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Virtual Invariants on Quot Schemes over Fano Surfaces

A dissertation submitted in partial satisfaction of the
requirements for the degree
Doctor of Philosophy

in

Mathematics

by

Daniel Schultheis

Committee in charge:

Professor Dragos Oprea, Chair
Professor Mark Gross
Professor Ken Intriligator
Professor Aneesh Manohar
Professor Justin Roberts

2012

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Chair

University of California, San Diego

2012

DEDICATION

To Dorothy and David Schultheis.

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ABSTRACT OF THE DISSERTATION

Virtual Invariants on Quot Schemes over Fano Surfaces

by

Daniel Schultheis
Doctor of Philosophy in Mathematics
University of California San Diego, 2012
Professor Dragos Oprea, Chair

Moduli theory, a subfield of algebraic geometry, focuses on computing geometric enumerative invariants. Building on earlier classical methods, algebraic geometers in the 20th century began approaching these questions using intersection theory, allowing many new and interesting examples.

The primary focus of this paper is the computation of several (virtual) enumerative invariants on the Quot scheme of rank $N-r$ quotients of $\mathcal{O}_S^N$ with Euler characteristic χ and first Chern class β over a Fano surface S, for which we use the shorthand notation Quot_S. This project is a natural generalization of the work of Aaron Bertram, Alina Marian, Dragos Oprea, and others, who considered the analogous problem on a Quot scheme over a smooth projective curve C and obtained the well known Vafa-Intriligator formula.

Our main computational technique will be virtual localization, a limiting procedure in intersection theory in which enumerative invariants are computed in terms of data from the fixed locus of a torus action. While such techniques would only yield results for toric surfaces such as $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$, we extend these results using cobordism classes to any surface S for which Quot_S admits a virtual fundamental class. In particular, our results hold over any Fano surface. The examples computed here outline a computational algorithm for further enumerative invariants on Quot_S.

Chapter 1

Introduction

Enumerative questions have interested algebraic geometers for hundreds of years. Classical methods were able to prove certain well known results, like the existence of 27 lines on a smooth cubic surface in $\mathbb{P}^3$. In the latter half of the 20th century new techniques emerged in moduli theory to consider these questions. From this standpoint, an enumerative invariant can be expressed as the intersection of certain classes in (co)homology with the fundamental class of the moduli space parametrizing all such objects.

In [LT] and [BF], the authors introduced virtual fundametal classes. A virtual fundamental class serves much the same role as a fundamental class, allowing the computation of (virtual) enumerative invariants even in certain cases when the moduli spaces fails to be smooth, reduced, or equidimensional.

The primary goal of this paper is to compute the first examples of enumerative invariants on the Quot scheme $\mathrm{Quot}_S(\mathbb{G}(r,N))_{\chi,\beta}$ over a Fano surface S, which we shall commonly abbreviate by Quot_S. This project directly generalizes prior work on $\mathrm{Quot}_C(\mathbb{G}(r,N),d)$ over a smooth projective curve C, which has been carried out in [B], [BDW], [ST], [MO1], and [MO2]. In particular, the methods of my thesis parallel those in [MO1], using virtual localization to obtain concrete intersection numbers.

The outline of this thesis is as follows. In chapter 2 we begin with a detailed explanation of the prior work on $\mathrm{Quot}_C(\mathbb{G}(r,N),d)$ in order to motivate the generalization over a surface S. We also discuss the use of virtual localization and pay

particular attention to [B] and [MO1].

Chapter 3 includes brief overviews of several important topics which are necessary to understanding this thesis. Specifically, we will overview equivariant cohomology, virtual fundamental classes, localization, and Hilbert and Quot schemes. These explanations are in no way comprehensive, but should provide enough guidance for the non-expert to navigate my methods and results. Readers familiar with these topics should feel free to skip ahead.

In chapter 4 we begin the process of generalizing to the Quot scheme over a surface. We prove the existence of a virtual fundamental class when S is Fano, compute the expected dimension, and specify a torus action for our later use in virtual localization. This chapter comprises the theoretical core of this thesis. The main result is Theorem 3, stated below.

Theorem. *The Quot scheme of quotients of the trivial bundle on a Fano surface S admits a perfect obstuction theory and a virtual fundamental class of the expected dimension e, which is given in Theorem 4.*

Chapters 5 and 6 deal with explicit computation of intersection products on Quot_S when S is one of the projective surfaces $\mathbb{P}^1 \times \mathbb{P}^1$ or $\mathbb{P}^2$. These results are the first enumerative invariants to be computed on a Quot scheme over a surface. The main outcome of Chapter 5 is the following theorem.

Theorem. *The Quot scheme $Quot_S(\mathbb{G}(1,3))_{4,0}$ has expected dimension 0 and*

$$\int_{[Quot_{\mathbb{P}^1\times\mathbb{P}^1}(\mathbb{G}(1,3))_{4,0}]^{vir}} 1 = 204 \qquad \textit{and} \qquad \int_{[Quot_{\mathbb{P}^2}(\mathbb{G}(1,3))_{4,0}]^{vir}} 1 = 270.$$

This result is proven in Theorem 5 for $\mathbb{P}^1 \times \mathbb{P}^1$ and in Theorem 6 for $\mathbb{P}^2$.

Chapter 6 details a more involved example in rank 2.

Theorem. *The Quot scheme $Quot_S(\mathbb{G}(2,8))_{9,0}$ again has expected dimension 0 and*

$$\int_{[Quot_{\mathbb{P}^1\times\mathbb{P}^1}(\mathbb{G}(2,8))_{9,0}]^{vir}} 1 = -1724160 \qquad \textit{and} \qquad \int_{[Quot_{\mathbb{P}^2}(\mathbb{G}(2,8))_{9,0}]^{vir}} 1 = -2680920.$$

These results are proven in Theorem 7 for $\mathbb{P}^1 \times \mathbb{P}^1$ and in Theorem 8 for $\mathbb{P}^2$.

Finally, chapter 7 summarizes these results and discusses the extent to which our methods may be generalized to other invariants and other surfaces. In particular, we prove the following generalization to other surfaces S in Theorem 10.

Theorem. *Let S be any surface for which $Quot_S(\mathbb{G}(1,3))_{4,0}$ admits a virtual fundamental class. Then*

$$\int_{[Quot_S(\mathbb{G}(1,3))_{4,0}]^{vir}} 1 = 39(c_1(K_S))^2 - 27c_2(K_S).$$

Under similar hypotheses

$$\int_{[Quot_S(\mathbb{G}(2,8))_{9,0}]^{vir}} 1 = -462600(c_1(K_S))^2 + 494160c_2(K_S).$$

Chapter 2

Motivation and Prior Work

We begin with an examination of the history of the enumerative theory of the Quot scheme.

In [BDW] and [B], Bertram, Daskalopoulos, and Wentworth were the first to explicitly examine the intersection theory of the Quot scheme $\mathrm{Quot}_C(\mathbb{G}(r,N),d)$ of degree d, rank $N-r$ quotients of $\mathcal{O}^N$ over a smooth projective curve C of genus g. More accurately, they sought to understand the quantum cohomology of the Grassmannian by investigating the intersection theory of the moduli space $Mor_C(\mathbb{G}(r,N),d)$ of morphisms to the Grassmannian. However, this moduli space is noncompact and in general singular, not of the expected dimension, and not projective. They dealt with many of these problems by instead compactifying $Mor_C(\mathbb{G}(r,N),d) \subseteq \mathrm{Quot}_C(\mathbb{G}(r,N),d)$ by mapping $f : C \to \mathbb{G}(r,N)$ to the locally free quotient

$$0 \to f^*S \to \mathcal{O}^N \to f^*Q \to 0$$

where

$$0 \to S \to \mathcal{O}^N \to Q \to 0$$

is the tautological sequence on $\mathbb{G}(r,N)$.

They showed that when the degree d was sufficiently large the Quot scheme is irreducible, generically reduced, and has the expected dimension. Furthmore, in this regime they showed that they could choose special representatives of their Chern classes for which there are no "intersections at infinity", that is, there are

no extra intersections occuring in the boundary of the compactification.

The primary approach focused on studying families of Quot schemes associated to families of curves degenerating to a nodal curve. This allowed the authors to exhibit a recursive formula inductively computing intersection products in terms of curves of lower genus. Up to a base case computation in genus 0, this completed the computation of intersection products in the so called a-classes and verified the Vafa-Intriligator formula. Here $a_i := c_i(\mathcal{E}|^{\vee}_{\mathrm{Quot}_C \times \{p\}})$ is a Chern class of the dual of the universal subbundle $\mathcal{E}$ on $\mathrm{Quot}_C \times C$, restricted to $\mathrm{Quot}_C \times \{p\}$ for a point $p \in C$. When $r = 2$, they were able to prove the base case as well.

This inductive argument was later completed by Siebert and Tian in [ST]. We will not need their methods in this paper, so we omit a detailed discussion. Interested readers should reference the original souce.

In [MO1], Marian and Oprea approached this problem again using the method of virtual localization. This thesis directly generalizes their approach, so we will review the method in some depth.

Let $\mathbb{C}^*$ be a one dimensional torus. Then one can define a $\mathbb{C}^*$ action on $\mathbb{C}^N$ by a diagonal matrix

$$\begin{bmatrix} t^{-\lambda_1} & & \\ & \ddots & \\ & & t^{-\lambda_N} \end{bmatrix}$$

so that the weights are just $-\lambda_1, \cdots, -\lambda_N$. We will call this torus action T. Now let

$$0 \to E \to \mathbb{C}^N \otimes \mathcal{O} \xrightarrow{p} F \to 0 \tag{2.1}$$

be a closed point in $\mathrm{Quot}_C(\mathbb{G}(r, N), d)$. Then the action of $\mathbb{C}^*$ on $\mathbb{C}^N$ descends to a torus action on $\mathrm{Quot}_C(\mathbb{G}(r, N), d)$ by the composition $p \circ T : \mathbb{C}^N \otimes \mathcal{O} \to F$.

Next, the authors showed that the fixed loci of this torus action consisted of those short exact sequences in which E decomposed as a sum of ideal sheaves of zero dimensional subschemes

$$E = \mathcal{I}_{z_1} \oplus \cdots \oplus \mathcal{I}_{z_r}$$

in which the sum of the lengths d_i of the z_i's is d. Furthermore, each $\mathcal{I}_{z_i}$ injects into a unique copy of $\mathcal{O}$. In this way, the full fixed locus is a products of Hilbert schemes of points on C, that is,

$$\mathrm{Sym}^{d_1} C \times \cdots \times \mathrm{Sym}^{d_r} C.$$

With this setup and a detailed computation of the virtual normal bundle of the fixed locus, the authors of [MO1] were also able to verify the Vafa-Intriligator formula using virtual localization. We state their formulation of the result here for reference.

Theorem 1. *Let $P(x_1, \ldots, x_r)$ be a polynomial of weighted degree equal to e, the expected dimension of $Quot_C(\mathbb{G}(r, N), d)$, where the variable x_i has degree i. Define*

$$J(x_1, \ldots, x_r) = N^r \cdot x_1^{-1} \ldots x_r^{-1} \cdot \prod_{1 \leq i < j \leq r} (x_i - x_j)^{-2}$$

Then,

$$\int_{[Quot_C(\mathbb{G}(r,N),d)]^{vir}} P(a_1, \ldots, a_r) = \pm \sum_{\lambda_1, \ldots, \lambda_r} R(\lambda_1, \ldots, \lambda_r) J^{g-1}(\lambda_1, \ldots, \lambda_r),$$

the sum being taken over all $\binom{N}{r}$ tuples $(\lambda_1, \ldots, \lambda_r)$ of distinct N^{th} roots of unity. Here R is the symmetric polynomial obtained by expressing $P(a_1, \ldots, a_r)$ in terms of the Chern roots of $\mathcal{E}|^{\vee}_{\mathrm{Quot}_C \times \{p\}}$.

While Marian and Oprea were also able to prove a few slight generalizations of this result over curves, the key observation is that a closed solution exists for intersections of a-classes with $[\mathrm{Quot}_C]^{vir}$. My work has been an attempt to generalize this process to Quot_S, an analogous Quot scheme over certain surfaces S. The examples in chapters 5 and 6 represent the beginnings of such a generalization.

Chapter 3

Necessary background

3.1 Equivariant cohomology

Our main intersection theoretic computations will make frequent use of equivariant cohomology. While this subject has a deep theory of its own, we will only need limited definitions and results for the majority of the work to follow. We review the necessary background here, all of which can be found in greater depth in [F].

Definition 1. *Let G be a Lie group acting on a space X. Let EG be a contractible space with a free G action and consider the quotient*

$$EG \times^G X \cong EG \times X/(e \cdot g, x) \sim (e, g \cdot x).$$

Then the equivariant cohomology $H^i_G(X)$ of X with respect to G is defined to be equal to be $H^i(EG \times^G X)$.

For our considerations, it suffices to think of $G = \mathbb{C}^*$ or $G = GL_n(\mathbb{C})$. This definition turns out to be independent of the choice of EG.

The most basic example, and the most useful for our purposes, is when $G = \mathbb{C}^*$ and X is a point. With $EG = \mathbb{C}^\infty \backslash \{0\}$ as a contractible space with the obvious $\mathbb{C}^*$-action, then $EG \times^G \{pt\} = EG/G = \mathbb{C}P^\infty$. Thus the equivariant cohomology ring of the point with respect to $\mathbb{C}^*$ is

$$H^*_{\mathbb{C}^*}(pt) = H^*(\mathbb{C}P^\infty) = \mathbb{Z}[h],$$

the polynomial ring in one variable. We will not need the precise definition of h. Throughout the rest of this paper, we will always use h to refer to the generator of this equivariant cohomology ring, though we will often use coefficients in $\mathbb{C}$ and not $\mathbb{Z}$. We emphasize that while the singular cohomology of a point is trivial, the equivariant cohomology is not.

Let $V = \mathbb{C}$ be an equivariant line bundle on the point, and let $\mathbb{C}^*$ act with weight a. That is, $t \cdot z = t^a z$. Then the first (equivariant) Chern class is $c_1^{\mathbb{C}^*}(V) = ah$. We will frequently use this result, along with the logical generalizations for higher dimensional vector spaces over a point.

3.2 Virtual fundamental classes

We begin by explaining the purpose of virtual fundamental classes. In classical enumerative algebraic geometry, one wants to compute an enumerative invariant such as the Euler characteristic of a space X, or the number of lines on a smooth cubic surface in $\mathbb{P}^3$. These invariants can frequently be computed by intersecting certain Chern classes with the fundamental class $[X]$ of X.

Roughly speaking, a virtual fundamental class takes the place of the usual fundamental class in the event that the moduli space fails to have a fundamental class because it is not smooth, reduced or equidimensional. Working with virtual fundamental classes allows the computation of enumerative invariants even when the moduli space is not well behaved, and can be thought of as an alternative to requiring that a moduli space be smooth.

For a more precise presentation of virtual classes, there are two formulations: that of Li and Tian [LT], and that of Behrend and Fantechi [BF]. We will keep to Li and Tian's approach and use their definitions.

Definition 2. *Let X be a scheme and $\mathcal{I}$ be a sheaf on X. Let $\mathcal{T}^1(\mathcal{I})$ and $\mathcal{T}^2 \otimes \mathcal{I}$ be the deformation and obstruction spaces, respectively, taken in the direction of $\mathcal{I}$. Then a perfect obstruction theory for X is a two term resolution of vector bundles $[\mathcal{A}_0 \to \mathcal{A}_1]$ such that for any $\mathcal{I}$, we have the short exact sequence*

$$0 \to \mathcal{T}^1(\mathcal{I}) \to \mathcal{A}_0 \otimes \mathcal{I} \to \mathcal{A}_1 \otimes \mathcal{I} \to \mathcal{T}^2 \otimes \mathcal{I} \to 0.$$

In the event that such a perfect obstruction theory exists, Chapter 3 of [LT] defines and constructs the virtual fundamental class. The existence of a virtual moduli class is quite a rare occurrence. Virtual fundamental classes are known to exist for certain Quot schemes and the moduli spaces associated to Gromov-Witten theory, Donaldson-Thomas theory, stable maps, stable quotients, and stable pairs, and very little else.

While proving the existence of $[X]^{vir}$ technically requires an explicit two term vector bundle resolution of the tangent obstruction complex, there is a general rule of thumb for when one expects the virtual class to exist. Let $\mathcal{T}_p^1$ and $\mathcal{T}_p^2$ be the tangent and obstruction spaces of X at the point p, respectively. If $dim\mathcal{T}_p^1 - dim\mathcal{T}_p^2$ is constant over all points, then one expects the virtual fundamental class to exist, even if $dim\mathcal{T}_p^1$ and $dim\mathcal{T}_p^2$ are not themselves constant.

We will be most interested in the Quot scheme $\mathrm{Quot}_S(\mathbb{G}(r,N))_{\chi,\beta}$ over a Fano surface S. General Quot schemes will be discussed in more depth in section 3.4. We will give a formal proof of the existence of a virtual fundamental class for this specific Quot scheme in section 4.1.

3.3 Localization and virtual localization

The primary computational tool used through this paper will be the method of Atiyah-Bott localization, and its virtual analog. We will skip the non-virtual case, and present only virtual localization. The following theorem is the primary result of [GP].

Theorem 2. *Let X be a space with a virtual fundamental class which admits a group action by an algebraic torus T. Let $\sum X_i$ be (the possibly disconnected components of) the fixed locus of T, let A be a vector bundle on X which restricts to A_i on X_i, and let $e(\mathcal{N}_i^{vir})$ be the Euler class of the virtual normal bundle of the inclusion $i : X_i \hookrightarrow X$. Then*

$$[X]^{vir} = i_* \sum \frac{[X_i]^{vir}}{e(\mathcal{N}_i^{vir})}$$

and (virtual) intersection products can be computed on the fixed locus by

$$\int_{[X]^{vir}} e(A) = \sum \int_{[X]^{vir}} \frac{e(A_i)}{e(\mathcal{N}_i^{vir})}.$$

In the presence of a particularly nice torus action, this allows intersection products to be passed to the fixed locus. Often, one can choose a torus action with only isolated fixed points. Of course, one must be able to compute and work with the virtual normal bundle. Many enumerative results in algebraic geometry rely heavily on this technique. Eventually we will use virtual localization to compute several virtual invariants on Quot_S. In order to motivate this approach, chapter 2 reviews the role of virtual localization in [MO1] in verifying the Vafa-Intriligator formula.

3.4 Hilbert and Quot schemes

Throughout this paper, the key spaces of interest will be various Hilbert and Quot schemes. We will not need the precise constuctions of these spaces, but it is worthwhile to give reasonably detailed definitions and point out the specific schemes arising later. The overview given here follows Nitsure's explanation in [FGIKNV], which in turn is an explanation of Grothendieck's original presentation in [G].

At their most basic level, Hilbert schemes and Quot schemes are moduli spaces, meaning that points in a moduli spaces parametrize some other objects of interest. However, the proper viewpoint to take requires a categorical explanation.

Let X be a scheme over a field k and consider the functor

$$\begin{aligned} \mathfrak{H}ilb_X : \{Sch\}^{op} &\to \{Sets\} \\ S &\mapsto \{Y \subset X \times S |\ Y \text{ is flat over } S\} \end{aligned}$$

from the opposite category of schemes to the category of sets. This functor turns out to be representable by a scheme, meaning that $\mathfrak{H}ilb_X$ is naturally isomorphic to the functor $Mor(-, H)$ for some scheme H. This scheme H is the Hilbert scheme Hilb_X.

As a set, Hilb_X parametrizes closed subschemes of X. Of course, the fact that Hilb_X is actually a scheme is not at all obvious, and implies that there is a great deal of additional structure to Hilb_X. One can show that the Hilbert scheme over X is stratified by the Hilbert polynomial Φ, meaning that Hilb_X breaks down into disjoint subschemes Hilb_X^Φ where the points in Hilb_X^Φ parametrize closed subschemes $Z \subset X$ with Hilbert polynomial equal to Φ. Thus

$$\mathrm{Hilb}_X = \coprod_\Phi \mathrm{Hilb}_X^\Phi.$$

Of particular interest is the case when Φ is equal to the constant $n \in \mathbb{Z}^+$. This Hilbert scheme, commonly written $X^{[n]}$, is called the Hilbert scheme of n points, and parametrizes dimension 0, length n subschemes of X. This is more complicated that simply asking for collections of n points in X because $X^{[n]}$ also includes schemes with "fat points" in which two or more points have collided to share support with nontrival scheme structure. Our primary interest in Hilbert schemes will be in Hilbert schemes of points.

Thinking in terms of short exact sequences, giving a point in Hilb_X is equivalent to giving a short exact sequence

$$0 \to \mathcal{I}_Z \to \mathcal{O}_X \to \mathcal{O}_Z \to 0$$

corresponding to the subscheme Z. This viewpoint has a natural generalization in which the structure sheaf $\mathcal{O}_X$ is replaced by any coherent sheaf F on X. The corresponding moduli space is called a Quot scheme. We will omit a detailed discussion of the functorial definition of a Quot scheme. Quot schemes are similarly stratified by Hilbert polynomials.

Various Quot schemes will show up throughout this paper, but most frequently $\mathrm{Quot}_C(\mathbb{G}(r,N),d)$. This scheme parametrizes degree d, rank $N-r$ quotients of $F = \mathcal{O}_C^N$ over the smooth projective curve $X = C$. The motivation for investigating this Quot scheme in particular is explained in chapter 2. Later on, we will also consider the analogous Quot scheme in which C is replaced by a Fano surface S.

Chapter 4

Generalizing to Quot schemes over surfaces

4.1 Fundamental changes and setup

We now shift to the main focus of the paper by moving to Quot_S. While we have already used this shorthand notation, we should actually specify which Quot scheme over S we mean to study. Of course, we need specify the surface S, as well as the integers r and N as before. For now, S is a general projective surface; later on we will need to restrict S in order to obtain meaningful results. Now that S is of higher dimension, we will need two pieces of data to determine which quotients we work with. Let

$$0 \longrightarrow E \longrightarrow \mathcal{O}^N \longrightarrow F \longrightarrow 0$$

be a sequence of sheaves on S with the rank of F equal to $N - r$. Let $\chi = \chi(F)$ be the Euler characteristic of F, and let $\beta = c_1(F)$ be the first Chern class. This data specifies the Quot scheme $\mathrm{Quot}_S(\mathbb{G}(r, N))_{\chi,\beta}$, which will be our moduli space of study for the next few chapters.

Before we can attempt to compute any virtual invariants, we need to verify that Quot_S even has a virtual fundamental class. This is by no means a trivial result, and in fact, $[\mathrm{Quot}_S]^{vir}$ only exists for certain surfaces S. The proof of the following theorem closely follows the proof of Theorem 1 in [MO1], with the

necessary modifications to work over S instead of C. In particular, the portion of the proof showing that $\mathcal{A}_1 := \mathcal{H}om_\eta(\mathcal{K}, \mathcal{F})$ is locally free has been heavily modified and expanded.

Theorem 3. *The Quot scheme of quotients of the trivial bundle on a Fano surface S admits a perfect obstuction theory and a virtual fundamental class of the expected dimension e, which is given in Theorem 4.*

Proof. Constructing the virtual fundamental class amounts to finding a two term resolution $[\mathcal{A}_0 \to \mathcal{A}_1]$ on Quot_S of the tangent-obstruction complex by vector bundles. More specifically, given any sheaf $\mathcal{I}$ on Quot_S, we need an exact sequence

$$0 \to \mathcal{T}^1(\mathcal{I}) \to \mathcal{A}_0 \otimes \mathcal{I} \to \mathcal{A}_1 \otimes \mathcal{I} \to \mathcal{T}^2 \otimes \mathcal{I} \to 0 \tag{4.1}$$

where $\mathcal{T}^1(\mathcal{I})$ and $\mathcal{T}^2 \otimes \mathcal{I}$ denote the tangent and obstruction spaces on Quot_S in the direction of $\mathcal{I}$, respectively. Complete definitions can be found in [LT]. We recall the well known fact that [BFl]

$$\mathcal{T}^1(\mathcal{I}) = \mathcal{H}om_\eta(\mathcal{E}, \mathcal{F} \otimes \eta^*\mathcal{I}), \quad \mathcal{T}^2 = \mathcal{E}xt^1_\eta(\mathcal{E}, \mathcal{F})$$

where $\eta : \mathrm{Quot}_S \times S \to \mathrm{Quot}_S$ is the projection.

We produce the resolution (4.1) by means of the Grothendieck embedding $i : \mathrm{Quot}_S \to G$ of a Quot scheme into a Grassmannian G. This embedding is part of the original constuction of the Quot scheme, and full details can be found either in [MO1] or [FGIKNV]. Let $m >> 0$, so that $\mathcal{E}(m)$ and $\mathcal{F}(m)$ are fiberwise generated by sections and

$$R^1\eta_*\mathcal{E}(m) = R^2\eta_*\mathcal{E}(m) = R^1\eta_*\mathcal{F}(m) = R^2\eta_*\mathcal{F}(m) = 0.$$

Next, let $\mathcal{K}$ be the sheaf on $\mathrm{Quot}_S \times S$ defined by the exact sequence

$$0 \to \mathcal{K} \to \eta^*(\eta_*\mathcal{E}(m)) \otimes \mathcal{O}(-m) \to \mathcal{E}(m) \to 0. \tag{4.2}$$

Then applying the functor $\mathcal{H}om_\eta(\cdot, \mathcal{F})$ to (4.2) produces the following sequence of sheaves on Quot_S:

$$\begin{array}{ccccccc} 0 \longrightarrow & \mathcal{H}om_\eta(\mathcal{E},\mathcal{F}) & \longrightarrow & (\eta_*\mathcal{E}(m))^\vee \otimes \eta_*\mathcal{F}(m) & \longrightarrow & \mathcal{H}om_\eta(\mathcal{K},\mathcal{F}) & \longrightarrow \\ & \mathcal{E}xt^1_\eta(\mathcal{E},\mathcal{F}) & \longrightarrow & 0 & \longrightarrow & \mathcal{E}xt^1_\eta(\mathcal{K},\mathcal{F}) & \longrightarrow \\ & \mathcal{E}xt^2_\eta(\mathcal{E},\mathcal{F}) & \longrightarrow & 0 & & & \end{array}$$

We will show that, up to a twist by $\eta_*\mathcal{I}$, the first four sheaves in this exact sequence give the desired resolution of the tangent-obstruction complex (4.1). This is essentially the same construction as for Quot_C in [MO1], although proving that $\mathcal{A}_1$ is a vector bundle is no longer immediate.

The second sheaf $(\eta_*\mathcal{E}(m))^\vee \otimes \eta_*\mathcal{F}(m)$ can be computed by the projection formula and equals the pullback $i^*\mathcal{T}G$ of the tangent bundle of G, so $\mathcal{A}_0 := i^*\mathcal{T}G$. We also know that $\mathcal{E}xt^2_\eta(\mathcal{K},\mathcal{F}) = 0$. If we can also show that $\mathcal{E}xt^1_\eta(\mathcal{K},\mathcal{F}) = 0$, then the third sheaf $\mathcal{A}_1 := \mathcal{H}om_\eta(\mathcal{K},\mathcal{F})$ will be locally free by cohomology and base change.

We therefore equivalently show that $\mathcal{E}xt^2_\eta(\mathcal{E},\mathcal{F}) = 0$. It suffices to show this fibrewise, so our goal is to show that over a closed point

$$[0 \to E \to \mathcal{O}^N \to F \to 0] \tag{4.3}$$

we have $\mathrm{Ext}^2(E,F) = 0$. Applying the functor $\mathrm{Ext}(E,\cdot)$ to (4.3) yields a long exact sequence ending with

$$\cdots \longrightarrow \mathrm{Ext}^2(E,E) \longrightarrow \mathrm{Ext}^2(E,\mathcal{O}^N) \longrightarrow \mathrm{Ext}^2(E,F) \longrightarrow 0. \tag{4.4}$$

Let K denote the canonical sheaf. Then by Serre duality and the Fano condition on S we find

$$\mathrm{Ext}^2(E,\mathcal{O}^N) \cong \oplus^N \mathrm{Ext}^2(E,\mathcal{O}) \cong \oplus^N \mathrm{Hom}(\mathcal{O}, E\otimes K)^\vee \hookrightarrow \oplus^N \mathrm{Hom}(\mathcal{O},K)^\vee = 0.$$

Going back to (4.4), this shows $\mathrm{Ext}^2(E,F) = 0$, and hence $\mathcal{A}_1 := \mathcal{H}om_\eta(\mathcal{K},\mathcal{F})$ is locally free.

The final step to complete the derivation of (4.1) is to observe that the argument goes through if we twist by $\eta_*\mathcal{I}$. That is, we apply the functor $\mathcal{H}om_\eta(\cdot, \mathcal{F}\otimes\eta^*\mathcal{I})$ to (4.2) instead of just $\mathcal{H}om_\eta(\cdot,\mathcal{F})$. This was done in [MO1] and works just as well for Quot_S. Then by [LT], the virtual fundamental class is constructed as

$$[\mathrm{Quot}_S]^{vir} = 0^!_{\mathcal{A}_1}[\mathcal{C}] \in H_{2e}(\mathrm{Quot}_S),$$

where $\mathcal{C}$ is a cone inside $\mathcal{A}_1$. This completes the proof. □

For the remainder of this paper, S will denote a Fano surface so that $[\mathrm{Quot}_S]^{vir}$ exists. This class of examples includes the projective spaces $\mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{P}^2$. The results of this theorem may not be optimal; it is entirely possible that that there are non-Fano surfaces which admit perfect obstruction theories. For example, as long as the dimension of $\mathrm{Ext}^2(E,F)$ is constant (not necessarily 0 as in the Fano case), we would expect the existence of a virtual fundamental class.

Next we will derive the expected dimension of $\mathrm{Quot}_S(\mathbb{G}(r,N))_{\chi,\beta}$. From there, we will choose several test cases for calculating virtual invariants. The specific computations will be carried out in the next two chapters.

Theorem 4. *Let K denote the canonical sheaf on S. Then $Quot_S(\mathbb{G}(r,N))_{\chi,\beta}$ has expected dimension $e = (N-r)r + \beta^2 - \left(\frac{N}{2}\right)\beta\cdot K + (N-2r)(N-\chi-r-\frac{1}{2}\beta\cdot K)$. As a special case, when $\beta = 0$ the expected dimension of $Quot_S(\mathbb{G}(r,N))_{\chi,0}$ is $e = (N-r)r + (N-2r)(N-\chi-r)$.*

Proof. We know that $virdim\mathrm{Quot}_S(\mathbb{G}(r,N))_{\chi,\beta} = \sum_j (-1)^j \mathcal{E}\mathrm{xt}^j(\mathcal{E},\mathcal{F})$ is equal to $\sum_j (-1)^j \mathrm{Ext}^j(E,F)$ which is constant on stalks, so it suffices to compute the virtual dimension on the stalks where the kernel E is the direct sum $\oplus_{i=1}^r \mathcal{I}_{z_i}(-D_i)$, with each direct summand injecting into a unique copy of $\mathcal{O}$. For notational simplicity, we assume that these summands inject into the first r copies of $\mathcal{O}$. Here the z_i denote subschemes of dimension 0 and D_i denote subschemes of dimension 1, meaning that $\sum_{i=1}^r D_i = \beta$. Over such a fiber the universal sequence restricts to

$$0 \longrightarrow \oplus_{i=1}^r \underbrace{\mathcal{I}_{z_i}(-D_i)}_{E_i} \longrightarrow \mathcal{O}^N \longrightarrow \oplus_{i=1}^r \underbrace{(\mathcal{O}_{z_i} \oplus \mathcal{O}_{D_i})}_{F_i} \oplus \mathcal{O}^{N-r} \longrightarrow 0.$$

Thus

$$e = \sum_{i=1}^r \sum_{k=1}^r \chi(E_i, F_k) + (N-r)\sum_{i=1}^r \chi(E_i, \mathcal{O}) \tag{4.5}$$

where $\chi(A,B) = \sum_j (-1)^j \mathrm{Ext}(A,B)$.

In order to compute this expected dimension, we make use of the exact sequences

$$0 \longrightarrow \mathcal{O}(-D_i) \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}_{D_i} \longrightarrow 0$$

and

$$0 \longrightarrow \mathcal{I}_{z_i} \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}_{z_i} \longrightarrow 0$$

along with the fact that $\chi(\cdot,\cdot)$ is additive on exact sequences. We will not exlicitly go through every application of this fact.

We begin with the terms

$$\begin{aligned}
\chi(E_i, F_k) &= \chi(\mathcal{I}_{z_i}(-D_i), \mathcal{O}_{z_k} \oplus \mathcal{O}_{D_k}) \\
&= \chi(\mathcal{I}_{z_i}(-D_i), \mathcal{O}_{z_k}) + \chi(\mathcal{I}_{z_i}(-D_i), \mathcal{O}_{D_k}) \\
&= \chi(\mathcal{I}_{z_i}, \mathcal{O}_{z_k}(D_i)) + \chi(\mathcal{I}_{z_i}, \mathcal{O}_{D_k}(D_i)) \\
&= \text{length}(z_k) + \chi(\mathcal{O}, \mathcal{O}_{D_k}(D_i)) - \chi(\mathcal{O}_{z_i}, \mathcal{O}_{D_k}(D_i)) \\
&= \text{length}(z_k) + \chi(\mathcal{O}, \mathcal{O}_{D_k}(D_i)) \\
&= \text{length}(z_k) + \chi(\mathcal{O}, \mathcal{O}(D_i)) - \chi(\mathcal{O}, \mathcal{O}(D_i - D_k)).
\end{aligned}$$

Let K be the canonical divisor and g_{D_i} the genus of D_i. Then by the adjunction formula,

$$\chi(\mathcal{O}, \mathcal{O}(-D_i)) = 1 - \chi(\mathcal{O}_{D_i}) = 1 - (1 - g_{D_i}) = g_{D_i} = 1 + \frac{1}{2} D_i \cdot (D_i + K).$$

Hence,

$$\begin{aligned}
\chi(E_i, F_k) =& \text{length}(z_k) + 1 + \frac{1}{2}(-D_i) \cdot (-D_i + K) \\
& - 1 - \frac{1}{2}(D_k - D_i) \cdot (D_k - D_i + K) \\
=& \text{length}(z_k) + D_i \cdot D_k - \frac{(D_k^2 + D_k \cdot K)}{2}
\end{aligned} \tag{4.6}$$

Next we consider the terms

$$\begin{aligned}
\chi(E_i, \mathcal{O}) &= \chi(\mathcal{I}_{z_i}(-D_i), \mathcal{O}) \\
&= \chi(\mathcal{I}_{z_i}, \mathcal{O}(D_i)) \\
&= \chi(\mathcal{O}, \mathcal{O}(D_i)) - \chi(\mathcal{O}_{z_i}, \mathcal{O}(D_i)) \\
&= 1 + \frac{1}{2}(-D_i) \cdot (-D_i + K) - \text{length}(z_i) \\
&= 1 + \frac{D_i^2}{2} - \frac{D_i \cdot K}{2} - \text{length}(z_i).
\end{aligned} \tag{4.7}$$

We now put together (4.5), (4.6), and (4.7) to obtain

$$\begin{aligned}
e =& \sum_{i=1}^{r}\sum_{k=1}^{r}\chi(E_i,F_k) + (N-r)\sum_{i=1}^{r}\chi(E_i,\mathcal{O}) \\
=& \sum_{i=1}^{r}\sum_{k=1}^{r}\left(\text{length}(z_k) + D_i\cdot D_k - \frac{D_k^2}{2} - \frac{D_k\cdot K}{2}\right) \\
&+ (N-r)\sum_{i=1}^{r}\left(1 + \frac{D_i^2}{2} - \frac{D_i\cdot K}{2} - \text{length}(z_i)\right) \\
=& r\cdot\text{total length}(z) + \beta^2 - \frac{r}{2}\cdot\sum_{k=1}^{r}D_k^2 - \frac{r}{2}\beta\cdot K \\
&+ (N-r)r + \left(\frac{N-r}{2}\right)\sum_{i=1}^{r}D_i^2 - \left(\frac{N-r}{2}\right)\beta\cdot K - (N-r)\cdot\text{total length}(z) \\
=& (N-r)r + \beta^2 + (2r-N)\cdot\text{total length}(z) + \left(\frac{N-2r}{2}\right)\sum_{i=1}^{r}D_i^2 - \left(\frac{N}{2}\right)\beta\cdot K.
\end{aligned} \tag{4.8}$$

This would seem to be a problem, as the virtual dimension appears to depend upon the particular decomposition of D_i's. To show that this is not the case, consider

$$\begin{aligned}
\sum_{i=1}^{r}\chi(\mathcal{O},E_i) &= \sum_{i=1}^{N}\chi(\mathcal{O},\mathcal{O}) - (\sum_{i=1}^{r}\chi(\mathcal{O},F_i) + \sum_{i=r+1}^{N}\chi(\mathcal{O},\mathcal{O})) \\
&= N - \chi.
\end{aligned}$$

However, we can alternately compute

$$\begin{aligned}
\sum_{i=1}^{r}\chi(\mathcal{O},E_i) &= \sum_{i=1}^{r}\chi(\mathcal{O},\mathcal{I}_{z_i}(-D_i)) \\
&= \sum_{i=1}^{r}\chi(\mathcal{O},\mathcal{O}(-D_i)) - \sum_{i=1}^{r}\chi(\mathcal{O},\mathcal{O}_{z_i}(-D_i)) \\
&= \sum_{i=1}^{r}\chi(\mathcal{O},\mathcal{O}(-D_i)) - \text{total length}(z) \\
&= \sum_{i=1}^{r}\left(1 + \frac{1}{2}D_i\cdot(D_i+K)\right) - \text{total length}(z) \\
&= r + \frac{1}{2}\beta\cdot K + \frac{1}{2}\sum_{i=1}^{r}D_i^2 - \text{total length}(z).
\end{aligned}$$

Combining these alternate computations of $\sum_{i=1}^{r} \chi(\mathcal{O}, E_i)$, we see

$$N - \chi = r + \frac{1}{2}\beta \cdot K + \frac{1}{2}\sum_{i=1}^{r} D_i^2 - \text{total length}(z)$$

or

$$(N-2r)(N - \chi - r - \frac{1}{2}\beta \cdot K) = \frac{N-2r}{2}\sum_{i=1}^{r} D_i^2 - (N-2r)\text{total length}(z).$$

When we substitute this equality into (4.8), we finally compute the expected dimension e as

$$e = (N-r)r + \beta^2 - \left(\frac{N}{2}\right)\beta \cdot K + (N-2r)(N - \chi - r - \frac{1}{2}\beta \cdot K). \tag{4.9}$$

The special case with $\beta = 0$ follows immediately. This completes the proof. □

We will now restrict our attention to the case when $\beta = 0$ for the remainder of this paper. This restriction is not an absolute necessity, but merely a convenience as we explore the simplest computations of enumerative invariants on Quot_S. A careful reading of the proof of Theorem 4 shows that when $\beta = 0$, the virtual dimension $e = (N-r)r + (N-2r)(N-\chi-r)$ can also be written as $e = (N-r)r + (N-2r)\cdot l$ where l is the total length of the zero dimensional subscheme. We will frequently refer this as the length of Quot_S.

In choosing data for the first explorations into this theory, we would like the expected dimension to be 0. In this case, $[\text{Quot}_S]^{vir}$ is simply a collection of points, and the intersection

$$[\text{Quot}_S]^{vir} \cap [1] = \int_{[\text{Quot}_S]^{vir}} 1$$

makes sense. In this way we avoid the additional complication of intersection with higher Chern classes.

One particularly nice example with expected dimension 0 occurs when $N = 3$, $r = 1$, and $\chi = 4$ (and $l = 2$). In chapter 5 we will compute $\int_{[Quot_S(\mathbb{G}(1,3))_{4,0}]^{vir}} 1$ when S is $\mathbb{P}^2$ or $\mathbb{P}^1 \times \mathbb{P}^1$. While this example is comparatively simple and serves well to illustrate our intended use of virtual localization, it is too simple to generalize well. To that end, we also need an example with $r \geq 2$.

Another example of expected dimension 0 occurs when $N = 8$, $r = 2$, and $\chi = 9$ (and $l = 3$). This example generalizes more readily, and we will explore this example in chapter 6.

4.2 Torus action and fixed loci

In parallel with the methods in [MO1], our next step is to pick a torus action for our localization. Unfortunately, we cannot use exactly their torus action inherited from the action on $\mathbb{C}^N$. Without going into great depth, this would lead to fixed loci consisting of products of Hilbert schemes of points on S, whose intersection theory proved more difficult than over a curve C, where this is just products of symmetric products of C. Therefore, we will refine this torus action with the hope of obtaining fixed loci consisting of isolated fixed points.

We will, in fact, use a product torus action. Let T_1 be the action of $\mathbb{C}^*$ on $\mathbb{C}^N$ as in [MO1]. If $S = \mathbb{P}^1 \times \mathbb{P}^1$ or $S = \mathbb{P}^2$ then we have another obvious C^* action S. This action extends to an action on $\mathcal{O}$ and we call this action T_2. Our new torus action T will be the product of these actions, so $\mathbb{C}^*$ acts on $\mathbb{C}^N$ by a diagonal matrix with weights $\lambda_1, \cdots, \lambda_N$, and also simultaneously on each copy of $\mathcal{O}$.

To explain this latter action in greater detail, let $S = \mathbb{P}^1 \times \mathbb{P}^1$ for a moment. Then in a neighborhood around the point $[1:0] \times [1:0]$, the structure sheaf is given locally by $\mathbb{C}[x, y]$, and T_2 has weight a on x and weight b on y.

This torus action is a refinement of T_1, so any point fixed by T lies within the fixed locus of T_1. When $S = \mathbb{P}^1 \times \mathbb{P}^1$, the only fixed points of T_2 are the "corners" of $\mathbb{P}^1 \times \mathbb{P}^1$, that is, the points $[1:0] \times [1:0]$, $[1:0] \times [0:1]$, $[0:1] \times [0:1]$, and $[0:1] \times [1:0]$. Thus, the fixed points of T are those short exact sequences

$$0 \to E \to \mathbb{C}^N \otimes \mathcal{O} \to F \to 0$$

in which E decomposes as a sum of ideal sheaves of zero dimensional subschemes

$$E = \mathcal{I}_{z_1} \oplus \cdots \oplus \mathcal{I}_{z_s} \oplus \mathcal{O}^{r-s}$$

supported only over the corners of $\mathbb{P}^1 \times \mathbb{P}^1$, and the sum of the lengths l_i of the z_i's is the total length l. There is a similar statement when $S = \mathbb{P}^2$, and support of the z_i lies over the three points $[1:0:0]$, $[0:1:0]$, and $[0:0:1]$.

Next, we need to determine the Euler class of the virtual normal bundle $e_T(N_i^{vir})$ for the fixed loci $\mathrm{Quot}_S^T \overset{i}{\hookrightarrow} \mathrm{Quot}_S$. The virtual normal bundle is the

moving part of the universal sequence

$$0 \to \mathcal{E} \to \mathcal{O}^N \to \mathcal{F} \to 0$$

on $\mathrm{Quot}_S \times S$, restricted to $\{\text{fixed points of } T\}\times S$.

Let

$$0 \to \mathcal{I}_{z_1} \oplus \cdots \oplus \mathcal{I}_{z_s} \oplus \mathcal{O}^{r-s} \to \mathcal{O}^N \to \mathcal{O}_{z_1} \oplus \cdots \oplus \mathcal{O}_{z_s} \oplus \mathcal{O}^{N-r} \to 0 \qquad (4.10)$$

be a point of the fixed locus. The presentation with the kernal injecting into the first r copies of $\mathcal{O}$ is merely a notational convenience. The equivariant normal bundle is just the product of the tangent minus obstruction spaces of (4.10). More concretely, over the fixed point corresponding to (4.10)

$$\begin{aligned} N^{vir} = &\oplus_{i=1}^{s} \oplus_{j=1}^{s} \left(\mathrm{Ext}^\bullet(\mathcal{I}_{z_i}, \mathcal{O}_{z_j})\right) \oplus_{i=1}^{s} \left(\mathrm{Ext}^\bullet(\mathcal{I}_{z_i}, \mathcal{O}^{N-r})\right) \\ &\oplus_{j=1}^{s} \left(\mathrm{Ext}^\bullet(\mathcal{O}^{r-s}, \mathcal{O}_{z_j})\right) \oplus \left(\mathrm{Ext}^\bullet(\mathcal{O}^{r-s}, \mathcal{O}^{N-r})\right). \end{aligned} \qquad (4.11)$$

We will be able to make this quite explicit in the examples of chapters 5 and 6.

The Euler class of (4.11) will just be the quotient of top Chern classes of each Ext group in the product, expressed in terms of the weights of T, namely a, b, and $\lambda_1 \cdots, \lambda_N$.

Chapter 5

Rank 1 example

We are finally ready to begin computing the intersection product

$$\int_{[Quot_S(\mathbb{G}(1,3))_{4,0}]^{vir}} 1$$

when $S = \mathbb{P}^1 \times \mathbb{P}^1$ or $S = \mathbb{P}^2$. We will start with $\mathbb{P}^1 \times \mathbb{P}^1$.

5.1 Computing $\int_{[Quot_{\mathbb{P}^1\times\mathbb{P}^1}(\mathbb{G}(1,3))_{4,0}]^{vir}} 1$

We begin by making the fixed loci description from (4.10) more explicit. Since the rank is 1 and the length $l = 2$, any fixed point in $\mathrm{Quot}_{\mathbb{P}^1\times\mathbb{P}^1}(\mathbb{G}(1,3))_{4,0}$ corresponds to a short exact sequence

$$0 \to \mathcal{I}_z \to \mathcal{O}^3 \to \mathcal{O}_z \oplus \mathcal{O}^2 \to 0$$

where z has length 2 and support over the fixed points of $\mathbb{P}^1 \times \mathbb{P}^1$.

Pictorially, we think of $\mathbb{P}^1 \times \mathbb{P}^1$ as a square, with $\mathbb{P}^1$'s as the edges and the corners as the fixed points. This is standard practice in algebraic geometry. Then the isolated fixed points correspond to Young tableaux at the corners with total length 2. Hence, up to rotation, the fixed loci are give by configurations in Figure 5.1.

More precisely, configurations (1), (2), and (3) each have four-fold rotation, while configuration (4) only has two-fold rotation. This gives a total of 14 configurations for the support of z. Of course, we also need to sum over the various copies

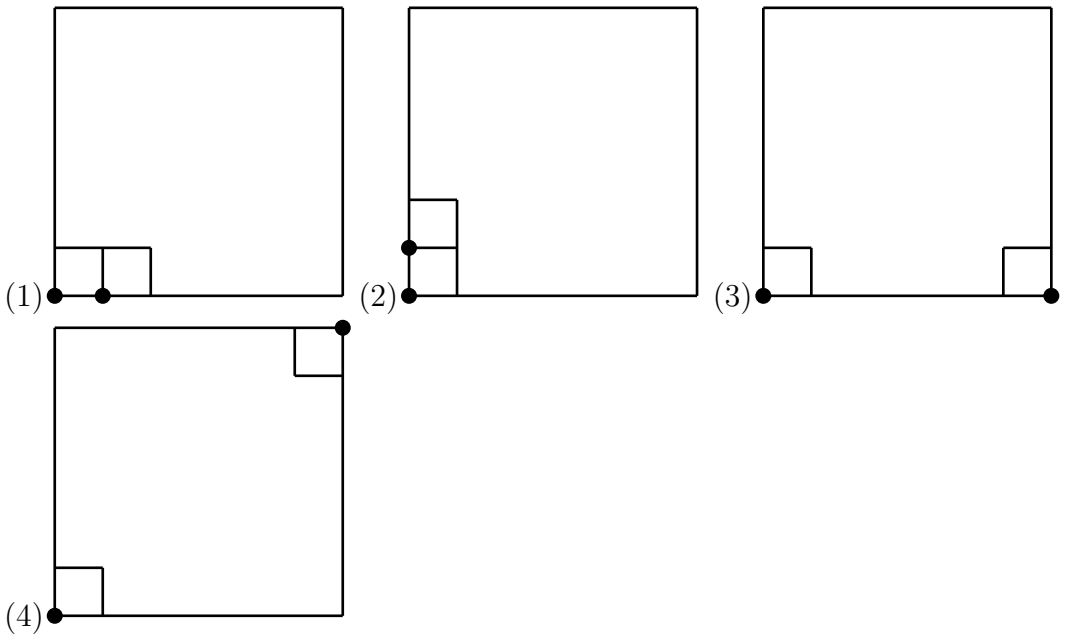


Figure 5.1: Fixed loci for rank 1 example over $\mathbb{P}^1 \times \mathbb{P}^1$

of $\mathcal{O}$ into which $\mathcal{I}_z$ can inject, which gives us 3 fixed loci for each configuration in Figure 5.1. Hence we have a total of 42 isolated fixed points, albeit with a great deal of symmetry.

Next we idenity the equivariant normal bundle. Using (4.11), we enter our specific enumerative data to obtain

$$N^{vir} = \mathrm{Ext}^\bullet(\mathcal{I}_z, \mathcal{O}_z) \oplus \mathrm{Ext}^\bullet(\mathcal{I}_{z_i}, \mathcal{O}^2).$$

Thus,

$$\begin{aligned}\int_{[Quot_{\mathbb{P}^1\times\mathbb{P}^1}(\mathbb{G}(1,3))_{4,0}]^{vir}} 1 &= \sum_{\text{fixed loci}} \int \frac{1}{e_T(N^{vir})} \\ &= \sum \int_{[\mathrm{pt}]} \frac{1}{e_T(\mathrm{Ext}^\bullet(\mathcal{I}_z, \mathcal{O}_z))} \cdot \frac{1}{e_T(\mathrm{Ext}^\bullet(\mathcal{I}_z, \mathcal{O}))} \cdot \frac{1}{e_T(\mathrm{Ext}^\bullet(\mathcal{I}_z, \mathcal{O}))} \\ &= \sum \int_{[\mathrm{pt}]} \frac{c_2(\mathrm{Ext}^1(\mathcal{I}_z, \mathcal{O}_z))}{c_4(\mathrm{Ext}^0(\mathcal{I}_z, \mathcal{O}_z))} \cdot \frac{c_2(\mathrm{Ext}^1(\mathcal{I}_z, \mathcal{O}))}{c_1(\mathrm{Ext}^0(\mathcal{I}_z, \mathcal{O}))} \cdot \frac{c_2(\mathrm{Ext}^1(\mathcal{I}_z, \mathcal{O}))}{c_1(\mathrm{Ext}^0(\mathcal{I}_z, \mathcal{O}))}. \quad (5.1)\end{aligned}$$

Here we have filled in the degrees of the top Chern classes of each Ext group, even though we have not yet derived them. The remainder of this section will be devoted to determining these degrees and computing the Chern classes explicitly for each fixed point configuration.

Most of these degrees are quite easy to derive. The tangent space $\mathrm{Ext}^0(\mathcal{I}_z, \mathcal{O}_z)$ and obstruction space $\mathrm{Ext}^1(\mathcal{I}_z, \mathcal{O}_z)$ of the Hilbert scheme of two points $S^{[2]}$ have expected dimensions 4 and 2, respectively, as shown in [N]. The only morphisms from $\mathcal{I}_z$ to $\mathcal{O}$ are given by scalar multiplication, so $\mathrm{Ext}^0(\mathcal{I}_z, \mathcal{O})$ has dimension 1. This only leaves the dimension of $\mathrm{Ext}^1(\mathcal{I}_z, \mathcal{O})$.

Let ω denote the class of a point in $\mathbb{P}^1 \times \mathbb{P}^1$ and recall that

$$\sum_{i=0}^{2} (-1)^i dim \mathrm{Ext}^i(\mathcal{I}_z, \mathcal{O})$$

is given by the Euler characteristic

$$\chi(\mathcal{I}_z, \mathcal{O}) = \int_{[\mathbb{P}^1 \times \mathbb{P}^1]} (1 - 2\omega)(1)(1 + \omega) = -1.$$

However, we already know that $\mathrm{Ext}^2(\mathcal{I}_z, \mathcal{O})$ vanishes by the Fano condition, and $dim \mathrm{Ext}^0(\mathcal{I}_z, \mathcal{O}) = 1$. Hence, $\mathrm{Ext}^i(\mathcal{I}_z, \mathcal{O})$ must be two dimensional. This gives the degrees of all relevant top Chern classes in (5.1).

Next we determine which of the Ext groups depend on the λ's, which depend on a and b, and which depend on both. Given that λ_i acts on the ith copy of $\mathcal{O}$, a group $\mathrm{Ext}(A, B)$ depends on the λ's if and only if A and B are the subsheaf and quotient of different copies of $\mathcal{O}$. Furthmore, $\mathrm{Ext}(A, B)$ will depend on a and b unless the only morphisms are scalar multiplication, as is the case with $\mathrm{Ext}^0(\mathcal{I}_z, \mathcal{O})$. We summarize these dependencies in Table 5.1.

Table 5.1: Weight dependency for rank 1 example over $\mathbb{P}^1 \times \mathbb{P}^1$

	depends on λ's	depends on a, b
$c_2(\mathrm{Ext}^1(\mathcal{I}_z, \mathcal{O}_z))$	no	yes
$c_4(\mathrm{Ext}^0(\mathcal{I}_z, \mathcal{O}_z))$	no	yes
$c_2(\mathrm{Ext}^1(\mathcal{I}_z, \mathcal{O}))$	yes	yes
$c_1(\mathrm{Ext}^0(\mathcal{I}_z, \mathcal{O}))$	yes	no

We establish the following conventions: $\mathcal{I}_z$ injects into the ith copy of $\mathcal{O}$ with weight λ_i. All other copies of $\mathcal{O}$ will be written with weight λ_k, so that we are taking a sum over k from 1 to 3, with $k \neq i$.

Before we begin computing the Chern classes of the various Ext groups, we make a brief note about equivariance. We will frequently make use of certain isomorphisms between Ext groups during our computations. Technically, we should be careful to check that any isomorphism used in this way is an *equivariant* isomorphism. In practice, any natural isomorphism will be equivariant. This includes isomorphisms induced by long exact sequences arising from the Snake Lemma, and Serre duality. The proofs of these facts are lengthy and overly technical, so we omit them. Unless otherwise mentioned, all isomorphisms will be assumed equivariant with respect to a and b *but not necessarily the* λ*'s.*

With that taken care of, we begin computing the Chern classes of each Ext group, for the four fixed point configurations in Figure 5.1.

$\underline{c_4(\mathrm{Ext}^0(\mathcal{I}_z, \mathcal{O}_z))}$

Note that $\mathrm{Ext}^0(\mathcal{I}_z, \mathcal{O}_z)$ is simply the tangent space of the Hilbert scheme of two points of $S = \mathbb{P}^1 \times \mathbb{P}^1$. Near the point $[1:0] \times [1:0]$, the torus T acts on the (local) structure sheaf $\mathbb{C}[x, y]$ by a and b. Proceeding counterclockwise around the square picture of $\mathbb{P}^1 \times \mathbb{P}^1$, T acts near $[0:1] \times [1:0]$ with weights $-a$ and b, near $[0:1] \times [0:1]$ with weights $-a$ and $-b$, and near $[1:0] \times [0:1]$ with weights a and $-b$. Then the Chern roots of $\mathrm{Ext}^0(\mathcal{I}_z, \mathcal{O}_z)$ are given in [N] (up to a global sign change), so we simply list the results.

(1) The four Chern roots are $-2ah$, $-bh$, $-ah$, $(a-b)h$, so the top Chern class is

$$c_4(\mathrm{Ext}^0(\mathcal{I}_z, \mathcal{O}_z)) = -2a^2b(a-b)h^4.$$

(2) The Chern roots are $-ah$, $-2bh$, $(b-a)h$, $-bh$, so

$$c_4(\mathrm{Ext}^0(\mathcal{I}_z, \mathcal{O}_z)) = -2ab^2(b-a)h^4.$$

(3) This is just a product the tangent spaces of the two disjoint points. Therefore the roots are

$$\underbrace{-ah, -bh}_{\text{first point}} \quad \underbrace{ah, -bh}_{\text{second point}}$$

so the Chern class is

$$c_4(\mathrm{Ext}^0(\mathcal{I}_z, \mathcal{O}_z)) = -a^2b^2h^4.$$

(4) Similarly, this is a product and the roots are

$$\underbrace{-ah, -bh}_{\text{first point}} \quad \underbrace{ah, bh}_{\text{second point}}$$

so the Chern class is

$$c_4(\mathrm{Ext}^0(\mathcal{I}_z, \mathcal{O}_z)) = a^2b^2h^4.$$

$\underline{c_1(\mathrm{Ext}^0(\mathcal{I}_z, \mathcal{O}))}$

This is actually the easiest Chern class to compute. There is no dependence on a or b, so only the λ's matter. Since $\mathrm{Ext}^0 = \mathrm{Hom}$, the weight of T on $\mathrm{Ext}^0(\mathcal{I}_z, \mathcal{O})$ is simply the weight on $\mathcal{O}$ minus the weight on $\mathcal{I}_z$. Thus for every fixed point configuration we find

$$c_1(\mathrm{Ext}^0(\mathcal{I}_z, \mathcal{O})) = (\lambda_k - \lambda_i)h.$$

$\underline{c_2(\mathrm{Ext}^1(\mathcal{I}_z, \mathcal{O}))}$

We already know that the λ dependence will be $(\lambda_k - \lambda_i)$ for all cases, so we will insert this at the end. To compute the dependence on a and b, consider the short exact sequence

$$0 \longrightarrow \mathcal{I}_z \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}_z \longrightarrow 0 \tag{5.2}$$

and apply the functor $\mathrm{Hom}(\mathcal{I}_z, _)$ to obtain the long exact sequence

$$\begin{array}{rlll} 0 \longrightarrow & \mathrm{Ext}^0(\mathcal{I}_z, \mathcal{I}_z) \longrightarrow & \mathrm{Ext}^0(\mathcal{I}_z, \mathcal{O}) \longrightarrow & \mathrm{Ext}^0(\mathcal{I}_z, \mathcal{O}_z) \longrightarrow \\ & \mathrm{Ext}^1(\mathcal{I}_z, \mathcal{I}_z) \longrightarrow & \mathrm{Ext}^1(\mathcal{I}_z, \mathcal{O}) \longrightarrow & \mathrm{Ext}^1(\mathcal{I}_z, \mathcal{O}_z) \longrightarrow \\ & \mathrm{Ext}^2(\mathcal{I}_z, \mathcal{I}_z) \longrightarrow & \mathrm{Ext}^2(\mathcal{I}_z, \mathcal{O}) \longrightarrow & \mathrm{Ext}^2(\mathcal{I}_z, \mathcal{O}_z) \longrightarrow \quad 0. \end{array}$$

Both $\mathrm{Ext}^0(\mathcal{I}_z, \mathcal{I}_z)$ and $\mathrm{Ext}^0(\mathcal{I}_z, \mathcal{O})$ consist only of maps given by scalar multiplication, so they are each one dimensional. Next, $\mathrm{Ext}^0(\mathcal{I}_z, \mathcal{O}_z)$ and $\mathrm{Ext}^1(\mathcal{I}_z, \mathcal{I}_z)$ are both the tangent space (one viewed as the tangent space of Hilbert scheme, one as the tangent space of the same Hilbert scheme considered as a Quot scheme), so they are four dimensional. Additionally, $\mathrm{Ext}^1(\mathcal{I}_z, \mathcal{O}_z)$ is the obstruction space, which makes it two dimensional, and $\mathrm{Ext}^2(\mathcal{I}_z, \mathcal{O})$ and $\mathrm{Ext}^2(\mathcal{I}_z, \mathcal{O}_z)$ vanish by the Fano condition. On page 23 we used Euler characteristics to prove that $\mathrm{Ext}^1(\mathcal{I}_z, \mathcal{O})$ was two dimensional, and a similar Euler characteristic computation shows that $\mathrm{Ext}^2(\mathcal{I}_z, \mathcal{I}_z)$ vanishes. Hence the dimensions are given by

$$\begin{array}{ccccccc} 0 \longrightarrow & 1 & \xrightarrow{\cong} & 1 & \longrightarrow & 4 & \xrightarrow{\cong} \\ & 4 & \longrightarrow & 2 & \xrightarrow{\cong} & 2 & \longrightarrow \\ & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow \quad 0 \end{array}$$

with the indicated isomorphisms. In particular, this shows that $\mathrm{Ext}^1(\mathcal{I}_z, \mathcal{O}) \cong \mathrm{Ext}^1(\mathcal{I}_z, \mathcal{O}_z)$ and this isomorphism is equivariant with respect to a and b.

If we apply the functor $\mathrm{Hom}(_, \mathcal{O}_z)$ to (5.2) then we can readily show that $\mathrm{Ext}^1(\mathcal{I}_z, \mathcal{O}_z) \cong \mathrm{Ext}^2(\mathcal{O}_z, \mathcal{O}_z)$. Using Serre duality as well, this implies that

$$\mathrm{Ext}^1(\mathcal{I}_z, \mathcal{O}) \cong \mathrm{Ext}^1(\mathcal{I}_z, \mathcal{O}_z) \cong \mathrm{Ext}^2(\mathcal{O}_z, \mathcal{O}_z) \cong \mathrm{Ext}^0(\mathcal{O}_z, \mathcal{O}_z \otimes K)^\vee$$

where K is the canonical sheaf on $\mathbb{P}^1 \times \mathbb{P}^1$. We can compute the Chern class of the last Ext group in this chain of isomorphisms.

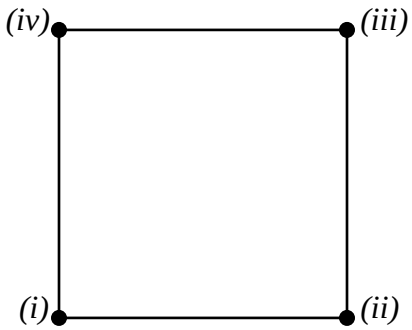


Figure 5.2: $\mathbb{P}^1 \times \mathbb{P}^1$ with labeled vertices

We begin by computing the Chern roots of K at the various fixed points given in Figure 5.2. In keeping with standard notation, we briefly redefine T to be the tangent space of $\mathbb{P}^1 \times \mathbb{P}^1$. Then over the point (i),

$$\begin{aligned} T \text{ has roots} \quad & -ah, -bh \\ T^\vee \text{ has roots} \quad & ah, bh \\ K = \wedge^2 T^\vee \text{ has the root} \quad & (a+b)h \end{aligned}$$

The roots for T are obtained from [N], adjusted for the global sign change. Pro-

ceeding to the other "vertices" of $\mathbb{P}^1 \times \mathbb{P}^1$, we get

$$\begin{aligned}
\text{over } (i): &\quad K = \wedge^2 T^\vee \text{ has the root } (a+b)h \\
\text{over } (ii): &\quad K = \wedge^2 T^\vee \text{ has the root } (-a+b)h \\
\text{over } (iii): &\quad K = \wedge^2 T^\vee \text{ has the root } (-a-b)h \\
\text{over } (iv): &\quad K = \wedge^2 T^\vee \text{ has the root } (a-b)h.
\end{aligned}$$

Now let's compute $c_2(\mathrm{Ext}^1(\mathcal{I}_z, \mathcal{O}))$ for our various fixed point configurations.

(1)

$$\begin{aligned}
K &\text{ has root } (a+b)h \\
\mathcal{O}_z \otimes K &\text{ has roots } (a+b)h, (2a+b)h \\
\mathrm{Hom}(\mathcal{O}_z, \mathcal{O}_z \otimes K) &\text{ has roots } (a+b)h, (2a+b)h \\
\mathrm{Ext}^1(\mathcal{I}_z, \mathcal{O}) \cong \mathrm{Ext}^1(\mathcal{I}_z, \mathcal{O}_z) &\cong \mathrm{Ext}^2(\mathcal{O}_z, \mathcal{O}_z) \cong \mathrm{Ext}^0(\mathcal{O}_z, \mathcal{O}_z \otimes K)^\vee \\
&\text{has roots } (-a-b)h, (-2a-b)h
\end{aligned}$$

In the end,

$$c_2(\mathrm{Ext}^1(\mathcal{I}_z, \mathcal{O})) = (\lambda_k - \lambda_i - a - b)(\lambda_k - \lambda_i - 2a - b)h^2.$$

(2) The method is sufficiently similar that we will simply list the Chern classes for the remaining cases. Here

$$c_2(\mathrm{Ext}^1(\mathcal{I}_z, \mathcal{O})) = (\lambda_k - \lambda_i - a - b)(\lambda_k - \lambda_i - a - 2b)h^2.$$

(3) In this case

$$c_2(\mathrm{Ext}^1(\mathcal{I}_z, \mathcal{O})) = (\lambda_k - \lambda_i - a - b)(\lambda_k - \lambda_i + a - b)h^2.$$

(4) In this case

$$c_2(\mathrm{Ext}^1(\mathcal{I}_z, \mathcal{O})) = (\lambda_k - \lambda_i - a - b)(\lambda_k - \lambda_i + a + b)h^2.$$

$\underline{c_2(\mathrm{Ext}^1(\mathcal{I}_z, \mathcal{O}_z))}$

Last, we need to compute $c_2(\mathrm{Ext}^1(\mathcal{I}_z, \mathcal{O}_z))$. Of course, we have already shown that the dependence on a and b is the same as for $c_2(\mathrm{Ext}^1(\mathcal{I}_z, \mathcal{O}))$, but $\mathrm{Ext}^1(\mathcal{I}_z, \mathcal{O}_z)$ does not depend on the λ's. Thus we can just remove the λ's from our results for $c_2(\mathrm{Ext}^1(\mathcal{I}_z, \mathcal{O}))$, and we are done. This completes all necessary Chern class computations.

Theorem 5. *The Quot scheme $Quot_S(\mathbb{G}(1,3))_{4,0}$ has expected dimension 0 and*

$$\int_{[Quot_{\mathbb{P}^1\times\mathbb{P}^1}(\mathbb{G}(1,3))_{4,0}]^{vir}} 1 = 204.$$

Proof. We already derived the expected dimension in Theorem 4. By virtual localization and (5.1),

$$\int_{[Quot_{\mathbb{P}^1\times\mathbb{P}^1}(\mathbb{G}(1,3))_{4,0}]^{vir}} 1 = \sum \int_{[\mathrm{pt}]} \frac{c_2(\mathrm{Ext}^1(\mathcal{I}_z, \mathcal{O}_z))}{c_4(\mathrm{Ext}^0(\mathcal{I}_z, \mathcal{O}_z))} \cdot \frac{c_2(\mathrm{Ext}^1(\mathcal{I}_z, \mathcal{O}))}{c_1(\mathrm{Ext}^0(\mathcal{I}_z, \mathcal{O}))} \cdot \frac{c_2(\mathrm{Ext}^1(\mathcal{I}_z, \mathcal{O}))}{c_1(\mathrm{Ext}^0(\mathcal{I}_z, \mathcal{O}))},$$

where the sum is taken over all 42 isolated fixed point. Each Chern class has been computed for all fixed points. Taking this finite sum by hand is possible, but it is neither quick nor particularly instructive. We therefore omit the actual summation and simply state the result. This completes the proof. Interested readers can verify the result of 204 using any mathematical computing software. □

5.2 Computing $\int_{[Quot_{\mathbb{P}^2}(\mathbb{G}(1,3))_{4,0}]^{vir}} 1$

Now we begin the same example on $\mathbb{P}^2$. Key things to remember here include the fact that the Todd class of $\mathbb{P}^2$ is still $(1+\omega)$, so all of the expected dimensions remain the same.

The torus action of $\mathbb{C}^*$ on $\mathbb{P}^2$ is

$$t \cdot [a_0 : a_1 : a_2] = [t^{\mu_0} a_0 : t^{\mu_1} a_1 : t^{\mu_2} a_2].$$

Thus, in a coordinate neighborhood around the point $[1 : 0 : 0]$, for example, where

the first coordinate is always nonzero, we can rescale the torus action as

$$\begin{aligned} t \cdot [a_0 : a_1 : a_2] &= [t^{\mu_0} a_0 : t^{\mu_1} a_1 : t^{\mu_2} a_2] \\ &= \left[1 : t^{\mu_1 - \mu_0} \frac{a_1}{a_0} : t^{\mu_2 - \mu_0} \frac{a_2}{a_0}\right] \end{aligned}$$

Hence, on this affine neighborhood the weights are $\mu_1 - \mu_0$ and $\mu_2 - \mu_0$. Similarly, on the affine neighborhood containing $[0 : 1 : 0]$ the weights are $\mu_0 - \mu_1$ and $\mu_2 - \mu_1$, and near $[0 : 0 : 1]$ the weights are $\mu_0 - \mu_2$ and $\mu_1 - \mu_2$.

More geometrically, we illustrate the weights around $\mathbb{P}^2$ in Figure 5.3.

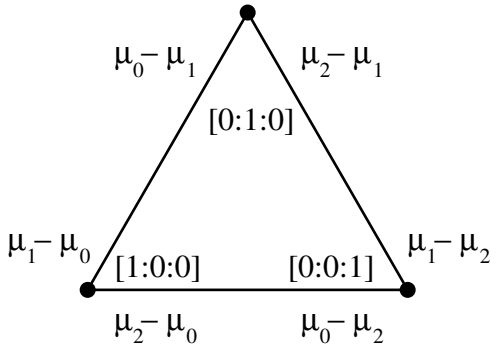


Figure 5.3: Fixed points and weights on $\mathbb{P}^2$

Ideally, this setup should display nice rotational symmetry, so let $\alpha = \mu_2 - \mu_0$, $\gamma = \mu_1 - \mu_2$, and $\beta = \mu_0 - \mu_1$. Then the weights can be relabeled as in Figure 5.4.

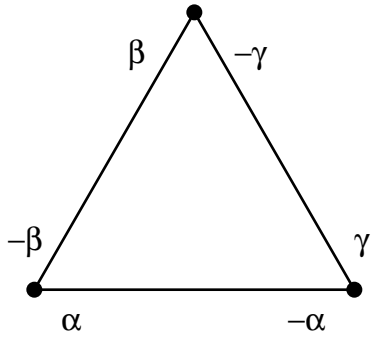


Figure 5.4: Relabeled weights on $\mathbb{P}^2$

From here on, we will use these Greek letters in place of the original weights μ, bearing in mind that $\alpha + \beta + \gamma = 0$. Most of the individual weights should remain

quite similar. We use λ's exactly as in the calculation on $\mathbb{P}^1 \times \mathbb{P}^1$, including the conventions on λ_i and λ_k.

The equivariant normal bundle consists of the exact same Ext groups, of the exact same dimensions, as in the calculation on $\mathbb{P}^1 \times \mathbb{P}^1$. Our next step must therefore be the determination of the fixed point loci. Up to three-fold rotational symmetry, any length two subscheme supported at the fixed points of $\mathbb{P}^2$ will be one given in Figure 5.5.

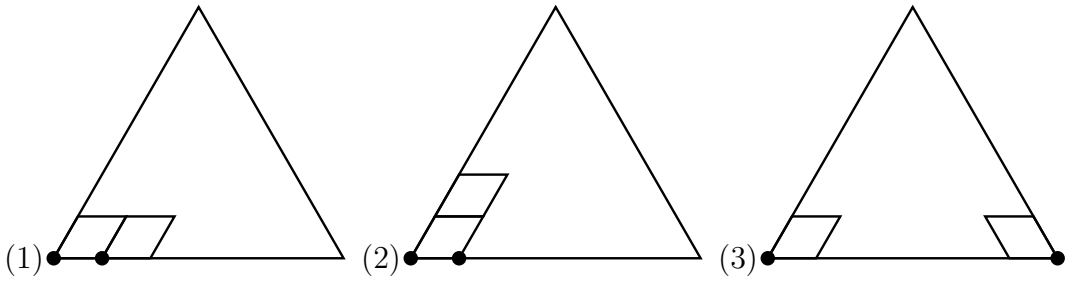


Figure 5.5: Fixed loci for rank 1 example over $\mathbb{P}^2$

Again, we also sum over the copies of $\mathcal{O}$ into which $\mathcal{I}_z$ can inject, which gives us three fixed loci for each configuration in Figure 5.5. Hence we have 27 isolated fixed points.

With all of these observations settled, we begin determining the various Chern classes over our fixed loci.

$\underline{c_4(\mathrm{Ext}^0(\mathcal{I}_z, \mathcal{O}_z))}$

(1) As before, this is simply the top Chern class of the tangent space to two points, which was determined in [N] in terms of the weights on x and y, which are α and $-\beta$, respectively. There is no dependence on the λ's. Thus the Chern roots are

$$-2\alpha h, \beta h, -\alpha h, (\alpha + \beta)h$$

and

$$c_4(\mathrm{Ext}^0(\mathcal{I}_z, \mathcal{O}_z)) = 2\alpha^2\beta(\alpha + \beta)h^4.$$

(2) Similarly, the Chern roots are

$$-\alpha h, 2\beta h, \beta h, (-\alpha - \beta)h$$

and

$$c_4(\mathrm{Ext}^0(\mathcal{I}_z, \mathcal{O}_z)) = 2\alpha\beta^2(\alpha+\beta)h^4.$$

(3) This is a product of tangent spaces of Hilbert schemes of points. Being careful to recall that the weights near the second point are γ and $-\alpha$, this implies that the Chern roots are

$$\underbrace{-\alpha h, \beta h}_{\text{first point}} \quad \underbrace{-\gamma h, \alpha h}_{\text{second point}}$$

and therefore

$$c_4(\mathrm{Ext}^0(\mathcal{I}_z, \mathcal{O}_z)) = \alpha^2\beta\gamma h^4.$$

$\underline{c_1(\mathrm{Ext}^0(\mathcal{I}_z, \mathcal{O}))}$

As in the example when $S = \mathbb{P}^1 \times \mathbb{P}^1$, this Chern class only depends on the λ's, so in all cases it is given by

$$c_1(\mathrm{Ext}^0(\mathcal{I}_z, \mathcal{O})) = (\lambda_k - \lambda_i)h.$$

$\underline{c_2(\mathrm{Ext}^1(\mathcal{I}_z, \mathcal{O}))}$

We make use of the exact same isomorphisms as before, which are equivariant with respect to α, β, and γ. Therefore we need to understand the weights on the canonical sheaf K.

$$\begin{aligned}
&\text{Over } [1:0:0], \quad K = \wedge^2 T^\vee \text{ has the root } (\alpha-\beta)h.\\
&\text{Over } [0:1:0], \quad K = \wedge^2 T^\vee \text{ has the root } (\gamma-\alpha)h.\\
&\text{Over } [0:0:1], \quad K = \wedge^2 T^\vee \text{ has the root } (\beta-\gamma)h.
\end{aligned}$$

From here, we can compute $c_2(\mathrm{Ext}^1(\mathcal{I}_z, \mathcal{O}))$ for each isolated fixed point. We will insert the dependence on the λ's at the end.

(1)

$$\begin{aligned}
K &\text{ has root } (\alpha-\beta)h\\
\mathcal{O}_z \otimes K &\text{ has roots } (\alpha-\beta)h, (2\alpha-\beta)h\\
\mathrm{Hom}(\mathcal{O}_z, \mathcal{O}_z \otimes K) &\text{ has roots } (\alpha-\beta)h, (2\alpha-\beta)h\\
\mathrm{Ext}^1(\mathcal{I}_z, \mathcal{O}) \cong \mathrm{Ext}^1(\mathcal{I}_z, \mathcal{O}_z) &\cong \mathrm{Ext}^2(\mathcal{O}_z, \mathcal{O}_z) \cong \mathrm{Ext}^0(\mathcal{O}_z, \mathcal{O}_z \otimes K)^\vee\\
&\text{has roots } (-\alpha+\beta)h, (-2\alpha+\beta)h
\end{aligned}$$

Therefore,

$$c_2(\mathrm{Ext}^1(\mathcal{I}_z, \mathcal{O})) = (\lambda_k - \lambda_i - \alpha + \beta)(\lambda_k - \lambda_i - 2\alpha + \beta)h^2.$$

(2) By an almost identical derivation

$$c_2(\mathrm{Ext}^1(\mathcal{I}_z, \mathcal{O})) = (\lambda_k - \lambda_i - \alpha + \beta)(\lambda_k - \lambda_i - \alpha + 2\beta)h^2.$$

(3) The Chern class is given by a product, so

$$c_2(\mathrm{Ext}^1(\mathcal{I}_z, \mathcal{O})) = (\lambda_k - \lambda_i - \alpha + \beta)(\lambda_k - \lambda_i - \gamma + \alpha)h^2.$$

$\underline{c_2(\mathrm{Ext}^1(\mathcal{I}_z, \mathcal{O}_z))}$

The dependence on a and b is the same as for $c_2(\mathrm{Ext}^1(\mathcal{I}_z, \mathcal{O}))$. We simply remove the λ's from our results for $c_2(\mathrm{Ext}^1(\mathcal{I}_z, \mathcal{O}))$, and the computation is complete.

Theorem 6. *Over the surface* $\mathbb{P}^2$*,*

$$\int_{[Quot_{\mathbb{P}^2}(\mathbb{G}(1,3))_{4,0}]^{vir}} 1 = 270.$$

Proof. As with $\mathbb{P}^1 \times \mathbb{P}^1$, we have computed all of the necessary Chern classes, making the derivation of this intersection number equivalent to finding the finite sum over the 27 isolated fixed points. There is nothing more to check. □

Chapter 6

Rank 2 example

The example in chapter 5 was quite straightforward, so much so that some of the complexity in the full virtual normal bundle (4.11) was lost. In order to observe all of these Ext groups coming into play, we will compute

$$\int_{[Quot_S(\mathbb{G}(2,8))_{9,0}]^{vir}} 1$$

when $S = \mathbb{P}^1 \times \mathbb{P}^1$ or $S = \mathbb{P}^2$. We will start with $\mathbb{P}^1 \times \mathbb{P}^1$.

6.1 Computing $\int_{[Quot_{\mathbb{P}^1\times\mathbb{P}^1}(\mathbb{G}(2,8))_{9,0}]^{vir}} 1$

We will use the same torus action as in section 5.1. Recall that any fixed point corresponding to the given data can be viewed as a length 3 subscheme of $\mathbb{P}^1 \times \mathbb{P}^1$. Therefore, the isolated fixed points correspond to short exact sequence of the two types

$$0 \longrightarrow \mathcal{I}_z \oplus \mathcal{O} \longrightarrow \mathcal{O}^8 \longrightarrow \mathcal{O}_z \oplus \mathcal{O}^6 \longrightarrow 0 \qquad (\text{length}(z) = 3)$$

or

$$0 \longrightarrow \mathcal{I}_z \oplus \mathcal{I}_w \longrightarrow \mathcal{O}^8 \longrightarrow \mathcal{O}_z \oplus \mathcal{O}_w \oplus \mathcal{O}^6 \longrightarrow 0$$
$$(\text{length}(z) = 1, \text{length}(w) = 2)$$

where the support of z and w lies on the fixed points of $\mathbb{P}^1 \times \mathbb{P}^1$.

We begin by focusing on the first type of fixed locus, in which z has length 3. Let

$$0 \longrightarrow E \longrightarrow \mathcal{O}^8 \longrightarrow F \longrightarrow 0$$

denote the generic form of the short exact sequence. Then there are $8 \cdot 7 = 56$ ways in which the direct summands of E can inject into separate copies of $\mathcal{O}$ in $\mathcal{O}^8$.

For each of these possibilities, we need to identify all of the fixed point configurations for $\mathcal{I}_z$. Equivalently, we identify all length 3 combinations of Young tableaux supported over the fixed "corners" of the $\mathbb{P}^1 \times \mathbb{P}^1$ square. These are given graphically in Figure 6.1.

Note that each of the fixed loci pictured also has four-fold rotational symmetry, for a total of 40 isolated fixed point configurations.

Next we need to specify the equivariant normal bundle. In parallel with (5.1), we derive

$$\begin{aligned}\int_{[Quot_{\mathbb{P}^1\times\mathbb{P}^1}(\mathbb{G}(2,8))_{9,0}]^{vir}} 1 &= \sum_{\text{fixed loci}} \int \frac{1}{e_T(N^{vir})} \\ &= \sum \int_{[\text{pt}]} \frac{c_3(\text{Ext}^1(\mathcal{I}_z,\mathcal{O}_z))}{c_6(\text{Ext}^0(\mathcal{I}_z,\mathcal{O}_z))} \cdot \frac{1}{\oplus_1^6 c_1(\text{Ext}^0(\mathcal{O},\mathcal{O}))} \\ &\qquad \cdot \frac{1}{c_3(\text{Ext}^0(\mathcal{O},\mathcal{O}_z))} \cdot \oplus_1^6 \left(\frac{c_3(\text{Ext}^1(\mathcal{I}_z,\mathcal{O}))}{c_1(\text{Ext}^0(\mathcal{I}_z,\mathcal{O}))}\right)\end{aligned} \tag{6.1}$$

Here the sums $\oplus_1^6$ are taken over the six copies of $\mathcal{O}$ in the quotient of the short exact sequence.

Next we determine which Ext groups depend on a and b, which depend on the λ's, and which depend on both. The only new additions from the rank 1 example in chapter 5 are $\text{Ext}^0(\mathcal{O},\mathcal{O})$ and $\text{Ext}^0(\mathcal{O},\mathcal{O}_z)$. Of course, any homomorphism from $\mathcal{O}$ to $\mathcal{O}$ is given by scalar multiplication, and $\text{Ext}^0(\mathcal{O},\mathcal{O}_z)$ is just $\mathcal{O}_z$ with the extra dependency on λ's added in. Thus we can summarize the dependencies in Table 6.1.

We will now calculate each Chern class for the various isolated fixed point configurations. We establish the following conventions: $\mathcal{I}_z$ injects into the ith copy

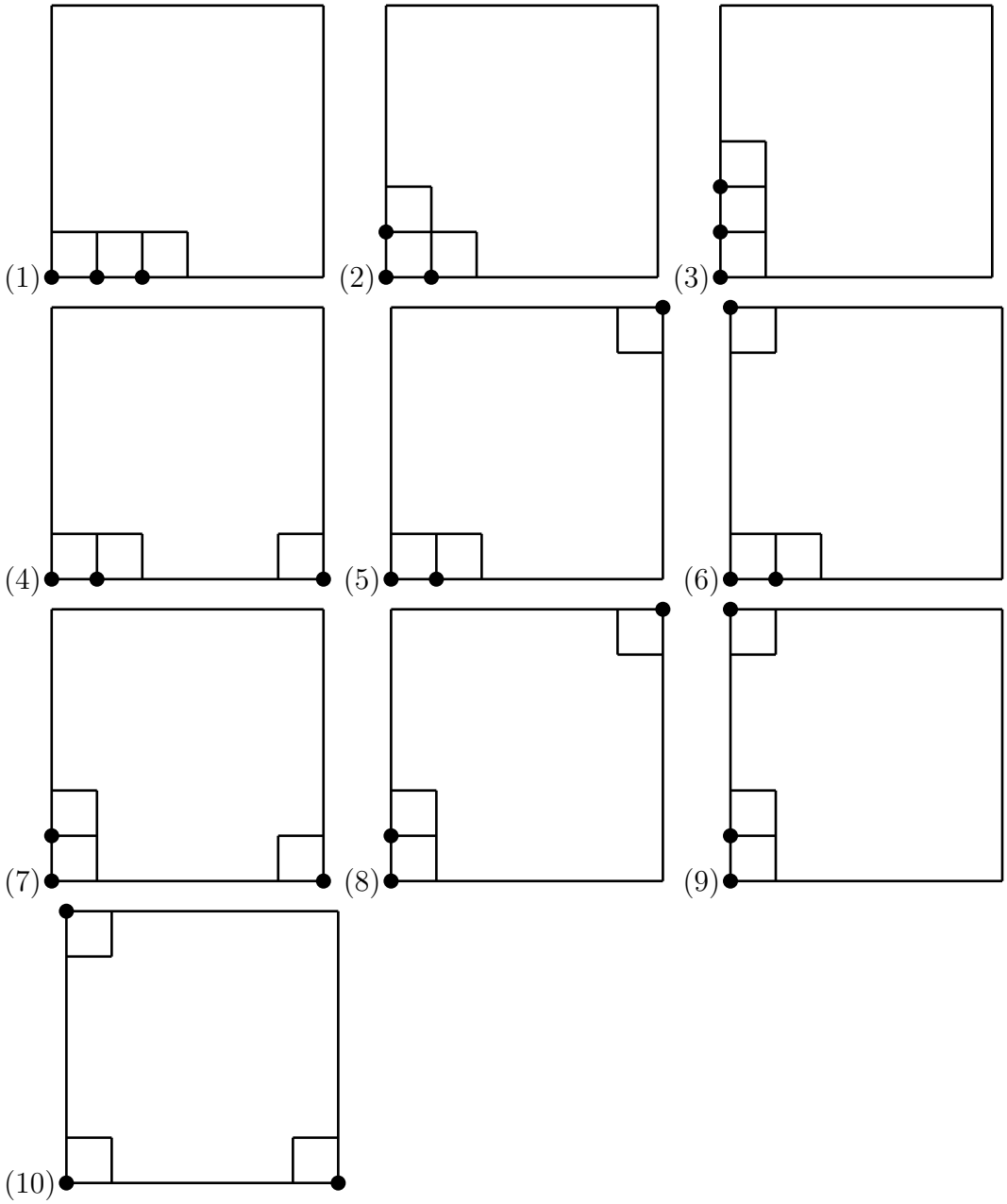


Figure 6.1: Fixed loci for rank 2 example over $\mathbb{P}^1 \times \mathbb{P}^1$

of $\mathcal{O}$ with weight λ_i. Furthermore in the subsheaf term $\mathcal{I}_z \oplus \mathcal{O}$, this $\mathcal{O}$ injects into the jth copy of $\mathcal{O}$ with weight λ_j. All other copies of $\mathcal{O}$ will be written with weight λ_k, so that we sum over k from 1 to 8, with $k \neq i$ and $k \neq j$.

$\underline{c_6(\mathrm{Ext}^0(\mathcal{I}_z, \mathcal{O}_z))}$

Recall that $\mathrm{Ext}^0(\mathcal{I}_z, \mathcal{O}_z)$ is the tangent space of the Hilbert scheme of points on

Table 6.1: Weight dependency when z has length 3 on $\mathbb{P}^1 \times \mathbb{P}^1$

	depends on λ's	depends on a, b
$c_3(\mathrm{Ext}^1(\mathcal{I}_z, \mathcal{O}_z))$	no	yes
$c_6(\mathrm{Ext}^0(\mathcal{I}_z, \mathcal{O}_z))$	no	yes
$c_1(\mathrm{Ext}^0(\mathcal{O}, \mathcal{O}))$	yes	no
$c_3(\mathrm{Ext}^0(\mathcal{O}, \mathcal{O}_z))$	yes	yes
$c_1(\mathrm{Ext}^0(\mathcal{I}_z, \mathcal{O}))$	yes	no
$c_3(\mathrm{Ext}^1(\mathcal{I}_z, \mathcal{O}))$	yes	yes

$\mathbb{C}^2$. The Chern roots are computed in [N], although with a global sign change. The roots depend only on the scheme structure of z. Note that there is no λ dependence. We will use the same numerical labels for the fixed point configurations that were established in Figure 6.1.

(1) Here the support is over one point and the ideal given locally by $\mathcal{I}_z = (x^3, y)$. Note that the weights on $\mathcal{I}_z$ are $3a$ and b, and the weights on $\mathcal{O}_z$ are 0, a, and $2a$. Thus the Chern roots are

$$-3ah, (2a-b)h, -2ah, (a-b)h, -ah, -bh.$$

Hence, the relevant Chern class is

$$c_6(\mathrm{Ext}^0(\mathcal{I}_z, \mathcal{O}_z)) = 6a^3b(b-a)(b-2a)h^6.$$

(2) From Nakajima (with global sign change corrected) we see that the Chern roots are

$$(b-2a)h, (a-2b)h, -ah, -bh, -ah, -bh$$

and the relevant Chern class is

$$c_6(\mathrm{Ext}^0(\mathcal{I}_z, \mathcal{O}_z)) = a^2b^2(2a-b)(2b-a)h^6.$$

(3) Pictorially, this fixed point configuration is just a reflection of (1), so the Chern roots will simply interchange the roles of a and b. Hence

$$c_6(\mathrm{Ext}^0(\mathcal{I}_z, \mathcal{O}_z)) = 6ab^3(a-b)(a-2b)h^6.$$

(4) This is a direct product of tangent spaces, so we can compute the roots separately:

$$\underbrace{-2ah, (a-b)h, -ah, -bh}_{\text{double point}}, \underbrace{ah, -bh}_{\text{single point}}.$$

Thus,

$$c_6(\mathrm{Ext}^0(\mathcal{I}_z, \mathcal{O}_z)) = -2a^3b^2(b-a)h^6.$$

(5) This is a direct product similar to the previous case, with only signs changing as we move the single point. Thus the Chern roots are

$$\underbrace{-2ah, (a-b)h, -ah, -bh}_{\text{double point}}, \underbrace{ah, bh}_{\text{single point}}$$

and the Chern class is

$$c_6(\mathrm{Ext}^0(\mathcal{I}_z, \mathcal{O}_z)) = 2a^3b^2(b-a)h^6.$$

(6) Again, the signs change on the single point. The roots are

$$\underbrace{-2ah, (a-b)h, -ah, -bh}_{\text{double point}}, \underbrace{-ah, bh}_{\text{single point}}$$

and therefore

$$c_6(\mathrm{Ext}^0(\mathcal{I}_z, \mathcal{O}_z)) = -2a^3b^2(b-a)h^6.$$

(7) Configurations (7), (8), and (9) are simply reflections of (6), (5), and (4), respectively. The reflection switches x and y, and thus we can switch the weights a and b in the computations. Hence,

$$c_6(\mathrm{Ext}^0(\mathcal{I}_z, \mathcal{O}_z)) = -2a^2b^3(a-b)h^6.$$

(8) As a reflection of (5), we compute

$$c_6(\mathrm{Ext}^0(\mathcal{I}_z, \mathcal{O}_z)) = 2a^2b^3(a-b)h^6.$$

(9) As a reflection of (4), we compute

$$c_6(\mathrm{Ext}^0(\mathcal{I}_z, \mathcal{O}_z)) = -2a^2b^3(a-b)h^6.$$

(10) Finally, the last case is a product of three tangent spaces of single points. The roots are

$$\underbrace{-ah, -bh}, \underbrace{ah, -bh}, \underbrace{-ah, bh}$$

and the Chern class is

$$c_6(\mathrm{Ext}^0(\mathcal{I}_z, \mathcal{O}_z)) = a^3b^3.$$

$\underline{c_1(\mathrm{Ext}^0(\mathcal{O}, \mathcal{O}))}$

This Chern class has no dependence on a or b. Thus, there is only one Chern root depending on the λ's alone. Since λ_j is the weight on the domain and λ_k is the weight on the target, we find

$$c_1(\mathrm{Ext}^0(\mathcal{O}, \mathcal{O})) = (\lambda_k - \lambda_j)h$$

for every fixed point configuration. As explained, j is fixed but k varies.

$\underline{c_1(\mathrm{Ext}^0(\mathcal{I}_z, \mathcal{O}))}$

Similarly, the (single) root of $\mathrm{Ext}^0(\mathcal{I}_z, \mathcal{O})$ is easy to compute. The only maps from $\mathcal{I}_z$ to $\mathcal{O}$ are scalar multiplication, so there is no dependence on a or b. Hence

$$c_1(\mathrm{Ext}^0(\mathcal{I}_z, \mathcal{O})) = (\lambda_k - \lambda_i)h$$

where i is fixed and k varies.

$\underline{c_3(\mathrm{Ext}^0(\mathcal{O}, \mathcal{O}_z))}$

Next we will compute $c_3(\mathrm{Ext}^0(\mathcal{O}, \mathcal{O}_z))$. The Chern roots depend on both the λ's and on a and b. However, the λ dependence will always be $(\lambda_i - \lambda_j)$. The dependence on a and b is the same as the dependence of $\mathcal{O}_z$ on a and b.

(1) The Chern roots are

$$(2a + \lambda_i - \lambda_j)h, (a + \lambda_i - \lambda_j)h, (\lambda_i - \lambda_j)h$$

and the Chern class is

$$c_3(\mathrm{Ext}^0(\mathcal{O}, \mathcal{O}_z)) = (2a + \lambda_i - \lambda_j)(a + \lambda_i - \lambda_j)(\lambda_i - \lambda_j)h^3.$$

(2) The Chern roots are

$$(a+\lambda_i-\lambda_j)h, (b+\lambda_i-\lambda_j)h, (\lambda_i-\lambda_j)h$$

and the Chern class is

$$c_3(\mathrm{Ext}^0(\mathcal{O},\mathcal{O}_z)) = (a+\lambda_i-\lambda_j)(b+\lambda_i-\lambda_j)(\lambda_i-\lambda_j)h^3.$$

(3) The Chern roots are

$$(2b+\lambda_i-\lambda_j)h, (b+\lambda_i-\lambda_j)h, (\lambda_i-\lambda_j)h$$

and the Chern class is

$$c_3(\mathrm{Ext}^0(\mathcal{O},\mathcal{O}_z)) = (2b+\lambda_i-\lambda_j)(b+\lambda_i-\lambda_j)(\lambda_i-\lambda_j)h^3.$$

(4)-(6) The Chern class is

$$c_3(\mathrm{Ext}^0(\mathcal{O},\mathcal{O}_z)) = (a+\lambda_i-\lambda_j)(\lambda_i-\lambda_j)(\lambda_i-\lambda_j)h^3.$$

(7)-(9) The Chern class is

$$c_3(\mathrm{Ext}^0(\mathcal{O},\mathcal{O}_z)) = (b+\lambda_i-\lambda_j)(\lambda_i-\lambda_j)(\lambda_i-\lambda_j)h^3.$$

(10) The Chern class is

$$c_3(\mathrm{Ext}^0(\mathcal{O},\mathcal{O}_z)) = (\lambda_i-\lambda_j)(\lambda_i-\lambda_j)(\lambda_i-\lambda_j)h^3.$$

$\underline{c_3(\mathrm{Ext}^1(\mathcal{I}_z,\mathcal{O}))}$

Last, have $c_3(\mathrm{Ext}^1(\mathcal{I}_z,\mathcal{O}_z))$ and $c_3(\mathrm{Ext}^1(\mathcal{I}_z,\mathcal{O}))$. These have the same dependence of their roots on a and b, but differing dependence on the λ's. Thus, we will only compute $c_3(\mathrm{Ext}^1(\mathcal{I}_z,\mathcal{O}))$ and omit the λ's for the other Chern class.

The weights on the canonical sheaf K were already computed over each fixed "vertex" on page 27.

(1) We will do one configuration in detail and then abbreviate the remainder.

$$\begin{aligned} K &\text{ has root } (a+b)h \\ \mathcal{O}_z \otimes K &\text{ has roots } (a+b)h, (2a+b)h, (3a+b)h \\ \mathrm{Hom}(\mathcal{O}_z, \mathcal{O}_z \otimes K) &\text{ has roots } (a+b)h, (2a+b)h, (3a+b)h \\ \mathrm{Ext}^1(\mathcal{I}_z, \mathcal{O}) \cong \mathrm{Ext}^1(\mathcal{I}_z, \mathcal{O}_z) \cong \mathrm{Ext}^2(\mathcal{O}_z, \mathcal{O}_z) &\cong \mathrm{Ext}^0(\mathcal{O}_z, \mathcal{O}_z \otimes K)^\vee \\ &\text{has roots } (-a-b)h, (-2a-b)h, (-3a-b)h. \end{aligned}$$

So in the end,

$$c_3(\mathrm{Ext}^1(\mathcal{I}_z, \mathcal{O})) = (\lambda_k - \lambda_i - a - b)(\lambda_k - \lambda_i - 2a - b)(\lambda_k - \lambda_i - 3a - b)h^3.$$

(2) By similar reasoning,

$$c_3(\mathrm{Ext}^1(\mathcal{I}_z, \mathcal{O})) = (\lambda_k - \lambda_i - a - b)(\lambda_k - \lambda_i - 2a - b)(\lambda_k - \lambda_i - a - 2b)h^3.$$

(3) The Chern class is

$$c_3(\mathrm{Ext}^1(\mathcal{I}_z, \mathcal{O})) = (\lambda_k - \lambda_i - a - b)(\lambda_k - \lambda_i - a - 2b)(\lambda_k - \lambda_i - a - 3b)h^3.$$

(4) We will need it repeatedly, so we compute the Chern root of $\mathrm{Ext}^1(\mathcal{I}_p, \mathcal{O})$ for a single point located at locus (i), using the position notation of Figure 5.2.

$$\begin{aligned} K &\text{ has root } (a+b)h \\ \mathcal{O}_p \otimes K &\text{ has root } (a+b)h \\ \mathrm{Ext}^1(\mathcal{I}_p, \mathcal{O}) &\text{ has root } (-a-b)h \end{aligned}$$

As this single point moves around $\mathbb{P}^1 \times \mathbb{P}^1$, we will simply need to flip signs on a or b, accordingly.

For configuration (4), the roots are

$$\underbrace{(\lambda_k - \lambda_i - a - b)h, (\lambda_k - \lambda_i - 2a - b)h}_{\text{double point}}, \underbrace{(\lambda_k - \lambda_i + a - b)h}_{\text{single point}}$$

and the Chern class is

$$c_3(\mathrm{Ext}^1(\mathcal{I}_z, \mathcal{O})) = (\lambda_k - \lambda_i - a - b)(\lambda_k - \lambda_i - 2a - b)(\lambda_k - \lambda_i + a - b)h^3.$$

(5) The Chern class is

$$c_3(\mathrm{Ext}^1(\mathcal{I}_z, \mathcal{O})) = (\lambda_k - \lambda_i - a - b)(\lambda_k - \lambda_i - 2a - b)(\lambda_k - \lambda_i + a + b)h^3.$$

(6) The Chern class is

$$c_3(\mathrm{Ext}^1(\mathcal{I}_z, \mathcal{O})) = (\lambda_k - \lambda_i - a - b)(\lambda_k - \lambda_i - 2a - b)(\lambda_k - \lambda_i - a + b)h^3.$$

(7) The Chern class is

$$c_3(\mathrm{Ext}^1(\mathcal{I}_z, \mathcal{O})) = (\lambda_k - \lambda_i - a - b)(\lambda_k - \lambda_i - a - 2b)(\lambda_k - \lambda_i + a - b)h^3.$$

(8) The Chern class is

$$c_3(\mathrm{Ext}^1(\mathcal{I}_z, \mathcal{O})) = (\lambda_k - \lambda_i - a - b)(\lambda_k - \lambda_i - a - 2b)(\lambda_k - \lambda_i + a + b)h^3.$$

(9) The Chern class is

$$c_3(\mathrm{Ext}^1(\mathcal{I}_z, \mathcal{O})) = (\lambda_k - \lambda_i - a - b)(\lambda_k - \lambda_i - a - 2b)(\lambda_k - \lambda_i - a + b)h^3.$$

(10) The Chern class is

$$c_3(\mathrm{Ext}^1(\mathcal{I}_z, \mathcal{O})) = (\lambda_k - \lambda_i - a - b)(\lambda_k - \lambda_i + a - b)(\lambda_k - \lambda_i - a + b)h^3.$$

Bearing in mind that $c_3(\mathrm{Ext}^1(\mathcal{I}_z, \mathcal{O}_z))$ is isomorphic $c_3(\mathrm{Ext}^1(\mathcal{I}_z, \mathcal{O}))$ but without the dependence on the λ's, this completes the listing of the Chern classes for the subcase in which z has length 3. We still need to do the case in which z has length 1 and w has length 2. We'll compute these next.

Here we continue the rank 2 example on $\mathbb{P}^1 \times \mathbb{P}^1$, focusing on the fixed points corresponding to short exact sequences of the type

$$0 \longrightarrow \mathcal{I}_z \oplus \mathcal{I}_w \longrightarrow \mathcal{O}^8 \longrightarrow \mathcal{O}_z \oplus \mathcal{O}_w \oplus \mathcal{O}^6 \longrightarrow 0.$$

The first step is to identify the possible configurations for fixed point loci. We will always assume that z has length 1 and w has length 2. We shade to indicate overlaps.

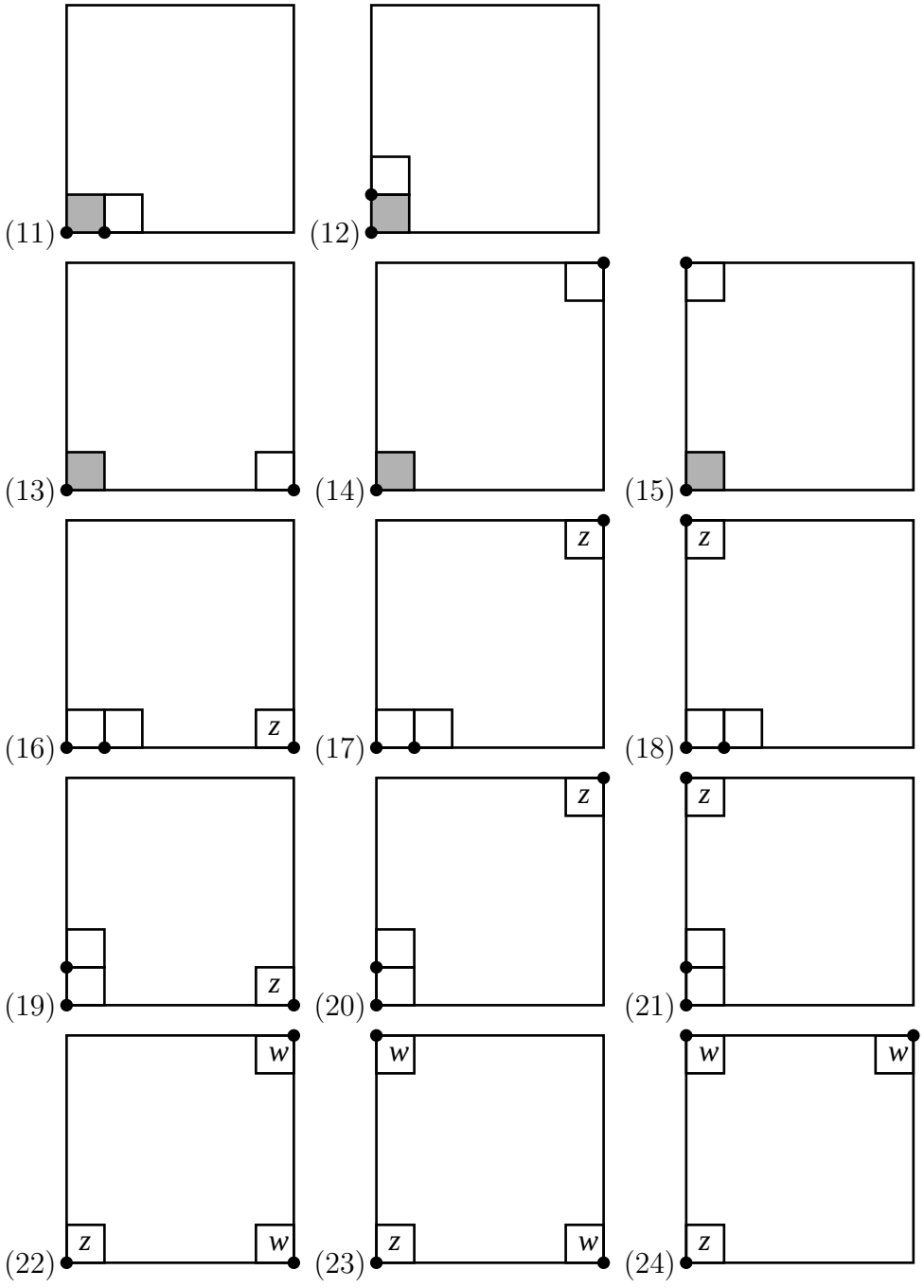


Figure 6.2: More fixed loci for rank 2 example over $\mathbb{P}^1 \times \mathbb{P}^1$

Again, each of the configurations shown in Figure 6.2 has four-fold rotation, for a total of $14 \cdot 4$ fixed point configurations for each of the $8 \cdot 7$ injections.

We establish conventions again: $\mathcal{I}_z$ injects into the ith copy of $\mathcal{O}$ and $\mathcal{I}_w$ injects into the jth copy of $\mathcal{O}$. All other copies of $\mathcal{O}$ will be written with weight λ_k where $1 \leq k \leq 8$ and $k \neq i$ and $k \neq j$. These are very similar to the conventions established in the case when z had length 3.

Next we need to find the specific dimensions of the Chern classes comprising the virtual normal bundle. We will be computing

$$\begin{aligned}
\int_{[\mathrm{Quot}]^{vir}} 1 &= \sum_{\text{fixed loci}} \int_{[\mathrm{pt}]} \frac{1}{e_T(N^{vir})} \\
&= \sum \int_{[\mathrm{pt}]} \frac{c_1(\mathrm{Ext}^1(\mathcal{I}_z, \mathcal{O}_z))}{c_2(\mathrm{Ext}^0(\mathcal{I}_z, \mathcal{O}_z))} \cdot \frac{c_2(\mathrm{Ext}^1(\mathcal{I}_w, \mathcal{O}_w))}{c_4(\mathrm{Ext}^0(\mathcal{I}_w, \mathcal{O}_w))} \cdot \frac{1}{e_{T_2}(N^{vir})} \\
&= \sum \int_{[\mathrm{pt}]} \frac{c_1(\mathrm{Ext}^1(\mathcal{I}_z, \mathcal{O}_z))}{c_2(\mathrm{Ext}^0(\mathcal{I}_z, \mathcal{O}_z))} \cdot \frac{c_2(\mathrm{Ext}^1(\mathcal{I}_w, \mathcal{O}_w))}{c_4(\mathrm{Ext}^0(\mathcal{I}_w, \mathcal{O}_w))} \cdot \frac{c(\mathrm{Ext}^1(\mathcal{I}_z, \mathcal{O}_w))}{c(\mathrm{Ext}^0(\mathcal{I}_z, \mathcal{O}_w))} \cdot \\
&\qquad \frac{c(\mathrm{Ext}^1(\mathcal{I}_w, \mathcal{O}_z))}{c(\mathrm{Ext}^0(\mathcal{I}_w, \mathcal{O}_z))} \cdot \oplus_1^6 \left(\frac{c_1(\mathrm{Ext}^1(\mathcal{I}_z, \mathcal{O}))}{c_1(\mathrm{Ext}^0(\mathcal{I}_z, \mathcal{O}))} \cdot \frac{c_2(\mathrm{Ext}^1(\mathcal{I}_w, \mathcal{O}))}{c_1(\mathrm{Ext}^0(\mathcal{I}_w, \mathcal{O}))} \right)
\end{aligned} \tag{6.2}$$

Note that the degrees of certain Chern classes are still omitted. We will find them next. We start with the $\mathrm{Ext}^\bullet(\mathcal{I}_z, \mathcal{O}_w)$ terms. Consider the short exact sequence

$$0 \longrightarrow \mathcal{I}_z \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}_z \longrightarrow 0.$$

If we apply the functor $\mathrm{Hom}(_, \mathcal{O}_w)$, we get the long exact sequence

$$\begin{array}{rccccc}
0 \longrightarrow & \mathrm{Ext}^0(\mathcal{O}_z, \mathcal{O}_w) \longrightarrow & \mathrm{Ext}^0(\mathcal{O}, \mathcal{O}_w) \longrightarrow & \mathrm{Ext}^0(\mathcal{I}_z, \mathcal{O}_w) \longrightarrow & & \\
& \mathrm{Ext}^1(\mathcal{O}_z, \mathcal{O}_w) \longrightarrow & \mathrm{Ext}^1(\mathcal{O}, \mathcal{O}_w) \longrightarrow & \mathrm{Ext}^1(\mathcal{I}_z, \mathcal{O}_w) \longrightarrow & & \\
& \mathrm{Ext}^2(\mathcal{O}_z, \mathcal{O}_w) \longrightarrow & \mathrm{Ext}^2(\mathcal{O}, \mathcal{O}_w) \longrightarrow & \mathrm{Ext}^2(\mathcal{I}_z, \mathcal{O}_w) \longrightarrow & 0 &
\end{array}$$

We know that $\mathrm{Ext}^1(\mathcal{O}, \mathcal{O}_w) = H^1(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}_w) = 0$ and $\mathrm{Ext}^2(\mathcal{O}, \mathcal{O}_w) = 0$. This in turn implies that $\mathrm{Ext}^2(\mathcal{I}_z, \mathcal{O}_w) = 0$. From here the long exact sequence implies that $\mathrm{Ext}^1(\mathcal{I}_z, \mathcal{O}_w) \cong \mathrm{Ext}^2(\mathcal{O}_z, \mathcal{O}_w)$, and this isomorphism is equivariant with respect to a and b, through not the λ's. The specific dimensions depend on the configuration.

Case 1: In configuration (11) or (12), w is a fat double point with the same support as z.

Case 2: In configurations (13)-(15), one point of w shares support with z, so we will abuse notation and write $w = z + p$.

Case 3: In configurations (16)-(24), the support of z and w are disjoint.

Case 1: For this case, z and w share support, which implies that we can handle all computations locally. Considering $\mathrm{Hom}(\mathcal{O}_z, \mathcal{O}_w)$, note that any homomorphism ϕ can be written locally as

$$\phi : \mathbb{C}[x,y]/(x,y) \to \mathbb{C}[x,y]/I$$

where $I = (x^2, y)$ (or $I = (x, y^2)$). These two possibilities are similar, so we address the first. If $\phi(1) = 1$, then $0 = \phi(0) = \phi(x) = x\phi(1) = x$, a contradiction. Hence any nonzero homomorphism is a scalar multiple of $\phi(1) = x$, and thus $dim\mathrm{Hom}(\mathcal{O}_z, \mathcal{O}_w) = 1$. Conversely, any homomorphism $\phi : \mathbb{C}[x,y]/I \to \mathbb{C}[x,y]/(x,y)$ is a scalar multiple of $\phi(1) = 1$ and using Serre duality shows that $dim\mathrm{Ext}^2(\mathcal{O}_z, \mathcal{O}_w) = 1$ as well.

As before, let ω denote the class of a point. Since the Euler characteristic $\chi(\mathcal{O}_z, \mathcal{O}_w) = \int_{[\mathbb{P}^1 \times \mathbb{P}^1]} \omega(2\omega)(1+\omega) = 0$, we also know that $dim\mathrm{Ext}^1(\mathcal{O}_z, \mathcal{O}_w) = 2$. Using $\chi(\mathcal{I}_z, \mathcal{O}_w) = \int_{[\mathbb{P}^1 \times \mathbb{P}^1]} (1-\omega)(2\omega)(1+\omega) = 2$ gives us enough information to fill in the dimensions of the long exact sequence:

$$\begin{array}{ccccccc} 0 & \longrightarrow & 1 & \longrightarrow & 2 & \longrightarrow & 3 \longrightarrow \\ & & 2 & \longrightarrow & 0 & \longrightarrow & 1 \longrightarrow \\ & & 1 & \longrightarrow & 0 & \longrightarrow & 0 \longrightarrow \quad 0. \end{array}$$

Case 2: In this case $\mathrm{Hom}(\mathcal{O}_z, \mathcal{O}_w) \cong \mathrm{Hom}(\mathcal{O}_z, \mathcal{O}_z)$ is one dimensional and

$$\mathrm{Ext}^2(\mathcal{O}_z, \mathcal{O}_w) \cong \mathrm{Hom}(\mathcal{O}_w, \mathcal{O}_z \otimes K) \cong \mathrm{Hom}(\mathcal{O}_w, \mathcal{O}_z) \cong \mathrm{Hom}(\mathcal{O}_z, \mathcal{O}_z)$$

is also one dimensional. The fact that the middle isomorphism is not equivariant is irrelevant, since we are only interested in finding dimensions. Again, this gives us enough information to fill in the dimensions in the long exact sequence:

$$\begin{array}{cccccc} 0 \longrightarrow & 1 & \longrightarrow & 2 & \longrightarrow & 3 \longrightarrow \\ & 2 & \longrightarrow & 0 & \longrightarrow & 1 \longrightarrow \\ & 1 & \longrightarrow & 0 & \longrightarrow & 0. \end{array}$$

Case 3: Last, when the support of z and w are disjoint, $\mathrm{Hom}(\mathcal{O}_z, \mathcal{O}_w) = 0$. By duality, $\mathrm{Ext}^2(\mathcal{O}_z, \mathcal{O}_w) = 0$ as well. Thus $\mathrm{Hom}(\mathcal{O}, \mathcal{O}_w) \cong \mathrm{Hom}(\mathcal{I}_z, \mathcal{O}_w)$ are two dimensional, and everything else vanishes, yielding the sequence of dimensions

$$\begin{array}{cccccc} 0 \longrightarrow & 0 & \longrightarrow & 2 & \longrightarrow & 2 \longrightarrow \\ & 0 & \longrightarrow & 0 & \longrightarrow & 0 \longrightarrow \\ & 0 & \longrightarrow & 0 & \longrightarrow & 0. \end{array}$$

Even though the previous analysis was to determine the degrees of the Chern classes of $\mathrm{Ext}^\bullet(\mathcal{I}_z, \mathcal{O}_w)$, much of it is also useful to find the degrees of the Chern classes of $\mathrm{Ext}^\bullet(\mathcal{I}_w, \mathcal{O}_z)$. The relevant long exact sequence is

$$\begin{array}{cccccc} 0 \longrightarrow & \mathrm{Ext}^0(\mathcal{O}_w, \mathcal{O}_z) & \longrightarrow & \mathrm{Ext}^0(\mathcal{O}, \mathcal{O}_z) & \longrightarrow & \mathrm{Ext}^0(\mathcal{I}_w, \mathcal{O}_z) \longrightarrow \\ & \mathrm{Ext}^1(\mathcal{O}_w, \mathcal{O}_z) & \longrightarrow & \mathrm{Ext}^1(\mathcal{O}, \mathcal{O}_z) & \longrightarrow & \mathrm{Ext}^1(\mathcal{I}_w, \mathcal{O}_z) \longrightarrow \\ & \mathrm{Ext}^2(\mathcal{O}_w, \mathcal{O}_z) & \longrightarrow & \mathrm{Ext}^2(\mathcal{O}, \mathcal{O}_z) & \longrightarrow & \mathrm{Ext}^2(\mathcal{I}_w, \mathcal{O}_z) \longrightarrow 0. \end{array}$$

Again, we note that $\mathrm{Ext}^1(\mathcal{O}, \mathcal{O}_z)$, $\mathrm{Ext}^2(\mathcal{O}, \mathcal{O}_z)$, and $\mathrm{Ext}^2(\mathcal{I}_w, \mathcal{O}_z)$ all vanish, just as before. We split into the same three cases, and note that the relevant Euler characteristics we will need are

$$\begin{aligned} \chi(\mathcal{O}_w, \mathcal{O}_z) &= \int_{[\mathbb{P}^1 \times \mathbb{P}^1]} (2\omega)\omega(1+\omega) = 0 \\ \chi(\mathcal{O}, \mathcal{O}_z) &= \int_{[\mathbb{P}^1 \times \mathbb{P}^1]} (1)\omega(1+\omega) = 1 \\ \chi(\mathcal{I}_w, \mathcal{O}_z) &= \int_{[\mathbb{P}^1 \times \mathbb{P}^1]} (1-2\omega)\omega(1+\omega) = 1. \end{aligned}$$

Case 1: In this case, since z and w share the same support, $dim\mathrm{Hom}(\mathcal{O}_w, \mathcal{O}_z) = dim\mathrm{Ext}^2(\mathcal{O}_w, \mathcal{O}_z) = 1$. This is enough to yield the long exact sequence of dimensions

$$\begin{array}{cccccc} 0 \longrightarrow & 1 & \longrightarrow & 1 & \longrightarrow & 2 \longrightarrow \\ & 2 & \longrightarrow & 0 & \longrightarrow & 1 \longrightarrow \\ & 1 & \longrightarrow & 0 & \longrightarrow & 0. \end{array}$$

Case 2: In this case, one point of w shares support with z, so $dim\text{Hom}(\mathcal{O}_w, \mathcal{O}_z) = 1$ and $dim\text{Ext}^2(\mathcal{O}_w, \mathcal{O}_z) = 1$. Thus, everything is determined and we get

$$\begin{array}{llllll} 0 \longrightarrow & 1 \longrightarrow & 1 \longrightarrow & 2 \longrightarrow \\ & 2 \longrightarrow & 0 \longrightarrow & 1 \longrightarrow \\ & 1 \longrightarrow & 0 \longrightarrow & 0 \longrightarrow & 0. \end{array}$$

Case 3: Since the support of z and w are completely disjoint, $dim\text{Hom}(\mathcal{O}_w, \mathcal{O}_z) = 0$ and $dim\text{Ext}^2(\mathcal{O}_w, \mathcal{O}_z) = 0$. From here, everything is determined and we have

$$\begin{array}{llllll} 0 \longrightarrow & 0 \longrightarrow & 1 \longrightarrow & 1 \longrightarrow \\ & 0 \longrightarrow & 0 \longrightarrow & 0 \longrightarrow \\ & 0 \longrightarrow & 0 \longrightarrow & 0 \longrightarrow & 0. \end{array}$$

Using this information, we can clarify the formula for the sum of intersection products in equation (6.2), yielding

$$\sum \int_{[\text{pt}]} \frac{c_1(\text{Ext}^1(\mathcal{I}_z, \mathcal{O}_z))}{c_2(\text{Ext}^0(\mathcal{I}_z, \mathcal{O}_z))} \cdot \frac{c_2(\text{Ext}^1(\mathcal{I}_w, \mathcal{O}_w))}{c_4(\text{Ext}^0(\mathcal{I}_w, \mathcal{O}_w))} \cdot \frac{c_{\blacktriangle}(\text{Ext}^1(\mathcal{I}_z, \mathcal{O}_w))}{c_{\blacktriangle+2}(\text{Ext}^0(\mathcal{I}_z, \mathcal{O}_w))} \cdot \frac{c_{\blacktriangle}(\text{Ext}^1(\mathcal{I}_w, \mathcal{O}_z))}{c_{\blacktriangle+1}(\text{Ext}^0(\mathcal{I}_w, \mathcal{O}_z))} \cdot \oplus_1^6 \left(\frac{c_1(\text{Ext}^1(\mathcal{I}_z, \mathcal{O}))}{c_1(\text{Ext}^0(\mathcal{I}_z, \mathcal{O}))} \cdot \frac{c_2(\text{Ext}^1(\mathcal{I}_w, \mathcal{O}))}{c_1(\text{Ext}^0(\mathcal{I}_w, \mathcal{O}))} \right) \tag{6.3}$$

where

$$\blacktriangle = \begin{cases} 1 & \text{for configurations } (11) - (15) \\ 0 & \text{for configurations } (16) - (24). \end{cases}$$

The dependencies of these classes on a and b and the λ's are given in Table 6.2.

We will not compute $c_1(\text{Ext}^1(\mathcal{I}_z, \mathcal{O}_z))$ separately since it is easier to simply compute $c_1(\text{Ext}^1(\mathcal{I}_z, \mathcal{O}))$ and then remove the dependence on λ. Similarly, we will not separately compute $c_2(\text{Ext}^1(\mathcal{I}_w, \mathcal{O}_w))$.

$\underline{c_2(\text{Ext}^0(\mathcal{I}_z, \mathcal{O}_z))}$

Note that this Chern class depends only on the position of z, and not on w at all.

(11)-(15) For each configuration we have $\mathcal{I}_z = (x, y)$ and $\mathcal{O}_z = \mathbb{C}[x, y]/(x, y)$. Thus the roots of $\text{Ext}^0(\mathcal{I}_z, \mathcal{O}_z)$ are $-ah$ and $-bh$, so the relevant Chern class is

$$c_2(\text{Ext}^0(\mathcal{I}_z, \mathcal{O}_z)) = abh^2.$$

Table 6.2: Weight dependency when z has length 1 and w has length 2 on $\mathbb{P}^1 \times \mathbb{P}^1$

	depends on λ's	depends on a, b
$c_1(\mathrm{Ext}^1(\mathcal{I}_z, \mathcal{O}_z))$	no	yes
$c_2(\mathrm{Ext}^0(\mathcal{I}_z, \mathcal{O}_z))$	no	yes
$c_2(\mathrm{Ext}^1(\mathcal{I}_w, \mathcal{O}_w))$	no	yes
$c_4(\mathrm{Ext}^0(\mathcal{I}_w, \mathcal{O}_w))$	no	yes
$c_{\blacktriangle}(\mathrm{Ext}^1(\mathcal{I}_z, \mathcal{O}_w))$	yes	only for (11)-(15)
$c_{\blacktriangle+2}(\mathrm{Ext}^0(\mathcal{I}_z, \mathcal{O}_w))$	yes	yes
$c_{\blacktriangle}(\mathrm{Ext}^1(\mathcal{I}_w, \mathcal{O}_z))$	yes	only for (11)-(15)
$c_{\blacktriangle+1}(\mathrm{Ext}^0(\mathcal{I}_w, \mathcal{O}_z))$	yes	yes
$c_1(\mathrm{Ext}^1(\mathcal{I}_z, \mathcal{O}))$	yes	yes
$c_1(\mathrm{Ext}^0(\mathcal{I}_z, \mathcal{O}))$	yes	no
$c_2(\mathrm{Ext}^1(\mathcal{I}_w, \mathcal{O}))$	yes	yes
$c_1(\mathrm{Ext}^0(\mathcal{I}_w, \mathcal{O}))$	yes	no

(16) The support of z has moved, so the roots are ah and $-bh$, and the Chern class is

$$c_2(\mathrm{Ext}^0(\mathcal{I}_z, \mathcal{O}_z)) = -abh^2.$$

The remaining cases are quite similar, so we will omit the Chern roots and simply state the results.

(17) $c_2(\mathrm{Ext}^0(\mathcal{I}_z, \mathcal{O}_z)) = abh^2$

(18) $c_2(\mathrm{Ext}^0(\mathcal{I}_z, \mathcal{O}_z)) = -abh^2$

(19) $c_2(\mathrm{Ext}^0(\mathcal{I}_z, \mathcal{O}_z)) = -abh^2$

(20) $c_2(\mathrm{Ext}^0(\mathcal{I}_z, \mathcal{O}_z)) = abh^2$

(21) $c_2(\mathrm{Ext}^0(\mathcal{I}_z, \mathcal{O}_z)) = -abh^2$

(22)-(24) $c_2(\mathrm{Ext}^0(\mathcal{I}_z, \mathcal{O}_z)) = abh^2$.

$\underline{c_4(\mathrm{Ext}^0(\mathcal{I}_w, \mathcal{O}_w))}$

These Chern classes depend only on w, and were therefore essentially computed on page 24 in the rank 1 example, since z had length 2 in that example. We simply state the results and refer the reader to that earlier example.

(11) $c_4(\mathrm{Ext}^0(\mathcal{I}_w, \mathcal{O}_w)) = -2a^2b(a-b)h^4$

(12) $c_4(\mathrm{Ext}^0(\mathcal{I}_w, \mathcal{O}_w)) = -2ab^2(b-a)h^4$

(13) $c_4(\mathrm{Ext}^0(\mathcal{I}_w, \mathcal{O}_w)) = -a^2b^2h^4$

(14) $c_4(\mathrm{Ext}^0(\mathcal{I}_w, \mathcal{O}_w)) = a^2b^2h^4$

(15) $c_4(\mathrm{Ext}^0(\mathcal{I}_w, \mathcal{O}_w)) = -a^2b^2h^4$

(16)-(18) $c_4(\mathrm{Ext}^0(\mathcal{I}_w, \mathcal{O}_w)) = -2a^2b(a-b)h^4$

(19)-(21) $c_4(\mathrm{Ext}^0(\mathcal{I}_w, \mathcal{O}_w)) = -2ab^2(b-a)h^4$

(22) $c_4(\mathrm{Ext}^0(\mathcal{I}_w, \mathcal{O}_w)) = -a^2b^2h^4$

(23) $c_4(\mathrm{Ext}^0(\mathcal{I}_w, \mathcal{O}_w)) = a^2b^2h^4$

(24) $c_4(\mathrm{Ext}^0(\mathcal{I}_w, \mathcal{O}_w)) = -a^2b^2h^4$.

$\underline{c_1(\mathrm{Ext}^0(\mathcal{I}_z, \mathcal{O}))}$

This Chern class does not depend on a or b, but only on the λ's, so in every case

$$c_1(\mathrm{Ext}^0(\mathcal{I}_z, \mathcal{O})) = (\lambda_k - \lambda_i)h.$$

$\underline{c_1(\mathrm{Ext}^0(\mathcal{I}_w, \mathcal{O}))}$

Similarly, this Chern class does not depend on a or b, so for fixed point configuration

$$c_1(\mathrm{Ext}^0(\mathcal{I}_w, \mathcal{O})) = (\lambda_k - \lambda_j)h.$$

$\underline{c_1(\mathrm{Ext}^1(\mathcal{I}_z, \mathcal{O}))}$

Next we move on to $c_1(\mathrm{Ext}^1(\mathcal{I}_z, \mathcal{O}))$. Recall that

$$\mathrm{Ext}^1(\mathcal{I}_z, \mathcal{O}) \cong \mathrm{Ext}^1(\mathcal{I}_z, \mathcal{O}_z) \cong \mathrm{Ext}^2(\mathcal{O}_z, \mathcal{O}_z) \cong \mathrm{Ext}^0(\mathcal{O}_z, \mathcal{O}_z \otimes K)^\vee$$

and at the point (i) in $\mathbb{P}^1 \times \mathbb{P}^1$ as in Figure 5.2. These isomorphisms are only equivariant with respect to a and b. Thus

$$\begin{aligned} K &\text{ has root } (a+b)h \\ \mathcal{O}_z \otimes K &\text{ has root } (a+b)h \\ \mathrm{Hom}(\mathcal{O}_z, \mathcal{O}_z \otimes K) &\text{ has root } (a+b)h \\ \mathrm{Ext}^1(\mathcal{I}_z, \mathcal{O}) \cong \mathrm{Ext}^1(\mathcal{I}_z, \mathcal{O}_z) \cong \mathrm{Ext}^2(\mathcal{O}_z, \mathcal{O}_z) &\cong \mathrm{Ext}^0(\mathcal{O}_z, \mathcal{O}_z \otimes K) \\ &\text{has root } (-a-b)h. \end{aligned}$$

The root only depends on the support of z, and we can straightforwardly evaluate the Chern class in all cases.

(11)-(15) $c_1(\mathrm{Ext}^1(\mathcal{I}_z, \mathcal{O})) = (-a-b+\lambda_k-\lambda_i)h$

(16) $c_1(\mathrm{Ext}^1(\mathcal{I}_z, \mathcal{O})) = (a-b+\lambda_k-\lambda_i)h$

(17) $c_1(\mathrm{Ext}^1(\mathcal{I}_z, \mathcal{O})) = (a+b+\lambda_k-\lambda_i)h$

(18) $c_1(\mathrm{Ext}^1(\mathcal{I}_z, \mathcal{O})) = (-a+b+\lambda_k-\lambda_i)h$

(19) $c_1(\mathrm{Ext}^1(\mathcal{I}_z, \mathcal{O})) = (a-b+\lambda_k-\lambda_i)h$

(20) $c_1(\mathrm{Ext}^1(\mathcal{I}_z, \mathcal{O})) = (a+b+\lambda_k-\lambda_i)h$

(21) $c_1(\mathrm{Ext}^1(\mathcal{I}_z, \mathcal{O})) = (-a+b+\lambda_k-\lambda_i)h$

(22)-(24) $c_1(\mathrm{Ext}^1(\mathcal{I}_z, \mathcal{O})) = (-a-b+\lambda_k-\lambda_i)h.$

This also gives us $c_1(\mathrm{Ext}^1(\mathcal{I}_z, \mathcal{O}_z))$ for each fixed point configuration if we remove the dependence on λ.

$\underline{c_2(\mathrm{Ext}^1(\mathcal{I}_w, \mathcal{O}))}$

Next we calculate $c_2(\mathrm{Ext}^1(\mathcal{I}_w, \mathcal{O}))$. These are just a relabeling of the computations on page 27. When the support of w is two distinct points, we have a product. When w is a fat double point, this has already been done at a subcase on page 27. We will just list the results.

(11) $c_2(\mathrm{Ext}^1(\mathcal{I}_w, \mathcal{O})) = (-a-b+\lambda_k-\lambda_j)(-2a-b+\lambda_k-\lambda_j)h^2$

(12) $c_2(\mathrm{Ext}^1(\mathcal{I}_w, \mathcal{O})) = (-a - b + \lambda_k - \lambda_j)(-a - 2b + \lambda_k - \lambda_j)h^2$

(13) $c_2(\mathrm{Ext}^1(\mathcal{I}_w, \mathcal{O})) = (-a - b + \lambda_k - \lambda_j)(a - b + \lambda_k - \lambda_j)h^2$

(14) $c_2(\mathrm{Ext}^1(\mathcal{I}_w, \mathcal{O})) = (-a - b + \lambda_k - \lambda_j)(a + b + \lambda_k - \lambda_j)h^2$

(15) $c_2(\mathrm{Ext}^1(\mathcal{I}_w, \mathcal{O})) = (-a - b + \lambda_k - \lambda_j)(-a + b + \lambda_k - \lambda_j)h^2$

(16)-(18) $c_2(\mathrm{Ext}^1(\mathcal{I}_w, \mathcal{O})) = (-a - b + \lambda_k - \lambda_j)(-2a - b + \lambda_k - \lambda_j)h^2$

(19)-(21) $c_2(\mathrm{Ext}^1(\mathcal{I}_w, \mathcal{O})) = (-a - b + \lambda_k - \lambda_j)(-a - 2b + \lambda_k - \lambda_j)h^2$

(22) $c_2(\mathrm{Ext}^1(\mathcal{I}_w, \mathcal{O})) = (a - b + \lambda_k - \lambda_j)(a + b + \lambda_k - \lambda_j)h^2$

(23) $c_2(\mathrm{Ext}^1(\mathcal{I}_w, \mathcal{O})) = (a - b + \lambda_k - \lambda_j)(-a + b + \lambda_k - \lambda_j)h^2$

(24) $c_2(\mathrm{Ext}^1(\mathcal{I}_w, \mathcal{O})) = (a + b + \lambda_k - \lambda_j)(-a + b + \lambda_k - \lambda_j)h^2$.

This also gives us $c_2(\mathrm{Ext}^1(\mathcal{I}_w, \mathcal{O}_w))$ for each fixed point configuration if we just drop the dependence on λ. Now all that remains are the "cross terms" involving both z and w.

$\underline{c_{\blacktriangle+2}(\mathrm{Ext}^0(\mathcal{I}_z, \mathcal{O}_w))}$

Next we will address the "cross terms" which involve both z and w. We will need to be rather careful here because even the dimensions change over the various configurations. We begin the cross terms with $c_{\blacktriangle+2}(\mathrm{Ext}^0(\mathcal{I}_z, \mathcal{O}_w))$.

When the support of z and w are the same, as in configurations (11) and (12), the problem is local and we can compute everything in local coordinates.

(11) Here $\mathcal{I}_z = (x, y)$ and $\mathcal{O}_w = \mathbb{C}[x, y]/(x^2, y)$. We should end up with 3 generating maps from $\mathcal{I}_z$ to $\mathcal{O}_w$ since $\mathrm{Ext}^0(\mathcal{I}_z, \mathcal{O}_w)$ is 3-dimensional. Naively, there are 4 generating maps ϕ, but a quick check shows that $\phi(y)$ cannot have a constant term. Thus, the (additive) generators are

$$\phi(y) = x$$
$$\phi(x) = 1$$
$$\phi(x) = x.$$

Therefore the roots for this configuration are

$$(a-b+\lambda_j-\lambda_i)h, (-a+\lambda_j-\lambda_i)h, (\lambda_j-\lambda_i)h.$$

Hence,

$$c_3(\mathrm{Ext}^0(\mathcal{I}_z,\mathcal{O}_w)) = (a-b+\lambda_j-\lambda_i)(-a+\lambda_j-\lambda_i)(\lambda_j-\lambda_i)h^3.$$

(12) This configuration is similar to the last. We have $\mathcal{I}_z = (x,y)$ and $\mathcal{O}_w = \mathbb{C}[x,y]/(x,y^2)$. The additive generating homomorphisms are

$$\begin{aligned}\phi(x) &= y\\ \phi(y) &= 1\\ \phi(y) &= y.\end{aligned}$$

Therefore the roots are

$$(b-a+\lambda_j-\lambda_i)h, (-b+\lambda_j-\lambda_i)h, (\lambda_j-\lambda_i)h$$

and

$$c_3(\mathrm{Ext}^0(\mathcal{I}_z,\mathcal{O}_w)) = (b-a+\lambda_j-\lambda_i)(-b+\lambda_j-\lambda_i)(\lambda_j-\lambda_i)h^3.$$

(13)-(15) For these configurations, the support of w shares one point in common with the support z. Thus, $\mathrm{Ext}^0(\mathcal{I}_z,\mathcal{O}_w)$ splits as a direct sum

$$\mathrm{Ext}^0(\mathcal{I}_z,\mathcal{O}_w) = \mathrm{Ext}^0(\mathcal{I}_z,\mathcal{O}_z) \oplus \mathrm{Ext}^0(\mathcal{I}_z,\mathcal{O}_p)$$

where $p \neq z$. We know the roots of the tangent space already, so we only need the (single) root of $\mathrm{Ext}^0(\mathcal{I}_z,\mathcal{O}_p)$. However, the only homomorphisms from $\mathcal{I}_z$ to $\mathcal{O}_p$ are constant maps. To see this, consider the long exact sequence

$$\begin{array}{lllll} 0 \longrightarrow & \mathrm{Ext}^0(\mathcal{O}_z,\mathcal{O}_p) \longrightarrow & \mathrm{Ext}^0(\mathcal{O},\mathcal{O}_p) \longrightarrow & \mathrm{Ext}^0(\mathcal{I}_z,\mathcal{O}_p) \longrightarrow & \\ & \mathrm{Ext}^1(\mathcal{O}_z,\mathcal{O}_p) \longrightarrow & \mathrm{Ext}^1(\mathcal{O},\mathcal{O}_p) \longrightarrow & \mathrm{Ext}^1(\mathcal{I}_z,\mathcal{O}_p) \longrightarrow & \\ & \mathrm{Ext}^2(\mathcal{O}_z,\mathcal{O}_p) \longrightarrow & \mathrm{Ext}^2(\mathcal{O},\mathcal{O}_p) \longrightarrow & \mathrm{Ext}^2(\mathcal{I}_z,\mathcal{O}_p) \longrightarrow & 0 \end{array}$$

which has dimensions

$$\begin{array}{ccccccc} 0 & \longrightarrow & 0 & \longrightarrow & 1 & \longrightarrow & 1 \longrightarrow \\ & & 0 & \longrightarrow & 0 & \longrightarrow & 0 \longrightarrow \\ & & 0 & \longrightarrow & 0 & \longrightarrow & 0 \longrightarrow \quad 0. \end{array}$$

Thus $\mathrm{Ext}^0(\mathcal{I}_z, \mathcal{O}_p) \cong \mathrm{Ext}^0(\mathcal{O}, \mathcal{O}_p)$, which has no dependence on a or b. This tells us that the roots of $\mathrm{Ext}^0(\mathcal{I}_z, \mathcal{O}_w)$ are

$$\underbrace{(-a+\lambda_j-\lambda_i)h, (-b+\lambda_j-\lambda_i)h}_{\text{shared support}}, \underbrace{(\lambda_j-\lambda_i)h}_{\text{disjoint support}}$$

and

$$c_3(\mathrm{Ext}^0(\mathcal{I}_z, \mathcal{O}_w)) = (-a+\lambda_j-\lambda_i)(-b+\lambda_j-\lambda_i)(\lambda_j-\lambda_i)h^3.$$

(16)-(18) For the remainder of the configurations, the support of z and w are disjoint. Hence, $dim\mathrm{Ext}^0(\mathcal{I}_z, \mathcal{O}_w) = 2$ and the key is that $\mathrm{Ext}^0(\mathcal{I}_z, \mathcal{O}_w) \cong \mathrm{Ext}^0(\mathcal{O}, \mathcal{O}_w)$. Thus, for these configurations the roots are

$$(a+\lambda_j-\lambda_i)h, (\lambda_j-\lambda_i)h$$

and

$$c_2(\mathrm{Ext}^0(\mathcal{I}_z, \mathcal{O}_w)) = (a+\lambda_j-\lambda_i)(\lambda_j-\lambda_i)h^2.$$

(19)-(21) The roots are

$$(b+\lambda_j-\lambda_i)h, (\lambda_j-\lambda_i)h$$

and

$$c_2(\mathrm{Ext}^0(\mathcal{I}_z, \mathcal{O}_w)) = (b+\lambda_j-\lambda_i)(\lambda_j-\lambda_i)h^2.$$

(22)-(24) The group splits as a direct sum with trivial dependence on a and b. Hence,

$$c_2(\mathrm{Ext}^0(\mathcal{I}_z, \mathcal{O}_w)) = (\lambda_j-\lambda_i)(\lambda_j-\lambda_i)h^2.$$

$\underline{c_\blacktriangle(\mathrm{Ext}^1(\mathcal{I}_z, \mathcal{O}_w))}$

When the support of z is contained in that of w we found that $\mathrm{Ext}^1(\mathcal{I}_z, \mathcal{O}_w)$ is 1-dimensional, and is 0-dimensional otherwise. In the 1-dimensional case, we make use of the isomorphisms

$$\mathrm{Ext}^1(\mathcal{I}_z, \mathcal{O}_w) \cong \mathrm{Ext}^2(\mathcal{O}_z, \mathcal{O}_w) \cong \mathrm{Ext}^0(\mathcal{O}_w, \mathcal{O}_z \otimes K)^\vee$$

Thus we compute the lone root as on page 49.

(11)-(15) $c_1(\mathrm{Ext}^1(\mathcal{I}_w, \mathcal{O}_z)) = (-a - b + \lambda_j - \lambda_i)h$

(16)-(24) $c_0(\mathrm{Ext}^1(\mathcal{I}_w, \mathcal{O}_z)) = 1$

$\underline{c_{\blacktriangle+1}(\mathrm{Ext}^0(\mathcal{I}_w, \mathcal{O}_z))}$

We determined this Ext group is 2-dimensional when the support of z is contained in the support of w and 1-dimensional when the supports are disjoint.

(11) The supports are identical, so the computation is local. The ideal sheaf is $\mathcal{I}_w = (x^2, y)$ and $\mathcal{O}_z = \mathbb{C}[x, y]/(x, y)$. Thus the roots are

$$(-2a + \lambda_i - \lambda_j)h, (-b + \lambda_i - \lambda_j)h$$

and

$$c_2(\mathrm{Ext}^0(\mathcal{I}_w, \mathcal{O}_z)) = (-2a + \lambda_i - \lambda_j)(-b + \lambda_i - \lambda_j)h^2.$$

(12) This case is also local, but now $\mathcal{I}_w = (x, y^2)$. Hence the roots are

$$(-a + \lambda_i - \lambda_j)h, (-2b + \lambda_i - \lambda_j)h$$

and

$$c_2(\mathrm{Ext}^0(\mathcal{I}_w, \mathcal{O}_z)) = (-a + \lambda_i - \lambda_j)(-2b + \lambda_i - \lambda_j)h^2.$$

(13)-(15) Configurations (13)-(15) are the most difficult. We have a partial overlap of support, so we cannot use only local methods, and we don't have any obvious isomorphisms from the long exact sequences. Instead, we will observe an inclusion which still allows us to complete the calculation.

From the long exact sequence, we have $\mathrm{Ext}^0(\mathcal{I}_w, \mathcal{O}_z) \cong \mathrm{Ext}^1(\mathcal{O}_w, \mathcal{O}_z)$. We also know that $\mathrm{Ext}^1(\mathcal{O}_z, \mathcal{O}_z) \hookrightarrow \mathrm{Ext}^1(\mathcal{O}_w, \mathcal{O}_z)$, but both of these spaces are 2-dimensional. Thus the inclusion is an isomorphism, so it suffices to compute the roots of $\mathrm{Ext}^1(\mathcal{O}_z, \mathcal{O}_z)$, which are in turn the roots of $\mathrm{Ext}^0(\mathcal{I}_z, \mathcal{O}_z)$, at least with respect to a and b. Hence the roots of $\mathrm{Ext}^0(\mathcal{I}_w, \mathcal{O}_z)$ are $(-a + \lambda_i - \lambda_j)h$ and $(-b + \lambda_i - \lambda_j)h$, so

$$c_2(\mathrm{Ext}^0(\mathcal{I}_w, \mathcal{O}_z)) = (-a + \lambda_i - \lambda_j)(-b + \lambda_i - \lambda_j)h^2.$$

(16)-(24) For the remaining configurations we use the isomorphism $\mathrm{Ext}^0(\mathcal{I}_w, \mathcal{O}_z) \cong \mathrm{Ext}^0(\mathcal{O}, \mathcal{O}_z)$. Thus, in each case $\mathrm{Ext}^0(\mathcal{I}_w, \mathcal{O}_z)$ is 1-dimensional with trivial dependence on a and b, so

$$c_1(\mathrm{Ext}^0(\mathcal{I}_w, \mathcal{O}_z)) = (\lambda_i - \lambda_j)h.$$

$\underline{c_{\blacktriangle}(\mathrm{Ext}^1(\mathcal{I}_w, \mathcal{O}_z))}$

Last we need to compute $c_{\blacktriangle}(\mathrm{Ext}^1(\mathcal{I}_w, \mathcal{O}_z))$, which will be similar to the computation of $c_{\blacktriangle}(\mathrm{Ext}^1(\mathcal{I}_z, \mathcal{O}_w))$. We will use the isomorphisms

$$\mathrm{Ext}^1(\mathcal{I}_w, \mathcal{O}_z) \cong \mathrm{Ext}^2(\mathcal{O}_w, \mathcal{O}_z) \cong \mathrm{Ext}^0(\mathcal{O}_z, \mathcal{O}_w \otimes K)^\vee.$$

(11) Everything is local, so observe that $\mathcal{O}_w = \mathbb{C}[x, y]/(x^2, y)$. Then

$$\begin{aligned} K &\text{ has root } (a+b)h \\ \mathcal{O}_w \otimes K &\text{ has roots } (a+b)h, (2a+b)h \\ \mathrm{Hom}(\mathcal{O}_z, \mathcal{O}_w \otimes K) &\text{ has roots } (2a+b)h \\ \mathrm{Ext}^1(\mathcal{I}_w, \mathcal{O}_z) \cong \mathrm{Ext}^2(\mathcal{O}_w, \mathcal{O}_z) &\cong \mathrm{Ext}^0(\mathcal{O}_z, \mathcal{O}_w \otimes K)^\vee \\ &\text{has root } (-2a-b)h. \end{aligned}$$

Thus

$$c_1(\mathrm{Ext}^1(\mathcal{I}_w, \mathcal{O}_z)) = (-2a - b + \lambda_i - \lambda_j)h.$$

(12) Again, the problem is local but with $\mathcal{O}_w = \mathbb{C}[x, y]/(x, y^2)$, so we similarly compute

$$c_1(\mathrm{Ext}^1(\mathcal{I}_w, \mathcal{O}_z)) = (-a - 2b + \lambda_i - \lambda_j)h.$$

(13)-(15) The point of non-overlap in the supports is irrelevant, so $\mathrm{Ext}^0(\mathcal{O}_z, \mathcal{O}_w \otimes K)^\vee \cong \mathrm{Ext}^0(\mathcal{O}_z, \mathcal{O}_z \otimes K)^\vee$ and thus

$$c_1(\mathrm{Ext}^1(\mathcal{I}_w, \mathcal{O}_z)) = (-a - b + \lambda_i - \lambda_j)h.$$

(16)-(24) For all of these configurations, $dim\mathrm{Ext}^1(\mathcal{I}_w, \mathcal{O}_z) = 0$, so

$$c_0(\mathrm{Ext}^1(\mathcal{I}_w, \mathcal{O}_z)) = 1.$$

This completes the computations of all necessary Chern classes, up to rotations.

Theorem 7. *The Quot scheme* $Quot_S(\mathbb{G}(2,8))_{9,0}$ *has expected dimension 0 and*

$$\int_{[Quot_{\mathbb{P}^1\times\mathbb{P}^1}(\mathbb{G}(2,8))_{9,0}]^{vir}} 1 = -1724160.$$

Proof. We derived the expected dimension in Theorem 4. By virtual localization, the result of the intersection product is given by (6.1) and (6.3), where the sum is over all 24 fixed point configurations, times four-fold rotational symmetry, times the 56 ways of injecting the subsheaf into $\mathcal{O}^8$. Not surprisingly, this finite sum is far too large to compute by hand, but any mathematical software yields the stated result. □

6.2 Computing $\int_{[Quot_{\mathbb{P}^2}(\mathbb{G}(2,8))_{9,0}]^{vir}} 1$

Now we begin the same example on $\mathbb{P}^2$. Recall that the Todd class of $\mathbb{P}^2$ is still $(1+\omega)$, so all of the expected dimensions remain the same. The torus action and labels for the weights will be the same as in section 5.2 and Figure 5.4.

In Figure 6.3 we outline the fixed point configurations in which z has length 3 and in Figure 6.4 we list the fixed point configurations in which z has length 1 and w has length 2.

Let's begin with the first case, in which the short exact sequence is

$$0 \longrightarrow \mathcal{I}_z \oplus \mathcal{O} \longrightarrow \mathcal{O}^8 \longrightarrow \mathcal{O}_z \oplus \mathcal{O}^6 \longrightarrow 0.$$

Then just as before, the Euler class of the virtual normal bundle is given by

$$\begin{aligned}\frac{1}{e_T(N^{vir})} =& \frac{c_3(\mathrm{Ext}^1(\mathcal{I}_z,\mathcal{O}_z))}{c_6(\mathrm{Ext}^0(\mathcal{I}_z,\mathcal{O}_z))}\cdot\frac{1}{\oplus_1^6 c_1(\mathrm{Ext}^0(\mathcal{O},\mathcal{O}))}\\ &\cdot\frac{1}{c_3(\mathrm{Ext}^0(\mathcal{O},\mathcal{O}_z))}\cdot\oplus_1^6\left(\frac{c_3(\mathrm{Ext}^1(\mathcal{I}_z,\mathcal{O}))}{c_1(\mathrm{Ext}^0(\mathcal{I}_z,\mathcal{O}))}\right).\end{aligned}$$

$\underline{c_6(\mathrm{Ext}^0(\mathcal{I}_z,\mathcal{O}_z))}$

For the explicit Chern classes, we will begin with $c_6(\mathrm{Ext}^0(\mathcal{I}_z,\mathcal{O}_z))$, being very careful of the signs on the weights from Figure 5.4. The computations heavily mirror those on $\mathbb{P}^1\times\mathbb{P}^1$.

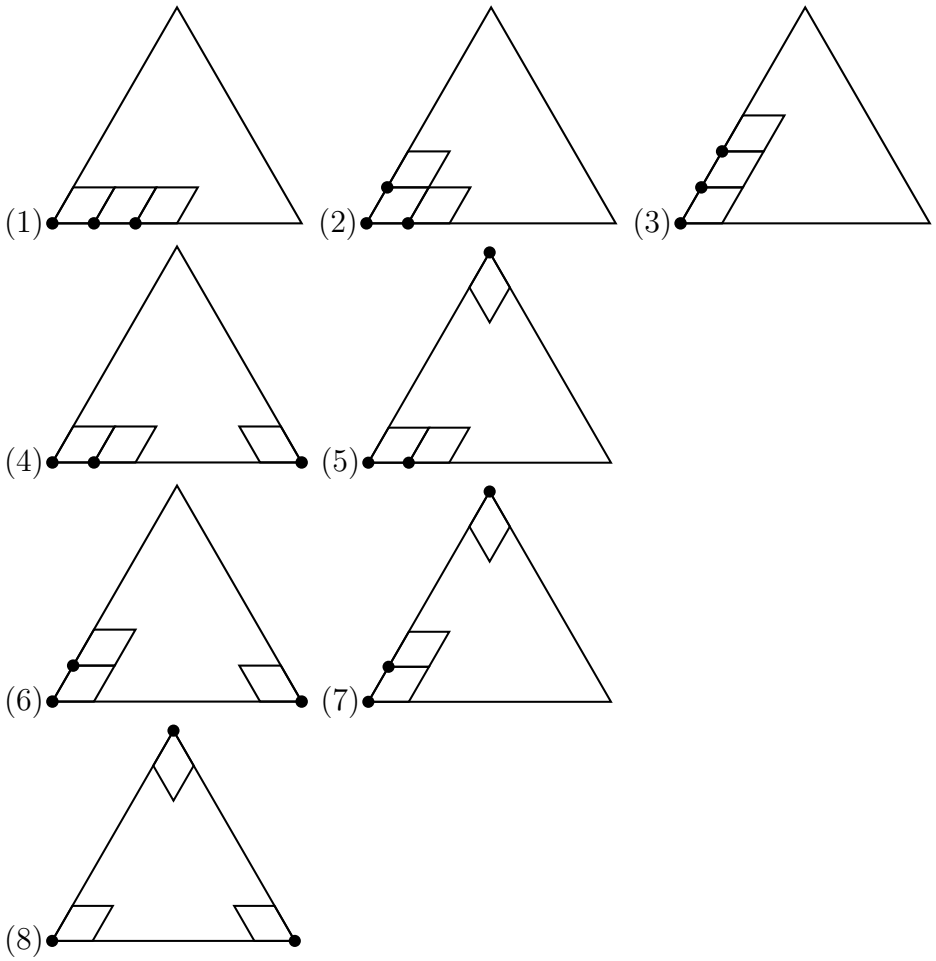


Figure 6.3: Fixed loci for rank 2 example over $\mathbb{P}^2$

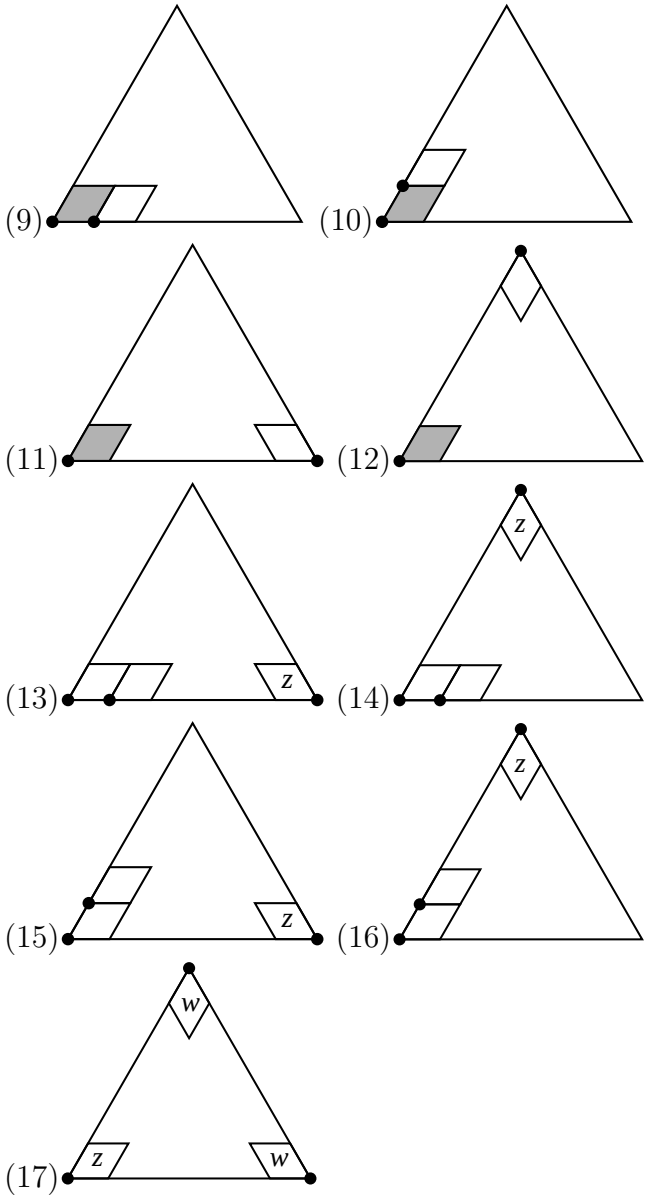


Figure 6.4: More fixed loci for rank 2 example over $\mathbb{P}^2$

(1) Here $\mathcal{I}_z = (x^3, y)$ and $\mathcal{O}_z = \mathbb{C}[x,y]/(x^3,y)$, so the roots are

$$-3\alpha h, \beta h, -2\alpha h, (\alpha+\beta)h, -\alpha h, (2\alpha+\beta)h$$

and

$$c_6(\mathrm{Ext}^0(\mathcal{I}_z, \mathcal{O}_z)) = -6\alpha^3\beta(\alpha+\beta)(2\alpha+\beta)h^6.$$

(2) Here $\mathcal{I}_z = (x^2, xy, y^2)$ and $\mathcal{O}_z = \mathbb{C}[x,y]/(x^2,xy,y^2)$, so the roots are

$$-\alpha h, \beta h, (\alpha+2\beta)h, (-\beta-2\alpha)h, -\alpha h, \beta h$$

and

$$c_6(\mathrm{Ext}^0(\mathcal{I}_z, \mathcal{O}_z)) = -\alpha^2\beta^2(\alpha+2\beta)(2\alpha+\beta)h^6.$$

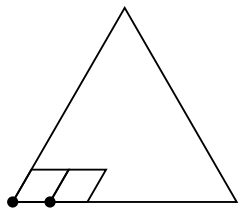
(3) Here $\mathcal{I}_z = (x, y^3)$ and $\mathcal{O}_z = \mathbb{C}[x,y]/(x,y^3)$, so the roots are

$$(-2\beta-\alpha)h, \beta h, (-\beta-\alpha)h, 2\beta h, -\alpha h, 3\beta h$$

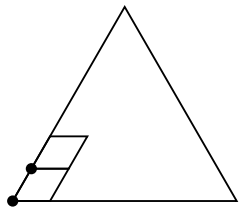
and

$$c_6(\mathrm{Ext}^0(\mathcal{I}_z, \mathcal{O}_z)) = -6\alpha\beta^3(\alpha+\beta)(\alpha+2\beta)h^6.$$

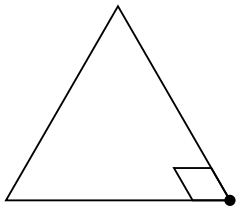
We could easily make mistakes if we attempt to assemble the next Chern classes without doing the component pieces. Intead, we shall calculate them ahead of time.



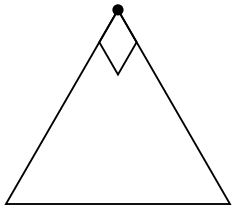
$c_4(\mathrm{Ext}^0(\mathcal{I}_\bullet, \mathcal{O}_\bullet))$ has roots $-2\alpha h$, βh, $-\alpha h$, $(\alpha+\beta)h$.



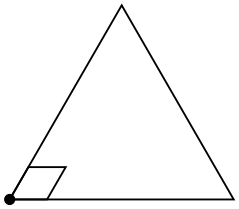
$c_4(\mathrm{Ext}^0(\mathcal{I}_\bullet, \mathcal{O}_\bullet))$ has roots $-\alpha h$, $2\beta h$, $(-\alpha-\beta)h$, βh.



$c_2(\mathrm{Ext}^0(\mathcal{I}_\bullet, \mathcal{O}_\bullet))$ has roots $-\gamma h$, αh.



$c_2(\mathrm{Ext}^0(\mathcal{I}_\bullet, \mathcal{O}_\bullet))$ has roots $-\beta h$, γh.



$c_2(\mathrm{Ext}^0(\mathcal{I}_\bullet, \mathcal{O}_\bullet))$ has roots $-\alpha h$, βh.

Now it will be easier to assemble $c_6(\mathrm{Ext}^0(\mathcal{I}_z, \mathcal{O}_z))$ for the remaining fixed point configurations, since they will just be direct sums.

(4) As a direct sum, this has roots

$$\underbrace{-2\alpha h, \beta h, -\alpha h, (\alpha+\beta)h}_{\text{double point}} \quad \underbrace{-\gamma h, \alpha h}_{\text{single point}}$$

so

$$c_6(\mathrm{Ext}^0(\mathcal{I}_z, \mathcal{O}_z)) = -2\alpha^3\beta\gamma(\alpha+\beta)h^6.$$

The remaining fixed point configurations are so easily assembled, we will simply list the results.

(5) $c_6(\mathrm{Ext}^0(\mathcal{I}_z, \mathcal{O}_z)) = -2\alpha^2\beta^2\gamma(\alpha+\beta)h^6$.

(6) $c_6(\mathrm{Ext}^0(\mathcal{I}_z, \mathcal{O}_z)) = -2\alpha^2\beta^2\gamma(\alpha+\beta)h^6$.

(7) $c_6(\mathrm{Ext}^0(\mathcal{I}_z, \mathcal{O}_z)) = -2\alpha\beta^3\gamma(\alpha+\beta)h^6$.

(8) $c_6(\mathrm{Ext}^0(\mathcal{I}_z, \mathcal{O}_z)) = -\alpha^2\beta^2\gamma^2(\alpha+\beta)h^6$.

$\underline{c_1(\mathrm{Ext}^0(\mathcal{O},\mathcal{O}))}$

Any homomorphism from $\mathcal{O}$ to $\mathcal{O}$ is given by scalar multiplication, so the Chern class is independent of α, β, and γ. Thus, for every fixed point configuration,

$$c_1(\mathrm{Ext}^0(\mathcal{O},\mathcal{O})) = (\lambda_k - \lambda_j)h.$$

$\underline{c_1(\mathrm{Ext}^0(\mathcal{I}_z,\mathcal{O}))}$

This is nearly identical to the previous Chern class. Any homomorphism from $\mathcal{I}_z$ to $\mathcal{O}$ is given by scalar multiplication, so the Chern class is independent of α, β, and γ. Thus, for every fixed point configuration,

$$c_1(\mathrm{Ext}^0(\mathcal{I}_z,\mathcal{O})) = (\lambda_k - \lambda_i)h.$$

$\underline{c_3(\mathrm{Ext}^0(\mathcal{O},\mathcal{O}_z))}$

Next we focus on computing $c_3(\mathrm{Ext}^0(\mathcal{O},\mathcal{O}_z))$. This computation is always local and splits with the support of z. The λ dependence will always be factors of $(\lambda_i - \lambda_j)$.

(1) The roots are $(2\alpha + \lambda_i - \lambda_j)h$, $(\alpha + \lambda_i - \lambda_j)h$, and $(\lambda_i - \lambda_j)h$, so

$$c_3(\mathrm{Ext}^0(\mathcal{O},\mathcal{O}_z)) = (2\alpha + \lambda_i - \lambda_j)(\alpha + \lambda_i - \lambda_j)(\lambda_i - \lambda_j)h^3.$$

(2) The roots are $(\alpha + \lambda_i - \lambda_j)h$, $(-\beta + \lambda_i - \lambda_j)h$, and $(\lambda_i - \lambda_j)h$, so

$$c_3(\mathrm{Ext}^0(\mathcal{O},\mathcal{O}_z)) = (\alpha + \lambda_i - \lambda_j)(-\beta + \lambda_i - \lambda_j)(\lambda_i - \lambda_j)h^3.$$

(3) The roots are $(-2\beta + \lambda_i - \lambda_j)h$, $(-\beta + \lambda_i - \lambda_j)h$, and $(\lambda_i - \lambda_j)h$, so

$$c_3(\mathrm{Ext}^0(\mathcal{O},\mathcal{O}_z)) = (-2\beta + \lambda_i - \lambda_j)(-\beta + \lambda_i - \lambda_j)(\lambda_i - \lambda_j)h^3.$$

(4)-(5) The roots are $(\alpha + \lambda_i - \lambda_j)h$, $(\lambda_i - \lambda_j)h$, and $(\lambda_i - \lambda_j)h$, so

$$c_3(\mathrm{Ext}^0(\mathcal{O},\mathcal{O}_z)) = (\alpha + \lambda_i - \lambda_j)(\lambda_i - \lambda_j)(\lambda_i - \lambda_j)h^3.$$

(6)-(7) The roots are $(-\beta + \lambda_i - \lambda_j)h$, $(\lambda_i - \lambda_j)h$, and $(\lambda_i - \lambda_j)h$, so

$$c_3(\mathrm{Ext}^0(\mathcal{O},\mathcal{O}_z)) = (-\beta + \lambda_i - \lambda_j)(\lambda_i - \lambda_j)(\lambda_i - \lambda_j)h^3.$$

(8) The roots are $(\lambda_i - \lambda_j)h$, $(\lambda_i - \lambda_j)h$, and $(\lambda_i - \lambda_j)h$, so

$$c_3(\mathrm{Ext}^0(\mathcal{O}, \mathcal{O}_z)) = (\lambda_i - \lambda_j)(\lambda_i - \lambda_j)(\lambda_i - \lambda_j)h^3.$$

$\underline{c_3(\mathrm{Ext}^1(\mathcal{I}_z, \mathcal{O}))}$

Next on the agenda we will calculate $c_3(\mathrm{Ext}^1(\mathcal{I}_z, \mathcal{O}))$. The λ-dependence will always be $\lambda_k - \lambda_i$. We also recall that (with λ's omitted), $K = \wedge^2 T^\vee$ has the root $(\alpha - \beta)h$ near the point $[1:0:0]$.

As before, we make use of the isomorphisms

$$\mathrm{Ext}^1(\mathcal{I}_z, \mathcal{O}) \cong \mathrm{Ext}^1(\mathcal{I}_z, \mathcal{O}_z) \cong \mathrm{Ext}^2(\mathcal{O}_z, \mathcal{O}_z) \cong \mathrm{Ext}^0(\mathcal{O}_z, K \otimes \mathcal{O}_z)^\vee.$$

Since any map from $\mathcal{O}_z$ is determined by the image of 1, this should just be the same as $(K \otimes \mathcal{O}_z)^\vee$.

(1) Here $\mathcal{O}_z$ has roots h, αh, and $2\alpha h$, so $K \otimes \mathcal{O}_z$ has roots $(\alpha - \beta)h$, $(2\alpha - \beta)h$, and $(3\alpha - \beta)h$. Hence,

$$c_3(\mathrm{Ext}^1(\mathcal{I}_z, \mathcal{O})) = (-\alpha+\beta+\lambda_k-\lambda_i)(-2\alpha+\beta+\lambda_k-\lambda_i)(-3\alpha+\beta+\lambda_k-\lambda_i)h^3.$$

The remainder of the configurations are quite similar, so we largely omit the explanations.

(2) $c_3(\mathrm{Ext}^1(\mathcal{I}_z, \mathcal{O})) = (-\alpha+\beta+\lambda_k-\lambda_i)(-2\alpha+\beta+\lambda_k-\lambda_i)(-\alpha+2\beta+\lambda_k-\lambda_i)h^3$

(3) $c_3(\mathrm{Ext}^1(\mathcal{I}_z, \mathcal{O})) = (-\alpha+\beta+\lambda_k-\lambda_i)(-\alpha+2\beta+\lambda_k-\lambda_i)(-\alpha+3\beta+\lambda_k-\lambda_i)h^3$

(4) The roots are

$$\underbrace{(-\alpha+\beta+\lambda_k-\lambda_i)h, (-2\alpha+\beta+\lambda_k-\lambda_i)h}, \underbrace{(-\gamma+\alpha+\lambda_k-\lambda_i)h}$$

so as a direct product, we have

$$c_3(\mathrm{Ext}^1(\mathcal{I}_z, \mathcal{O})) = (-\alpha+\beta+\lambda_k-\lambda_i)(-2\alpha+\beta+\lambda_k-\lambda_i)(-\gamma+\alpha+\lambda_k-\lambda_i)h^3.$$

(5) $c_3(\mathrm{Ext}^1(\mathcal{I}_z, \mathcal{O})) = (-\alpha+\beta+\lambda_k-\lambda_i)(-2\alpha+\beta+\lambda_k-\lambda_i)(-\beta+\gamma+\lambda_k-\lambda_i)h^3$

(6) $c_3(\mathrm{Ext}^1(\mathcal{I}_z, \mathcal{O})) = (-\alpha+\beta+\lambda_k-\lambda_i)(-\alpha+2\beta+\lambda_k-\lambda_i)(-\gamma+\alpha+\lambda_k-\lambda_i)h^3$

(7) $c_3(\mathrm{Ext}^1(\mathcal{I}_z,\mathcal{O})) = (-\alpha+\beta+\lambda_k-\lambda_i)(-\alpha+2\beta+\lambda_k-\lambda_i)(-\beta+\gamma+\lambda_k-\lambda_i)h^3$

(8) $c_3(\mathrm{Ext}^1(\mathcal{I}_z,\mathcal{O})) = (-\alpha+\beta+\lambda_k-\lambda_i)(-\gamma+\alpha+\lambda_k-\lambda_i)(-\beta+\gamma+\lambda_k-\lambda_i)h^3.$

We observe that $c_3(\mathrm{Ext}^1(\mathcal{I}_z,\mathcal{O}_z))$ is isomorphic to $c_3(\mathrm{Ext}^1(\mathcal{I}_z,\mathcal{O}))$ without the dependence on λ, so we have completed the Chern classes when z has length 3. Next we proceed to the case when z has length 1 and w has length 2. The fixed point configurations for this case are illustrated in Figure 6.4 on page 57.

We begin our analysis of this case by recalling that

$$\frac{1}{e_T(N^{vir})} = \frac{c_1(\mathrm{Ext}^1(\mathcal{I}_z,\mathcal{O}_z))}{c_2(\mathrm{Ext}^0(\mathcal{I}_z,\mathcal{O}_z))}\cdot\frac{c_2(\mathrm{Ext}^1(\mathcal{I}_w,\mathcal{O}_w))}{c_4(\mathrm{Ext}^0(\mathcal{I}_w,\mathcal{O}_w))}\cdot\frac{c_{\blacktriangle}(\mathrm{Ext}^1(\mathcal{I}_z,\mathcal{O}_w))}{c_{\blacktriangle+2}(\mathrm{Ext}^0(\mathcal{I}_z,\mathcal{O}_w))}\cdot \frac{c_{\blacktriangle}(\mathrm{Ext}^1(\mathcal{I}_w,\mathcal{O}_z))}{c_{\blacktriangle+1}(\mathrm{Ext}^0(\mathcal{I}_w,\mathcal{O}_z))}\cdot\oplus_1^6\left(\frac{c_1(\mathrm{Ext}^1(\mathcal{I}_z,\mathcal{O}))}{c_1(\mathrm{Ext}^0(\mathcal{I}_z,\mathcal{O}))}\cdot\frac{c_2(\mathrm{Ext}^1(\mathcal{I}_w,\mathcal{O}))}{c_1(\mathrm{Ext}^0(\mathcal{I}_w,\mathcal{O}))}\right).$$

Many of these Chern classes we can simply list based on prior computations.

$\underline{c_2(\mathrm{Ext}^0(\mathcal{I}_z,\mathcal{O}_z))}$

(9)-(12) $c_2(\mathrm{Ext}^0(\mathcal{I}_z,\mathcal{O}_z)) = (-\alpha h)(\beta h) = (-\alpha\beta)h^2$

(13) $c_2(\mathrm{Ext}^0(\mathcal{I}_z,\mathcal{O}_z)) = (\alpha h)(-\gamma h) = (-\alpha\gamma)h^2$

(14) $c_2(\mathrm{Ext}^0(\mathcal{I}_z,\mathcal{O}_z)) = (-\beta h)(\gamma h) = (-\beta\gamma)h^2$

(15) $c_2(\mathrm{Ext}^0(\mathcal{I}_z,\mathcal{O}_z)) = (\alpha h)(-\gamma h) = (-\alpha\gamma)h^2$

(16) $c_2(\mathrm{Ext}^0(\mathcal{I}_z,\mathcal{O}_z)) = (-\beta h)(\gamma h) = (-\beta\gamma)h^2$

(17) $c_2(\mathrm{Ext}^0(\mathcal{I}_z,\mathcal{O}_z)) = (-\alpha h)(\beta h) = (-\alpha\beta)h^2.$

$\underline{c_4(\mathrm{Ext}^0(\mathcal{I}_w,\mathcal{O}_w))}$

The pieces necessary for each of these configurations were computed on page 58.

(9) $c_4(\mathrm{Ext}^0(\mathcal{I}_w,\mathcal{O}_w)) = 2\alpha^2\beta(\alpha+\beta)h^4$

(10) $c_4(\mathrm{Ext}^0(\mathcal{I}_w,\mathcal{O}_w)) = 2\alpha\beta^2(\alpha+\beta)h^4$

(11) $c_4(\mathrm{Ext}^0(\mathcal{I}_w,\mathcal{O}_w)) = (-\alpha h)(\beta h)(-\gamma h)(\alpha h) = \alpha^2\beta\gamma h^4$

(12) $c_4(\mathrm{Ext}^0(\mathcal{I}_w, \mathcal{O}_w)) = (-\alpha h)(\beta h)(-\beta h)(\gamma h) = \alpha\beta^2\gamma h^4$

(13)-(14) $c_4(\mathrm{Ext}^0(\mathcal{I}_w, \mathcal{O}_w)) = 2\alpha^2\beta(\alpha+\beta)h^4$

(15)-(16) $c_4(\mathrm{Ext}^0(\mathcal{I}_w, \mathcal{O}_w)) = 2\alpha\beta^2(\alpha+\beta)h^4$

(17) $c_4(\mathrm{Ext}^0(\mathcal{I}_w, \mathcal{O}_w)) = (-\gamma h)(\alpha h)(-\beta h)(\gamma h) = \alpha\beta\gamma^2 h^4$.

$\underline{c_1(\mathrm{Ext}^0(\mathcal{I}_z, \mathcal{O}))}$

Given the independence of this Chern class on α, β, and γ, for every fixed point configuration we have

$$c_1(\mathrm{Ext}^0(\mathcal{I}_z, \mathcal{O})) = (\lambda_k - \lambda_i)h.$$

$\underline{c_1(\mathrm{Ext}^0(\mathcal{I}_w, \mathcal{O}))}$

Similarly, $c_1(\mathrm{Ext}^0(\mathcal{I}_w, \mathcal{O}))$ is independent of α, β, and γ, so

$$c_1(\mathrm{Ext}^0(\mathcal{I}_w, \mathcal{O})) = (\lambda_k - \lambda_j)h.$$

$\underline{c_1(\mathrm{Ext}^1(\mathcal{I}_z, \mathcal{O}))}$

Next we compute $c_1(\mathrm{Ext}^1(\mathcal{I}_z, \mathcal{O}))$. Note that the λ-dependence is always $\lambda_k - \lambda_i$. Additionally, the single root will be the same as that of $K^\vee$.

(9)-(12) $c_1(\mathrm{Ext}^1(\mathcal{I}_z, \mathcal{O})) = (-\alpha + \beta + \lambda_k - \lambda_i)h$

(13) $c_1(\mathrm{Ext}^1(\mathcal{I}_z, \mathcal{O})) = (-\gamma + \alpha + \lambda_k - \lambda_i)h$

(14) $c_1(\mathrm{Ext}^1(\mathcal{I}_z, \mathcal{O})) = (-\beta + \gamma + \lambda_k - \lambda_i)h$

(15) $c_1(\mathrm{Ext}^1(\mathcal{I}_z, \mathcal{O})) = (-\gamma + \alpha + \lambda_k - \lambda_i)h$

(16) $c_1(\mathrm{Ext}^1(\mathcal{I}_z, \mathcal{O})) = (-\beta + \gamma + \lambda_k - \lambda_i)h$

(17) $c_1(\mathrm{Ext}^1(\mathcal{I}_z, \mathcal{O})) = (-\alpha + \beta + \lambda_k - \lambda_i)h$.

This also takes care of $c_1(\mathrm{Ext}^1(\mathcal{I}_z, \mathcal{O}_z))$ since we just lose the dependence on λ.

$\underline{c_2(\mathrm{Ext}^1(\mathcal{I}_w, \mathcal{O}))}$

The λ-dependence is always $\lambda_k - \lambda_j$. This computation was largely done on page 61.

(9) $(-\alpha+\beta+\lambda_k-\lambda_j)(-2\alpha+\beta+\lambda_k-\lambda_j)h^2$

(10) $(-\alpha+\beta+\lambda_k-\lambda_j)(-\alpha+2\beta+\lambda_k-\lambda_j)h^2$

(11) $(-\alpha+\beta+\lambda_k-\lambda_j)(-\gamma+\alpha+\lambda_k-\lambda_j)h^2$

(12) $(-\alpha+\beta+\lambda_k-\lambda_j)(-\beta+\gamma+\lambda_k-\lambda_j)h^2$

(13)-(14) $(-\alpha+\beta+\lambda_k-\lambda_j)(-2\alpha+\beta+\lambda_k-\lambda_j)h^2$

(15)-(16) $(-\alpha+\beta+\lambda_k-\lambda_j)(-\alpha+2\beta+\lambda_k-\lambda_j)h^2$

(17) $(-\gamma+\alpha+\lambda_k-\lambda_j)(-\beta+\gamma+\lambda_k-\lambda_j)h^2$.

Dropping the λ's also computes $c_2(\mathrm{Ext}^1(\mathcal{I}_w,\mathcal{O}))$. Now all that remains to compute are the cross terms.

$\underline{c_{\blacktriangle+2}(\mathrm{Ext}^0(\mathcal{I}_z,\mathcal{O}_w))}$

We begin the cross terms by computing $c_{\blacktriangle+2}(\mathrm{Ext}^0(\mathcal{I}_z,\mathcal{O}_w))$. Since Ext^0 is just Hom, this is a rather naive computation. There are several subcases, as on page 43, corresponding to the overlap of the supports of z and w. The dependence on λ is $\lambda_j-\lambda_i$.

(9) This is a completely local calculation, so $\mathcal{I}_z=(x,y)$ and $\mathcal{O}_w=\mathbb{C}[x,y]/(x^2,y)$. Thus the generating homomorphisms are given by $\phi(y)=x$, $\phi(x)=1$, and $\phi(x)=x$. Hence,

$$c_3(\mathrm{Ext}^0(\mathcal{I}_z,\mathcal{O}_w))=(\alpha+\beta+\lambda_j-\lambda_i)(-\alpha+\lambda_j-\lambda_i)(\lambda_j-\lambda_i)h^3.$$

(10) Similarly, $\mathcal{I}_z=(x,y)$ and $\mathcal{O}_w=\mathbb{C}[x,y]/(x,y^2)$ and the generating homomorphisms are $\phi(x)=y$, $\phi(y)=1$, and $\phi(y)=y$. Hence,

$$c_3(\mathrm{Ext}^0(\mathcal{I}_z,\mathcal{O}_w))=(-\alpha-\beta+\lambda_j-\lambda_i)(\beta+\lambda_j-\lambda_i)(\lambda_j-\lambda_i)h^3.$$

(11)-(12) Recall that for a point p distinct from z, $\mathrm{Ext}^0(\mathcal{I}_z,\mathcal{O}_p)\cong\mathrm{Ext}^0(\mathcal{O},\mathcal{O}_p)$, which has no dependence on α, β, or γ. Therefore for these configurations, $\mathrm{Ext}^0(\mathcal{I}_z,\mathcal{O}_w)\cong\mathrm{Ext}^0(\mathcal{I}_z,\mathcal{O}_z)\oplus\mathrm{Ext}^0(\mathcal{I}_z,\mathcal{O}_p)$. Therefore,

$$c_3(\mathrm{Ext}^0(\mathcal{I}_z,\mathcal{O}_w))=(-\alpha+\lambda_j-\lambda_i)(\beta+\lambda_j-\lambda_i)(\lambda_j-\lambda_i)h^3.$$

For the remaining cases, the support of z and w are disjoint, so $\mathrm{Ext}^0(\mathcal{I}_z, \mathcal{O}_w)$ is 2-dimensional. If w is two disjoint points, then we are done. If w is a double point, then we use the fact that $\mathrm{Ext}^0(\mathcal{I}_z, \mathcal{O}_w) \cong \mathrm{Ext}^0(\mathcal{O}, \mathcal{O}_w)$.

(13)-(14) $c_2(\mathrm{Ext}^0(\mathcal{I}_z, \mathcal{O}_w)) = (\alpha + \lambda_j - \lambda_i)(\lambda_j - \lambda_i)h^2$

(15)-(16) $c_2(\mathrm{Ext}^0(\mathcal{I}_z, \mathcal{O}_w)) = (-\beta + \lambda_j - \lambda_i)(\lambda_j - \lambda_i)h^2$

(17) $c_2(\mathrm{Ext}^0(\mathcal{I}_z, \mathcal{O}_w)) = (\lambda_j - \lambda_i)(\lambda_j - \lambda_i)h^2$.

$\underline{c_{\blacktriangle}(\mathrm{Ext}^1(\mathcal{I}_z, \mathcal{O}_w))}$

Next we move on to $c_{\blacktriangle}(\mathrm{Ext}^1(\mathcal{I}_z, \mathcal{O}_w))$. When the supports of z and w are disjoint, the dimension is 0, and when they overlap the dimension is 1. In the 1-dimensional case,

$$\mathrm{Ext}^1(\mathcal{I}_z, \mathcal{O}_w) \cong \mathrm{Ext}^2(\mathcal{O}_z, \mathcal{O}_w) \cong \mathrm{Ext}^0(\mathcal{O}_w, K \otimes \mathcal{O}_z)^\vee.$$

Any such homomorphism is determined by the image of 1, so this is isomorphic to $K^\vee$.

(9)-(12) $c_1(\mathrm{Ext}^1(\mathcal{I}_z, \mathcal{O}_w)) = (-\alpha + \beta + \lambda_j - \lambda_i)h$

(13)-(17) $c_0(\mathrm{Ext}^1(\mathcal{I}_z, \mathcal{O}_w)) = 1$.

$\underline{c_{\blacktriangle+1}(\mathrm{Ext}^0(\mathcal{I}_w, \mathcal{O}_z))}$

When the supports of z and w overlaps this is 2-dimensional. When the supports are disjoint this is 1-dimensional.

(9) This is local with $\mathcal{I}_w = (x^2, y)$ and $\mathcal{O}_z = \mathbb{C}[x, y]/(x, y)$. Thus,

$$c_2(\mathrm{Ext}^0(\mathcal{I}_w, \mathcal{O}_z)) = (-2\alpha + \lambda_i - \lambda_j)(\beta + \lambda_i - \lambda_j)h^2.$$

(10) Here $\mathcal{I}_w = (x, y^2)$ and $\mathcal{O}_z = \mathbb{C}[x, y]/(x, y)$ so

$$c_2(\mathrm{Ext}^0(\mathcal{I}_w, \mathcal{O}_z)) = (-\alpha + \lambda_i - \lambda_j)(2\beta + \lambda_i - \lambda_j)h^2.$$

For the next cases when z overlaps partially with w, the key is that

$$\mathrm{Ext}^0(\mathcal{I}_w, \mathcal{O}_z) \cong \mathrm{Ext}^1(\mathcal{O}_w, \mathcal{O}_z) \hookleftarrow \mathrm{Ext}^1(\mathcal{O}_z, \mathcal{O}_z) \cong \mathrm{Ext}^0(\mathcal{I}_z, \mathcal{O}_z).$$

Since the inclusion is of the same dimension, it is an isomorphism as well. This isomorphism can also be shown with tensor products, but there is no need to justify it twice.

(11)-(12) $c_2(\mathrm{Ext}^0(\mathcal{I}_w, \mathcal{O}_z)) = (-\alpha + \lambda_i - \lambda_j)(\beta + \lambda_i - \lambda_j)h^2$

(13)-(17) The supports are disjoint so $\mathrm{Ext}^0(\mathcal{I}_w, \mathcal{O}_z) \cong \mathrm{Ext}^0(\mathcal{O}, \mathcal{O}_z)$. Thus,

$$c_1(\mathrm{Ext}^0(\mathcal{I}_w, \mathcal{O}_z)) = (\lambda_i - \lambda_j)h.$$

$\underline{c_\blacktriangle(\mathrm{Ext}^1(\mathcal{I}_w, \mathcal{O}_z))}$

Last we have $c_\blacktriangle(\mathrm{Ext}^1(\mathcal{I}_w, \mathcal{O}_z))$. We can always make use of the isomorphisms

$$\mathrm{Ext}^1(\mathcal{I}_w, \mathcal{O}_z) \cong \mathrm{Ext}^2(\mathcal{O}_w, \mathcal{O}_z) \cong \mathrm{Ext}^0(\mathcal{O}_z, K \otimes \mathcal{O}_w)^\vee.$$

(9) This is local. Here $\mathcal{O}_z = \mathbb{C}[x, y]/(x, y)$ and $\mathcal{O}_w = \mathbb{C}[x, y]/(x^2, y)$, so

$$\begin{aligned} K \text{ has root} \quad & (\alpha - \beta)h \\ K \otimes \mathcal{O}_w \text{ has roots} \quad & (\alpha - \beta)h, (2\alpha - \beta)h \\ \mathrm{Ext}^0(\mathcal{O}_z, K \otimes \mathcal{O}_w) \text{ has root} \quad & (2\alpha - \beta)h \end{aligned}$$

Thus

$$c_1(\mathrm{Ext}^1(\mathcal{I}_w, \mathcal{O}_z)) = (-2\alpha + \beta + \lambda_i - \lambda_j)h.$$

(10) Similarly, $c_1(\mathrm{Ext}^1(\mathcal{I}_w, \mathcal{O}_z)) = (-\alpha + 2\beta + \lambda_i - \lambda_j)h$.

(11)-(12) Non-overlapping support is not relevant, so we have the isomorphism $\mathrm{Ext}^0(\mathcal{O}_z, K \otimes \mathcal{O}_w)^\vee \cong \mathrm{Ext}^0(\mathcal{O}_z, K \otimes \mathcal{O}_z)^\vee$. Hence,

$$c_1(\mathrm{Ext}^1(\mathcal{I}_w, \mathcal{O}_z)) = (-\alpha + \beta + \lambda_i - \lambda_j)h.$$

(13)-(17) $c_0(\mathrm{Ext}^1(\mathcal{I}_w, \mathcal{O}_z)) = 1$.

This is all the necessary Chern classes.

Theorem 8. *Over the surface* $\mathbb{P}^2$*,*

$$\int_{[Quot_{\mathbb{P}^2}(\mathbb{G}(2,8))_{9,0}]^{vir}} 1 = -2680920.$$

Proof. We have computed all of the necessary Chern classes, making the derivation of this intersection number equivalent to finding the finite sum over the many isolated fixed points. This sum is too large to compute concisely by hand, but can be verified using any mathematical software. There is nothing more to check. □

Chapter 7

Summary and Generalizing S

The theorems in chapters 5 and 6 explicitly compute several intersections products on $\text{Quot}_{\mathbb{P}^1\times\mathbb{P}^1}$ and $\text{Quot}_{\mathbb{P}^2}$. While this is far from the closed form of the Vafa-Intriligator formula in Theorem 1, we do have the makings of a computational algorithm for further intersection products on Quot_S. The specified torus actions will always yield isolated fixed points, and the equivariant normal bundle will always be given by (4.11). Furthermore, we have demonstrated useful isomorphisms and injections in the various long exact sequences which make the Chern classes of the various Ext groups calculable. Thus, while the computations may become lengthy and unwieldy for other invariants on Quot_S, this algorithmic approach holds valid.

The last item we will address is a generalization to other surfaces S. Every computation so far has relied quite explicitly on the natural torus actions on $\mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{P}^2$. We will prove that any enumerative invariant on Quot_S can be computed in terms of the same invariant computed on $\text{Quot}_{\mathbb{P}^1\times\mathbb{P}^1}$ and $\text{Quot}_{\mathbb{P}^2}$, at least when S admits a virtual fundamental class. This result will extend even to non-toric surfaces and potentially to Quot schemes over S whose virtual fundamental classes might be constructed by means other than Theorem 3. We provide a bit of motivation before stating the precise theorem.

It is well known that the complex cobordism ring $\Omega = \Omega^U \otimes \mathbb{Q}$ with rational coefficients is a polynomial ring generated by the cobordism classes $[\mathbb{P}^i]$ for positive i. In [EGL], the authors proved the following as Theorem 3.1.

Theorem. *Let f be a polynomial in the Chern classes of the following sheaves on $S^{[n+1]} \times S^k$:*

$$pr_0^* T_{n+1},\ pr_{0i}^* I_{n+1},\ pr_{ij}^* \mathcal{O}_\triangle,\ pr_i^* T_S \quad for\ any\ 1 \leq i, j \leq k.$$

Then there is a polynomial $\tilde{f}$, depending only f, in the Chern classes of the analogously defined sheaves on $S^{[n]} \times S^{k+1}$ such that

$$\int_{S^{[n+1]} \times S^k} f = \int_{S^{[n]} \times S^{k+1}} \tilde{f}.$$

Here, for any $I \subset \{0, 1, \ldots, k\}$, the morphism pr_I is the projection from $S^{[n+1]} \times S^k$ onto the factors indexed by I. Successive application of this theorem says that intersection products in the given classes over $S^{[n]}$ can be computed over S instead, and since $[\mathbb{P}^1 \times \mathbb{P}^1]$ and $[\mathbb{P}^2]$ generate Ω_2, it suffices to know the intersection products for these surfaces. The authors themselves slightly generalized this result to also include Chern classes in $pr_i^* F$ and $pr_0^* F^{[n+1]}$ where F is a locally free sheaf on S and $F^{[n+1]}$ is an associated tautological sheaf defined over $S^{[n+1]}$.

We will explain all of this dense notational momentarily, but let us explain the relevance of this theory first. Instead of using the given torus actions on Quot_S, consider for a moment using only the torus action from [MO1] which acts diagonally on $\mathbb{C}^N \otimes \mathcal{O}$. As explained in chapter 4, the fixed loci would be products $S^{[n_1]} \times \cdots \times S^{[n_l]}$ of Hilbert schemes of points on S. Furthermore, the equivariant normal bundle of this action would be polynomials in the Chern classes of the relative $\mathcal{E}\mathrm{xt}_p$ sheaves. Even though we did not use this torus action for explicit computation, we will use it and an analog of Theorem 3.1 from [EGL] to show that any intersection product of Quot_S with a polynomial f in the Chern classes of the $\mathcal{E}\mathrm{xt}$ sheaves appearing in (4.11) depends only on the Chern classes of S and the same intersection product on $\mathrm{Quot}_{\mathbb{P}^1 \times \mathbb{P}^1}$ and $\mathrm{Quot}_{\mathbb{P}^2}$.

We now introduce the notation and minor results from [EGL] which we will need. Let $\Sigma_n \subset S^{[n]} \times S$ be the universal family parametrized by $S^{[n]}$. Let I_n be its ideal sheaf and $\mathcal{O}_n$ be its structure sheaf on $S^{[n]} \times S$. For $I \subset \{1, 2, 3\}$ let π_I denote the projection of $S^{[n]} \times S^{[m]} \times S$ onto the factors indexed by I. Furthermore, define $\mathcal{L} := \mathcal{O}_{I_n}(1)$.

The inductive step in [EGL] makes use of the incidence variety $S^{[n,n+1]}$. Let $\psi : S^{[n,n+1]} \to S^{[n+1]}$ and $\phi : S^{[n,n+1]} \to S^{[n]}$ be the obvious projection maps. Given any map f, let f_S denote the maps $f \times \mathrm{id}_S$. Then there are short exact sequences

$$0 \to \psi_S^* I_{n+1} \to \phi_S^* I_n \to j_*\mathcal{L} \to 0$$
$$0 \to j_*\mathcal{L} \to \psi_S^* \mathcal{O}_{n+1} \to \phi_S^* \mathcal{O}_n \to 0$$

Finally, given a morphism $f : X \to Y$ of smooth projective varieties, the K-theoretic pushforward is $f_! : K(X) \to K(Y)$, mapping $F \mapsto \sum_i (-1)^i R^i f_*(F)$. Let $f^!$ denote the K-theoretic pullback.

We can now state our key lemma.

Lemma 1. *The following relation holds in $K(S^{[n,n+1]} \times S^{[m]})$:*

$$\begin{aligned}
&\psi^!((\pi_{12})_!\mathcal{E}xt^\bullet((\pi_{13})^! I_{n+1}, (\pi_{23})^!\mathcal{O}_m)) \\
&\quad = -\psi^! H^\bullet(K_S) \otimes (\mathcal{O}_{S^{[n]}\times S^{[m]}}) + \psi^!(\pi_1)^!(K_S^{[n]}) \\
&\qquad - \phi^!(\pi_{12})_!((\pi_{13})^! I_n^\vee \cdot (\pi_{23})^! I_m) + (\pi_{13})^!\mathcal{L}^\vee \cdot \rho^! \omega_S^\vee \cdot (\pi_{23})^! I_m
\end{aligned}$$

Proof. The main object of study is now $S^{[n,n+1]} \times S^{[m]}$ instead of $S^{[n,n+1]}$. Rather than throw out the notation of [EGL] and introduce new clutter, we abuse notation slightly and still write ψ, and ϕ when we really mean $\psi \times \mathrm{id}_{S^{[m]}}$ and $\phi \times \mathrm{id}_{S^{[m]}}$. Furthermore, define $\rho : S^{[n,n+1]} \times S^{[m]} \to S$ as the map which remembers the special point in an element of the incidence variety.

In $K(S^{[n]} \times S^{[m]})$ we know

$$\begin{aligned}
&(\pi_{12})_!\mathcal{E}xt^\bullet((\pi_{13})^! I_n, (\pi_{23})^!\mathcal{O}_m) \\
&\quad = (\pi_{12})_!\mathcal{E}xt^\bullet((\pi_{13})^! I_n, \mathcal{O}) - (\pi_{12})_!\mathcal{E}xt^\bullet((\pi_{13})^! I_n, (\pi_{23})^! I_m) \\
&\quad = -(\pi_{12})_!\mathcal{E}xt^\bullet(\mathcal{O}, (\pi_{13})^! I_n \otimes (\pi_3)^! K_S) - (\pi_{12})_!((\pi_{13})^! I_n^\vee \cdot (\pi_{23})^! I_m) \\
&\quad = -(\pi_{12})_!\mathcal{E}xt^\bullet(\mathcal{O}, (\pi_3)^! K_S) + (\pi_{12})_!\mathcal{E}xt^\bullet(\mathcal{O}, (\pi_{13})^!\mathcal{O}_n \otimes (\pi_3)^! K_S) \\
&\qquad - (\pi_{12})_!((\pi_{13})^! I_n^\vee \cdot (\pi_{23})^! I_m) \\
&\quad = -H^\bullet(K_S) \otimes (\mathcal{O}_{S^{[n]}\times S^{[m]}}) + (\pi_1)^!(K_S^{[n]}) - (\pi_{12})_!((\pi_{13})^! I_n^\vee \cdot (\pi_{23})^! I_m). \qquad (7.1)
\end{aligned}$$

Note that $H^\bullet(K_S)$ depends only on S, and $K_S^{[n]}$ was already accounted for in [EGL]. Thus, it suffices to focus our attention on the remaining piece

$$T_{n,m} := (\pi_{12})_!((\pi_{13})^! I_n^\vee \cdot (\pi_{23})^! I_m). \qquad (7.2)$$

Next, we observe that in $K(S^{[n,n+1]} \times S^{[m]})$ we have

$$\psi^! T_{n+1,m} = \phi^! T_{n,m} - (\pi_{13})^! \mathcal{L}^\vee \cdot \rho^! \omega_S^\vee \cdot (\pi_{23})^! I_m. \tag{7.3}$$

The proof is functionally identical to the proof of Proposition 2.3 from [EGL]. In an effort to minimize confusion due to excess notation, we omit a detailed explanation. Then substituting (7.2) and (7.3) into (7.1) finishes the proof. □

We now introduce our slight generalization of Theorem 3.1 from [EGL]. In order to accomodate the extra $S^{[m]}$ term, we redefine pr_I to be the projection from $S^{[n+1]} \times S^{[m]} \times S^k$ onto the factors indexed by I, where $I \subset \{0, 1, \ldots, k+1\}$.

Theorem 9. *Let F be any locally free sheaf on S. Let f be a polynomial in the Chern classes of the following sheaves on $S^{[n+1]} \times S^{[m]} \times S^k$:*

$$pr_0^* T_{n+1},\ pr_{0i}^* I_{n+1},\ pr_{ij}^* \mathcal{O}_\triangle,\ pr_i^* T_S,\ pr_0^* F^{[n+1]},\ pr_{01}^* T_{n+1,m} \quad for\ 2 \le i, j \le k+1.$$

Then there is a polynomial $\tilde{f}$, depending only f, in the Chern classes of the analogously defined sheaves on $S^{[n]} \times S^{[m]} \times S^{k+1}$ such that

$$\int_{S^{[n+1]} \times S^{[m]} \times S^k} f = \int_{S^{[n]} \times S^{[m]} \times S^{k+1}} \tilde{f}.$$

Proof. With the introduction of the $S^{[m]}$ term, the only new statement here is the inclusion of Chern classes depending on $pr_{01}^* T_{n+1,m}$. However, the proof is completely analogous to that of Theorem 3.1 in [EGL]: one passes through the incidence variety $S^{[n,n+1]}$ by means of the recursive relationship proved in Lemma 1. Of course, the recursion in the lemma also introduces other terms depending on $pr_i^* T_S$ and $pr_0^* F^{[n+1]}$, but these terms were dealt with in [EGL]. □

It is worth taking a moment to consider the implications of this theorem. The intersection products computed in Chapters 5 and 6 were only for the projective surfaces $\mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{P}^2$. We made heavy use of the natural torus actions on these two surfaces. However, using the torus action on Quot_S from $\mathbb{C}^N \otimes \mathcal{O}$ only, localization says that the intersection product over Quot_S is equal to another product over sums of Hilbert schemes of points on S.

Let's now make this explicit for the intersection products over $Quot_S(\mathbb{G}(1,3))_{4,0}$ and $Quot_S(\mathbb{G}(2,8))_{9,0}$.

Theorem 10. *Let S be any surface for which $Quot_S(\mathbb{G}(1,3))_{4,0}$ admits a virtual fundamental class. Then*

$$\int_{[Quot_S(\mathbb{G}(1,3))_{4,0}]^{vir}} 1 = 39(c_1(K_S))^2 - 27c_2(K_S).$$

Under similar hypotheses

$$\int_{[Quot_S(\mathbb{G}(2,8))_{9,0}]^{vir}} 1 = -462600(c_1(K_S))^2 + 494160c_2(K_S).$$

Proof. We know that the class $[S] \in \Omega$ is a linear combination of $[\mathbb{P}^1 \times \mathbb{P}^1]$ and $[\mathbb{P}^2]$. It follows from Theorem 9 that

$$\int_{[Quot_S(\mathbb{G}(1,3))_{4,0}]^{vir}} 1 = a \int_{[Quot_{\mathbb{P}^1\times\mathbb{P}^1}(\mathbb{G}(1,3))_{4,0}]^{vir}} 1 + b \int_{[Quot_{\mathbb{P}^2}(\mathbb{G}(1,3))_{4,0}]^{vir}} 1$$

for some constants a and b. Equivalently, there exist some constants d and e such that

$$\int_{[Quot_S(\mathbb{G}(1,3))_{4,0}]^{vir}} 1 = d \cdot (c_1(K_S))^2 + e \cdot c_2(K_S). \tag{7.4}$$

We can solve for d and e by using Theorems 5 and 6 to obtain the system of equations

$$204 = 8d + 4e$$
$$270 = 9d + 3e$$

from which $d = 39$ and $e = -27$. The rank 2 example follows similarly using the results of Theorems 7 and 8. □

Theorem 10 demonstrates the power of working with cobordism classes. Even without a torus action on S, the computability of an intersection product over both $\mathrm{Quot}_{\mathbb{P}^1\times\mathbb{P}^1}$ and $\mathrm{Quot}_{\mathbb{P}^2}$ is enough to provide explicit results as long as $[\mathrm{Quot}_S]^{vir}$ exists.

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