UNIVERSITY OF CALIFORNIA, SAN DIEGO

Simple Lie Algebras, Algebraic Prolongations and Contact Structures

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by

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The dissertation of Orest Bucicovschi is approved, and it is acceptable in quality and form for publication on microfilm:

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ABSTRACT OF THE DISSERTATION

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by

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The story starts with the result of Mukai that every complex simple finite dimensional Lie algebra has a faithful realization as a subalgebra of an algebra of polynomials with the Legendre bracket. Every such realization is determined by a unique polynomial of degree 4. This generalizes the result of Cartan that found a 14-dimensional vector space of polynomials in 5 variables which is a Lie algebra of type G_2 with respect to the Legendre bracket.

To prove his result Mukai uses the notion of algebraic prolongation of a negatively graded Lie algebra. He observes that the algebraic prolongation of a graded Heisenberg Lie algebra of dimension $2d+1$ is the algebra of polynomials in $2d+1$ variables with the Legendre bracket.

We present a different approach to Mukai's result by using the geometry of generalized flag varieties and coadjoint orbits. We made the connection between the embedding of the simple Lie algebras into the Legendre algebra and the contact structure on the minimal nilpotent orbit on of the simple algebra. The existence is due to the fact that the big cell of the minimal orbit is a Heisenberg group and the contact structure has a canonical representative 1-form over it.

We show that the polynomial of degree 4 determining the embedding is up to a precisely determined factor the generator of the algebra of invariants of a particular representation considered by Wallach and Gross [GW96] in connection with quaternionic real forms of complex Lie algebras (the so called invariant of

degree 4). This invariant of degree 4 is also used by Wallach [Wal03] in the study of quaternionic discrete series, where an element is *generic* precisely when the invariant does not vanish at it.

We provide a complete proof of the result of Tanaka that the algebraic prolongation of the negative graded Heisenberg is the Legendre algebra using results on cohomology of Lie algebras by Wallach.

We interpreted the Legendre algebra as a dual Verma module and provided an intrinsic reason for the existence of the multiplicative structure. Moreover, we give a uniform description of the algebraic prolongation of a class of negatively graded algebras.

Related to the invariant of degree 4, we show, using results of Kac-Popov and Sato-Kimura that the situation arising from the minimal nilpotent orbit is the only one of a symplectic representation with a free algebra of invariants.

Boothby proved the converse that every complex compact simply connected homogenous contact manifold is the projectivized minimal nilpotent orbit of a unique complex simple Lie algebra. We give a new simplified proof of Boothby's result. Moreover we give a description of homogenous contact manifolds, relating them to nilpotent orbits.

We mention that there is an overlap of some of our results with results of Landsberg, see [Lan08], [LR07].

1 Introduction- Description of the result of Mukai

We will be studying questions in Lie algebras, Lie groups and contact geometry. The background on Lie algebras and Lie groups comes mostly from [Ser06], [Ser01]. For contact and symplectic structures we refer to [MS04].

In [Muk98] Mukai proved that every simple Lie algebra over the complex numbers can be realized as a subalgebra of an algebra of polynomials with the Legendre bracket. This generalizes a result of Cartan that found a 14 dimensional subspace of an algebra of polynomials in 5 variables with the Legendre bracket.

In this chapter we describe the result of Mukai. The Legendre algebra and the notion of algebraic prolongation is introduced.

In chapter 2 we introduce the notion of contact structure. The Legendre algebra naturally arises from a contact structure on the Heisenberg group.

In chapter 3 we study homogenous contact complex manifolds. We show that the minimal projectivized nilpotent orbit in a simple Lie algebra has a contact structure. Boothby showed conversely that every compact simply connected homogenous contact manifold is the projectivized minimal orbit of a unique complex simple Lie algebra. We give a new simplified proof of Boothby's result. Moreover we give a general description of homogenous contact manifolds relating them to nilpotent orbits.

In chapter 4 we show that the polynomial of degree 4 determining the embedding is the same polynomial of degree 4 considered by Wallach and Gross in connection with quaternionic real forms of simple complex Lie algebras (the so called invariant of degree 4). Related to the invariant of degree 4 we show using

result of Brion and Popov that the situation arising from the minimal nilpotent orbit is the only one of a symplectic representation with a free algebra of invariants.

In chapter 5 we interpret the Legendre algebra as a dual Verma module and provide an intrinsic reason for the existence of the multiplicative structure. Using results on cohomology of Lie algebras of Wallach we give a uniform description of the algebraic prolongation of a class of negatively graded algebras.

1.1 The Longest Root

1.1.1 Existence

For background on root systems we recommend [Bou02]. We also follow the material from [GW96].

Let R a *reduced* root system on the vector space V. Let $B = \{\alpha_1, \ldots, \alpha_l\}$ a basis of R. We know that every positive root has an expression

$$\alpha = \sum_{i=1}^{l} m_i(\alpha)\alpha_i \tag{1.1}$$

where $m_i(\alpha)$ are natural numbers depending on α.
Assertion: There exists a positive root

$$\beta = \sum m_i(\beta)\alpha_i \tag{1.2}$$

such that

$$m_i(\beta) \geq m_i(\alpha) \tag{1.3}$$

for all $1 \leq i \leq l$. To show this consider a root β *maximal* with respect to the order $\succ$ defined as follows: For α, α' in R, $\alpha \succ \alpha'$ if

$$m_i(\alpha) \geq m_i(\alpha') \tag{1.4}$$

for all i. We will show that any other maximal root in fact equals β. That will prove that β is the largest element for this order.

Let $(\cdot,\cdot)$ a positive definite bilinear form on V invariant under the Weyl group of the root system. For instance take

$$(v_1, v_2) \colon = \sum_{\alpha^\vee \in \hat{R}} \alpha^\vee(v_1) \cdot \alpha^\vee(v_2) \tag{1.5}$$

where $\hat{R} \subset V^*$ is the *dual* root system. Such an invariant form is unique up to multiplication by a scalar since R is an irreducible root system.

Recall that if for two roots α, β if $(\alpha, \beta) < 0$ then $\alpha + \beta$ is a root. It follows that

$$(\beta, \alpha_j) \geq 0 \tag{1.6}$$

for all j.

Consider the expansion

$$\beta = \sum_{i=1}^{n} m_i(\beta)\alpha_i \tag{1.7}$$

We show first that $m_i(\beta) > 0$ for all i. Assume the contrary. We have thus a nontrivial partition $\{1, \ldots, n\} = \{i \mid m_i(\beta) > 0\} \cup \{i \mid m_i(\beta) = 0\}$

Let j such that $m_j(\beta) = 0$. Recall that for any two distinct basic roots α_i, α_j we have

$$(\alpha_i, \alpha_j) \leq 0 \tag{1.8}$$

We have now

$$(\beta, \alpha_j) = \sum_{m_i > 0} m_i(\beta)(\alpha_i, \alpha_j) \leq 0 \tag{1.9}$$

with equality if and only if $(\alpha_i, \alpha_j) = 0$ for all i such that $m_i(\beta) > 0$. Because of (1.6) we conclude we have indeed equality. Therefore $(\alpha_i, \alpha_j) = 0$ for all i, j such that $m_i(\beta) > 0$ and $m_j(\beta) = 0$. But that means that the root system R is reducible, contradiction.

Now consider another maximal root γ. Again, like for β, we have $m_i(\gamma) > 0$ We will end up showing that $\gamma = \beta$. Indeed, we have again like in (1.6)

$$(\gamma, \alpha_i) \geq 0 \tag{1.10}$$

for all i. Since the α_i's generate the full vector space and the bilinear form $(,)$ is nondegenerate at least one of the inequalities above is strict. But then it follows

$$(\gamma, \beta) = \sum_{i=1}^{l} m_i(\gamma, \alpha_i) > 0 \tag{1.11}$$

We now conclude $\gamma = \beta$ or $\beta - \gamma$ is a root. The second possibility implies $\gamma \succ \beta$ or $\beta \succ \gamma$, which is not the case, since β, γ are maximal. Conclude $\gamma = \beta$.

Thus, β is the largest element.

1.1.2 Basic Properties

Let $\beta = \sum_{i=1}^{l} m_i\alpha_i$ be the largest root, as above. Let $\alpha = \sum_{i=1}^{l} n_i\alpha_i$ be another root. We have

$$(\beta, \alpha) = \sum_{i=1}^{n} n_i(\beta, \alpha_i) \tag{1.12}$$

Since as before $(\beta, \alpha_i) \geq 0$ for all i we conclude :

$(\beta, \alpha) = 0$ if and only if $(\beta, \alpha_i) = 0$ for all i such that $n_i \neq 0$.

Let's normalize the scalar product $(\cdot, \cdot)$ such that $(\beta, \beta) = 2$. This determines uniquely the invariant scalar product $(\cdot, \cdot)$. Let $\beta^\vee$ the dual root of β. We have then for any α root

$$\alpha(\beta^\vee) = \frac{2(\beta, \alpha)}{(\beta, \beta)} = (\beta, \alpha) \tag{1.13}$$

Again like in (1.12) we have

$$(\beta, \alpha) = \sum_{i=1}^{n} n_i(\beta, \alpha_i) \leq \sum_{i=1}^{n} m_i(\beta, \alpha_i) = (\beta, \beta) \tag{1.14}$$

We now show that

$$(\alpha, \alpha) \leq (\beta, \beta) \tag{1.15}$$

There exists (a unique) w in the Weyl group of R such that $w\alpha$ is in positive chamber C. Since w is an isometry $(w\alpha, w\alpha) = (\alpha, \alpha)$ so we may assume α itself is in the positive chamber, that is $(\alpha, \alpha_i) \geq 0$ for all i. We have thus:

$$(\alpha, \alpha) = \sum_{i=1}^{n} n_i(\alpha, \alpha_i) \leq \sum_{i=1}^{n} m_i(\alpha, \alpha_i) = (\alpha, \beta) \tag{1.16}$$

From (1.14) and (1.16) it follows that $(\alpha, \alpha) \leq (\beta, \beta)$. This is true for all roots α. Thus β can also be called the longest root. An observation is in place. An invariant scalar product on V is unique up to isomorphism. So we can partition the set of roots into "equal length" subsets. It turns out that the Weyl group of R acts transitively on the subsets of equal length. If α is a root largest length then there exists a unique ordering of R such that α is in the positive chamber associated with this order. Then α will be the longest root for this ordering, in the sense above. So the condition is : in the positive chamber and (α, α) longest. For simply laced root system β is characterized as the only root in the positive Weyl chamber. The example of the root system of type G_2 shows this is not true in general.

For any α root we have

$$\alpha(\beta^\vee) = \frac{2(\beta, \alpha)}{(\beta, \beta)} = (\beta, \alpha) \tag{1.17}$$

Now we have

$$(\beta, \alpha)^2 \leq (\beta, \beta) \cdot (\alpha, \alpha) \tag{1.18}$$

and

$$(\alpha, \alpha) \leq (\beta, \beta) \tag{1.19}$$

and so

$$|\alpha(\beta^\vee)| = |(\beta, \alpha)| \leq (\beta, \beta) = 2 \tag{1.20}$$

with equality if and only if $\alpha = \pm\beta$. Moreover $\beta^\vee(\alpha)$ is an integer. We conclude that

$$\alpha(\beta^\vee) = (\beta, \alpha) \in \{-1, 0, 1\} \tag{1.21}$$

for all roots α different from $\pm\beta$.

Moreover, $\alpha(\beta^\vee)$, $\beta(\alpha^\vee)$ are integers of the same sign and if α is not proportional to β their product is at most 3. We conclude that $\hat{\beta}(\alpha) = (\beta, \alpha) \in \{-1, 0, 1\}$. Therefore, if $(\beta, \alpha) = 1$ then α is positive and $\beta - \alpha$ is again a positive root with $(\beta, \beta - \alpha) = 1$.

Let $\rho = \frac{1}{2} \sum_{\alpha \in R^+} \in V$ half the sum of the positive roots. Since for every basic root α_i we have $s_{\alpha_i}(\rho) = \rho - \alpha_i$ we have

$$(\rho, \alpha_i) = \frac{(\alpha_i, \alpha_i)}{2}, \quad i = 1, 2, \ldots l \tag{1.22}$$

In particular we have

$$(\rho, \beta) = \frac{1}{2} \sum_{i=1}^{l} m_i(\beta) \cdot (\alpha_i, \alpha_i). \tag{1.23}$$

The following is a result from [GW96], Prop. 1.3.

Proposition 1.1.1. *(1) If $\alpha \in R^+$ then $(\alpha, \beta) \in \{0, 1, 2\}$. If $(\alpha, \beta) = 2$ then $\alpha = \beta$.*

(2) Let $\Sigma^+ := \{\alpha \in R^+ \mid (\alpha, \beta) = 1\}$. Then $\alpha \mapsto \beta - \alpha$ gives a fixed-point free involution of Σ^+. In particular the number of elements of Σ^+ is even $= 2d$.

(3) We have $(\rho, \beta) = d + 1$ where d is the integer defined above.

Proof. Parts (1) and (2) are proved above. For (3) consider

$$\rho = \frac{1}{2} \left(\sum_{\alpha \in R^+, (\alpha, \beta) = 0} \alpha + \sum_{\alpha \in \Sigma^+} \alpha + \beta \right) \tag{1.24}$$

Therefore we have

$$(\rho, \beta) = \frac{1}{2} \text{ Card } (\Sigma^+) + \frac{1}{2}(\beta, \beta) = d + 1 \tag{1.25}$$

□

Now using (1.23) we can determine the integer d for all the types of irreducible (reduced) root systems using the tables from [Bou02] Planches. One checks that if R is simply laced then $d = h - 2$ where h is the Coxeter number of g.

1.1.3 The Parabolic Subset

As before, R is an irreducible root system with a choice of a basis and hence of a positive part R^+. Let β the largest root of R (see the previous subsection).

Consider the subset of R defined as

$$\mathcal{P} = \{\alpha \in R \mid (\beta, \alpha) \geq 0\} \tag{1.26}$$

It is clearly closed under addition and $\mathcal{P} \cup (-\mathcal{P}) = R$, so $\mathcal{P}$ is a parabolic subset of R. It can also be defined as follows: Consider the partition $B = \{\alpha_1, \ldots, \alpha_n\} = \{\alpha_i \mid (\beta, \alpha_i) = 0\} \cup \{\alpha_i \mid (\beta, \alpha_i) = 0\} = B_0 \cup B_1$. Then $\mathcal{P}$ consists of all the positive

Table 1.1: The integer d for different root systems

R		d
A_l	$l \geq 1$	$l-1$
B_l	$l \geq 2$	$2l-3$
C_l	$l \geq 3$	$l-1$
D_l	$l \geq 4$	$2l-4$
G_2		2
F_4		7
E_6		10
E_7		16
E_8		28

roots and of the negative combinations of basic roots in B_0. One inclusion is clear. Conversely, let α such that $(\beta, \alpha) \geq 0$. Assume that $n_i < 0$ for some i such that with $(\beta, \alpha_i) > 0$. Then α is a negative root and all the coefficients are ≤ 0. But then using also (1.6) we get $(\beta, \alpha) < 0$, contradiction.

1.2 The Heisenberg Parabolic

1.2.1 The Setup

Let $\mathfrak{g}$ a finite dimensional *simple* Lie algebra over $\mathbb{C}$, $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$ and R the root system of $\mathfrak{g}$ with respect to $\mathfrak{h}$. We have the root space decomposition of $\mathfrak{g}$

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}^{\alpha} \tag{1.27}$$

Let $\mathfrak{b}$ a Borel subalgebra of $\mathfrak{g}$ containing $\mathfrak{h}$. Let $B = \{\alpha_1, \ldots, \alpha_l\}$ the positive basis associated with $\mathfrak{b}$.

1.2.2 Standard Parabolic Subalgebras

The conjugacy classes of parabolic subalgebras of $\mathfrak{g}$ are in $1-1$ correspondence with parabolic subalgebras of $\mathfrak{g}$ containing the subalgebra $\mathfrak{b}$. Now any Lie subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ containing $\mathfrak{b}$ is given by a unique *parabolic* subset $\mathcal{P}$ of R containing the subset of positive roots R^+.

$$\mathfrak{p} = \mathfrak{h} \oplus \bigoplus_{\alpha \in P} \mathfrak{g}^\alpha \tag{1.28}$$

The parabolic subsets of R can be characterized as subsets $\mathcal{P}$ of R that are closed under addition (α, $\beta \in \mathcal{P}$ and $\alpha + \beta \in R$ implies $\alpha + \beta \in \mathcal{P}$) and such that $\mathcal{P} \cup (-\mathcal{P}) = R$ – see [Bou02]. Now the parabolic subsets of R that contain R_+ are in $1-1$ correspondence with partitions

$$B = B_0 \cup B_1 \tag{1.29}$$

as follows. Let $\mathcal{P}$ the subset of R defined by

$$\mathcal{P} = \{\alpha \in R \mid \alpha = \sum n_i \alpha_i \text{ such that } n_i \geq 0 \text{ for all } i \text{ with } \alpha_i \in B_1 \} \tag{1.30}$$

Then P is a parabolic subset of R. It can also be characterized as

$$\mathcal{P} = R^+ \cup \{\text{negative combinations of elements of } B_0\} \tag{1.31}$$

Then $\mathfrak{p}$ is given by

$$\mathfrak{p} = h \oplus \bigoplus_{\alpha \in \mathcal{P}} \mathfrak{g}^\alpha \tag{1.32}$$

Consider now the partition $\mathcal{P} = \mathcal{P}_l \cup \mathcal{P}_n$ where

$$\mathcal{P}_l = \{\alpha \in \mathcal{P} \mid n_i = 0 \text{ for all } \alpha_i \in B_1\} \tag{1.33}$$

and

$$\mathcal{P}_n = \{\alpha \in \mathcal{P} \mid n_i > 0 \text{ for some } \alpha_i \in B_1\} \tag{1.34}$$

We have the Levi decomposition of p

$$\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{n} \tag{1.35}$$

where

$$\mathfrak{l} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \mathcal{P}_l} g^{\alpha} \tag{1.36}$$

is a Levi component of $\mathfrak{p}$ and

$$\mathfrak{n} = \bigoplus_{\alpha \in \mathcal{P}_n} g^{\alpha} \tag{1.37}$$

is the nilpotent radical of $\mathfrak{p}$.

1.2.3 The Existence of the Heisenberg Parabolic

For the definition of Heisenberg Lie algebra and Heisenberg Lie group see (3.5.1).

The Lie algebra $\mathfrak{g}$ is simple. So its root system R is irreducible. Consider β the longest root of R with respect to the basis B. Let $B = B_0 \cup B_1$ the associated partition

$$B_0 = \{\alpha_i \in B \mid (\beta, \alpha_i) = 0\} \tag{1.38}$$

$$B_1 = \{\alpha_i \in B \mid (\beta, \alpha_i) > 0\} \tag{1.39}$$

Except for R of type A_1, when β is the unique positive root we have

$$B_1 = \{\alpha_i \in B \mid (\beta, \alpha_i) = 1\} \tag{1.40}$$

We have the associated parabolic subset $P \subset R$ with

$$\mathcal{P} = \{\alpha \in R \mid (\beta, \alpha) \geq 0\} \tag{1.41}$$

Let

$$\mathfrak{p} = h \oplus \bigoplus_{\alpha \in \mathcal{P}} \mathfrak{g}^{\alpha} \tag{1.42}$$

the standard parabolic subalgebra coresponding to P. The partition of $\mathcal{P}$ that determines the Levi decomposition of $\mathfrak{p}$ is

$$\mathcal{P} = \{\alpha \in R \mid (\beta, \alpha) = 0\} \cup \{\alpha \in R \mid (\beta, \alpha) > 0\} \tag{1.43}$$

The Levi decomposition of $\mathfrak{p}$ is therefore

$$\mathfrak{p} = \left(\mathfrak{h} \oplus \bigoplus_{(\beta,\alpha)=0} \mathfrak{g}^{\alpha} \right) \oplus \left(\bigoplus_{(\beta,\alpha)>0} \mathfrak{g}^{\alpha} \right) \tag{1.44}$$

Consider the element $\beta^{\vee}$ in $[\mathfrak{g}^{\beta}, \mathfrak{g}^{-\beta}]$ such that $\beta(\beta^{\vee}) = 2$. $\beta^{\vee}$ can be considered as the dual root of β, that is $\hat{\beta}$. Now $ad(\beta^{\vee})$ acts semisimply on g and we have the decomposition:

$$\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2 \tag{1.45}$$

where $\mathfrak{g}_i = \{X \in \mathfrak{g} \mid ad(\beta^{\vee})(X) = iX\}$. Thus $\mathfrak{g}$ becomes a graded Lie algebra. Moreover we have for the parabolic subalgebra defined above

$$\mathfrak{p} = (\mathfrak{g}_0) \oplus (\mathfrak{g}_1 \oplus \mathfrak{g}_2) \tag{1.46}$$

as its Levi decomposition

Let's show that the nilpotent radical of $\mathfrak{p}$ which is $\mathfrak{n} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ is a Heisenberg Lie algebra. Indeed let α a root, $\alpha \neq \beta$ and $(\alpha, \beta) > 0$. Then $\beta - \alpha$ is again a root and $(\beta - \alpha, \beta) = 1$. We have $[\mathfrak{g}^{\alpha}, \mathfrak{g}^{\beta-\alpha}] = \mathfrak{g}^{\beta}$. Moreover, $[\mathfrak{g}^{\beta}, \mathfrak{g}^{\alpha}] = 0$ for all roots α in $\mathcal{P}_1$ and so $\mathfrak{g}^{\beta}$ is the center of $\mathfrak{n}$.

Note that the Heisenberg parabolic $\mathfrak{p}$ can be characterized as

$$\mathfrak{p} = \{X \in \mathfrak{g} \mid [X, X_{\beta}] \in \mathbf{C} \cdot X_{\beta}\} \tag{1.47}$$

1.2.4 The Uniqueness of the Heisenberg Parabolic

Let $\mathfrak{p}$ a standard parabolic subalgebra such that the nilpotent radical of $\mathfrak{p}$ is a Heisenberg algebra. We will show that $\mathfrak{p}$ is given by the construction above.

Indeed, let first $\mathfrak{p}$ an arbitrary parabolic. Since $\sup(\beta) = R^{+}$ we have $\mathfrak{g}^{\beta} \subset \mathfrak{n}$, the nilpotent radical of p. Moreover, $\beta + \alpha$ is not a root for any positive root α, in particular for any root α in $\mathcal{P}_l$. We conclude that $\mathfrak{g}^{\beta}$ is contained in the center of $\mathfrak{n}$ for every standard parabolic subalgebra $\mathfrak{p}$.

Assume moreover that $\mathfrak{n}$ is Heisenberg. Then the center of $\mathfrak{n}$ is exactly $\mathfrak{g}^{\beta}$. Let now α, α' in $\mathcal{P}_n$. We have therefore $[g^{\alpha}, g^{\alpha'}] \subset g^{\beta}$. It follows that for all α, α' in $\mathcal{P}_n$ we have $\alpha + \alpha' = \beta$ or $\alpha + \alpha'$ is not a root, that is $[\mathfrak{g}^{\alpha}, \mathfrak{g}^{\alpha'}] = \mathfrak{g}^{\beta}$ or $[\mathfrak{g}^{\alpha}, \mathfrak{g}^{\alpha'}] = 0$.

Moreover, since for every $\alpha \in \mathcal{P}_n \backslash \{\beta\}$ we have $\mathfrak{g}^\alpha$ is not in the center of $\mathfrak{n}$ there exists α' in $\mathcal{P}_n$ such that $\alpha + \alpha' = \beta$. Conclude : For every α in $\mathcal{P}_n \backslash \{\beta\}$ we have $\beta - \alpha$ (also a root and) in $\mathcal{P}_n$.

Now recall that for any positive root α that is not β we have $\alpha(\beta^\vee) = (\beta, \alpha) \in \{0, 1\}$. Also, if α and $\beta - \alpha$ are positive roots then $(\beta, \alpha) + (\beta, \beta - \alpha) = (\beta, \beta) = 2$. It follows that both (β, α) and $(\beta, \beta - \alpha)$ equal 1. We conclude : $\mathcal{P}_n \backslash \{\beta\} \subset \{\alpha \mid (\beta, \alpha) = 1\}$

We'll prove now the opposite inclusion. Let α with $(\beta, \alpha) = 1$. Then $\alpha' := \beta - \alpha$ is again a positive root and $\alpha + \alpha' = \beta$. Now α and α' are positive roots and so $\sup(\alpha) \cup \sup(\alpha') = \sup(\alpha + \alpha') = \sup(\beta) = R_+$. It follows that at least one of α, α' are in P_1. But by the above, if α is in $\mathcal{P}_n$ then also α' is (and the same for α'). We conclude that in any case α is in $\mathcal{P}_n$.

We conclude : $\mathcal{P} = \{\alpha \in R \mid (\beta, \alpha) = \beta^\vee(\alpha) \geq 0\}$ and hence $\mathfrak{p}$ is unique.

1.2.5 The case of complex simple Lie groups

Let G a complex connected simple Lie group. Its Lie algebra $\mathfrak{g}$ is a complex simple Lie algebra. A parabolic subgroup P of G is called a Heisenberg parabolic if its unipotent radical H is a Heisenberg Lie group. Since $Lie(H) = \mathfrak{n}$ where $\mathfrak{n}$ is the nilpotent radical of $\mathfrak{p} = Lie(P)$ it follows that a parabolic subgroup of G is Heisenberg parabolic if and only if its Lie algebra is Heisenberg parabolic. By the existence and uniqueness result above, there exists a unique up to conjugacy Heisenberg parabolic subgroup P of G.

1.3 The algebraic prolongation of the Heisenberg algebra

1.3.1 Algebraic prolongation

The notion of algebraic prolongation was introduced by Tanaka [Tan70]

Let $\mathfrak{n} = \oplus_{i<0} \mathfrak{n}_i$ a negatively graded Lie algebra. Then there exists a $\mathbb{Z}$-graded Lie algebra $\mathcal{C}(\mathfrak{n})$, called the algebraic prolongation of $\mathfrak{n}$ and a morphism of neg-

atively graded Lie algebras $\mathfrak{n} \to \mathcal{C}(\mathfrak{n})_{<0}$ with the following property: For every $\mathbf{Z}$-graded Lie algebra $\mathfrak{g} = \oplus_{i \in \mathbf{Z}} \mathfrak{g}_i$ and a morphism of graded Lie algebras

$$\mathfrak{n} \to \mathfrak{g}_{<0} \tag{1.48}$$

there exists a unique extension

$$\mathfrak{g} \to \mathcal{C}(\mathfrak{n}) \tag{1.49}$$

to a morphism of $\mathbb{Z}$-graded Lie algebras

$$\begin{array}{ccc} \mathfrak{n} & \longrightarrow & \mathcal{C}(\mathfrak{n}) \\ & \searrow & \uparrow \exists! \\ & & \mathfrak{g} \end{array} \tag{1.50}$$

1.3.2 Construction

The construction under the additional hypothesis that $\mathfrak{n}$ is generated by $\mathfrak{n}_{-1}$ and the centralizer of $\mathfrak{n}_{-1}$ in $\mathfrak{n}$ equals $\mathfrak{n}_{\text{lowest degree}}$ (this is what Tanaka calls "regular")(NOTE: Weaker hypothesis are possible- However we are mostly interested in the case $\mathfrak{n}$ = negatively graded Heisenberg). In this case the map $\mathfrak{n} \to \mathcal{C}(\mathfrak{n})_{<0}$ is an embedding.

$\mathcal{C}(\mathfrak{n})$ is constructed inductively as follows : First, define

$\mathfrak{n}_0 =$ Graded derivations of $\mathfrak{n}$. Assume that we have built already $\mathfrak{n}_0$, $\mathfrak{n}_1$,... $\mathfrak{n}_m$ and the bilinear brackets $[X, Y]$ for $X \in \mathfrak{n}$ and $Y \in \mathfrak{n} \oplus_{l \leq m} \mathfrak{n}_l$. Then define $\mathfrak{n}_{m+1}$: $=$ {Linear maps of degree $m+1 \phi$: $\mathfrak{n} \to \oplus_{l \leq m} \mathfrak{n}_l$ such that $\phi([X, Y]) = [\phi X, Y] + [X, \phi Y]$ for all $X, Y \in \mathfrak{n}$}

Moreover, define the bracket $[X, Y]$ for $X \in \mathfrak{n}$, $Y \in \mathfrak{n}_{m+1}$ as $[X, Y] = -Y(X)$ (recall that Y is a linear map as above!).

We have defined $\mathcal{C}(\mathfrak{n})$ as a *vector space*. We define now the brackets for $X \in \mathfrak{n}_m$, $Y \in \mathfrak{n}_n$ as the linear map given by $[[X, Y], Z] = [[X, Z], Y] + [X, [Y, Z]]$ for all $X \in \mathfrak{n}_m$, $Y \in \mathfrak{n}_n$ and $Z \in \mathfrak{n}$. This is done by induction on $m + n$, taking into account that the bracket is already defined for m, n negative integers.

One notices that at each step the grade is taken care of. Moreover, one checks that with this bracket $\mathcal{C}(\mathfrak{n})$ becomes a Lie algebra and that is has the universal property stated above.

1.3.3 The Legendre Algebra

We define the Legendre algebra in $2d+1$ variables to be the Lie algebra of polynomials $\mathcal{L} = \mathcal{L}_d = \mathbb{C}[z, q_1, \ldots, q_d, p_1, \ldots, p_d]$ with the Legendre bracket for bracket. The Legendre bracket of two functions $f(z,q,p)$ and $g(z,q,p)$ is given by

$$\{f,g\} = (f - E_p f)\frac{\partial g}{\partial z} - (g - E_p g)\frac{\partial f}{\partial z} + (f,g) \tag{1.51}$$

where E_p is the Euler operator with respect to the p variables

$$E_p f = \sum_{i=1}^{n} p_i \frac{\partial f}{\partial p_i} \tag{1.52}$$

and (f,g) is the Poisson bracket with respect to the variables (q_i, p_i)

$$(f,g) = \sum_{i=1}^{d} \left(\frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial q_i}\frac{\partial f}{\partial p_i} \right) \tag{1.53}$$

Equivalently we have

$$\{f,g\} = f\frac{\partial g}{\partial z} - g\frac{\partial f}{\partial z} + \sum_{i=1}^{d} \left(\frac{df}{dq_i}\frac{\partial g}{\partial p_i} - \frac{dg}{dq_i}\frac{\partial f}{\partial p_i} \right) \tag{1.54}$$

where

$$\frac{d}{dq_i} = \frac{\partial f}{dq_i} + p_i \frac{\partial}{\partial z} \tag{1.55}$$

Some particular cases:

$$\{f,1\} = -\frac{\partial f}{\partial z} \tag{1.56}$$

$$\begin{aligned} \{f,q_i\} &= -q_i\frac{\partial f}{\partial z} - \frac{\partial f}{\partial p_i} \\ \{f,p_i\} &= \frac{\partial f}{\partial q_i} \end{aligned} \tag{1.57}$$

$$\{f,z\} = f - E_p f - z\frac{\partial f}{\partial z}$$

1.3.4 The Grading

Note that $\mathcal{L}_d$ is also an associative algebra for the usual multiplication of polynomials We grade $\mathcal{L}_d$ by

$$\deg q_i = \deg p_j = 1 \tag{1.58}$$

$$\deg z = 2 \tag{1.59}$$

We note that if f, g are (weighted) *homogenous* polynomials then

$$\deg\{f,g\} = \deg f + \deg g - 2 \tag{1.60}$$

Now let's modify the grading by requiring

$$\text{grade} f = \deg f - 2 \tag{1.61}$$

for a homogenous f . That means for instance

$$\begin{aligned} \text{grade } 1 &= -2 \\ \text{grade } q_i = \text{grade } p_j &= -1 \\ \text{grade } z &= 0 \end{aligned}$$

Thus the equation (1.60) becomes

$$\text{grade}\{f,g\} = \text{grade } f + \text{grade } g \tag{1.62}$$

Thus $\mathcal{L} = \mathbb{C}[q_1,\dots,q_d,p_1,\dots,p_d,z]$ with $\{,\cdot,\cdot\}$ and the grade becomes a *graded* Lie algebra

$$\mathcal{L} = \oplus_{m\geq -2}\mathcal{L}_m \tag{1.63}$$

We note that $n = \mathcal{L}_{<0} = \mathcal{L}_{-2} \oplus \mathcal{L}_{-1}$ is a graded Heisenberg algebra of dimension $2d+1$.

1.3.5 Scaling Element

The *scaling element* is

$$\tilde{z} = 2z - \sum_{i=1}^{d} q_i p_i \tag{1.64}$$

Let g any (weighted - see (1.58)) homogenous element in L. Then we have

$$\{\tilde{z},g\} = (\deg g - 2)g = \text{grade} g \cdot g \tag{1.65}$$

Indeed we have

$$\{\tilde{z},g\} = (\tilde{z} - E_p\tilde{z})\frac{\partial g}{\partial z} - (g - E_p g)\cdot 2 - \sum_i p_i \frac{\partial g}{\partial p_i} + \sum_i q_i \frac{\partial g}{\partial q_i} =$$

$$= 2z\frac{\partial g}{\partial z} + \sum_i p_i \frac{\partial g}{\partial p_i} + \sum_i q_i \frac{\partial g}{\partial q_i} - 2g = (\deg g - 2)\cdot g = \text{grade}\cdot g$$

Now $\tilde{z}$ is the *unique* scaling element. Indeed, let f a scaling element.

- f scaling for $g = 1$ implies (see (1.56)) $\frac{\partial f}{\partial z} = 2$
- f scaling for $g = p_i$ implies $\frac{\partial f}{\partial q_i} = -p_i$
- f scaling for $g = q_i$ implies (using also $\frac{\partial f}{\partial z} = 2$) $\frac{\partial f}{\partial p_i} = -q_i$

We conclude that

$$f = \tilde{z} + c$$

where c is a constant. But the $\{c, q\} = 0$ for all g. That implies

$$c \cdot \frac{\partial g}{\partial z}$$

for all g. We conclude (for $g = z$) that $c = 0$. (Or we could have used $\{c, \tilde{z}\} = -2c = 0$)

1.3.6 The Nondegeneracy

Consider our graded Lie algebra

$$\mathcal{L} = \oplus_{m \geq -2} \mathcal{L}_m \tag{1.66}$$

We have

$$\mathcal{L}_{-2} = \mathbb{C} \text{ and}$$
$$\mathcal{L}_{-1} = \langle q_1, \ldots, q_d, p_1, \ldots, p_d \rangle$$

For all $m \geq -1$ we have a map

$$\mathcal{L}_m \times \mathcal{L}_{-1} \to \mathcal{L}_{-m-1} \tag{1.67}$$

Let's determine the left kernel of this map, that is

$$\{f \in \mathcal{L} \mid \{f, \mathcal{L}_{-1}\} = 0\} \tag{1.68}$$

From (1.56) it follows that

$$-q_i \frac{\partial f}{\partial z} - \frac{\partial f}{\partial p_i} = 0 \tag{1.69}$$
$$\frac{\partial f}{\partial q_i} = 0 \tag{1.70}$$

Now take $\frac{\partial}{\partial q_i}$ of the first equation and we get

$$-\frac{\partial f}{\partial z} - q_i \frac{\partial^2 f}{\partial q_i \partial z} - \frac{\partial^2 f}{\partial q_i \partial p_i} = 0$$

Now using (1.70) and the symmetry of second derivatives we get

$$\frac{\partial f}{\partial z} = 0 \tag{1.71}$$

and thus

$$\frac{\partial f}{\partial p_i} = 0 \tag{1.72}$$

Conclude all the partial derivatives of f are zero and thus f is a constant, $f \in \mathcal{L}_{-2}$. We conclude that for all $m \geq -1$ the map (1.67) is nondegenerate

1.3.7 The group of automorphisms of the Legendre algebra

We will determine the group of automorphisms of $\mathcal{L}$ as a graded Lie algebra. Every automorphism of $\mathcal{L}$ restricts to an automorphism of $\mathcal{L}_{<0}$, that is, an automorphism of the graded Heisenberg Lie algebra $\mathfrak{n} = \mathfrak{n}_{-2} \oplus \mathfrak{n}_{-1}$. Conversely, since by (1.3.3) $\mathcal{L}$ is the algebraic prologation of $\mathfrak{n}$, every automorphism of $\mathfrak{n}$ extends uniquely to an automorphism of $\mathcal{L}$. Therefore we have

Proposition 1.3.1. *The group of automorphisms of $\mathcal{L}$ is isomorphic to the group of automorphisms of $\mathfrak{n}$*

Note: This result is valid for any algebraic prologation, the group of automorphisms being the group of automorphisms of the negative part.

Now the group of automorphisms of $\mathfrak{n}$– the Heisenberg algebra is $GSP(\mathfrak{n}_{-1})$, hence so is . the group of automorphisms of the Legendre algebra

Let's first describe the subgroup of automorphisms given by $Sp(\mathfrak{n}_{-1})$

Theorem 1.3.2. *Let $T\colon \mathfrak{n}_{-1} \to \mathfrak{n}_{-1}$ a linear symplectic automorphism of $\mathfrak{n}_{-1}$. Consider the automorphism $\tilde{T}$ of $\mathcal{L}$ as a graded* associative *algebra that is T on $\mathfrak{n}_{-1}$ and takes $\tilde{z}$ to itself. Then $\tilde{T}$ is an automorphism of $\mathcal{L}$ as a graded Lie algebra – the unique extension of T.*

Proof. Note that $\tilde{T}$ is the unique extension of T as morphism of graded Lie algebras since (see below) $\mathcal{L}$ is the algebraic prolongation of $\mathfrak{n}$. To show that $\tilde{T}$ is indeed a morphism of Lie algebras note that $\tilde{T}$ is a graded automorphism and moreover, for every $l \in \mathfrak{n}_{-1}$ the map $ad(l) = \{l, \cdot\}$ is a derivation of $\mathcal{L}$ as a an associative algebra and as a Lie algebra. Moreover, the centralizer of $\mathfrak{n}_{-1}$ in $\mathcal{L}$ is $\mathfrak{n}_{-2}$. Now the result follows by induction on degrees. □

As an example let T be given by $p_i \mapsto q_i$, $q_i \mapsto -p_i$. Let also $z \mapsto z - \sum p_i q_i$. Extend this multiplicatively. We get an automorphism of $\mathcal{L}$ as a graded Lie algebra, which is of order 4. Note that $\tilde{z} = 2z - \sum p_i q_i$ is invariant, like for every automorphism of $\mathcal{L}$. We observe now that $GSp(\mathfrak{n}_{-1}) = Sp(\mathfrak{n}_{-1}) \times \mathbb{C}^\times / \pm 1$. Also, $\mathbb{C}^\times$ acts on $\mathcal{L}$ by $\mathbb{C}^\times \ni s \mapsto$ multiplication by s^m on $\mathcal{L}_m$. Let $T \in GSp(\mathfrak{n}_{-1})$. Then $(Tu, Tv) = t^2 \cdot (u, v)$ for some $t \in \mathbb{C}^\times$. Then $T = t \cdot T_0$ where T_0 $in Sp(\mathfrak{n}_{-1})$. The automorphism of $\mathcal{L}$ determined by T is given by

$$p_i^a q_j^b \tilde{z}^c \mapsto \frac{(T_0 p_i)^a (T_0 q_i)^b \tilde{z}^c}{t^m} \tag{1.73}$$

for $a + b + 2c = m + 2$

1.3.8 The Cohomology Groups $H^i(\mathfrak{n}, \mathcal{L})$

Let $\mathfrak{n} = \mathcal{L}_{-2} \oplus \mathcal{L}_{-1}$ the Heisenberg algebra, subalgebra of $\mathcal{L}$. Let $\mathfrak{a}$ the center of h, it is 1-dimensional. Now $\mathfrak{g}/\mathfrak{a}$ is a $2d$-dimensional abelian subalgebra. We have the Hochschild-Serre spectral sequence

$$E_2^{p,q} = H^p(\mathfrak{n}/\mathfrak{a}, H^q(\mathfrak{a}, \mathcal{L})) \Rightarrow H^n(\mathfrak{n}, \mathcal{L}) \tag{1.74}$$

Now the action of $\mathfrak{a} = \mathbb{C} \cdot 1$ on $\mathcal{L}$ is given by

$$1 \cdot f = \frac{\partial f}{\partial z} \tag{1.75}$$

The cohomology of a 1-dimensional Lie algebra acting by a linear operator T are given by

$$H^0(\mathfrak{a}, \mathcal{L}) = \ker T \tag{1.76}$$

$$H^1(\mathfrak{a}, \mathcal{L}) = \text{coker } T \tag{1.77}$$

$$H^q(\mathfrak{a}, \mathcal{L}) = 0 \text{ for } q \geq 2 \tag{1.78}$$

Now $\ker T = \ker \frac{\partial}{\partial z} = \mathbb{C}[p_i, q_j]$ and $\operatorname{coker} T = \operatorname{coker} \frac{\partial}{\partial z} = 0$ (since we can integrate with respect to z. It follows that in the spectral sequence above at step 2 the only nonzero column is column $(E_2^{p,0})_{p\geq 0}$. It follows that the spectral sequence degenerates at step 2 and so we have

$$H^n(\mathfrak{n}, \mathcal{L}) = H^n(\mathfrak{n}/\mathfrak{a}, H^0(a, \mathcal{L})) = H^n(\mathbb{C}^{2n}, \mathbb{C}[p_i, q_j]) \tag{1.79}$$

where $\mathbb{C}^{2n}$ is the commutative Lie algebra with basis p_i, q_j. Moreover the action of $\mathbb{C}^{2n}$ on $\mathbb{C}[p_i, q_j]$ is given by partial derivatives (with some signs that can be absorbed). Now we use Poincare Lemma for differential forms with polynomial coefficients (that is what the standard cohomology complex will be). We conclude:

$$H^{\geq 1}(\mathfrak{n}, \mathcal{L}) = 0 \tag{1.80}$$

$$H^0(\mathfrak{n}, \mathcal{L}) = \mathbb{C} \tag{1.81}$$

1.3.9 The Legendre Algebra and the Heisenberg Algebra

Rather than use the construction of Tanaka we show directly that

Theorem 1.3.3. *$\mathfrak{n} \to \mathcal{L}$ is the algebraic prolongation of the Heisenberg algebra.*

Proof. et $\mathbf{Z}$-graded Lie algebra $\mathfrak{g} = \oplus_{i\in \mathbf{Z}} \mathfrak{g}_i$ and a morphism of graded Lie algebras

$$\mathfrak{n} \to \mathfrak{g}_{<0} \tag{1.82}$$

We will show this under the additional assumption that the above morphism is an isomorphism, that is the negative part of $\mathfrak{g}$ is $\mathfrak{n}$. This is always the case when $\mathfrak{g}$ is a simple Lie algebra over $\mathbb{C}$ and $\mathfrak{n}$ is the nilpotent radical of Heisenberg parabolic. We the map $\mathfrak{n} \to \mathcal{L}$ extends uniquely to a morphism of graded Lie algebras $\phi\colon \mathfrak{g} \to \mathcal{L}$. We define ϕ inductively on each $\mathfrak{g}_i$. First, ϕ is already defined for $i < 0$ – indeed, ϕ is just the identity on the negative part of $\mathfrak{g}$ which is $\mathfrak{n}$. Assume we have defined already ϕ on $\oplus_{j<i}\mathfrak{g}_j$ such that $\phi([H, Y]) = [H, \phi(Y)]$. Take $X \in \mathfrak{g}_i$. Now for all H in h we have $[H, X] \in \oplus_{j<i}\mathfrak{g}_j$ since $\mathfrak{n}$ is the negative part of $\mathfrak{g}$. Therefore $\phi([H, X])$ is already defined What should $\phi(X)$ be? We must have $[\phi(H), \phi(X)] = \phi([H, X])$,

that is $[H, \phi(X)] = \phi([H, X])$ Consider the map $\alpha \colon \mathfrak{n} \to \mathcal{L}$, $\alpha(H) = \phi([H, X])$. We have

$$\alpha([H_1, H_2]) = \phi([[H_1, H_2], X]) = \phi([H_1, [H_2, X]]) - \phi([H_2, [H_1, X]]) = \\ (\text{ by induction hypothesis}) = [H_1, \alpha(H_2)] - [H_2, \alpha(H_1)]$$

The above equality says α is a 1-cocycle of the $\mathfrak{n}$-module $\mathcal{L}$. Since the 1-cohomology is zero the cocycle α is a coboundary, that is there exists Y in $\mathcal{L}$ such that $\alpha(H) = \phi([H, X]) = [H, Y]$ for all H in h. Since the centralizer of $\mathfrak{n}$ in $\mathcal{L}$ is $\mathfrak{n}_{-2}$ it follows that Y is unique modulo $\mathfrak{n}_{-2}$. By degree considerations it follows that there exists a unique Y with $\deg Y = \deg X$ such that $[H, Y] = \phi([H, X])$ for all $H \in h$. Define therefore $\phi(X) = Y$. Thus we construct ϕ inductively. We check now ϕ is a morphism of Lie algebras. Linearity follows easily from the uniqueness of the construction. Let now X_1, X_2 in $\mathfrak{g}$ homogenous elements. We show that $\phi([X_1, X_2]) = [\phi(X_1), \phi(X_2)]$ inductively on $\dim X_1 + \dim X_2$. When $\dim X_1$, $\dim X_2$ negative they are in $\mathfrak{n}$ so the equality holds. Let now X_1, X_2 in g. We have $\phi([H, [X_1, X_2]]) = [H, \phi([X_1, X_2])]$ for all H in h, by the construction of ϕ. Now $\phi([H, [X_1, X_2]]) = \phi([[H, X_1], X_2]) + \phi([X_1, [H, X_2]])$. Now the elements of $\mathfrak{g}$ $[H, X_1]$, $[H, X_2]$ have degree smaller than respectively X_1, X_2. By induction hypothesis we have $\phi([[H, X_1], X_2]) = [\phi([H, X_1]), \phi(X_2)]$ and $\phi([X_1, [H, X_2]]) = [\phi(X_1), \phi([H, X_2])]$ Now we have $\phi([H, X_i]) = [H, \phi(X_i)]$ for $i = 1, 2$. We conclude by adding all up:

$$\phi([H, [X_1, X_2]]) = [[H, \phi(X_1)], X_2] + [\phi(X_1), [H, \phi(X_2)]] = [H, [\phi(X_1), \phi(X_2)]]$$

and therefore

$$[H, \phi([X_1, X_2])] = [H, [\phi(X_1), \phi(X_2)]]$$

Since the centralizer of $\mathfrak{n}$ in $\mathcal{L}$ is contained in $\mathfrak{n}$ we conclude

$$\phi([X_1, X_2]) = [\phi(X_1), \phi(X_2)]$$

□

1.4 The result of Mukai

Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$. Let G a complex Lie group with Lie algebra $\mathfrak{g}$. Consider the adjoint action of G on the projective space $\mathbb{P}(\mathfrak{g})$. (note that this action factors through the adjoint group G_{ad} so the choice of G between the adjoint group G_{ad} and simply connected group G_{sc} is in fact not important at this point). From [CM93] we know that the action has a unique closed orbit which belongs to the closure of any other orbit. If $X \in \mathfrak{g}$ then $\bar{X}$ belongs to the minimal orbit if and only if there exists a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ and a choice of positive subsystem $R^+ \subset R$, the root system of $(\mathfrak{g}, \mathfrak{h})$ such that $\bar{X}$ belongs to $\mathfrak{g}^\beta$ where β is the largest root of R. Moreover the choice of h and order on R is unique. We can understand this as follows: $\mathfrak{g}$ with the adjoint action is an irreducible representation of G because g is simple. The orbit of a highest weight vector, that is a nonzero element in $\mathfrak{g}^\beta$ is the minimal closed orbit in $\mathbb{P}(g)$. Alternatively, $\bar{X}$ is in the minimal orbit if and only if $\mathbb{C} \cdot X$ is the center of the nilpotent radical of a Heisenberg parabolic.

Consider thus $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$, R the root system of $(\mathfrak{g}, \mathfrak{h})$, a choice of a positive subsystem R^+. Let $\mathfrak{p}$ the unique standard Heisenberg parabolic (parabolic with the nilpotent radical a Heisenberg algebra). Recall that $\mathfrak{p}$ is obtained as follows: Let β the largest root of R. We have $[\mathfrak{g}^\beta, \mathfrak{g}^{-\beta}]$ a 1-dimensional subspace of $\mathfrak{h}$. Let $\beta^\vee$ in $[\mathfrak{g}^\beta, \mathfrak{g}^{-\beta}]$ uniquely determined by : $\beta(\beta^\vee) = 2$. We have the eigenvalue decomposition of $\mathfrak{g}$ relative to $ad(\beta^\vee)$ $\mathfrak{g} = \oplus_{i=-2}^{2} \mathfrak{g}_i$. Then $\mathfrak{p} = \oplus_{i \geq 0} \mathfrak{g}_i$ and its nilpotent radical is $\mathfrak{g}_1 \oplus \mathfrak{g}_2$, a Heisenberg algebra. Moreover, the opposite parabolic $\oplus_{i \leq 0} \mathfrak{g}_i$ has again as nilpotent radical a Heisenberg algebra $\mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$ Thus $\mathfrak{g}$ is a graded Lie algebra with the negative part $\mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} = \mathfrak{n} = \mathfrak{n}_{-2} \oplus \mathfrak{n}_{-1}$ a graded Heisenberg algebra. Let $2d+1$ the dimension of this Heisenberg algebra (d is determined as in Table 1.2). Now we know that the algebraic prolongation of the Heisenberg algebra of dimension $2d+1$ is the Legendre algebra in $2d+1$ variables $\mathcal{L}_d$. Therefore

Theorem 1.4.1. *([Muk98]) there exists an embedding of graded Lie algebras $\mathfrak{g} \to \mathcal{L}_d = \mathcal{L}$ that is an isomorphims from $\mathfrak{g}_{<0}$ to $\mathcal{L}_{<0}$.*

Let $X_\beta \in \mathfrak{g}^\beta$, $X_{-\beta} \in \mathfrak{g}^{-\beta}$ such that $(X_\beta, X_{-\beta}, \beta^\vee)$ form an $sl(2)$ triple. We can choose the mapping $g_{<0}$ to $\mathcal{L}_{<0}$ by taking $X_{-\beta} \mapsto 1 \in \mathcal{L}_{-2}$. Now $\beta^\vee$ is the scaling element for the graded Lie algebra $\mathfrak{g}$. Therefore it will map to the unique scaling element of $\mathcal{L}$ which is $\tilde{z} = 2z - \sum_{i=1}^d q_i p_i$. Since $[X_\beta, X_{-\beta}] = \beta^\vee$, X_β maps to an element in $\mathcal{L}_2$ f such that $\{f, 1\} = \tilde{z}$. From (1.56) we have $\{f, 1\} = -\frac{\partial f}{\partial z}$. Therefore $\frac{\partial f}{\partial z} = -2z + \sum_{i=1}^d q_i p_i$ and so

$$f = -(z - \frac{1}{2}\sum_{i=1}^d q_i p_i)^2 + F(p, q) \tag{1.83}$$

where $F(q, p)$ is a homogenous polynomial of degree 4 in p_i, q_i. Therefore the embedding $\mathfrak{g} \to \mathcal{L}$ is thus determined by a form of degree 4 that we call the Mukai form. We will later on analyse further this form of degree 4 making the connection to another form of degree 4 defined by Gross and Wallach.

Note that our polynomial of degree 4 differs from Mukai's by a factor of -4.

1.5 Calculations for $sl(n, \mathbb{C})$

We determine explicitly the mapping from the Lie algebra $sl(n, \mathbb{C})$ to $\mathcal{L}_d$ given by Mukai's theorem. We have $d = n - 2$ so $n = d + 2$. In fact we will find a map ϕ from $gl(d + 2, \mathbb{C})$ to $\mathcal{L}_d$. Let E_{ij} the standard basis of $gl(d + 2, \mathbb{C})$ where $0 \leq i, j \leq d + 1$. We use the theorem on the existence and uniqueness of the extension morphism of graded Lie algebras that takes E_{i0} to p_i, $E_{d+1,j}$ to q_j and $E_{d+1,0}$ to 1. Let $a_{ij} = \phi(E_{ij})$ the image of E_{ij}. We have

$$\begin{aligned} a_{i,0} &= p_i \quad \text{for } 1 \leq i \leq d \\ a_{d+1,j} &= q_j \quad \text{for } 1 \leq i \leq d \\ a_{d+1,0} &= 1 \end{aligned}$$

Determination of a_{ij} for $1 \leq i, j \leq d$:

We have $[E_{i,j}, E_{d+1,0}] = 0$ and so $\{a_{ij}, a_{d+1,0}\} = 0$. Therefore $\{a_{ij}, 1\} = -\frac{\partial a_{ij}}{\partial z} = 0$ and so a_{ij} is a polynomial in p's and q's only.

For $1 \leq k \leq d$, $k \neq j$ we have

$$[E_{ij}, E_{k,0}] = 0$$

and so

$$\{a_{ij}, p_k\} = 0$$
$$\frac{\partial a_{ij}}{\partial q_k} = 0$$

Similarly, for $1 \leq l \leq d$, $k \neq i$

$$\{a_{ij}, q_l\} = 0$$
$$\frac{\partial a_{ij}}{\partial p_l} = 0$$

Now we have

$$[E_{ij}, E_{j,0}] = E_{i,d+1}$$

and so

$$\{a_{ij}, a_{j,0}\} = a_{i,d+1}$$

that is

$$\{a_{ij}, p_j\} = p_i$$
$$\frac{\partial a_{ij}}{\partial q_j} = p_i$$

Similarly we have

$$[E_{ij}, E_{d+1,i}] = -E_{n+1,i}$$

and so

$$\{a_{ij}, a_{n+1,i}\} = -a_{n+1,j}$$

that is

$$\{a_{ij}, q_i\} = -q_j$$
$$-\frac{\partial a_{ij}}{\partial p_i} = -q_j$$

Here is the complete list of the partial derivatives of a_{ij}

$$\begin{aligned}
\frac{\partial a_{ij}}{\partial z} &= 0 \\
\frac{\partial a_{ij}}{\partial p_l} &= 0 \quad \text{for } l \neq i \\
\frac{\partial a_{ij}}{\partial q_k} &= 0 \quad \text{for } k \neq j \\
\frac{\partial a_{ij}}{\partial p_i} &= q_j \\
\frac{\partial a_{ij}}{\partial q_j} &= p_i
\end{aligned}$$

Since a_{ij} has no constant term (being a polynomial of weighted degree 2) it follows that

$$a_{ij} = p_i q_j \tag{1.84}$$

Determination of a_{00} :

We have

$$[E_{00}, E_{d+1,0}] = -E_{d+1,0}$$

and so

$$\begin{aligned}
\{a_{00}, 1\} &= -1 \\
-\frac{\partial a_{00}}{\partial z} &= -1
\end{aligned}$$

For $1 \leq i \leq d$ we have

$$[E_{00}, E_{d+1,i}] = 0$$

and so

$$\begin{aligned}
\{a_{00}, a_{d+1,i} &= 0 \\
\{a_{00}, q_i\} &= -q_i \frac{\partial a_{00}}{\partial z} - \frac{\partial a_{00}}{\partial p_i}
\end{aligned}$$

and so

$$\frac{\partial a_{00}}{\partial p_i} = -q_i$$

For $1 \leq j \leq d$ we have

$$[E_{00}, E_{j,0}] = -E_{j0}$$

and so

$$\{a_{00}, a_{j,0}\} = -a_{j,0}$$
$$\{a_{00}, p_j\} = \frac{\partial a_{00}}{\partial q_j} = -p_j$$

Here is the complete list of the partial derivatives of a_{00}

$$\begin{aligned} \frac{\partial a_{00}}{\partial z} &= 1 \\ \frac{\partial a_{00}}{\partial p_j} &= -q_j \\ \frac{\partial a_{00}}{\partial q_i} &= -p_i \end{aligned}$$

Since a_{ij} has no constant term (being a polynomial of weighted degree 2) it follows that

$$a_{00} = z - \sum_{i=1}^{d} p_i q_i \tag{1.85}$$

Determination of $a_{n+1,n+1}$:

We have

$$[E_{d+1,d+1}, E_{d+1,0}] = E_{d+1,0}$$

and so

$$\{a_{d+1,d+1}, a_{d+1,0}\} = a_{d+1,0}$$

and so

$$\{a_{d+1,d+1}, 1\} = -1$$
$$\frac{\partial a_{d+1,d+1}}{\partial z} = -1$$

For $1 \leq i \leq d$ we have

$$[E_{d+1,d+1}, E_{i,0}] = 0$$

and so

$$\{a_{d+1,d+1}, a_{i0}\} = 0$$

that is

$$\{a_{d+1,d+1}, p_i\} = 0$$
$$\frac{\partial a_{d+1,d+1}}{\partial q_i} = 0$$

For $1 \leq j \leq n$ we have

$$[E_{d+1,d+1}, E_{d+1,j}] = E_{d+1,j}$$

and so

$$\{a_{d+1,n+1}, a_{d+1,j}\} = a_{n+1,j}$$
$$\{a_{d+1,d+1}, q_j\} = 0$$
$$-q_j \frac{\partial a_{d+1,d+1}}{\partial z} - \frac{\partial a_{d+1,d+1}}{\partial p_j} = -q_j$$

therefore

$$\frac{\partial a_{d+1,d+1}}{\partial p_j} = 0$$

Here is the complete list of the partial derivatives of $a_{n+1,n+1}$

$$\begin{aligned} \frac{\partial a_{d+1,d+1}}{\partial z} &= -1 \\ \frac{\partial a_{d+1,d+1}}{\partial p_j} &= 0 \\ \frac{\partial a_{d+1,d+1}}{\partial q_i} &= 0 \end{aligned}$$

Since $a_{d+1,d+1}$ has no constant term (being a polynomial of weighted degree 2) it follows that

$$a_{d+1,d+1} = -z \tag{1.86}$$

Determination of a_{0i} for $1 \leq i \leq d$:

We have

$$[E_{0i}, E_{d+1,0}] = -E_{d+1,i}$$

and so

$$\{a_{0i}, a_{d+1,0}\} = -a_{d+1,i}$$
$$\{a_{0i}, 1\} = -q_i$$
$$-\frac{\partial a_{0i}}{\partial z} = -q_i$$

We have

$$[E_{0i}, E_{i0}] = E_{00} - E_{ii}$$

and so

$$\{a_{0i}, a_{i0}\} = a_{00} - a_{ii}$$
$$\{a_{0i}, p_i\} = z - \sum_{l=1}^{d} p_l q_l - p_i q_i$$
$$\frac{\partial a_{0i}}{\partial q_i} = z - \sum_{l=1}^{d} p_l q_l - p_i q_i$$

For $1 \leq j \leq d$, $j \neq i$ we have

$$[E_{0i}, E_{j0}] = -E_{ji}$$

and so

$$\{a_{0i}, a_{j0}\} = -a_{ji}$$
$$\{a_{0i}, p_j\} = p_j q_i$$
$$\frac{\partial a_{0i}}{\partial q_j} = -p_j q_i$$

We have

$$[E_{0i}, E_{d+1,j}] = 0$$

and so

$$\{a_{0i}, a_{d+1,j}\} = 0 \{a_{0i}, q_j\} = 0$$
$$-q_j \frac{\partial a_{0i}}{\partial z} - \frac{\partial a_{0i}}{\partial p_j} = 0$$

and so

$$\frac{\partial a_{0i}}{\partial p_j} = -q_i q_j$$

Here is the complete list of partial derivatives of a_{0i} :

$$\begin{aligned}
\frac{\partial a_{0i}}{\partial z} &= q_i \\
\frac{\partial a_{0i}}{\partial q_i} &= z - \sum_{l=1}^{n} p_l q_l - p_i q_i \\
\frac{\partial a_{0i}}{\partial q_j} &= -p_j q_i \quad \text{for } j \neq i \\
\frac{\partial a_{0i}}{\partial p_j} &= -q_i q_j
\end{aligned}$$

We conclude

$$a_{0i} = (z - \sum p_l q_l) q_i \tag{1.87}$$

Determination of $a_{i,n+1}$ for $1 \leq i \leq n$:

We have

$$[E_{i,d+1}, E_{d+1,0}] = E_{i,0}$$

and so

$$\begin{aligned}
\{a_{i,d+1}, a_{d+1,0}\} &= -a_{i0} \\
\{a_{i,d+1}, 1\} &= -p_i \\
-\frac{\partial a_{i,d+1}}{\partial z} &= p_i
\end{aligned}$$

We have

$$[E_{i,d+1}, E_{j,0}] = 0$$

and so

$$\begin{aligned}
\{a_{i,d+1}, a_{j,0}\} &= 0 \\
\{a_{i,d+1}, p_j\} &= 0 \\
\frac{\partial a_{i,d+1}}{\partial q_j} &= 0
\end{aligned}$$

We have

$$[E_{i,d+1}, E_{d+1,i}] = E_{ii} - E_{d+1,d+1}$$

and so

$$\{a_{i,d+1}, a_{d+1,i}\} = a_{ii} - a_{d+1,d+1}$$
$$\{a_{i,d+1}, q_i\} = p_i q_i + z$$
$$-q_i \frac{\partial a_{i,d+1}}{\partial z} - \frac{\partial a_{i,d+1}}{\partial p_i} = p_i q_i + z$$

and so

$$\frac{\partial a_{i,d+1}}{\partial p_i} = -z$$

For $1 \leq j \leq d$, $j \neq i$ we have

$$[E_{i,d+1}, E_{d+1,j}] = E_{ij}$$

and so

$$\{a_{i,d+1}, a_{d+1,j}\} = a_{ij}$$
$$\{a_{i,d+1}, q_j\} = p_i q_j$$
$$-q_j \frac{\partial a_{i,d+1}}{\partial z} - \frac{\partial a_{i,n+1}}{\partial p_j} = p_i q_j$$

and so

$$\frac{\partial a_{i,d+1}}{\partial p_j} = 0$$

Here is the list of the partial derivatives of $a_{i,d+1}$

$$\begin{aligned}
\frac{\partial a_{i,d+1}}{\partial z} &= -p_i \\
\frac{\partial a_{i,d+1}}{\partial q_j} &= 0 \\
\frac{\partial a_{i,d+1}}{\partial p_i} &= -z \\
\frac{\partial a_{i,d+1}}{\partial p_j} &= 0 \quad \text{for } j \neq i
\end{aligned}$$

It follows that

$$a_{i,d+1} = -p_i z \tag{1.88}$$

Determination of $a_{0,n+1}$:

We have

$$[E_{0,d+1}, E_{d+1,0}] = E_{00} - E_{d+1,d+1}$$

and so

$$\begin{aligned} \{a_{0,d+1}, a_{d+1,0}\} &= a_{00} - a_{d+1,d+1} \\ \{a_{0,d+1}, 1\} &= 2z - \sum p_l q_l \\ -\frac{\partial a_{0,n+1}}{\partial z} &= 2z - \sum p_l q_l \end{aligned}$$

We have

$$[E_{0,d+1}, E_{i,0}] = -E_{i,d+1}$$

and so

$$\begin{aligned} \{a_{0,d+1}, a_{i,0}\} &= -a_{i,d+1} \\ \{a_{0,d+1}, p_i\} &= p_i z \\ \frac{\partial a_{0,d+1}}{\partial q_i} &= p_i z \end{aligned}$$

We have

$$[E_{0,d+1}, E_{d+1,i}] = -E_{0,i}$$

and so

$$\begin{aligned} \{a_{0,d+1}, a_{d+1,i}\} &= a_{0i} \\ \{a_{0,d+1}, q_i\} &= (z - \sum p_l q_l) q_i \\ -q_i \frac{\partial a_{0,d+1}}{\partial z} - \frac{\partial a_{0,d+1}}{\partial p_i} &= (z - \sum p_l q_l) q_i \end{aligned}$$

and so

$$\frac{\partial a_{0,d+1}}{\partial p_i} = q_i z$$

The list of all the partial derivatives of $a_{0,n+1}$ is

$$\begin{aligned} \frac{\partial a_{0,d+1}}{\partial z} &= -2z + \sum p_l q_l \\ \frac{\partial a_{0,d+1}}{\partial p_i} &= q_i z \\ \frac{\partial a_{0,d+1}}{\partial q_i} &= p_i z \end{aligned}$$

We conclude that

$$a_{0,n+1} = -z(z - \sum p_l q_l) \tag{1.89}$$

Therefore the element X_β with $\beta = e_0 - e_{d+1}$, that is $E_{0,d+1}$ maps to $a_{0,n+1} = -z(z - \sum_{i=1}^{d} p_i q_i)$ in $\mathcal{L}_d$.

We have $z(z - \sum_{i=1}^{d} p_l q_l) = -(z - \frac{1}{2}\sum_{i=1}^{d} p_i)^2 + \frac{1}{4}(\sum_{i=1}^{d} p_i q_i)^2$. Therefore in the case of the simple Lie algebra of type A the polynomial of degree 4 is (up to a constant) $(\sum_{i=1}^{d} p_i q_i)^2$.

Note that

- $E_{ij} \mapsto \alpha_i \beta_j$ where

$$\begin{aligned} \alpha_0 &= z - \sum p_i q_i \\ \alpha_i &= p_i \\ \alpha_{d+1} &= 1 \\ \beta_0 &= 1 \\ \beta_j &= q_j \\ \beta_{d+1} &= -z \end{aligned}$$

- $E_{0,0} + E_{1,1} + \ldots E_{n+1,n+1} \mapsto 0$

An explicit form of the map of Lie algebras $gl(d+2, \mathbb{C}) \to \mathcal{L}_d$ is: Let (c_{ij}) an element in $gl(d+2, \mathbb{C})$. Then

$$(c_{ij}) \mapsto (z - \sum p_i q_i, p_1, \ldots, p_d, 1) \cdot (c_{ij}) \cdot (1, q_1, \ldots, q_d, -z)^t \tag{1.90}$$

The following table gives the mapping $gl(4, \mathbb{C}) \to \mathcal{L}_2$

The description of the map is: $E_{ij} \mapsto$ the element (i, j) of the matrix.

Table 1.2: The mapping of $gl(4,\mathbb{C})$

$z-\sum p_iq_i$	$(z-\sum p_iq_i)q_1$	$(z-\sum p_iq_i)q_2$	$-z(z-\sum p_iq_i)$
p_1	p_1q_1	p_1q_2	$-p_1z$
p_2	p_2q_1	p_2q_2	$-p_2z$
1	q_1	q_2	$-z$

2 Contact structures

2.1 Definition, Examples

2.1.1 Contact Forms

All manifolds can be real or complex. Let M a manifold of odd dimension $2n+1$. A 1-form α defined on an open subset of M is called a *contact form* if

$$\alpha \wedge (d\alpha)^n \neq 0 \tag{2.1}$$

(it has no zeroes). Example : $M = \mathbf{R}^{2n+1}$ with coordinates $(z, q_1, \dots q_n)$.Take the 1-form α

$$\alpha = dz - \sum_{i=1}^{n} p_i dq_i \tag{2.2}$$

We have

$$d\alpha = dz + \sum_{i=1}^{n} dq_i \wedge dp_i \tag{2.3}$$

We get

$$\alpha \wedge (d\alpha)^n = n! dz \wedge dq_1 \wedge dp_1 \dots \wedge dq_n \wedge dp_n \tag{2.4}$$

Let $\beta = f \cdot \alpha$. We get $d\beta = df \wedge \alpha + f \cdot d\alpha$. We conclude

$$\beta \wedge (d\beta)^n = f^{n+1} \cdot \alpha \wedge (d\alpha)^n \tag{2.5}$$

Note that the condition (2.25) is equivalent to $d\alpha$ is a nondegenerate skewsymmetric form on $\ker \alpha$ at each point. This follows from the

Lemma 2.1.1. *(Linear Algebra) Let V a vector space (finite dimensional) $\alpha \neq 0$ a 1-form on V and η a m-form on V $(m \geq 1$). Let $V_0 \colon = \{v \in V \mid \alpha(v) = 0\}$. Then $\eta_{|V_0} \neq 0$ if and only if $\alpha \wedge \eta \neq 0$*

1

Proof. Let $e_1^* = \alpha$, $e_2^*, \ldots, e_n^*$ a basis of V^*, $e_1, \ldots, e_n$ the dual basis of V. Then $V_0 = \text{span}\{e_2, \ldots, e_n\}$ and the restrictions of $e_2^*, \ldots, e_m$ are the dual basis of . Write

$$\eta = \sum a_{i_1,\ldots,i_m} e_{i_1}^* \wedge \ldots \wedge e_{i_m}^* \tag{2.6}$$

... □

2.1.2 Definition of Contact Manifolds

Let M an $2n+1$ dimensional manifold. Take for every x in M a codimension one subspace of $T_x(M)$ in a smooth manner. We get a codimension 1 subbundle of the tangent bundle $B \subset T(M)$. Let $L \subset T^*(M)$ the orthogonal complement of B. L is a line bundle over M. We say that B is a *contact structure* on M if all local nonzero sections of L are contact forms. From (2.5) this is equivalent to: B can be given locally by contact forms.

Definition 2.1.2. A contact manifold is a manifold with a contact structure.

Note: It may be impossible to give B globally by a 1-form, that is L may not have a global section without zeros , that is L may not be a trivial bundle. By theorem of About (see [MS04]) every contact manifold is locally isomorphic to $\mathbb{R}$ with the standard contact form (2.2).

2.1.3 The role of the cotangent bundle

Let M a manifold. Consider the cotangent manifold $T^*(M)$. It has a tautological 1-form that in canonical coordinates is given by

$$\theta = \sum p_i dq_i \tag{2.7}$$

How is this constructed? Consider P a point in $T^*(M)$. It projects to a point in M. Now P is (x, ϕ) where ϕ is a linear form on $T_x(M)$. The canonical map $T^*(M) \to M$ induces a map between tangent spaces $T_p(T^*(M)) \to T_x(M)$. Consider the pullback of ϕ under this map. It is a linear map on $T_P(T^*(M))$. We thus get

a tautological 1-form θ on $T^*(M)$. It has the following universality property (a "universal 1-form"). For every 1-form β on M there exists a unique map $\bar{\beta}\colon M \to T^*(M)$ such that β is the pullback of θ by the map $\bar{\beta}$. Indeed, take $\bar{\beta}$ as the section of the bundle $T^*(M) \to M$ given by β (!).

Moreover, $T^*(M)$ has a canonical symplectic form

$$\omega = d\theta = \sum dp_i \wedge dq_i \tag{2.8}$$

In regard to this, what are the Lagrangian submanifolds of $T^*(M)$ that are images of sections (1-forms) $\beta\colon M \to T^*(M)$? The condition is that ω restricts to a zero form. Now β is an imbedding so the condition is equivalent to $\beta^*(\omega) \equiv 0$. But $\omega = d\theta$ so $\beta^*(\omega) = \beta^*(d\theta) = d\beta^*(\theta) = d\beta$ where we now view β as a 1-form. Thus the Lagrangian submanifolds correspond to closed 1-forms.

Let now M is manifold and L a line subbundle of $T^*(M)$. Let $L^\times = L\backslash\{\text{zero section}\}$ the associated $\mathbb{C}^\times$ bundle. The local sections of $L^\times$ define a 1-dimensional subbundle of M. When is this a contact structure on M?

Proposition 2.1.3. *$L^\times$ defines a contact structure on M if and only if $d\theta_{|L^\times}$ is a symplectic form on $L^\times$.*

Proof. Note that $L^\times$ is a principal $\mathbb{C}^\times$ bundle with a 1-form $\theta_{|L^\times}$ (where θ is the canonical 1-form on $T^*(M)$ such that if $t \in \mathbb{C}^\times$ then for the push forward under the diffeomorphism t we have $t_*\theta = t \cdot \theta$. Conversely, assume we have a principal $\mathbb{C}^\times$ bundle B over M and θ a form without zeroes on B such that $t_*\theta = t \cdot \theta$. Then $B \to M$ with θ is isomorphic to $L^\times \to M$ with $\theta_{|L^\times}$ where θ is the canonical 1-form on $T^*(M)$.

Let α a local section of $L^\times \to M$. Then locally $\theta_{|L^\times}$ is $t \cdot \alpha$. We have

$$d(t \cdot \alpha) = dt \wedge \alpha + t \wedge d\alpha$$

Conclude:

$$d(t \cdot \alpha)^{n+1} = nt^n dt \wedge \alpha \wedge (d\alpha)^n$$

Therefore α is a contact form if and only if $d\theta_{|L^\times}$ is a symplectic form ∎

In fact we have the (apparently more general) statement.

Proposition 2.1.4. *Let assume a principal $\mathbb{C}^\times$ bundle B over M and θ a form without zeroes on B such that $t_*\theta = t \cdot \theta$. Then the pullback of θ by sections of $B \to M$ define a contact structure on M if and only if $d\theta$ is a symplectic form on B.*

2.2 Contact Vector Fields and Contact Hamiltonians

2.2.1 The Reeb vector associated to a pair (α, η)

Let V a vector space of odd dimension (I guess the field can have characteristic 2). Let η a 2-form of maximal rank $2n$. The kernel of η is defined as $\{v \in V \mid v \cdot \eta = 0\}$. This turns out to be the kernel of η considered as an alternating form on V. It is a 1-dimensional subspace of V. Let α a 1-form on M such that the following equivalent conditions are satisfied

- $\alpha \wedge \eta^n \neq 0$
- η restricted to the kernel of α is nondegenerate
- α restricted to the kernel of η is nonzero.

There exists thus a unique w in the kernel of η such that $\alpha(w) = 1$. We call such w the *Reeb vector* associated to the pair (α, η). The defining equations of w are

$$w \cdot \alpha = \alpha(w) = 1 \tag{2.9}$$

$$w \cdot \eta = 0 \tag{2.10}$$

Example: If η is as above of maximal rank then there exists $2n$ linearly independent elements in V^* such that

$$\eta = e_2^* \wedge e_3^* + \ldots + e_{2n}^* \wedge e_{2n+1}^* \tag{2.11}$$

Assume that (α, η) are as above. Then α is not in the span of $e_2, \ldots, e_{2n}$. Therefore we can write $\alpha = e_1^*$ such that $e_1^*, e_2^*, \ldots, e_{2n+1}^*$ are a basis of V^*. Consider $e_1, \ldots e_n$ the dual basis in V. Then the kernel of η is generated by e_1. The Reeb vector associated to the pair (α, η) is e_1.

2.2.2 The Reeb vector field associated to a contact form

Let α a contact form on M. Then $(\alpha, d\alpha)$ form a pair as above at each point of M (for the tangent space of M at this point). Consider the vector field Y of Reeb vectors at each point. We get the *Reeb vector field*. The defining equations of Y are

$$Y \cdot \alpha = \alpha(Y) = 1 \tag{2.12}$$

$$Y \cdot d\alpha = 0 \tag{2.13}$$

Let X another vector field. We have

$$d\alpha(X, Y) = X(\alpha(Y)) - Y(\alpha(X)) - \alpha([X, Y]) \tag{2.14}$$

Since Y is the Reeb vector field associated to α we get

$$d\alpha(X, Y) = 0$$

$$X(\alpha(Y)) = 0$$

Conclude

$$Y(\alpha(X)) = \alpha([Y, X]) \tag{2.15}$$

2.2.3 Another definition of the Reeb vector field

Let α a contact (1-)form on M (defined locally). For every function f on there exists a unique function $v(f)$ such that

$$v(f) \cdot \alpha \wedge (d\alpha)^n = df \wedge (d\alpha)^n \tag{2.16}$$

This is because $\alpha \wedge (d\alpha)^n$ is a volume form.

Fact : $f \mapsto v(f)$ is a derivation. Indeed let f, g functions on M. Then

$$\begin{aligned} v(fg) \wedge (d\alpha)^n = d(fg) \wedge (d\alpha)^n = (df \cdot g + dg \cdot f) \wedge (d\alpha)^n = \\ = g \cdot df \wedge (d\alpha)^n + f \cdot dg \wedge (d\alpha)^n = \\ = g \cdot v(f) \cdot \alpha \wedge (d\alpha)^n + f \cdot v(g) \cdot \alpha \wedge (d\alpha)^n = \end{aligned}$$

Conclude

$$v(fg) = v(f) \cdot g + f \cdot v(g)$$

that is , v is a vector field.

We show that v is the Reeb vector field associated to the form α , that is the following is true

$$v \cdot \alpha = 1 \tag{2.17}$$

$$v \cdot d\alpha = 0 \tag{2.18}$$

Indeed, the equality (2.16) can be rewritten as

$$df(v) \cdot \alpha \wedge (d\alpha)^n = df \wedge (d\alpha)^n$$

and implies by linearity

$$\beta(v) \cdot \alpha \wedge (d\alpha)^n = \beta \wedge (d\alpha)^n \tag{2.19}$$

for all 1-forms β. In particular, for $\beta = \alpha$ we get

$$\alpha(v) \cdot \alpha \wedge (d\alpha)^n = \alpha \wedge (d\alpha)^n$$

and so $\alpha(v) = v \cdot \alpha = 1$.

Now $v \cdot d\alpha = 0$ is equivalent to $d\alpha(w, v) = 0$ for all vector fields w, or that the 1-form $d\alpha(w, \cdot)$ is zero on v. Consider thus $\beta = d\alpha(w, \cdot)$ in the equality (2.19). We get

$$\beta(v) \cdot \alpha \wedge (d\alpha)^n = d\alpha(w, v) \cdot \alpha \wedge (d\alpha)^n = d\alpha(w, \cdot) \wedge (d\alpha)^n$$

But the last product is zero since $(d\alpha)^n$ is a top-form on the support subspace of α. Thus $v \cdot d\alpha = 0$

2.2.4 Lie derivatives

Recall the formula for the Lie derivative applied to differential forms

$$\mathcal{L}_X = d \circ \iota_X + \iota_X \circ d \tag{2.20}$$

Apply this for the 1-form α. We get

$$\mathcal{L}_X(\alpha) = d(\alpha(X)) + d\alpha(X, \cdot) \tag{2.21}$$

or, if Z is another vector field

$$\mathcal{L}_X(\alpha)(Z) = d(\alpha(X))(Z) + d\alpha(X, Z) = Z(\alpha(X)) + d\alpha(X, Z) \tag{2.22}$$

Now we also have

$$\mathcal{L}_X(\alpha)(Z) = X(\alpha(Z)) - \alpha([X, Z]) \tag{2.23}$$

Note these two equations above are compatible since by the formula for the exterior derivative

$$d\alpha(X, Z) = X(\alpha(Z)) - Z(\alpha(X)) - \alpha([X, Z]) \tag{2.24}$$

2.2.5 Contact Vector Fields - Definitions

Let M a contact manifold. A vector field X on M is called contact the local 1-parameter group of diffeomorphisms of M determined by X leaves invariant the field of hyperplanes of the contact structures. Let α a contact vector field defining locally the contact structure. Then X is a contact vector field if

$$\mathcal{L}_X(\alpha) = g \cdot \alpha \tag{2.25}$$

for some function g on M.

Example: The Reeb vector associated to α is a contact vector field. Indeed:

$$\mathcal{L}_Y(\alpha) = d\iota_Y(\alpha) + \iota_Y(d\alpha) \tag{2.26}$$

Now

$$\begin{aligned} \iota_Y(\alpha) &= 1 \qquad \text{and} \\ \iota_Y(d\alpha) &= 0 \end{aligned}$$

Conclude

$$\mathcal{L}_Y(\alpha) = 0 \tag{2.27}$$

2.2.6 More Formulas

Let X a contact field, α a contact form and Y the Reeb vector field associated to α. Recall the equation (2.25) and (2.23) applied to $Z := Y$ we get

$$\mathcal{L}_X(\alpha)(Y) = g \cdot \alpha(Y) = g$$
$$\mathcal{L}_X(\alpha)(Y) = X(\alpha(Y)) - \alpha([X,Y]) = -\alpha([X,Y]) = \alpha([Y,X])$$

Conclude : If X is a contact vector field then in the equation (2.25) we have (using also (2.15))

$$g = \alpha([Y,X]) = Y(\alpha(X)) \tag{2.28}$$

This is consistent with the calculation for the contact vector field Y. For this we have $g = 0 = \alpha([Y,Y])$ Thus

$$\mathcal{L}_X(\alpha) = Y(\alpha(X)) \cdot \alpha \tag{2.29}$$

2.2.7 Contact Hamiltonians

Let X a contact vector field as above (2.25). Fix a contact form α. The *contact Hamiltonian* of X (with respect to α) denoted by H is defined by

$$H = \alpha(X) \tag{2.30}$$

We see that

$$\mathcal{L}_X(\alpha) = Y(H) \cdot \alpha \tag{2.31}$$

Now use (2.22)and get

$$X \cdot \alpha = H \tag{2.32}$$
$$X \cdot d\alpha = -dH + Y(H) \cdot \alpha \tag{2.33}$$

Let's examine the above equations. They basically say that

$$\mathcal{L}_X(\alpha) = X \cdot d\alpha + d(X \cdot \alpha) = Y(H) \cdot \alpha \tag{2.34}$$

Now if for a vector field X we have $d(X \cdot \alpha) + X \cdot d\alpha$ a multiple of α then X is a contact vector field and we have the above equations.

Now, given a function H on M there exists a unique contact vector field X such that $H = \alpha(X)$.

Indeed there exists a unique vector field X such that

$$\begin{aligned} X \cdot \alpha &= H \\ X \cdot d\alpha &= -dH + \text{multiple of } \alpha \end{aligned}$$

and this vector field X is a contact vector field. For, let ξ the contact distribution on M. There exists a unique vector field Z in ξ such that

$$Z \cdot d\alpha|_\xi = -dH|\xi \tag{2.35}$$

We have thus

$$\begin{aligned} Z \cdot \alpha &= 0 \\ Z \cdot d\alpha &= -dH + \text{multiple of } \alpha \end{aligned}$$

Recall now that for the Reeb vector field Y we have

$$\begin{aligned} Y \cdot \alpha &= 1 \\ Y \cdot d\alpha &= 0 \end{aligned}$$

We conclude that for the vector field It follows that the unique solution is $X = Z + HY$ (Y the Reeb vector field of α - see (2.12)).

Now if for a vector field X $d(X \cdot \alpha) + X \cdot d\alpha$ is a multiple of α then X is a contact vector field. The multiple of α will be uniquely determined (follows from (2.31)) – it's $Y(H) \cdot \alpha$

2.2.8 Contact Vector Field with a given Hamiltonian

Let M with contact form

$$\alpha = dz - \sum_{i=1}^{n} p_i dq_i \tag{2.36}$$

What is the contact vector field with the Hamiltonian H? We have

$$d\alpha = \sum_{i=1}^{n} dq_i \wedge dp_i$$

The Reeb vector field is

$$Y = \frac{\partial}{\partial z} \tag{2.37}$$

Let's find first, like in the proof above the (unique) vector field Z such that

$$Z \cdot \alpha = 0$$

$$Z \cdot d\alpha = -dH + \text{multiple of } \alpha$$

Let

$$Z = a\frac{\partial}{\partial z} + \sum_{i=1}^{n} b_i \frac{\partial}{\partial p_i} + \sum_{i=1}^{n} c_i \frac{\partial}{\partial q_i}$$

We have

$$Z \cdot \alpha = a - \sum_{i=1}^{n} p_i c_i = 0$$

and

$$Z \cdot d\alpha = -\sum_{i=1}^{n} b_i dq_i + \sum_{i=1}^{n} c_i dp_i$$

Now

$$dH = \frac{\partial H}{\partial z} dz + \sum_{i=1}^{n} \frac{\partial H}{\partial p_i} dp_i + \sum_{i=1}^{n} \frac{\partial H}{\partial q_i} dq_i \tag{2.38}$$

We know already what the multiple of α in the equation above is

$$Z \cdot d\alpha = -dH + Y(H) \cdot \alpha \tag{2.39}$$

or

$$Z \cdot d\alpha = -dH + \frac{\partial H}{\partial z}\alpha \tag{2.40}$$

Conclude

$$-\sum_{i=1}^{n} b_i dq_i + \sum_{i=1}^{n} c_i dp_i = -\left(\frac{\partial H}{\partial z} dz + \sum_{i=1}^{n} \frac{\partial H}{\partial p_i} dp_i + \sum_{i=1}^{n} \frac{\partial H}{\partial q_i} dq_i \right) + \tag{2.41}$$

$$+\frac{\partial H}{\partial z} \cdot \left(dz - \sum_{i=1}^{n} p_i dq_i \right) \tag{2.42}$$

Conclude

$$b_i = p_i \frac{\partial H}{\partial z} + \frac{\partial H}{\partial q_i}$$

$$c_i = -\frac{\partial H}{\partial p_i}$$

$$a = -\sum_{i=1}^{n} p_i \frac{\partial H}{\partial p_i}$$

Now to get the contact vector field add the correcting term $H \cdot Y$. Conclude

$$X = \left(H - \sum_{i=1}^{n} p_i \frac{\partial H}{\partial p_i} \right) \frac{\partial}{\partial z} + \sum_{i=1}^{n} \left(\frac{\partial H}{\partial q_i} + p_i \frac{\partial H}{\partial z} \right) \frac{\partial}{\partial p_i} - \sum_{i=1}^{n} \frac{\partial H}{\partial p_i} \frac{\partial}{\partial q_i}$$

or, if we use short form summation

$$X = (H - p\frac{\partial H}{\partial p})\frac{\partial}{\partial z} + (\frac{\partial H}{\partial q} + p\frac{\partial H}{\partial z})\frac{\partial}{\partial p} - \frac{\partial H}{\partial p}\frac{\partial}{\partial q} \tag{2.43}$$

2.2.9 The Legendre Bracket

Let M a contact manifold. The contact vector fields form a Lie subalgebra of the Lie algebra of vector fields on M. Let α a contact form. Then to every contact vector field X corresponds a function on M $\alpha(X)$. This gives a linear bijection between contact vector fields and functions, where the inverse map is given by the formula (2.43). We get in this way a structure of Lie algebra on the functions on M. To functions H_1, H_2 on M corresponds the function $\{H_1, H_2\} := \alpha([X_1, X_2])$. What is the explicit formula?

First recall some formulas of tensor calculus. Let X_1, X_2 vector fields on M. We have

$$\mathcal{L}_{X_1}\alpha(X_2) = X_1(\alpha(X_2)) - \alpha([X_1, X_2]) \tag{2.44}$$

Put instead of X_2 in the above formula the Reeb vector field associated to the contact form α. We get

$$\mathcal{L}_{X_1}\alpha(Y) = X_1(\alpha(Y)) - \alpha([X_1, Y]) \tag{2.45}$$

Also recall that

$$0 = d\alpha(X_1, Y) = X_1(\alpha(Y)) - Y(\alpha(X_1) - \alpha([X_1, Y]) \tag{2.46}$$

and so

$$\mathcal{L}_{X_1}\alpha(Y) = \alpha([Y, X_1]) = Y(\alpha(X_1)) \tag{2.47}$$

(we also use (2.15)) Now assume that X_1 is a contact vector field. Then since $\alpha(Y) = 1$

$$\mathcal{L}_{X_1}\alpha = \mathcal{L}_{X_1}\alpha(Y)\alpha \tag{2.48}$$

and so

$$\mathcal{L}_{X_1}\alpha(X_2) = Y(\alpha(X_1))\alpha(X_2) \tag{2.49}$$

Now using the equalities above we get

$$X_1(\alpha(X_2)) - \alpha([X_1, X_2]) = Y(\alpha(X_1))\alpha(X_2) \tag{2.50}$$

and so

$$\alpha([X_1, X_2]) = X_1(\alpha(X_2)) - Y(\alpha(X_1))\alpha(X_2) \tag{2.51}$$

Assume now that both X_1, X_2 are contact vector fields with Hamiltonians H_1, H_2. Then the Legendre bracket of H_1, H_2 is the Hamiltonian of $[X_1, X_2]$ that is , $\alpha([X_1, X_2])$. We get

$$\{H_1, H_2\} = X_1(H_2) - Y(H_1) \cdot H_2 \tag{2.52}$$

2.2.10 Legendre bracket for the standard contact structure

Let the contact form be as before

$$\alpha = dz - \sum_{i=1}^{n} p_i dq_i \tag{2.53}$$

. Recall that the Reeb vector field is $Y = \frac{\partial}{\partial z}$ Let H_1 , H_2 functions. The associated contact vector fields are

$$X_i = (H_i - p\frac{\partial H}{\partial p})\frac{\partial}{\partial z} + (\frac{\partial H_i}{\partial q} + p\frac{\partial H_i}{\partial z})\frac{\partial}{\partial p} - \frac{\partial H_i}{\partial p}\frac{\partial}{\partial q} \tag{2.54}$$

for $i = 1, 2$. We get

$$\{H_1, H_2\} = X_1(H_2) - Y(H_1) \cdot H_2 = \tag{2.55}$$
$$= (H_1 - p\frac{\partial H_1}{\partial p})\frac{\partial H_2}{\partial z} + (\frac{\partial H_1}{\partial q} + p\frac{\partial H_1}{\partial z})\frac{\partial H_2}{\partial p} - \frac{\partial H_1}{\partial p}\frac{\partial H_2}{\partial q} - \frac{\partial H_1}{\partial z}H_2$$

We get the important formula

$$\{H_1, H_2\} = (H_1 - p\frac{\partial H_1}{\partial p})\frac{\partial H_2}{\partial z} - (H_2 - p\frac{\partial H_2}{\partial p})\frac{\partial H_1}{\partial z} + \tag{2.56}$$
$$+\frac{\partial H_1}{\partial q}\frac{\partial H_2}{\partial p} - \frac{\partial H_2}{\partial q}\frac{\partial H_1}{\partial p}$$

3 Homogenous Contact Manifolds

3.1 Homogenous contact structures

We show that homogenous contact structures are related to coadjoint orbits.

3.1.1 The setup

Assume we have an invariant contact structure on the homogenous manifold G/P. We get a linear form $\omega \neq 0$ on $\mathfrak{g}$ the Lie algebra of the group G (up to proportionality) such that

- $\omega(\mathfrak{p}) = 0$
- $\mathrm{Ad}(P)(\omega) \subset \mathbb{C}^{\times} \cdot \omega$

It is thus important to study coadjoint orbits.

3.1.2 A result on coadjoint orbits

Let $\mathfrak{g}$ be a Lie algebra. Consider the representation of $\mathfrak{g}$ on the space of linear forms on $\mathfrak{g}$, denoted by $\mathfrak{g}^*$.
Let $\omega \neq 0$ in $\mathfrak{g}^*$. Define the following two subalgebras of $\mathfrak{g}$:

$$\mathfrak{p}^0 = \{X \in \mathfrak{g} \mid ad(X)\omega = 0\} \tag{3.1}$$

$$\mathfrak{p} = \{X \in \mathfrak{g} \mid \ ad(X)\omega \in \mathbb{C}\omega\} \tag{3.2}$$

We have the following

Proposition 3.1.1. *The followind assertions are equivalent:*

1. $\omega_{|\mathfrak{p}} = 0$
2. $\omega_{|\mathfrak{p}^0} = 0$
3. *there exists* Z *in* $\mathfrak{g}$ *such that* $Z \cdot \omega = \omega$
4. $\dim \mathfrak{p} = \dim \mathfrak{p}^0 + 1$

Proof. We have a representation of $\mathfrak{p}$ on $\mathbb{C}$ by $X \mapsto c$ if $ad(X)\omega = c\omega$ with kernel $\mathfrak{p}^0$. Hence $\dim \mathfrak{p} \leq \dim \mathfrak{p}^0 + 1$. Thus 3) and 4) are equivalent. Clearly 1) $\Rightarrow$ 2).

We now prove 3) $\Rightarrow$ 1). Let Z such that $Z \cdot \omega = \omega$. Then $\omega([X, Z]) = \omega(X)$ for all $X \in \mathfrak{g}$. Let now H be such that $H \cdot \omega = 0$. Thus means that $\omega([X, H]) = 0$ for all X. In particular it's true for Z as above. We conclude $\omega(H) = 0$. Thus $\omega_{|\mathfrak{p}^0} = 0$. Since $\mathfrak{p} = \mathfrak{p}^0 \oplus \mathbb{C}Z$ we still need $\omega(Z) = 0$. But by the above $\omega(Z) = \omega([Z, Z]) = \omega(0) = 0$.

2) $\Rightarrow$ 3) We use the followind lemma from linear algebra which is the linear version of Nullstellensatz: Let V a finite dimensional vector space and ϕ_i, ϕ functionals on V such that $\cap \ker \phi_i \subset \ker \phi$. Then there exists i_1, ..., i_n and scalars a_1, ..., a_n such that $\phi = \sum a_{i_l} \phi_{i_l}$. Indeed, by finite dimensionality there exist i_1, ..., i_n such that $\cap_{l=1}^n \ker \phi_{i_l} \subset \ker \phi$. Now consider the linear map from V to k^n, $\Phi = (\phi_{i_1}, \ldots \phi_{i_n})$. We have $\ker \Phi \subset \ker \phi$ and therefore there exists a linear map $T : \text{Image}(\Phi) \to k$ such that $\phi = T \circ \Phi$. Extend T to a linear map from k^n to k, given by $(a_1, \ldots, a_n)$ and the equality is still valid. Thus $\phi = \sum a_{i_l} \phi_{i_l}$

Let now ω in $\mathfrak{g}$ such that : $\omega([X, H] = 0$ for all $X \in \mathfrak{g}$ implies $\omega(H) = 0$. Consider now the family of linear functionals on $\mathfrak{g}$ $\phi_X(\cdot) = \omega([X, \cdot]$ and $\phi(\cdot) = \omega(\cdot)$. We have the conditions of the lemma above and therefore $\phi = \sum a_l \phi_{X_l}$ that is $\omega(Y) = \sum a_l \omega([X_l, Y])$. Take $Z = -(\sum a_l X_l)$ and we have $Z \cdot \omega = \omega$. □

3.1.3 Homogenous contact manifolds- first case

Let ω in $\mathfrak{g}^*$ satisfying the equivalent conditions in Proposition (3.1.1). Consider the adjoint representation of G on $\mathfrak{g}^*$ and the corresponding representation on $\mathbb{P}(\mathfrak{g}^*)$- the projective space associated to $\mathfrak{g}^*$. Let P the stabilizer of $[\omega]$ - an element

of $\mathbb{P}(\mathfrak{g}^*)$ and P^0 the stabilizer of ω. Consider the Lie algebras of these subgroups of G. By the above we have $\dim \mathfrak{p} = \dim \mathfrak{p}^\circ + 1$. Therefore we have a $\mathbb{C}^\times$ bundle

$$P/P^0 \to G/P^0 \to G/P \tag{3.3}$$

Since $\omega_{|\mathfrak{p}^0} = 0$ we get a left invariant form on G/P^0 whose exterior differential is the Kostant-Kirillov form hence nondegenerate (see [Kir04]). From above $\omega_{|\mathfrak{p}} = 0$. The left translations of $\ker(\omega)$ give a contact structure on G/P.

We show that this is the only possibility up to a covering map, that is, if the form ω endows G/L with a contact structure then the Lie algebra of L equals $\mathfrak{p}$.

Let $\mathfrak{l} = \mathrm{Lie}(L)$. We have $\mathfrak{l} \subset \mathfrak{p}$. Let $\mathfrak{l}^0 = \mathfrak{l} \cap \mathfrak{p}^0$. We have $\mathfrak{l}^0 = \mathrm{Lie}(L \cap P^0)$. We distinguish two cases:

Case 1. $\dim \mathfrak{l} = \dim \mathfrak{l}^0 + 1$.

We have the $\mathbb{C}^\times$ principal bundle

$$\mathbb{C}^\times \simeq L/L^0 \to G/L^0 \to G/L \tag{3.4}$$

Now ω gives a contact form on G/L if and only if $d\omega$ gives a symplectic form on G/L^0. But this is equivalent to $\omega([\cdot,\cdot])$ is a nondegenerate form on $\mathfrak{g}/\mathfrak{l}^0$. Therefore $\mathfrak{l}^0 \supset \mathfrak{p}^0$. This implies $\mathfrak{l}^0 = p^0$ and so $\mathfrak{l} = \mathfrak{p}$.

Case 2. $\mathfrak{l}^0 = \mathfrak{l}$.

That means L and L^0 have the same identity connected component. The map $G/L^0 \to G/L$ is a covering map, so locally a diffeomorphism. Hence if ω gives a contact structure on G/L it also gives a contact structure on G/L^0. Since $\dim G - \dim L^0$ odd, $\dim G - \dim P^0$ even, and $L^0 \subset P^0$ we have $\dim \mathfrak{p}^0/\mathfrak{l}^0 \geq 1$. At $\bar{e} \in G/L^0$ $\ker \omega$ contains $\mathfrak{p}^0/\mathfrak{l}^0$ which is in the kernel of $d\omega$. But this contradicts the fact that ω give a contact form on G/L^0.

3.1.4 Homogenous contact manifolds - second case

Assume now ω does not satisfy the equivalent conditions Proposition (3.1.1). Then $\omega(\mathfrak{p}^\circ) \neq 0$. Define

$$\mathfrak{p}^1 = \mathfrak{p}^\circ \cap \ker(\omega) \tag{3.5}$$

Now $\mathfrak{p}^0$ is a subalgebra of $\mathfrak{g}$ so $[\mathfrak{p}^0, \mathfrak{p}^0] \subset \mathfrak{p}^0$. Moreover, by the definition of $\mathfrak{p}^0$ we have $[\mathfrak{p}^0, \mathfrak{g}] \subset \ker \omega$. It follows that $[\mathfrak{p}^0, \mathfrak{p}^0] \subset \mathfrak{p}^1$ so in particular, $\mathfrak{p}^1$ is an ideal of $\mathfrak{p}^0$. Let P^1 the connected subgroup of G with Lie algebra $\mathfrak{p}^1$. Assume moreover that P^1 is closed. We get a fibration

$$P^0/P^1 \to G/P^1 \to G/P^0 \tag{3.6}$$

over the coadjoint orbit of ω. In this case G/P^1 is again a contact homogenous manifold. Indeed, ω gives a globally defined 1-form on G/P^1. It is enough to check the contact form condition at $\bar{e}$. But from (3.5) $\ker d\omega \cap \ker \omega = \mathfrak{p}^1/\mathfrak{p}^1 = 0$.

Note: It is possible that P^1 is not closed.

Conversely, if ω gives a contact structure on G/L then $\mathrm{Lie}(L) = \mathfrak{p}^1$. The proof is similar to the first case.

3.2 Contact structure on nilpotent orbits

Let G a connected reductive Lie group, in our context meaning its Lie algebra $\mathfrak{g}$ is reductive. This is equivalent to the existence of a nondegenerate symmetric bilinear form $(\cdot, \cdot)$ on $\mathfrak{g}$ that is $\mathfrak{g}$-invariant, that is $([X, Y], Z) + (Y, [X, Z]) = 0$ for all X, Y, Z in $\mathfrak{g}$. This form gives a G-equivariant isomorphism $\mathfrak{g} \to \mathfrak{g}^*$. Therefore we have a correspondence between adjoint and coadjoint orbits.

Let X in $\mathfrak{g}$ and $\omega_X = (X, \cdot)$ the corresponding linear form on $\mathfrak{g}$. We see that ω_X satisfies the equivalent conditions of (3.1.1) if and only if there exists Z in $\mathfrak{g}$ such that $[Z, X] = X$. Now we have the following

Lemma 3.2.1. *Let $\mathfrak{g}$ a reductive Lie algebra and X in $\mathfrak{g}$. The following assertions are equivalent*

1. *There exists Z in $\mathfrak{g}$ such that $[Z, X] = X$*
2. *X is nilpotent*

Proof. 1) $\Rightarrow$ 2) is standard linear algebra and 2) $\Rightarrow$ 1) follows from Jacobson-Morozov theorem [CM93]. $\square$

It follows now from (3.1.1) that if G is a connected reductive group and X a nonzero nilpotent element then the orbit of $[X]$ in $\mathbb{P}(\mathfrak{g})$ has a contact structure invariant under G.

3.3 Contact structure on the minimal nilpotent orbit

Let G a complex connected simple Lie group with Lie algebra $\mathfrak{g}$. Thus $\mathfrak{g}$ is a complex simple Lie algebra. Let $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$. Consider the root decomposition of $\mathfrak{g}$ corresponding to the pair $(\mathfrak{g}, \mathfrak{h})$:

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}^{\alpha}$$

with R the root system of $(\mathfrak{g}, \mathfrak{h})$. Let $\mathfrak{b}$ a Borel subalgebra of G. Then $N_G(\mathfrak{b})$ is a Borel subgroup of G. Recall now that a parabolic subgroup of G is a subgroup of G containing a conjugate of B and a standard parabolic subgroup of G is a subgroup of G containing B. The standard parabolic subgroups of G are into $1-1$ correspondence with the Lie subalgebras of $\mathfrak{g} \colon = \mathrm{Lie}(G)$ containing $\mathfrak{b} \colon = \mathrm{Lie}(\mathfrak{b})$ by the Lie correspondence. Every parabolic subgroup P is connected and we have $P = N_G(\mathfrak{p})$, $P = N_G(P)$ where $\mathfrak{p} = Lie(P)$. Moreover, every parabolic subgroup of G is *algebraic*. Note there exists a unique structure of complex algebraic group on G that gives the holomorphic structure on G and G is an affine complex algebraic variety (recall G is a *complex* simple Lie group). Also P is a closed complex algebraic subvariety. Moreover, G/P is a projective algebraic variety and so G/P is compact. One may ask whether if L is a closed Lie subgroup of G such that G/P is compact then is L a parabolic subgroup? The answer is negative. However, it is still true if we assume moreover that L is algebraic. The reason for this is that G/P, being a quasiprojective complex algebraic variety and compact in the classical topology is then *projective*.

Let $\mathfrak{p}$ be he unique Heisenberg parabolic subgalgebra of $\mathfrak{g}$ containing $\mathfrak{b}$ and P its corresponding Lie subgroup of G. Recall that $\mathfrak{p}$ is determined as follows: Let

β the largest root of R corresponding to the ordering R_+. We have thus

$$\mathfrak{p} = \mathfrak{h} \oplus \bigoplus_{\alpha(\beta^\vee)\geq 0} \mathfrak{g}^\alpha \tag{3.7}$$

Note that for α root we have $(\alpha, \beta) \geq 0 \iff [X_\alpha, X_\beta] = 0$. Indeed $\alpha + \beta$ *is not* a root for all α root such that $\alpha(\beta^\vee) \geq 0$ and $\alpha + \beta$ *is* a root or zero for all α root such that $\alpha(\beta^\vee) < 0$. (See the subsection on the Largest Root).

Let $X_\beta \in \mathfrak{g}^\beta$. Let P^0 the centralizer of X_β. that is

$$P^0 = \{\mathfrak{g} \in G \mid Ad(g)(X_\beta) = X_\beta\}$$

The Lie algebra of P^0 is

$$\mathfrak{p}^0 = \{X \in \mathfrak{g} \mid [X, X_\beta] = 0\}$$

We note that

$$\mathfrak{p}^0 = Ker(\beta) \oplus \bigoplus_{\beta^\vee(\alpha)\geq 0} \mathfrak{g}^\alpha \tag{3.8}$$

since it's a standard fact for any root vector that the centralizer in $\mathfrak{g}$ has the form

$$Z_\mathfrak{g}(X_\beta) = Ker(\beta) \oplus \bigoplus_{\alpha+\beta\notin R} \mathfrak{g}^\alpha$$

(think of $[\mathfrak{g}^\beta, \mathfrak{g}^\alpha] = \mathfrak{g}^{\beta+\alpha}$ or 0 so a shift by β in the root system) and hence the conclusion.

Thus $\mathfrak{p}^0$ is a codimension 1 subalgebra of $\mathfrak{p}$, the Heisenberg parabolic. Since P is the largest possible subgroup of G with Lie algebra $\mathfrak{p}$ (in fact there are no others!) then P^0 is a codimension 1 subgroup of P.

Then P^0 is a subgroup of P of codimension 1 and we can see that the quotient group P/P^0 is isomorphic to $\mathbb{C}^\times$. Note that β extends to a homomorphism $\beta\colon \mathfrak{p} \to \mathbb{C}$ with kernel $\mathfrak{p}^0$. The exponential of β gives the isomorphism $P/P^0 \simeq \mathbb{C}^\times$.

Let 1-form ω is given by

$$\omega(X) = (X_\beta, X) \tag{3.9}$$

for $X \in \mathfrak{g}$, where $(,)$ is an invariant bilinear form on $\mathfrak{g}$ (for example the Killing form).

It follows that P is the stabilizer of $\bar{\omega}$ in $\mathbb{P}(\mathfrak{g})$ and P^0 is the stabilizer of ω. This situates us in the first case of homogenous contact manifolds. Thus ω determines a homogenous contact structure on G/P. Note that G/P is a compact and simply connected manifold. The compactness follows from the fact that P is a parabolic subgroup. For simply connectedness, note that the orbit in $\mathbb{P}(\mathfrak{g})$ does not depend on the connected simple Lie group G with Lie algebra $\mathfrak{g}$. Hence, we can consider for G the simply connected one. Now P as a parabolic subgroup is connected. It follows that G/P is simply connected.

3.4 Boothby's Theorem

Homogenous manifolds are manifolds(real or complex) on which a Lie group acts transitively. They are of the form G/H where G is a Lie group and H is a closed subgroup.

The following theorem is due to Boothby ([Boo61]) and is a converse to the result about minimal nilpotent orbits. We will give a proof of this result in (3.4.2) using some results on homogenous contact manifolds.

Theorem 3.4.1. *Let M a compact complex simply connected homogenous contact manifold. Then M is isomorphic with the projectivized adjoint orbit of a complex connected simple Lie group G. Thus, M is given as G/P where P is a Heisenberg parabolic subgroup of G.*

Moreover, the complex Lie group G is uniquely determined up to local isomorphism. Indeed, Wolf showed [Wol65] that the connected component of the group of all contact automorphism of M is G_{ad}- the adjoint group of G. Hence, we have a $1-1$ correspondence between compact complex simply connected contact manifolds and complex simple Lie algebras.

3.4.1 Results on homogenous manifolds

We have an easy lemma

Lemma 3.4.2. *Let G be a Lie group and let H be a closed subgroup such that the homogenous space G/H is connected. Then the identity component of G acts transitively on G/H.*

Proof. We have to show that $G^0 \cdot H = G$, that is, in each coset of G^0 in G there is an element of H, or equivalently, H intersects every connected component of G. Let g an element of G. In the space G/H there exists a path $\bar{\sigma}$ from $\bar{g}$ to $\bar{e}$. The fibration $G \to G/H$ has the homotopy lifting property (see [Ste99]). Hence there exists a lift σ of $\bar{\sigma}$ from g to some h in H

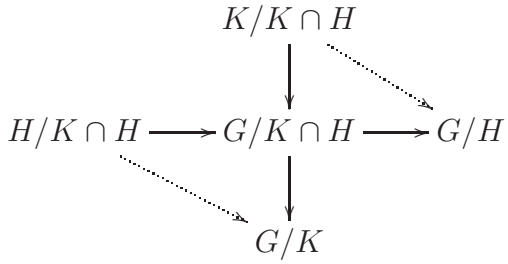
□

Another result of the same kind is due to Montgomery [Mon50].

Lemma 3.4.3. *Let G a connected Lie group and G/H a compact homogenous space. Then a maximal compact subgroup of G acts transitively on G/H.*

Proof. All the maximal compact subgroups of G are conjugate (see [Hel01]). So it's enough to prove the result for a particular one. (the argument : K is transitive on G/H iff $KH = G$. But this is equivalent to $gKg^{-1}H = G$ or $Kg^{-1}H = G$. Now write $g^{-1} = kh$ and we are left with $KkhH = G$ which is true – may also think in terms of homogenous spaces - different points correspond to conjugate subgroups)

Now, take L a maximal compact subgroup of H. Let K a maximal compact subgroup containing L . Then we have $L = K \cap H$. Consider the diagram



$K/K \cap H$ embeds into the connected manifold G/H. To show that the map is surjective (and so a diffeomorphism) it's enough to show the two manifolds have the same dimension.

Let $d = \dim G/H$. The top dimensional homology group $H_d(G/H, \mathbb{Z}/2\mathbb{Z})$ is nonzero. since G/H is compact .

The horizontal line in the diagram above is a fiber bundle with fiber diffeomorphic to $\mathbb{R}^m$ for some m (see [Hel01]). From the general theory (see [Ste99]) this fiber bundle has a section that is a retract of $G/K \cap H$. We conclude that $H_*(G/H, \mathbb{Z}/2\mathbb{Z})$ injects into $H_*(G/K \cap H, \mathbb{Z}/2\mathbb{Z})$.

The vertical line in the diagram above is a fiber bundle with contractible base . It follows that the fiber bundle is trivial and so $G/K \cap H$ is diffeomorphic to $G/K \times K/K \cap H$.

We conclude that $H_*(G/K \cap H, \mathbb{Z}/2\mathbb{Z}) = H_*(K/K \cap H, \mathbb{Z}/2\mathbb{Z})$.

From the above it follows that $H_d(K/K \cap H, \mathbb{Z}/2\mathbb{Z}) \neq 0$ and so $\dim K/K \cap H \geq d$. □

The following two lemmas are due to Wang [Wan54].

Lemma 3.4.4. *Let K a compact connected Lie group and K/L a compact homogenous space with first homotopy group $\pi_1(K/L)$ finite. Then $[K, K]$ the maximal connected semisimple subgroup of K acts transitively on K/L.*

Proof. Let J be the maximal connected semisimple subgroup of G. Then J is a normal subgroup and K/J is a torus. It follows that JL is a closed subgroup of K and K/JL is a torus. We have to show that K/JL is a trivial group in fact. Consider the fiber bundle

$$JL/L \to K/L \to K/JL$$

JL/L is connected as a quotient of J. It follows that we have a surjective map

$$\pi_1(K/L) \to \pi_1(K/JL)$$

Hence $\pi_1(K/JL)$ is also finite. But K/JL is a torus so a free group of rank $\dim(K/JL)$. It follows that K/JL is the trivial group □

We have a generalization of this result

Lemma 3.4.5. *Let G a connected Lie group and G/H a compact homogenous space with first homotopy group $\pi_1(G/H)$ finite. Then a maximal connected semisimple subgroup of G acts transitively on G/H.*

Proof. A maximal connected semisimple subgroup of G is determined by the semisimple part of a Levi decomposition of the Lie algebra $\mathfrak{g}$ of G. Any two such subgroups are conjugate so it's enough to show the result for any of them. Let first K a maximal compact subgroup of G. By lemma (3.4.3) K acts transitively on G/H. Since $\pi_1(G/H)$ finite by lemma (3.4.4) the maximal semisimple subgroup of K again acts transitively on G/H. This subgroup is contained in a maximal semisimple subgroup of G, which therefore acts transitively on G/H. □

3.4.2 Proof of Boothby's result

Let M a compact complex simply connected contact manifold. By a result of Bochner and Montgomery (see [Kob95]) the group of analytic diffeomorphisms of M is a *complex* Lie group (the compactness of M is essential). The subgroup leaving invariant the contact structure can be given by complex equations and is thus a *complex* Lie subgroup of the group of diffeomorphisms of M. Thus we can write $M = G/P$ where G is a complex Lie group and G acts by contact diffeomorphisms on M. Moreover, by the results above we may assume that G is connected and semisimple. Now we can apply the results of (3.1). Let $\omega \neq 0$ in $\mathfrak{g}^*$ endowing G/P with the invariant contact structure . We show that we are not in case considered in (3.1.4)) . Indeed, assume the contrary. Then we have a contact structure on G/P^1. Now G/P being simply connected and $\mathrm{Lie}(P) = \mathrm{Lie}(P^1)$ it follows that G/P is a cover of G/P^1. It follows that G/P^1 is compact. Now we have a bundle $G/P^1 \to G/P^0$ where G/P^0 is the orbit of ω in $\mathfrak{g}^*$. But this implies the orbit of ω is again compact and thus $\omega = 0$, contradiction. We observe

Lemma 3.4.6. *Let $\mathfrak{g}$ be a complex semisimple Lie algebra and X in $\mathfrak{g}$ such that the orbit of X in $\mathfrak{g}$ under the adjoint action is compact. Then $X = 0$.*

Proof. Consider G a complex connected Lie group with Lie algebra $\mathfrak{g}$. Then G has a natural complex algebraic group structure and the adjoint action of G on g is algebraic (see [OV90]). It follows that the orbit of X is a quasiaffine subvariety of $\mathfrak{g}$ (see [Bor91]). Being also compact it is necessarily projective (see [Mum95]). But a connected projective subvariety of $\mathfrak{g}$ consists of one point. Therefore $X = 0$ □

Therefore we are in the first case (see (3.1.3)). G/P is a cover of a projectivized (co)adjoint orbit of ω. Moreover, $\omega = (X, \cdot)$ for some X nonzero nilpotent (correspondence between coadjoint and adjoint orbits). Thus X is nilpotent and its projectivized orbit is compact. It follows that X is in the minimal nilpotent orbit. Now the minimal nilpotent orbit is itself simply connected because it is the quotient of a simply connected group by a parabolic subgroup which is connected. Therefore $M = G/P$ is the minimal projectivized nilpotent orbit of a complex simple Lie algebra.

3.4.3 A result of Wolf

The following result of Wolf ([Wol65] 2.5) recovers $\mathfrak{g}$ from the its minimal projectivized adjoint orbit as a complex contact manifold.

Theorem 3.4.7. *Let G a connected simple complex Lie group, P a Heisenberg parabolic. Consider the simply connected compact complex contact manifold G/P. Then the connected component of the group of automorphisms of G/P is G_{ad}, the adjoint group of G.*

For the proof we will need several lemmas.

Define a parabolic subgroup of a complex Lie group any subgroup containing a maximal connected solvable subgroup (Borel subgroup). These are defined not only for reductive complex groups but for general complex Lie groups. Now a quotient by a parabolic subgroup is compact. We have the partial converse due to Tits (see [Tit63]):

Proposition 3.4.8. *Let A a complex Lie group, E a closed complex Lie subgroup such that A/E is compact. Then the normalizer $N(E_0)$ of the unit connected component E_0 of E in A is a parabolic subgroup, that is, it contains a Borel subgroup.*

Proof. Note that E normalizes E_0 so we have $E_0 \subset E \subset N(E_0)$. Let $\mathfrak{a}$, $\mathfrak{e}$ the Lie algebras of A, E. Let $e = \dim E = \dim \mathfrak{e}$. Consider the action of A on $\mathbb{P}(\wedge^e(\mathfrak{a}))$ coming from the adjoint action of A on $\mathfrak{a}$. Let $p = [\wedge^e(\mathfrak{e})]$ the point corresponding to the subspace $\mathfrak{e}$ of $\mathfrak{a}$. The stabilizer of p is $N(E_0)$. We therefore have a map

of analytic manifolds $A/N(E_0) \to \wedge^e(\mathfrak{a})$. Since the source is a compact manifold by a theorem of Remmert (see [Whi72]) the image - that is $\mathcal{O}$, the orbit of p - is a compact analytic subvariety of $\mathbb{P}(\wedge^e(\mathfrak{a}))$. Moreover, since A acts transitively the orbit of p has no singular points. Now using a theorem of Chow ([Cho49]) we conclude that $\mathcal{O}$ is a projective subvariety of $\mathbb{P}(\wedge^e(\mathfrak{a}))$.

Let now B a connected solvable subgroup of A (for example a Borel subgroup). The image B_1 of B in $GL(\mathfrak{a})$ under the adjoint action may not an algebraic subgroup but B_2 the Zariski closure of the image will be again a solvable connected algebraic subgroup of $GL(\mathfrak{a})$. Moreover, B_1 leaves invariant the Zariski closed subset $\mathcal{O} \subset \mathbb{P}(\wedge^e(\mathfrak{a}))$. It follows that B_2 also leaves invariant $\mathcal{O}$. It follows that B_2 is a solvable connected algebraic group acting on the projective variety $\mathcal{O}$ and so it has a fixed point $a \cdot p$ (see [Bor91]). We conclude that B is contained in a conjugate $aN(E_0)a^{-1}$ of $N(E_0)$. Therefore, $N(E_0)$ contains a conjugate of B and so it is a parabolic subgroup. □

We note the following well known fact.

Lemma 3.4.9. *Let G a complex connected semisimple Lie group and V a finite dimensional representation of G. Let v in $V\backslash\{0\}$ such that the orbit $G[v]$ is closed in $\mathbb{P}(V)$. Then v is a highest weight vector and thus $Span(Gv)$ is an irreducible subrepresentation of V.*

Proof. G has a natural complex algebraic group structure of and the action $G \times \mathbb{P}(V) \to \mathbb{P}(V)$ is algebraic. It follows that the orbit $G[v]$ is a quasiprojective subvariety of $\mathbb{P}(V)$. and being closed in the standard topology it follows that $G[v]$ is a Zariski closed subset of $\mathbb{P}(V)$ and so a projective variety. Let B a Borel subgroup of G. The action of B on the projective variety $G[v]$ has a fixed point gv by Borel's fixed point theorem (see [Bor91]). Thus $g^{-1}Bg$ leaves invariant $[v]$ and so v is a highest weight vector for the Borel subgroup $g^{-1}Bg$. It follows now (see [Ser01]) that $Span(Gv)$ is an irreducible subrepresentation of V. □

The following lemma and its proof was suggested by Wallach. Let G a complex semisimple Lie group. Recall ([FH91] p. 388) that for every parabolic subgroup

of G there exist a finite dimensional irreducible representation V of G with highest weight vector v such that the stabilizer of $[v]$ in $\mathbb{P}(V)$ is P. We have

Lemma 3.4.10. *Let G a complex semisimple Lie group, V an irreducible representation of G with highest weight vector v and P the parabolic subgroup of G the stabilizer of $[v]$ in $\mathbb{P}(V)$. Let G' a semisimple subgroup of G such that G' acts transitively on the homogenous space G/P. Then the restriction of the representation V to G' is irreducible.*

Proof. Consider V as a representation of G'. The orbit of $[v]$ in $\mathbb{P}(V)$ is $G'[v] = G[v] = G/P$ and therefore closed. From (3.4.9) it follows that v is a highest weight vector for V as representation of V' and therefore $Span(G'v)$ is an irreducible representation of G'. But $G'v = Gv$ and $Span(Gv) = V$ since V is an irreducible representation of G. It follows that V is an irreducible representation of G'. □

We have now the following consequence

Proposition 3.4.11. *Let G a complex connected simple Lie group and P a Heisenberg parabolic. Let G' a semisimple subgroup of G such that G' act transitively on G/P. Then $G' = G$*

Proof. Consider the adjoint representation of G on $\mathfrak{g}$. Now P is the stabilizer of $[X]$ in $\mathbb{P}(\mathfrak{g})$ where X is a highest weight vector in $\mathfrak{g}$, that is a minimal nilpotent. Using the lemma (3.4.10) we conclude that $\mathfrak{g}$ is an irreducible representation of G'. But $\mathfrak{g}'$ is a nonzero subrepresentation of $\mathfrak{g}$. We conclude that $\mathfrak{g}' = \mathfrak{g}$ and so $G' = G$. □

We have the following lemma concerning groups of automorphisms of flag varieties. Note that for G complex connected semisimple Lie group, P a parabolic subgroup, the group $\mathrm{Aut}(G/P)$ of automorphisms of G/P as a complex variety is a complex Lie group that naturally contains G_{ad} – the adjoint group of G .

Lemma 3.4.12. *Let G a complex connected semisimple Lie group, P a parabolic subgroup of G and A a connected Lie subgroup of $Aut(G/P)$ containing G_{ad}. Then A is semisimple.*

Proof. We may assume $G = G_{ad}$ – a connected group of adjoint type, by replacing G with G_{ad} and P by its image in G_{ad}. Then G acts faithfully on G/P and so we have $G \hookrightarrow A$. We have $G/P = A/E$ where E is the stabilizer of the point $\bar{e}$ and $P = G \cap E$. Since $G/P = A/E$ is simply connected we conclude that E is connected. Let $N = N(E)$ the normalizer of E in A. We have $A/N = G/(N \cap G)$. Now $N \cap G$ normalizes E and so $E \cap G = P$. However P is a parabolic subgroup and so its own normalizer. We conclude $N \cap G = P$. But then $A/N = G/P = A/E$ and so $N = E$. Let now R the solvable radical of A (the largest connected solvable normal subgroup of A). From the result of Tits above (3.4.8) it follows that a conjugate of R is contained in N, and so, R being normal, R is contained in $N = E$. But then R acts trivially on A/E. Since A is a group of transformations of $M = A/E$ we conclude $R = \{1\}$. Therefore A is semisimple. □

We can now give the proof of (3.4.7).

Proof. Again we may assume that G is of adjoint type, that is $G = G_{ad}$. Like in the proof of the theorem of Boothby in (3.4.2) the group of automorphisms of G/P as a complex contact manifold is a complex Lie group, denoted $A(M)$. Let $A\colon = A_0(M)$ the connected component of $A(M)$. We have $G \subset A$. From (3.4.12) it follows that A is semisimple. Let E the stabilizer of $\bar{e}$ in A. Since $A/E = G/P$ is a compact homogenous contact manifold it follows (see (3.1.3)) that A is simple and E is a Heisenberg parabolic. Since the semisimple subgroup G that acts transitively on A/E we conclude using (3.4.11) that $G = A$. □

Note: One can show that in general if G is a complex simple Lie group of adjoint type and P is a parabolic subgroup then the group of automorphisms of G/P as a *complex manifold* equals G in "most cases". For a list of exceptions, see [Tit63], [Dem77]. A remarkable exception is the minimal projectivized nilpotent orbit of the group $PSp(2n, \mathbb{C})$ which is diffeomorphic to $\mathbb{P}^{2n-1}(\mathbb{C})$. The group of automorphism as a complex contact variety is $PSp(2n, \mathbb{C})$ by the above, but the group of automorphisms as a complex variety is $PSL(2n, \mathbb{C})$.

3.5 The role of the Heisenberg Group

3.5.1 Definitions

A reference for Heisenberg groups and algebras is [Kir04].

A Heisenberg Lie algebra is a complex finite dimensional Lie algebra $\mathfrak{n}$ such that $[\mathfrak{n},[\mathfrak{n},\mathfrak{n}]]=0$ and the center of $\mathfrak{n}$ is 1-dimensional. It follows that the skew-symmetric bilinear map $\mathfrak{n}\times\mathfrak{n}\to[\mathfrak{n},\mathfrak{n}]$ is nondegenerate. All Heisenberg Lie algebras of the same dimension are isomorphic. We can get a Heisenberg Lie algebra as follows: Let V a complex vector space of even dimension and $B\colon V\times V\to\mathbb{C}$ a bilinear skewsymmetric nondegenerate form. Then $\mathfrak{n}=\mathbb{C}\oplus V$ with the bracket $[(a,v),(a',v')]=(B(v,v'),0)$ is a Heisenberg Lie algebra.

A Heisenberg group over $\mathbb{C}$ is a complex connected and simply connected Lie group H with its Lie algebra $Lie(H)$ a Heisenberg algebra. All Heisenberg Lie groups of the same dimension are isomorphic. We can get a Heisenberg Lie algebra as follows: Let V a complex vector space of even dimension and $B\colon V\times V\to\mathbb{C}$ a bilinear form such that the skewsymmetric form $(v,v')\mapsto B(v,v')-B(v',v)$ is nondegenerate. Then $H=\mathbb{C}\oplus V$ with multiplication $[(a,v)\cdot(a',v')]=(a+a'+B(v,v'),v+v')$ is a Heisenberg Lie group.

Let H a Heisenberg Lie group and $\mathfrak{n}\colon=Lie(H)$ its Lie algebra. The exponential map $\exp\colon\mathfrak{n}\to H$ is a diffeomorphism. The multiplication in logarithmic coordinates is given by

$$\exp(X)\cdot\exp(X')=\exp(X+X'+\frac{1}{2}[X,X']) \tag{3.10}$$

This also follows from the Campbell-Hausdorff formula using $[\mathfrak{n},[\mathfrak{n},\mathfrak{n}]]=0$

3.5.2 The Heisenberg group as a group of matrices

Recall that if G is a complex simple Lie group there exists a unique up to conjugacy parabolic subgroup of G such that its unipotent radical is a Heisenberg Lie group (existence and uniqueness of the Heisenberg parabolic).

In the case $G = SL(n, \mathbb{C})$ we get the unipotent radical of a Heinseberg parabolic (the opposite of the standar one) H the group of matrices of form

$$x = \begin{pmatrix} 1 & & & & \\ q_1 & 1 & & & \\ \vdots & & \ddots & & \\ q_n & & & 1 & \\ z & p_1 & \dots & p_n & 1 \end{pmatrix} \tag{3.11}$$

Denote this also by

$$x = (p_1, \dots, p_n, q_1, \dots, q_n, z) \tag{3.12}$$

If we have also

$$x' = (p'_1, \dots, p'_n, q'_1, \dots, q'_n, z) \tag{3.13}$$

then

$$x \cdot x' = x'' = (p''_1, \dots, p''_n, q''_1, \dots, q''_n, z'') \tag{3.14}$$

where

$$\begin{aligned} p''_i &= p_i + p'_i \\ q''_i &= q_i + q'_i \\ z'' &= z + z' + \sum_j p_j \cdot q'_j \end{aligned}$$

Let $\mathfrak{n}$ be the Lie algebra of the Lie group H. It is a Heisenberg Lie algebra and it consists of matrices of form

$$\mathbf{X} = \begin{pmatrix} 0 & & & & \\ q_1 & 0 & & & \\ \vdots & & \ddots & & \\ q_n & & & 0 & \\ z & p_1 & \dots & p_n & 0 \end{pmatrix} \tag{3.15}$$

Denote this also by

$$X = \{p_1, \dots, p_n, q_1, \dots, q_n, z\} \tag{3.16}$$

The bracket in the Lie algebra $\mathfrak{n}$ is

$$[X, X'] = X'' \tag{3.17}$$

where

$$X'' = \{0, \ldots, 0, \sum_{i=1}^{n} (p_i q_i' - p_i' q_i)\} \tag{3.18}$$

3.5.3 The Canonical 1-form

Consider the imbedding of H into $G = SL(n, \mathbb{C})$. The (left invariant) Maurer-Cartan form is $\omega = X^{-1} \cdot dX$. Let P the Heisenberg parabolic subgroup of $SL(n, \mathbb{C})$, the stabilizer of $[E_{1n}]$ in $\mathbb{P}(sl(n, \mathbb{C})$. The contact 1-form on G/P is $\phi \circ \omega$ where $\phi \colon sl(n, \mathbb{C}) \to \mathbb{C}$, $A \overset{\phi}{\mapsto} Tr(E_{1n} \cdot A)$. Consider the restriction of this $1-form$ to $\mathfrak{h} = Lie(H)$. Let x a generic element of H as in (3.11). We have the Maurer-Cartan form on H

$$\mathbf{X} = \begin{pmatrix} 1 & & & & \\ -q_1 & 1 & & & \\ \vdots & & \ddots & & \\ -q_n & & & 1 & \\ -z + \sum p_i q_i & -p_1 & \ldots & -p_n & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & & & & \\ dq_1 & 0 & & & \\ \vdots & & \ddots & & \\ dq_n & & & 0 & \\ dz & dp_1 & \ldots & dp_n & 0 \end{pmatrix} \cdot$$

We get the contact 1-form on H

$$\alpha = dz - \sum p_i dq_i \tag{3.19}$$

3.5.4 Calculations in logarithmic coordinates

The pullback of the contact form α from H to h is the contact 1-form on h given in coordinates p_i, q_j, z by

$$\beta = d(z + \frac{1}{2} \sum_{i=1}^{n} p_i q_i) - \sum p_i dq_i = dz - \frac{1}{2} \sum (p_i dq_i - q_i dp_i) \tag{3.20}$$

3.5.5 Right Invariant Vector Fields

This is to recall some standard constructions for right invariant vector fields on subgroups of $GL(n,\mathbb{C})$. Let A in $M(n,\mathbb{C})$. Consider the right invariant vector field

$$(X_A(f))(x) = \frac{d}{dt} f((I+tA)x)_{|t=0} \tag{3.21}$$

(see [GW98])

Let E_{ij} is an elementary matrix. Let $x = \sum_{kl} x_{kl} E_{kl}$. We have

$$x + tE_{ij}x = x + tE_{ij}x_{jl}E_{jl} = x + tE_{il}x_{jl}$$

Conclude:

$$X_{E_{ij}} = \sum_l x_{jl} \frac{\partial}{\partial_{il}} \tag{3.22}$$

3.5.6 The Heisenberg group - big cell in the minimal nilpotent orbit

Let G a complex connected simple Lie group. Let P a Heisenberg parabolic subgroup. We have showed that the complex compact simply connected manifold G/P has a homogenous contact structure. Now consider N_- the nilpotent radical of an opposite parabolic P_- of P. N_- is a Heisenberg group and from the theory of Bruhat decomposition we have an embedding

$$N_- \to G/P \tag{3.23}$$

with the image a (Zariski) dense open subset of G/P. The canonical contact structure on G/P restricts to a left invariant contact structure on N_- that can be given by a 1-form (3.19). The action of G on G/P induces a map of Lie algebras $\Phi\colon \mathfrak{g} \to Vect(G/P)$ where $Vect(G/P)$ is the (infinite-dimensional) Lie algebra of vector fields on G/P. Since G leaves invariant the contact structure on G/P, for every X in $\mathfrak{g}$ the vector field $\Phi(X)$ is a contact vector field on G/P. This restricts to a contact vector field on N_-. Now we have a morphism from the contact vector fields on N_- to functions on N_- given by the contact Hamiltonian.

We thus get a morphism of Lie algebras from $\mathfrak{g}$ to the algebra of functions on N_- with the Legendre bracket. Moreover, the restriction of the map from $\mathfrak{n}_-$ maps to polynomials, in fact to $\mathcal{L}_{<0}$. It follows that we get another realization of the extension $\mathfrak{g} \to \mathcal{L}$.

4 The invariant of degree 4

4.1 A symplectic representation

Let $\mathfrak{g}$ a finite dimensional simple Lie algebra over $\mathbb{C}$. Let $\mathfrak{h}$ be a Cartan subalgebra, R the root system associated to $(\mathfrak{g}, \mathfrak{h})$.
Fix a root order of R. Let β be the largest root corresponding to this ordering (see (1.1)). Let $\beta^\vee$ the coroot associated to β. We have $\beta^\vee \in [\mathfrak{g}^\beta, \mathfrak{g}^{-\beta}] \subset \mathfrak{h}$. Then $ad(\beta^\vee)$ acts semisimply on $\mathfrak{g}$ with eigenvalues in the set $\{-2, -1, 0, 1, 2\}$. Let

$$\mathfrak{g}_i = \{X \in \mathfrak{g} \mid ad(\beta^\vee)(X) = i \cdot X\} \tag{4.1}$$

Since $ad(\beta^\vee)$ is a derivation of $\mathfrak{g}$ we get a grading of the Lie algebra $\mathfrak{g}$

$$\mathfrak{g} = \bigoplus_{i=-2}^{2} \mathfrak{g}_i \tag{4.2}$$

Let $\mathfrak{p} = \oplus_{i \geq 0} \mathfrak{g}_i$. $\mathfrak{p}$ is a parabolic subalgebra with nilpotent radical $\mathfrak{n} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ which is a Heisenberg algebra. Moreover, $\mathfrak{p}$ is the unique parabolic subalgebra of $\mathfrak{g}$ containing $\mathfrak{b}$ and having for the nilpotent radical a Heinseberg algebra.
Now $\mathfrak{g}_0$ is a Levi component of $\mathfrak{p}$. We have

$$\mathfrak{g}_0 = \mathfrak{h} \oplus \bigoplus_{\hat{\beta}(\alpha)=0} \mathfrak{g}^\alpha \tag{4.3}$$

Decompose $\mathfrak{g}_0$ as follows

$$\mathfrak{g}_0 = \mathbb{C}\beta^\vee \oplus \mathfrak{m} \tag{4.4}$$

where

$$\mathfrak{m} = \ker(\beta) \oplus \bigoplus_{\hat{\beta}(\alpha)=0} \mathfrak{g}^\alpha \tag{4.5}$$

We have the following result (see [GW96] p. 76) :

Proposition 4.1.1. *1. The Lie algebra $\mathfrak{m}$ is reductive with Cartan subalgebra $\mathfrak{m} \cap \mathfrak{h} = \ker \beta \subset \mathfrak{h}$. The simple roots of $\mathfrak{m} \cap \mathfrak{h}$ on $\mathfrak{m} \cap \mathfrak{b}$ consists of those α_i in B which are orthogonal to β.*

2. $\mathfrak{m}$ acts trivially on the root spaces $\mathfrak{g}^{\beta}$ and $\mathfrak{g}^{-\beta}$ so acts symplectically on $\mathfrak{g}_{-1}$ and $\mathfrak{g}_1$

The reductive algebra $\mathfrak{m}$ can be determined from the extended Dynkin diagram of $\mathfrak{g}$ ([GW96]). The vertices of the usual Dynkin diagram correspond to the roots in the basis B. The extended vertex corresponds to the lowest root $-\beta$. Vertices of the diagram corresponding to roots α, α' are connected if $\langle \alpha, \alpha' \rangle \neq 0$. The Dynkin diagram of $\mathfrak{m}$ is obtained by removing all the vertices in the subset of B of S which are connected to the extended vertex. Note that S consists of one vertex and $\mathfrak{m}$ is semisimple, except when $\mathfrak{g}$ of type A_l, $l = d + 1 \geq 2$ when $S = \{\alpha_1, \alpha_l\}$ and $\mathfrak{m} = gl_d$. The table (4.1) describes the $\mathfrak{m}$'s for different $\mathfrak{g}$'s.

Note that m is the intersection of the stabilizer of X_β and $X_{-\beta}$. The representations of $\mathfrak{m}$ on $\mathfrak{g}_1$ and g_{-1} are isomorphic. Denote by V either of them. The representation V is symplectic of dimension $2d$ and is irreducible if $\mathfrak{g}$ is not of type A_l, $l \geq 2$. As in [GW98] let $M_{\mathbb{C}}$ the complex connected Lie group with Lie algebra $\mathfrak{m}$ that acts faithfully on $\mathfrak{g}_{-1}$ (and $\mathfrak{g}_1$). The table (4.1) (see [GW98]) describe the group $M_{\mathbb{C}}$ and the representation V for different simple Lie algebras $\mathfrak{g}$.

4.2 The invariant of degree 4

Gross and Wallach [GW96] show that the algebra of invariants $S^{\bullet}(V)^M$ is a polynomial algebra in one generator except when $\mathfrak{g}$ is of type C_l:

Proposition 4.2.1. *The algebra of invariants in $S^{\bullet}(V)$ is given by*

$$S^{\bullet}(V)^M = \begin{cases} \mathbb{C} & \text{if } \mathfrak{g} \text{ is of type } C_l,\ l \geq 1\ , \\ \mathbb{C}[f_2] & \text{if } \mathfrak{g} \text{ is of type } A_l,\ l \geq 2\ , \\ \mathbb{C}[f_4] & \text{if } \mathfrak{g} \text{ is not of type } C \text{ or } A, \end{cases} \tag{4.6}$$

where $f_2 \in S^2(V)$ has degree 2, and $f_4 \in S^2(V)$ has degree 4.

Table 4.1: The reductive subalgebra m for different g's

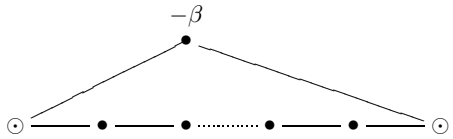
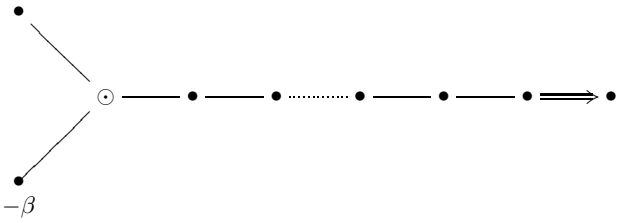
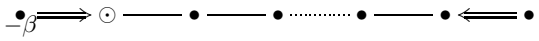
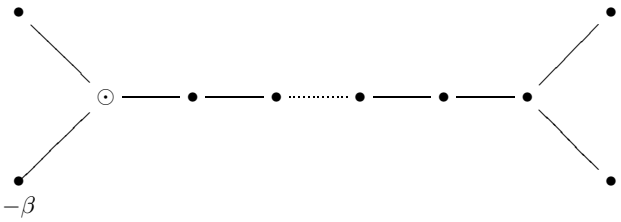
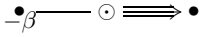
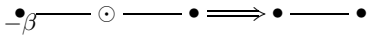
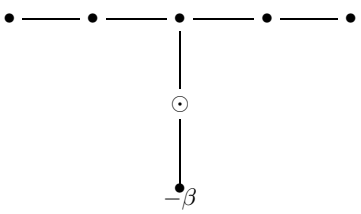
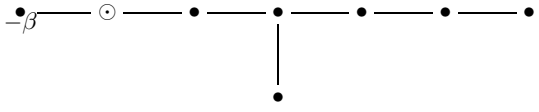
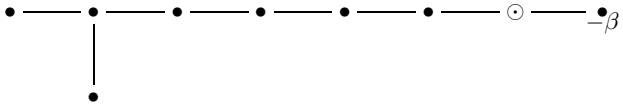
$\mathfrak{g}$	Extended diagram with vertices in S circled	$\mathfrak{m}$
$A_l = sl_{d+2}$		gl_d
$B_l = so_{d+4}$ d odd		$sl_2 \oplus so_d$
$C_l = sp_{2d+2}$		sp_{2d}
$D_l = so_{d+4}$ d even		$sl_2 \oplus so_d$
G_2		sl_2
F_4		sp_6
E_6		sl_6
E_7		so_{12}
E_8		E_7

Table 4.2: The Representation V of $M_{\mathbb{C}}$

$\mathfrak{g}$	$M_{\mathbb{C}}$	V
A	$GL_d(\mathbb{C})$	$\mathbb{C}^d \oplus (\mathbb{C}^d)^*$
B	$SL_2(\mathbb{C}) \times SO_d(\mathbb{C})$	$\mathbb{C}^2 \otimes \mathbb{C}^d$
C	$Sp_{2d}(\mathbb{C})$	$\mathbb{C}^{2d}$
D	$SL_2(\mathbb{C}) \times SO_d(\mathbb{C})/\Delta\mu_2$	$\mathbb{C}^2 \otimes \mathbb{C}^d$
G_2	$SL_2(\mathbb{C})$	$S^3(\mathbb{C}^2)$ of dim 4
F_4	$Sp_6(\mathbb{C})$	$(\wedge^3\mathbb{C}^6)_0$ of dim 14
E_6	$SL_6(\mathbb{C})/\mu_3$	$\wedge^3\mathbb{C}^6$ of dim 20
E_7	$Spin_{12}(\mathbb{C})/\mu_2$	$\frac{1}{2}-$ spin of dim 32
E_8	$E_7(\mathbb{C})$	minuscule of dim 56

Also, $M_{\mathbb{C}}$ has an open dense orbit $\mathcal{O}$ on $\mathbb{P}(V)$ and we have in the case $\mathfrak{g}$ not of type C we have $[X] \in \mathcal{O}$ if and only if $f(X) \neq 0$ where f is the minimal invariant ([Wal03]).

One can describe easily the minimal invariant in the classical cases. Since the representation V is self-dual it's enough to give an invariant form on V of degree 2 or 4.

If $\mathfrak{g}$ is of type A we have $M_{\mathbb{C}} = GL_d = GL(W)$ and $V = W \oplus W^*$. Let $\{w_i\}$ a basis of V and $\{w_i^*\}$ the dual basis in W^*. Write v in V as $v = \sum a_i w_i + b_i w_i^*$. The invariant quadratic form is

$$f_2^*(v) = \sum a_i b_i \tag{4.7}$$

If $\mathfrak{g}$ is of type B or D, $M_{\mathbb{C}}$ is a quotient of $SL_2 \times SO_d$ and $V = U \otimes W$. Let $\{u_1, u_2\}$ a symplectic basis for U and $(\ ,\ )$ the orthogonal form on W. Any v in V is written uniquely as $v = u_1 \otimes w_1 + u_2 \otimes w_2$ with w_1, w_2 in W. The invariant quartic form is

$$f_4^*(v) = (w_1, w_1)(w_2, w_2) - (w_1, w_2)^2 \tag{4.8}$$

If $\mathfrak{g}$ is of type C then $M_{\mathbb{C}} = Sp_{2d}(\mathbb{C})$ and $V = \mathbb{C}^{2d}$ and so $M_{\mathbb{C}}$ acts transitively on $V\backslash\{0\}$, hence $S^\bullet(V)^M = \mathbb{C}$

In [GW96] Gross and Wallach introduce a certain canonically defined element f of the symmetric algebra $S^\bullet(V)$ that is invariant under the action of $M_{\mathbb{C}}$.

Let $X \in \mathfrak{g}_{-1}$. Take X_β in $\mathfrak{g}^\beta$, $X_{-\beta}$ in $\mathfrak{g}^{-\beta}$ such that $[X_\beta, X_{-\beta}] = \beta^\vee$. Since $\mathfrak{g}$ is a graded Lie algebra and $X_\beta \in g_2$ we have $ad(X)^4 X_\beta \in \mathfrak{g}_{-2}$. Therefore ([GW96]) there exists a constant depending on X $f(X)$ such that

$$ad(X)^4(X_\beta) = f(X) \cdot X_{-\beta} \tag{4.9}$$

Since $X \mapsto ad(X)^4$ is homogenous of degree 4, $f(X)$ is a polynomial of degree 4 in X. Moreover f is an invariant of the action of $\mathfrak{m}$ on $\mathfrak{g}_{-1}$. Indeed, write the polarized equality

$$ad(X_1) \circ \ldots \circ ad(X_4)(X_\beta) = B(X_1, X_2, X_3, X_4) \cdot X_{-\beta} \tag{4.10}$$

for $X_i \in \mathfrak{g}_{-1}$. Now take $Y \in m$ and apply $ad(Y)$ to the above equality Since $ad(Y)(X_\beta) = 0$ and $ad(Y)(X_{-\beta}) = 0$ we conclude that the multilinear form B is invariant under the action of $\mathfrak{m}$. Therefore f is invariant under the action of $\mathfrak{m}$ and so invariant under $M_{\mathbb{C}}$.

It follows that if $\mathfrak{g}$ is of type C we have $f = 0$, if $\mathfrak{g}$ is of type A then $f = f_2^2$ and in the other cases $f = f_4$.

Note that if $\mathfrak{g}$ is not of type C we can reduce the calculation of f to the case $\mathfrak{g}$ of type A_2 as follows. Let X in $\mathfrak{g}_{-1}$ such that $f(X) \neq 0$. Then (citeWa03) the Lie algebra generated by X and X_β is isomorphic to $sl(3, \mathbb{C})$. We get therefore an imbedding $sl(3, \mathbb{C}) \to \mathfrak{g}$ compatible with the grading $\mathfrak{g} = \oplus \mathfrak{g}_i$. The restriction of the invariant f to $sl(3, \mathbb{C})_{-1}$ will equal the invariant $f_{sl(3,\mathbb{C})}$ corresponding to this copy of $sl(3, \mathbb{C})$.

If $\mathfrak{g}$ is not of type A or C the algebra of invariants is freely generated by a quartic form f_4. The generator can be given as follows: Recall that in this case $\mathfrak{m}$ is simple and V is irreducible, since $\mathfrak{g}$ is not of type A. Then (see [Wal03]and also [LM02])) $Sym^2(V)$ decomposes as

$$Sym^2(V) = V^{(2)} \oplus \mathfrak{m} \tag{4.11}$$

where $V^{(2)}$ is the Cartan square of V and $\mathfrak{m}$ is the adjoint representation of $\mathfrak{m}$. Let p the projection of $Sym^2(V)$ onto $\mathfrak{m}$.

Proposition 4.2.2. *(see also [Wal03]) If B is the Killing form of $\mathfrak{m}$ we have*

$$f(X) = f_4^*(X) = B(p(X^2), p(X^2)) \tag{4.12}$$

Proof. Since B is an invariant bilinear form on $\mathfrak{m}$ we conclude that $X \mapsto B(p(X^2), p(X^2))$ is an invariant of degree 4. It is nonzero since m appears in the decomposition (4.11) . Now we use the characterization of the algebra of invariants and we conclude the result. □

Note that if $\mathfrak{g}$ is of type C we have $Sym^2(V) = V^{(2)} = \mathfrak{m}$

We have now an important converse:

Theorem 4.2.3. *Let m a complex Lie algebra and V a finite dimensional faithfull irreducible representation of m such that*

- *the representation V is symplectic*
- *the algebra of invariants $Sym^{\mathfrak{m}}(V) \subset Sym(V)$ is either $\mathbb{C}$ or freely generated by one element f*
- *we have the inclusion*

$$Sym^2(V) \subset V^{(2)} \oplus \mathfrak{m} \tag{4.13}$$

Then $\mathfrak{m}$ with the representation V is obtained from a simple Lie algebra $\mathfrak{g}$ of type not A by the above procedure.

Proof. Since $\mathfrak{m}$ has an irreducible faithful representation it follows that $\mathfrak{m}$ is reductive. The center of $\mathfrak{m}$ acts by scalars (by Schur lemma) and faithfully and so $\dim Z(\mathfrak{m}) \leq 1$. However, since m acts symplectically it follows that $\mathfrak{m}$ is semisimple.

Let M the semisimple connected subgroup of $GL(V)$ with the Lie algebra $\mathfrak{m}$. We distinguish two cases:

Case 1. The algebra of invariants of M is $\mathbb{C}$.

It will follow that M is simple. Indeed, otherwise we have $M = M_1 \times M_2$ where M_1, M_2 semisimple and $V = V_1 \otimes V_2$ where $\dim V_i \geq 2$. . Now, the representation V is symplectic and we may assume that V_1 is symplectic and V_2 is orthogonal.

Therefore $M_1 \subset Sp(V_1)$ and $M_2 \subset SO(V_2)$. However, the representation $V_1 \otimes V_2$ has an $Sp(V_1) \times SO(V_2)$ has an invariant of degree 4 and therefore it has an $M = M_1 \times M_2$ invariant of degree 4, contradiction. We conclude that M is simple.

Now using the list of [KPV76] we see that the only case when M is simple and the M is $\mathbb{C}$ is the case of the groups with an open orbit on V (locally transitive in their terminology). with the following possibilities:

1. $SL_n(\mathbb{C})$ with the standard representation (or its dual) for $n \geq 2$
2. $SL_n(\mathbb{C})$ with $\wedge^2(\mathbb{C}^n)$ (or its dual) for n odd
3. $Sp_n(\mathbb{C})$ with the standard representation
4. $Spin_{10}(\mathbb{C})$ with the half-spin representation

Of these only the ones in 1) for $n = 2$ and 3) are self dual. We obtain $M = Spn(\mathbb{C})$ with the standard representation. This comes from $\mathfrak{g}$ a simple algebra of type C.

Case 2. The algebra of invariants of M is freely generated by a form f.

We conclude (see [Kra84]) that the codimension of a generic closed orbit of M on V is 1. Consider the reductive group $\mathbb{C}^\times \times M \subset GL(V)$. It is a reductive group acting irreducibly on V. From the above it follows that $\mathbb{C}^\times \times M$ has an open orbit on V. Therefore V is a prehomogenous vector space for $\mathbb{C}^\times \times M$ (see [SK77]). Now V is equivalent to a unique reduced prehomogenous representation. We have a list of all the *reduced* ones. Let's assume that V is not reduced. Consider a castling transform (see again [SK77]) from V as representation of $M \times \mathbb{C}^\times = M \times GL(1)$. We must have $M = M_1 \times SL(m)$ and so $M \times GL(1) = M_1 \times GL(m)$. Moreover $V = V_1 \otimes \mathbb{C}^m$. Therefore $M = M_1 \times SL(m)$. Moreover, the representation $\mathbb{C}^m$ of $SL(m)$ is self dual and therefore $m = 1$ or $m = 2$. Since we have a castling transformwe conclude $m = 2$. The castling transform will be from $V = V_1 \otimes \mathbb{C}^m$ to $V_1^* \otimes \mathbb{C}^{n-m}$ as representation of $M_1 \times GL(n-m)$ where $n = \dim V_1$. Since we assume that V is not reduced we must have $n - m = 1$ and so $n = 3$. We conclude that $V_1^* \otimes \mathbb{C}^{n-m}$ as representation of $M_1 \times GL(n-m)$ is reduced. We conclude using the list of Sato and Kimura pp. 141-142 that $M_1 = SO(3)$ and $V_1 = \mathbb{C}^3$. Therefore $M = SO(3) \times SL(2)$ with the representation $V = \mathbb{C}^3 \otimes \mathbb{C}^2$.

We are left with the case V is a reduced representation of $\mathbb{C}^\times \times M$.

The case when M was simple can be analysed using again the tables from [KPV76] and we see that the only cases when $Sym^\bullet(V^*)$ is freely generated by one element (under our hypotheses we have $V \simeq V^*$) are

1. $SO_n(\mathbb{C})$ $(n \neq 4)$ with the standard representation,
2. $SL_n(\mathbb{C})$ with $\wedge^2(\mathbb{C}^n)$ for n even
3. $SL_n(\mathbb{C})$ with $\wedge^3(\mathbb{C}^n)$ for $n = 6, 7, 8$
4. $SL_2(\mathbb{C})$ with $Sym^3(\mathbb{C}^2)$
5. $Sp_6(\mathbb{C})$ with $\wedge^3_0(\mathbb{C}^6)$
6. $Spin_n(\mathbb{C})$ with the ($\frac{1}{2}$ -) spin representation for $n = 7, 9, 11, 12, 14$
7. $E_6(\mathbb{C})$, $E_7(\mathbb{C})$, $G_2(\mathbb{C})$ with the the representations of smallest dimension

We will go through the list and check that M with the representation V is either coming from a simple Lie algebra (see Table 4.1) or is not a symplectic representation or we have the case $Spin_{11}(\mathbb{C})$ with the spin representation.

Recall ([FH91]) that a symplectic representation V is self-dual ($V \simeq V^*$). A self-dual representation is either symplectic (it has an nondegenerate alternating invariant bilinear form) or orthogonal (it has a symmetric invariant bilinear form).

The standard representation $\mathbb{C}^n$ of $SO_n(\mathbb{C})$ has (obviously) an invariant symmetric bilinear form and hence is not symplectic.

The representation $\wedge^2(\mathbb{C}^n)$ of $SL_n(\mathbb{C})$ has as dual $\wedge^{n-2}(\mathbb{C}^n)$. Hence $\wedge^2(\mathbb{C}^n)$ is self dual only for $n = 4$. However, $\wedge^2(\mathbb{C}^4)$ is an orthogonal representation of $SL_4(\mathbb{C})$ and therefore not symplectic.

$\wedge^3(\mathbb{C}^n)$ is self-dual only for $n = 6$. This situation occurs in the table 4.1 for the simple Lie algebra $\mathfrak{g}$ of type G_2.

$Sp_6(\mathbb{C})$ with $\wedge^3_0(\mathbb{C}^6)$ occurs again in the table.

For n natural number the spin representation of $Spin_{2n+1}(\mathbb{C})$ is self dual. It is orthogonal if $n \equiv 0, 3(\mod 4)$ and symplectic if $n \equiv 1, 2(\mod 4)$. Also the

half-spin representations of $Spin_{2n}(\mathbb{C})$ are dual to each other if n is odd, symplectic if $n \equiv 2(\mod 4)$ and orthogonal if $n \equiv 0(\mod 4)$. (see [FH91]). The half-spin representation of $Spin_{12}(\mathbb{C})$ occurs in the table 4.1. (note that there are 2 half-spin representations and one is taken into the other by an outer automorphism of of $Spin_{12}(\mathbb{C})$).

$Sym^3(\mathbb{C}^2)$ and $\wedge^3_0(\mathbb{C}^6)$ again occur in the table.

The 2 representations of minimal degree 27 of $E_6(\mathbb{C})$ are one dual of the other, hence not self dual. The representation of minimal degree 7 of $G_2(\mathbb{C})$ is orthogonal and hence not symplectic.

$E_7(\mathbb{C})$ with its minuscule representation of dimension 56 occurs in the table 4.1.

Therefore the only possible exception is $Spin_{11}(\mathbb{C})$ with V the spin representation of dimension 32. We have however

$$Sym^2(V) = V^{(2)} \oplus spin_{11}(\mathbb{C}) \oplus \mathbb{C}^{11} \tag{4.14}$$

where $V^{(2)}$ is the Cartan square of V, $spin_{11}$ is the adjoint representation of $Spin_{11}(\mathbb{C})$ and $\mathbb{C}^{11}$ is the standard 11 dimensional representation of $SO_{11}(\mathbb{C})$.

We consult now again the list of Sato and Kimura (the case : the dimension of the center is 1) The case when M was simple was analysed already. We are left with analysing

Case 1) (item 1 on the list) $M = M_1 \times SL(2)$ with $V = V_1 \otimes \mathbb{C}^2$ and V_1 is a 2-dimensional irreducible representation of M. It follows that $M_1 = SL(2)$ and so the representation is not orthogonal

Case 2) (item 17 on the list) $M = Spin(7) \times SL(2)$. The decomposition is not right, there is an extra term.

Case 3) (item 22 on the list) $M = Spin(11)$. The decomposition is not right again

Case 4) (item 28 on the list) $M = E_6 \times SL(2)$. Again the decomposition is not right.

In the other cases from the list I the representation is not symplectic or confirms the theorem. For the list II we only have to consider $M = Sp(n) \times SO(3)$, n even.

However, for $n > 2$ the algebra of invariants is not free since $\dim Sym^4(\mathbb{C}^n \otimes \mathbb{C}^3) = 1$ but $\dim Sym^8(\mathbb{C}^n \otimes \mathbb{C}^3) = 2$.

For the list III there are no relative invariants.

□

4.3 Calculations for type A_l

We calculate from definition the invariant of Gross and Wallach for $\mathfrak{g}$ of type A.

Let $\mathfrak{g} = sl(n,k)$.Take

$$X = x_2E_{12} + x_3E_{13} + \ldots + x_{n-1}E_{1,n-1} + y_2E_{2,n} + y_3E_{3,n} + \ldots + y_{n-1}E_{n-1,n} \quad (4.15)$$

We have

$$ad(X)(E_{n,1}) = [X, E_{n,1}] = -x_2E_{n,2} - x_3E_{n,3} - \ldots - x_{n-1}E_{n,n-1} + y_2E_{21} + \ldots + y_{n-1}E_{n,1} \quad (4.16)$$

$$\begin{aligned} ad(X)^2(E_{n,1}) &= ad(X)(-\sum_{l=2}^{n-1} x_lE_{n,l} + \sum_{l=2}^{n-1} y_lE_{l,1}) = \\ &= \sum_{l=2}^{n-1} x_ly_l(E_{11} - E_{ll}) - 2\sum_{k\neq l} y_kx_lE_{kl} - \sum_{l=2}^{n-1} x_ly_l(E_{ll} - E_{nn}) = \\ &= \left(\sum_{k=2}^{n-1} x_ky_k\right) \cdot (E_{11} + E_{nn}) - 2\sum_{k,l=2}^{n-1} x_ly_kE_{kl} \end{aligned} \quad (4.17)$$

$$\begin{aligned} ad(X)^3E_{n,1} = [X, \left(\sum_{k=2}^{n-1} x_ky_k\right) \cdot (E_{11} + E_{nn}) - 2\sum_{k,l=2}^{n-1} x_ky_kE_{kl}] = \\ - \left(\sum_{k=2}^{n-1} x_ky_k\right) \sum_{l=2}^{n-1} x_lE_{1l} - 2\left(\sum_{k=2}^{n-1} x_ky_k\right) \sum_{l=2}^{n-1} x_lE_{1l} + \\ + \left(\sum_{k=2}^{n-1} x_ky_k\right) \sum_{l=2}^{n-1} y_lE_{ln} + 2 \cdot \left(\sum_{k=2}^{n-1} x_ky_k\right) \sum_{l=2}^{n-1} y_lE_{ln} = \\ = -3\left(\sum_{k=2}^{n-1} x_ky_k\right) \sum_{l=2}^{n-1} x_lE_{1l} + 3\left(\sum_{k=2}^{n-1} x_ky_k\right) \sum_{l=2}^{n-1} y_lE_{ln} \end{aligned}$$

Finally

$$ad(X)^4(E_{n1}) = [X, -3\left(\sum_{k=2}^{n-1} x_k y_k\right)\sum_{l=2}^{n-1} x_l E_{1l} + 3\left(\sum_{k=2}^{n-1} x_k y_k\right)\sum_{l=2}^{n-1} y_l E_{ln}] = \\ = 6\left(\sum_{k=2}^{n-1} x_k y_k\right)^2 E_{1,n} \quad (4.18)$$

We see therefore that

$$f(X) = 6\left(\sum_{k=2}^{n-1} x_k y_k\right)^2 \quad (4.19)$$

4.4 Link with the Mukai form

The negative part of $\mathfrak{g}$, $\mathfrak{g}_{-1} \oplus \mathfrak{g}_{-2}$ is a Heisenberg algebra. It follows that there exists a unique morphism from $\mathfrak{g}$ to the polynomial algebra $\mathcal{P} = \mathbb{C}[p_i, q_j, z]$ with the Legendre bracket such that $X_{-\beta} \mapsto 1$ and $\mathfrak{g}_{-1}$ maps to $\oplus\mathbb{C}p_i \oplus \oplus\mathbb{C}q_j$.
Now $\beta^\vee$ maps to $2z - \sum p_i q_i$, the unique scaling element. Denote this by $\tilde{z}$.
Moreover, X_β maps to a homogenous polynomial P of degree 4 (see (1.83) which can be written as

$$P = -\frac{1}{4}(2z - \sum p_i q_i)^2 + F(p_i, q_j) \quad (4.20)$$

for some polynomial of degree 4 $F(p_i, q_j)$ in p_i, q_j. Indeed, we have

$$[X_\beta, X_{-\beta}] = \beta^\vee$$

and so

$$\{P, 1\} = 2z - \sum p_i q_i$$

But we have in general $\{P, 1\} = \frac{\partial P}{\partial z}$.

Now let l a linear form in p_i, q_j. Then we have

$$ad(l)^4(\tilde{z}^2) = 0 \quad (4.21)$$

in the Lie algebra $\mathcal{P}$. Indeed, for any l linear polynomial in p_i, q_j the map

$$P \mapsto \{l, P\} = ad(l)(P)$$

is a derivation. Therefore we have

$$ad(l)^4(\tilde{z}^2) = \sum_{k=0}^{4} \binom{4}{k} ad(l)^k(\tilde{z}) \cdot ad(l)^k(\tilde{z})$$

In the sum above, for each $0 \leq k \leq 4$ at least one of the numbers k, $4-k$ is ≥ 2. Now we have

$$ad(l)^2(\tilde{z}) = ad(l)(ad(l)(\tilde{z})) = ad(l)(-\text{grade}(l) \cdot l) = 0$$

the last equality since $\tilde{z}$ is a scaling element for the Lie algebra $\mathcal{P}$.
In fact we have

$$\begin{aligned} ad(l)(\tilde{z}^2) &= -2l\tilde{z} \\ ad(l)^2(\tilde{z}^2) &= 2l^2 \\ ad(l)^3(\tilde{z}^2) &= 0 \end{aligned}$$

Moreover , for any homogenous form of degree 4 in $\mathfrak{p}_i$, q_j we have

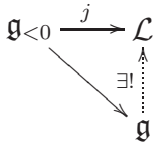
$$ad(l)^4(F) = c \cdot F(l) \tag{4.22}$$

for some constant c that depends only on n.
Now compare this with the equality (4.9). If follows that:

Theorem 4.4.1. *The invariant of degree* 4 *of Gross and Wallach coincides up to a constant with the polynomial of degree* 4 *of Mukai.*

4.5 Another proof for the invariant of degree 4

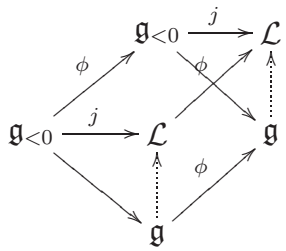
Let $\mathfrak{g}$ a simple Lie algebra with the grading like in (4.2). By (1.4.1) we have a commutative diagram of graded Lie algebras



(4.23)

where j is an isomorphism onto $\mathcal{L}_{<0}$. Let now ϕ an automorphism of the *graded* Lie algebra $\mathfrak{g} = \oplus_{i=-2}^{2} \mathfrak{g}_i$. Now ϕ induces an automorphism of $g_{<0} = \mathfrak{n} = \mathfrak{n}_{-2} \oplus \mathfrak{n}_{-1}$

and therefore an automorphism of $\mathcal{L}$. By the universal property of $\mathfrak{g}_{<0} \to \mathcal{L}$ we have the commutative diagram :



(4.24)

Assume now $\phi = Ad(m)$ where $m \in M_{\mathbb{C}}$. Then ϕ is a symplectic automorphism of $\mathfrak{g}_{-1} = V$. This extends uniquely to an automorphism of the graded Lie algebra $\mathcal{L}$. This will be the uniquely automorphism of $\mathcal{L}$ as an *associative* algebra that takes $\tilde{z}$ to itself and on $\mathbb{C}[p_i, q_j] \simeq S^{\cdot}(V)$ is given by the (extension of) m.

Let X_β a vector in $\mathfrak{g}^\beta$ where β is the largest root. Since $m \in M_{\mathbb{C}}$ it follows that m fixes X_β in $\mathfrak{g}$. Let the image of X_β in $\mathcal{L}$ be $-\frac{1}{4}\tilde{z}^2 + F(p_i, q_j)$ (see (1.83)) . Since m fixes X_β in $\mathfrak{g}$ it follows from (4.24) that the extension of m to $\mathcal{L}$ fixes the image of X_β. We conclude that m invariates the form $F(p_i, q_j)$. Therefore the form $F(p_i, q_j)$ is an invariant of $M_{\mathbb{C}}$. Since the algebra of invariants has dimension ≤ 1 in degree 4 it follows that the form F coincides with the invariant of Gross and Wallach. This proves the theorem (4.4.1).

Note that we can obtain all the complex simple Lie algebras except the ones of type A as subalgebras of the Legendre algebra $\mathcal{L}$ in the following way: Consider the symplectic vector space $V = \mathcal{L}_{-1}$ and $\mathfrak{m}$ a Lie subalgebra of $sp(\mathcal{L}_{-1})$ such that $\mathfrak{m}$ acts irreducibly and the algebra of invariants of $\mathfrak{m}$ is freely generated by one form or equals $\mathbb{C}$. Assume moreover the condition (4.13) . Let $F(p_i, q_j)$ an invariant of degree 4 (unique up to multiplication by a constant). Then the Lie subalgebra of $\mathcal{L}$ generated by $\mathcal{L}_{-1}$ and $-\frac{1}{4}\tilde{z}^2 + F(p_i, q_j)$ is a simple Lie algebra.

5 Another form for the Legendre Algebra

5.1 Some general algebraic results

The universality property of the tensor product of modules provides two pairs of adjoint functors. The setup is as follows:

Let M be an (A, B) module, N a (B, C) module and let P be an (A, D) module. Then the following are naturally isomorphic as (C, D) modules

$$Hom_B(N, Hom_A(M, P)) \simeq Hom_A(M \otimes_B N, P) \tag{5.1}$$

We have two important particular cases of this

I. $B \to A$ morphism of rings and $M = A$. Our isomorphism becomes

$$Hom_B(N, Hom_A(A, P)) \simeq Hom_A(A \otimes_B N, P) \tag{5.2}$$

Note that $Hom_A(A, P)$ as a B module is P with scalars restricted from A to B, while $A \otimes_B N$ is the extension of scalars from B to A applied to N.

II. $A \to B$ morphism of rings and $M = B$. Our isomorphism becomes

$$Hom_B(N, Hom_A(B, P)) \simeq Hom_A(B \otimes_B N, P) \tag{5.3}$$

Note that $B \otimes_B N$ is the restriction of scalars from B to A applied to N and $Hom_A(B, P)$ is the coextension of scalars from A to B.

5.2 Coinduced modules for Lie algebras

Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{C}$ and let $\mathfrak{p}$ be a subalgebra. Then we have the inclusion of enveloping algebras $\mathcal{U}(\mathfrak{p}) \subset \mathcal{U}(\mathfrak{g})$. Let N a $\mathcal{U}(\mathfrak{g})$-module and P a $\mathcal{U}(\mathfrak{p})$-module. Then we have the isomorphism

$$Hom_{\mathcal{U}(\mathfrak{g})}(N, Hom_{\mathcal{U}(\mathfrak{p})}(\mathcal{U}(\mathfrak{g}), P)) \simeq Hom_{\mathcal{U}(\mathfrak{p})}(N, P) \tag{5.4}$$

$Hom_{\mathcal{U}(\mathfrak{p})}(\mathcal{U}(\mathfrak{g}), P)$ is called the coinduced module from $\mathfrak{p}$ to $\mathfrak{g}$ of the $\mathfrak{p}$-module P.

5.3 An important particular case

Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{C}$ and let $\mathfrak{p}$ be a subalgebra. Take $N = \mathfrak{g}$, a $\mathcal{U}(\mathfrak{g})$-module and $P = \mathbb{C}$ with a $\mathcal{U}(\mathfrak{p})$-module action given by a linear map $-\beta\colon \mathfrak{p} \to \mathbb{C}$, notation $P = \mathbb{C}_{-\beta}$. We conclude that here exists an isomorphism

$$Hom_{\mathcal{U}(\mathfrak{g})}(g, Hom_{\mathcal{U}(\mathfrak{p})}(\mathcal{U}(\mathfrak{g}), \mathbb{C}_{-\beta})) \simeq Hom_{\mathcal{U}(\mathfrak{p})}(g, \mathbb{C}_{-\beta}) \tag{5.5}$$

Let's make the correspondence explicit. There exists a bijection between $\mathcal{U}(\mathfrak{g})$-maps from $\mathfrak{g}$ to $Hom_{\mathcal{U}(\mathfrak{p})}(\mathcal{U}(\mathfrak{g}), \mathbb{C}_{-\beta})$ and linear maps $f\colon g \to \mathbb{C}$ such that $f([Y, X]) = -\beta(Y) \cdot f(X)$ for all $Y \in p$ and $X \in \mathfrak{g}$. Note that it's possible that the only such linear map f is the zero map. The previous condition means that $f \in \mathfrak{g}^*$ is an eigenvector for $\mathfrak{p}$ with weight β (or zero). Analysing the isomorphism (5.3) we obtain the following: To f as above corresponds the map $F\colon \mathfrak{g} \to Hom_{\mathcal{U}(\mathfrak{p})}(\mathcal{U}(\mathfrak{g}), \mathbb{C}_{-\beta})$ given by $F(X)(x) = f(x \cdot X)$ for all $X \in \mathfrak{g}$ and $x \in \mathbf{U}(\mathfrak{g})$. Note that for $x = X_1 \cdot \ldots \cdot X_m \in \mathcal{U}(\mathfrak{g})$ and $X_i \in \mathfrak{g}$ we have $x \cdot X = (X_1 X_2 \ldots X_m) \cdot X = [X_1, [X_2, \ldots [X_m, X]] \ldots]$

5.4 The case of a parabolic subalgebra

We continue with notations as above, assume moreover $\mathfrak{g}$ is a simple Lie algebra and $\mathfrak{p}$ is a parabolic subalgebra. The Killing form of $\mathfrak{g}$, denoted by $\langle , \rangle$ is nondegenerate. Therefore, any f in $\mathfrak{g}^*$ is given by $f(\cdot) = \langle E, \cdot \rangle$ for a unique $E \in \mathfrak{g}$. Now $f \in \mathfrak{g}^*$ is an eigenvector for $\mathfrak{p}$ with weight β if and only if E is an eigenvector for $\mathfrak{p}$

with weight β. (Note that the adjoint representation of g is self dual, consequence of the nondegeneracy of the Killing form). Assume this is the case. Let $\mathfrak{b}$ the unique Borel subalgebra contained in $\mathfrak{p}$. Then E is a highest weight vector for the adjoint representation of g. Therefore, $E = E_\beta$ where β is the largest root of $(\mathfrak{g}, \mathfrak{b})$. Also we conclude that $\mathfrak{p}$ is contained in the Heisenberg parabolic corresponding to the longest root β.

5.5 The case of Heisenberg Parabolic

Assume all as above and $\mathfrak{p}$ *is* a Heisenberg parabolic. Therefore we have an $\mathfrak{g}$-equivariant map

$$\mathfrak{g} \to Hom_{\mathcal{U}(\mathfrak{p})}(\mathcal{U}(\mathfrak{g}), \mathbb{C}_{-\beta}) \tag{5.6}$$

given by

$$X \mapsto \phi_X(\cdot), \quad \phi_X(x) = \langle E_\beta, x \cdot X \rangle \tag{5.7}$$

Note that the $\mathfrak{g}$ module action on $Hom_{\mathcal{U}(\mathfrak{p})}(\mathcal{U}(\mathfrak{g}), \mathbb{C}_{-\beta})$ is given by:

$$(Y \cdot \phi)(x) = \phi(x \cdot Y) \tag{5.8}$$

for $Y \in \mathfrak{g}$, $x \in \mathcal{U}(\mathfrak{g})$. See [Wal69]

Let $\mathfrak{n}$ be nilpotent algebra . For any n-module M denote by $M_{[}n]$ the submodule of M consisting of elements annihilated by a large enough power $\mathcal{U}(\mathfrak{n})_+^l$ of the augmentation ideal $\mathcal{U}(\mathfrak{n})_+$ of $\mathcal{U}(\mathfrak{n})$. We note that the image of $\mathfrak{g}$ under the map (5.6) lies in $Hom_{\mathcal{U}(\mathfrak{p})}(\mathcal{U}(\mathfrak{g})_{[n]}$ for any $\mathfrak{n}$ subalgebra of $\mathfrak{g}$ consisting of nilpotent elements. Indeed, the map (5.6) is $\mathfrak{g}$-equivariant and $\mathfrak{g} = \mathfrak{g}_{[n]}$.

Moreover the map (5.6) is an imbedding. Indeed, let $X \in \mathfrak{g}$ such that $\phi_X(\cdot) = 0$. It follows that $\langle E_\beta, x \cdot X \rangle = 0$ for all $x \in \mathcal{U}(\mathfrak{g})$. Now $\mathfrak{g}$ is irreducible as a $\mathcal{U}(\mathfrak{g})$-module and so $\mathcal{U}(\mathfrak{g}) \cdot X = 0$ or $= \mathfrak{g}$. Since $\langle , \rangle$ is nondegenerate we conclude $\mathcal{U}(\mathfrak{g}) \cdot X = 0$ and so $X = 0$.

We conclude that we have a $\mathfrak{g}$-equivariant embedding

$$\mathfrak{g} \to Hom_{\mathcal{U}(\mathfrak{p})}(\mathcal{U}(\mathfrak{g}), \mathbb{C}_{-\beta})_{[n]} \tag{5.9}$$

5.6 Identification with the dual of the nilpotent radical

Let $\mathfrak{p}$ as above a Heisenberg parabolic subalgebra of g and $\mathfrak{n} = \mathfrak{n}_-$ the nilpotent radical of the opposite of $\mathfrak{p}$. $\mathfrak{n}$ is a Heisenberg algebra itself and we have $\mathfrak{n} \oplus p = \mathfrak{g}$. It follows that $\mathcal{U}(\mathfrak{g}) = \mathcal{U}(\mathfrak{p}) \otimes \mathcal{U}(\mathfrak{n})$. Therefore:

$$Hom_{\mathcal{U}(\mathfrak{p})}(\mathcal{U}(\mathfrak{g}), \mathbb{C}_{-\beta}) \simeq Hom(\mathcal{U}(\mathfrak{n}), \mathbb{C}) \tag{5.10}$$

as $\mathcal{U}(\mathfrak{n})$-modules and therefore

$$Hom_{\mathcal{U}(\mathfrak{p})}(\mathcal{U}(\mathfrak{g}), \mathbb{C}_{-\beta})_{[n]} \simeq Hom(\mathcal{U}(\mathfrak{n}), \mathbb{C})_{[n]} \tag{5.11}$$

We conclude: There exists an imbedding

$$\mathfrak{g} \to Hom_{\mathbb{C}}(\mathcal{U}(\mathfrak{n}), \mathbb{C})_{[n]} \tag{5.12}$$

5.7 The annihilation of cohomology

The $\mathfrak{n}$-module $Hom_{\mathbb{C}}(\mathcal{U}(\mathfrak{n}), \mathbb{C})_{[n]}$ (in fact a $\mathfrak{g}$-module as we have seen above) has zero cohomology in dimension > 0. This follows from a more general result of Wallach ([Wal88]). More generally let n a finite dimensional nilpotent Lie algebra and M an n-module. Define as above $M_{[n]}$. Let W is a complex vector space. Then $Hom(\mathcal{U}(\mathfrak{n}), W)$ has a structure of n module (i.e. $\mathcal{U}(n)$-module). Define like in [Wal69] $N(W) = Hom(\mathcal{U}(\mathfrak{n}), W)_{[n]}$. Then we have

Lemma 5.7.1. *With the hypotheses above we have $H^i(\mathfrak{n}, N(W)) = 0$ for all complex vector spaces W and for all $i > 0$.*

Proof. Note that by additivity of the cohomology it's enough to prove for $\dim W = 1$ i.e. for $W = \mathbb{C}$. However, the proof works as well for any W. We prove by induction on $\dim \mathfrak{n}$. Assume first that $\dim \mathfrak{n} = 1$. It is known that $H^i(\mathfrak{n}, N(W)) = 0$ for $i > \dim \mathfrak{n}$. Thus we only have to prove that $H^1(\mathfrak{n}, N(W)) = 0$. Let X a basis of $\mathfrak{n}$. From the definition of cohomology we must show that $N(W) = X \cdot N(W)$.

Let f in $N(W)$. That means $f\colon \mathcal{U}(\mathfrak{n}) \to W$ linear and $f(X^k) = 0$ for $k \geq k_0$. Define now $g\colon \mathcal{U}(\mathfrak{n}) \to W$, $g(1)$ arbitrary (say $= 0$) and $g(X^{k+1}) = f(X^k)$. We have $X \cdot g = f$.

Assume that the result is true for $\dim \mathfrak{n} = d$. Let now $\dim \mathfrak{n} = d + 1$. Since $\mathfrak{n}$ is nilpotent we have $[\mathfrak{n}, \mathfrak{n}] \subset \mathfrak{n}$ strictly. Take $\mathfrak{n}_1$ a subspace of $\mathfrak{n}$ of codimension 1 containing $[\mathfrak{n}, \mathfrak{n}]$ and $X \in \mathfrak{n}$ such that $\mathfrak{n} = \mathbb{C}X \oplus \mathfrak{n}_1$. Then $\mathfrak{n}_1$ is an ideal of $\mathfrak{n}$. By P-B-W theorem (see [Ser06]) we have $\mathcal{U}(\mathfrak{n}) = \oplus_{k\geq 0} X^k \mathcal{U}(\mathfrak{n}_1)$.

Since $\mathfrak{n}_1$ is an ideal of $\mathfrak{n}$ we have a Hochschild-Serre spectral sequence

$$E_2^{p,q} = H^p(\mathfrak{n}/\mathfrak{n}_1, H^q(\mathfrak{n}_1, N(W))) \Rightarrow H^i(\mathfrak{n}, N(W)) \tag{5.13}$$

By the induction hypothesis we have $H^q(\mathfrak{n}_1, N(W)) = 0$ for $q > 0$. It follows that

$$H^i(\mathfrak{n}, N(W)) = H^i(\mathfrak{n}/\mathfrak{n}_1, H^0(n_1, N(W))) \tag{5.14}$$

Now $H^0(\mathfrak{n}_1, N(W)) = N(W)^{\mathfrak{n}_1} = Hom(\mathcal{U}(\mathfrak{n}), W)^{\mathfrak{n}_1}_{[n]} = Hom(\mathcal{U}(\mathfrak{n}/\mathfrak{n}_1), W)_{[\mathfrak{n}/\mathfrak{n}_1]}$ Now apply the induction hypothesis for $\mathfrak{n}/\mathfrak{n}_1$. We conclude that $H^i(\mathfrak{n}, N(W)) = 0$ for $i > 0$. □

5.8 A new realization of the algebraic prolongation

Let again $\mathfrak{n}$ a nilpotent algebra.

Lemma 5.8.1. *Let M be an n-module such that $M = M_{[n]}$. Assume moreover that $H^1(\mathfrak{n}, M) = 0$. Then M is isomorphic to $N(M^{\mathfrak{n}})$. In particular, we have $H^i(\mathfrak{n}, M) = 0$ for all $i > 0$.*

For proof we use the following result of Wallach ([Wal88])

Lemma 5.8.2. *Let M, V in $\mathcal{N}$. Assume that $H^1(\mathfrak{n}, V) = 0$.*

- *If A is in $Hom_{\mathbb{C}}(M^{\mathfrak{n}}, V^{\mathfrak{n}})$ then there exists $T \in Hom_{\mathfrak{n}}(M, V)$ extending A.*
- *If A is injective then T is also injective*

- *If A is bijective and if $H^1(\mathfrak{n}, M) = 0$ then T is surjective (and so bijective)*

Let now $\mathfrak{n}$ a negatively graded Lie algebra of finite dimension with the center concentrated in the lowest degree $Z(\mathfrak{n}) = \mathfrak{n}_{j_0}$. Consider the algebraic completion of $\mathfrak{n}$ $\mathcal{C}(\mathfrak{n})$ (see part [2]). By the construction of $\mathcal{C}(\mathfrak{n})$ it follows that $H^1(\mathfrak{n}, \mathcal{C}(n))$ is concentrated in negative degree. Assume moreover that all the homogenous derivations of $\mathfrak{n}$ of degree < 0 are inner. It follows that $H^1(n, \mathcal{C}(n)) = 0$. Note now that $\mathcal{C}(n)$ and $Hom(\mathcal{U}(n), Z(n))_{[n]}$ are both $\mathfrak{n}$-modules with the first cohomology $= 0$. Using the above proposition we conclude

Theorem 5.8.3. *The algebraic prolongation of $\mathfrak{n}$ is naturally isomorphic to $Hom(\mathcal{U}(\mathfrak{n}), Z(\mathfrak{n}))_{[n]}$*

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