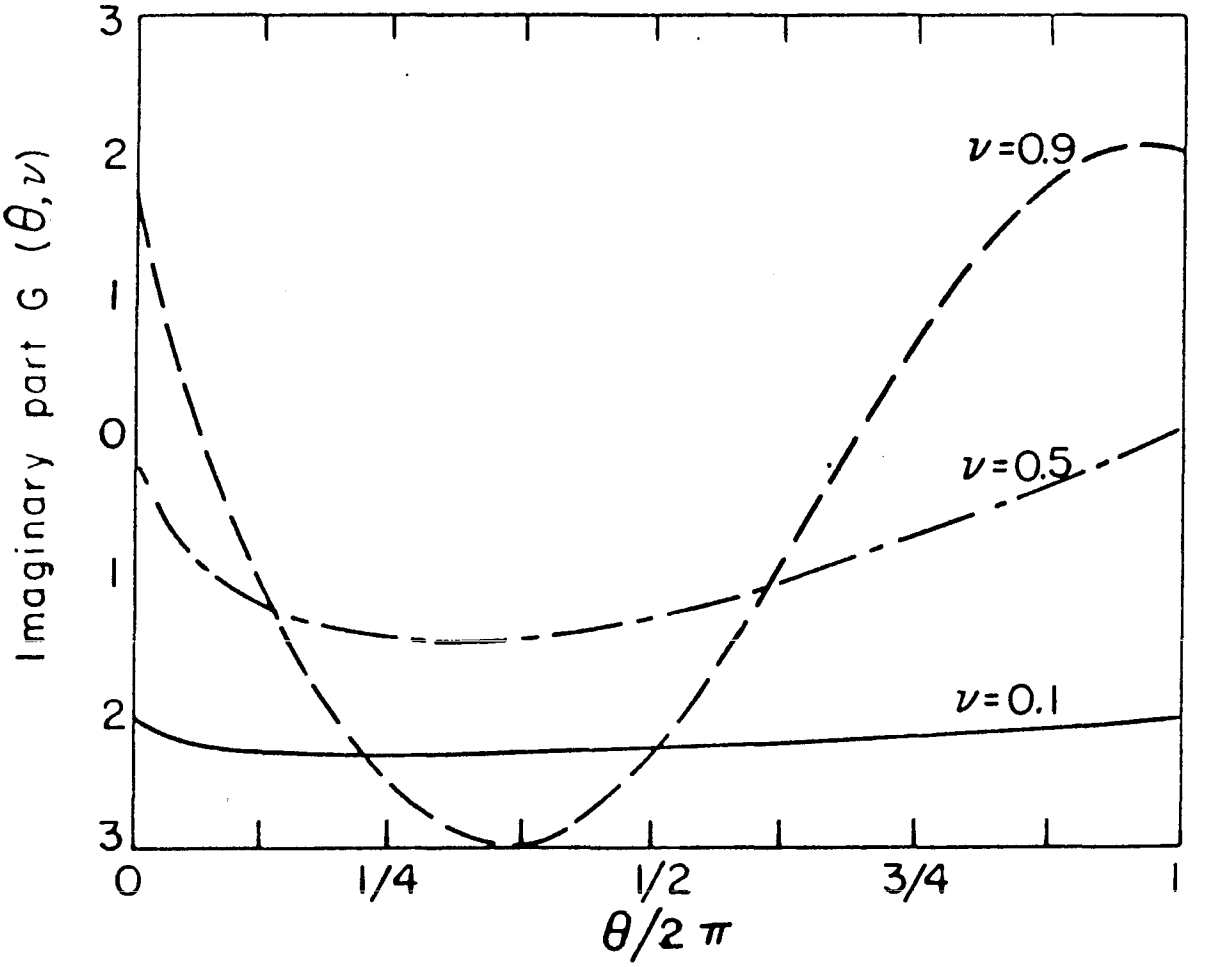
Imaginary part G (θ, ν)
3
2
1
0
1
2
3
0
1/4
1/2
3/4
1
θ/2 π
ν=0.9
ν=0.5
ν=0.1
MUB-10860

Fig. 3a

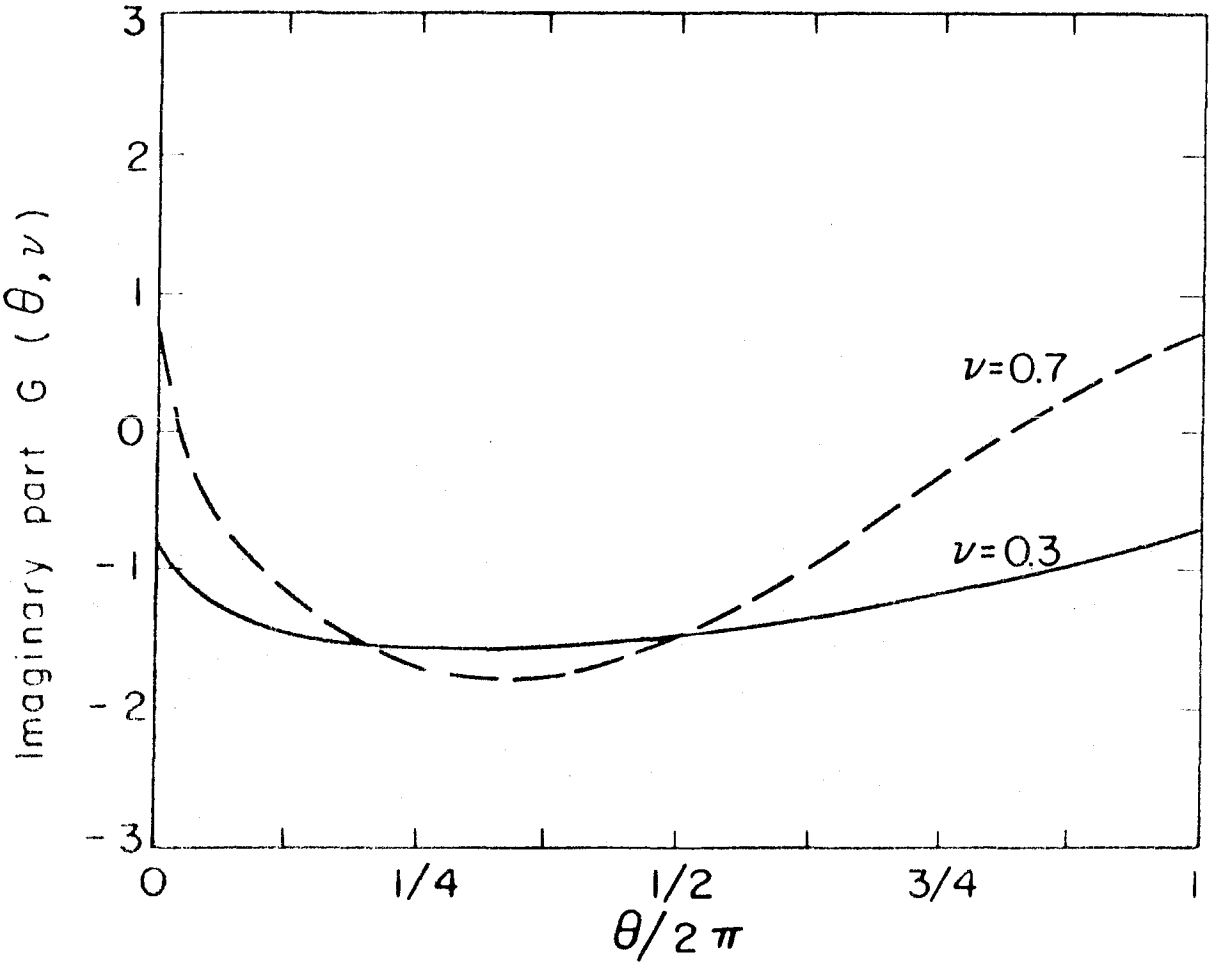

Fig. 3b