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Skew-Linked Partitions and a Representation-Theoretic Model for k-Schur Functions

by

Li-Chung Chen

A dissertation submitted in partial satisfaction of the

requirements for the degree of

Doctor of Philosophy

in

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University of California, Berkeley

Committee in charge:

Professor Mark Haiman, Chair
Professor Nicolai Reshetikhin
Professor Alistair Sinclair

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Abstract

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In 2001, Lapointe, Lascoux, and Morse discovered a class of symmetric functions $s_\lambda^{(k)}(z;t)$ called k-Schur functions, where k is a positive integer and λ is a k-bounded partition. These functions have many properties similar to Schur functions and were motivated by a conjectured refinement of the Macdonald positivity conjecture. We describe a representation-theoretic model for k-Schur functions by studying the combinatorics of special pairs of partitions called skew-linked partitions. En route we also study nonnegative integer matrices with specified row and column sums. These data allow us to construct "small" modules of $\mathbb{C}[x_1,\ldots,x_n]*S_n$ that are generalizations of Garsia-Procesi modules. We describe properties of k-Schur functions that can be deduced from these modules.

Contents

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Chapter 1

Background

1.1 Partitions and Tableaux

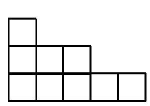
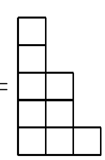
A *composition* is a sequence $\alpha = (\alpha_1, \alpha_2, \ldots)$ of nonnegative integers whose sum is finite (denoted $|\alpha|$, the size of α). A *partition* $\lambda = (\lambda_1, \lambda_2, \ldots)$ is a composition satisfying $\lambda_1 \geq \lambda_2 \geq \ldots$. Thus each composition has a unique rearrangement into a partition. If $n = |\lambda|$, we say that λ is a *partition of* n, denoted $\lambda \vdash n$. A partition λ is said to be *k-bounded* if $\lambda_1 \leq k$ (so all parts of λ are at most k).

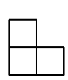
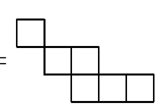
The *length* of λ, denoted $\ell(\lambda)$, is the number of nonzero entries. Note that λ begins with $\ell(\lambda)$ nonzero terms and ends with an infinite sequence of zeroes; the finite subsequence of nonzero terms is the usual definition of a paritition, but adding infinite zeroes is a convenient convention. One important statistic of a partition is $n(\lambda) = \sum_i (i-1)\lambda_i$.

A partition λ may be represented by a *(Ferrers) diagram*, consisting of the squares in the two-dimensional grid at coordinates $\{(i, j) \in \mathbb{N} \times \mathbb{N} : j \leq \lambda_i\}$. Thus the ith row contains λ_i squares and the rows line up on the left. The *conjugate partition* of λ, denoted λ', is the partition whose diagram is the transpose (with respect to the main diagonal) of the diagram of λ.

A *skew shape* λ/ν consists of two partitions λ, ν satisfying $\nu_i \leq \lambda_i$ for all i. Equivalently, the diagram of ν is a subset of the diagram of λ. The diagram of λ/ν is obtained by deleting the squares of the diagram of ν from the diagram of λ, i.e. the squares (i, j) satisfying $i \geq 1$ and $\nu_i < j \leq \lambda_i$. This is called a *skew diagram*.

Example 1.1.1

$\lambda = (5, 3, 1) =$  $\lambda' = (3, 2, 2, 1, 1) =$ 

$\nu = (2, 1) =$  $\lambda/\nu = (5, 3, 1)/(2, 1) =$ 

The *main hook length* of a partition λ is $h_M(\lambda) = \lambda_1 + \ell(\lambda) - 1$. The *hook length* of a square in a diagram is one plus the number of squares strictly to its north plus the number of squares strictly to its east. In other words, the main hook length is the hook length of the square at $(1, 1)$. A partition is said to be a *k-core* if no square of its diagram has hook length exactly k.

The *dominance order* is a partial order on partitions of the same size, defined by $\lambda \geq \mu$ iff $\lambda_1+\ldots+\lambda_j \geq \mu_1+\ldots+\mu_j$ for all j. It can be shown that $\lambda \geq \mu$ iff $\mu' \geq \lambda'$. A *lowering operator* transforms a composition α into the composition $(\alpha_1, \ldots, \alpha_{i-1}, \alpha_i-1, \alpha_{i+1}+1, \alpha_{i+2}, \alpha_{i+3}, \ldots)$ for some fixed i. It is easy to show that for partitions λ and μ, $\lambda \geq \mu$ iff there exists a (possibly empty) sequence of lowering operators transforming λ into μ with intermediate results also being partitions. The dominance order on compositions is similarly defined.

The *lexicographical order* is a total order on compositions defined by $\lambda > \mu$ iff there exists j such that $\lambda_j > \mu_j$ and $\lambda_i = \mu_i$ for $i = 1, \ldots, j-1$. However, when we compare two partitions (or compositions), the notation $\geq$ will refer to dominance order unless otherwise specified.

A *tableau* T of shape λ (resp. λ/ν) is obtained by filling each square of the diagram of λ (resp. λ/ν) with a positive integer. For a square (i, j) in the diagram of λ, $T_{(i,j)}$ denotes the letter that occurs in square (i, j) in T.

A tableau T is *standard* if each of $1, 2, \ldots,$(size of T) occurs once in the fillings, and the fillings are strictly increasing left to right in each row and bottom to top in each column. A tableau T is *semistandard* if its fillings are weakly increasing left to right in each row and strictly increasing bottom to top in each column. The *weight* $wt(T)$ of a tableau T is the composition $(\alpha_1, \alpha_2, \ldots)$, where α_i is the number of occurrences of i in T. Let $SSYT(\lambda, \alpha)$ (resp. $SSYT(\lambda/\nu, \alpha)$) denote the set of semistandard tableaux of shape λ (resp. λ/ν) and weight α. Also, let $SSYT(\lambda) = \cup_\alpha SSYT(\lambda, \alpha)$.

Example 1.1.2 *Standard tableau:* $\begin{ytableau} 5 \\ 2 & 6 & 7 \\ 1 & 3 & 4 & 8 & 9 \end{ytableau} \in SYT((5, 3, 1))$, *row-reading word* 526713489

Semistandard tableau: $\begin{ytableau} 6 \\ 3 & 3 & 4 \\ 1 & 1 & 1 & 2 & 6 \end{ytableau} \in SSYT((5, 3, 1), (3, 1, 2, 1, 0, 2))$,
row-reading word 633411126

Semistandard skew tableau: $\begin{ytableau} 1 \\ \none & 1 & 3 \\ \none & \none & 2 & 2 & 4 \end{ytableau} \in SSYT((5, 3, 1)/(2, 1), (2, 2, 1, 1))$, *row-reading word* 113224, $T_{113224} = \begin{ytableau} 3 \\ 1 & 1 & 2 & 2 & 4 \end{ytableau}$ *because* $113224 \equiv 311224$

The *Kostka number* $K_{\lambda,\alpha}$ is the size of $SSYT(\lambda, \alpha)$. For partition μ with $|\lambda| = |\mu|$, we have

- $K_{\lambda,\mu} \geq 1$ if $\lambda \geq \mu$
- $K_{\lambda,\mu} = 1$ if $\lambda = \mu$
- $K_{\lambda,\mu} = 0$ if $\lambda \ngeq \mu$.

A *word* is a finite sequence of letters taken from the alphabet of positive integers. The *weight* of a word is the composition $(\mu_1, \mu_2, \ldots)$ where μ_i is the number of occurrences of i in the word. Two words are related by a *Knuth relation* in the following scenarios (where u, v are words and a, b, c are letters):

1. $uacbv \equiv ucabv$ if $a \leq b < c$
2. $ubacv \equiv ubcav$ if $a < b \leq c$.

The transitive closure of the Knuth relations forms the *Knuth equivalence* on the set of words.

The *row-reading word* $w(T)$ of a tableau T is obtained by reading the entries of T from left to right and top to bottom. For any word w, it can be shown that there exists a unique tableau T_w of non-skew shape such that w and $w(T_w)$ are Knuth equivalent. In particular this is true for the row-reading word of a skew tableau U, yielding the *straightening operation* mapping U to the non-skew tableau $T_{w(U)}$. See above for examples.

1.2 Charge

For words w whose weights are partitions, there is a unique *charge function* c satisfying the following properties:

1. For letter $a \neq 1$ and word w, $c(wa) = c(aw) + 1$.
2. If w's letters are in weakly decreasing order, then $c(w) = 0$.
3. Charge is invariant under Knuth equivalence.

The *charge of a tableau* T is defined to be $c(w(T))$. There is an explicit algorithm to compute charge.

Algorithm 1.2.1 *Given a word whose weight is a partition, label its letters in the following way.*

1. *At the beginning, all letters are unlabelled.*
2. *Set $\ell = 0$. Starting from the end of the word and scanning backward, give label ℓ to the first unlabelled 1, to the first unlabelled 2 following this 1, to the first unlabelled 3 following this 2, and so on.*
3. *When the next higher letter (say p) is not found, start again at the end of the word and increment ℓ by 1. Give label ℓ to the first unlabelled p, the first unlabelled $p+1$ following this p, and so on.*
4. *Keep scanning, incrementing ℓ as necessary, until one of each letter has been labelled.*
5. *Repeat steps (2)-(4) on the unlabelled letters. Repeat this as many times as necessary (each time resetting ℓ to 0) until all letters have been labelled.*

6. Then the charge is the sum of the labels.

Example 1.2.2 $T = \begin{array}{lllll} \boxed{3} & \boxed{6} & & & \\ \boxed{2} & \boxed{2} & & & \\ \boxed{1} & \boxed{1} & \boxed{3} & \boxed{4} & \boxed{5} \end{array}$ *has word* $w = 362211345$. *The labelling passes are*

$\mathbf{3}_0 6 2 \mathbf{2}_0 1 \mathbf{1}_0 345 \quad 3_0 6 2 2_0 1 1_0 3 \mathbf{4}_1 5 \quad 3_0 \mathbf{6}_2 2 2_0 1 1_0 3 4_1 \mathbf{5}_2 \quad 3_0 6_2 \mathbf{2}_0 2_0 \mathbf{1}_0 1_0 3 4_1 5_2 \quad 3_0 6_2 2_0 2_0 1_0 1_0 \mathbf{3}_1 4_1 5_2$

Charge $= 0 + 2 + 0 + 0 + 0 + 0 + 1 + 1 + 2 = 6$.

1.3 Catabolism

We say that a tableau T is *d-catabolizable* if for $i = 1, 2, \ldots, d$, the ith smallest letter that occurs in T all appear in the ith row. (Vacuously every tableau is 0-catabolizable.) Notice that every nonempty tableau is 1-catabolizable. If T is d-catabolizable, we define $\mathrm{Cat}_d(T)$ as follows. Let U be the tableau obtained by deleting rows $1, \ldots, d$ from T. Let V be the (skew-shaped) tableau obtained by deleting rows $d+1, d+2, \ldots$ from T and also removing the squares containing the d smallest letters that occur in T. Then set $\mathrm{Cat}_d(T) = T_{w(V)w(U)}$. (Vacuously $\mathrm{Cat}_0(T) = T$.)

We can also define multiple-step catabolism recursively. Let $d_1, d_2, \ldots, d_r \geq 0$. If $r > 1$, a tableau T is said to be *$d_1, \ldots, d_r$-catabolizable* if T is d_1-catabolizable and $\mathrm{Cat}_{d_1}(T)$ is $d_2, \ldots, d_r$-catabolizable. In this case, set $\mathrm{Cat}_{d_1, d_2, \ldots, d_r}(T) = \mathrm{Cat}_{d_r} \mathrm{Cat}_{d_{r-1}} \ldots \mathrm{Cat}_{d_1}(T)$.

Because 0-catabolism is vacuous, we may define catabolism with respect to a composition: Suppose composition α has the form $(\alpha_1, \ldots, \alpha_r, 0, 0, \ldots)$. Then T is *α-catabolizable* if T is $\alpha_1, \ldots, \alpha_r$-catabolizable, in which case $\mathrm{Cat}_\alpha(T)$ is defined as $\mathrm{Cat}_{\alpha_1, \ldots, \alpha_r}(T)$.

Finally, for convenience we define a normalized version of catabolism: If T is d-catabolizable and the smallest letter occurring in $\mathrm{Cat}_d(T)$ is j, then let $\overline{\mathrm{Cat}}_d(T)$ be the result of subtracting every letter in $\mathrm{Cat}_d(T)$ by $j - 1$. (Usually $j = d + 1$.) Clearly this does not affect any catabolizabilities.

Example 1.3.1

$$d = 2 \quad T = \begin{array}{lllll} \boxed{7} & & & & \\ \boxed{3} & \boxed{5} & \boxed{6} & & \\ \boxed{2} & \boxed{2} & \boxed{4} & \boxed{8} & \\ \boxed{1} & \boxed{1} & \boxed{1} & \boxed{3} & \boxed{4} \end{array} \quad U = \begin{array}{lll} \boxed{7} & & \\ \boxed{3} & \boxed{5} & \boxed{6} \end{array} \quad V = \begin{array}{ll} \boxed{4} & \boxed{8} \\ & \boxed{3} \; \boxed{4} \end{array}$$

$$\mathrm{Cat}_2(T) = T_{w(V)w(U)} = T_{48347356} = \begin{array}{llll} \boxed{8} & & & \\ \boxed{4} & \boxed{4} & \boxed{7} & \\ \boxed{3} & \boxed{3} & \boxed{5} & \boxed{6} \end{array} \quad \overline{\mathrm{Cat}}_2(T) = \begin{array}{llll} \boxed{6} & & & \\ \boxed{2} & \boxed{2} & \boxed{5} & \\ \boxed{1} & \boxed{1} & \boxed{3} & \boxed{4} \end{array}$$

$\mathrm{Cat}_2(T)$ *is 2-catabolizable:* $U = \boxed{8}$ $\;V = \begin{array}{ll} \boxed{7} & \\ & \boxed{6} \end{array}$ $\;\mathrm{Cat}_2(\mathrm{Cat}_2(T)) = T_{w(V)w(U)} = T_{768} = \begin{array}{ll} \boxed{7} & \\ \boxed{6} & \boxed{8} \end{array}$

Hence T *is* $2, 2$*-catabolizable with* $\mathrm{Cat}_{2,2}(T) = \mathrm{Cat}_2(\mathrm{Cat}_2(T)) = \begin{array}{ll} \boxed{7} & \\ \boxed{6} & \boxed{8} \end{array}$.

Furthermore, T *is* $2, 2, 1$*-catabolizable with* $\mathrm{Cat}_{2,2,1}(T) = \mathrm{Cat}_1(\mathrm{Cat}_2(\mathrm{Cat}_2(T))) = \begin{array}{l} \boxed{8} \\ \boxed{7} \end{array}$.

1.4 Representation Theory of S_n

Proposition 1.4.1 *Let $R = \mathbb{C}S_n$ be the group ring. It is well known that the irreducible R-modules are naturally indexed by partitions μ of n. This irreducible is denoted V_μ. The trivial representation 1 and the sign representation ε are $V_{(n)}$ and $V_{(1^n)}$, respectively.*

Definition 1.4.2 *Fix a standard tableau T_0 of shape $\mu \vdash n$. Suppose $\mathfrak{C}_i$ is the set of letters in the ith row of T_0. For the rest of this paper, fix the* Young subgroup $S_\mu = S_{\mathfrak{C}_1} \times S_{\mathfrak{C}_2} \times \ldots$. *On a Young subgroup, let 1 and ε denote the trivial representation and the sign representation, respectively.*

Definition 1.4.3 *Let P and Q be the subgroups of R preserving the rows and columns of T_0, respectively. Then $P = S_\mu$. Set $a_\mu = \sum_{g\in P} g = \sum_{g \in S_\mu} g$, $b_\mu = \sum_{g\in Q}(-1)^g g$, $c_\mu = a_\mu b_\mu$, and $\tilde{c}_\mu = b_\mu a_\mu$.*

Fulton-Harris [5] uses the *Young Symmetrizer* c_μ, but similar results hold using $\tilde{c}_\mu$:

Proposition 1.4.4 *1. $R\tilde{c}_\mu \cong Rc_\mu = V_\mu$, and $\tilde{c}_\mu^2 = n_\mu \tilde{c}_\mu$ with nonzero $n_\mu \in \mathbb{C}$.*

2. $\tilde{c}_\mu R \tilde{c}_\mu = \mathbb{C}\tilde{c}_\mu$. If $\lambda \neq \mu$ then $\tilde{c}_\mu V_\lambda = \tilde{c}_\mu R \tilde{c}_\lambda = 0$.

Proof

1. Right multiplication by b_μ and a_μ maps $R\tilde{c}_\mu$ and Rc_μ into each other. One composite map acts as right multiplication by c_μ on Rc_μ, which by [5] Lemma 4.26 is multiplication by a nonzero scalar n_μ. This establishes the isomorphism. The other composite map must also be multiplication by n_μ, establishing the second fact.

2. Let $\hat{}$ be the anti-involution of R generated by $g \mapsto g^{-1}, g \in S_n$. Then $\widehat{\tilde{c}_\mu} = \hat{a}_\mu \hat{b}_\mu = a_\mu b_\mu = c_\mu$. So for $v \in R$, $\widehat{\tilde{c}_\mu v \tilde{c}_\mu} = c_\mu \hat{v} c_\mu \in \mathbb{C}c_\mu$ by [5] Lemma 4.23(2). Applying $\hat{}$ to both sides, we get $\tilde{c}_\mu v \tilde{c}_\mu \in \mathbb{C}\tilde{c}_\mu$. But $\tilde{c}_\mu \cdot 1 \cdot \tilde{c}_\mu = n_\mu \tilde{c}_\mu \neq 0$, so $\tilde{c}_\mu R \tilde{c}_\mu = \mathbb{C}\tilde{c}_\mu$.

   If $\mu > \lambda$ in lexicographical order, then $\tilde{c}_\mu R \tilde{c}_\lambda = b_\mu (a_\mu R b_\lambda) a_\lambda = b_\mu \cdot 0 \cdot a_\lambda$ by [5] Lemma 4.23(1). Suppose now $\lambda > \mu$ in lexicographical order. Then for $v \in R$ we have $\widehat{\tilde{c}_\mu v \tilde{c}_\lambda} = a_\lambda (b_\lambda \hat{v} a_\mu) b_\mu = 0$ again by [5] Lemma 4.23(1).

$\square$

Corollary 1.4.5 *The multiplicity of V_μ in an R-module M is* $\dim_{\mathbb{C}} \tilde{c}_\mu M$.

Proposition 1.4.6 *(Specht module presentation). For a standard tableau T of shape λ, define* Garnir element $e_T = \prod_{i=1}^{\lambda_1} \prod_{1 \le p < q \le \lambda_i'} y_{T(i,q)} - y_{T(1,p)} \in \mathbb{C}[y_1, \ldots, y_n]$, *the product of the Vandermondes of the columns. Let the* Specht module *be the span of e_T over all standard tableaux T of shape λ. Then the Specht module is an S_n-submodule of $\mathbb{C}[\mathbf{y}]$ that is isomorphic to V_λ.*

Then $V_\lambda \otimes_{\mathbb{C}} \mathbb{C}[\mathbf{x}]$ embeds in $\mathbb{C}[\mathbf{y}] \otimes_{\mathbb{C}} \mathbb{C}[\mathbf{x}] \cong \mathbb{C}[\mathbf{y}, \mathbf{x}]$, so it is convenient to drop the $\otimes$ and regard elements as polynomials. Then S_n acts on $V_\lambda \otimes_{\mathbb{C}} \mathbb{C}[\mathbf{x}]$ by acting on both sets of variables simultaneously.

We record some facts about induced modules from Young subgroups.

Proposition 1.4.7 *1. Let S_n-module U_μ be the $\mathbb{C}$-span of all monomials x^u where u is a permutation of $(0^{\mu_1}1^{\mu_2}2^{\mu_3}\ldots)$. Then $U_\mu \cong Ra_\mu \cong 1 \uparrow_{S_\mu}^{S_n} \cong \oplus_\eta \left({V_\eta}^{\oplus K_{\eta,\mu}}\right)$. Also by definition $U_\mu \subset \mathbb{C}[\mathbf{x}]_{n(\mu)}$. Since $K_{\mu,\mu} = 1$, U_μ contains a unique copy of V_μ.*

2. $\varepsilon \uparrow_{S_{\lambda'}}^{S_n} \cong Rb_\lambda \cong \oplus_\gamma \left({V_\gamma}^{\oplus K_{\gamma',\lambda'}}\right)$.

This leads to an alternate presentation of irreducible S_n-modules.

Proposition 1.4.8 *Let $V = \varepsilon \uparrow_{S_{\lambda'}}^{S_n}$ and $W = 1 \uparrow_{S_\lambda}^{S_n}$. Then there is a unique (up to a constant) S_n-homomorphism*

$$V \underset{\phi}{\to} W$$

and the irreducible V_λ is the image of ϕ.

Proof

From $V \cong \oplus_\gamma \left({V_\gamma}^{\oplus K_{\gamma',\lambda'}}\right)$, the irreducible V_μ occurs in V iff $\mu' \geq \lambda'$ iff $\lambda \geq \mu$ (because otherwise $K_{\mu',\lambda'} = 0$). From $W \cong \oplus_\eta \left({V_\eta}^{\oplus K_{\eta,\lambda}}\right)$, the irreducible V_μ occurs in W iff $\mu \geq \lambda$ (because otherwise $K_{\mu,\lambda} = 0$). Hence for V_μ to occur in both V and W, we must have $\lambda \geq \mu \geq \lambda$ and $\lambda = \mu$. Now V_μ occurs exactly once in each of V and W because $K_{\lambda',\lambda'} = 1 = K_{\lambda,\lambda}$. Thus the result follows from Schur's lemma.

□

1.5 Symmetric Function Theory

In the next several sections, we review facts about symmetric functions as given in Macdonald's book [18].

Let Λ be the space of symmetric functions (strictly speaking, power series) in $z_1, z_2, \ldots$ with coefficients in $\mathbb{Q}(q,t)$. It has several important bases. For a composition α, let z^α denote the monomial ${z_1}^{\alpha_1}{z_2}^{\alpha_2}\ldots$. For a partition λ, let m_λ denote the sum of the monomials z^α for all permutations α of λ. Then m_λ is symmetric and $\{m_\lambda : \lambda \text{ is a partition}\}$ forms the *monomial basis* of Λ.

The *complete homogeneous function* h_n is the sum of monomials z^α over all compositions α of n. The *elementary symmetric function* e_n is the sum of monomials $z_{i_1}z_{i_2}\cdots z_{i_n}$ over all choices of $1 \leq i_1 < i_2 < \ldots < i_n$. The *power sum* p_n is ${z_1}^n + {z_2}^n + {z_3}^n + \ldots$. For a partition λ, set $h_\lambda = h_{\lambda_1}h_{\lambda_2}\cdots$, $e_\lambda = e_{\lambda_1}e_{\lambda_2}\cdots$, and $p_\lambda = p_{\lambda_1}p_{\lambda_2}\cdots$. Then $\{h_\lambda : \lambda \text{ is a partition}\}$, $\{e_\lambda : \lambda \text{ is a partition}\}$, $\{p_\lambda : \lambda \text{ is a partition}\}$ are bases of Λ.

For a partition λ, the *Schur function* s_λ is defined as $\sum_{T \in SSYT(\lambda)} z^{wt(T)}$. It has the special cases $s_{(n)} = h_n$ and $s_{(1^n)} = e_n$ (where (1^n) denotes the partition with n parts of 1).

It is not immediately obvious that s_λ is symmetric. One way to see this is to define an action of S_∞ on words (and then on $SSYT(\lambda)$) as follows. Let σ_i denote the transposition $(i \;\; i+1)$. It shall act on a word w by affecting only w's subword u consisting of the letters i and $i+1$. In u, keep deleting occurrences of subwords $(i+1)i$ until no longer possible; the end result is of the form $i\,i \ldots i\;(i+1)\;(i+1)\ldots(i+1) = i^r(i+1)^s$. Replace it with $i^s(i+1)^r$, undo the deletions, and reinsert back into w in the corresponding places. Set $\sigma_i w$ to be the final result. It can be checked that the σ_i actions satisfy the Coxeter relations and preserve Knuth equivalence, so they induce a well-defined action of S_∞ on semistandard tableaux. But if w is the row-reading word of a tableau of shape λ, then $\sigma_i w$ is easily shown to be the row-reading word of another tableau of shape λ. Hence in fact S_∞ acts on $SSYT(\lambda)$.

Example 1.5.1 *Consider applying σ_2 to* $T = \begin{array}{l} \boxed{4} \\ \boxed{3}\boxed{3}\boxed{4} \\ \boxed{2}\boxed{2}\boxed{2}\boxed{3}\boxed{3}\boxed{3}\boxed{4} \\ \boxed{1}\boxed{1}\boxed{1}\boxed{1}\boxed{2}\boxed{2}\boxed{2}\boxed{2}\boxed{3}\boxed{3}\boxed{3}\boxed{3} \end{array}$ *with* $w(T) = 43342223334111122223333$. *The occurrences of letters* 2,3 *form the subword* 3322233322223333. *Denote the occurrences of* 32 *by parentheses pairs* $\big(3(32)2\big)2\Big(3\big(3(32)2\big)2\Big)23333$. *What remains is* 223333, *which is replaced with* **222233**, *resulting in* $\big(3(32)2\big)\mathbf{2}\Big(3\big(3(32)2\big)2\Big)\mathbf{22233}$. *Reinserting into the main word yields* 43342223334111122222233, *which is the row-reading word of*

$$\sigma_2 T = \begin{array}{l} \boxed{4} \\ \boxed{3}\boxed{3}\boxed{4} \\ \boxed{2}\boxed{2}\boxed{2}\boxed{3}\boxed{3}\boxed{3}\boxed{4} \\ \boxed{1}\boxed{1}\boxed{1}\boxed{1}\boxed{2}\boxed{2}\boxed{2}\boxed{2}\boxed{2}\boxed{2}\boxed{3}\boxed{3} \end{array}.$$

By construction $wt(\tau(T)) = \tau(wt(T))$ for $\tau \in S_\infty$, so τ induces a bijection between $SSYT(\lambda, \alpha)$ and $SSYT(\lambda, \tau(\alpha))$ for any composition α. Thus for a partition μ, the contribution to $\sum_{T \in SSYT(\lambda)} z^{wt(T)}$ from tableaux whose weights are permutations of μ is exactly $K_{\lambda,\mu} m_\mu$. Hence $s_\lambda = \sum_{\text{partition } \mu} K_{\lambda,\mu} m_\mu$ is symmetric.

The set of Schur functions $\{s_\lambda : \lambda \text{ is a partition}\}$ forms a very important basis of Λ. Define the *Hall inner product* on Λ so that the Schur function is an orthonormal basis with respect to it, i.e. $\langle s_\lambda, s_\mu \rangle = \delta_{\lambda\mu}$. This inner product makes the monomial symmetric functions m_λ dual to the complete homogeneous functions h_μ, i.e. $\langle m_\lambda, h_\mu \rangle = \delta_{\lambda\mu}$. Thus $s_\lambda = \sum_\mu K_{\lambda,\mu} m_\mu$ imples $h_\mu = \sum_\lambda K_{\lambda,\mu} s_\lambda$.

Because $\{p_\lambda : \lambda \text{ is a partition}\}$ is a basis of Λ, Λ is freely generated by the power sums p_i. Thus a $\mathbb{Q}(q,t)$-algebra homomorphism from Λ to a $\mathbb{Q}(q,t)$-algebra is uniquely definable by specifying the images of the power sums p_i.

Let A be a formal Laurent series with rational coefficients in indeterminates $a_1, a_2, \ldots$ (possibly including q and t). Define $p_d[A]$ to be the result of replacing each indeterminate a_i in A by ${a_i}^d$. By the above remarks, there is a unique $\mathbb{Q}(q,t)$-algebra homomorphism from Λ to $\mathbb{Q}((a_1, a_2, \ldots))$ that sends p_d to $p_d[A]$ for all d. This homomorphism is called *plethystic substitution*, and we denote the image of $f \in \Lambda$ by $f[A]$.

Set $Z = z_1 + z_2 + \ldots$. Then $p_d[Z] = {z_1}^d + {z_2}^d + \ldots$ is the power-sum p_d, so plethystic substitution by [Z] is the identity map on Λ, i. e. $f[Z] = f(z)$ for all $f \in \Lambda$. Similarly set $X = x_1 + x_2 + \ldots$ and $Y = y_1 + y_2 + \ldots$.

1.6 Symmetric Functions in the Representation Theory of S_n

Schur functions arise naturally from the representation theory of S_n. Let χ^λ be the irreducible character corresponding to V_λ. Then $s_\lambda = \frac{1}{n!}\sum_{\omega\in S_n}\chi^\lambda(\omega)p_{\tau(\omega)}$, where $\tau(\omega)$ is the partition whose parts are the lengths of the disjoint cycles of the permutation ω.

Definition 1.6.1 *The* Frobenius characteristic map *from class functions on S_n to symmetric functions is given by $F_\chi = \frac{1}{n!}\sum_{\omega\in S_n}\chi(\omega)p_{\tau(\omega)}$.*

Proposition 1.6.2 *The Frobenius characteristic map sends the irreducible character χ^λ to s_λ.*

An S_n-representation may be analyzed via its Frobenius characteristic. If $F_{\mathrm{char}\,A} = \sum_\lambda C_\lambda s_\lambda$ for $C_\lambda \in \mathbb{N}$, then A contains C_λ copies of V_λ for each λ.

Definition 1.6.3 *Let $A = \oplus_r A_r$ be a graded S_n-module with each A_r finite-dimensional. Then define the* Frobenius series *of A as $F_A(z;t) = \sum_r t^r F_{\mathrm{char}\,A_r}$. If $A = \oplus_{r,s}A_{r,s}$ is doubly graded, define its Frobenius series as $F_A(z;q,t) = \sum_{r,s} t^r q^s F_{\mathrm{char}\,A_{r,s}}$.*

Consequently, the Frobenius series of a graded (resp. doubly graded) S_n-module is automatically in $\mathbb{N}[t]\{s_\lambda : \lambda \text{ is a partition of } n\}$ (resp. $\mathbb{N}[t,q]\{s_\lambda : \lambda \text{ is a partition of } n\}$).

Lemma 1.6.4
1. *Suppose A is an S_n-module and A' is an $S_{n'}$-module. Let $B = (A\otimes_{\mathbb{C}} A')\uparrow_{S_n\times S_{n'}}^{S_{n+n'}}$. Then $F_{\mathrm{char}\,B}(z) = F_{\mathrm{char}\,A}(z)F_{\mathrm{char}\,A'}(z)$.*
2. *Suppose A is a graded S_n-module and A' is a graded $S_{n'}$-module. Let $B = (A\otimes_{\mathbb{C}} A')\uparrow_{S_n\times S_{n'}}^{S_{n+n'}}$. Then $F_B(z;t) = F_A(z;t)F_{A'}(z;t)$.*

Proof

(1) is shown in [5] Exercise 4.41. (2) follows from (1) because $B_r = \oplus_{i+j=r}(A_i\otimes_{\mathbb{C}} A'_j)\uparrow_{S_n\times S_{n'}}^{S_{n+n'}}$ and $F_{\mathrm{char}\,B_r}(z) = \sum_{i+j=r} F_{\mathrm{char}\,A_i}(z)F_{\mathrm{char}\,A'_j}(z)$.

□

1.7 Hall-Littlewood polynomials

For positive integer j, define its t-analog and t-factorial as follows: $[j]_t = \frac{1-t^j}{1-t} = t^{j-1} + t^{j-2} + \ldots + 1$, $[j]_t! = [j]_t[j-1]_t\ldots[1]_t$.

For partition $\mu = (1^{\alpha_1}, 2^{\alpha_2}, \ldots)$ and $d > \ell(\mu)$, define the *Hall-Littlewood polynomial* $P_\mu(z_1,\ldots,z_d;t) = \frac{1}{\prod_{i\geq 0}[\alpha_i]_t!}\sum_{\omega\in S_n}\left(z^\mu\frac{\prod_{i<j}(1-tz_j/z_i)}{\prod_{i<j}(1-z_j/z_i)}\right)$. This is indeed a polynomial symmetric in the z_i's. Furthermore, it is stable with respect to changing the number of variables z_i's, so we may extend to infinitely many z_i's to obtain $P_\mu(z_1,z_2,\ldots;t)$.

Set $Q_\mu(z;t) = (1-t)^{\ell(\mu)} P_\mu(z;t) \prod_i [\alpha_i]_t!$. Then the *transformed Hall-Littlewood polynomial* $\tilde{H}_\mu$ is defined as $\tilde{H}_\mu(z;t) = Q_\mu[Z/(1-t);t]$. The *cocharge transformed Hall-Littlewood polynomial* H_μ is defined as $t^{n(\mu)} H_\mu(z;t^{-1})$. It has specialization $H_\mu(z;1) = h_\mu(z)$.

Define the *Kostka-Foulkes polynomials* $K_{\lambda,\mu}(t)$ as the coefficients in the expansion $H_\mu(z;t) = \sum_{\lambda\mu} K_{\lambda,\mu}(t) s_\lambda(z)$. Because $H_\mu(z;1) = h_\mu(z)$, we have $K_{\lambda,\mu}(1) = K_{\lambda,\mu}$, the Kostka numbers. This has a combinatorial interpretation via the charge formula for $K_{\lambda,\mu}(t)$ of Lascoux and Schutzenberger: $K_{\lambda,\mu}(t) = \sum_{T \in SSYT(\lambda,\mu)} t^{n(\mu)-c(T)}$. [18]

We may relate H_μ to the *Garsia-Procesi module* [6], defined as follows.

Proposition 1.7.1 *There is a unique copy of V_μ in $\mathbb{C}[\mathbf{y}]_{n(\mu)}$ and no copy of V_μ in $\mathbb{C}[\mathbf{y}]_{<n(\mu)}$.*

Proof

The Specht module presentation of V_μ embeds directly into $\mathbb{C}[\mathbf{y}]_{n(\mu)}$, so there is at least one copy of V_μ. Conversely, suppose some copy of V_μ is embedded into $\mathbb{C}[\mathbf{y}]_{\le n(\mu)}$. It must contains a one-dimensional subspace $\mathbb{C}f$ on which the Young subgroup $S_{\mu'} = S_{\mathfrak{C}_1} \times S_{\mathfrak{C}_2} \times \dots$ acts as the sign representation, because such subspace exists in the Specht module presentation (e.g. spanned by the appropriate Garnir element). Since f is a polynomial, the sign representation $S_{\mu'}$-action implies that f is divisible by factors $y_i - y_{i'}$ for distinct i, i' in some $\mathfrak{C}_j$. These factors are coprime, so f must be divisible by their product, which has degree $n(\mu)$. Since $f \in \mathbb{C}[\mathbf{y}]_{\le n(\mu)}$, f must be homogeneous of degree $n(\mu)$ and is uniquely determined up to scalar. Hence $\mathbb{C}[\mathbf{y}]_{n(\mu)}$ has at most one copy of V_μ and $\mathbb{C}[\mathbf{y}]_{<n(\mu)}$ has none.

□

Corollary 1.7.2 *There exists a unique largest homogeneous S_n-invariant ideal $I_\mu \subset \mathbb{C}[\mathbf{y}]$ having zero intersection with the unique copy of V_μ in $\mathbb{C}[\mathbf{y}]_{n(\mu)}$.*

Proof

Due to uniqueness of V_μ in $\mathbb{C}[\mathbf{y}]_{n(\mu)}$, V_μ has a direct summand N (namely the sum of the other isotypic components) so that for a S_n-submodule $M \subset \mathbb{C}[\mathbf{y}]$, $M \cap V_\mu = 0$ iff $M \cap \mathbb{C}[\mathbf{y}]_{n(\mu)} \subset N$. The set of homogeneous ideals satisfying this condition is closed under summation, so there is a unique largest ideal with the desired property.

□

Definition 1.7.3 *[6] The* Garsia-Procesi module *R_μ is defined to be $\mathbb{C}[\mathbf{y}]/I_\mu$. It is naturally graded by total degree in the y_i's.*

Theorem 1.7.4 *[1] [6] The Frobenius series of R_μ is $F_{R_\mu}(z;t) = H_\mu(z;t)$.*

1.8 Macdonald polynomials

The *transformed Macdonald polynomials* $H_\mu(z;q,t)$ are uniquely characterized by the conditions [8]

1. $H_\mu[(1-q)Z;q;t] \in \mathbb{Q}(q,t)\{s_\lambda : \lambda \geq \mu\}$

2. $H_\mu[(1-t)Z;q;t] \in \mathbb{Q}(q,t)\{s_\lambda : \lambda \geq \mu'\}$

3. $H_\mu[1;q;t] = 1$.

The same notation H_μ is used as for Hall-Littlewood polynomials because $H_\mu(z;0,t) = H_\mu(z;t)$. Define the *Kostka-Foulkes polynomials* $K_{\lambda,\mu}(q,t)$ as the coefficients in the expansion $H_\mu(z;q,t) = \sum_{\lambda\mu} K_{\lambda,\mu}(q,t)s_\lambda(z)$. Then $K_{\lambda,\mu}(0;t) = K_{\lambda,\mu}(t)$, justifying the use of the same name and notation.

The *Macdonald positivity conjecture*, which states that $K_{\lambda,\mu}(q;t) \in \mathbb{Z}[q,t]$, remained open for more than a decade until Haiman proved it using algebraic geometry and representation theory [7]. Let $(a_1,b_1),\ldots,(a_n,b_n)$ be the coordinates of the squares in the diagram of μ. Set $\Delta_\mu(\mathbf{x},\mathbf{y}) = \det({x_i}^{a_i}{y_i}^{b_i})_{i,j=1}^n$. Let $J_\mu \subset \mathbb{C}[\mathbf{x},\mathbf{y}]$ be the homogeneous ideal that contains an element f iff $f(\frac{\partial}{\partial x_1},\frac{\partial}{\partial y_1},\ldots,\frac{\partial}{\partial x_n},\frac{\partial}{\partial y_n})\Delta_\mu = 0$. Then the Frobenius series of $R_\mu = \mathbb{C}[\mathbf{x},\mathbf{y}]/J_\mu$ is the transformed Macdonald polynomial $H_\mu(z;q,t)$, proving Macdonald positivity.

1.9 k-Schur functions

In 2001, Lapointe, Lascoux, and Morse discovered a class of symmetric functions $s_\lambda^{(k)}(z;t)$ called *k-Schur functions* [13], where k is a positive integer and λ is a k-bounded partition (meaning $\lambda_1 \leq k$). The motivation is a conjectured refinement of the Macdonald positivity conjecture:

1. $s_\mu^{(k)}(z;t) = s_\mu(z) + \sum_{\lambda>\mu} v_{\lambda\mu}^{(k)}(t)s_\lambda(z)$ with $v_{\lambda\mu}^{(k)}(t) \in \mathbb{N}[t]$;

2. for k-bounded μ, $H_\mu(z;q,t) = \sum_\lambda K_{\lambda\mu}^{(k)}(q,t)s_\lambda^{(k)}(z;t)$ with $K_{\lambda\mu}^{(k)}(q,t) \in \mathbb{N}[q,t]$.

The name k-Schur function comes from many similar properties to Schur functions, such as variants of Pieri rules and Littlewood-Richardson rules. These properties, as well as the connections to important classes of symmetric polynomials such as Hall-Littlewood polynomials and Macdonald polynomials, suggest that k-Schur functions serve a fundamental role in the theory of symmetric polynomials. The earliest two attempts to characterize k-Schur functions will be described below.

1.10 k-Schur functions via catabolism

The *k-split* of a k-bounded partition λ is the unique sequence of partitions $\lambda^{\to k} = (\lambda^{(1)},\lambda^{(2)},\ldots,\lambda^{(r)})$ whose concatenation is λ and whose main hook lengths are k except $h_M(\lambda^{(r)}) \leq k$. Pictorally, the diagram of λ is partitioned horizontally to produce as many subdiagrams with main hook length k as possible. See examples below.

A *horizontal r-strip* is a skew diagram of size r with at most one cell in each column. Given a tableau T whose largest letter is m, let R_rT be the set of tableaux that can be obtained by adding a horizontal r-strip of the letter $m+1$ to T in all possible ways. (In other words, each such tableau U would subsume T, and the additional cells contain letter $m+1$ and form a horizontal r-strip.) Let $\mathbb{B}_r T = \sigma_1\sigma_2\ldots\sigma_m R_r T$. For a set A of tableaux, set $\mathbb{B}_r A = \cup_{T\in A}\mathbb{B}_r T$. This is the *promotion operator*.

Given a sequence of partitions $S = (\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(r)})$ and a set A of tableaux, define $\mathbb{P}_S A = \{T \in A : T \text{ is } \ell(\lambda^{(1)}), \ell(\lambda^{(2)}), \ldots, \ell(\lambda^{(r)})\text{-catabolizable}\}$. This is the *filtering operator*.

Let $\mathbb{A}_0^{(k)}$ be the set consisting of the empty tableau. The *Lapointe-Lascoux-Morse super atom* of a k-bounded partition λ is defined recursively via $\mathbb{A}_\lambda^{(k)} = \mathbb{P}_{\lambda^{\to k}}\mathbb{B}_{\lambda_1}\mathbb{A}^{(k)}_{(\lambda_2,\lambda_3,\ldots)}$. The k-Schur function is defined as $s_\lambda^{(k)}(z;t) = \sum_{T\in\mathbb{A}_\lambda^{(k)}} t^{c(T)} s_{\mathrm{shape}(T)}(z)$. [13]

Example 1.10.1

$$T = (4,3,2,2,2,1,1,1) = \begin{array}{l} \square \\ \square \\ \square \\ \square\square \\ \square\square \\ \square\square \\ \square\square\square \\ \square\square\square\square \end{array} \qquad\qquad k=4$$

$$\lambda^{\to k} = \left(\begin{array}{l}\square\square\square\square\end{array}, \begin{array}{l}\square\square\\ \square\square\square\end{array}, \begin{array}{l}\square\\ \square\square\\ \square\square\end{array}, \begin{array}{l}\square\\ \square\end{array} \right) = ((4),(3,2),(2,2,1),(1,1))$$

Example 1.10.2 *If* $T = \begin{array}{l} 2 \\ 1\,1\,2 \end{array}$ *then*

$$R_4T = \left\{ \begin{array}{l} 3\\ 2\,3\,3\\ 1\,1\,2\,3 \end{array}, \begin{array}{l} 3\\ 2\,3\\ 1\,1\,2\,3\,3 \end{array}, \begin{array}{l} 3\\ 2\\ 1\,1\,2\,3\,3\,3 \end{array}, \begin{array}{l} 2\,3\,3\\ 1\,1\,2\,3\,3 \end{array}, \begin{array}{l} 2\,3\\ 1\,1\,2\,3\,3\,3 \end{array}, \begin{array}{l} 2\\ 1\,1\,2\,3\,3\,3\,3 \end{array} \right\}$$

$$\mathbb{B}_4T = \sigma_1\sigma_2R_4T =$$

$$\left\{ \begin{array}{l} 3\\ 2\,2\,3\\ 1\,1\,1\,1 \end{array}, \begin{array}{l} 3\\ 2\,3\\ 1\,1\,1\,1\,2 \end{array}, \begin{array}{l} 3\\ 2\\ 1\,1\,1\,1\,2\,3 \end{array}, \begin{array}{l} 2\,2\,3\\ 1\,1\,1\,1\,3 \end{array}, \begin{array}{l} 2\,3\\ 1\,1\,1\,1\,2\,3 \end{array}, \begin{array}{l} 2\\ 1\,1\,1\,1\,2\,3\,3 \end{array} \right\}$$

Example 1.10.3

$$\lambda = (2,2,1) = \begin{array}{l}\square\\ \square\square\\ \square\square\end{array} \qquad k=3 \qquad \lambda^{\to k} = \left(\begin{array}{l}\square\square\\ \square\square\end{array}, \square \right) = ((2,2),(1))$$

$$\mathbb{A}^{(2)}_{(1)} = \mathbb{P}_{((1))}\mathbb{B}_1\mathbb{A}_0^{(2)} = \{\boxed{1}\}$$

$$\mathbb{A}^{(2)}_{(2,1)} = \mathbb{P}_{((2,1))}\mathbb{B}_2\mathbb{A}^{(2)}_{(1)} = \mathbb{P}_{((2,1))}\mathbb{B}_2\{\boxed{1}\} = \mathbb{P}_{((2,1))}\left\{ \begin{array}{l} 2\\ 1\,1 \end{array}, \begin{array}{l} 1\,1\,2 \end{array} \right\} = \left\{ \begin{array}{l} 2\\ 1\,1 \end{array} \right\}$$

$$\mathbb{A}^{(2)}_{(2,2,1)} = \mathbb{P}_{((2,2),(1))}\mathbb{B}_2\mathbb{A}^{(2)}_{(2,1)} = \mathbb{P}_{((2,2),(1))}\left\{ \begin{array}{l} 3\\ 2\,2\\ 1\,1 \end{array}, \begin{array}{l} 3\\ 2\\ 1\,1\,2 \end{array}, \begin{array}{l} 2\,2\\ 1\,1\,3 \end{array}, \begin{array}{l} 2\\ 1\,1\,2\,3 \end{array} \right\} = \left\{ \begin{array}{l} 3\\ 2\,2\\ 1\,1 \end{array}, \begin{array}{l} 2\,2\\ 1\,1\,3 \end{array} \right\}$$

$$s_\lambda^{(k)}(z;t) = s^{(2)}_{(2,2,1)}(z;t) = \sum_{T\in\mathbb{A}^{(2)}_{(2,2,1)}} t^{c(T)} s_{shape(T)}(z) = s_{(2,2,1)}(z) + ts_{(3,2)}(z)$$

1.11 k-Schur functions via vertex operators

Define operator $B_j = \sum_{i=0}^{\infty} s_{i+j}(z) s_i[Z(t-1)]^{\perp}$, where $\perp$ means adjoint with respect to the Hall inner product (i. e. $\langle s_i[Z(t-1)]^{\perp} f, g\rangle = \langle f, s_i[Z(t-1)]g\rangle$ for all f, g). This operator has the property of adding an entry to the Hall-Littlewood polynomial: $H_\lambda(z;t) = B_{\lambda_1} H_{\lambda_2,\lambda_3,\ldots}(z;t)$. For a partition λ of length m, define $B_\lambda = \prod_{1 \le i < j \le m}(1 - te_{ij}) B_{\lambda_1} \ldots B_{\lambda_m}$, where e_{ij} acts on a product of B operators by $e_{ij}(B_{\mu_1} \ldots B_{\mu_m}) = B_{\mu_1} \ldots B_{\mu_i+1} \ldots B_{\mu_j-1} \ldots B_m$.

If λ has k-split $\lambda^{\to k} = (\lambda^{(1)}, \ldots, \lambda^{(r)})$, define the *$k$-split polynomial* to be $G_\lambda(z;t) = B_{\lambda^{(1)}} B_{\lambda^{(2)}} \ldots B_{\lambda^{(r)}} 1$. It turns out that the k-split polynomials form a basis of $\Delta^{(k)} = \mathbb{Q}(q,t)\{H_\mu(z;t) : \mu_1 \le k\}$. Thus we can define a *projection operator* $T_j^{(k)}$ on $\Delta^{(k)}$ by $T_j^{(k)} G_\lambda(z;t) = \begin{cases} G_\lambda(z;t) & \text{if } \lambda_1 = j \\ 0 & \text{otherwise} \end{cases}$.

In [14], Lapointe and Morse proposed a second definition of k-Schur functions as $s_\lambda^{(k)}(z;t) = T_{\lambda_1}^{(k)} B_{\lambda_1} T_{\lambda_2}^{(k)} B_{\lambda_2} \ldots T_{\lambda_{\ell(\lambda)}}^{(k)} B_{\lambda_{\ell(\lambda)}} 1$. It is conjectured but not known that this definition coincides with the definition in the previous section. Both definitions involve promotion and filtering.

1.12 $SSYT(\lambda, \mu)$ and s_λ for dominant weights λ, μ

Throughout almost all of this document, λ and μ are assumed to be partitions. But for Section 5.4, it is convenient to work with dominant weights for GL_m, which are m-tuples $(\alpha_1, \ldots, \alpha_m) \in \mathbb{Z}^m$ with $\alpha_1 \ge \alpha_2 \ge \ldots \ge \alpha_m$. In this section, assume λ and μ are dominant weights for GL_m.

Definition 1.12.1 *Let $r = \max\{-\lambda_m, -\mu_m, 0\}$. Then $\hat{\lambda} = \lambda + (r, \ldots, r)$ and $\hat{\mu} = \mu + (r, \ldots, r)$ are partitions. Define $SSYT(\lambda, \mu) = SSYT(\hat{\lambda}, \hat{\mu})$. If λ, μ are partitions, then $r = 0$ and we recover the original definition of $SSYT(\lambda, \mu)$.*

Definition 1.12.2 *Let $r = \max\{-\lambda_m, 0\}$. Then $\lambda + (r, \ldots, r)$ is a partition. Define $s_\lambda(z_1, \ldots, z_m) = (x_1 x_2 \cdots x_m)^{-r} s_{\lambda+(r,\ldots,r)}(z_1, \ldots, z_m)$. If λ is a partition, then $r = 0$ and we recover the original definition of Schur functions.*

Lemma 1.12.3 *The character of the irreducible highest weight representation of GL_m with highest weight λ is $s_\lambda(x_1, \ldots, x_m)$.*

Proof

By the Weyl character formula, the desired character is $\det({z_i}^{\lambda_j+m-j})_{i,j=1}^m / \det({z_i}^{m-j})_{i,j=1}^m$. By multiplying row i of the numerator by ${z_i}^r$ for all i, we obtain the usual formula for $s_{\lambda+(r,\ldots,r)}(z_1, \ldots, z_m)$.

$\square$

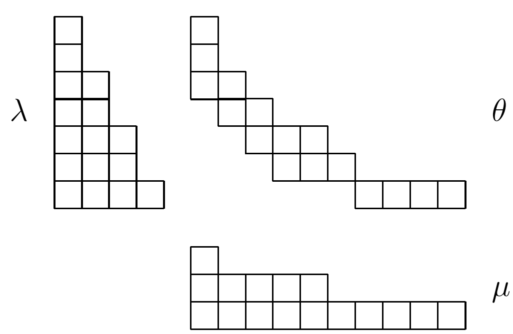
Chapter 2

Combinatorics of Skew-Linked Partitions

2.1 Definition of Skew-Linked Partitions

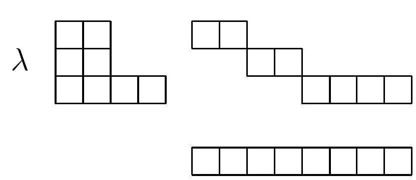
Definition 2.1.1 *Partitions λ and μ are* skew linked, *written $\lambda \xrightarrow{\theta} \mu$, if there exists a skew diagram θ with the same row lengths (in order) as λ and the same column lengths (in order) as μ (meaning column lengths $\mu'_1, \mu'_2, \ldots$). The notation $\lambda \to \mu$ means there exists a skew diagram θ with $\lambda \xrightarrow{\theta} \mu$. Conversely, a skew diagram with weakly decreasing row and column lengths is called a* skew-linking shape.

Example 2.1.2



Lemma 2.1.3 *1. Every partition is linked to itself: $\lambda \xrightarrow{\lambda} \lambda$.*

2. Every λ is linked to the one-row partition (n).



3. If $\lambda \xrightarrow{\theta} \mu$, then $\lambda \leq \mu$ in the dominance partial ordering on partitions.

4. Transpose symmetry: $\lambda \xrightarrow{\theta} \mu$ *if and only if* $\mu' \xrightarrow{\theta'} \lambda'$.

The proof of the above lemma is trivial and is omitted.

Lemma 2.1.4 *If* $\lambda \xrightarrow{\theta} \mu$*, then* θ *is uniquely determined by* λ *and* μ*. Hence skew-linked pairs of partitions correspond bijectively to skew-linking shapes.*

Proof

Suppose $\lambda \xrightarrow{\theta} \mu$ and $\lambda \xrightarrow{\eta} \mu$ with $\theta \neq \eta$, with $\theta = \alpha/\beta$ and $\eta = \hat{\alpha}/\hat{\beta}$. Take the largest i such that the ith rows of θ and η differ in position. Because the ith rows have the same length λ_i, we may assume without loss of generality that $\beta_i < \hat{\beta}_i$. Let $j = \beta_i + 1 \leq \hat{\beta}_i$.

Now the pth row of θ contains a square at position j iff $\beta_p < j \leq \alpha_p$, and similarly for η. Thus $\mu_j = |\{p : \beta_p < j \leq \alpha_p\}| = |\{p : \hat{\beta}_p < j \leq \hat{\alpha}_p\}|$. By assumption θ and η have the same rows $i+1, i+2, \ldots, \ell(\lambda)$, so $|\{p : p \leq i, \beta_p < j \leq \alpha_p\}| = \mu_j - |\{p : p > i, \beta_p < j \leq \alpha_p\}| = \mu_j - |\{p : p > i, \beta_p < j \leq \alpha_p\}| = |\{p : p \leq i, \hat{\beta}_p < j \leq \hat{\alpha}_p\}|$.

Now the ith row of θ contains a square at position j. But the ith row of η does not, so neither do rows $1, 2, \ldots, i-1$ of η. Thus $\{p : p \leq i, \beta_p < j \leq \alpha_p\}$ is nonempty while $\{p : p \leq i, \hat{\beta}_p < j \leq \hat{\alpha}_p\}$ is empty, a contradiction.

□

We can also give a constructive version of the above argument. Given any two partitions λ, μ of the same size, we try to construct a skew-linking θ by adding rows of lengths $\lambda_{\ell(\lambda)}, \ldots, \lambda_2, \lambda_1$ in succession. In the base case, row $\ell(\lambda)$'s leftmost square must be in column 1. Suppose by induction that we have built a skew diagram α^i/β^i from rows of lengths $\lambda_{\ell(\lambda)}, \ldots, \lambda_i$.

We claim that the leftmost square of the $(i-1)$th row must be in column j, where j is the smallest value such that the jth column length of α^i/β^i is smaller than μ'_j. There can be no other choice: If we place the $(i-1)$th row further to the left, then α^{i-1}/β^{i-1}'s $(j-1)$th column will be too long. If we place it further to the right, then the $(i-1)$th row and any future rows will not intersect the jth column, implying that the final skew shape's jth column will be too short. Therefore, each row's placement is strictly determined.

We also see that if there exist i and j such that jth column of α^i/β^i has length greater than μ'_j, then λ and μ are not skew-linked. Thus the above procedure tests for skew-linkedness along the way of constructing the skew-linking shape.

2.2 Row Labels

Definition 2.2.1 *We say that a row (say the rth row) within a skew-shape* $\theta = \alpha/\beta$ begins at i *and* ends at i' *if* $\beta_r = i$ *and* $\alpha_r = i'$*. In other words, the row's leftmost square is in column* $i+1$ *and the row's rightmost square is in column* i'*. (By convention the leftmost column of* θ *is column 1.)*

Lemma 2.2.2 *Let θ be a skew-linking shape. Then for each $j > 0$, θ has at least as many rows ending at j as rows beginning at j. The latter rows are all below the former rows, and the former rows all have the same length. Analogous statements hold if we switch the roles of rows and columns.*

Proof

Suppose p rows end at j and q rows begin at j. The p rows are all the rows that contribute a square to the jth column of θ but not to the $(j+1)$th column. The q rows do the opposite. Therefore, $p - q = \mu'_j - \mu'_{j+1} \geq 0$.

The rows beginning at j are below the rows ending at j by the definition of a skew shape. The latter rows end at the same place, so their lengths are weakly increasing by the definition of a skew shape. But the lengths are also weakly decreasing because θ is a skew-linking shape. Hence the lengths are equal.

The same statements for columns follow by transpose symmetry.

$\square$

Definition 2.2.3 *Let $\theta = \alpha/\beta$ be a skew-linking shape. The* labelling scheme *attached to a skew-linking shape is defined as follows. We will attach a label (i, j) to each row of θ. These labels will be distinct and we will denote by $r_{i,j}$ the number such that the $r_{i,j}$-th row has label (i, j). At the same time we will define positive integers $p_1, \ldots, p_s$ such that the set of labels is $\{(1, 1), \ldots, (1, p_1), (2, 1), \ldots, (2, p_2), \ldots, (s, 1), \ldots, (s, p_s)\}$.*

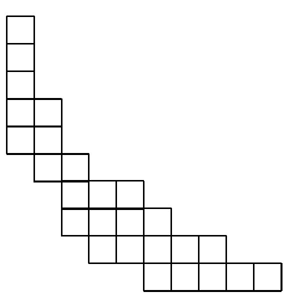
We proceed inductively. Give label $(1, 1)$ to row $r_{1,1} = 1$. Having assigned labels $(1, 1)$ through $(1, j)$ inductively, if $\beta_{r_{1,j}} = 0$, then set $p_1 = j$ and assign no further labels of the form $(1, \cdot)$. Otherwise by Lemma 2.2.2 we may choose the smallest $r_{1,j+1}$ such that $\alpha_{r_{1,j+1}} = \beta_{r_{1,j}}$. In this case, give label $(1, j + 1)$ to row $r_{1,j+1}$. Thus we have given out labels $(1, 1), (1, 2), \ldots, (1, p_1)$ to rows $r_{1,1}, \ldots, r_{1,p_1}$.

Next, if any rows remain unlabelled, we ignore the already labelled rows and repeat the above for labels $(2, 1), \ldots, (2, p_2)$: Give label $(2, 1)$ to the bottommost unlabelled row, and so on. Again success is ensured by Lemma 2.2.2. Next, if any rows remain unlabelled, we we start over with labels $(3, 1), \ldots, (3, p_3)$ and so on. Eventually all rows will be labelled.

The label (i, j) is really a tuple of two different labels, clumped together for convenience. The i is the first label, while j is the second label.

If no row has label (i, j), then by convention we set $r_{i,j} = \infty$ and $\alpha_{r_{i,j}} = \beta_{r_{i,j}} = 0$. We may also label the columns of α/β similarly, using the transposed version of Lemma 2.2.2. Below we will assume that every skew-linking shape's rows and columns are labelled as above.

Example 2.2.4



$p_1 = 3 : r_{1,1} = 1, r_{1,2} = 4, r_{1,3} = 6$
$p_2 = 3 : r_{2,1} = 2, r_{2,2} = 5, r_{2,3} = 8$
$p_3 = 2 : r_{3,1} = 3, r_{3,2} = 7$
$p_4 = 1 : r_{4,1} = 9$
$p_5 = 1 : r_{5,1} = 10$

$\nu^1 = (5,5,4,1,1), \nu^2 = (3,2,2), \nu^3 = (2,1); \gamma^1 = (5,3,2), \gamma^2 = (5,2,1), \gamma^3 = (4,2), \gamma^4 = (1), \gamma^5 = (1)$ *(ν's and γ's are defined in following sections)*

If $(i_1, j_1), \ldots, (i_t, j_t)$ are the rows beginning at $q > 0$, then the labelling procedure implies that the bottom t rows ending at q are labelled $(i_1, j_1+1), \ldots, (i_t, j_t+1)$ in some order. The remaining rows ending at q are not matched up on the right, so they must have 1 as second label; there are $\mu'_q - \mu'_{q+1}$ such rows (see proof of Lemma 2.2.2). In fact all rows with second label 1 must arise this way for some q.

What are their first labels? Each row $(i,1)$ is the lowest among all rows still unlabelled, which include rows $(i',1)$ for all $i' > i$. Thus $r_{1,1} < r_{2,1} < r_{3,1} < \ldots$. Now because α is a partition, rows ending at q are above rows ending at $> q$. Thus rows with second label 1 are labelled in the following order: {rows ending at α_1}, {rows ending at $\alpha_1 - 1$}, {rows ending at $\alpha_1 - 2$}, and so on. In particular, before the rows ending at q are labelled, the number of second labels of 1 already used is $\sum_{j \geq q+1} (\mu'_q - \mu'_{q+1}) = \mu'_{q+1}$. Thus the $\mu'_q - \mu'_{q+1}$ rows ending at q with second label 1 receive the first labels $\mu'_{q+1}+1, \mu'_{q+1}+2, \ldots, \mu'_q$.

There is a cleaner way of saying this. To find the row with label $(i,1)$, find the rightmost column of length at least i. Then the row containing this column's ith square (from bottom to top) has label $(i,1)$. For instance, if $\mu'_3 = 7$ and $\mu'_4 = 5$, then the top two squares of the third column are in rows $(7,1)$ and $(6,1)$.

Definition 2.2.5 *(Graphical Interpretation) The* directed graph representation of the labelling scheme *of α/β is defined as follows: For each row create a vertex that is labelled by the label of the row as well the length of the row. If vertices (i,j) and $(i,j+1)$ exist, create a directed edge from the former to the latter. Then G is a union of directed paths of the form $(i,1), \lambda_{r_{i,1}} \to (i,2), \lambda_{r_{i,2}} \to \cdots \to (i,p_i), \lambda_{r_{i,p_i}}$.*

Lemma 2.2.6 *Consider a different labelling scheme where the label $(i,j+1)$ is given to* any *row ending at where row (i,j) begins, as opposed to the the first such row (as above). Then graph G is constant regardless of choices made.*

Proof

By Lemma 2.2.2 and induction, at each step the choice is among rows that are vertical translates. Since the graph only records the length of each row (i,j), we get the same graph regardless of the choice of the vertical translate each time.

□

We make an important remark that $\sum_j \lambda_{r_{i,j}} = \mu_i$. In other words, the lengths of the rows with first label i sum to μ_i. The proof will be in Lemma 2.3.7.

2.3 Interweaving Property and Definition of ν^i

Proposition 2.3.1 *(Betweenness) Suppose $a < b$ and row $(b,1)$ exists.*

1. *There exists a unique q such $r_{a,q} < r_{b,1} < r_{a,q+1}$ (with our convention that $r_{i,j} = \infty$ for unused labels (i,j)).*

2. *For $j = 1, \ldots, p_b$, we have $r_{a,q+j-1} < r_{b,j}$.*

3. *For $j = 1, \ldots, p_b - 1$, row $(a, q+j)$ exists and $\beta_{r_{b,j}} \geq \beta_{r_{a,q+j}}$ and $\alpha_{r_{b,j}} \geq \alpha_{r_{a,q+j}}$. If row $(a, q+p_b)$ exists, then these inequalities hold for $j = p_b$ too.*

4. *For $j = 1, \ldots, p_b - 1$, either $r_{b,j} < r_{a,q+j}$ or the two rows are vertical translates of each other. This also holds in the case $j = p_b$ if row $(a, q+p_b)$ exists.*

5. *$p_a = p_b + q - 1$ or $p_b + q$.*

6. *For $j = 1, \ldots, p_b$, $\lambda_{r_{a,q+j}} \leq \lambda_{r_{b,j}} \leq \lambda_{r_{a,q+j-1}}$. This also holds for larger j because $\lambda_\infty = 0$ by convention.*

Proof

Row $(b,1)$ was unlabelled when row $(a,1)$ was labelled, so $r_{b,1} > r_{a,1}$. Since $r_{a,1} < r_{a,2} < \ldots, < r_{a,p_a}$, q may (and must) be chosen as the largest value of q such that $r_{b,1} > r_{a,q}$. From this it follows that row (a,q) exists and $r_{b,1} < r_{a,q+1}$ (because r values are distinct except at ∞).

We show (2) and (3) by induction in the cases $j = 1, \ldots, p_b - 1$. If $p_b = 1$ then there is nothing to prove. Suppose $p_b > 1$. Then the case $j = 1$ for (2) follows from (1). Now (1) and $p_b > 1$ imply $\beta_{r_{a,q}} \geq \beta_{r_{b,1}} > 0$, so row $(a, q+1)$ exists. Since $r_{b,1} < r_{a,q+1}$, (3) holds for $j = 1$ and the base case is proved.

Suppose (2) and (3) hold for some $j < p_b - 1$. Then $r_{a,q+j-1} < r_{b,j}$, so $\alpha_{r_{b,j+1}} = \beta_{r_{b,j}} \geq \beta_{r_{a,q+j-1}} = \alpha_{r_{a,q+j}}$. If equality holds, then by Lemma 2.2.2, rows $(b, j+1)$ and $(a, q+j)$ are vertical translates. Since the label $(a, q+j)$ was given before $(b, j+1)$, $r_{a,q+j} < r_{b,j+1}$. If equality fails, then we also have $r_{a,q+j} < r_{b,j+1}$ because α is weakly decreasing. This gives (2).

Now $r_{a,q+j} < r_{b,j+1}$ and $j+1 < p_b$ imply $\beta_{r_{a,q+j}} \geq \beta_{r_{b,j+1}} > 0$, so row $(a, q+j+1)$ exists. By the inductive hypothesis for (3), $\alpha_{r_{b,j+1}} = \beta_{r_{b,j}} \geq \beta_{r_{a,q+j}} = \alpha_{r_{a,q+j+1}}$. If equality holds, then Lemma 2.2.2 says that rows $(b, j+1)$ and $(a, q+j+1)$ are vertical translates, so $\beta_{r_{b,j+1}} = \beta_{r_{a,q+j+1}}$. If equality fails, then $r_{b,j+1} < r_{a,q+j+1}$ because α is weakly decreasing, so $\beta_{r_{b,j+1}} \geq \beta_{r_{a,q+j+1}}$. Thus (3) holds and induction is complete.

In particular, (2) and (3) hold for $j = p_b - 1$. Thus the argument two paragraphs above also applies to $j = p_b - 1$, so (2) also holds for $j = p_b$ and is completely proved. The argument one paragraph above applies to $j = p_b - 1$ except for the first sentence, where the existence of row $(a, q+j+1)$ is shown. Thus if row $(a, q+p_b)$ exists, then the entire argument is valid and proves (3) for $j = p_b$. Hence (3) is completely proved too. Now (4) follows from (3) because α and β are weakly decreasing.

Note that row $(a, q+p_b-1)$ exists by (3). If row $(a, q+p_b)$ does not exist, then $p_a = q + p_b - 1$. If row $(a, q+p_b)$ exists, then $0 = \beta_{r_{b,p_b}} \geq \beta_{r_{a,q+p_b}}$ by (3). Hence $\beta_{r_{a,q+p_b}} = 0$ and $p_a = q + p_b$, so (5) is proved.

Lastly, the right inequality of (6) follows from (2) because row lengths are weakly decreasing. For cases $j = 1, \ldots, p_b - 1$, the left inequality of (6) follows from (4) because row lengths are weakly decreasing and vertical translates have the same length. For the case $j = p_b$, if row $(a, q + p_b)$ exists, then the left inequality again follows from (4). If row $(a, q + p_b)$ does not exist, by convention $\lambda_{r_{a,q+p_b}} = \lambda_\infty = 0$ and the left inequality still holds.

□

Corollary 2.3.2 *Suppose* $1 \le a < b$. *Then* $p_a \ge p_b$, $r_{a,j} < r_{b,j}$ *for* $j = 1, \ldots, p_b$, *and* $r_{a,j} \le r_{b,j}$ *for* $j > p_b$.

Proof

Choose q according to Proposition 2.3.1 (1). Then $p_a \ge p_b + q - 1 \ge p_b$. For $j = 1, \ldots, p_b$, $r_{a,j} < r_{a,q+j-1} \le r_{b,j}$ by $j \le q + j - 1$ and by Proposition 2.3.1 (2). If $j > p_b$, then $r_{b,j} = \infty$ and the inequality holds.

□

Proposition 2.3.3 *Suppose* α/β *is a skew-linking shape. Then* $\nu^1 = (\lambda_{r_{1,1}}, \lambda_{r_{2,1}}, \lambda_{r_{3,1}}, \ldots)$ *is a partition. If we remove from* α/β *the rows* $r_{i,1}$ *for all* i*, the resulting skew diagram* $\hat{\alpha}/\hat{\beta}$ *is once again a skew-linking shape. Furthermore, row* (i, j) *of* $\hat{\alpha}/\hat{\beta}$ *was originally row* $(i, j+1)$ *in* α/β.

Proof

By Corollary 2.3.2, $r_{i,1}$ is weakly increasing in i. Thus $\lambda_{r_{i,1}}$ is weakly decreasing and ν^1 is a partition.

Because α/β has weakly decreasing row lengths, so does $\hat{\alpha}/\hat{\beta}$. Now take $j \ge 1$. By the discussion about rows with second label 1 in the previous section, the number of such rows ending at j is $\mu'_j - \mu'_{j+1}$. Let p be the number of rows with second label 1 that intersect both columns j and $j+1$. Let q be the number of rows beginning at j with second label 1. Then $\hat{\alpha}/\hat{\beta}$'s column j has length $\mu'_j - (\mu'_j - \mu'_{j+1}) - p = \mu'_{j+1} - p$ and its column $j+1$ has length $\mu'_{j+1} - p - q \le \mu'_{j+1} - p$. Therefore, $\hat{\alpha}/\hat{\beta}$ has weakly decreasing column lengths and is a skew-linking shape.

Let B be the set of rows of α/β with second label ≥ 2, i.e. the rows of $\hat{\alpha}/\hat{\beta}$. By Corollary 2.3.2, for $i \ge 1$ and $j \ge 2$ we know that $r_{i,j} \ge r_{1,j} \ge r_{1,2}$. Thus row $(1, 2)$ is the highest remaining row in B and gets the label $(1, 1)$ in $\hat{\alpha}/\hat{\beta}$.

Suppose by induction that the original row $(1, j)$ gets label $(1, j-1)$ for $j = 2, \ldots, p$. B has a more restricted choice for the next label than does α/β. But the original row $(1, j+1)$ is still unlabelled and hence the lowest row among all valid choices, so it gets label (i, j). Thus the rows with original first label 1 get relabelled in the manner claimed.

Now that the original rows $(1, j)$ are removed from B for all j, by the above argument (with $i \ge 2$ instead of $i \ge 1$) the highest remaining row is the original row $(2, 2)$, so it gets label $(2,1)$ in $\hat{\alpha}/\hat{\beta}$. By a similar induction as above, the original row $(2, j+1)$ gets label $(2, j)$ in $\hat{\alpha}/\hat{\beta}$ for all j.

Therefore, continuing this argument by induction, all rows in B get labelled in $\hat{\alpha}/\hat{\beta}$ in the manner claimed.

□

Corollary 2.3.4 *Let $q \geq 1$. By Proposition 2.3.3 and induction, removing rows with original second labels $\leq q$ from a skew-linking shape α/β produces another skew-linking shape whose row (i, j) is the original row $(i, j+q)$.*

Definition 2.3.5 *For skew-linking shape α/β, let $\nu^j = (\lambda_{r_{1,j}}, \lambda_{r_{2,j}}, \lambda_{r_{3,j}}, \ldots)$, a partition by Corollary 2.3.2. Define $\nu = (|\nu^1|, |\nu^2|, |\nu^3|, \ldots)$. We consider ν^j and ν to be associated with α/β.*

Lemma 2.3.6 *ν is a partition of n. The statistic $n(\nu) = \sum_i (i-1)\,\nu_i = \sum_i (i-1)\,|\nu^i|$ is equal to $|\beta|$, the number of "missing boxes."*

Proof

Suppose $i < i'$. By Corollary 2.3.2, $r_{i,j} \leq r_{i',j}$ for all j, so $\nu_i = \sum_j \lambda_{r_{i,j}} \geq \sum_j \lambda_{r_{i',j}} = \nu_{i'}$. Now every row in α/β is labelled, so $\sum_i \nu_i = \sum_i \sum_j \lambda_{r_{i,j}} = \sum_i \lambda_i = n$. Thus ν is a partition of n.

Fix i. By construction, rows $r_{i,j}, r_{i,j+1}, \ldots$ combined have exactly one square in each of columns $1, 2, \ldots, \beta_{r_{i,j}}$ (and intersect no other column). Hence $\beta_{r_{i,j}} = \sum_{j'>j} \lambda_{r_{i,j'}}$ for each j. Thus $\sum_j \beta_{r_{i,j}} = \sum_j \sum_{j'>j} \lambda_{r_{i,j'}} = \sum_{j'} \sum_{j<j'} \lambda_{r_{i,j'}} = \sum_{j'} (j'-1)\lambda_{r_{i,j'}}$. Now vary over i to get $|\beta| = \sum_i \sum_j \beta_{r_{i,j}} = \sum_i \sum_{j'} (j'-1)\lambda_{r_{i,j'}} = \sum_{j'} (j'-1) \sum_i \lambda_{r_{i,j'}} = \sum_{j'} (j'-1)|\nu^{j'}| = \sum_{j'} (j'-1)\nu_{j'} = n(\nu)$.

□

Lemma 2.3.7 *$\cup_i \nu^i = \lambda$ and $\sum_i \nu^i = \mu$.*

Proof

The first equality is clear from the construction of ν^i's. For the second equality, note that the rows $(i,1), (i,2), \ldots, (i, p_i)$ combine to contribute one square to each of columns $1, 2, \ldots, \sum_j \lambda_{r_{i,j}}$. Thus $\mu'_q =$ the qth column length of $\alpha/\beta = |\{i : q \leq \sum_j \lambda_{r_{i,j}}\}|$. By Corollary 2.3.2, $(\sum_j \lambda_{r_{1,j}}, \sum_j \lambda_{r_{2,j}}, \ldots)$ is a partition, and note it also has qth column length $|\{i : q \leq \sum_j \lambda_{r_{i,j}}\}|$. Therefore $\mu = (\sum_j \lambda_{r_{1,j}}, \sum_j \lambda_{r_{2,j}}, \ldots) = \sum_j (\lambda_{r_{1,j}}, \lambda_{r_{2,j}}, \ldots) = \sum_j \nu^j$.

□

2.4 Definitions and Properties of γ^i and $\bar{A}$

Definition 2.4.1 *Let $\gamma^i = (\lambda_{r_{i,1}}, \lambda_{r_{i,2}}, \ldots)$, the lengths of rows with first label i. Each γ^i is a partition because $r_{i,j}$ weakly increases as j increases. We have containments $\gamma^1 \supset \gamma^2 \supset \ldots$ by Corollary 2.3.2. Clearly $\cup_i \gamma^i = \lambda$. By Lemma 2.3.7, $\mu_i = |\gamma^i|$ for all i.*

Proposition 2.4.2 *Let $\tilde{\alpha}/\tilde{\beta}$ and $\hat{\alpha}/\hat{\beta}$ be the results of removing rows with first label 1 and removing the first row from α/β, respectively, from α/β.*

1. *$\tilde{\alpha}/\tilde{\beta}$ and $\hat{\alpha}/\hat{\beta}$ are skew-linking shapes, so we may label their rows.*

2. *Row (i,j) in $\tilde{\alpha}/\tilde{\beta}$ corresponds to row $(i+1,j)$ in α/β.*

3. *Suppose $p_1 > 1$ (i.e. $\beta \neq \emptyset$). Let $\{\hat{\gamma}^i\}$ correspond to $\hat{\alpha}/\hat{\beta}$. Then there exists q such that $\hat{\gamma}^i = \gamma^{i+1}$ for $i = 1, \ldots, q-1$, $\hat{\gamma}^q = (\gamma^1_2, \gamma^1_3, \ldots)$, and $\hat{\gamma}^i = \gamma^i$ for $i \geq q+1$.*

Proof

The new skew shapes' row length sequences are subsequences of α/β's, so they remain weakly decreasing. We removed one square from every column of α/β to obtain $\tilde{\alpha}/\tilde{\beta}$, so the latter still has weakly decreasing column lengths. We removed one square from each of the $\alpha_1 - \beta_1$ rightmost columns of α/β to obtain $\hat{\alpha}/\hat{\beta}$. This also preserves the weakly decreasing property of column lengths. Hence (1) holds. (2) follows directly from the labelling procedure.

For (3), by assumption row $(1,2)$ exists in α/β. By Lemma 2.2.2 and the remarks about rows with second label 1 in the previous section, the set of rows $(i,1)$ in $\hat{\alpha}/\hat{\beta}$ correspond to the set consisting of rows $(i,1)$ for $i \geq 2$ in α/β and another row c that is a vertical translate of row $(1,2)$ in α/β. The latter set is in order $r_{2,1} < \ldots < r_{q,1} < c < r_{q+1,1} < \ldots$ for some $q \geq 1$ (depending on where c inserts into $r_{2,1} < r_{3,1} < \ldots$). Hence

- row $(i,1)$ in $\hat{\alpha}/\hat{\beta}$ corresponds to row $(i+1,1)$ in α/β for $i = 1, 2, \ldots, q-1$;
- row $(q,1)$ in $\hat{\alpha}/\hat{\beta}$ corresponds to row c in α/β;
- row $(i,1)$ in $\hat{\alpha}/\hat{\beta}$ corresponds to row $(i,1)$ in α/β for $i \geq q+1$.

According to Definition 2.2.5, let G and $\hat{G}$ be the graphs for α/β and $\hat{\alpha}/\hat{\beta}$ respectively. We may consider G and $\hat{G}$ to be constructed in the way of Lemma 2.2.6 (where we may choose among vertical translates) and still retain the same graphs, so that instead of row c we choose row $(1,2)$ instead. Then deleting the first row preserves much of the graph structure: To get from G to $\hat{G}$, we delete vertex $(1,1)$ and relabel the components' starting vertices by the above correspondence (and where $(1,2)$ becomes $(q,1)$). As a result, the other vertices also get relabelled appropriately, and we have for $j \geq 1$:

- row (i,j) in $\hat{\alpha}/\hat{\beta}$ has the same length as row $(i+1,j)$ in α/β for $i = 1, 2, \ldots, q-1$;
- row (q,j) in $\hat{\alpha}/\hat{\beta}$ has the same length as row $(1,j+1)$ in α/β;

- row (i,j) in $\hat{\alpha}/\hat{\beta}$ has the same length as row (i,j) in α/β for $i \geq q+1$.

This is exactly the condition stated in (3).

$\square$

Proposition 2.4.3 *Suppose* $1 \leq a < b$ *and* $1 \leq p < q$.

1. *If* $(\gamma^b)'_q \geq 1$, *then* $(\gamma^a)'_p - (\gamma^a)'_q$ *and* $(\gamma^b)'_p - (\gamma^b)'_q$ *differ by no more than 1.*
2. *If* $(\gamma^b)'_q = 0$, *then* $(\gamma^a)'_p - (\gamma^a)'_q \geq (\gamma^b)'_p - (\gamma^b)'_q - 1$.

Proof

In both cases note that $(\gamma^a)'_p - (\gamma^a)'_q$ is the number of rows in γ^a (i.e. first label a) of lengths $p, p+1, \ldots, q-1$. Also, $(\gamma^a)'_q$ is the number of rows with first label a and length $\geq q$. Similar is true for b. Choose the unique s such that $r_{a,s} < r_{b,1} < r_{a,s+1}$, as in Proposition 2.3.1 (1). Below we repeatedly use Proposition 2.3.1 (6): For all $j \geq 1$, $\lambda_{r_{a,s+j}} \leq \lambda_{r_{b,j}} \leq \lambda_{r_{a,s+j-1}}$.

1. In this case, row $(b,1)$ has length $\geq q$. Let t,u be the largest values such that rows (b,t) and (b,u) have lengths $\geq q$ and $\geq p$, respectively. We investigate the value of $\lambda_{r_{a,i}}$ by considering four cases:

   - Use $j = t$ above to get $\lambda_{r_{b,t}} \leq \lambda_{r_{a,t+s-1}}$. By construction $q \leq \lambda_{r_{b,t}}$. Thus for $i = 1, \ldots, t+s-1$, $q \leq \lambda_{r_{a,t+s-1}} \leq \lambda_{r_{a,i}}$.
   - Use $j = t+1$ above and the definition of t to get $\lambda_{r_{a,t+s+1}} \leq \lambda_{r_{b,t+1}} < q$. Use $j = u$ above and the definition of u to get $p \leq \lambda_{r_{b,u}} \leq \lambda_{r_{a,u+s-1}}$. Thus for $i = t+s+1, \ldots, u+s-1$ (possibly vacuous), $p \leq \lambda_{r_{a,u+s-1}} \leq \lambda_{r_{a,i}} \leq \lambda_{r_{a,t+s+1}} < q$.
   - Use $j = u+1$ above and the definition of u to get $\lambda_{r_{a,u+s+1}} \leq \lambda_{r_{b,u+1}} < p$. Thus for $i \geq u+s+1$, $\lambda_{r_{a,i}} \leq \lambda_{r_{a,u+s+1}} < p$.
   - The remaining cases are $i = t+s$ and $u+s$.

   Hence the values of i where $p \leq \lambda_{r_{a,i}} < q$ are $t+s+1, \ldots, u+s-1$ and possibly $t+s$ and/or $u+s$. Recall that $(\gamma^a)'_p - (\gamma^a)'_q$ is the number of such values i.

   Now by construction, $p \leq \lambda_{r_{b,i}} < q$ iff $t+1 \leq i \leq u$, so $(\gamma^b)'_p - (\gamma^b)'_q = u-t$. If $u > t$, then $\{t+s+1, \ldots, u+s-1\}$ has size $u-t-1$, so $u-t-1 \leq (\gamma^a)'_p - (\gamma^a)'_q \leq u-t+1$. If $u = t$, then $\{t+s+1, \ldots, u+s-1\}$ has size 0 and $t+s$ and $u+s$ are the same case, so $(\gamma^a)'_p - (\gamma^a)'_q$ is 0 or 1. In all cases, we see that $(\gamma^a)'_p - (\gamma^a)'_q$ and $(\gamma^b)'_p - (\gamma^b)'_q$ differ by no more than 1.

2. In this case, row $(b,1)$ has length $< q$. If it has length $< p$ as well, then all rows (b,i) have length $< p$, so $(\gamma^b)'_p - (\gamma^b)'_q - 1 = -1 < (\gamma^a)'_p - (\gamma^a)'_q$.

   Now consider the case row $(b,1)$ has length $\geq p$. Then we may define u to be the largest value such that row (b,u) has length $\geq p$. Use $j = 1$ above to get $\lambda_{r_{a,s+1}} \leq \lambda_{r_{b,1}} < q$. Use $j = u$ above and the definition of u to get $p \leq \lambda_{r_{b,u}} \leq \lambda_{r_{a,u+s-1}}$. Thus for

$i = s+1, \ldots, u+s-1$, $p \leq \lambda_{r_{a,u+s-1}} \leq \lambda_{r_{a,i}} \leq \lambda_{r_{a,s+1}} < q$. The set $\{s+1, \ldots, u+s-1\}$ has size $u-1$ (since $u \geq 1$), so $(\gamma^a)'_p - (\gamma^a)'_q \geq u-1$. But the values of i where $p \leq \lambda_{r_{b,i}} < q$ are $1, \ldots, u$, so $(\gamma^b)'_p - (\gamma^b)'_q = u$ and the desired inequality holds.

□

Definition 2.4.4 *Given* $\lambda \xrightarrow{\theta} \mu$, *construct* γ^i *as above and associate to* θ *the* $\mathbb{N} \times \mathbb{N}$ *matrix* $\bar{A}$ *given by* $\bar{A}_{i,j} = (\gamma^i)'_j$.

Lemma 2.4.5 $\bar{A}$ *is a plane partition with layers* $(\nu^1, \nu^2, \ldots)$, *i.e.* $\nu^p_i = |\{j : A_{i,j} \geq p\}|$.

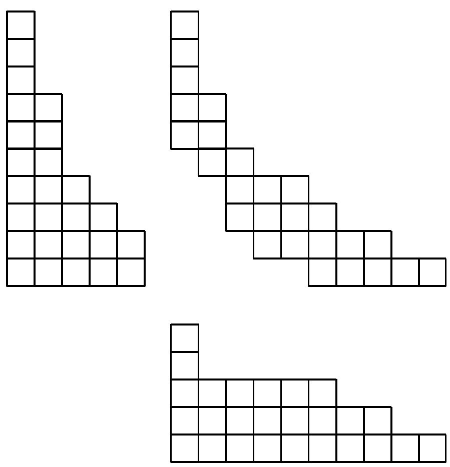
Proof

The containments $\gamma^1 \supset \gamma^2 \supset \ldots$ imply that $(\gamma^1)' \supset (\gamma^2)' \supset \ldots$ and that $\bar{A}$ is a plane partition. Now $A_{i,j} \geq p \iff (\gamma^i)'_j \geq p \iff \gamma^i_p \geq j \iff$ row (i,p) has length $\geq j$. Thus $|\{j : A_{i,j} \geq p\}| = $ length of row $(i,p) = \nu^p_i$.

□

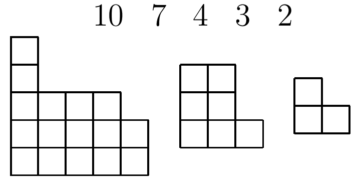
Example 2.4.6

λ



μ

$$A = \begin{array}{cc} \begin{pmatrix} 3 & 3 & 2 & 1 & 1 \\ 3 & 2 & 1 & 1 & 1 \\ 2 & 2 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} & \begin{matrix} 10 \\ 8 \\ 6 \\ 1 \\ 1 \end{matrix} \\ \begin{matrix} 10 & 7 & 4 & 3 & 2 \end{matrix} & \end{array}$$



ν^1 ν^2 ν^3

Chapter 3

Connections to Nonnegative Integer Matrices of Specified Row and Column Sums

3.1 Nonnegative Integer Matrices and Tableaux

Definition 3.1.1 *Given compositions α, β, let $\mathbb{M}(\alpha, \beta)$ be the set of finitely-supported $\mathbb{N} \times \mathbb{N}$ matrices A with nonnegative integer entries and row and column sums according to α, β, i.e. $\sum_j A_{i,j} = \alpha_i$ and $\sum_i A_{i,j} = \beta_j$ for all i, j. Evidently A is just an $\ell(\beta) \times \ell(\alpha)$ matrix, but the extra zero rows and columns are for convenience.*

Definition 3.1.2 *(Tableau interpretation) In the case β is a partition, we may regard each $A \in \mathbb{M}(\alpha, \beta)$ as a tableau of shape β' with $A_{i,j}$ copies of i in the jth column, disregarding the order within each column. Then A records the multiplicities, and the letter i occurs α_i times in total (i.e. the tableau has weight α). Note that the tableau in this interpretation needs not be semistandard.*

3.2 Balanced Matrices

Definition 3.2.1 *A matrix A is* balanced *if it is $\mathbb{N} \times \mathbb{N}$ with nonnegative integer entries such that for any i, i', j, j' with $i \neq i'$ and $j \neq j'$, if $A_{i,j} \neq 0 \neq A_{i',j'}$ then $A_{i,j} + A_{i',j'} \leq A_{i,j'} + A_{i',j} + 1$. Let $\mathbb{M}_b(\alpha, \beta)$ denote the set of balanced matrices in $\mathbb{M}(\alpha, \beta)$.*

Lemma 3.2.2 *If A is balanced, then so is the transpose A^T.*

Proof

Suppose i, i', j, j' with $i \neq i'$ and $j \neq j'$ and $A^T_{i,j} \neq 0 \neq A^T_{i',j'}$. Then $A_{j,i} \neq 0 \neq A_{j',i'}$, so by balancedness $A_{j,i} + A_{j',i'} \leq A_{j,i'} + A_{j',i} + 1$. It follows that $A^T_{i,j} + A^T_{i',j'} = A_{j,i} + A_{j',i'} \leq A_{j,i'} + A_{j',i} + 1 = A^T_{i,j'} + A^T_{i',j} + 1$, so A^T is balanced.

□

Lemma 3.2.3 *If A is balanced, then for $i \neq i'$, either $A_{i,j} \geq A_{i',j}$ for all j or $A_{i',j} \geq A_{i,j}$ for all j. In other words, every pair of rows are comparable in the product order $\geq_p$. Every pair of columns are comparable in the product order too.*

Proof

Suppose rows i and i' are incomparable in the product order. Then there exist j, j' such that $A_{i,j} > A_{i',j} \geq 0$ and $0 \leq A_{i,j'} < A_{i',j'}$. Then $A_{i,j} \neq 0 \neq A_{i',j'}$ and $A_{i,j} + A_{i',j'} \geq (A_{i,j'} + 1) + (A_{i',j} + 1) > A_{i,j'} + A_{i',j} + 1$, contradicting the assumption that A is balanced. The last statement follows from Lemma 3.2.2 and using the above argument for A^T instead of A.

□

Lemma 3.2.4 *Suppose A is balanced and that there exist p and q such that $A_{p,j} \geq 1$ for $j = 1, \ldots, q$ and $A_{i,j} = 0$ when $j > q$. Define matrix B with the same entries as A except that $B_{p,j} = A_{p,j} - 1$ for $j = 1, \ldots, q$. Then B is balanced.*

Proof

By construction B is also an $\mathbb{N} \times \mathbb{N}$ matrix with nonnegative integer entries. Suppose i, i', j, j' with $i \neq i'$ and $j \neq j'$ and $B_{i,j} \neq 0 \neq B_{i',j'}$. Then $j, j' \leq q$. The balancedness inequality for B follows from the inequality for A by either keeping both sides the same (if $i \neq p \neq i'$) or subtracting one from each side (if $i = p \neq i'$ or $i' = p \neq i$).

□

Lemma 3.2.5 *If λ, μ are partitions and $A \in \mathbb{M}_b(\mu, \lambda')$, then A has weakly decreasing rows and columns (i.e. is a plane partition).*

Proof

Suppose $A_{i,j} < A_{i,j'}$ for some $j < j'$ and some i. Now $\sum_{i'}(A_{i',j} - A_{i',j'}) = \lambda'_j - \lambda'_{j'} \geq 0$ and $A_{i,j} - A_{i,j'} < 0$, so there exists $i' \neq i$ such that $A_{i',j} - A_{i',j'} > 0$. Then columns j and j' are incomparable in the product order, contradicting Lemma 3.2.3. Hence A has weakly decreasing rows.

Suppose $A_{i,j} < A_{i',j}$ for some $i < i'$ and some j. Now $\sum_{j'}(A_{i,j'} - A_{i',j'}) = \mu_i - \mu_{i'} \geq 0$ and $A_{i,j} - A_{i',j} < 0$, so there exists $j' \neq j$ such that $A_{i,j'} - A_{i',j'} > 0$. Then rows i and i' are incomparable in the product order, contradicting Lemma 3.2.3. Hence A has weakly decreasing columns.

□

3.3 Connections Between Balanced Matrices and Skew-Linked Partitions

Proposition 3.3.1 *Let $\bar{A}$ be the matrix associated to $\lambda \to \mu$ (see Definition 2.4.4). Then $\bar{A} \in \mathbb{M}_b(\mu, \lambda')$.*

Proof

Definition 2.4.1 notes that $|\gamma^i| = \mu_i$, so $\sum_j \bar{A}_{i,j} = \sum_j (\gamma^i)'_j = |(\gamma^i)'| = |\gamma^i| = \mu_i$. Definition 2.4.1 also notes that $\cup_i \gamma^i = \lambda$, so we can equate column lengths to get $\lambda'_j = \sum_i (\gamma^i)'_j = \sum_i \bar{A}_{i,j}$. Thus $\bar{A} \in \mathbb{M}(\mu, \lambda')$.

Suppose we have $i \neq i'$ and $j \neq j'$ and $A_{i,j} \neq 0 \neq A_{i',j'}$. Without loss of generality, assume $i < i'$. If $j < j'$, then the balancedness inequality follows from Proposition 2.4.3 case 1, using $a = i, b = i', p = j, q = j'$. If $j > j'$, then use $a = i, b = i', p = j', q = j$; in either case of Proposition 2.4.3, we have $\bar{A}_{i,j'} - \bar{A}_{i,j} \geq \bar{A}_{i',j'} - \bar{A}_{i',j} - 1$, which implies balancedness.

□

Theorem 3.3.2 *Suppose λ, μ are partitions of n.*

1. *$\mathbb{M}_b(\mu, \lambda')$ is nonempty iff $\lambda \to \mu$.*
2. *Suppose $\lambda \xrightarrow{\theta} \mu$. Then $\mathbb{M}_b(\mu, \lambda') = \{\bar{A}\}$, where $\bar{A}$ is given in Definition 2.4.4.*

Proof

We prove both parts by strong induction on n. The base case $n = 1$ is clear. Suppose the claims are true for $1, 2, \ldots, n-1$.

Consider the case $\lambda = (n)$. Note $\lambda \xrightarrow{\theta} \mu$ iff $\theta = (n)$ and thus iff $\mu = (n)$. Suppose $A \in \cup_\mu \mathbb{M}_b(\mu, \lambda')$. Then A has column sums $(1, 1, \ldots)$ and is a plane partition by Lemma 3.2.5, so A must have all zeroes except $A_{1,j} = 1$ for $j = 1, \ldots, n$. Since A is balanced, $\{A\} = \cup_\mu \mathbb{M}_b(\mu, \lambda')$. This A has row sums $(n, 0, \ldots)$, so $\mathbb{M}_b(\mu, \lambda') = \{A\}$ if $\mu = (n)$ and $\emptyset$ if $\mu \neq (n)$, and (1) holds. In the case $\lambda \to \mu$ (i.e. $\mu = (n)$),we have $\gamma^{(1)} = (n)$ and $\gamma^{(p)} = \emptyset$ for $p \geq 2$, so $\bar{A}$ equals the above A and (2) holds.

Now consider the case $\ell(\lambda) \geq 2$ and let $A \in \mathbb{M}_b(\mu, \lambda')$. By Lemma 3.2.5 A is a plane partition, so $A_{1,j} \geq A_{i,j}$ for all i. Since $\sum_i A_{i,j} = \lambda'_j = 0$ iff $j > \lambda_1$, we must have $A_{1,j} > 0$ for $j = 1, \ldots, \lambda_1$ and $A_{i,j} = 0$ when $j > \lambda_1$. Define matrix B with the same entries as A except that $B_{1,j} = A_{1,j} - 1$ for $j = 1, \ldots, \lambda_1$. Then B is balanced by Lemma 3.2.4.

Let $\rho = (\mu_1 - \lambda_1, \mu_2, \mu_3, \ldots)$ and $\eta = (\lambda_2, \lambda_3, \ldots)$. Then $\eta'_j = \lambda'_j - 1$ for $j = 1, \ldots, \lambda_1$ and $\eta'_j = 0$ for $j > \lambda_1$, so $B \in \mathbb{M}_b(\rho, \eta')$. There exists unique q such that $\mu_q > \mu_1 - \lambda_1 \geq \mu_{q+1}$ because $\mu_1 - \lambda_1 \in [0, \mu_1) = \sqcup_q [\mu_{q+1}, \mu_q)$. Then $\hat{\mu} = (\mu_2, \ldots, \mu_q, \mu_1 - \lambda_1, \mu_{q+1}, \mu_{q+2}, \ldots)$ is the partition that is a permutation of ρ. Define matrix C by $C_{i,j} = B_{i+1,j}$ for $i = 1, \ldots, q-1$, $C_{q,j} = B_{1,j}$, and $C_{i,j} = B_{i,j}$ for $j \geq q+1$. Then $C \in \mathbb{M}(\hat{\mu}, \eta')$ because $B \in \mathbb{M}(\rho, \eta')$. Now B is balanced and C is obtained by permuting the rows of B, so it is clear from the definition that C is also balanced. Hence $C \in \mathbb{M}_b(\hat{\mu}, \eta')$.

Since $|\hat{\mu}| = |\eta| = n - \lambda_1 < n$ and $|\eta| \geq \lambda_2 > 0$, the inductive hypothesis of (1) implies that $\eta \overset{\hat{\alpha}/\hat{\beta}}{\rightarrow} \hat{\mu}$ for some $\hat{\alpha}/\hat{\beta}$. By the inductive hypothesis of (2), C is the matrix produced from $\eta \overset{\hat{\alpha}/\hat{\beta}}{\rightarrow} \hat{\mu}$ via Definition 2.4.4. Since $\hat{\alpha}_1$ is the sum of the lengths of rows with first label 1, $\hat{\alpha}_1 = \sum_j C_{1,j}$. Since $\hat{\beta}_1$ is the sum of the lengths of rows with first label 1 and second label > 1, $\hat{\beta}_1 = \sum_j \max(C_{1,j} - 1, 0)$.

Define $\alpha = (\mu_1, \hat{\alpha}_1, \hat{\alpha}_2, \ldots)$ and $\beta = (\mu_1 - \lambda_1, \hat{\beta}_1, \hat{\beta}_2, \ldots)$. If $q = 1$, then $\hat{\beta}_1 \leq \hat{\alpha}_1 = \sum_j C_{1,j} = \sum_j B_{1,j} = \rho_1 = \mu_1 - \lambda_1$, so α and β are partitions. If $q \geq 2$, then $\hat{\alpha}_1 = \sum_j C_{1,j} = \sum_j B_{2,j} = \rho_2 = \mu_2 \leq \mu_1$, so α is a partition. Now $\hat{\beta}_1 = \sum_j \max(C_{1,j} - 1, 0) = \sum_j \max(B_{2,j} - 1, 0) = \sum_j \max(A_{2,j} - 1, 0) \leq \sum_j \max(A_{1,j} - 1, 0) = \sum_{j=1}^{\lambda_1} (A_{1,j} - 1) = \left(\sum_{j=1}^{\lambda_1} A_{1,j}\right) - \lambda_1 = \left(\sum_{j=1} A_{1,j}\right) - \lambda_1 = \mu_1 - \lambda_1$, so β is a partition. Hence in all cases $\theta = \alpha/\beta$ is a valid skew-shape.

Notice θ is just adding a row to $\hat{\alpha}/\hat{\beta}$. The latter has row lengths η, so θ's row lengths are $(\lambda_1, \eta_1, \eta_2, \ldots) = \lambda$. Set $\delta_j = \begin{cases} 1 & \text{if } \mu_1 - \lambda_1 < j \leq \mu_1 \\ 0 & \text{otherwise} \end{cases}$. Set $f(a, j) = \begin{cases} 1 & \text{if } a \geq j \\ 0 & \text{otherwise.} \end{cases}$ Then θ's jth column length $= \delta_j + \hat{\alpha}/\hat{\beta}$'s jth column length $= \delta_j + \hat{\mu}'_j = \delta_j + \sum_i f(\hat{\mu}_i, j) = \delta_j + f(\mu_1 - \lambda_1, j) + \sum_{i \geq 2} f(\mu_i, j) = f(\mu_1, j) + \sum_{i \geq 2} f(\mu_i, j) = \sum_{i \geq 1} f(\mu_i, j) = \mu'_j$. Therefore $\lambda \overset{\theta}{\rightarrow} \mu$.

Hence $\mathbb{M}_b(\mu, \lambda')$ being nonempty implies $\lambda \rightarrow \mu$. Now in the above argument, if $\lambda \rightarrow \mu$, then the inductive hypothesis says that C is uniquely determined. Since B can be recovered from C by q (which is a function of μ and λ), B is also uniquely determined. Thus A is uniquely determined and $|\mathbb{M}_b(\mu, \lambda')| \leq 1$ when $\lambda \rightarrow \mu$. But $\bar{A} \in \mathbb{M}_b(\mu, \lambda')$ by Proposition 3.3.1. Therefore (1) and (2) hold and induction is complete.

□

Notice that this theorem gives a bijection between skew-linked pairs and balanced matrices whose row and column sums are partitions. Clearly this bijection is compatible with transposing both the skew-linking shape and the matrix.

3.4 Duality Between Row and Column Labels

Recall all of the above arguments and definitions have valid dual versions if we exchange the roles of rows and columns. In particular, we may define partition $\tilde{\nu}^j$ so that its ith column has the same length as column (i, j). Theorem 3.3.2 has a surprising consequence.

Corollary 3.4.1 *$\tilde{\nu}^j = \nu^j$ for all j. Deleting the rows with second label 1 produces the same skew diagram as deleting the columns with second label 1.*

Proof

By Theorem 3.3.2 and Lemma 2.4.5, the unique matrix in $\mathbb{M}_{\mu,\lambda'}$ is a plane partition with sections given by the ν^j's. The transposed versions of Theorem 3.3.2 and Lemma 2.4.5 imply that the unique matrix in $\mathbb{M}_{\mu,\lambda'}$ is a plane partition with sections given by the $\tilde{\nu}^j$'s. Hence $\tilde{\nu}^j = \nu^j$ for all j.

By Corollary 2.3.4, deleting the rows with second label 1 produces a skew-linking shape $\hat{\theta}$ whose ν^i equals the original ν^{i+1} for all i. The transposed version of Corollary 2.3.4 says that deleting the columns with second label 1 produces a skew-linking shape $\tilde{\theta}$ whose $\tilde{\nu}^i$ equals the original $\tilde{\nu}^{i+1}$ for all i. By the first part of this corollary, the ν^i of $\tilde{\theta}$ equals the original ν^{i+1} for all i. Thus $\hat{\theta}$ and $\tilde{\theta}$ have the same ν^i's. By Lemma 2.3.7, $\hat{\theta}$ and $\tilde{\theta}$ have the same row and column sums. By Lemma 2.1.4, $\hat{\theta} = \tilde{\theta}$.

□

3.5 Minimizing the Degree of Matrices

Definition 3.5.1 *Given a matrix A with nonnegative integer entries, define its* degree *as* $d(A) = \sum_{i,j} \binom{A_{i,j}}{2}$.

Definition 3.5.2 *(Swap operation) Given $A \in \mathbb{M}(\alpha, \beta)$, if $i \neq i'$ and $j \neq j'$ and $A_{i,j} \neq 0 \neq A_{i',j'}$, then in the tableau interpretation of A (see Definition 3.1.2), letter j occurs in column i and letter j' occurs in column i'. Define a* swap *between letter j in column i and letter j' in column i' as the tableau operation changing one copy of j in column i to j', and changing one copy of j' in column i' to j. The new tableau corresponds to the matrix $\hat{A}$ that is identical to A except that $\hat{A}_{i,j} = A_{i,j} - 1$, $\hat{A}_{i',j} = A_{i',j} + 1$, $\hat{A}_{i,j'} = A_{i,j'} + 1$, $\hat{A}_{i',j'} = A_{i',j'} - 1$.*

Lemma 3.5.3 *If A is balanced, then any swap increases the degree of A.*

Proof

With A and $\hat{A}$ as in Definition 3.5.2, a simple calculation shows that $d(\hat{A}) - d(A) = A_{i',j} + A_{i,j'} - A_{i,j} - A_{i',j'} + 2 \geq 1$ by the definition of balanced.

□

Theorem 3.5.4 *Suppose μ, λ are partitions of n. Then $\min_{A \in \mathbb{M}(\mu,\lambda')} d(A)$ is achieved by a unique matrix A iff $\lambda \to \mu$. In the case $\lambda \overset{\alpha/\beta}{\to} \mu$, the minimum is uniquely achieved by $\bar{A}$ of Definition 2.4.4, and $d(\bar{A}) = |\beta|$.*

Proof

Suppose the minimum is achieved by a unique matrix A. Suppose A is not balanced. Then there exist $i \neq i'$ and $j \neq j'$ such that $A_{i,j} \neq 0 \neq A_{i',j'}$ and $A_{i,j} + A_{i',j'} \geq A_{i,j'} + A_{i',j} + 2$. Define matrix B to be the same as A except $B_{i,j} = A_{i,j} - 1$, $B_{i',j'} = A_{i',j'} - 1$, $B_{i,j'} = A_{i,j'} + 1$, and $B_{i',j} = A_{i',j} + 1$. Then $B \in \mathbb{M}(\mu, \lambda')$. But $d(A) - d(B) = \binom{A_{i,j}}{2} - \binom{A_{i,j}-1}{2} + \binom{A_{i',j'}}{2} - \binom{A_{i',j'}-1}{2} + \binom{A_{i,j'}}{2} - \binom{A_{i,j'}+1}{2} + \binom{A_{i',j}}{2} - \binom{A_{i',j}+1}{2} = (A_{i,j} - 1) + (A_{i',j'} - 1) - A_{i,j'} - A_{i',j} \geq 0$, and $A \neq B$. Thus A cannot uniquely achieve the minimum value for $d(\cdot)$, a contradiction. Hence $A \in \mathbb{M}_b(\mu, \lambda')$. By Theorem 3.3.2 this implies $\lambda \to \mu$, so the forward implication is proved.

For the reverse implication, start with any $A \in \mathbb{M}(\mu, \lambda')$ and regard it (by Definition 3.1.2) as a tableau W of shape λ and weight μ. We claim that there is a sequence of degree-nonincreasing swaps on W to transform A into $\bar{A}$, and we prove this claim by induction on the number of rows of α/β. In the case of one row, $\alpha/\beta = (n)/\emptyset$ and $\lambda = \mu = (n)$, so $\mathbb{M}(\mu, \lambda') = \{\bar{A}\}$ and the claim is trivially true.

In the general case, the first step is to show that if some column in W does not contain 1, then there is a sequence of degree-nonincreasing swaps that reduces the number of columns in which 1 does not appear. Suppose no such sequence exists. Let B, C, D denote the sets of (indices of) columns in which 1 appears exactly once, zero times, and at least twice respectively. By assumption C is nonempty. Note that $|B \cup C \cup D| =$ total number of columns $= \ell(\lambda') = \lambda_1$.

Define $C_0, C_1, \ldots$ and $L_0, L_1, \ldots$ inductively as follows. Set $C_0 = C$ and let L_0 be the set of letters (none of them being 1) that appear in the columns of C. Then L_0 is nonempty because C is nonempty. Having defined up through C_p and L_p, let C_{p+1} be the columns in $B \setminus \cup_{j=0}^{p} C_j$ that are missing at least one of the letters in $\cup_{j=0}^{p} L_j$. Let L_{p+1} be the set of letters in the columns of C_{p+1} that are not in $\{1\} \cup (\cup_{j=0}^{p} L_j)$.

By construction the L_j's are pairwise disjoint and do not contain 1. Also, the set of letters appearing in the columns of C_p is a subset of $\{1\} \cup (\cup_{j=0}^{p} L_j)$. Hence for $j > j'$ and letter $z \in L_j$, $z \notin \{1\} \cup (\cup_{j=0}^{j'} L_j)$ and z does not appear in any of the columns of $C_{j'}$.

Because the C_j's are disjoint, there exists a smallest value of u such that $C_u = \emptyset$ (so $u \geq 1$). Then $L_u = \emptyset$ too, so $B \setminus \cup_{j=0}^{u} C_j = B \setminus \cup_{j=0}^{u-1} C_j$ and $\cup_{j=0}^{u} L_j = \cup_{j=0}^{u-1} L_j$. Hence $C_{u+1} = \emptyset$ (being the set satisfying the exact same conditions as C_u) and $L_{u+1} = \emptyset$ and so on. Thus $C_j = \emptyset = L_j$ for all $j \geq u$. Let $C' = \cup_{j=0}^{u} C_j \supset C_0 = C$, $B' = B \setminus C'$, and $L = \cup_{j=0}^{u} L_j$. Then each column in B' contains every letter in L, and each column in C' contains only 1's and letters in L.

Let $z \in L$ be arbitrary. Then $z \in L_t$ for some $t \leq u$, so there exists some column $q \in C_t$ containing z. If $t > 0$, then column q contains no copy of some letter $z' \in \cup_{i=0}^{t-1} L_i$. Say $z' \in L_{t'}$ with $t' < t$. Then there exists some column $q' \in C_{t'}$ containing z'. If $t' > 0$, then column q' contains no copy of some letter $z'' \in \cup_{i=0}^{t'-1} L_i$, so $z'' \in L_{t''}$ for some $t'' < t'$. This continues a finite number of steps. Hence there exist $0 = t_0 < t_1 < \ldots < t_m = t$ and letter $z_j \in L_j$ contained in column $q_j \in C_j$ ($j = 0, \ldots, m$) such that z_j does not appear in column q_{j-1} for $j = 1, \ldots, m$, and $z_m = z$ and $q_m = q$. Because the L_j's are disjoint, the z_j's are distinct. Because the C_j's are disjoint, the q_j's are distinct.

Let Δ denote the following sequence of swaps (possibly empty if $m = 0$):

- $z = z_m$ in column $q = q_m$ with z_{m-1} in column q_{m-1}
- z in column q_{m-1} with z_{m-2} in column q_{m-2}
- z in column q_{m-2} with z_{m-3} in column q_{m-3}

  $\ldots$

- z in column q_1 with z_0 in column q_0

We saw above that the z_j's and q_j's are disjoint, so this sequence of swaps is legal if z_j appears in column q_j for $j = 0, \ldots, m$ in the original tableau, which is true by construction. The effect of the swaps is that z goes from column q_m to q_{m-1} to q_{m-2} to $\ldots$ to q_0.

We investigate the change in degree caused by each swap. Suppose $0 \leq j \leq m-1$ and we are about to swap z in column q_{j+1} (arrived there via the previous swap) with z_j in column q_j. Since $z \in L_t = L_{t_m}$ and $q_j \in C_{t_j}$ and $t_m > t_j$, by remarks four paragraphs above, z does not appear in column q_j originally. By construction z_j appears in column q_j and not in column q_{j-1} originally. All these conditions are still true currently because the previous swaps do not involve column q_j or the letter z_j. Thus by Lemma 3.5.3 the current swap causes a degree change of $\leq 0+0-1-1+2 = 0$, i.e. these swaps do not increase the degree.

Let $p \in D$ be arbitrary. After performing swap sequence Δ, we perform swap Ψ, which swaps z in column q_0 with a 1 in column p. This is legal because column $p \in D$ originally contains at least two 1's, and no previous swaps involved column p because $q_j \in L_j \subset B \cup C$ for all j and $p \in D$. Note that no previous swaps involved the letter 1 because $z_j \neq 1$ for all j, so column $q_0 \in C_0 = C$ still has no 1's. Also note that originally column p has $A_{1,p}$ copies of 1's and $A_{z,p}$ copies of z's. Hence by Lemma 3.5.3, swap Ψ causes a degree change of $\leq A_{z,p} + 0 - A_{1,p} - 1 + 2 = A_{z,p} - A_{1,p} + 1$.

But before swap Ψ, column p contains at least two 1's while column q_0 contains no 1's. Hence Δ followed by Ψ reduces the number of columns with no 1's. By assumption, this sequence cannot consist entirely of degree-nonincreasing swaps. But the swaps in Δ are degree-nonincreasing, so Ψ must increase the degree. Therefore, $A_{z,p} \geq A_{1,p}$ for arbitrary $p \in D$ and $z \in L$.

Now we string together many observations to derive a contradiction. For any $p \in B' \subset B$ and $z \in L$, z occurs in column p by the definition of B', so $A_{z,p} \geq 1 = A_{1,p}$. Thus $A_{z,p} \geq A_{1,p} \geq 1$ for any $p \in D \cup B'$ and $z \in L$. This means that for $p \in D \cup B'$, $\lambda'_p = \sum_i A_{i,p} \geq A_{1,p} + \sum_{z \in L} A_{z,p} \geq 1 + |L|$. Note that $D \cup B' = (B \cup C \cup D) \setminus C'$. Hence

$$i \leq |L| + 1 \text{ implies } \lambda_i \geq |D \cup B'| = |B \cup C \cup D| - |C'| = \lambda_1 - |C'|.$$

For any i, $r_{i,1} \geq i$ in α/β, so $\lambda_{r_{i,1}} \leq \lambda_i$. By Corollary 2.3.2 and Definition 2.4.1, $\mu_i = \sum_j \gamma^i_j = \lambda_{r_{i,1}} + \sum_{j>1} \gamma^i_j \leq \lambda_i + \sum_{j>1} \gamma^1_j = \lambda_i + \sum_j \gamma^1_j - \gamma^1_1 = \lambda_i + \mu_1 - \lambda_1$, implying

$$\mu_i - \mu_1 \leq \lambda_i - \lambda_1 \text{ for all } i.$$

By the above facts, for $z \in L$,

$$\begin{aligned}
\sum_{j \in C'} A_{z,j} &= \sum_{j \in B \cup C \cup D} A_{z,j} - \sum_{j \in D \cup B'} A_{z,j} = \mu_z - \sum_{j \in D \cup B'} A_{z,j} \leq \mu_z - \sum_{j \in D \cup B'} A_{1,j} \\
&= \mu_z - \left(\sum_{j \in B \cup C \cup D} A_{1,j} - \sum_{j \in D \cup B'} A_{1,j} \right) = \mu_z - \mu_1 + \sum_{j \in C'} A_{1,j} \\
&= \mu_z - \mu_1 + \sum_{j \in C' \setminus C} 1 + \sum_{j \in C} 0 = \mu_z - \mu_1 + |C'| - |C| < \lambda_z - \lambda_1 + |C'|.
\end{aligned}$$

Now each column in C' contains only 1's and letters in L, and at most one copy of 1 because $C' \subset B \cup C$. Thus for $j \in C'$, $\lambda'_j = \sum_z A_{z,j} = A_{1,j} + \sum_{z \in L} A_{z,j} \leq 1 + \sum_{z \in L} A_{z,j}$. If $q_1 > q_2 > \ldots > q_{|C'|}$ are the columns of C', then $q_1 \leq \lambda_1$ and $q_i \leq \lambda_1 + 1 - i$. If $z_1 < z_2 < \ldots < z_{|L|}$ are the letters in L, then $z_1 \geq 2$, so $z_i \geq i + 1$ for all i. Also note $L \neq \emptyset$ because $L_0 \neq \emptyset$.

From the above observations,

$$\begin{aligned}
\sum_{j \in C'} \lambda'_j &= \sum_{i=1}^{|C'|} \lambda'_{q_i} \geq \sum_{i=1}^{|C'|} \lambda'_{\lambda_1+1-i} = \sum_{j=\lambda_1+1-|C'|}^{\lambda_1} \lambda'_j = \sum_{j \geq \lambda_1+1-|C'|} \lambda'_j \\
&= \sum_i \max(\lambda_i - (\lambda_1 - |C'|), 0) \geq \sum_{i=1}^{|L|+1} \max(\lambda_i - \lambda_1 + |C'|, 0) \\
&= |C'| + \sum_{i=2}^{|L|+1} (\lambda_i - \lambda_1 + |C'|) \geq |C'| + \sum_{i=1}^{|L|} (\lambda_{z_i} - \lambda_1 + |C'|) \\
&= |C'| + \sum_{z \in L} (\lambda_z - \lambda_1 + |C'|) > |C'| + \sum_{z \in L} \sum_{j \in C'} A_{z,j} = \sum_{j \in C'} (1 + \sum_{z \in L} A_{z,j}) \geq \sum_{j \in C'} \lambda'_j.
\end{aligned}$$

This is a contradiction. Therefore, starting with any $A \in \mathbb{M}(\mu, \lambda')$, there exists a sequence of degree-nonincreasing swaps that reduces the number of columns in which 1 does not appear. Hence we may perform more swap sequences until this number is reduced to zero. Let $\tilde{W}$ be the new tableau produced, with corresponding matrix $\tilde{A} \in \mathbb{M}(\mu, \lambda')$. Then 1 appears in every column of $\tilde{W}$, and $\tilde{A}_{1,j} \geq 1$ for all j.

In fact, the above contradiction shows that there exist $p \in D$ and $z \in L$ such that $A_{z,p} < A_{1,p}$. For these p and z, we construct swaps Δ and Ψ. Then Ψ does not increase the degree because $A_{z,p} < A_{1,p}$, so Δ and Ψ form an explicit degree-nonincreasing swap sequence that reduces the number of columns in which 1 does not appear.

We may assume that the first row of $\tilde{W}$ consist of ones. Now find q according to Proposition 2.4.2. construct $\dot{W}$ from $\tilde{W}$ in the following way:

1. Remove the first row of $\tilde{W}$.

2. Simultaneously rename the letters so that all original 1's becomes q's, and all original j's becomes $j - 1$'s for $j = 2, 3, \ldots, q$.

3. The letters $q + 1, q + 2, \ldots$ are unaffected.

Then $\dot{W}$ has shape $\hat{\lambda} = (\lambda_2, \lambda_3, \ldots)$ and weight $\hat{\mu} = (\mu_2, \ldots, \mu_q, \mu_1 - \lambda_1, \mu_{q+1}, \mu_{q+2}, \ldots)$. Let $\dot{A} \in \mathbb{M}(\hat{\mu}, \hat{\lambda}')$ correspond to $\dot{W}$. Let $\phi : \mathbb{M}(\hat{\mu}, \hat{\lambda}') \to \mathbb{M}(\mu, \lambda')$ be the operation of moving the qth row to the first row, sliding the original rows 1 through $q - 1$ down by one row, and finally incrementing the matrix entries $(1, 1), \ldots, (1, \lambda_1)$ by 1. Then $\phi(\dot{A}) = \tilde{A}$.

By Proposition 2.4.2, $\hat{\mu}$ and $\hat{\lambda}'$ are exactly the row and column lengths of $\hat{\alpha}/\hat{\beta}$, the skew-linking shape obtained by removing the first row of α/β. Per Definition 2.4.4, construct matrix $\hat{A} \in \mathbb{M}(\hat{\mu}, \hat{\lambda}')$ associated to $\hat{\alpha}/\hat{\beta}$, and let $\hat{W}$ be the corresponding tableau. Because

$\hat{\alpha}/\hat{\beta}$ has fewer rows than α/β, by the inductive hypothesis there exists a degree-nonincreasing swap sequence Ξ to transform $\dot{W}$ to $\hat{W}$ (and hence $\dot{A}$ to $\hat{A}$).

By Proposition 2.4.2(3), $\phi(\hat{A}) = \bar{A}$. At the level of tableaux, ϕ changes letters $1, 2, \ldots, q-1, q$ to $2, 3, \ldots, q, 1$ correspondingly and adds a length λ_1 row of 1's. Since Ξ transforms $\dot{W}$ to $\hat{W}$, we may construct swap sequence Υ to transform $\phi(\dot{A}) = \tilde{A}$ to $\phi(\hat{A}) = \bar{A}$ by performing the same swap sequence on the larger tableau, but changing the roles of letters $1, 2, \ldots, q-1, q$ to $2, 3, \ldots, q, 1$ correspondingly. Furthermore, corresponding swaps between Ξ and Υ produce the same change in degree because the formula given in Lemma 3.5.3 is invariant under incrementing matrix entries $(1, 1), \ldots, (1, \lambda_1)$ by 1. Thus Υ is also degree-nonincreasing.

In summary, the concatenation of swap sequences Δ, Ψ, Υ is a degree-nonincreasing sequence Ω that changes the original A to $\tilde{A}$ and then to $\bar{A}$. Therefore, $d(A) \geq d(\bar{A})$ and $\bar{A}$ minimizes degree. Now if $A \neq \bar{A}$, then Ω is nonempty. Its last swap takes some matrix $\ddot{A}$ to $\bar{A}$, so the reverse swap takes $\bar{A}$ to $\ddot{A}$. Now $\bar{A}$ is balanced, and by Lemma 3.5.3 any swap increases the degree. Hence $d(A) \geq d(\ddot{A}) > d(\bar{A})$ and $\bar{A}$ uniquely minimizes degree.

Now recall $\bar{A}_{i,j} = (\gamma^i)'_j$. Define $\chi_{i,j,p} = \begin{cases} 1 & \text{if } (\gamma^i)'_j \geq p \\ 0 & \text{otherwise} \end{cases}$. Note that $(\gamma^i)'_j \geq p$ is equivalent to $\lambda_{r_{i,p}} \geq j$ (i.e. row (i,p) has length $\geq j$). Then from Definition 2.3.5 and Lemma 2.3.6,

$$\begin{aligned} d(\bar{A}) &= \sum_{i,j\geq 1} \binom{(\gamma^i)'_j}{2} = \sum_{i\geq 1}\sum_{j\geq 1}\sum_{p\geq 1}(p-1)\chi_{i,j,p} = \sum_{i\geq 1}\sum_{p\geq 1}\left((p-1)\sum_{j\geq 1}\chi_{i,j,p}\right) \\ &= \sum_{i\geq 1}\sum_{p\geq 1}(p-1)\lambda_{r_{i,p}} = \sum_{p\geq 1}(p-1)\left(\sum_{i\geq 1}\lambda_{r_{i,p}}\right) = \sum_{p\geq 1}(p-1)|\nu^p| = |\beta|. \end{aligned}$$

□

3.6 More on Minimizing the Degree of Matrices

Definition 3.6.1 *For partitions λ, μ of n, define $d(\lambda, \mu) = \min_{A\in\mathbb{M}(\mu,\lambda')} d(A)$.*

Lemma 3.6.2 *(Transpose symmetry) $d(\lambda, \mu) = d(\mu', \lambda')$.*

Proof

By definition $d(\lambda, \mu) = \min_{A\in\mathbb{M}(\mu,\lambda')} d(A)$ and $d(\mu', \lambda') = \min_{A\in\mathbb{M}(\lambda',\mu)} d(A)$. Because taking transpose is a degree-preserving bijection between $\mathbb{M}(\mu, \lambda')$ and $\mathbb{M}(\lambda', \mu)$, the minimum degrees are the same.

□

Lemma 3.6.3 *If $\eta \geq \mu$ in dominance order and $B \in \mathbb{M}(\eta, \lambda')$, then there exists $C \in \mathbb{M}(\mu, \lambda')$ with $d(B) \geq d(C)$. If $\eta > \mu$ and C has weakly decreasing columns, then $d(B) > d(C)$.*

Proof

Because $\eta \geq \mu$, there exists a (possibly empty) sequence of lowering operators $R_1, \ldots, R_m$ such that $\mu = R_m \ldots R_2 R_1 \eta$ and that $\eta^{(t)} = R_t \ldots R_2 R_1 \eta$ is a partition for all t. Then $\eta^{(0)} = \eta$ and $\eta^{(m)} = \mu$.

We claim that we can inductively construct matrices $A^{(0)}, A^{(1)}, \ldots, A^{(m)}$ with $A^{(t)} \in \mathbb{M}(\eta^{(t)}, \lambda')$ for $t = 0, \ldots, m$ and $d(A^{(t-1)}) \geq d(A^{(t)})$ for $t = 1, \ldots, m$. For the base case, set $A^{(0)} = B$.

Suppose such $A^{(0)}, A^{(1)}, \ldots, A^{(t)}$ have been constructed. Because $\eta^{(t+1)} = R_{t+1}\eta^{(t)}$, there exists i such that $\eta_i^{(t+1)} = \eta_i^{(t)} - 1$, $\eta_{i+1}^{(t+1)} = \eta_{i+1}^{(t)} + 1$, and $\eta_p^{(t+1)} = \eta_p^{(t)}$ for $p \neq i, i+1$. In particular it means $\sum_j A_{i,j}^{(t)} - A_{i+1,j}^{(t)} = \eta_i^{(t)} - \eta_{i+1}^{(t)} = 2 + \eta_i^{(t+1)} - \eta_{i+1}^{(t+1)} \geq 2$. Thus there exists j with $A_{i,j}^{(t)} - A_{i+1,j}^{(t)} \geq 1$. Define $A^{(t+1)}$ to be identical to $A^{(t)}$ except that $A_{i,j}^{(t+1)} = A_{i,j}^{(t)} - 1$ and $A_{i+1,j}^{(t+1)} = A_{i+1,j}^{(t)} + 1$. Then $A^{(t)} \in \mathbb{M}(\eta^{(t)}, \lambda')$ implies $A^{(t+1)} \in \mathbb{M}(R_{t+1}\eta^{(t)}, \lambda') = \mathbb{M}(\eta^{(t+1)}, \lambda')$. Furthermore, $d(A^{(t)}) - d(A^{(t+1)}) = \binom{A_{i,j}^{(t)}}{2} + \binom{A_{i+1,j}^{(t)}}{2} - \binom{A_{i,j}^{(t)}-1}{2} - \binom{A_{i+1,j}^{(t)}+1}{2} = A_{i,j}^{(t)} - A_{i+1,j}^{(t)} - 1 \geq 0$. Induction is complete.

Now set $C = A^{(m)} \in \mathbb{M}(\eta^{(m)}, \lambda') = \mathbb{M}(\mu, \lambda')$. Then $d(B) = d(A^{(0)}) \geq d(A^{(1)}) \geq \ldots \geq d(A^{(m)}) = d(C)$.

Finally, consider the case $\eta > \mu$ and C has weakly decreasing columns. Then $m \geq 1$ and $A^{(m-1)}$ exists. Let i be the index used to construct $A^{(m)}$ from $A^{(m-1)}$. Then $d(A^{(m-1)}) - d(A^{(m)}) = \binom{A_{i,j}^{(m)}+1}{2} + \binom{A_{i+1,j}^{(m)}-1}{2} - \binom{A_{i,j}^{(m)}}{2} - \binom{A_{i+1,j}^{(m)}}{2} = A_{i,j}^{(m)} - A_{i+1,j}^{(m)} + 1 = C_{i,j} - C_{i+1,j} + 1 > 0$. Therefore, $d(B) = d(A^{(0)}) \geq \ldots \geq d(A^{(m-1)}) > d(A^{(m)}) = d(C)$.

$\square$

Proposition 3.6.4 *Suppose $\lambda \to \mu$ and $\eta > \mu$ in dominance order. Then $d(\lambda, \eta) > d(\lambda, \mu)$.*

Proof

Choose $B \in \mathbb{M}(\eta, \lambda')$ so that $d(B) = d(\lambda, \eta)$. By Lemma 3.6.3, there exists $C \in \mathbb{M}(\mu, \lambda')$ with $d(B) \geq d(C)$. Recall by Theorem 3.5.4 that $\min_{A \in \mathbb{M}(\mu, \lambda')} d(A) = d(\lambda, \mu)$ is achieved by a unique matrix $\bar{A}$. If $C \neq \tilde{A}$, then $d(\lambda, \eta) = d(B) \geq d(C) > d(\lambda, \mu)$. If $C = \bar{A}$, then C has weakly decreasing columns by Lemma 2.4.5, so $d(B) > d(C)$ by Lemma 3.6.3. Hence $d(\lambda, \eta) \geq d(B) > d(C) = d(\lambda, \mu)$, so $d(\lambda, \eta) > d(\lambda, \mu)$ in all cases.

$\square$

Proposition 3.6.5 *Let $\lambda \to \mu$. Suppose $\gamma \leq \lambda$ and $\eta \geq \mu$ in dominance order. Then $d(\gamma, \eta) \geq d(\lambda, \mu)$, with equality iff $\gamma = \lambda$ and $\eta = \mu$.*

Proof

Choose $B \in \mathbb{M}(\eta, \gamma')$ so that $d(B) = d(\gamma, \eta)$. Now $B^T \in \mathbb{M}(\gamma', \eta)$ and $\gamma' \geq \lambda'$, so by Lemma 3.6.3 there exists $C \in \mathbb{M}(\lambda', \eta)$ such that $d(B^T) \geq d(C)$. Because $C^T \in \mathbb{M}(\eta, \lambda')$

and $\eta \geq \mu$, by Lemma 3.6.3 again there exists $D \in \mathbb{M}(\mu, \lambda')$ such that $d(C^T) \geq d(D)$. Thus $d(\gamma, \eta) = d(B) = d(B^T) \geq d(C) = d(C^T) \geq d(D) \geq d(\lambda, \mu)$.

Now we show that if $(\gamma, \eta) \neq (\lambda, \mu)$, then at least one inequality in the above chain is strict. Recall by Theorem 3.5.4 that $\min_{A \in \mathbb{M}(\mu, \lambda')} d(A) = d(\lambda, \mu)$ is achieved by a unique matrix $\bar{A}$. There are several cases.

1. $\eta > \mu$ and $D \neq \bar{A}$: Then $d(D) > d(\lambda, \mu)$.
2. $\eta > \mu$ and $D = \bar{A}$: By Lemma 2.4.5, $\bar{A}$ has weakly decreasing columns, so $d(C^T) > d(D)$ by Lemma 3.6.3.
3. $\eta = \mu$: Then $\gamma < \lambda$. Since $\mu' \to \lambda'$ and $\gamma' > \lambda'$, by Proposition 3.6.4 we have $d(\mu', \gamma') > d(\mu', \lambda')$. Thus by transpose symmetry (Lemma 3.6.2), $d(\gamma, \eta) = d(\gamma, \mu) = d(\mu', \gamma') > d(\mu', \lambda') = d(\lambda, \mu)$.

Therefore, equality occurs iff $\gamma = \lambda$ and $\eta = \mu$.

$\square$

Chapter 4

Modules Associated with Skew-Linked Partitions

4.1 Graded Dual, Co-Generation, and Co-Freeness

Definition 4.1.1 *For a graded* $\mathbb{C}[\mathbf{x}]$*-module* M*, let* M^* *denote its* graded dual*, defined as* $\oplus_d (M_d)^*$*. Note* M^* *is also a graded* $\mathbb{C}[\mathbf{x}]$*-module, but with reverse grading:* $(M^*)_j = (M_{-j})^*$*. Explicitly, if* $|\alpha| = i$ *and* $w \in (M^*)_j = (M_{-j})^*$*, then let* $\mathbf{x}^\alpha w \in (M^*)_{i+j} = (M_{-i-j})^*$ *be given by* $(\mathbf{x}^\alpha w)(v) = w(\mathbf{x}^\alpha v)$*.*

Note that if each M_d is finite-dimensional, then $(M^*)^* \cong M$. In particular, this is true for $\mathbb{C}[\mathbf{x}]$ and $\mathbb{C}[\mathbf{x}]^*$.

Definition 4.1.2 *Suppose we have* $\mathbb{C}[\mathbf{x}]$*-module* M*,* $\mathbb{C}$*-module* W*, and* $\mathbb{C}$*-homomorphism* $M \xrightarrow{\pi} W$*. We say that* (W, π) co-generates M *if whenever we have a commutative diagram*

$$\begin{tikzcd} M' \arrow[d, "\pi'"'] & \\ W & M \arrow[l, "\pi"] \arrow[ul, "\sigma"'] \end{tikzcd}$$

with $\mathbb{C}[\mathbf{x}]$*-module* M'*,* $\mathbb{C}$*-homomorphism* π'*, and* $\mathbb{C}[\mathbf{x}]$*-homomorphism* σ*, we also have* σ *injective.*

The same picture with arrows reversed and "injective" replaced by "surjective" is clearly a diagrammatic form of the statement that W generates M. Hence in particular a graded $\mathbb{C}[\mathbf{x}]$-module M is co-generated by W iff its graded dual is generated by W^*.

Typically M and W are spaces of linear functionals and π is a projection. We can also define co-generation of an $\mathbb{C}[\mathbf{x}] * S_n$-module M by requiring M, M', N to have compatible S_n actions and σ to be S_n-equivariant.

Proposition 4.1.3 *A pair* (W, π) *co-generates* M *iff no nonzero* $\mathbb{C}[\mathbf{x}]$*-submodule* N *of* M *is contained in* $\operatorname{Ker} \pi$*.*

Proof

Suppose (W, π) co-generates M and N is a $\mathbb{C}[\mathbf{x}]$-submodule of M that is contained in $\operatorname{Ker}\pi$. Then we may factor π as $W \leftarrow M/N \overset{j}{\leftarrow} M$. By co-generation, j is injective, so $0 = \operatorname{Ker} j = N$.

Conversely, suppose no nonzero $\mathbb{C}[\mathbf{x}]$-submodule N of M is contained in $\operatorname{Ker}\pi$, and suppose we have a commutative diagram as given in Definition 4.1.2. If $v \in \operatorname{Ker}\sigma$, then $\pi v = \pi'\sigma v = 0$. Thus $\operatorname{Ker}\sigma$ is a $\mathbb{C}[\mathbf{x}]$-submodule of M that is contained in $\operatorname{Ker}\pi$. By assumption $\operatorname{Ker}\sigma = 0$, so σ is injective.

$\square$

Definition 4.1.4 *A graded $\mathbb{C}[\mathbf{x}]$-module M is* co-free *if its graded dual M^* is a free $\mathbb{C}[\mathbf{x}]$-module.*

In particular, if W is a finite-dimensional $\mathbb{C}$-module, then $W \otimes_{\mathbb{C}} \mathbb{C}[\mathbf{x}]^*$ is co-free because its graded dual is isomorphic to $W^* \otimes_{\mathbb{C}} \mathbb{C}[\mathbf{x}]$.

4.2 Modules $M_{\lambda,\mu}$ Associated to Skew-Linked Partitions

Let $\mathbb{C}[\mathbf{x}] * S_n$ be the semidirect product. Then a $\mathbb{C}[\mathbf{x}] * S_n$ module may be regarded as a $\mathbb{C}[x_1, \ldots, x_n]$ module with compatible S_n action. Which $\mathbb{C}[\mathbf{x}] * S_n$ modules can be characterized in a similar fashion as Proposition 1.4.8?

Throughout this chapter, we will assume that λ and μ (and other relevant partitions) are partitions of n.

Proposition 4.2.1 *Let $V = \left(\varepsilon \uparrow_{S_{\lambda'}}^{S_n}\right) \otimes \mathbb{C}[\mathbf{x}]$, the free $\mathbb{C}[\mathbf{x}]$-module on our previously considered induced S_n module. Let $W = \left(1 \uparrow_{S_\mu}^{S_n}\right) \otimes \mathbb{C}[\mathbf{x}]^*$, a co-free $\mathbb{C}[\mathbf{x}]$-module on an induced S_n-module, but we may have $\mu \neq \lambda$. Let $\bar{d}(\lambda, \mu)$ be the smallest degree d such that there is a nonzero S_n-module homomorphism*

$$\psi\colon \left(\varepsilon \uparrow_{S_{\lambda'}}^{S_n}\right) \otimes \mathbb{C}[\mathbf{x}]_d \to 1 \uparrow_{S_\mu}^{S_n} .$$

Suppose further that λ and μ are such that ψ is unique up to a constant at $d = \bar{d}(\lambda, \mu)$. Then up to a constant, there is a unique nonzero $\mathbb{C}[\mathbf{x}] * S_n$ homomorphism, homogeneous of degree zero

$$\phi\colon V \to W[-\bar{d}(\lambda, \mu)].$$

Its image $M_{\lambda,\mu}$ is the unique nonzero graded $\mathbb{C}[\mathbf{x}] * S_n$ module for which there exists the diagram

$$V \twoheadrightarrow M_{\lambda,\mu} \hookrightarrow W[-\bar{d}(\lambda, \mu)]$$

of homogeneous degree-zero $\mathbb{C}[\mathbf{x}] * S_n$ homomorphisms.

Proof

Because V is the free $\mathbb{C}[\mathbf{x}]$ module generated over its degree zero component $\varepsilon \uparrow_{S_{\lambda'}}^{S_n}$, the uniqueness (up to a constant) of ϕ is equivalent to the uniqueness (up to a constant) of a nonzero S_n-homomorphism from $\varepsilon \uparrow_{S_{\lambda'}}^{S_n}$ to $W[-\bar{d}(\lambda,\mu)]_0 = W_{-\bar{d}(\lambda,\mu)} = \left(1 \uparrow_{S_\mu}^{S_n}\right) \otimes \mathbb{C}[\mathbf{x}]^*_{-\bar{d}(\lambda,\mu)} \cong \left(1 \uparrow_{S_\mu}^{S_n}\right) \otimes \mathbb{C}[\mathbf{x}]_{\bar{d}(\lambda,\mu)}$.

Now the uniqueness (up to a constant) of nonzero ψ implies $\langle \left(\varepsilon \uparrow_{S_{\lambda'}}^{S_n}\right) \otimes \mathbb{C}[\mathbf{x}]_{\bar{d}(\lambda,\mu)}$, $1 \uparrow_{S_\mu}^{S_n} \rangle_{S_n} = 1$. It follows that $\langle \varepsilon \uparrow_{S_{\lambda'}}^{S_n}, \left(1 \uparrow_{S_\mu}^{S_n}\right) \otimes \mathbb{C}[\mathbf{x}]_{\bar{d}(\lambda,\mu)} \rangle_{S_n} = 1$. This implies the uniqueness (up to a constant) of a nonzero S_n-homomorphism from $\varepsilon \uparrow_{S_{\lambda'}}^{S_n}$ to $\left(1 \uparrow_{S_\mu}^{S_n}\right) \otimes \mathbb{C}[\mathbf{x}]_{\bar{d}(\lambda,\mu)}$, as desired.

The last statement holds because any module occurring in such a diagram must be the image of some map $V \to W[-\bar{d}(\lambda,\mu)]$, which we just showed is unique up to a constant.

$\square$

Theorem 4.2.2 *1. The necessary and sufficient condition for the hypotheses of Proposition 4.2.1 to hold is that λ be skew-linked to μ (and $|\lambda| = |\mu| = n$). Hence in this case, we call $M_{\lambda,\mu}$ the skew-linked module associated to $\lambda \to \mu$.*

2. In that case, $\bar{d}(\lambda,\mu) =$ top degree of $M_{\lambda,\mu} = n(\gamma) = |\beta|$, where the skew diagram linking λ to μ is $\theta = \alpha/\beta$.

3. Moreover, the degree zero and top degree components of $M_{\lambda,\mu}$ are irreducible S_n modules isomorphic to V_λ and V_μ, respectively.

To prove this theorem, we will need several preliminary results. The proof of this theorem is located after Lemma 4.2.7.

Proposition 4.2.3 *We have $\bar{d}(\lambda,\mu) = \min_{A \in \mathbb{M}(\mu,\lambda')} d(A) = d(\lambda,\mu)$ (see Definition 3.6.1). Furthermore,*

$$\mathrm{Hom}_{S_n}\left(\left(\varepsilon \uparrow_{S_{\lambda'}}^{S_n}\right) \otimes \mathbb{C}[\mathbf{x}]_{\bar{d}(\lambda,\mu)},\ 1 \uparrow_{S_\mu}^{S_n} \right)$$

has dimension 1 iff the minimizing matrix A is unique, which occurs iff λ is skew-linked to μ. If $\lambda \overset{\alpha/\beta}{\to} \mu$, then the minimizing matrix is $\bar{A}$ (see Theorem 3.5.4) and $\bar{d}(\lambda,\mu) = d(\bar{A}) = |\beta|$.

Proof

Recall $S_\mu = S_{\mathfrak{C}_1} \times S_{\mathfrak{C}_2} \times \ldots$ from Definition 1.4.2. Fix the Young subgroup $S_{\lambda'} = S_{\mathfrak{B}_1} \times S_{\mathfrak{B}_2} \times \ldots$. Then the alphabet is partitioned $\{1,\ldots,n\} = \sqcup_j \mathfrak{B}_j = \sqcup_i \mathfrak{C}_i$, and we have sizes $|\mathfrak{B}_j| = \lambda'_j$ and $|\mathfrak{C}_i| = \mu_i$ for all i,j.

Note that

$$\dim_{\mathbb{C}} \mathrm{Hom}_{S_n}\left(\left(\varepsilon \uparrow_{S_{\lambda'}}^{S_n}\right) \otimes \mathbb{C}[\mathbf{x}]_d,\ 1 \uparrow_{S_\mu}^{S_n} \right) \neq 0 \quad \Longleftrightarrow$$

$$\begin{aligned} 0 &\neq \langle \varepsilon \uparrow_{S_{\lambda'}}^{S_n} \otimes \mathbb{C}[\mathbf{x}]_d,\ 1 \uparrow_{S_\mu}^{S_n} \rangle_{S_n} \\ &= \langle \left(\varepsilon \uparrow_{S_{\lambda'}}^{S_n} \otimes \mathbb{C}[\mathbf{x}]_d \right) \downarrow_{S_\mu}^{S_n},\ 1 \rangle_{S_\mu} \\ &= \dim_{\mathbb{C}} \left\{ \text{vectors in } \varepsilon \uparrow_{S_{\lambda'}}^{S_n} \otimes \mathbb{C}[\mathbf{x}]_d \text{ fixed by } S_\mu \right\} \\ &= \dim_{\mathbb{C}} \left(\sum_{h \in S_\mu} h \right) \left(\varepsilon \uparrow_{S_{\lambda'}}^{S_n} \otimes \mathbb{C}[\mathbf{x}]_d \right) \end{aligned}$$

The above also holds if "$\neq 0$" is replaced by "$= 1$".

We claim that if $d < \min_{A \in \mathbb{M}(\mu,\lambda')} d(A)$, then $\left(\sum_{h \in S_\mu} h\right) \left(\varepsilon \uparrow_{S_{\lambda'}}^{S_n} \otimes \mathbb{C}[\mathbf{x}]_d \right) = 0$. Note $\varepsilon \uparrow_{S_{\lambda'}}^{S_n} \otimes \mathbb{C}[\mathbf{x}]_d = \frac{\mathbb{C}S_n \cdot e}{\langle \sigma e - (-1)^\sigma e : \sigma \in S_{\lambda'} \rangle} \otimes \mathbb{C}[\mathbf{x}]_d$ is spanned by $\{ge \otimes \mathbf{x}^u : g \in S_n, |u| = d\}$. Fix some $g \in S_n, |u| = d$. Then $gS_{\lambda'}g^{-1} = S_{g(\mathfrak{B}_1)} \times S_{g(\mathfrak{B}_2)} \times \ldots$. Note that the matrix A with entries $A_{i,j} = |g(\mathfrak{B}_j) \cap \mathfrak{C}_i|$ has row sums μ and column sums λ'.

Suppose for any i, j and any distinct $a, b \in g(\mathfrak{B}_j) \cap \mathfrak{C}_i$ we have $u_a \neq u_b$. Then $\sum_{a \in g(\mathfrak{B}_j) \cap \mathfrak{C}_i} u_a \geq \binom{|g(\mathfrak{B}_j) \cap \mathfrak{C}_i|}{2}$, so $d = |u| \geq \sum_{i,j} \binom{|g(\mathfrak{B}_j) \cap \mathfrak{C}_i|}{2} = d(A)$, a contradiction. Thus there exist some i, j and some distinct $a, b \in g(\mathfrak{B}_j) \cap \mathfrak{C}_i$ such that $u_a = u_b$. This means that the transposition (ab) is in $gS_{\lambda'}g^{-1} \cap S_\mu$ and fixes $\mathbf{x}^u$. But then $\left(\sum_{h \in S_\mu} h\right) (ge \otimes \mathbf{x}^u) = \left(\sum_{h \in S_\mu} h \cdot (ab)\right) (ge \otimes \mathbf{x}^u) = \left(\sum_{h \in S_\mu} h\right) ((ab)ge \otimes (ab)\mathbf{x}^u) = -\left(\sum_{h \in S_\mu} h\right) (ge \otimes \mathbf{x}^u)$. Thus $\left(\sum_{h \in S_\mu} h\right) (ge \otimes \mathbf{x}^u) = 0$. But $ge \otimes \mathbf{x}^u$ was an arbitrary spanning element of $\varepsilon \uparrow_{S_{\lambda'}}^{S_n} \otimes \mathbb{C}[\mathbf{x}]_d$. Hence $\left(\sum_{h \in S_\mu} h\right) \left(\varepsilon \uparrow_{S_{\lambda'}}^{S_n} \otimes \mathbb{C}[\mathbf{x}]_d \right) = 0$ and the claim is proved.

Now consider the case $d = \min_{A \in \mathbb{M}(\mu,\lambda')} d(A)$. Choose $A \in \mathbb{M}(\mu, \lambda')$ that minimizes $d(A)$. Pick an appropriate $g \in S_n$ so that $|g(\mathfrak{B}_j) \cap \mathfrak{C}_i| = A_{i,j}$ for all i, j. Construct monomial $\mathbf{x}^u$ so that for each (i, j), $\{u_a : a \in g(\mathfrak{B}_j) \cap \mathfrak{C}_i\} = \{0, 1, \ldots, |g(\mathfrak{B}_j) \cap \mathfrak{C}_i| - 1\}$. Then $|u| = \sum_{i,j} \sum_{a \in g(\mathfrak{B}_j) \cap \mathfrak{C}_i} u_a = \sum_{i,j} \binom{|g(\mathfrak{B}_j) \cap \mathfrak{C}_i|}{2} = \sum_{i,j} \binom{A_{i,j}}{2} = d(A)$. Let $S_u = \{\tau \in S_n : \tau u = u\}$. Then by construction $gS_{\lambda'}g^{-1} \cap S_\mu \cap S_u = \{1\}$.

If we write $\left(\sum_{h \in S_\mu} h\right) (ge \otimes \mathbf{x}^u) = \sum_w z_w \otimes \mathbf{x}^w$ with $z_w \in \varepsilon \uparrow_{S_{\lambda'}}^{S_n}$, then $z_u = \sum_{h \in S_\mu \cap S_u} ge$. Suppose $h, h' \in S_\mu \cap S_u$ and $h'g \in hgS_{\lambda'}$. Then $h' \in h(gS_{\lambda'}g^{-1})$, so $h^{-1}h' \in gS_{\lambda'}g^{-1} \cap S_\mu \cap S_u = \{1\}$ and $h = h'$. This means that $\sum_{h \in S_\mu \cap S_u} hg$ consists of elements in different left cosets of $S_n/S_{\lambda'}$. Thus $z_u = \sum_{h \in S_\mu \cap S_u} ge$ is a sum of linearly independent elements in $\varepsilon \uparrow_{S_{\lambda'}}^{S_n} = \frac{\mathbb{C}S_n \cdot e}{\langle \sigma e - (-1)^\sigma e : \sigma \in S_{\lambda'} \rangle}$ and is nonzero. This implies $\left(\sum_{h \in S_\mu} h\right) (ge \otimes \mathbf{x}^u) \neq 0$ too. Since $|u| = d(A) = d$, we have $\left(\sum_{h \in S_\mu} h\right) \left(\varepsilon \uparrow_{S_{\lambda'}}^{S_n} \otimes \mathbb{C}[\mathbf{x}]_d \right) \neq 0$. Therefore, $\bar{d}(\lambda, \mu) = \min_{A \in \mathbb{M}(\mu,\lambda')} d(A)$.

Next we need to show that for $d = \min_{A \in \mathbb{M}(\mu,\lambda')} d(A)$, $\dim_{\mathbb{C}} \mathrm{Hom}_{S_n} \left(\left(\varepsilon \uparrow_{S_{\lambda'}}^{S_n} \right) \otimes \mathbb{C}[\mathbf{x}]_d,\ 1 \uparrow_{S_\mu}^{S_n} \right) = 1$ iff the minimizing matrix A is unique. By the second paragraph of this proof, we need to show that $\dim_{\mathbb{C}} \left(\sum_{h \in S_\mu} h\right) \left(\varepsilon \uparrow_{S_{\lambda'}}^{S_n} \otimes \mathbb{C}[\mathbf{x}]_d \right) = 1$ iff the minimizing matrix A is unique.

First consider the case that there exist distinct matrices $A, \tilde{A}$ minimizing $d(\cdot)$. Similar to above, pick $g, \tilde{g} \in S_n$ so that $|g(\mathfrak{B}_j) \cap \mathfrak{C}_i| = A_{i,j}$ and $|\tilde{g}(\mathfrak{B}_j) \cap \mathfrak{C}_i| = \tilde{A}_{i,j}$ for all i, j.

Then $gS_{\lambda'}g^{-1} = S_{g(\mathfrak{B}_1)} \times S_{g(\mathfrak{B}_2)} \times \ldots$ and $\tilde{g}S_{\lambda'}\tilde{g}^{-1} = S_{\tilde{g}(\tilde{\mathfrak{B}})_1} \times S_{\tilde{g}(\tilde{\mathfrak{B}})_2} \times \ldots$. Also similar to above, construct monomials $\mathbf{x}^u$ and $\mathbf{x}^{\tilde{u}}$ so that for each (i,j), $\{u_a : a \in g(\mathfrak{B}_j) \cap \mathfrak{C}_i\} = \{0, 1, \ldots, |g(\mathfrak{B}_j) \cap \mathfrak{C}_i| - 1\}$ and $\{\tilde{u}_a : a \in \tilde{g}(\mathfrak{B}_j) \cap \mathfrak{C}_i\} = \{0, 1, \ldots, |\tilde{g}(\mathfrak{B}_j) \cap \mathfrak{C}_i| - 1\}$. Then $|u| = d(A)$ and $|\tilde{u}| = d(\tilde{A})$ as before.

By assumption there exist i, j so that $A_{i,j} \neq \tilde{A}_{i,j}$. Suppose $h, \tilde{h} \in S_u$ satisfy $hg \in \tilde{h}\tilde{g}S_{\lambda'}$. Because $S_{\lambda'}$ fixes $\mathfrak{B}_i$, the sets $Q = hg(\mathfrak{B}_i)$ and $\tilde{Q} = \tilde{h}\tilde{g}(\mathfrak{B}_i)$ are equal. By construction, $g(\mathfrak{B}_i)$ contains $A_{i,j}$ elements of $\mathfrak{C}_j$. Since $h \in S_u$, Q also contains $A_{i,j}$ elements of $\mathfrak{C}_j$. Similarly, $\tilde{Q}$ contains $\tilde{A}_{i,j}$ elements of $\mathfrak{C}_j$. This contradicts $Q = \tilde{Q}$ because $A_{i,j} \neq \tilde{A}_{i,j}$. Thus for any $h, \tilde{h} \in S_u$, hg and $\tilde{h}\tilde{g}$ are in different left cosets of $S_n/S_{\lambda'}$.

Since $\varepsilon \uparrow_{S_{\lambda'}}^{S_n} \otimes \mathbb{C}[\mathbf{x}]_d = \frac{\mathbb{C}S_n \cdot e}{\langle \sigma e - (-1)^\sigma e : \sigma \in S_{\lambda'} \rangle} \otimes \mathbb{C}[\mathbf{x}]_d = \oplus_{\text{left cosets } \sigma S_{\lambda'}} (\mathbb{C}\sigma e \otimes \mathbb{C}[\mathbf{x}]_d)$, we see that $\left(\sum_{h \in S_\mu} h\right)(ge \otimes \mathbf{x}^u)$ and $\left(\sum_{h \in S_\mu} h\right)(\tilde{g}e \otimes \mathbf{x}^{\tilde{u}})$ belong in different direct summands of $\varepsilon \uparrow_{S_{\lambda'}}^{S_n} \otimes \mathbb{C}[\mathbf{x}]_d$. Now the argument four paragraphs above shows that $\left(\sum_{h \in S_\mu} h\right)(ge \otimes \mathbf{x}^u) \neq 0$ and $\left(\sum_{h \in S_\mu} h\right)(\tilde{g}e \otimes \mathbf{x}^{\tilde{u}}) \neq 0$. Therefore, $\left(\sum_{h \in S_\mu} h\right)(ge \otimes \mathbf{x}^u)$ and $\left(\sum_{h \in S_\mu} h\right)(\tilde{g}e \otimes \mathbf{x}^{\tilde{u}})$ are linearly independent and $\dim_{\mathbb{C}} \left(\sum_{h \in S_\mu} h\right)\left(\varepsilon \uparrow_{S_{\lambda'}}^{S_n} \otimes \mathbb{C}[\mathbf{x}]_d\right) \geq 2$.

Now consider the case that the minimizing matrix A is unique. We claim that for any $g, \tilde{g} \in S_n$ and $u, \tilde{u}$ with $|u| = |\tilde{u}| = d(A)$, $v = \left(\sum_{h \in S_\mu} h\right)(ge \otimes \mathbf{x}^u)$ and $\tilde{v} = \left(\sum_{h \in S_\mu} h\right)(\tilde{g}e \otimes \mathbf{x}^{\tilde{u}})$ are linearly dependent. If either is zero then we are done. We saw above that if there exist i, j and some distinct $a, b \in g(\mathfrak{B}_j) \cap \mathfrak{C}_i$ such that $u_a = u_b$, then $v = 0$. Thus we may assume no such occurrence exists, and also for the $\tilde{v}$ version. In that case, we saw that $|u|$ is at least the degree of the matrix with (i,j) entry equal to $|g(\mathfrak{B}_j) \cap \mathfrak{C}_i|$, and this matrix is in $\mathbb{M}(\mu, \lambda')$. Since $|u| = d(A)$, by unique minimization this matrix must be A. Hence $|g(\mathfrak{B}_j) \cap \mathfrak{C}_i| = A_{i,j}$ for all i, j, and similarly $|\tilde{g}(\mathfrak{B}_j) \cap \mathfrak{C}_i| = A_{i,j}$. Furthermore, we can only have equality $|u| = d(A) = |\tilde{u}|$ if in fact $\{u_a : a \in g(\mathfrak{B}_j) \cap \mathfrak{C}_i\} = \{0, 1, \ldots, A_{i,j} - 1\} = \{\tilde{u}_a : a \in \tilde{g}(\mathfrak{B}_j) \cap \mathfrak{C}_i\}$.

This last fact allows us to define $\tau \in S_n$ as follows: For $a \in \{1, \ldots, n\}$, there exist unique i, j such that $a \in g(\mathfrak{B}_j) \cap \mathfrak{C}_i$. Set $\tau(a)$ to be the *unique* letter in $\tilde{g}(\mathfrak{B}_j) \cap \mathfrak{C}_i$ such that $\tilde{u}_{\tau(a)} = u_a$. Then $\tau(\mathbf{x}^u) = \mathbf{x}^{\tilde{u}}$.

Since τ maps $g(\mathfrak{B}_j) \cap \mathfrak{C}_i$ bijectively to $\tilde{g}(\mathfrak{B}_j) \cap \mathfrak{C}_i$, τ permutes within each $\mathfrak{C}_i$ and is in S_μ. Furthermore, $\tau g(\mathfrak{B}_j) = \cup_j \tau(g(\mathfrak{B}_j) \cap \mathfrak{C}_i) = \cup_j \tilde{g}(\mathfrak{B}_j) \cap \mathfrak{C}_i = \tilde{g}(\mathfrak{B}_j)$, so τg and $\tilde{g}$ are in the same left coset of $S_n/S_{\lambda'}$. Thus we have $v = \left(\sum_{h \in S_\mu} h\right)(ge \otimes \mathbf{x}^u) = \left(\sum_{h \in S_\mu} h\tau\right)(ge \otimes \mathbf{x}^u) = \left(\sum_{h \in S_\mu} h\right)(\tau ge \otimes \tau(\mathbf{x}^u)) = (-1)^{\tilde{g}^{-1}\tau g}\left(\sum_{h \in S_\mu} h\right)(\tilde{g}e \otimes \mathbf{x}^{\tilde{u}}) = \pm\tilde{v}$, and the claim is proved. It follows that $\dim_{\mathbb{C}} \left(\sum_{h \in S_\mu} h\right)\left(\varepsilon \uparrow_{S_{\lambda'}}^{S_n} \otimes \mathbb{C}[\mathbf{x}]_d\right) \leq 1$. But above we already exhibited a nonzero element in this space, so the dimension is exactly 1.

Therefore, $\mathrm{Hom}_{S_n}\left(\left(\varepsilon \uparrow_{S_{\lambda'}}^{S_n}\right) \otimes \mathbb{C}[\mathbf{x}]_{\bar{d}(\lambda,\mu)},\ 1 \uparrow_{S_\mu}^{S_n}\right)$ has dimension 1 iff the minimizing matrix A is unique. By Theorem 3.5.4, this is equivalent to λ being skew-linked to μ.

The last statement comes from Theorem 3.5.4.

$\square$

Lemma 4.2.4 *Suppose $\lambda \to \mu$. Then*

$$\mathrm{Hom}_{S_n}\big(\varepsilon \uparrow_{S_{\lambda'}}^{S_n},\ \left(1 \uparrow_{S_\mu}^{S_n}\right) \otimes \mathbb{C}[\mathbf{x}]_d\big)$$

is zero for $d < \bar{d}(\lambda,\mu)$ and has dimension 1 for $d = \bar{d}(\lambda,\mu)$.

Proof

This follows from Proposition 4.2.3 and the following observations for any S_n-modules M, N, P:

- $\mathrm{Hom}_{S_n}(M,N) = 0$ iff $\langle M,N\rangle_{S_n} = 0$.
- $\dim_{\mathbb{C}} \mathrm{Hom}_{S_n}(M,N) = 1$ iff $\langle M,N\rangle_{S_n} = 1$.
- $\langle M \otimes P, N\rangle_{S_n} = \langle M, N \otimes P\rangle_{S_n}$.

$\square$

Proposition 4.2.5 *Suppose $\lambda \to \mu$. By Proposition 4.2.3, there exists unique (up to a constant) nonzero $\psi \in \mathrm{Hom}_{S_n}\big(\left(\varepsilon \uparrow_{S_{\lambda'}}^{S_n}\right) \otimes \mathbb{C}[\mathbf{x}]_{\bar{d}(\lambda,\mu)},\ 1 \uparrow_{S_\mu}^{S_n}\big)$. By Lemma 4.2.4 and $\mathbb{C}[\mathbf{x}]_{\bar{d}(\lambda,\mu)} \cong_{S_n} \mathbb{C}[\mathbf{x}]^*_{-\bar{d}(\lambda,\mu)}$, there exists unique (up to a constant) nonzero $\xi \in \mathrm{Hom}_{S_n}\big(\varepsilon \uparrow_{S_{\lambda'}}^{S_n}, \left(1 \uparrow_{S_\mu}^{S_n}\right) \otimes \mathbb{C}[\mathbf{x}]^*_{-\bar{d}(\lambda,\mu)}\big)$. By Proposition 1.4.7, $\varepsilon \uparrow_{S_{\lambda'}}^{S_n}$ contains a unique copy of V_λ, and $1 \uparrow_{S_\mu}^{S_n}$ contains a unique copy of V_μ.*

1. *With respect to these embeddings, ψ restricted to $V_\lambda \otimes \mathbb{C}[\mathbf{x}]_{\bar{d}(\lambda,\mu)}$ is nonzero, has the same image as ψ, and maps into V_μ (and thus onto because V_μ is irreducible). Furthermore, ψ restricted to $V_\lambda \otimes \mathbb{C}[\mathbf{x}]_{\bar{d}(\lambda,\mu)}$ is the unique (up to a constant) S_n-homomorphism from $V_\lambda \otimes \mathbb{C}[\mathbf{x}]_{\bar{d}(\lambda,\mu)}$ to V_μ.*
2. *With respect to these embeddings, ξ restricted to V_λ is nonzero, has the same image as ξ, and maps into $V_\mu \otimes \mathbb{C}[\mathbf{x}]_{\bar{d}(\lambda,\mu)}$. Furthermore, ξ restricted to V_λ is the unique (up to a constant) S_n-homomorphism from V_λ to $V_\mu \otimes \mathbb{C}[\mathbf{x}]^*_{-\bar{d}(\lambda,\mu)}$.*

Proof

Proposition 1.4.7 states that $\varepsilon \uparrow_{S_{\lambda'}}^{S_n} \cong \oplus_{\gamma \le \lambda}\left({V_\gamma}^{\oplus K_{\gamma',\lambda'}}\right)$ and $1 \uparrow_{S_\mu}^{S_n} \cong \oplus_{\eta \ge \mu}\left({V_\eta}^{\oplus K_{\eta,\mu}}\right)$. Thus $\mathrm{Hom}_{S_n}\big(\left(\varepsilon \uparrow_{S_{\lambda'}}^{S_n}\right) \otimes \mathbb{C}[\mathbf{x}]_{\bar{d}(\lambda,\mu)},\ 1 \uparrow_{S_\mu}^{S_n}\big) \cong \oplus_{\gamma \le \lambda, \eta \ge \mu} \mathrm{Hom}_{S_n}\left(V_\gamma \otimes \mathbb{C}[\mathbf{x}]_{\bar{d}(\lambda,\mu)}, V_\eta\right)^{\oplus K_{\gamma',\lambda'} K_{\eta,\mu}}$.

Suppose $\gamma \le \lambda$ and $\eta \ge \mu$ with at least one inequality. Then by Proposition 4.2.3 and Proposition 3.6.5, $\bar{d}(\gamma,\eta) = d(\gamma,\eta) > d(\lambda,\mu)$. Hence by Proposition 4.2.3, $\mathrm{Hom}_{S_n}\big(\left(\varepsilon \uparrow_{S_{\gamma'}}^{S_n}\right) \otimes \mathbb{C}[\mathbf{x}]_{\bar{d}(\lambda,\mu)},\ 1 \uparrow_{S_\eta}^{S_n}\big) = 0$. But $V_\gamma \otimes \mathbb{C}[\mathbf{x}]_{\bar{d}(\lambda,\mu)}$ embeds into $\varepsilon \uparrow_{S_{\gamma'}}^{S_n} \otimes \mathbb{C}[\mathbf{x}]_{\bar{d}(\lambda,\mu)}$ and V_η embeds into $1 \uparrow_{S_\eta}^{S_n}$. Thus $\mathrm{Hom}_{S_n}\left(V_\gamma \otimes \mathbb{C}[\mathbf{x}]_{\bar{d}(\lambda,\mu)}, V_\eta\right) = 0$.

So within $\mathrm{Hom}_{S_n}\big(\left(\varepsilon \uparrow_{S_{\lambda'}}^{S_n}\right) \otimes \mathbb{C}[\mathbf{x}]_{\bar{d}(\lambda,\mu)},\ 1 \uparrow_{S_\mu}^{S_n}\big)$, all components are zero except $\mathrm{Hom}_{S_n}\left(V_\lambda \otimes \mathbb{C}[\mathbf{x}]_{\bar{d}(\lambda,\mu)}, V_\mu\right)$ (where $K_{\lambda',\lambda'} K_{\mu,mu} = 1$). Because

$\mathrm{Hom}_{S_n}\big(\left(\varepsilon\uparrow_{S_{\lambda'}}^{S_n}\right)\otimes\mathbb{C}[\mathbf{x}]_{\bar{d}(\lambda,\mu)},\ 1\uparrow_{S_\mu}^{S_n}\big)$ has dimension 1, we have proven the desired statements about ψ.

For ξ, use the fact $\mathbb{C}[\mathbf{x}]_{\bar{d}(\lambda,\mu)}\cong_{S_n}\mathbb{C}[\mathbf{x}]^*_{-\bar{d}(\lambda,\mu)}$, use the above arguments except with $\mathbb{C}[\mathbf{x}]_{\bar{d}(\lambda,\mu)}$ tensored on the other side, and use Lemma 4.2.4 instead of Proposition 4.2.3.

□

Corollary 4.2.6 *Suppose $\lambda\to\mu$.*

1. *$d=\bar{d}(\lambda,\mu)$ is the smallest degree d such that $V_\lambda\otimes\mathbb{C}[\mathbf{x}]_d$ contains a copy of the irreducible V_μ. At $d=\bar{d}(\lambda,\mu)$, there is exactly one copy.*
2. *$d=\bar{d}(\lambda,\mu)$ is the smallest degree d such that $V_\mu\otimes\mathbb{C}[\mathbf{x}]_d$ contains a copy of the irreducible V_λ. At $d=\bar{d}(\lambda,\mu)$, there is exactly one copy.*
3. *Define ν as in Definition 2.3.5. Then $\langle V_\lambda\otimes V_\nu, V_\mu\rangle_{S_n}=1$.*

Proof

Proposition 4.2.5 shows that there is a unique (up to a constant) S_n-homomorphism $V_\lambda\otimes\mathbb{C}[\mathbf{x}]_{\bar{d}(\lambda,\mu)}\to V_\mu$, so $V_\lambda\otimes\mathbb{C}[\mathbf{x}]_{\bar{d}(\lambda,\mu)}$ contains exactly one copy of V_μ. If $d<\bar{d}(\lambda,\mu)$, then $\mathrm{Hom}_{S_n}\big(\left(\varepsilon\uparrow_{S_{\lambda'}}^{S_n}\right)\otimes\mathbb{C}[\mathbf{x}]_d,\ 1\uparrow_{S_\mu}^{S_n}\big)=0$. Since $V_\lambda\otimes\mathbb{C}[\mathbf{x}]_d$ embeds into $\varepsilon\uparrow_{S_{\lambda'}}^{S_n}\otimes\mathbb{C}[\mathbf{x}]_d$ and V_μ embeds into $1\uparrow_{S_\mu}^{S_n}$, we have $\mathrm{Hom}_{S_n}\big(V_\lambda\otimes\mathbb{C}[\mathbf{x}]_d,\ V_\mu\big)=0$. Thus $V_\lambda\otimes\mathbb{C}[\mathbf{x}]_d$ contains no copy of V_μ for $d<\bar{d}(\lambda,\mu)$, and (1) holds.

Now $\langle V_\lambda\otimes\mathbb{C}[\mathbf{x}]_d, V_\mu\rangle_{S_n}=\langle V_\lambda, V_\mu\otimes\mathbb{C}[\mathbf{x}]_d\rangle_{S_n}$. If $d=\bar{d}(\lambda,\mu)$, then LHS $=1$ by (1), so RHS $=1$ and $V_\mu\otimes\mathbb{C}[\mathbf{x}]_{\bar{d}(\lambda,\mu)}$ contains exactly one copy of V_λ. If $d<\bar{d}(\lambda,\mu)$, then LHS $=0$ by (1), so RHS $=0$ and $V_\mu\otimes\mathbb{C}[\mathbf{x}]_{\bar{d}(\lambda,\mu)}$ contains no copy of V_λ. Thus (2) holds.

For (3), revisit the proof of Proposition 4.2.3. In the case that the minimizing A is unique (and thus $A=\bar{A}$ by Theorem 3.5.4), we saw that if $\left(\sum_{h\in S_\mu}h\right)(ge\otimes\mathbf{x}^u)$ is nonzero, then $\{u_a : a\in g(\mathfrak{B}_j)\cap\mathfrak{C}_i\}=\{0,1,\ldots,A_{i,j}-1\}$ for all i,j. Recall $\bar{A}_{i,j}=(\gamma^i)'_j$. Then for $i,j,p\geq 1$, $p-1\in\{0,1,\ldots,A_{i,j}-1\}\iff \bar{A}_{i,j}\geq p\iff \lambda_{r_{i,p}}\geq j$. So for fixed i and p, the number of occurrences of $p-1$ in the multiset $\cup_j\{0,1,\ldots,A_{i,j}-1\}$ is $\lambda_{r_{i,p}}$. Thus the number of occurrences of $p-1$ in the multiset $\{u_1,u_2,\ldots,u_n\}=\cup_i\cup_j\{0,1,\ldots,A_{i,j}-1\}$ is $\sum_i\lambda_{r_{i,p}}=\nu_i$. Therefore, u is a permutation of $(0^{\nu_1}1^{\nu_2}2^{\nu_3}\ldots)$.

Let $\tilde{W}$ be the span of degree-$d(\bar{A})$ monomials $\mathbf{x}^u$ such that u is not a permutation of $(0^{\nu_1}1^{\nu_2}2^{\nu_3}\ldots)$. Then the preceding paragraph implies that $\varepsilon\uparrow_{S_{\lambda'}}^{S_n}\otimes\tilde{W}$ is annihilated by $\left(\sum_{h\in S_\mu}h\right)=a_\mu$. Thus, $V_\lambda\otimes\tilde{W}\subset\varepsilon\uparrow_{S_{\lambda'}}^{S_n}\otimes\tilde{W}$ is also annihilated by $\tilde{c}_\mu=b_\mu a_\mu$ and contains no copy of V_μ by Corollary 1.4.5. Now $d(\bar{A})=\bar{d}(\lambda,\mu)$ and $\mathbb{C}[\mathbf{x}]_{\bar{d}(\lambda,\mu)}=\tilde{W}\oplus U_\nu$ (see Proposition 1.4.7). Thus (1) implies $1=\langle V_\lambda\otimes\mathbb{C}[\mathbf{x}]_{\bar{d}(\lambda,\mu)},V_\mu\rangle_{S_n}=\langle V_\lambda\otimes\tilde{W},V_\mu\rangle_{S_n}+\langle V_\lambda\otimes U_\nu,V_\mu\rangle_{S_n}=\langle V_\lambda\otimes U_\nu,V_\mu\rangle_{S_n}$.

By Proposition 1.4.7, U_ν contains irreducible V_η only if $\eta\geq\nu$. Take $\eta\geq\neq\nu$. We have embeddings $V_\eta\subset U_\eta\subset\mathbb{C}[\mathbf{x}]_{n(\eta)}$. Now $n(\eta)<n(\nu)=\bar{d}(\lambda,\mu)$, so by (1), $V_\lambda\otimes\mathbb{C}[\mathbf{x}]_{n(\eta)}$ contains no copy of V_μ. Thus neither does $V_\lambda\otimes V_\eta$. By Proposition 1.4.7 we have $1=\langle V_\lambda\otimes U_\nu,V_\mu\rangle_{S_n}=\sum_{\eta\geq\nu}K_{\eta,\nu}\langle V_\lambda\otimes V_\eta,V_\mu\rangle_{S_n}=K_{\nu,\nu}\langle V_\lambda\otimes V_\nu,V_\mu\rangle_{S_n}=\langle V_\lambda\otimes V_\nu,V_\mu\rangle_{S_n}$.

∎

Lemma 4.2.7 *1. If $\kappa \in \mathbb{C}[\mathbf{x}]^* \setminus \{0\}$, then there exists monomial $\mathbf{x}^u$ such that $\mathbf{x}^u\kappa \in \mathbb{C}[\mathbf{x}]^*_0 \setminus \{0\}$.*

2. *For any S_n-module M, consider $M \otimes \mathbb{C}[\mathbf{x}]^*$ as a graded module over $\mathbb{C}[\mathbf{x}] * S_n$. If κ is a nonzero element in $\mathbb{M} \otimes \mathbb{C}[\mathbf{x}]^*$, then there exists monomial $\mathbf{x}^u$ such that $\mathbf{x}^u\kappa$ is a nonzero element in $(M \otimes \mathbb{C}[\mathbf{x}]^*)_0$.*

Proof

1. Since $\kappa \neq 0$, we may write $\kappa = \sum_{i=d}^{0} \kappa_i$ for $\kappa_i \in \mathbb{C}[\mathbf{x}]^*_i$, $d \leq 0$, and $\kappa_d \neq 0$. Since κ_d is a nonzero linear functional on $\mathbb{C}[\mathbf{x}]_{-d}$, there exists monomial $\mathbf{x}^u \in \mathbb{C}[\mathbf{x}]_{-d}$ such that $\kappa_d(\mathbf{x}^u) \neq 0$. Then $\mathbf{x}^u\kappa_d$ is a nonzero element in $\mathbb{C}[\mathbf{x}]^*_0$. Now for $i > d$, $\mathbf{x}^u\kappa_i$ has degree $i - d > 0$ and is thus zero. Therefore, $\mathbf{x}^u\kappa = \mathbf{x}^u\kappa_d$ is a nonzero element in $\mathbb{C}[\mathbf{x}]^*_0$.

2. Let $\mathfrak{B}$ be a basis of M over $\mathbb{C}$. Then $M \otimes \mathbb{C}[\mathbf{x}]^* = \oplus_{v\in\mathfrak{B}}\mathbb{C}v \otimes \mathbb{C}[\mathbf{x}]^*$, where each $\mathbb{C}v \otimes \mathbb{C}[\mathbf{x}]^* \cong \mathbb{C}[\mathbf{x}]^*$ as a $\mathbb{C}[\mathbf{x}]$-module and we may apply part (1) to it.

   Now κ may be written as $\sum_{v\in\mathfrak{B}'} z_v$ for $z_v \in (\mathbb{C}v \otimes \mathbb{C}[\mathbf{x}]^*)\setminus\{0\}$ and nonzero finite subset $\mathfrak{B}' \subset \mathfrak{B}$. For each z_v, choose $\mathbf{x}^{u_v}$ according to part (1). Choose $v' \in \mathfrak{B}'$ with $|u_{v'}| = d = \max_{v\in\mathfrak{B}'} |u_v|$. Then for all $v \in \mathfrak{B}'$, $\mathbf{x}^{u_{v'}} z_v$ either has positive degree and is zero, or has degree zero. Thus $\mathbf{x}^{u_{v'}} z_v \in \mathbb{C}v \otimes \mathbb{C}[\mathbf{x}]^*_0$. Now $(M \otimes \mathbb{C}[\mathbf{x}]^*)_0 = \oplus_{v\in\mathfrak{B}} (\mathbb{C}v \otimes \mathbb{C}[\mathbf{x}]^*_0)$, so the sum $\mathbf{x}^{u_{v'}}\kappa = \sum_{v\in\mathfrak{B}'} \mathbf{x}^{u_{v'}} z_v$ is zero iff all summands are zero. But the summand $\mathbf{x}^{u_{v'}} z_{v'}$ is nonzero, so $\mathbf{x}^{u_{v'}}\kappa$ is a nonzero degree zero element.

∎

Now we are ready to prove Theorem 4.2.2.

Proof

Statement (1) follows from Proposition 4.2.3. For the rest of the proof, assume $\lambda \overset{\alpha/\beta}{\to} \mu$.

The argument in Proposition 4.2.1 shows that at degree zero, ϕ is a nonzero S_n-homomorphism ϕ_0 from $\varepsilon \uparrow_{S_{\lambda'}}^{S_n}$ to $\left(1 \uparrow_{S_\mu}^{S_n}\right) \otimes \mathbb{C}[\mathbf{x}]^*_{-\bar{d}(\lambda,\mu)}$. So ϕ_0 is a nonzero scalar multiple of the ξ in Proposition 4.2.5. By Proposition 4.2.5 (2), ϕ_0 restricted to V_λ is nonzero and has the same image as ϕ_0. Now since V_λ is irreducible, $\mathrm{Ker}(\phi_0|_{V_\lambda}) = 0$. Thus $V_\lambda \cong \mathrm{Im}(\phi_0|_{V_\lambda}) = \mathrm{Im}(\phi_0) = (M_{\lambda,\mu})_0$.

In particular, there exists $v \in \varepsilon \uparrow_{S_{\lambda'}}^{S_n} = V_0$ such that $\phi_0(v) \in (M_{\lambda,\mu})_0$ is nonzero. By Lemma 4.2.7 applied to $\phi_0(v)$, there exists monomial $\mathbf{x}^u$ such that $\phi(\mathbf{x}^u v) = \mathbf{x}^u\phi_0(v)$ is a nonzero element of $\left(1 \uparrow_{S_\mu}^{S_n}\right) \otimes \mathbb{C}[\mathbf{x}]^*_0 = W[-\bar{d}(\lambda,\mu)]_{\bar{d}(\lambda,\mu)}$. Thus $\mathbf{x}^u v$ has degree $\bar{d}(\lambda,\mu)$. Hence $\phi_{\bar{d}(\lambda,\mu)} : V_{\bar{d}(\lambda,\mu)} \to W_0$ is nonzero, and it has image isomorphic to V_μ by Proposition 4.2.5. Since $(M_{\lambda,\mu})_{\bar{d}(\lambda,\mu)}$ is the image of $\phi_{\bar{d}(\lambda,\mu)}$, it is also ismorphic to V_μ.

Now for $d > \bar{d}(\lambda,\mu)$, $(M_{\lambda,\mu})_d \subset W[-\bar{d}(\lambda,\mu)]_d = 0$. So $M_{\lambda,\mu}$ has top degree at $\bar{d}(\lambda,\mu)$ and statement (3) holds. Statement (2) also holds by Theorem 3.5.4.

□

We end by summarizing the relationships between $d, \bar{d}, \bar{A}, \lambda, \mu$.

Proposition 4.2.8 *1. For any partitions λ, μ of size n,*

$$\bar{d}(\lambda, \mu) = d(\lambda, \mu) = \min_{A \in \mathbb{M}(\mu, \lambda')} d(A)$$

is the minimum degree d where there exists nonzero S_n-homomorphism

$$\psi \colon \left(\varepsilon \uparrow_{S_{\lambda'}}^{S_n} \right) \otimes \mathbb{C}[\mathbf{x}]_d \to 1 \uparrow_{S_\mu}^{S_n} .$$

2. *If furthermore $\lambda \xrightarrow{\alpha/\beta} \mu$, then $d(\bar{A}) = d(\lambda, \mu) = |\beta|$ uniquely achieves the minimum degree, and ψ exists uniquely up to a constant.*

4.3 Properties of Skew-Linked Modules

We propose two alternate and closely related formulations of $M_{\lambda,\mu}$. The second and third formulations will be very useful.

Proposition 4.3.1 *Suppose $\lambda \to \mu$. Recall $M_{\lambda,\mu}$ is defined to be the image of ϕ in Proposition 4.2.1.*

1. *Up to a constant, there is a unique nonzero $\mathbb{C}[\mathbf{x}] * S_n$ homomorphism, homogeneous of degree zero*
$$\tilde{\phi} \colon V_\lambda \otimes \mathbb{C}[\mathbf{x}] \to \left(V_\mu \otimes \mathbb{C}[\mathbf{x}]^* \right) [-\bar{d}(\lambda, \mu)].$$
Its image is the previously defined $M_{\lambda,\mu}$. Furthermore, $M_{\lambda,\mu}$ is the unique nonzero graded $\mathbb{C}[\mathbf{x}] * S_n$ module for which there exists the diagram
$$V_\lambda \otimes \mathbb{C}[\mathbf{x}] \twoheadrightarrow M_{\lambda,\mu} \hookrightarrow \left(V_\mu \otimes \mathbb{C}[\mathbf{x}]^* \right) [-\bar{d}(\lambda, \mu)]$$
of homogeneous degree-zero $\mathbb{C}[\mathbf{x}] * S_n$ homomorphisms.

2. *By Corollary 4.2.6, $V_\lambda \otimes \mathbb{C}[\mathbf{x}]_{\bar{d}(\lambda,\mu)}$ contains a unique copy of V_μ. Denote this copy by N. Let $I_{\lambda,\mu}$ be the largest homogeneous $(\mathbb{C}[\mathbf{x}] * S_n)$-submodule of $V_\lambda \otimes \mathbb{C}[\mathbf{x}]$ with zero intersection with N. Then $I_{\lambda,\mu} = \mathrm{Ker}(\tilde{\phi})$, so $M_{\lambda,\mu} = \mathrm{Im}(\tilde{\phi}) \cong \left(V_\lambda \otimes \mathbb{C}[\mathbf{x}] \right) / I_{\lambda,\mu}$.*

3. *Recall that $\varepsilon \uparrow_{S_{\lambda'}}^{S_n}$ contains a unique copy of V_λ and decomposes as $V_\lambda \oplus L$ for a unique S_n-submodule L. Let $J_{\lambda,\mu}$ be the largest homogeneous $(\mathbb{C}[\mathbf{x}] * S_n)$-submodule of $V = \left(\varepsilon \uparrow_{S_{\lambda'}}^{S_n} \right) \otimes \mathbb{C}[\mathbf{x}]$ with zero intersection with N. Then $\mathrm{Ker}(\phi) = J_{\lambda,\mu} = I_{\lambda,\mu} \oplus (L \otimes \mathbb{C}[\mathbf{x}])$, so $M_{\lambda,\mu} = \mathrm{Im}(\phi) = V / J_{\lambda,\mu}$.*

Proof

By Corollary 4.2.6 (2), there exists a unique (up to constant) S_n-homomorphism $V_\lambda = (V_\lambda \otimes \mathbb{C}[\mathbf{x}])_0 \to (V_\mu \otimes \mathbb{C}[\mathbf{x}^*])\,[-\bar{d}(\lambda,\mu)]_0 \cong V_\mu \otimes \mathbb{C}[\mathbf{x}]_{\bar{d}(\lambda,\mu)}$. Since $V_\lambda \otimes \mathbb{C}[\mathbf{x}]$ is freely generated over $\mathbb{C}[\mathbf{x}]$ by V_λ, the desired $\tilde{\phi}$ also exists uniquely up to a constant.

By Proposition 4.2.5 and uniquness of $\tilde{\phi}_0$, $\tilde{\phi}_0$ is a nonzero scalar multiple of ξ restricted to $V_\lambda \subset \varepsilon \uparrow_{S_{\lambda'}}^{S_n}$. But ϕ_0 is a nonzero scalar multiple of ξ (see the proof of Theorem 4.2.2). So by Proposition 4.2.5 again, $(M_{\lambda,\mu})_0 = \mathrm{Im}(\phi_0) = \mathrm{Im}(\xi) = \mathrm{Im}(\xi|_{V_\lambda}) = \mathrm{Im}(\tilde{\phi}_0)$. Now $M_{\lambda,\mu} = \mathrm{Im}(\phi)$ is generated by $(M_{\lambda,\mu})_0 = \mathrm{Im}(\phi_0)$ because the domain of ϕ is generated by its degree zero component. For the same reason, $\mathrm{Im}(\tilde{\phi})$ is generated by $\mathrm{Im}(\tilde{\phi}_0)$. Therefore, $M_{\lambda,\mu} = \mathrm{Im}(\tilde{\phi})$ and $\tilde{\phi}$ factors through $M_{\lambda,\mu}$ in the desired diagram. For the uniqueness in the last statement in (1), note that the middle term in any such diagram is the image of some map $V_\lambda \otimes \mathbb{C}[\mathbf{x}] \to (V_\mu \otimes \mathbb{C}[\mathbf{x}^*])\,[-\bar{d}(\lambda,\mu)]$, which we showed above is unique up to a constant.

Now N is the V_μ-isotypic component of $V_\lambda \otimes \mathbb{C}[\mathbf{x}]_{\bar{d}(\lambda,\mu)}$. Let N' be the direct sum of all other isotypic components. Then $V_\lambda \otimes \mathbb{C}[\mathbf{x}]_{\bar{d}(\lambda,\mu)} = N \oplus N'$. Suppose N'' is an S_n-submodule of $V_\lambda \otimes \mathbb{C}[\mathbf{x}]_{\bar{d}(\lambda,\mu)}$. Then $N'' = (N \cap N'') \oplus (N' \cap N'')$, so $N \cap N'' = 0$ iff $N' \cap N'' = N''$ iff $N'' \subset N'$. Hence for homogeneous $(\mathbb{C}[\mathbf{x}] * S_n)$-submodules $I \subset V_\lambda \otimes \mathbb{C}[\mathbf{x}]$, $I \cap N = 0$ iff $\big(I \cap (V_\lambda \otimes \mathbb{C}[\mathbf{x}]_{\bar{d}(\lambda,\mu)})\big) \cap N = 0$ iff $I \cap (V_\lambda \otimes \mathbb{C}[\mathbf{x}]_{\bar{d}(\lambda,\mu)}) \subset N'$. Since the last condition is preserved under infinite sums of submodules, there exists a largest homogeneous submodule $I_{\lambda,\mu}$ satisfying the last condition. Therefore, $I_{\lambda,\mu}$ is well-defined.

Recall $\mathrm{Im}(\tilde{\phi}_{\bar{d}(\lambda,\mu)}) = (M_{\lambda,\mu})_{\bar{d}(\lambda,\mu)} \cong V_\mu$ by Theorem 4.2.2. Since N is the unique copy of V_μ in $V_\lambda \otimes \mathbb{C}[\mathbf{x}]_{\bar{d}(\lambda,\mu)}$, $\tilde{\phi}$ maps N isomorphically onto its image and maps N' to 0. Thus $\mathrm{Ker}(\tilde{\phi}) \cap N = 0$ and $\mathrm{Ker}(\tilde{\phi}) \subset I_{\lambda,\mu}$.

Conversely, suppose $I_{\lambda,\mu} \backslash \mathrm{Ker}(\tilde{\phi})$ is nonempty, and pick some v in the set. By homogeneity of $I_{\lambda,\mu}$ and $\mathrm{Ker}(\tilde{\phi})$, we may assume v is homogeneous of say degree d. Since $\tilde{\phi}(v) \neq 0$, $d \leq \bar{d}(\lambda,\mu)$. By Lemma 4.2.7, there exists monomial $\mathbf{x}^u$ of degree $\bar{d}(\lambda,\mu) - d$ such that $\tilde{\phi}(\mathbf{x}^u v) = \mathbf{x}^u \tilde{\phi}(v)$ is a nonzero element of $V_\mu \otimes \mathbb{C}[\mathbf{x}]^*_0$. Thus $\mathbf{x}^u v \notin N'$ (since $\tilde{\phi}(N') = 0$). But $\mathbf{x}^u v \in I_{\lambda,\mu} \cap (V_\lambda \otimes \mathbb{C}[\mathbf{x}]_{\bar{d}(\lambda,\mu)})$, so $I_{\lambda,\mu} \cap (V_\lambda \otimes \mathbb{C}[\mathbf{x}]_{\bar{d}(\lambda,\mu)}) \not\subset N'$, contradicting the construction of $I_{\lambda,\mu}$. Therefore, $\mathrm{Ker}(\tilde{\phi}) = I_{\lambda,\mu}$ and we have proved (2).

For (3), the last two paragraphs in the proof of Proposition 4.2.5 show that $\mathrm{Hom}_{S_n}(L, \left(1 \uparrow_{S_\mu}^{S_n}\right) \otimes \mathbb{C}[\mathbf{x}]^*_{-\bar{d}(\lambda,\mu)}) = 0$. Thus $\phi|_L = 0$ and all parts of (3) follow from (2).

□

Definition 4.3.2 *(Skew-linked characters) Given $\lambda \overset{\alpha/\beta}{\to} \mu$, define $P_{\lambda,\mu}(z;t)$ to be the Frobenius characteristic of $M_{\lambda,\mu}$.*

Proposition 4.3.3 *Suppose $\lambda \overset{\alpha/\beta}{\to} \mu$.*

1. *We may write*
$$P_{\lambda,\mu}(z;t) = t^{|\beta|} s_\mu(z) + \sum_{\eta \neq \lambda,\mu} f_\eta(t) s_\eta(z) + s_\lambda(z)$$
where each $f_\eta(t) \in t\mathbb{N}[t]$ has degree $< |\beta| = \bar{d}(\lambda,\mu)$.

2. *If $\lambda = \mu$, then $P_{\lambda,\mu}(z,t) = s_\lambda(z)$.*

3. *If $\mu = (n)$, then $P_{\lambda,\mu}(z,t) = t^{n(\lambda)} H_\lambda(z;t^{-1})$.*

4. *$P_{\mu',\lambda'}(z;t)$ is also defined because $\mu' \overset{\alpha'/\beta'}{\to} \lambda'$ by Lemma 2.1.3. It has the form*

$$P_{\mu',\lambda'}(z;t) = t^{|\beta|}\omega P_{\lambda,\mu}(z;t^{-1}) = s_{\mu'}(z) + t^{|\beta|}\sum_{\eta\neq\lambda,\mu} f_\eta(t^{-1}) s_{\eta^t}(z) + t^{|\beta|} s_{\lambda'}(z).$$

5. *$M_{\lambda,\mu}$ has a unique (up to a constant) $S_{\lambda'}$-antisymmetric element, which occurs in the bottom degree and generates $M_{\lambda,\mu}$. $M_{\lambda,\mu}$ also has a unique (up to a constant) S_μ-invariant linear function, which occurs in the top degree and co-generates $M_{\lambda,\mu}$.*

Proof

(2) immediately follows from Theorem 4.2.2. Most of (1) is shown by Theorem 4.2.2, but it remains to show that V_μ and V_λ do not appear in degrees $1,\ldots,|\beta|-1$ of $M_{\lambda,\mu}$.

Suppose $1 \le d < |\beta| = \bar{d}(\lambda,\mu)$. Then $V \otimes \mathbb{C}[\mathbf{x}]_d$ does not contain V_μ. But $(M_{\lambda,\mu})_d$ is an S_n-quotient of $V \otimes \mathbb{C}[\mathbf{x}]_d$, so $(M_{\lambda,\mu})_d$ also does not contain V_μ. Now $\langle V_\lambda, W[-|\beta|]_d\rangle_{S_n} = \langle V_\lambda, V_\mu \otimes \mathbb{C}[\mathbf{x}]_{|\beta|-d}\rangle_{S_n} = \langle V_\mu, V_\lambda \otimes \mathbb{C}[\mathbf{x}]_{|\beta|-d}\rangle_{S_n} = 0$ since $|\beta|-d < |\beta|$. Thus $W[-|\beta|]_d$ contains no V_λ. But $(M_{\lambda,\mu})_d$ is an S_n-submodule of $W[-|\beta|]_d$, so $(M_{\lambda,\mu})_d$ also contains no V_λ. Hence (1) is proved.

Now for any S_n-module M, $\langle \varepsilon \otimes \left(\varepsilon \uparrow_{S_{\lambda'}}^{S_n}\right), M\rangle_{S_n} = \langle \varepsilon \uparrow_{S_{\lambda'}}^{S_n}, \varepsilon \otimes M\rangle_{S_n} = \langle \varepsilon, (\varepsilon \otimes M)\downarrow_{S_{\lambda'}}^{S_n}\rangle_{S_{\lambda'}} = \langle \varepsilon, \varepsilon \otimes \left(M \downarrow_{S_{\lambda'}}^{S_n}\right)\rangle_{S_{\lambda'}} = \langle \varepsilon\otimes\varepsilon, M\downarrow_{S_{\lambda'}}^{S_n}\rangle_{S_{\lambda'}} = \langle 1, M\downarrow_{S_{\lambda'}}^{S_n}\rangle_{S_{\lambda'}} = \langle 1\uparrow_{S_{\lambda'}}^{S_n}, M\rangle_{S_n}$. Thus $\varepsilon \otimes \left(\varepsilon \uparrow_{S_{\lambda'}}^{S_n}\right) \cong 1 \uparrow_{S_{\lambda'}}^{S_n}$. Similarly, $\langle \varepsilon \otimes \left(1 \uparrow_{S_\mu}^{S_n}\right), M\rangle_{S_n} = \langle 1\uparrow_{S_\mu}^{S_n}, \varepsilon\otimes M\rangle_{S_n} = \langle 1, (\varepsilon\otimes M)\downarrow_{S_\mu}^{S_n}\rangle_{S_\mu} = \langle 1, \varepsilon \otimes \left(M\downarrow_{S_\mu}^{S_n}\right)\rangle_{S_\mu} = \langle \varepsilon, M\downarrow_{S_\mu}^{S_n}\rangle_{S_\mu} = \langle \varepsilon\uparrow_{S_\mu}^{S_n}, M\rangle_{S_n}$. Thus $\varepsilon \otimes \left(1\uparrow_{S_\mu}^{S_n}\right) \cong \varepsilon\uparrow_{S_\mu}^{S_n}$.

By Proposition 4.2.1, ϕ factors as $V \twoheadrightarrow M_{\lambda,\mu} \hookrightarrow W[-\bar{d}(\lambda,\mu)]$, where $V = \left(\varepsilon \uparrow_{S_{\lambda'}}^{S_n}\right) \otimes \mathbb{C}[\mathbf{x}]$ and $W = \left(1 \uparrow_{S_\mu}^{S_n}\right) \otimes \mathbb{C}[\mathbf{x}]^*$. Suppose we take the graded dual of this diagram, then tensor with ε, then shift degree by $|\beta|$. The result is the diagram

$$\varepsilon \otimes W^* \twoheadrightarrow \varepsilon \otimes M^*_{\lambda,\mu}[-|\beta|] \hookrightarrow (\varepsilon \otimes V^*)[-|\beta|].$$

Now $\varepsilon \otimes W^* \cong \varepsilon \otimes \left(1\uparrow_{S_\mu}^{S_n}\right)\otimes \mathbb{C}[\mathbf{x}] \cong \left(\varepsilon \uparrow_{S_\mu}^{S_n}\right) \otimes \mathbb{C}[\mathbf{x}]$. Also, $\varepsilon \otimes V^* \cong \varepsilon \otimes \left(\varepsilon\uparrow_{S_{\lambda'}}^{S_n}\right) \otimes \mathbb{C}[\mathbf{x}]^* \cong \left(1 \uparrow_{S_{\lambda'}}^{S_n}\right) \otimes \mathbb{C}[\mathbf{x}]^*$. Hence the above diagram is

$$\left(\varepsilon \uparrow_{S_{(\mu')'}}^{S_n}\right) \otimes \mathbb{C}[\mathbf{x}] \twoheadrightarrow \varepsilon \otimes M^*_{\lambda,\mu}[-|\beta|] \hookrightarrow \left(\left(1 \uparrow_{S_{\lambda'}}^{S_n}\right)\otimes \mathbb{C}[\mathbf{x}]^*\right)[-|\beta|].$$

But $\mu' \overset{\alpha'/\beta'}{\to} \lambda'$ by Lemma 2.1.3. Because $|\beta'| = |\beta|$, the last statement of Proposition 4.2.1 implies that $M_{\mu',\lambda'} \cong \varepsilon \otimes M^*_{\lambda,\mu}[-|\beta|]$. The implies (4) because taking graded dual reverses the degrees, while tensoring with ε changes each copy of V_η to $V_{\eta'}$.

Recall that the Garsia-Procesi module R_λ is the quotient $\mathbb{C}[\mathbf{x}]/I_\lambda$, where I_λ is the largest homogeneous S_n-invariant ideal having zero intersection with the unique copy of V_λ in $\mathbb{C}[\mathbf{x}]_{n(\lambda)}$. By a similar argument as Proposition 4.3.1, R_λ fits into a diagram of nonzero homogeneous degree-zero $\mathbb{C}[\mathbf{x}] * S_n$ homomorphisms

$$\mathbb{C}[\mathbf{x}] = V_{(n)} \otimes \mathbb{C}[\mathbf{x}] \twoheadrightarrow R_\lambda \hookrightarrow (V_\lambda \otimes \mathbb{C}[\mathbf{x}]^*)\,[-n(\lambda)].$$

Dualizing this produces the diagram

$$V_\lambda \otimes \mathbb{C}[\mathbf{x}] \twoheadrightarrow R_\lambda^*[-n(\lambda)] \hookrightarrow \left(V_{(n)} \otimes \mathbb{C}[\mathbf{x}]^*\right)[-n(\lambda)].$$

Now $\lambda \overset{\hat{\alpha}/\hat{\beta}}{\to} (n)$ with $|\hat{\beta}| = n(\lambda)$, so by the last statement of Proposition 4.2.1, we must have $M_{\lambda,(n)} \cong R_\lambda^*[-n(\lambda)]$. Recall from Theorem 1.7.4 that the Frobenius series of R_λ is $H_\lambda(z;t)$. Dualizing reverses the degrees, so $M_{\lambda,(n)}$ has Frobenius series $t^{n(\lambda)} H_\lambda(z;t^{-1})$. This proves (3).

For the first part of (5), note that the $S_{\lambda'}$-antisymmetric subspace of $M_{\lambda,\mu}$ is $b_\lambda M_{\lambda,\mu}$. Now the multiplicity of V_λ in an S_n-module W is $\dim_{\mathbb{C}} c_\lambda W$ by [5]. By part (1), this dimension is 0 except at the bottom degree. Since $c_\lambda = a_\lambda b_\lambda$, b_λ kills $M_{\lambda,\mu}$ above the bottom degree. The bottom degree consists of a copy of V_λ, which has a one-dimensional $S_{\lambda'}$-antisymmetric subspace. Therefore, $M_{\lambda,\mu}$ has a unique (up to a constant) $S_{\lambda'}$-antisymmetric element w, and it is at the bottom degree.

By Proposition 4.3.1(1), we may choose $\tilde{w} \in (V_\lambda \otimes \mathbb{C}[\mathbf{x}])_0 \cong V_\lambda$ so that $\tilde{\phi}(\tilde{w}) = w$. Then $\tilde{w} \neq 0$, so it generates V_λ and thus $V_\lambda \otimes \mathbb{C}[\mathbf{x}]$. A generator of the domain maps to a generator of the image, so w generates $M_{\lambda,\mu}$.

For the second part of (5), note that the S_μ-symmetric subspace of $M_{\lambda,\mu}$ is $a_\mu M_{\lambda,\mu}$. By part (1) and Corollary 1.4.5, $\tilde{c}_\mu = b_\mu a_\mu$ kills $M_{\lambda,\mu}$ below the top degree, so a_μ does too. The top degree consists of a copy of V_μ, so it has a one-dimensional S_μ-symmetric subspace. Therefore, $M_{\lambda,\mu}$ has a unique (up to a constant) S_μ-symmetric element w'. Fix a projection $M_{\lambda,\mu} \overset{\pi}{\twoheadrightarrow} \mathbb{C}w'$.

Let $\tilde{M}$ be a nonzero $\mathbb{C}[\mathbf{x}] * S_n$-submodule of $M_{\lambda,\mu}$. Choose some nonzero $v \in \tilde{M}$. Since $M_{\lambda,\mu} \subset W = \left(1 \uparrow_{S_\mu}^{S_n}\right) \otimes \mathbb{C}[\mathbf{x}]^*$, Lemma 4.2.7 implies that for some monomial $\mathbf{x}^u$, $\mathbf{x}^u v$ is a nonzero element in the top degree of W. But $\mathbf{x}^u v \in \tilde{M}$ too, so $\tilde{M}$ intersects the top degree of $M_{\lambda,\mu}$. Since the top degree is an irreducible S_n-module, $w' \in \tilde{M}$ and $\tilde{M} \not\subset \operatorname{Ker}\pi$. Therefore, by Definition 4.1.2 $M_{\lambda,\mu}$ is co-generated by w' (or more precisely, by $\mathbb{C}w'$ and π).

□

4.4 Relationship to k-Schur Functions

Proposition 4.4.1 *[13] Given a k-bounded partition λ, there exists a unique skew shape $\theta = \alpha/\beta$ such that*

1. *θ has the same corresponding row lengths as λ;*

2. *α is a $(k+1)$-core; and*

3. *$\beta \subset \alpha$ consists of the set of boxes with hook length $> k+1$ in α.*

Furthermore, θ has weakly decreasing column lengths that form a k-bounded partition λ^{ω_k} called the k-conjugate *of λ. The operation of taking the k-conjugate is an involution on the set of k-bounded partitions.*

Corollary 4.4.2 *If λ is k-bounded, then it is skew-linked to the transpose of its k-conjugate via the unique skew shape given in Proposition 4.4.1.*

Our primary interest in the skew-linked modules is the following:

Conjecture 4.4.3 *Suppose λ is k-bounded. Then $P_{\lambda,(\lambda^{\omega_k})'}(z;t)$ equals the k-Schur function $s_\lambda^{(k)}(z;t)$ of Lascoux, Lapointe, and Morse.*

The k-Schur functions currently have a number of different definitions, no pair of which have been proven to coincide. (Two of them were reviewed in Sections 1.10 and 1.11.) But there is strong computational evidence that they do coincide. Here are some properties of k-Schur functions that follow from various definitions or are computationally observed:

1. Suppose λ is k-bounded and skew-linked to $\mu = (\lambda^{\omega_k})'$ by α/β. Then $s_\lambda^{(k)}(z;t)$ can be written in the form
$$t^{|\beta|}s_\mu(z) + \sum_{\eta \neq \lambda,\mu} f_\eta(t) s_\eta(z) + s_\lambda(z)$$
where each $f_\eta(t) \in t\mathbb{N}[t]$ has degree $< |\beta|$.

2. $s_{\lambda^{\omega_k}}^{(k)}(z;t) = t^{|\beta|}\omega(s_\lambda^{(k)}(z;t^{-1}))$, where ω is the involution of Λ exchanging h_ν with e_ν and s_ν with $s_{\nu'}$.

3. If all hook lengths in λ are at most k, then $\lambda^{\omega_k} = \lambda$ and $s_\lambda^{(k)}(z;t) = s_\lambda(z)$.

These match the properties of the skew-linked characters $P_{\lambda,\mu}(z;t)$ in the case $\mu = (\lambda^{\omega_k})'$ (see Proposition 4.3.3). Thus we consider $P_{\lambda,\mu}(z;t)$ to be a (conjectured) generalization of k-Schur functions.

4.5 Investigating the Unique Copy of V_μ

Lemma 4.5.1 *Let T be a standard tableau of shape λ. Suppose there exist distinct letters $p, q \in \{1, \ldots, n\}$ that appear in the same column of T and the same row of T_0. Also suppose monomial $\mathbf{x}^u$ satisfies $u_p = u_q$. Then $\tilde{c}_\mu e_T \mathbf{x}^u = 0$.*

Proof

Suppose such p, q exist. Then (p,q) preserves $\mathbf{x}^u$. It swaps two elements in the same column of T and negates e_T. Thus $(p,q)e_T\mathbf{x}^u = -e_T\mathbf{x}^u$. Now $\{(), (p,q)\}$ is a subgroup of Q, so we can find left coset representatives $g_1, \ldots, g_t$. Then $\tilde{c}_\mu e_T \mathbf{x}^u = \sum_i b_\mu g_i(1+(p,q))e_T\mathbf{x}^u = 0$.

□

Recall $\mathfrak{C}_i$ is the set of letters in the jth row of T_0, and $S_\mu = S_{\mathfrak{C}_1} \times S_{\mathfrak{C}_2} \times \ldots$ from Definition 1.4.2 was used to define $a_\mu = \sum_{g \in S_\mu} g$.

Definition 4.5.2 *Suppose* $\lambda \overset{\alpha/\beta}{\to} \mu$. *Consider pairs* $(T, \mathbf{x}^u)$ *where* T *is a standard tableau of shape* λ *and* $\mathbf{x}^u$ *has degree* $|\beta| = d(\bar{A})$. *Let* $\mathfrak{B}_{T,j}$ *be the set of letters in the* j*th column of* T. *Let* Ω *be the set of pairs* $(T, \mathbf{x}^u)$ *satisfying:*

1. $|\mathfrak{B}_{T,j} \cap \mathfrak{C}_i| = \bar{A}_{i,j}$ *for all* i, j;
2. $\{u_a : a \in \mathfrak{B}_{T,j} \cap \mathfrak{C}_i\} = \{0, 1, \ldots, \bar{A}_{i,j} - 1\}$ *for all* i, j.

Let Ω' *be the set of all other pairs* $(T, \mathbf{x}^u)$.

Proposition 4.5.3 1. *If* $(T, \mathbf{x}^u) \in \Omega$, *then* $\tilde{c}_\mu e_T \mathbf{x}^u \neq 0$.

2. *If* $(T, \mathbf{x}^u) \in \Omega'$, *then* $\tilde{c}_\mu e_T \mathbf{x}^u = 0$.

Proof

Suppose T is a standard tableau of shape λ and $\tilde{c}_\mu e_T \mathbf{x}^u \neq 0$. By Lemma 4.5.1, for any i, j, $\{u_a : a \in \mathfrak{B}_{T,j} \cap \mathfrak{C}_i\}$ contains distinct elements. The matrix A defined by $A_{i,j} = |\mathfrak{B}_{T,j} \cap \mathfrak{C}_i|$ is in $\mathbb{M}_{\mu,\lambda'}$. So $d(\bar{A}) = |u| \geq \sum_{i,j} \binom{|\mathfrak{B}_{T,j} \cap \mathfrak{C}_i|}{2} = d(A) \geq d(\bar{A})$, and every step must be an equality. This implies that $A = \bar{A}$ and $\{u_a : a \in \mathfrak{B}_{T,j} \cap \mathfrak{C}_i\} = \{0, 1, \ldots, |\mathfrak{B}_{T,j} \cap \mathfrak{C}_i| - 1\} = \{0, 1, \ldots, A_{i,j} - 1\} = \{0, 1, \ldots, \bar{A}_{i,j} - 1\}$. Hence $(T, \mathbf{x}^u) \in \Omega$ and (2) holds.

Conversely, suppose $(T, \mathbf{x}^u), (\tilde{T}, \mathbf{x}^{\tilde{u}}) \in \Omega$. Define $\tau \in S_n$ as follows: For $a \in \{1, \ldots, n\}$, there exist unique i, j such that $a \in \mathfrak{B}_{T,j} \cap \mathfrak{C}_i$. By Definition 4.5.2(2), there is a unique letter $\tau(a)$ in $\mathfrak{B}_{\tilde{T}_i} \cap \mathfrak{C}_i$ such that $\tilde{u}_{\tau(a)} = u_a$. Since τ is injective, it is a permutation. By construction $\tau(\mathbf{x}^u) = \mathbf{x}^{\tilde{u}}$.

Since τ maps $\mathfrak{B}_{T,j} \cap \mathfrak{C}_i$ bijectively to $\mathfrak{B}_{\tilde{T},j} \cap \mathfrak{C}_i$, τ permutes within each $\mathfrak{C}_i$ and is in S_μ. Furthermore, $\tau \mathfrak{B}_{T,j} = \cup_j \tau(\mathfrak{B}_{T,j} \cap \mathfrak{C}_i) = \cup_j (\mathfrak{B}_{\tilde{T},j} \cap \mathfrak{C}_i) = \mathfrak{B}_{\tilde{T},j}$. Thus τT and $\tilde{T}$ have the same sets of letters in corresponding columns. Hence $\tau e_T = e_{\tau T} = \pm e_{\tilde{T}}$. Thus we have $\tilde{c}_\mu e_T \mathbf{x}^u = b_\mu \left(\sum_{h \in S_\mu} h\tau\right) e_T \mathbf{x}^u = b_\mu \left(\sum_{h \in S_\mu} h\right) (\tau(e_T)\tau(\mathbf{x}^u)) = \pm b_\mu a_\mu (e_{\tilde{T}} \mathbf{x}^{\tilde{u}}) = \pm \tilde{c}_\mu e_{\tilde{T}} \mathbf{x}^{\tilde{u}}$.

By Corollary 4.2.6 and Corollary 1.4.5, $\dim_{\mathbb{C}} \tilde{c}_\mu (V_\lambda \otimes \mathbb{C}[\mathbf{x}]_{d(\bar{A})}) = 1$. Now $V_\lambda \otimes \mathbb{C}[\mathbf{x}]_{d(\bar{A})} = \operatorname{span}_{\mathbb{C}}\{e_T \mathbf{x}^u : (T, u) \in \Omega\} \oplus \operatorname{span}_{\mathbb{C}}\{e_T \mathbf{x}^u : (T, u) \in \Omega'\}$. By (2), $\dim_{\mathbb{C}} \operatorname{span}_{\mathbb{C}}\{\tilde{c}_\mu e_T \mathbf{x}^u : (T, u) \in \Omega'\} = 0$, so we must have $\dim_{\mathbb{C}} \operatorname{span}_{\mathbb{C}}\{\tilde{c}_\mu e_T \mathbf{x}^u : (T, u) \in \Omega\} = 1$. Above we showed that $\{\tilde{c}_\mu e_T \mathbf{x}^u : (T, u) \in \Omega\}$ consists of terms that are $\pm$ of each other. Thus all of them are nonzero, and (1) holds.

The astute reader will notice that this is very similar to the same argument used in Proposition 4.2.3.

□

The following proposition gets us closer to understanding the second formulation of $M_{\lambda,\mu}$ in Proposition 4.3.1:

Proposition 4.5.4 *Suppose $\lambda \to \mu$ and $(T,u) \in \Omega$. Then $a_\mu e_T \mathbf{x}^u$ generates the unique copy of V_μ in $V_\lambda \otimes_{\mathbb{C}} \mathbb{C}[\mathbf{x}]_{d(\bar{A})}$ as an S_n-module.*

Proof

By Wedderburn theory, $R = \mathbb{C}S_n$ is the direct sum of isotypic components M_γ with $M_\gamma M_\sigma = 0$ if $\gamma \neq \sigma$ and $M_\gamma M_\gamma = M_\gamma$. Now $\oplus_{\beta \geq \mu}({V_\beta}^{\oplus K_{\beta,\mu}}) \cong U_\mu \cong Ra_\mu \subset R = \oplus_\gamma M_\gamma$, so Ra_μ has isotypic decomposition $Ra_\mu = \oplus_{\beta \geq \mu} W_\beta$ with $W_\beta \subset M_\beta$.

The irreducible $V_\alpha = Rc_\alpha$ has submodule $Ra_\mu V_\alpha$, so $Ra_\mu V_\alpha$ is either 0 or V_α. Suppose $\alpha \not\geq \mu$. Then $Ra_\mu V_\alpha \cong Ra_\mu R c_\alpha = \sum_{\beta \geq \mu} W_\beta R c_\alpha \subset \sum_{\beta \geq \mu} M_\beta M_\alpha = 0$. Therefore, $Ra_\mu V_\alpha = 0$ for $\alpha \not\geq \mu$, while $Ra_\mu V_\alpha$ is 0 or V_α if $\alpha \geq \mu$.

In other words, if $\tilde{N}$ is any S_n-module, then any irreducible $V_\eta \subset Ra_\mu \tilde{N}$ satisfies $\eta \geq \mu$. For $v \in \tilde{N}$, $Ra_\mu v$ is a submodule of $Ra_\mu \tilde{N}$, so $Ra_\mu v$ can only contain copies of irreducibles V_η for $\eta \geq \mu$.

In particular, take $\tilde{N} = V_\lambda \otimes_{\mathbb{C}} \mathbb{C}[\mathbf{x}]_{d(\bar{A})}$ and $v = e_T \mathbf{x}^u \in \tilde{N}$. For $\eta \geq\neq \mu$, we have $\bar{d}(\lambda,\eta) = d(\lambda,\eta) > d(\lambda,\mu) = d(\bar{A})$ (see Proposition 3.6.4, Proposition 4.2.8). So $\tilde{N}$ contains no copy of V_η, and neither does $\tilde{N}$'s submodule $Ra_\mu e_T \mathbf{x}^u$. But we noted earlier that $Ra_\mu v$ can only contain copies of irreducibles V_η for $\eta \geq \mu$. Therefore, any irreducible contained in $Ra_\mu e_T \mathbf{x}^u$ must be isomorphic to V_μ.

Now $b_\mu(a_\mu e_T \mathbf{x}^u) = \tilde{c}_\mu e_T \mathbf{x}^u \neq 0$ by Proposition 4.5.3, so $Ra_\mu e_T \mathbf{x}^u \neq 0$. But $Ra_\mu e_T \mathbf{x}^u$ is a submodule of $\tilde{N}$, which contains a unique copy of V_μ. Therefore $Ra_\mu e_T \mathbf{x}^u$ must equal this unique copy of V_μ.

□

4.6 Containment Conditions

Proposition 4.6.1 *Suppose $\lambda \xrightarrow{\theta} \mu$ and $\lambda \xrightarrow{\tilde{\theta}} \tilde{\mu}$. Recall from Proposition 4.3.1 that $M_{\lambda,\mu} \cong (V_\lambda \otimes_{\mathbb{C}} \mathbb{C}[\mathbf{x}])/I_{\lambda,\mu} \cong V/J_{\lambda,\mu}$ and $M_{\lambda,\tilde{\mu}} \cong (V_\lambda \otimes_{\mathbb{C}} \mathbb{C}[\mathbf{x}])/I_{\lambda,\tilde{\mu}} \cong V/J_{\lambda,\tilde{\mu}}$. Write $P_{\lambda,\tilde{\mu}}(z;t) - P_{\lambda,\mu}(z;t) = \sum_\eta f_\eta(t) s_\eta(z)$, $f_\eta(t) \in \mathbb{Z}[t]$. Then the following are equivalent:*

1. *There exists a degree-zero surjective $\mathbb{C}[\mathbf{x}] * S_n$-homomorphism $M_{\lambda,\tilde{\mu}} \twoheadrightarrow M_{\lambda,\mu}$ (i.e. $M_{\lambda,\mu}$ is a quotient of $M_{\lambda,\tilde{\mu}}$).*

2. *$f_\eta(t) \in \mathbb{N}[t]$ for all η.*

3. *The coefficient of $t^{d(\lambda,\mu)}$ in $f_\mu(t)$ is 0.*

4. *$I_{\lambda,\tilde{\mu}} \subset I_{\lambda,\mu}$.*

5. *$J_{\lambda,\tilde{\mu}} \subset J_{\lambda,\mu}$.*

Proof

Note that $\{t^j s_\eta(z) : j \geq 0, \text{ partition } \eta\}$ is an $\mathbb{N}$-basis of the set of Frobenius series of singly-graded S_n-modules. Thus it makes sense to take the coefficient of $t^j s_\eta(z)$ in any such Frobenius series, and this coefficient will be exactly the number of copies of V_η in degree j of the module.

(1) $\Longrightarrow$ (2): $P_{\lambda,\tilde{\mu}}(z;t) - P_{\lambda,\mu}(z;t) = F_{\mathrm{Ker}(M_{\lambda,\tilde{\mu}} \twoheadrightarrow M_{\lambda,\mu})}(z;t) \in \mathbb{N}[t]\{s_\eta(z)\}$.

(2) $\Longrightarrow$ (3): The coefficient of $t^{d(\lambda,\mu)} s_\mu(z)$ in $P_{\lambda,\mu}(z;t)$ is 1 by Proposition 4.3.3. Thus the coefficient of $t^{d(\lambda,\mu)} s_\mu(z)$ in $P_{\lambda,\tilde{\mu}}(z;t)$ is at least 1.

The latter coefficient is the number of copies of V_μ in degree $d(\lambda,\mu)$ of $M_{\lambda,\tilde{\mu}}$, which is a graded S_n-quotient of $V_\lambda \otimes_{\mathbb{C}} \mathbb{C}[\mathbf{x}]$. But $V_\lambda \otimes_{\mathbb{C}} \mathbb{C}[\mathbf{x}]$ has exactly one copy of V_μ in degree $d(\lambda,\mu)$ by Corollary 4.2.6. Hence the coefficient is exactly 1.

The coefficient of $t^{d(\lambda,\mu)}$ in $f_\mu(t)$ is the difference of the above two coefficients and thus must be 0.

(3) $\Longrightarrow$ (4): The coefficient of $t^{d(\lambda,\mu)} s_\mu(z)$ in $P_{\lambda,\mu}(z;t)$ is 1 by Proposition 4.3.3. Thus the coefficient of $t^{d(\lambda,\mu)} s_\mu(z)$ in $P_{\lambda,\tilde{\mu}}(z;t)$ is also 1.

Now $M_{\lambda,\tilde{\mu}} \cong (V_\lambda \otimes_{\mathbb{C}} \mathbb{C}[\mathbf{x}])/I_{\lambda,\tilde{\mu}}$. Since $V_\lambda \otimes_{\mathbb{C}} \mathbb{C}[\mathbf{x}]$ has exactly one copy of V_μ (called N) in degree $d(\lambda,\mu)$ (by Corollary 4.2.6), $I_{\lambda,\tilde{\mu}}$ must have zero copies of V_μ in degree $d(\lambda,\mu)$. In other words, $I_{\lambda,\tilde{\mu}}$ does not intersect N. By the construction in Proposition 4.3.3, $I_{\lambda,\tilde{\mu}} \subset I_{\lambda,\mu}$.

(4) $\Longrightarrow$ (1): The inclusion $I_{\lambda,\tilde{\mu}} \subset I_{\lambda,\mu}$ naturally induces the required

$$M_{\lambda,\tilde{\mu}} \cong (V_\lambda \otimes_{\mathbb{C}} \mathbb{C}[\mathbf{x}])/I_{\lambda,\tilde{\mu}} \twoheadrightarrow (V_\lambda \otimes_{\mathbb{C}} \mathbb{C}[\mathbf{x}])/I_{\lambda,\mu} \cong M_{\lambda,\mu}.$$

(4) $\Longleftrightarrow$ (5): This follows from $J_{\lambda,\mu} = I_{\lambda,\mu} \oplus (L \otimes \mathbb{C}[\mathbf{x}])$ and $J_{\lambda,\tilde{\mu}} = I_{\lambda,\tilde{\mu}} \oplus (L \otimes \mathbb{C}[\mathbf{x}])$ (see the last part of Proposition 4.3.1).

$\square$

Before proving the next theorem, we need a slightly different version of the previous section. Fix the Young subgroup $S_{\lambda'} = S_{\mathfrak{B}_1} \times S_{\mathfrak{B}_2} \times \ldots$ to be used in $\varepsilon \uparrow_{S_{\lambda'}}^{S_n}$. Fix the Young subgroup $S_\mu = S_{\mathfrak{C}_1} \times S_{\mathfrak{C}_2} \times \ldots$ used to define $a_\mu = \sum_{g \in S_\mu} g$. Note $V = \varepsilon \uparrow_{S_{\lambda'}}^{S_n} \otimes \mathbb{C}[\mathbf{x}] = \frac{\mathbb{C}S_n \cdot e}{\langle \sigma e - (-1)^\sigma e : \sigma \in S_{\lambda'} \rangle} \otimes \mathbb{C}[\mathbf{x}]$. We will write its elements in the form $ge \otimes \mathbf{x}^u$ for $g \in S_n$.

Definition 4.6.2 *For $g \in S_n$ and monomial $\mathbf{x}^u$, we define $w(g,u,j)$ to be the composition whose ith part is the size of $\{p : p \in g(\mathfrak{B}_j)$ and $u_p = i-1\}$.*

Lemma 4.6.3 *Suppose $g \in S_n$ and u, v are compositions. Then $w(g,u,j) \geq w(g,u+v,j)$ in dominance order (of compositions).*

Proof

Let $r \geq 1$. By definition $\sum_{j=1}^r w(g,u,j)$ is the size of $\{p : p \in g(\mathfrak{B}_j)$ and $u_p \leq r-1\}$, while $\sum_{j=1}^r w(g,u+v,j)$ is the size of $\{p : p \in g(\mathfrak{B}_j)$ and $u_p + v_p \leq r-1\}$. The latter set is a subset of the former set, so $\sum_{j=1}^r w(g,u,j) \geq \sum_{j=1}^r w(g,u+v,j)$ for any r.

$\square$

Definition 4.6.4 *Suppose $\lambda \stackrel{\alpha/\beta}{\to} \mu$ with corresponding matrix A as in Definition 2.4.4. For each j, define $\eta^{(j)} = (A_{1,j}, A_{2,j}, \ldots)$, which is a partition by Lemma 2.4.5.*

Definition 4.6.5 *Suppose $\lambda \stackrel{\alpha/\beta}{\to} \mu$ with corresponding matrix A as in Definition 2.4.4. Consider pairs $(g, \mathbf{x}^u)$ where $g \in S_n$ and $|u| = |\beta|$. Let Π be the set of pairs $(g, \mathbf{x}^u)$ satisfying:*

1. *$|g(\mathfrak{B}_j) \cap \mathfrak{C}_i| = A_{i,j}$ for all i, j;*
2. *$\{u_a : a \in g(\mathfrak{B}_j) \cap \mathfrak{C}_i\} = \{0, 1, \ldots, A_{i,j} - 1\}$ for all i, j.*

Let Δ be the set of pairs $(g, \mathbf{x}^u)$ satisfying $w(g, u, j) = \left(\eta^{(j)}\right)'$ for all j. Then $\Pi \subset \Delta$.

Lemma 4.6.6 *For any $h \in S_{\lambda'}$ and $\tau \in S_n$, $(g, \mathbf{x}^u) \in \Pi \Longrightarrow (gh, \mathbf{x}^u) \in \Pi$ and $(g, \mathbf{x}^u) \in \Delta \Longrightarrow (\tau gh, \tau(\mathbf{x}^u)) \in \Delta$.*

Proof

Note that $gh(\mathfrak{B}_j) = g(\mathfrak{B}_j)$ for all j, which immediately proves the first statement. Now $\tau(\mathbf{x}^u) = \mathbf{x}^{\tau u}$. Observe that $[p \in \tau gh(\mathfrak{B}_j)$ and $(\tau u)_p = i - 1]$ is equivalent to $[\tau^{-1} p \in g(\mathfrak{B}_j)$ and $u_{\tau^{-1}p} = i - 1]$. Hence the action of τ^{-1} produces a bijection that shows $w(\tau gh, \tau u, j) = w(g, u, j)$ for all j, and the second statement is proved.

$\square$

Definition 4.6.7 *Define the natural projection*

$$\pi : \mathbb{C}S_n \otimes \mathbb{C}[\mathbf{x}] \to V = \varepsilon \uparrow_{S_{\lambda'}}^{S_n} \otimes \mathbb{C}[\mathbf{x}] = \frac{\mathbb{C}S_n \cdot e}{\langle \sigma e - (-1)^\sigma e : \sigma \in S_{\lambda'} \rangle} \otimes \mathbb{C}[\mathbf{x}].$$

Note that π is a $\mathbb{C}[\mathbf{x}] * S_n$-homomorphism. Let

$$W = \oplus_{(g, \mathbf{x}^u) \in \Delta} \mathbb{C}(g \otimes \mathbf{x}^u) \subset \mathbb{C}S_n \otimes \mathbb{C}[\mathbf{x}].$$

Lemma 4.6.8 *Suppose $(g, \mathbf{x}^u) \in \Delta$. Then $\pi^{-1}(ge \otimes \mathbf{x}^u) \subset W$.*

Proof

Any element of $\pi^{-1}(ge \otimes \mathbf{x}^u)$ is a $\mathbb{C}$-linear combination of vectors of the form $gh \otimes \mathbf{x}^u$ where $h \in S_{\lambda'}$. But $(gh, \mathbf{x}^u) \in \Delta$ by Lemma 4.6.6, so such vectors are in W.

$\square$

Proposition 4.6.9 *If $(g, \mathbf{x}^u) \in \Pi$, then $a_\mu(ge \otimes \mathbf{x}^u)$ generates the unique copy of V_μ in $V_{|\beta|}$.*

Proof

Within the proof of Proposition 4.2.3, we showed that $a_\mu(ge \otimes \mathbf{x}^u) \neq 0$. Now reuse the proof of Proposition 4.5.4 except with the following replacements:

- Take $\tilde{N} = V_{|\beta|}$, which also has no copy of V_τ for $\tau \geq\neq \mu$ because $\bar{d}(\lambda, \tau) > \bar{d}(\lambda, \mu)$.
- Take $v = ge \otimes \mathbf{x}^u$.
- $Ra_\mu(ge \otimes \mathbf{x}^u) \neq 0$ is used in the last paragraph.

$\square$

Theorem 4.6.10 *Suppose* $\lambda \xrightarrow{\theta} \mu$ *and* $\lambda \xrightarrow{\tilde{\theta}} \tilde{\mu}$. *Let* $A = (a_{i,j})$ *and* $\tilde{A} = (\tilde{a}_{i,j})$ *be the corresponding integer matrices (Definition 2.4.4). Then the conditions in Proposition 4.6.1 imply that*

$$\sum_{i=1}^{r} \tilde{a}_{i,j} \geq \sum_{i=1}^{r} a_{i,j} \text{ for every } j, r \geq 1.$$

Proof

Per Definition 4.6.4, let $\eta^{(j)} = (a_{1,j}, a_{2,j}, \ldots)$ and $\tilde{\eta}^{(j)} = (\tilde{a}_{1,j}, \tilde{a}_{2,j}, \ldots)$. Then the inequalities to be shown are equivalent to $\tilde{\eta}^{(j)} \geq \eta^{(j)}$ for all j.

Per Proposition 4.3.1, let N be the unique copy of V_μ in $V_{d(\lambda,\mu)}$, and let $\tilde{N}$ be the unique copy of $V_{\tilde{\mu}}$ in $V_{d(\lambda,\tilde{\mu})}$. Let J be the $\mathbb{C}[\mathbf{x}] * S_n$-submodule of V generated by N. Clearly J is homogeneous. Now J has nonzero intersection with N, so by definition $J \not\subset J_{\lambda,\mu}$.

By assumption $J_{\lambda,\tilde{\mu}} \subset J_{\lambda,\mu}$, so $J \not\subset J_{\lambda,\tilde{\mu}}$. By definition, J has nonzero intersection with $\tilde{N}$. This implies $\tilde{N} \subset J$ because $\tilde{N}$ is irreducible. Now J is generated by the degree $d(\lambda, \mu)$ generator N, while $\tilde{N}$ is at degree $d(\lambda, \mu)$. Hence

- $d(\lambda, \tilde{\mu}) \geq d(\lambda, \mu)$.
- Any element in $\tilde{N} \subset J$ can be written in the form $\sum_i f_i(x) v_i$ for some $v_i \in N$ and $f_i(x) \in \mathbb{C}[\mathbf{x}]_{d(\lambda,\tilde{\mu})-d(\lambda,\mu)}$.

By Proposition 4.6.9, N is generated as an S_n-module by elements of the form $a_\mu(ge \otimes \mathbf{x}^u)$, where $(g, \mathbf{x}^u) \in \Pi$. Because $\Pi \subset \Delta$, the second part of Lemma 4.6.6 implies that $N \subset \mathrm{span}_{\mathbb{C}}\{ge \otimes \mathbf{x}^u : (g, u) \in \Delta\}$. Therefore, the second bullet above implies

$$\tilde{N} \subset \mathrm{span}_{\mathbb{C}}\{ge \otimes \mathbf{x}^u \mathbf{x}^v : (g, u) \in \Delta, |v| = d(\lambda, \tilde{\mu}) - d(\lambda, \mu)\}.$$

Define the tilde versions of Π, Δ, and W in the analogous way. Pick some $(\hat{g}, \mathbf{x}^{\hat{u}}) \in \tilde{\Pi} \subset \tilde{\Delta}$. Then by Proposition 4.6.9, $\tilde{N}$ contains nonzero element $w = a_{\tilde{\mu}}(\hat{g}e \otimes \mathbf{x}^{\hat{u}}) = \sum_{h \in S_\mu} h\hat{g}e \otimes h(\mathbf{x}^{\hat{u}})$. By Lemma 4.6.6 and Lemma 4.6.8, $\pi^{-1}(w) \in \tilde{W}$.

By above remarks, we may write w in the form $\sum_p C_p g^{(p)} e \otimes \mathbf{x}^{u^{(p)}+v^{(p)}}$, where $C_p \in \mathbb{C}$, $(g^{(p)}, \mathbf{x}^{u^{(p)}}) \in \Delta$, $|v^{(p)}| = d(\lambda, \tilde{\mu}) - d(\lambda, \mu)$. Let $v = \sum_p C_p g^{(p)} \otimes \mathbf{x}^{u^{(p)}+v^{(p)}} \in V$. Then $\pi(v) = w$, so $v \in \pi^{-1}(w) \in \tilde{W}$. Because $w \neq 0$, the definition of $\tilde{W}$ implies that there exists p such that $(g^{(p)}, \mathbf{x}^{u^{(p)}+v^{(p)}}) \in \tilde{\Delta}$.

By the definitions of Δ and $\tilde{\Delta}$ and by Lemma 4.6.3, $\left(\eta^{(j)}\right)' = w(g^{(p)}, u^{(p)}, j) \geq w(g^{(p)}, u^{(p)} + v^{(p)}, j) = \left(\tilde{\eta}^{(j)}\right)'$ in dominance order for all j. Hence $\eta^{(j)} \leq \tilde{\eta}^{(j)}$ in dominance order for all j.

□

Conjecture 4.6.11 *The converse of Theorem 4.6.10 also holds.*

Since $\mu_i = \sum_j a_{i,j}$ and $\tilde{\mu}_i = \sum_j \tilde{a}_{i,j}$ for all i, the condition in Theorem 4.6.10 implies that $\tilde{\mu} \geq \mu$ in dominance order. Unfortunately this condition is too weak. Consider $\lambda = (2,2,1,1,1), \mu = (4,1,1,1), \tilde{\mu} = (4,3)$. Then $\lambda \to \mu$ and $\lambda \to \tilde{\mu}$. It can be checked that conditions 1 and 2 in Proposition 4.6.1 fail. But $A = \begin{pmatrix} 2 & 2 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \end{pmatrix}$ and $\tilde{A} = \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix}$, so the condition in Theorem 4.6.10 also fails.

Chapter 5

Conjectures

5.1 Category of k-bounded resolutions

Throughout this section, tensor products are over $\mathbb{C}$ unless otherwise noted.

Definition 5.1.1 *Let $\mathcal{M}_n^{(k)}$ be the category of graded $\mathbb{C}[x_1, \ldots, x_n] * S_n$-modules M that have an S_n-equivariant resolution*

$$F_m \to \cdots \to F_1 \to F_0 \to M \to 0$$

so that each F_i has the form $W_i \otimes \mathbb{C}[x_1, \ldots, x_n]$, where W_i is an S_n-module that contains only irreducible representations V_κ corresponding to k-bounded κ's. An irreducible module *in $\mathcal{M}_n^{(k)}$ is a nonzero module that has no proper nonzero submodules in $\mathcal{M}_n^{(k)}$.*

Conjecture 5.1.2 *The irreducible modules in $\mathcal{M}_n^{(k)}$ are exactly the modules $M_{\lambda,(\lambda^{\omega_k})'}$ for k-bounded $\lambda \vdash n$.*

Note that $\mathcal{M}_n^{(k)}$ is not an Abelian subcategory of the category of $\mathbb{C}[x_1, \ldots, x_n] * S_n$-modules. But $\mathcal{M}_n^{(k)}$ has the property that if it contains two terms of a three-term exact sequence of $\mathbb{C}[x_1, \ldots, x_n] * S_n$-modules, then it also contains the third term.

Definition 5.1.3 *Let $\mathcal{M}_n$ denote the category of graded $\mathbb{C}[x_1, \ldots, x_n]*S_n$-modules. If $p+q = n$, let $R_{p,q} = (\mathbb{C}[x_1, \ldots, x_p] * S_p) \otimes (\mathbb{C}[y_1, \ldots, y_q] * S_q) = \mathbb{C}[x_1, \ldots, x_p, y_1, \ldots, y_q] * (S_p \times S_q)$. It can be regarded as a subring of $\mathbb{C}[x_1, \ldots, x_n] * S_n$ by identifying $y_i = x_{i+p}$. Restricting to this subring induces a functor $\mathcal{M}_n \to \mathcal{M}_{p,q}$, where $\mathcal{M}_{p,q}$ is the category of $R_{p,q}$-modules. Denote this functor as $\mathrm{Res}_{p,q}$.*

Definition 5.1.4 *Let $\mathcal{M}_{p,q}^{(k)} \subset \mathcal{M}_{p,q}$ be the subcategory of modules M with an $(S_p \times S_q)$-equivariant resolution*

$$F_m \to \cdots \to F_1 \to F_0 \to M \to 0$$

so that each F_i has the form $\oplus_j (M_j \otimes \mathbb{C}[x_1, \ldots, x_p] \otimes N_j \otimes \mathbb{C}[y_1, \ldots, y_q])$, where M_j is an irreducible S_p-module corresponding to a k-bounded partition and N_j is an irreducible S_q-module corresponding to a k-bounded partition.

Lemma 5.1.5 *The functor* $\mathrm{Res}_{p,q}$ *sends* $\mathcal{M}_n^{(k)}$ *into* $\mathcal{M}_{p,q}^{(k)}$.

Proof

We verify that applying $\mathrm{Res}_{p,q}$ to the resolution given in Definition 5.1.1 yields a resolution of the appropriate form. First of all, restriction is exact, so the new sequence is an $(S_p \times S_q)$-equivariant resolution. The irreducible $(S_p \times S_q)$-modules are of the form $V_\tau \otimes V_\eta$ for $\tau \vdash p, \eta \vdash q$. Take the irreducible decomposition $V_\kappa \downarrow_{S_p\times S_q}^{S_n} = \oplus_{\tau\vdash p,\eta\vdash q}(V_\tau \otimes V_\eta)^{\oplus c^\kappa_{\tau,\eta}}$. Then it suffices to check that if κ is k-bounded and $c^\kappa_{\tau,\eta}$ is nonzero, then τ, η are k-bounded.

Now $c^\kappa_{\tau,\eta} = \langle V_\kappa \downarrow_{S_p\times S_q}^{S_n}, V_\tau \otimes V_\eta\rangle_{S_p\times S_q} = \langle V_\kappa, V_\tau \otimes V_\eta \uparrow_{S_p\times S_q}^{S_n}\rangle_{S_n} = \langle s_\kappa(z), s_\tau(z)s_\eta(z)\rangle$. The second equlity is by Frobenius reciprocity, while the third equality is by Frobenius characteristic and Lemma 1.6.4. Hence $c^\kappa_{\tau,\eta}$ is the Littlewood-Richardson coefficient with the same notation. By [18]I(5.3) and I(5.7), $c^\kappa_{\tau,\eta}$ is zero unless $\tau \subset \kappa$ and $\eta \subset \kappa$. Thus if κ is k-bounded and $c^\kappa_{\tau,\eta}$ is nonzero, then τ and η are also k-bounded.

□

Conjecture 5.1.6 *The irreducible modules in* $\mathcal{M}_{p,q}^{(k)}$ *have the form* $M \otimes N$*, where* $M \in \mathcal{M}_p^{(k)}$ *and* $N \in \mathcal{M}_q^{(k)}$ *are irreducible modules in the respective categories.*

Proposition 5.1.7 *Suppose that Conjecture 5.1.2 and (for (2)) Conjecture 5.1.6 are true.*

1. *For* $\ell > k$, $s_\lambda^{(k)}(z;t) \in \mathbb{N}[t]\{s_\mu^{(\ell)}(z;t) : \mu \text{ is } \ell\text{-bounded}, |\mu| = |\lambda|\}$.
2. $s_\lambda^{(k)}[Y+Z;t] \in \mathbb{N}[t]\{s_\mu^{(k)}(y;t)s_\gamma^{(k)}(z;t) : \mu, \gamma \text{ are } k\text{-bounded}, |\mu| + |\gamma| = |\lambda|\}$.
3. $s_\lambda^{(k)}(z;t)s_\mu^{(\ell)}(z;t) \in \mathbb{N}[t]\{s_\gamma^{(k+\ell)}(z;t) : \gamma \text{ is } (k+\ell)\text{-bounded}, |\gamma| = |\lambda| + |\mu|\}$.
4. *For* k*-bounded* μ, $H_\mu(z;q,t)$ *can be written as* $\sum_\lambda K_{\lambda\mu}^{(k)}(q,t)s_\lambda^{(k)}(z;t)$ *for* $K_{\lambda\mu}^{(k)}(q,t) \in \mathbb{N}[q,t]$. *The coefficients* $K_{\lambda\mu}^{(k)}(q,t)$ *are called the* k-Kostka-Foulkes polynomials.

Proof

For (1), because $M_{\lambda,(\lambda^{\omega_k})'} \in \mathcal{M}_{|\lambda|}^{(k)}$, by definition $M_{\lambda,(\lambda^{\omega_k})'} \in \mathcal{M}_{|\lambda|}^{(\ell)}$ also. The composition factors of $M_{\lambda,(\lambda^{\omega_k})'}$ in $\mathcal{M}_{|\lambda|}^{(\ell)}$ are of the form $M_{\mu,(\mu^{\omega_\ell})'}$ for ℓ-bounded μ of the same size as λ, so the required positivity follows.

For (2), we recall [18]1§7 Example 26: Suppose U is a finite-dimensional S_n-module. For each pair p, q with $p+q = n$, express $U \downarrow_{S_p\times S_q}^{S_n}$ as $\oplus_i(V^{(p,i)} \otimes W^{(q,i)})$, where each $V^{(p,i)}$ is an S_p-module and each $W^{(q,i)}$ is an S_q-module. (For instance, the irreducible decomposition is of this form.) Then

$$F_{\mathrm{char}\,U}[Y+Z] = \sum_{p+q=n}\sum_i F_{\mathrm{char}\,V^{(p,i)}}[Y]F_{\mathrm{char}\,W^{(q,i)}}[Z].$$

Take $M = M_{\lambda,(\lambda^{\omega_k})'}$, which is in $\mathcal{M}_n^{(k)}$ by Conjecture 5.1.2. By Lemma 5.1.5, $\mathrm{Res}_{p,q} M$ has the form $\oplus_i (M^{(p,i)} \otimes N^{(q,i)})$ for some $M^{(p,i)} \in \mathcal{M}_p$ and $N^{(q,i)} \in \mathcal{M}_q$. We examine what happens at a particular degree r: As S_n-modules, $M_r \downarrow^{S_n}_{S_p \times S_q} = (\mathrm{Res}_{p,q} M)_r = \oplus_{d+e=r} \oplus_i (M_d^{(p,i)} \otimes N_e^{(q,i)})$.

Summing over the degrees and applying [18]1§7 Example 26 on each M_r, we get

$$\begin{aligned}
s_\lambda^{(k)}[Y+Z;t] &= F_M[Y+Z] = \sum_r t^r F_{\mathrm{char}\, M_r}[Y+Z] \\
&= \sum_r t^r \sum_{p+q=n} \sum_{d+e=r} \sum_i F_{\mathrm{char}\, M_d^{(p,i)}}[Y] F_{\mathrm{char}\, N_e^{(q,i)}}[Z] \\
&= \sum_{p+q=n} \sum_i \sum_{d,e} t^d F_{\mathrm{char}\, M_d^{(p,i)}}[Y] t^e F_{\mathrm{char}\, N_e^{(q,i)}}[Z] \\
&= \sum_{p+q=n} \sum_i F_{M^{(p,i)}}[Y] F_{N^{(q,i)}}[Z].
\end{aligned}$$

But $M^{(p,i)} \in \mathcal{M}_p$ and $N^{(q,i)} \in \mathcal{M}_q$, so their Frobenius series are $\mathbb{N}[t]$-linear combinations of p-Schur functions and q-Schur functions, respectively. Therefore, (2) is proved.

For (3), take $M = M_{\lambda,(\lambda^{\omega_k})'}$ and $N = M_{\mu,(\mu^{\omega_\ell})'}$. Let $p = |\lambda|$ and $q = |\mu|$. According to Conjecture 5.1.2, there exist resolutions

$$\cdots \xrightarrow{e_2} E_1 \xrightarrow{e_1} E_0 \xrightarrow{e_0} M \to 0$$

$$\cdots \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} N \to 0$$

with $E_i = U_i \otimes \mathbb{C}[x_1, \ldots, x_p]$, $U_i \cong \oplus_{k\text{-bounded } \tau \vdash p} \left(V_\tau{}^{\oplus a_{i,\tau}}\right)$, $F_j = W_j \otimes \mathbb{C}[y_1, \ldots, y_q]$, $W_j \cong \oplus_{\ell\text{-bounded } \eta \vdash q} \left(V_\eta{}^{\oplus b_{j,\eta}}\right)$, and $E_i = 0$, $F_j = 0$ for sufficiently large i, j.

Consider the double complex of $R_{p,q}$-modules

$$\begin{array}{ccccccc}
\cdots \to & E_0 \otimes F_2 & \xrightarrow{id\otimes f_2} & E_0 \otimes F_1 & \xrightarrow{id\otimes f_1} & E_0 \otimes F_0 \\
& \uparrow e_1\otimes id & & \uparrow e_1\otimes id & & \uparrow e_1\otimes id \\
\cdots \to & E_1 \otimes F_2 & \xrightarrow{id\otimes f_2} & E_1 \otimes F_1 & \xrightarrow{id\otimes f_1} & E_1 \otimes F_0 \\
& \uparrow e_2\otimes id & & \uparrow e_2\otimes id & & \uparrow e_2\otimes id \\
\cdots \to & E_2 \otimes F_2 & \xrightarrow{id\otimes f_2} & E_2 \otimes F_1 & \xrightarrow{id\otimes f_1} & E_2 \otimes F_0 \\
& \uparrow & & \uparrow & & \uparrow \\
& \vdots & & \vdots & & \vdots
\end{array}$$

Its *total complex* $C_\bullet = \left[\cdots \xrightarrow{d_2} C_2 \xrightarrow{d_1} C_1 \xrightarrow{d_1} C_0\right]$ is given by $C_r = \oplus_{i+j=r}(E_i \otimes F_j)$ and $d_r(a \otimes b) = (e_i a) \otimes b + (-1)^i a \otimes (f_j b)$ for $a \in E_i$, $b \in F_j$.

Note that every $\mathbb{C}$-module is free and hence flat. In particular, every row and every column of the double complex are exact. It is a standard fact that these properties imply that the total complex $C_\bullet$ is exact. Now we append $C_0 = E_0 \otimes F_0 \xrightarrow{e_0 \otimes f_0} M \otimes N \to 0$ to $C_\bullet$. It is still exact because e_0, f_0 are surjective (so that $e_0 \otimes f_0$ is also surjective and

$\ker(e_0 \otimes f_0) = (\operatorname{Ker} e_0) \otimes F_0 + E_0 \otimes (\operatorname{Ker} f_0))$. Lastly, we apply the exact functor $\uparrow_{S_p \times S_q}^{S_{p+q}}$ to produce an exact sequence in $\mathcal{M}_{p+q}$

$$\cdots \to C'_2 \to C'_1 \to C'_0 \to (M \otimes N) \uparrow_{S_p \times S_q}^{S_{p+q}} \to 0$$

where $C'_r = \oplus_{i+j=r}(E_i \otimes F_j) \uparrow_{S_p \times S_q}^{S_{p+q}}$.

Now $(E_i \otimes F_j) \uparrow_{S_p \times S_q}^{S_{p+q}}$ is

$$\bigoplus_{\substack{k\text{-bounded } \tau \vdash p \\ \ell\text{-bounded } \eta \vdash q}} \left((V_\tau \times V_\eta) \uparrow_{S_p \times S_q}^{S_{p+q}} \right)^{\otimes a_{i,\tau} b_{j,\eta}} \otimes \mathbb{C}[x_1, \ldots, x_p, y_1, \ldots, y_q].$$

By Lemma 1.6.4, the Frobenius characteristic of $(V_\tau \times V_\eta) \uparrow_{S_p \times S_q}^{S_{p+q}}$ is $s_\tau(z)s_\eta(z)$. Because τ is k-bounded and η is ℓ-bounded, the Schur expansion of $s_\tau(z)s_\eta(z)$ only contains $s_\kappa(z)$ for $(k+\ell)$-bounded $\kappa \vdash p+q$ by the Littlewood-Richardson rule. Hence C'_r has the form $W \otimes \mathbb{C}[x_1, \ldots, x_p, y_1, \ldots, y_q]$ with W containing only irreducible S_{p+q}-representations V_κ corresponding to k-bounded κ's.

Because $E_i = 0$ and $F_j = 0$ for sufficiently large i, j, $C'_r = 0$ for sufficiently large r. Therefore, $(M \otimes N) \uparrow_{S_p \times S_q}^{S_{p+q}} \in \mathcal{M}_{p+q}^{(k+\ell)}$. By Conjecture 5.1.2, its Frobenius series is an $\mathbb{N}[t]$-linear combination of $s_\gamma^{(k+\ell)}(z;t)$ for $(k+\ell)$-bounded $\gamma \vdash p+q$. We are done because by Lemma 1.6.4, the Frobenius series of $(M \otimes N) \uparrow_{S_p \times S_q}^{S_{p+q}}$ is $F_M(z;t)F_N(z;t) = s_\lambda^{(k)}(z;t)s_\mu^{(\ell)}(z;t)$.

For (4), it is known in [18] that the $\tilde{R}_\mu$ in section 1.8, considered as a $\mathbb{C}[\mathbf{y}] * S_n$-module (forgetting the $\mathbb{C}[\mathbf{x}]$ action), is in $\mathcal{M}_{|\mu|}^{(k)}$ for k-bounded μ. Because $\tilde{R}_\mu$ has Frobenius series $H_\mu(z;q,t)$, the result follows.

□

5.2 Catabolism

Currently not much is understood about catabolism. Here are some conjectural basic properties about catabolism. Some of them motivate the conjectures in the ensuing sections.

Definition 5.2.1 *We say that a composition β is a* refinement *of a composition α if there exists an increasing sequence of integers $0 = c_0 < c_1 < c_2 < \ldots$ such that $\alpha_i = \sum_{j=c_{i-1}+1}^{c_i} \beta_j$ for all i.*

Definition 5.2.2 *Let T,U be semistandard tableaux. We say that U is a* cyclage *of T if there exist letter $a \neq 1$ and word w such that $T = T_{wa}$ and $U = T_{aw}$. Notice the definition of charge says that $c(U) = c(T) - 1$.*

Conjecture 5.2.3 *Let α, β be compositions and let T be a nonempty semistandard tableau.*

1. *If T is α-catabolizable and β is a refinement of α, then T is β-catabolizable.*

2. *If U is a cyclage of T and T is α-catabolizable, then U is α-catabolizable.*

3. *Suppose m is the largest letter in T and U is the result of deleting the squares with the letter d from $\sigma_{m-1}\sigma_{m-2}\ldots\sigma_1 T$. Then U is α-catabolizable iff* $\mathrm{Cat}_1(T)$ *is α-catabolizable.*

4. *The number of α-catabolizable tableaux of weight μ and shape λ is the Littlewood-Richardson coefficient*

$$c^{\lambda}_{\mu^{(1)},\mu^{(2)},\ldots} = \langle s_\lambda, s_{\mu^{(1)}} s_{\mu^{(2)}} \ldots \rangle,$$

where $\mu^{(1)}$ consists of the first α_1 parts of μ, $\mu^{(2)}$ consists of the next α_2 parts of μ, and so on.

5.3 Skew-linked tableau atoms

Let $R_+ = \{(i,j) : 1 \le i < j \le m\}$, an upper-triangular shaped set. For a composition τ of m, let $S_\tau = \{(i,j) \in R_+ :$ there exists r such that $i \le \tau_1 + \tau_2 + \ldots + \tau_r < j\}$. The conceptual way to obtain S_τ is to start with R_+ and remove triangular subregions of side lengths $\tau_1, \tau_2, \ldots$ along the main diagonal.

A subset $S \subset R_+$ is an *upper order ideal* if $(i,j) \in S$ implies $\{1,2,\ldots,i\} \times \{j, j+1, \ldots, m\} \subset S$. Note that S_τ is an upper order ideal. We say that a tableau T is S-*catabolizable* if T is τ-catabolizable for every composition τ of m satisfying $S \subset S_\tau$.

Notice that for compositions τ and σ, $S_\sigma \subset S_\tau$ iff τ is a refinement of σ. Thus if Conjecture 5.2.3(1) holds, then S_σ-catabolizability is the same as σ-catabolizability. This was part of the motivation for the definition of S-catabolizability.

Suppose that $\lambda \overset{\alpha/\beta}{\to} \mu$ and $m = \ell(\lambda)$. Define a *row-chaining function* $f_{\lambda,\mu} : \{1,\ldots,m\} \to \{1,\ldots,m\} \cup \{\infty\}$ as follows. If $\beta_i = 0$, then set $f_{\lambda,\mu}(i) = \infty$. If $\beta_i > 0$, suppose r,s are the largest integers such that $\beta_r = \beta_i$ and $\alpha_s = \beta_i$, respectively. Then set $f_{\lambda,\mu}(i) = s - r + i$. Define $Q_{\lambda,\mu} = \{(i,j) \in R_+ : j \ge f_{\lambda,\mu}(i)\}$. Because f is a weakly increasing function, $Q_{\lambda,\mu}$ is an upper order ideal.

Definition 5.3.1 *The* skew-linked tableau atom *is defined as*

$$\mathbb{A}_{\lambda,\mu} = \cup_\tau \{T \in SSYT(\tau,\lambda) : T \textit{ is } Q_{\lambda,\mu}\textit{-catabolizable}\}.$$

Conjecture 5.3.2

1. *The Frobenius series of $M_{\lambda,\mu}$ equals $\sum_{T \in \mathbb{A}_{\lambda,\mu}} t^{c(T)} s_{shape(T)}(z)$.*

2. *In the case λ is k-bounded and $\mu = (\lambda^{\omega_k})'$, $\mathbb{A}_{\lambda,\mu}$ coincides with the $\mathbb{A}^{(k)}_\lambda$ that Lapointe, Lascoux, and Morse used to define k-Schur functions originally.*

It is easily seen that the skew-linked tableau atoms can be defined recursively.

Proposition 5.3.3 *Let $\lambda \overset{\alpha/\beta}{\to} \mu$. By removing the first (bottommost) r rows of α/β, we obtain a skew-linking shape that links $\lambda^r = (\lambda_{r+1}, \lambda_{r+2}, \ldots)$ with some partition μ^r. Then $\mathbb{A}_{\lambda,\mu}$ is the set of tableaux of weight λ such that for every $r = 1, 2, \ldots, f_{\lambda,\mu}(1)$, T is r-catabolizable and $\overline{\mathrm{Cat}}_r(T) \in \mathbb{A}_{\lambda^r,\mu^r}$.*

In the case λ is k-bounded and $\mu = (\lambda^{\omega_k})'$, we have $\mu^r = ((\lambda^r)^{\omega_k})'$. Hence if part 2 of Conjecture 5.3.2 holds, then the Lapointe-Lascoux-Morse atoms $\mathbb{A}_\lambda^{(k)}$ can be defined recursively among themselves using catabolism. This is similar but different from the original definition of $\mathbb{A}_\lambda^{(k)}$.

It appears that the full set of catabolism conditions is not required in defining skew-linked tableau atoms.

Conjecture 5.3.4 *Let $\lambda \overset{\alpha/\beta}{\to} \mu$. Define λ^r and μ^r as in Proposition 5.3.3. Let b be the smallest integer such that $\alpha_b = \beta_1$. Let i be any integer such that $b \leq i \leq f_{\lambda,\mu}(1)$. Then $\underline{\mathbb{A}_{\lambda,\mu}}$ is the set of tableaux of weight λ such that for $r = 1$ and $r = i$, T is r-catabolizable and $\overline{\mathrm{Cat}}_r(T) \in \mathbb{A}_{\lambda^r,\mu^r}$.*

The part about $r = 1$ is motivated by the definition of the Lapointe-Lascoux-Morse atoms because of Conjecture 5.2.3(3), which implies that we can redefine the Lapointe-Lascoux-Morse atoms purely using catabolism conditions.

5.4 Generalization of Borel-Weil-Bott

Another conjecture relates the skew-linked modules to cohomology and is a generalization of the Borel-Weil-Bott theorem. Let G be the general linear group GL_m and let B be the subgroup of upper-triangular matrices. Then G/B is the flag variety. Let $\mathfrak{g} = Lie(G) = \mathfrak{gl}_m$ and $\mathfrak{b} = Lie(B) = \{\text{upper-triangular matrices}\}$. Let $\mathfrak{n}$ be the set of strictly upper-triangular matrices. Since $\mathfrak{n}$ is a Lie algebra ideal in $\mathfrak{b}$, the adjoint action of B on $\mathfrak{n}$ induces a B-module structure on $\mathfrak{n}$. For any B-module L, define $G \times_B L$ as the orbit space $(G \times L)/B$ under the action $(g, v)b = (gb, b^{-1}v)$. Then L is a G-equivariant vector bundle over G/B with bundle map $(g, v)B \mapsto gB$.

Note that every B-submodule $\mathfrak{j} \subset \mathfrak{n}$ is generated by an upper order ideal of positive roots, i.e. there is an upper order ideal $S \subset R_+$ such that $\mathfrak{j} = \{M \in \mathfrak{gl}_m : M_{ij} = 0 \text{ if } (i,j) \notin S\}$. We say that a tableau T is $\mathfrak{j}$-*catabolizable* if it is S-catabolizable.

Let $\mathbb{C}_\lambda$ be the 1-dimensional B-module with dominant weight λ. Let w_0 be the permutation $12\ldots m \mapsto m \ldots 21$. Set $\mathcal{L}_\lambda = G \times_B \mathbb{C}_{w_0(\lambda)}$, which is a line bundle over G/B. Then Borel-Weil-Bott says:

Theorem 5.4.1 *1. $H^0(G/B, \mathcal{L}_\lambda)$ is the irreducible G-module with highest weight λ.*

2. $H^i(G/B, \mathcal{L}_\lambda) = 0$ for $i > 0$.

Let $\tilde{\mathcal{L}}_\lambda$ be the pullback of $\mathcal{L}_\lambda$ to $G \times_B \mathfrak{j}$. It can be shown that

Lemma 5.4.2 *$H^i(G \times_B \mathfrak{j}, \tilde{\mathcal{L}}_\lambda) \cong H^i(G/B, (G \times_B Sym(\mathfrak{j}^*)) \otimes \mathcal{L}_\lambda)$.*

Conjecture 5.4.3 *1. $H^0(G \times_B \mathfrak{j}, \tilde{\mathcal{L}}_\lambda)$ has graded character*

$$\sum_{dominant\ \tau \in \mathbb{Z}^m} \sum_{\substack{T \in SSYT(\tau,\lambda) \\ T \text{ is } \mathfrak{j}\text{-catabolizable}}} t^{c(T)} s_\tau(z_1, \ldots, z_m).$$

Here $SSYT(\tau, \lambda)$ and $s_\tau(z_1, \ldots, z_m)$ are as defined in Section 1.12.

2. $H^i(G \times_B \mathfrak{j}, \tilde{\mathcal{L}}_\lambda)$ *for* $i > 0$.

Borel-Weil-Bott is the special case where $\mathfrak{j} = 0$. To see this, note that $T \in SSYT(\tau, \lambda)$ is 0-catabolizable iff the entire tableau can be catabolized, meaning that T must be *super-standard* (row i contains only letter i for all i), $\tau = \lambda$, and $c(T) = 0$. Hence the formula in Conjecture 5.4.3 reduces to $s_\lambda(z_1, \ldots, z_m)$, the graded character of irreducible G-module with highest weight λ.

In the case $\mathfrak{j}$ is the Lie algebra of the parabolic subgroup $P = P_\eta \subset B$ for a composition η of m, we have $S = S_\eta$ (as defined in the previous section). Conjecturally S_η-catabolizability is the same as η-catabolizability, so by Lemma 5.4.2, Conjecture 5.4.3 becomes the situation in [19]. In particular, Conjecture 5.4.3(1) becomes a conjecture of Shimozono-Weyman and Conjecture 5.4.3(2) becomes a conjecture of Broer [2][3][4].

Corollary 5.4.4 *Let* $\rho = (m-1, m-2, \ldots, 0)$. *Define operators* J *and* π *on* $\mathbb{C}(z_1, \ldots, z_m)$

$$J(f) = \sum_{\omega \in S_m} (-1)^\omega \omega f$$

$$\pi(f) = J(\mathbf{z}^\rho f)/J(\mathbf{z}^\rho).$$

If Conjecture 5.4.3(2) holds, then the graded character of $H^0(G \times_B \mathfrak{j}, \tilde{\mathcal{L}}_\lambda) = H^0(G/B, (G \times_B Sym(\mathfrak{j}^*)) \otimes \mathcal{L}_\lambda)$ *is*

$$\pi\left(\prod_{e_{ij} \in \mathfrak{j}} \frac{1}{1 - tz_i/z_j} \mathbf{z}^\lambda\right).$$

Proof

(Sketch) By Bott's formula, $\pi\left(\prod_{e_{ij}\in\mathfrak{j}} \frac{1}{1-tz_i/z_j}\mathbf{z}^\lambda\right)$ is the character of $\sum_{i\geq 0}(-1)^i[H^0(G/B, (G \times_B Sym(\mathfrak{j}^*)) \otimes \mathcal{L}_\lambda)]$, an element in the Grothendieck group of graded G-modules. By Lemma 5.4.2 and Conjecture 5.4.3(2), all terms drop out except for the zero term.

□

Corollary 5.4.5 *Let* $\rho = (m-1, m-2, \ldots, 0)$. *Define a linear functional* Ψ *as follows. For weight* α, *if* $\alpha + \rho$ *is regular, find the unique* $\omega \in S_m$ *so that* $\omega(\alpha + \rho)$ *and set* $\Psi(\mathbf{z}^\alpha) = (-1)^{\ell(\omega)} s_{\omega(\alpha+\rho)-\rho}(z_1, \ldots, z_m)$. *Otherwise set* $\Psi(\mathbf{z}^\alpha) = 0$. *Then* $\pi = \Psi$. *If Conjecture 5.4.3 holds, then*

$$\sum_{\text{dominant } \tau \in \mathbb{Z}^m} \sum_{\substack{T \in SSYT(\tau,\lambda) \\ T \text{ is } \mathfrak{j}\text{-catabolizable}}} t^{c(T)} s_\tau(z_1, \ldots, z_m) = \Psi\left(\prod_{e_{ij}\in\mathfrak{j}} \frac{1}{1 - tz_i/z_j}\mathbf{z}^\lambda\right).$$

Proof

In view of Conjecture 5.4.3 and Corollary 5.4.4, we only need to show $\pi = \Psi$. Now $\pi(\mathbf{x}^\alpha) = \det({z_i}^{\alpha_j+\rho_j})_{i,j=1}^m / \det({z_i}^{\rho_j})_{i,j=1}^m$. If $\alpha+\rho$ is not regular, then the numerator is 0. If $\alpha+\rho$ is regular and $\omega(\alpha+\rho)$ is dominant, then $\omega(\alpha+\rho)-\rho$ is also dominant. Thus $\pi(\mathbf{x}^\alpha) = (-1)^\omega \det({z_i}^{\omega(\alpha+\rho)_j})_{i,j=1}^m / \det({z_i}^{\rho_j})_{i,j=1}^m = (-1)^\omega s_{\omega(\alpha+\rho)-\rho}$ by the Weyl character formula and Lemma 1.12.3.

□

Let $\tilde{\Lambda} = \mathbb{Q}(q,t)[z_1, {z_1}^{-1}, \ldots, z_m, {z_m}^{-1}]]^{S_m}$, the space of symmetric Laurent polynomials in $z_1, \ldots, z_m$ with coefficients in $\mathbb{Q}(q,t)$. Then $\{s_\tau(z_1, \ldots, z_m) : \text{dominant weight } \tau \in \mathbb{Z}^m\}$ is a basis for $\tilde{\Lambda}$ (see Section 1.12). Let $\tilde{\Lambda}_+$ be the subspace generated by $\{s_\tau(z_1, \ldots, z_m) : \text{dominant weight } \tau \in \mathbb{N}^m\}$, i.e. partitions τ with at most m parts. For $f \in \tilde{\Lambda}$, the *polynomial part of* f is the image of projecting f into $\tilde{\Lambda}_+$ along the Schur basis of $\tilde{\Lambda}$.

The representation-theoretic interpretation is that a finite-dimensional rational representation M of GL_m has character χ in $\tilde{\Lambda}$, and taking the polynomial part of χ results in the character of the polynomial representation component of M.

Corollary 5.4.6 *Assuming Conjecture 5.3.2 and Conjecture 5.4.3 hold, if* $\lambda \xrightarrow{\alpha/\beta} \mu$, $m \geq \ell(\lambda)$, *and* $\mathfrak{j} = \{M \in \mathfrak{gl}_m : M_{ij} = 0 \text{ if } (i,j) \notin Q_{\lambda,\mu}\} \subset \mathfrak{n}$, *then the Frobenius series of* $M_{\lambda,\mu}$ *is obtained by extending the polynomial part of the right hand side of Corollary 5.4.5 to infinitely many variables.*

Proof

In view of Conjecture 5.3.2 and Corollary 5.4.5, $F_{M_{\lambda,\mu}}(t; z_1, \ldots, z_m, 0, \ldots)$ is the polynomial part of the right hand side of Corollary 5.4.5. Hence it suffices to check that the Schur expansion of $F_{M_{\lambda,\mu}}(t; z)$ contains no $s_\tau(z)$ with $\ell(\tau) > m$. Now if $s_\tau(z)$ occurs in the expansion, then by Conjecture 5.3.2 there exists some $\mathfrak{j}$-catabolizable tableau in $SSYT(\tau, \lambda)$. But $SSYT(\tau, \lambda)$ nonempty implies that $\ell(\tau) \leq \ell(\lambda) \leq m$, so we are done.

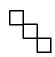
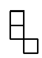
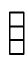
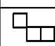
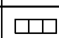
□

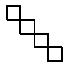
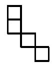
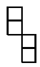
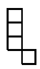
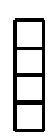
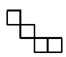
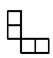
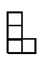
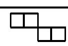
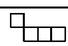
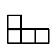
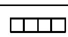
Finally, recall the correspondence between B-modules M and G-equivariant vector bundles X on G/B given by $M \mapsto G \times_B M$ and $X \mapsto$ fiber of X at the point $B/B \in G/B$. Thus $G \times_B \mathfrak{j}$ runs over all G-invariant subbundles of $G \times_B \mathfrak{n}$. We claim that $G \times_B \mathfrak{n}$ is the cotangent bundle $T^*(G/B)$. By the above correspondence, it suffices to show that $\mathfrak{n}$ is isomorphic to the fiber of $T^*(G/B)$ at B/B, which is $(\mathfrak{gl}_m/\mathfrak{b})^*$.

By modding out the $\mathbb{C}I_m$ (I_m = identity matrix), we get $(\mathfrak{gl}_m/\mathfrak{b})^* = (\mathfrak{sl}_m/\mathfrak{b}')^*$, where $\mathfrak{b}' = \mathfrak{b} \cap \mathfrak{sl}_m$. Identify $(\mathfrak{sl}_m/\mathfrak{b}')^*$ as $\mathfrak{b}'^\perp$, the space of functionals in ${\mathfrak{sl}_m}^*$ that annihilate $\mathfrak{b}'$. Recall the nondegenerate Killing form $\mathfrak{sl}_m \otimes \mathfrak{sl}_m \xrightarrow{K} \mathbb{C}$, which gives an identification of $\mathfrak{sl}_m$ with ${\mathfrak{sl}_m}^*$. Under this identification, $\mathfrak{n}$ corresponds to $\mathfrak{b}'^\perp$ because $K(\mathfrak{n}, \mathfrak{b}') = 0$ and $\dim \mathfrak{n} + \dim \mathfrak{b}'^\perp = m^2 - 1 = \dim \mathfrak{sl}_m$. The correspondence commutes with the adjoint action of $\mathfrak{b}'$ because $K(ad(x)y, z) = -K(y, ad(x)z)$, so $\mathfrak{n} \cong (\mathfrak{gl}_m/\mathfrak{b})^*$ as modules over $B \cap SL_m$ and hence over B.

Chapter 6

Tables for $P_{\lambda,\mu}(z;t)$ in Terms of Schur Functions

k-Schur for $k=$	θ	λ	μ	3	21	1^3	Matrix A
1		1^3	3	t^3	t^2+t	1	$\begin{pmatrix} 3 \end{pmatrix}$
2		1^3	21		t	1	$\begin{pmatrix} 2 \\ 1 \end{pmatrix}$
≥ 3		1^3	1^3			1	$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$
2		21	3	t	1		$\begin{pmatrix} 2 & 1 \end{pmatrix}$
≥ 3		21	21		1		$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$
≥ 3		3	3	1			$\begin{pmatrix} 1 & 1 & 1 \end{pmatrix}$

k-Schur for $k =$	θ	λ	μ	4	31	2^2	21^2	1^4	Matrix A
1		1^4	4	t^6	$t^5+t^4+t^3$	t^4+t^2	t^3+t^2+t	1	$\begin{pmatrix}4\end{pmatrix}$
		1^4	31		t^3	t^2	t^2+t	1	$\begin{pmatrix}3\\1\end{pmatrix}$
2		1^4	2^2			t^2	t	1	$\begin{pmatrix}2\\2\end{pmatrix}$
3		1^4	21^2				t	1	$\begin{pmatrix}2\\1\\1\end{pmatrix}$
≥ 4		1^4	1^4					1	$\begin{pmatrix}1\\1\\1\\1\end{pmatrix}$
		21^2	4	t^3	t^2+t	t	1		$\begin{pmatrix}3 & 1\end{pmatrix}$
$2, 3$		21^2	31		t		1		$\begin{pmatrix}2 & 1\\1 & 0\end{pmatrix}$
≥ 4		21^2	21^2				1		$\begin{pmatrix}1 & 1\\1 & 0\\1 & 0\end{pmatrix}$
2		2^2	4	t^2	t	1			$\begin{pmatrix}2 & 2\end{pmatrix}$
≥ 3		2^2	2^2			1			$\begin{pmatrix}1 & 1\\1 & 1\end{pmatrix}$
3		31	4	t	1				$\begin{pmatrix}2 & 1 & 1\end{pmatrix}$
≥ 4		31	31		1				$\begin{pmatrix}1 & 1 & 1\\1 & 0 & 0\end{pmatrix}$
≥ 4		4	4	1					$\begin{pmatrix}1 & 1 & 1 & 1\end{pmatrix}$

λ	μ	5	41	32	31^2	2^21	21^3	1^5
1^5	5	t^{10}	t^9+t^8 $+t^7+t^6$	$t^8+t^7+t^6$ $+t^5+t^4$	$t^7+t^6+2t^5$ $+t^4+t^3$	$t^6+t^5+t^4$ $+t^3+t^2$	t^4+t^3 $+t^2+t$	1
1^5	41		t^6	t^5+t^4	$t^5+t^4+t^3$	$t^4+t^3+t^2$	t^3+t^2+t	1
1^5	32			t^4	t^3	t^3+t^2	t^2+t	1
1^5	31^2				t^3	t^2	t^2+t	1
1^5	2^21					t^2	t	1
1^5	21^3						t	1
1^5	1^5							1
21^3	5	t^6	$t^5+t^4+t^3$	$t^4+t^3+t^2$	t^3+t^2+t	t^2+t	1	
21^3	41		t^3	t^2	t^2+t	t	1	
21^3	32			t^2	t	t	1	
21^3	31^2				t		1	
21^3	21^3						1	
2^21	5	t^4	t^3+t^2	t^2+t	t	1		
2^21	41		t^2	t	t	1		
2^21	32			t		1		
2^21	2^21					1		
31^2	5	t^3	t^2+t	t	1			
31^2	41		t		1			
31^2	31^2				1			
32	5	t^2	t	1				
32	32			1				
41	5	t	1					
41	41		1					
5	5	1						

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