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Fragility of CVaR in portfolio optimization

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Abstract We evaluate conditional value-at-risk (CVaR) as a risk measure in data-driven portfolio optimization. We show that portfolios obtained by solving mean-CVaR and global minimum CVaR problems are unreliable due to estimation errors of CVaR and/or the mean, which are aggravated by optimization. This problem is exacerbated when the tail of the return distribution is made heavier. We conclude that CVaR, a *coherent* risk measure, is fragile in portfolio optimization due to estimation errors.

Keywords: portfolio optimization, conditional value-at-risk, expected shortfall, TailVaR, coherent measures of risk, mean-CVaR optimization, mean-variance optimization, global minimum CVaR, global minimum variance, estimation errors

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1 Introduction

Conditional value-at-risk[1] (abbrev. CVaR) has gained considerable attention in the financial risk management literature as a viable risk measure. CVaR at level β refers to the conditional expectation of losses in the top $100(1-\beta)\%$, and is anticipated to be a superior risk measure to value-at-risk (VaR), which, at level β, refers to the threshold level for losses in the top $100(1-\beta)\%$. At a time when the use of VaR is partly blamed for the 2007-2008 financial crisis[2], CVaR is more appealing than VaR because it takes into account the contribution from the very rare but very large losses. Formally, CVaR is a "coherent" risk measure, in that it satisfies [Pflug (2000), Acerbi and Tasche (2001)] the four coherence axioms[3] of Artzner et al. (1999).

There have been many studies on CVaR once its coherence was established. In statistics/econometrics, there have been studies about CVaR estimation [Scaillet (2004), Brown (2007) and Chen (2008), to name a few]. Another line of work has been in incorporating CVaR as a risk measure in portfolio optimization, led by Rockafellar and Uryasev (2000) and Krokhmal, Palmquist and Uryasev (2002) [and independently Bertsimas, Lauprete and Samarov (2004)], who developed numerical methods for computing optimal portfolios with CVaR in the objective or in the constraint. These papers demonstrate CVaR portfolio optimization from a purely data-driven approach, i.e. the investor optimizes her portfolio based on empirical estimates of mean and CVaR.

If the underlying return distribution is multivariate normal and the investor knows its parameters, then the portfolio that minimizes CVaR with an expected return R is equivalent to the portfolio that minimizes variance (or VaR) with the same expected return R [Rockafellar and Uryasev (2000), De Giorgi (2002) and Bertsimas et al. (2004)]. As a consequence, the frontiers of mean-variance and mean-CVaR portfolios coincide if plotted on the same scale. The same authors also consider the case where the investor does not know the model and computes optimal portfolios purely based on historical data. Using real market data, they empirically show that the *empirical frontiers*[4] of mean-variance and mean-CVaR portfolios are very similar for the (different) assets and time periods under consideration. This merely indicates that the data used were approximately multivar! iate normal, and deems

[1] Also known as Mean/Expected Shortfall, Mean/Expected Excess Loss, or Tail VaR.

[2] See the New York Times article Risk Mismanagement (Jan. 2, 2009) by Joe Nocera, for an account of the use of VaR as a risk measure in the financial industry.

[3] Translation invariance, subadditivity, positive homogeneity and monotonicity. VaR violates subadditivity [Artzner, Delbaen, Eber and Heath (1999), Embrechts, Resnick and Samorodnitsky (1999)], i.e. diversification can result in greater risk.

[4] See Sec. (2) for a precise definition.

the use of CVaR over variance in a 'normal' market unnecessary[5]. Nevertheless, the proponents of CVaR argue that its usefulness will be evident when the return distributions deviate from normality; in particular when they have fat left tails, as is often the case in 'crisis' periods [Litzenberger and Modest (2008)]. However, to date, no studies on the use of CVaR as a risk measure in such a market have been done to validate this claim.

One important point that has been omitted by Artzner et al. (and subsequent works on CVaR) is the role of estimation errors in the computation of a risk measure, and their effect on decision-making, such as portfolio optimization. CVaR may be coherent, but a large number of observations are needed to estimate it accurately because it is a tail statistic. However, in financial risk management, historical data older than 5 years are rarely used because underlying distributions are non-stationary over a long period of time. Thus from a practical perspective, data-driven portfolio optimization that involves estimated statistics is subject to estimation errors that may be very significant. Such observations in the context of mean-variance optimization have been made by Jorion (1985), Michaud (1989), Broadie (1993) and Chopra and Ziemba (1993), where the quality of the solution was shown to be greatly affected by estimati! on errors of the mean. We believe that an understanding of the impact of estimation errors in CVaR (as well as other tail risk measures) is important given its increasing popularity.

The goal of this paper is thus to provide an objective analysis of the use of CVaR as a risk measure in data-driven portfolio optimization. Specifically, we set out to answer the following questions:

- How do estimation errors affect data-driven portfolio optimization that minimize CVaR as an objective?
- Is CVaR a reliable risk measure, in terms of estimation errors, for portfolio optimization in a heavy-tailed market?

To address these questions, we look at the mean-CVaR frontiers associated with the solution of empirical mean-CVaR problems constructed from data generated under different market models. We will first show that these empirical mean-CVaR frontiers vary wildly when both mean and CVaR are empirically estimated (call this problem EMEC, for *empirical mean-empirical CVaR*). Of course, such a variation may be due to the well-known problem of estimation errors of the mean. To isolate the effect of the mean, we also consider (i) global minimum CVaR portfolio optimization (GMC) and (ii) mean-CVaR problem where the true mean is known (TMEC for

[5] Of the four papers, only De Giorgi correctly identifies this point.

true mean-empirical CVaR). For comparative purposes, we also analyze *empirical mean-empirical variance* (EMEV), *global minimum variance* (GMV) and *true mean-empirical variance* (TMEV) problems.

To see the effect of the tail behavior of return distributions, we do the analysis mentioned above for three different market scenarios: excess return distributions are multivariate normal ($\mathcal{M}1$), mixture of multivariate normal and negative exponential tail ($\mathcal{M}2$), or mixture of multivariate normal and one-sided power tail ($\mathcal{M}3$). Such mixture distributions represent a normal market that undergoes a shock with a small probability, with increasing heaviness in the tail.

The details of the evaluation methodology can be found in Sec. (2), of the optimization problems in Sec. (3) and simulation results and discussion in Sec. (4).

2 Evaluation Methodology

We consider single-period portfolio optimization with n risky assets. We denote the excess returns of the assets by the random vector $\mathbf{X} = [X_1, \ldots, X_n]'$. To see how estimation errors affect EMEC portfolio optimization, we employ the following procedure[6]:

1. Choose β (usually, 95% or 99%) and a model $\mathcal{M}$ for the distribution of the underlying assets. For example, $\mathcal{M}$ could be a multivariate normal distribution with parameters $(\boldsymbol{\mu}, \mathbf{V})$.

2. Simulate asset returns $\mathbf{D} = [\mathbf{x}_1 \ldots, \mathbf{x}_q]$ for a time period of size q under $\mathcal{M}$. This is the historical data the investor observes.

3. Fix a portfolio return level R. Compute the optimal solution of the EMEC$(R; \mathbf{D}, \beta)$ problem (see Sec. (3) for details on the optimization). For the same data set $\mathbf{D}$, vary the range of R to compute a family of optimal portfolios. Note this family is random since the input data $\mathbf{D}$ is random.

4. For each portfolio in the family of optimal portfolios computed in Step 3, compute its expected (excess) return and CVaR under the true model $\mathcal{M}$, and plot the resulting mean-CVaR values. This generates a curve, the *empirical frontier*, representing the mean and CVaR of the portfolios computed in Step 3 under the true model $\mathcal{M}$. We do this because we are interested in the

[6] We employ a similar procedure for TMEC/TMEV and GMC/GMV portfolio optimization; the only difference is in the optimization problem we solve in Step 3.

true performance of the EMEC portfolios. Note the empirical frontier is also random.

5. Repeat Steps 3-4 for EMEV($R; \mathbf{D}, \beta$) portfolio optimization.

6. Repeat Steps 2-5 many times (50 in our study), each time with fresh input data $\mathbf{D}$. We can now compare the distribution of EMEC and EMEV empirical frontiers.

To see the effect of the tail of return distributions, we consider market models $\mathcal{M}$ with increasingly heavier one-sided tail; the exact characterizations of these markets are in Sec. (4.2). Next, we provide details of the optimization.

3 Data-driven Portfolio Optimization

3.1 Data-driven mean-variance portfolio optimization

The optimal data-driven mean-variance portfolio $\boldsymbol{\pi}^*_{MV}$ is given by solving the quadratic program:

$$\min_{\boldsymbol{\pi}} \quad \boldsymbol{\pi}' V^{(q)} \boldsymbol{\pi} \tag{1a}$$

$$s.t. \quad \sum_{i=1}^{n} \pi_i = 1 \tag{1b}$$

$$\boldsymbol{\pi}' \mathbf{g} \geq R, \tag{1c}$$

where $V^{(q)}$ is the sample covariance matrix computed from the observed data. For the EMEV problem, $\mathbf{g} = q^{-1} \sum_{i=1}^{q} \mathbf{x}_i$, i.e. the sample mean, and for the TMEV problem, $\mathbf{g} = E(\mathbf{X})$. For the GMV problem, we omit the constraint (1c).

3.2 Data-driven mean-CVaR portfolio optimization

The optimal data-driven mean-CVaR portfolio $\boldsymbol{\pi}^*_{CVaR}$ is given by solving:

$$\min_{\boldsymbol{\pi}, \alpha} \quad \alpha + \frac{1}{q(1-\beta)} \sum_{i=1}^{q} (-\boldsymbol{\pi}' \mathbf{x}_i - \alpha)^+ \tag{2a}$$

$$s.t. \quad \sum_{i=1}^{n} \pi_i = 1 \tag{2b}$$

$$\boldsymbol{\pi}' \mathbf{g} \geq R, \tag{2c}$$

where $\mathbf{D} = [\mathbf{x}_1, \ldots, \mathbf{x}_q]$ are vectors of observed asset returns. We know from Rockafellar and Uryasev (2000) that (2) can be transformed into a linear program. Again, for the EMEC problem, $\mathbf{g} = q^{-1} \sum_{i=1}^{q} \mathbf{x}_i$, and for the TMEC problem, $\mathbf{g} = E(\mathbf{X})$, and for the GMC problem, we omit the constraint (2c).

The objective (2a) is a sample estimate of the following[7]:

Theorem (Rockafellar and Uryasev (2000)). *Let* $\mathbf{X}$ *be a vector of iid asset returns, with a continuous cdf and a density function* f*. Then CVaR at level* β *of a portfolio* $\boldsymbol{\pi}$ *can be expressed as an optimization problem:*

$$\text{CVaR}(\boldsymbol{\pi}; \beta) = \min_{\alpha \in R} \;\; \alpha + \frac{1}{1-\beta} \int_{\mathbf{x} \in \mathbf{R}^n} (-\alpha - \boldsymbol{\pi}'\mathbf{x})^+ f(\mathbf{x}) d\mathbf{x}.$$

4 Results and Discussion

4.1 Returns ∼ multivariate normal

4.1.1 Model Description

We first consider the case where excess returns of $n = 5$ assets have a multivariate normal distribution:

$$\mathbf{X} \sim N(\boldsymbol{\mu}, \boldsymbol{V}) \qquad (\mathcal{M}1)$$

where[8]

$$\begin{aligned} \boldsymbol{\mu} &= \begin{bmatrix} 26.11, & 25.21, & 28.90, & 28.68, & 24.18 \end{bmatrix} \times 10^{-4}, \\ \boldsymbol{V} &= \begin{bmatrix} 3.715, & 3.730, & 4.420, & 3.606, & 3.673 \\ 3.730, & 3.908, & 4.943, & 3.732, & 3.916 \\ 4.420, & 4.943, & 8.885, & 4.378, & 5.010 \\ 3.606, & 3.732, & 4.378, & 3.930, & 3.789 \\ 3.673, & 3.916, & 5.010, & 3.799, & 4.027 \end{bmatrix} \times 10^{-4}. \end{aligned}$$

The histogram for the 10,000 sample returns of X_1 is shown in Fig. (2a).

[7] For a fixed portfolio $\boldsymbol{\pi}$, the sample estimate (2a) is a non-parametric estimator for the portfolio's CVaR. It can be shown that this estimator is related to another popular non-parametric estimator of CVaR, and both are weakly consistent and asymptotically normal with data size q. Note, however, that this does not say that the empirical CVaR of the optimized portfolio is asymptotically normal. See Lim, Shanthikumar and Vahn (n.d.) for details.

[8] The vector $\boldsymbol{\mu}$ and the matrix $\boldsymbol{V}$ are the sample mean and covariance matrix of 299 monthly excess returns of 5 stock indicies (NYA, GSPC, IXIC,DJI,OEX) from the period spanning August 3, 1984 to June 1, 2009.

4.1.2 Empirical mean-Empirical CVaR (EMEC) problem

As described in Sec. (2), we generate data using model $\mathcal{M}1$ and solve for EMEC and EMEV portfolios for a number of target expected returns R. We then simulate the returns for these portfolios under model $\mathcal{M}1$ to compute the true expected return and true CVaR, and generate the empirical frontiers in Fig. (1). Note we could have equally chosen true variance as the common risk scale. We also emphasize that the frontiers are not observed by the investor herself; they show the variability in the performance of empirical portfolio optimization under the true model.

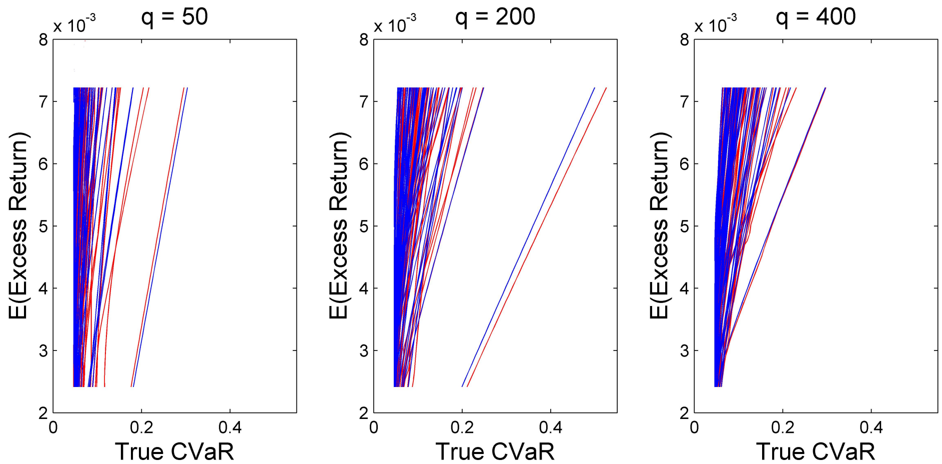

Figure 1: The frontiers of empirical mean-empirical CVaR (red) and empirical mean-empirical variance (blue) portfolios under model $\mathcal{M}1$. All scales are bps/mth.

The EMEC and EMEV frontiers both vary wildly; for example, with $q = 50$ (≈ 4 years), the range of expected excess return of a portfolio with CVaR $= 1000$ bps/mth per dollar invested[9] is greater than the range 25–72 bps/mth, which translates to a relative excess return ratio greater than $(1.0072^{12} - 1)/(1.0025^{12} - 1) \approx 300\%$ per year. The performance of EMEC and EMEV portfolios are very similar in that the positions and the spread of the frontiers are very similar. This is not very surprising, since, as previously mentioned, theoretical mean-variance and mean-CVaR problems yield equivalent solutions if the underlying asset returns have a multivariate normal distribution. Thus in a normal market, the EMEC problem is subject to large estimation errors of the mean, as is the EMEV problem. The same shortcomings apply to markets with heavier tails, as estimating the mean becomes more difficult as the underlyi! ng distribution becomes more irregular.

[9] We will omit 'per dollar invested' hereafter.

However, we cannot attribute the variation of the empirical frontiers solely to errors of the mean. What if we remove the expected return constraint, or assume the investor knows the true mean of asset returns? Removing the expected return constraint is a natural extension of EMEC and EMEV problems[10], and allows us to compare errors of CVaR and variance without errors of the mean. On the other hand, assuming the investor knows the true mean is an idealization of the situation where the investor has a good estimate of the mean obtained from alternatives to the sample average, e.g. based on CAPM [Huang and Litzenberger (1988)], forecasts of exceptional returns [Grinold and Kahn (2000)], factor models [Fama and French (1993)], or incorporating investor knowledge via Black-Litterman [Black and Litterman (1991)]. In this regard, the TMEC model will allow us to evaluate whether the hard work put into estimating the mean can be destroyed by the errors associated with estimating CVaR. Hence, for the rest of the paper, we consider (i) GMC/GMV and (ii) TMEC/TMEV problems.

4.1.3 Global minimum CVaR (GMC) problem

We plot expected return vs. true CVaR of portfolios that solve the GMC problem (red) for $\mathcal{M}1$ in Fig. 4(a). For comparative purposes, GMV portfolios are also plotted (blue). In Table 1, we give the ranges for CVaR and expected return values corresponding to the solutions of GMC/GMV problems from our 50 simulations. We observe that GMV portfolios outperform GMC portfolios in that most blue stars are found on the left of the red stars (i.e. more accurate) and also are less spread out (i.e. more precise). The GMC portfolios are not precise at all; e.g. for our 50 simulations, when $q = 50$, the range of expected return is 9.15–43.2 bps/mth (481% per year relative difference) and for CVaR is 465–723 bps/mth.

What are the origins of the variation in the GMC empirical frontiers? In order to distinguish between the effects of inherent estimation errors and the optimization procedure, we plot in Fig. 3(a) *Perceived CVaR* and *Random Empirical CVaR* against *True CVaR*. By *Perceived CVaR* we mean the optimal objective from solving the GMC problem, i.e. the CVaR value perceived by the investor. By *Random Empirical CVaR* we mean the empirical CVaR[11] of portfolios that are randomly drawn[12] from the set of portfolios that satisfy the constraint (2b). *True CVaR* refers to the true CVaR value of a portfolio. Thus the scatter plot of Random Empirical CVaR

[10]The GMV problem is currently being studied as an alternative to the EMEV problem [see Jagannathan and Ma (2003), DeMiguel, Garlappi, Nogales and Uppal (2009)].

[11]Defined by (2a).

[12]Note the drawing procedure was not uniform.

against True CVaR (blue) gives an indication of the inherent estimation errors *before* optimization, and the scatter plot of Perceived CVaR against True CVaR (re! d) gives an indication of how the optimization affects the already present estimation errors. We observe that empirical CVaR values of randomly drawn, unoptimized portfolios are slightly biased in the direction of underestimating the true CVaR[13]. However, this underestimation is aggravated by the optimization procedure, as can be seen by the position of the red dots— they are generally further below the black line (perfect estimation) than the blue dots. In Table 1, we highlight this phenomenon by listing the ranges of Perceived CVaR as well as the true CVaR for the GMC problem.

4.1.4 True mean-Empirical CVaR (TMEC) problem

Figure 5(a) shows TMEV empirical frontiers (blue) and TMEC($\beta = 0.99$) empirical frontiers (red) for different data sizes $q = 50, 200, 400$ (months). We also plot the theoretical mean-variance (equivalently, mean-CVaR) frontiers in green (left-edge of the blue curves). In Table 2, we give the ranges for CVaR and expected return values corresponding to the solutions of TMEC/TMEV problems from our 50 simulations. In all three instances, the spreads of the TMEV empirical frontiers are significantly smaller than the TMEC empirical frontiers, and the TMEV empirical frontiers lie on the left-most side of the TMEC curves, closer to the theoretical frontier. Thus a portfolio manager who wishes to find the optimal TMEC portfolio should solve the TMEV problem to get a portfolio with higher accuracy and precision. For example, a manager with risk level CVaR = 1000 bps/mth and $q = 50$ using TMEV optimization generates an average excess re! turn of 56.6 bps/mth, but a manager with the same risk level using TMEC optimization generates 52.3 bps/mth— $\approx 8.5\%$ per year higher excess return, on average. Furthermore, the TMEV manager is more reliable; e.g. for our 50 simulations and $q = 50$, the range of true CVaR is 613–691 bps/mth whereas for the TMEC manager it is 621–942 bps/mth for the same expected return of 40 bps/mth.

As with the GMC problem, the variation in the TMEC portfolios is due to inherent estimation errors of CVaR coupled with the effect of optimization. In Fig. 3(b) we plot the Perceived CVaR of GMC portfolios in red, and empirical CVaR of random portfolios that satisfy (2b) in blue; notice the red dots are generally further below the black line (perfect estimation) than the blue dots. This is further verified by the CVaR (per) column in Table 2. This tells us that the TMEC investor can substantially underestimate the true CVaR value of her 'optimal' portfolio and be

[13] This is because the sample estimator from (2a) is biased in the direction of underestimation [Lim et al. (n.d.)].

exposed to more risk than suggested by her perceived value.

Lastly, we comment on the difference between the performance of TMEC and TMEV empirical frontiers. Theoretically, the mean-CVaR and mean-variance empirical frontiers coincide; thus the discrepancy in the observed performance suggests that estimation errors of CVaR are more significant than of variance. This is not surprising since the mean vector and the covariance matrix are minimal sufficient statistics of a multivariate normal model. Thus minimizing portfolio variance only contains errors of the covariance matrix, which are less significant than errors of the mean, whereas minimizing portfolio CVaR contains errors of both the mean vector and the covariance matrix.

4.2 Returns $\sim$ multivariate normal + heavy loss tail

Recall that one of our objective is to evaluate CVaR portfolio optimization in markets with heavier tails. We now present analysis of GMC/GMV and TMEC/TMEV problems for two such markets.

4.2.1 Negative exponential Tail

Let us consider returns being driven by a mixture of multivariate normal and negative exponential distributions, such that with a small probability, all assets suffer a perfectly correlated exponential-tail loss. Formally,

$$\mathbf{X} \sim (1 - I(\epsilon))N(\boldsymbol{\mu}, \mathbf{V}) + I(\epsilon)(Y\mathbf{e} + \mathbf{f}), \tag{$\mathcal{M}2$}$$

where $I(\epsilon)$ is a Bernoulli random variable with parameter ϵ, $\mathbf{e}$ is a $n \times 1$ vector of ones, and $\mathbf{f} = [f_1, \ldots, f_n]'$ is a $n \times 1$ vector of constants, and Y is a negative exponential random variable with density

$$P(Y = y) = \begin{cases} \lambda e^{\lambda y}, & \text{if } y \leq 0 \\ 0 & \text{otherwise.} \end{cases}$$

In our simulations, we consider $\epsilon = 0.05$ (i.e. one shock every ≈ 1.7 years), $f_i = \mu_i - \sqrt{V_{ii}}$ and $\lambda = 10$. The histograms for $10,000$ sample returns of X_1 is shown in Fig. (2b).

4.2.2 One-sided power tail

Finally, we consider returns being driven by a mixture of multivariate normal and one-sided power distribution, such that with a small probability, all assets suffer a perfectly correlated power-tail loss. Formally,

$$\mathbf{X} \sim (1 - I(\epsilon))N(\boldsymbol{\mu}, \mathbf{V}) + I(\epsilon)Z(\gamma)\mathbf{f}, \tag{$\mathcal{M}3$}$$

where $I(\epsilon)$ is a Bernoulli random variable with parameter ϵ, $\mathbf{f} = [f_1, \ldots, f_n]'$ is a $n \times 1$ vector of constants, and $Z(\gamma)$ is a random variable defined for $\gamma \geq 1$ such that

$$P(Z(\gamma) = z) = \begin{cases} (\gamma - 1)(-z)^{-\gamma} & \text{if } z \leq -1 \\ 0 & \text{otherwise.} \end{cases}$$

Note $\mathbf{X}$ under $\mathcal{M}3$ has finite variance for $\gamma > 3$. In our simulations, we consider $\epsilon = 0.05$, $f_i = \mu_i - 5\sqrt{V_{ii}}$, and $\gamma = 3.5$. The histogram for $10,000$ sample returns of X_1 is shown in Fig.(2c).

4.3 Discussion of results

4.3.1 Global minimum CVaR (GMC) problem

The expected return vs. true CVaR of GMC and GMV portfolios under models $\mathcal{M}1$–3 are plotted in Fig. (4). In Table 1, we give the ranges for CVaR and expected return values corresponding to the solutions of GMC and GMV problems from our 50 simulations. In $\mathcal{M}1$ and $\mathcal{M}2$, the GMV portfolios perform better than GMC portfolios in that the blue stars are further to the left (i.e. more accurate) and have substantially smaller vertical and horizontal spreads (i.e. more precise) than the red stars. In $\mathcal{M}3$, the GMV portfolios are slightly more accurate and precise than GMC portfolios when $q = 50$, but the GMC portfolios converge faster with increasing data size. However, as financial data older than 5 years is rarely used in practice, $q = 50$ presents the most realistic scenario, and in this case, the GMV problem is clearly more reliable than the GMC problem across all models. In addition, when $q = 50$, t! he investor substantially underestimates the CVaR value— for our 50 simulations, True CVaR range is 2736–10610 bps/mth, whereas Perceived CVaR range is 1053–4957 bps/mth for the same expected return 40 bps/mth.

4.3.2 True mean-Empirical CVaR (TMEC) problem

The empirical frontiers of TMEC and TMEV portfolios under models $\mathcal{M}1$–3 are plotted in Fig. (5). In Table 2, we give the ranges for CVaR and expected return values corresponding to the solutions of TMEC/TMEV problems from our 50 simulations. For comparison, we also provide the ranges for EMEC/EMEV problems under $\mathcal{M}1$. Across all models, the TMEV empirical frontiers perform better than TMEV empirical frontiers in that the blue curves are further to the left and have smaller spreads than the red curves. There are differences between the models, however— the TMEV most out-perform TMEC portfolios in $\mathcal{M}2$, whereas in $\mathcal{M}3$, the out-performance of TMEV empirical frontiers at $q = 50$ diminishes with increasing data. However, as previously mentioned, $q = 50$ presents the most realistic scenario, and in this case, the TMEV problem is clearly more reliable than the TMEC problem across all models. The inve! stor remains too optimistic when $q = 50$— for our 50 simulations, True CVaR range is 1123–1778 bps/mth, whereas Perceived CVaR range is 461–916 bps/mth for the same expected return 40 bps/mth.

We can also see the effect of assuming the investor knows the true mean— from the $\mathcal{M}1$ columns in Table 2 and comparing Fig. (1) and Fig. (5(a)), we see that the variation of both mean-variance and mean-CVaR empirical frontiers decrease substantially when the true mean is known. However, even if the investor estimates the mean exactly, estimation errors in CVaR can significantly affect the reliability of empirical frontiers, as is the case for $\mathcal{M}2$.

5 Conclusion

In this paper, we have empirically evaluated CVaR as a risk measure in data-driven portfolio optimization. We have focused on two goals: to see the effects of estimation errors on portfolio optimization that minimize CVaR as an objective, and to determine whether such optimization is reliable in heavy-tailed markets. We summarize our findings below.

- Estimation errors of the mean plague the empirical mean-empirical CVaR problem, just as the empirical mean-empirical variance problem. As the difficulty in estimating the mean is well-known, we focused on global minimum CVaR/variance and true mean-empirical CVaR/variance problems for the rest of the paper.

- The portfolio solutions of GMC and TMEC problems are unreliable due to statistical errors associated with estimating CVaR coupled with the effect of optimization, which tends to amplify the statistical errors. An investor who minimizes CVaR is thus likely to find a portfolio that is far from being efficient with substantially underestimated CVaR; her risk exposure is likely to be substantially higher than she might otherwise realize.

- The analysis of the TMEC problem show that any benefits of accurately estimating expected return are destroyed by estimation errors of CVaR.

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Appendix

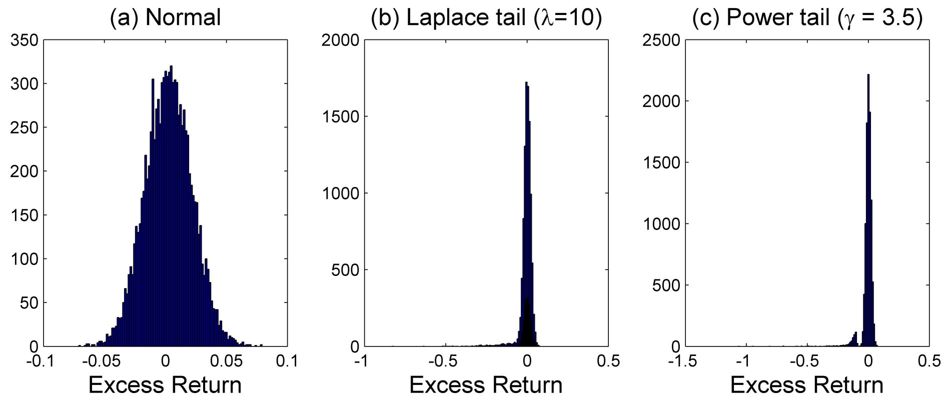
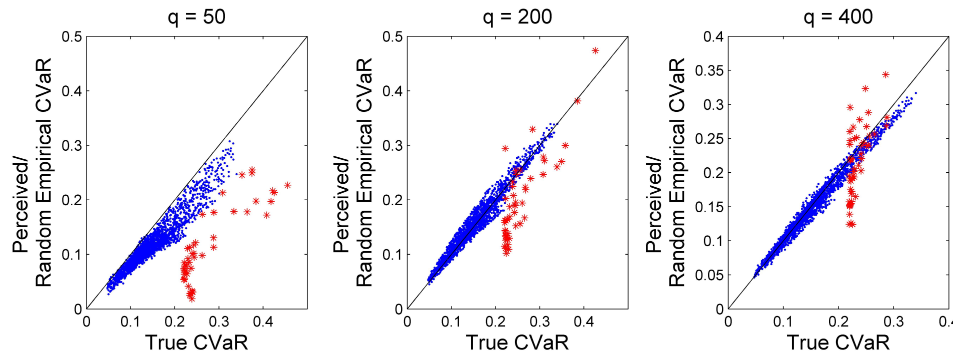
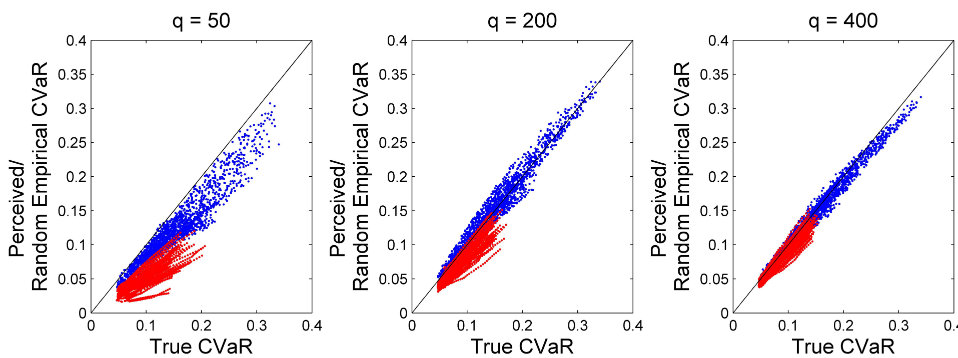


Figure 2: Histogram for 10,000 samples of X_1 (bps/mth) under model (a) $\mathcal{M}1$, (b) $\mathcal{M}2$ and (c) $\mathcal{M}3$.

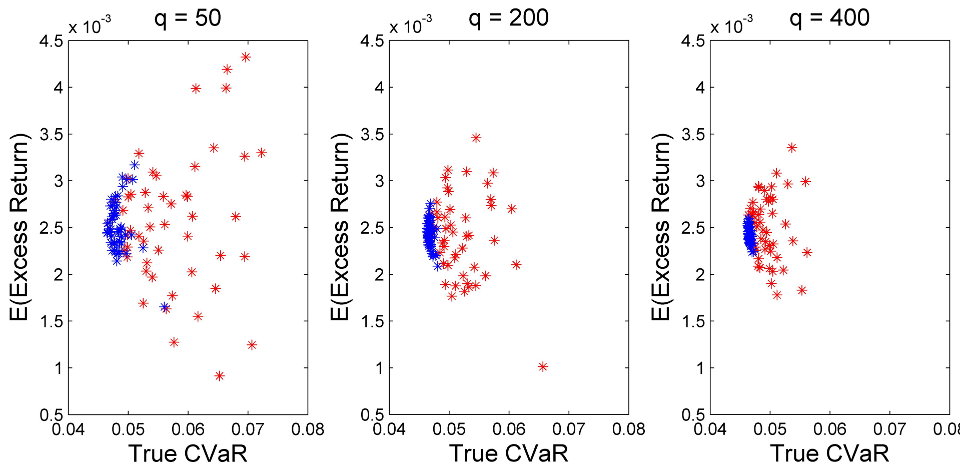


(a)

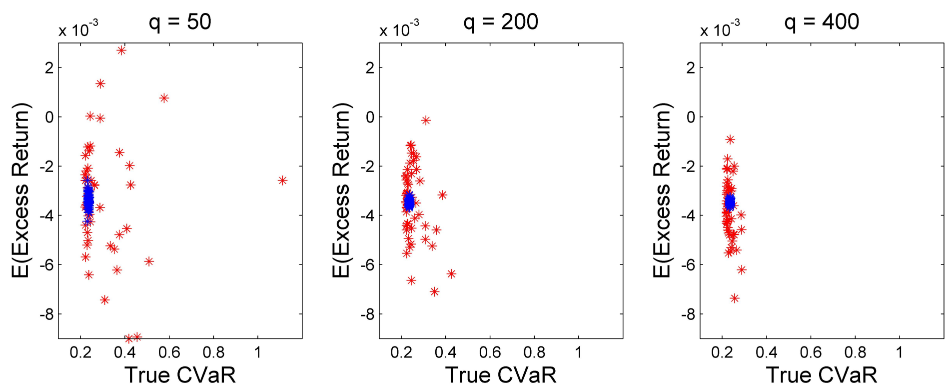


(b)

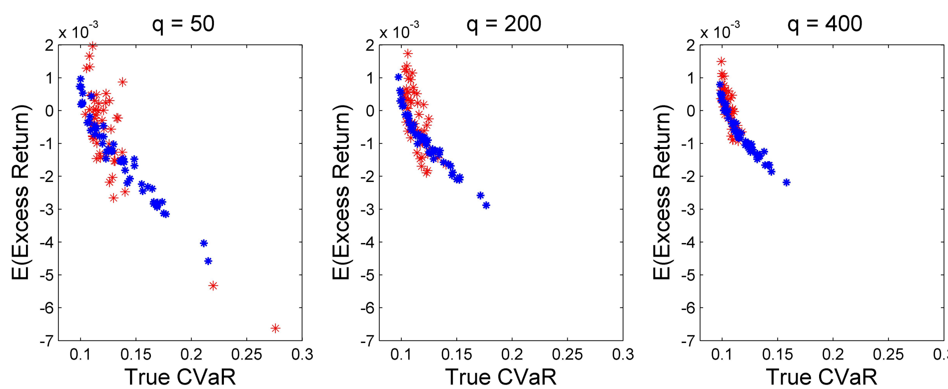
Figure 3: Random Empirical CVaR vs. True CVaR (blue) and Perceived CVaR vs. True CVaR (red) for (a) true mean-empirical CVaR problem and (b) global minimum CVaR problem under model $\mathcal{M}1$. All scales are bps/mth.



(a)



(b)



(c)

Figure 4: The empirical frontiers of global minimum variance (blue) and global minimum CVaR (red) portfolios under models (a) $\mathcal{M}1$ (b) $\mathcal{M}2$ and (c) $\mathcal{M}3$. All scales are bps/mth.

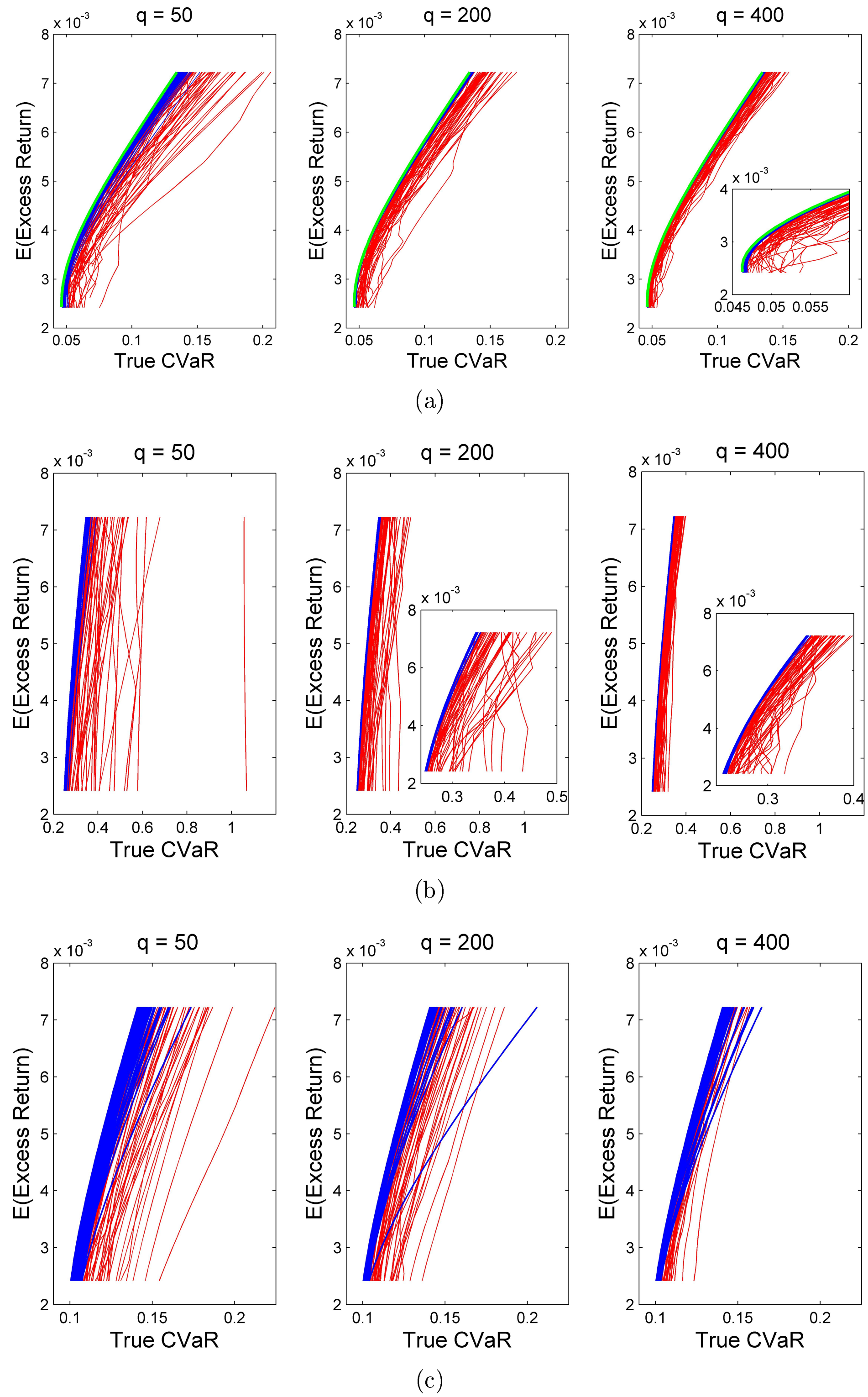


Figure 5: The empirical frontiers of true mean-empirical variance (blue) and true mean-empirical CVaR (red) portfolios under models (a) $\mathcal{M}1$ (b) $\mathcal{M}2$ and (c) $\mathcal{M}3$. In (a), we also plot the theoretical mean-variance (equivalently, mean-CVaR) frontier in green (found at the left-most edge of the blue curves). All scales are bps/mth.

Table 1: Expected return and CVaR (true and perceived) ranges (bps/mth) for 50 GMC and GMV portfolios. In parenthesis is the size of the range.

	$\mathcal{M}1$			$\mathcal{M}2$			$\mathcal{M}3$		
	GMV	GMC (true)	GMC (per)	GMV	GMC (true)	GMC (per)	GMV	GMC (true)	GMC (per)
CVaR									
$q = 50$	465–560	465–723	165–380	2298–2391	2204–11120	188–4706	990–2151	1043–2758	255–687
	(96)	(257)	(257)	(93)	(8916)	(4518)	(1161)	(1715)	(432)
$q = 200$	463–481	466–657	314–494	2332–2374	2202–4258	1020–4738	974–1764	1007–1403	480–837
	(18)	(191)	(180)	(42)	(2056)	(3718)	(792)	(396)	(357)
$q = 400$	463–472	464–562	377–497	2339–2376	2194–2876	1240–3439	980–1580	988–1177	555–946
	(9)	(98)	(121)	(37)	(682)	(2199)	(600)	(189)	(391)
Exp. Return									
$q = 50$	16.5–31.7	9.15–43.2	-14.9–84.7	-42.4– -25.8	-90.0–27.0	-318–426	-45.8–9.63	-66.3–19.6	-183– 48.1
	(15.2)	(34.1)	(99.6)	(16.6)	(117)	(744)	(55.4)	(85.9)	(232)
$q = 200$	20.9–27.5	10.1–34.6	-11.4–37.8	-36.9– -31.6	-70.9– -1.43	-34.4–98.5	-28.8–10.2	-18.9– 17.3	-19.7– -34.8
	(6.67)	(24.5)	(49.2)	(5.30)	(69.5)	(132.8)	(39.0)	(36.3)	(54.6)
$q = 400$	22.4–25.9	17.8–33.5	8.31–41.5	-36.9– -32.5	-73.6– -9.21	-52.9–8.03	-21.8–7.94	-9.30–14.9	-20.4– -5.65
	(3.55)	(15.7)	(33.2)	(4.40)	(64.4)	(61.0)	(29.8)	(24.2)	(14.8)

Table 2: CVaR (true and perceived) ranges (bps/mth) for 50 mean-risk empirical frontiers at fixed expected return levels. In parenthesis is the size of the range.

	EMEV/EMEC		TMEV/TMEC								
	$\mathcal{M}1$		$\mathcal{M}1$			$\mathcal{M}2$			$\mathcal{M}3$		
Exp. Return	TMEV	TMEC	TMEV	TMEC (true)	TMEC (per)	TMEV	TMEC (true)	TMEC (per)	TMEV	TMEC (true)	TMEC (per)
40bps/mth											
$q = 50$	470–2210	490–2150	613–691	621–942	171–588	2721–2992	2736–10610	1053–4957	1110–1242	1123–1778	469–916
	(1740)	(1660)	(78)	(321)	(417)	(271)	(7874)	(3903)	(132)	(655)	(447)
$q = 200$	460–2970	490–3120	612–624	624–910	426–697	2720–2772	2739–4423	1759–4998	1109–1291	1113–1518	686–1159
	(2510)	(2630)	(12)	(286)	(271)	(52)	(1684)	(3239)	(182)	(405)	(472)
$q = 400$	470–1200	470–1250	612–620	612–709	484–692	2720–2745	2728–3388	1904–3940	1110–1179	1112–1299	846–1288
	(730)	(780)	(8)	(97)	(208)	(25)	(660)	(2036)	(69)	(187)	(443)
60bps/mth											
$q = 50$	460–2720	470–2650	1039–1179	1056–1747	220–963	3139–3542	3161–10570	1282–5031	1287–1530	1294–2072	565–1100
	(2260)	(2180)	(140)	(691)	(742)	(403)	(7409)	(3750)	(243)	(779)	(536)
$q = 200$	480–4220	520–4440	1038–1063	1042–1294	692–1182	3140–3213	3155–4538	2002–5126	1286–1740	1288–1719	814–1362
	(3740)	(3920)	(25)	(252)	(490)	(73)	(1383)	(3124)	(455)	(431)	(549)
$q = 400$	550–2280	560–2300	1038–1055	1045–1183	798–1186	3139–3175	3148–3555	2393–4122	1286–1450	1287–1454	1008–1492
	(1730)	(1740)	(17)	(138)	(388)	(36)	(406)	(1729)	(164)	(167)	(484)