Lawrence Berkeley National Laboratory

BARRELET ZEROS and ELASTIC pi+p PARTICLE WAVES

Author
Chew, D.M.

Publication Date
1976-06-01

Invited talk to the Topical Conference on Baryon Resonances, Oxford, England, July 5 - 9, 1976; and Contribution to the 17th International Conference on High Energy Physics, Tbilisi, USSR, July 15 - 21, 1976

LBL-4851*
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D. M. Chew and M. Urban

June 1976

Prepared for the U. S. Energy Research and Development Administration under Contract W-7405-ENG-48

*This paper has also been mistakenly referenced as LBL-4856.

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Ussr, (july 1976).

LBL-4856

BARRELET ZEROS AND ELASTIC π^+p PARTIAL WAVES*

D. M. Chew†, and M. Urban
Lawrence Berkeley Laboratory
Berkeley, California 94720

Talk presented by D. M. Chew

Abstract

A procedure is proposed for constructing low-order partial-wave amplitudes from a knowledge of Barrelet zeros near the physical region. The method is applied to the zeros already obtained for elastic π^+p scattering data between 1.2 and 2.2 GeV cm. energies. The partial waves emerge with errors that are straight-forwardly related to the accuracy of the data and satisfy unitarity without any constraint being imposed. There are significant differences from the partial waves obtained by other methods; this can be partially explained by the fact that no previous partial-wave analysis has been able to solve the discrete ambiguity. The cost of the analysis is much less.

*This report was done with support from the United States Energy Research and Development Administration.

†On leave from the University of Paris VI, Paris France.

1. Introduction

It has been shown by Barrelet that zeros near the physical region of $0 + \frac{1}{2} \to 0 + \frac{1}{2}$ amplitudes can be systematically determined from experimental measurements of differential cross section and polarization[1]. The Barrelet proposal has been successfully implemented[2] for $\pi^+p \to \pi^+p$ elastic scattering between 1.1 and 2.2 GeV center of mass energy where resolution of the discrete ambiguity (related to that of Minami) was achieved by the requirements that zero trajectories be analytic functions of energy and that causality demands negative imaginary parts for pole positions in the complex energy plane. Since pole properties (spin, parity and position in the complex energy plane) turn out to be by-products of Barrelet-zero analysis[1,2], it is natural to ask what further information about the amplitude is implied by a knowledge of nearby zeros. We have attempted to construct partial waves of order less than or equal to the order corresponding to the number of nearby zeros. We assume that all ambiguity about the location of such zeros has been removed, as in π^+p scattering[2], and illustrate our method with the latter special case.

I. Principle of the method

As in Refs. (1) and (2), we use the variable $w=e^{i\theta}$ and discuss an amplitude $F(w)$ whose modulus squared in the physical region is $\Sigma(w)$-- a quantity equal to $\frac{d\sigma}{d\Omega}(1 + P)$ for $0 < \theta < \pi$ and equal to $\frac{d\sigma}{d\Omega}(1 - P)$ for $\pi < \theta < 2\pi$. If the nearby zeros of $F(w)$ have been determined to be at the positions w_i (i.e., the discrete ambiguity has been solved in a close study of the data[2]), what can we say about the amplitude $F(w)$?

For a total of N nearby zeros, we propose the following approximate formula for the amplitude:

$$F(w) \simeq \sqrt{\left(\frac{d\sigma}{d\Omega}\right)_{\theta=0}} \; \frac{e^{i\phi}}{w^{E(N/2)}} \prod_{i=1}^{N} \left(\frac{w-w_i}{1-w_i}\right) \qquad (1)$$

where $E(x)$ means the integer part of x.*. General principles allow the

*In fact, as observed in Ref. 2, the zeros always seem to enter by pairs into the domain of convergence of the polynomial representation.

the phase ϕ to be an analytic function of w, (with the usual possible singularities on the real axis), corresponding to a continuous ambiguity; We have investigated the linear form: $\phi = \phi_0 [1 + A(\cos\theta - 1) + B\sin\theta]$, where A and B are coefficients which have been allowed to vary in order to study the uncertainties due to the continous ambiguity, (i.e. the "mathematical errors"[1]). In the $\Delta(1232)$ mass region, we have observed that only for A and B < 1%, was unitarity satisfied within the statistical errors (see Figs 1 and 2 and Paragraph III below). We assume that such a small possible value for the parameters A and B of $(\Delta\phi/\phi_0)$ would be difficult to explain from physical models and so we set ϕ equal to a real constant-- which is seen to be the forward direction (w=1) phase (obtained from dispersion relations[3]) in view of the way in which the other factors of formula (1) have been arranged. In Ref. (1), Barrelet discusses a formula of the type of (1) where the power $\frac{N}{2}$ is replaced by an undetermined integer n. We tentatively arrive at the choice $n = \frac{N}{2}$ by invoking the principal of "lack of sufficient reason." That is positive and negative powers of w deserve a priori equal attention. Our choice is supported by Table I which shows what happens in the 1900 resonance region (where N=6) when $\frac{N}{2}$ is replaced by an integer other than 3: the resonant wave is not longer F37, but some other wave with the same naturality; the F37 wave might even completely disappear (for n = 4) and some of the waves will strongly (by more than 50%) violate unitarity.

The partial-wave representation of the amplitude through the orthogonal pseudopolynomials $R_{J\varepsilon}(w)$,

$$F(w) = \sum_{\substack{\varepsilon \pm 1 \\ J=\frac{1}{2}}}^{\infty} T_{J\varepsilon} \cdot R_{J\varepsilon}(w) \qquad (3)$$

has the inverse:

$$T_{J\varepsilon} = \frac{\lambda^{-1}}{2(2J+1)} \int_{\gamma} F(w)\, \overline{R_{J\varepsilon}}(w)\, |dz| \qquad (4)$$

where the integration is over the unit circle γ in the w plane. Formula (1) may be substituted into formula (4), so that a determination of N zeros translates into an approximate determination of those partial-wave amplitudes $T_{J\varepsilon}$ for which $J \leq (N+1)/2$.

A striking advantage of our approach is that statistical errors of

the data may be straightforwardly converted into errors of partial-wave amplitudes. This can be done step by step by standard linear methods[4], first obtaining the errors on the moments of the experimentally - measured distributions, thence the errors of the zero locations and finally the errors of the partial-wave amplitudes. Since several stages are involved, and one might worry that neglected higher order correlations somehow accumulate, we performed the following simulations at two different energies (chosen to be in the $\Delta(1232)$ and $\Delta(1900)$ mass regions). The experimental distribution, with the stated errors, was converted into a large number (50 to 100) of different distributions by replacing each data point of $\frac{d\sigma}{d\Omega}$ and P with randomly selected points within the error interval (weighted by a gaussian distribution whose standard deviation is given by the error bar). From each such distribution, moments, zeros and finally partial-wave amplitudes were computed. At each stage a "cloud" of (50 to 100) points emerged, the dimensions of the cloud indicating the uncertainty. Fig.1a shows the clouds of zeros at $\sim$ 1900 MeV in the w plane, compared to the error ellipses calculated by standard (linear) formulas[4], Fig.1b makes a similar comparison for the partial wave amplitudes in the $\Delta(1232)$ and $\Delta(1900)$ mass region. One observes that except for the wave F37 where the calculated errors seem too small*, about 70% of the individual points (considered to represent the density of probability of the real errors) lie inside the rectangle of the error bars, verifying that the approximate formulas are adequate (in some particular cases the fraction is smaller than 70%-see footnote below.) Fig 3 shows the zero trajectories with errors in the (Ret, $\sqrt{s}$) plane. The poorest quality data is seen to lie in the 1450-1700 mass region.

II. The Experimental Data

By a procedure described in Ref. (2) and summarized in Table II,[2] we selected 75 different energies at which differential cross-section and polarization measurements were available.[5] At each energy, the nearby zeros were located by Barrelet's method of moment analysis of

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*This result can be related to the fact that as pointed out in Ref. (4), the standard linear formulas break down when some zeros are very close to the physical region. Such is the case in the $\Delta(1900)$ mass region.

the experimental distributions, the discrete ambiguity being resolved by the requirement of causality in conjunction with smooth behavior in energy. Column 4 of Table III[2] indicates the critical points at which zero trajectories cross the unit circle in the w complex plane, column 5 showing at each energy which zeros are inside and which are outside the unit circle. Also shown in Table II[2] is a measure of goodness of fit of the experimental distribution by the finite-order polynomial that corresponds to the N nearby zeros determined from the Barrelet method (column 5). The fit is satisfactory except at a few isolated energies, where we presume that undiagnosed systematic errors are present in the data.

As explained in Ref. (1,2) the Barrelet method establishes the number N of determinable (nearby) zeros by finding the number of moments that are non-zero within the errors. Polynomials of order higher than N do not significantly improve the representation of the data. The accuracy of the data turns out to be such that N=2 near the Δ(1232), and N=6 near the Δ(1900). The N zeros at each energy determine the partial-wave amplitudes $T_{J\varepsilon}$ through Formulae (4) and (1).

III. Results

Figure 4 shows the determined partial waves for the energy interval 1.1 GeV $< \sqrt{s} <$ 2.1 GeV, first in an Argand plot, and then through projections Im $T_{J\varepsilon}$ and Re $T_{J\varepsilon}$ as a function of $\sqrt{s}$ with the errors displayed.[6] Fig. 5 is similar but with the elimination of those individual energies where the error on either Im $T_{J\varepsilon}$ or Re $T_{J\varepsilon}$ turned out larger than 0.05. Before discussing the individual partial waves we draw attention to certain general features.

The unitarity constraint is satisfied within errors even though no requirement thereof has been imposed. In the elastic region near the Δ(1232) the three determined waves, S_{31}, P_{31} and P_{33} all lie on the Argand circle. (Figures 1,2,4 and 5c). All waves at all energies are either on or inside the circle.

In certain energy regions the partial-wave errors are so large as to make impossible the accurate deduction of resonance positions and widths. The only accurately-determined resonances are the Δ(1232) and Δ(1900). Furthermore, except for the Δ(1232), we do not find simple Breit-Wigner forms.

Let us now compare, case by case, each of our partial waves with the results of previous analyses as given in the Particle Data Group (PDG) tables.[7] No fit has so far been performed so thatour results, though qualitative, are still model independant.

S31: We find a resonance between 1400 and 1700 MeV, an interval expanded in Fig.6 , although accurate data are so sparse in this region that the resonance parameters are poorly determined. Presumably this resonance is the Δ(1650) which has achieved four-star status in the PDG tables[7]. Although our results give a hint of further structure between 1800 and 1900 MeV, the motion here in the Argand diagram is clockwise, so we cannot be dealing with a simple isolated resonance.

P31: We have no clear evidence for a resonance here. The most promising candidate for a counter-clockwise loop in the Argand diagram occurs near 1650 MeV, where no previous analyses have found anything. If this is a resonance the width would be rather narrow. In the region near 1910, where the PDG tables list a three-star resonance[7], we find no structure, as the motion in the Argand plot is clockwise also.

P33: In addition to the Δ(1232), all of whose properties from our analysis are in agreement with other work, we find a structure near 1900 MeV that is compatible with counter-clockwise Argand motion. If this is indeed a resonance we find its parameters to be $E_R \simeq$ 1900 MeV, with a rather narrow width. Litchfield, updating the PDG tables [7] for the London Conference[8] lists a one-star Δ(1900); but the PDG tables[7] also list a one-star Δ(1690) for which we see no evidence, as the motion is clockwise around 1900 MeV.

D33: Here we find a resonance E_R = 1700 MeV, with a rather large width which may be identified with the three-star Δ(1670) in the PDG tables. In addition we find sharp and strong structure near 1900 MeV that cannot be interpreted as a single isolated resonance. Further work will be done in an attempt to clarify the meaning of this striking and puzzling structure.

D35: We find a structure near 1900 MeV that may be related to teh two-star Δ(1960) listed in the PDG tables. Our parameters are $E_R \sim$1900 MeV, the width being <100 MeV. Our results lead to no other significant resonance candidates.

F35: We find within errors a large counter clockwise loop between 1600 and 2000 MeV, the mass being $E_R \sim$1800 MeV; the width (< 200 MeV) could presumably be determined from a fit with a Breit-Wigner surimposed on a background varying linearly with energy.

F37: Here we find strong resonance behavior between 1700 and 2000 MeV, with a max at E_R ~1900MeV, unmistakeably corresponding to the well-known (four-star) Δ(1950). The unsymmetrical shape of our resonance does not correspond to simple Breit-Wigner, however, a point that we are investigating further. A possible source of the anomalous shape of our F37 (1900) resonance is a systematic inaccuracy of polarization measurements near the forward direction in the resonance region. The zero corresponding to an observed minimum in P near cosθ=.8, turns out not to be located in the expected region, whereas all five other zeros have the anticipated locations[1,2]. We are led to suspect that the forward-direction polarization measurement has a systematic error larger than indicated by the quoted errors. Table IV summarizes these results.

The differences between the partial wave amplitudes obtained in this paper and those determined by conventional partial wave analysis (CPWA) may be attributed to differences in the zero trajectories. In Ref. (9), for example, the zeros corresponding to the 1974 Saclay partial-wave[10] analysis were computed. Although the total number of nearby zeros turned out to be the same as found in the direct Barrelet zero analysis of Ref. (2), which constitutes the basis for the present paper, there are significant differences between the two sets of trajectories. In particular the critical points -- where trajectories cross the physical region -- are different. The Saclay CPWA zeros do not consistently obey the smoothness and causality requirements that were involved in Ref. (2); the different set of critical points corresponds to a different resolution of the discrete ambiguity. This type of mistake could explain why ,by lack of resolution,no narrow signal has been detected so far by any CPWA. Table V[9] compares the location (either inside or outside the unit circle) of the zeros from the 1974 Saclay CPWA with the zeros determined directly from the data. A separate column identifies those trajectories where there is disagreement with respect to the discrete ambiguity. Except between 1615MeV and 1720MeV there is always disagreement for at least one trajectory. We are presently engaged in a similar comparison of all previous π^+p partial-wave analyses.

A further aspect of the 1974 Saclay PWA is displayed in Fig. 7 ,

where we have plotted as a function of P_{lab} the radius of the z-plane ellipse passing through each of the zeros. We also show the radius corresponding to the ρ-pole, which presumably determines the effective domain of convergence of a polynomial representation. At most energies (between 1.1 and 2.0 GeV) there occur, well inside the domain of convergence, either two or six zeros that can be put into correspondence with the zeros determined directly from the data, (above P_{lab} = 1.9 GeV/c, we can observe four more stable zeros inside the domain of convergence). The remaining zeros are unstable from one energy to the next and are located either near the boundary of, or outside the domain of convergence. In other words, PWA finds the correct number of nearby zeros(and should be able to get a good fit to the data with it); its major problems is the discrete ambiguity.

To summarize, we have found a procedure for determing partial-wave amplitudes via Barrelet zeros with the following advantages:

(1) With data of sufficient accuracy the discrete ambiguity may be resolved.

(2) The procedure is model independent.

(3) The errors in the data translate unambiguously, via errors in the zeros, into errors of the partial-wave amplitudes.

(4) Since the analysis is carried out at each energy independently, a small number of bad measurements does not distort the entire analysis.

(5) The procedure is inexpensive.

Although unitarity is not imposed as an a priori requirement, the partial waves we have found for the π^+p elastic amplitude satisfy this constraint. The quality of our fit to the data is excellent, the ratio of χ^2 to the number of data points being less than 1 in 2/3 of the cases. Furthermore,if the discrete ambiguity is correctly resolved from the analysis of the data[2],we fail to find evidence for the Δ(1900)S31,the Δ(1910)P31 and the Δ(1690)P33.

Acknowledgements

We are deeply grateful to E Barrelet who, after having introduced us to the secrets of the zeros, has been very helpful with criticism of the ZPWA. We sincerely thank Prof R Ely for his kind encouragement and suggestions. Together with G Gidal and W Michael, his comments and questions in open-minded discussions have provided the necessary stimulation to carry this work through.

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Table Caption

I. Justification of the factor n=N/2 for the amplitude in formula (1).

II. Experimental data used for this analysis (Ref.2).

III. Resolution of the discrete ambiguity in π^+p elastic data (Ref.2).

IV. Summary of the results found in this Zero Partial-wave Analysis(ZPWA) compared to the Particle Data Group Table of the status of the Baryon (Ed.1974 [7] identical to the 1976 Edition).

V. Summary of the conclusions of the study of a Conventional Partial-wave Analysis[10],and comparison with the results of Ref.2 and this analysis.

Figure Captions

1. Comparison of the errors as calculated by standard linear methods (ellipse) with the clouds of points as given by the simulation method,
   (a) for the zeros in the w-plane, in the $\Delta(1900)$ mass region.*
   (b) for the $T_{J\varepsilon}$ in the $\Delta(1232)$ and the $\Delta(1900)$ mass regions.

2. Study of the continuum ambiguity in the $\Delta(1232)$ mass region showing the variation of the coefficients A and B of the parametrization of the phase ϕ.

3. Elastic $\pi^+ p$ zero trajectories in the (Re t, $\sqrt{s}$) plane (Ref 2).

4. Partial -wave amplitudes without any cut:
   a)S31, b)P31, c)P33, d)D33, e)D35, f)F35, g)F37.

5. Partial-wave amplitudes with $\delta Re(T_{J\varepsilon})$ and $\delta Im(T_{J\varepsilon})$ smaller than 0.05:
   a)S31, b)P31, c)P33, d)D33, e)D35, f)F35, g)F37.

6. Partial-wave S31 without cut on the errors, between 1500 and 1700 MeV.

7. Comparison of the location of the zeros of a CPWA[10], with the domain of convergence of the series which approximates $\Sigma(w)$: radius R_z of the ellipse (in the z-plane) going through each zero of a CPWA[10], vs Plab (Ref 9).

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* arbitrarily positioned outside the unit circle of the physical region in the w-plane (Table III of reference (2) indicates that in fact the 6 zeros of the $\Delta(1900)$ mass region are all inside this circle, in particular for the ZPWA).

Table I. Justification of formula (1) for the amplitude: influence of the factor n.

n	Waves with $J_{MAX} = \frac{N+1}{2} = \frac{7}{2}$, $\varepsilon = +1$	OUT	IN (to be added to the previous waves)
0	S31, Pw1, P33, D33, D35, F35, F37	-	G37, G39, H39, H311, I3 11, I3 13
1	S31, P31, P33, D33, D35, F35, F37	-	G37, G39, H39, H3 11
2	S31, P31, P33, D33, D35, F35, F37		G37, [G39]
3	S31, P31, P33, D33, D35, F35, [F37]	-	-
4	~~S31~~, P31, P33, D33, [D35], F35	F37	G37
5	~~S31~~, ~~P31~~, [P33], D33 D35, F35, F37	-	G37, H39
6	[~~S31~~], ~~P31~~, ~~P33~~, D33, D35, F35, F37	-	G37, G39, H39, I3 11

□ = waves presenting a resonance behavior at M=1900 MeV.

/ = waves violating unitanity by more than 50%

Table II(a)

No. of set	PLAB (GeV/c)	√s (MeV)	No. of zeros	χ^2/NPts	Differential PLAB	Differential Ref.	Polarization PLAB	Polarization Ref.	ΔPLAB	Results: Critical Points	Results: Comments on data
1	0.174	1148	2	0.3	0.174	A57	0.188	A76	14		
2	0.195	1162	2	0.25	0.195	G57	0.188	A76	7		
3	0.219	1178	2	0.28	0.219	L61	0.188	A76	31		
4	0.246	1196	2	0.7	0.246	B73	0.242	A76	26		
5	0.272	1214	2	0.6	0.272	B73	0.272	A76	0		
6	0.303	1235	2	0.46	0.303	B73	0.303	A76	0		
7	0.326	1251	2	0.36	0.326	B73	0.303	A76	23		
8	0.422	1317	2	1.9	0.427	O65	0.417	G73	10		
9	0.505	1372	4	0.74	0.490	O65	0.519	G73	29		
10	0.532			1.5	0.532	B67	0.532	G67	0		
11	0.597	1432	6	0.3	0.590	B72	0.603	M75	13	B	
12	0.616	1444	6	1.35	0.614	B67	0.617	M75	3		
13	0.647	1463	6	0.74	0.645	B72	0.660	M75	15		
14	0.687	1488	6	0.75	0.700	B72	0.674	M75	26		
15	0.704	1500	6	0.89	0.700	B72	0.708	M75	8	C,E	Precise measurements needed in the very forward and backward directions for definitive conclusions regarding : -(1) the trajectories (A),(D),(I),(F) -(2) the critical points of (E) and (F). Also complete measurements needed at the values of s where the 2nd critical point of (C) is about to occur.
16	0.756	1530	6	0.75	0.759	M74	0.759	M75	7	C	
17	0.764		6	0.93	0.752	B72	0.776	M75	24		
18	0.802	1558	6	0.32	0.800	H72	0.803	M75	3		
19	0.805	1560	6	0.43	0.807	B72	0.803	M75	4	F	
20	0.810	1563	6	1.36	0.807	B72	0.820	M75	10		
	0.814	1565	6	1.57	0.807	B72	0.820	M75	13		
21	0.852	1587	6	0.38	0.850	D68	0.853	M75	3		
22	0.882	1605	6	1.01	0.892	H72	0.072	M75	20		
23	0.900	1615	6	0.63	0.892	H72	0.907	M75	15		
24	0.915	1624	6	0.66	0.903	H72	0.907	M75	16		
25	0.925	1630	6	1.15	0.923	H72	0.927	M75	4		
26	0.954	1646	6	0.8	0.954	H72	0.973	M75	1		
27	0.964	1652	6	1.01	0.954	H72	0.974	M75	20		
28	1.027		6		1.033	H72	1.020	M75	13		
29	1.043	1695	6	0.79	1.033	H72	1.054	M75	21	E	
30	1.063		6		1.072	H72	1.059	M75	18		
31	1.095		6		1.072	H72	1.078	M75	6		
32	1.105	1729	6	0.59	1.113	H72	1.097	M75	16		
33	1.118	1736	6	0.66	1.113	H72	1.122	M75	9		
34	1.150	1753	6	0.59	1.154	H72	1.146	M75	8		
35	1.182	1770	6	0.82	1.193	H72	1.171	M75	22		
36	1.200	1780	6	0.59	1.193	H72	1.207	M75	14		
37	1.207	1783	6	0.42	1.204	A74	1.207	M75	0		
38	1.235	1798	6	1.43	1.235	H72	1.234	M75	1		
39	1.241	1800	6	0.8	1.235	H72	1.246	M75	11		
40	1.274	1818	6	0.93	1.273	H72	1.294	M75	1		
41	1.277	1820	6	0.75	1.280	K71(1)	1.274	M75	6		
42	1.289	1826	6	0.75	1.280	K71(2)	1.298	M75	18		
43	1.325	1844	6	1.18	1.324	H72	1.326	M75	2		
44	1.333	1848	6	2.47	1.340	K71	1.326	M75	14		New measurements of the polarisation needed in the forward direction, in very narrow bins of cosθ as the present polarisation around P=-1 is very badly determined . Until then, the (B) trajectory and critical point have dubious values in θ.
45	1.349	1856	6	2.65	1.340	K71(5)	1.357	M75	17	B,A	
46	1.365	1864	6	1.34	1.374	A72	1.357	M75	17		
	1.366		6		1.345	A74	1.357	M75	18		
47	1.404	1884	6	1.14	1.400	K71(6)	1.407	M75	7		
48	1.432	1897	6	0.89	1.425	H72	1.439	M75	14		
	1.435	1900	6	2.33	1.430	K71(7)	1.439	M75	9		
49	1.445	1904	6	1.36	1.451	H72	1.439	M75	12		
50	1.468	1915	6	0.76	1.460	A74	1.476	M75	16		
	1.475	1918	6	0.49	1.473	H72	1.476	M75	3		
51	1.485	1923	6	0.83	1.495	A74	1.476	M75	19		
52	1.501		6		1.495	A74	1.507	M75	12		
53	1.533	1946	6	1.26	1.530	A74	1.535	M75	5		
	1.534	1947	6	1.03	1.533	H72	1.535	M75	2		
54	1.543	1951	6	1.06	1.550	K71(8)	1.535	M75	15	A	
55	1.560	1959	6	0.92	1.550	K71(9)	1.569	M75	19		
56	1.599	1978	6	0.53	1.594	H72	1.604	M75	10		
57	1.602	1979	6	0.64	1.600	A74	1.604	M75	4		More data needed in the very forward and backward directions, for the polarisation in particular, in order to determine with some accuracy the next 4 trajectories (G),(H),(I),(J), and the critical point (E).
58	1.680	2016	6	0.62	1.680	K71(10)	1.680	M75	0		
59	1.685	2018	6	0.53 0.71	1.690	K71(11) A74	1.680	M75	10		
60	1.690		6		1.690	A74	1.690	M75	0		
61	1.769	2057	6	0.38	1.770	K71(12)	1.768	M75	2		
					1.770	A74	1.768	M75	2		
62	1.777	2060	6	0.56	1.770	K71(13)	1.783	M75	13		
63	1.792		6		1.800	A74	1.783	M75	17		
64	1.831	2085	6	1.27	1.840	K71(14)	1.822	M75	18	E	
65	1.837	2087	6	0.61	1.840	K71(15)	1.833	M75	7		
66	1.872				1.869	A74	1.874	M75	5		

Table III [2]

Resolution of the discrete ambiguity from elastic π^+p data

P lab (GeV/c)	data set	$\sqrt{s}$ (MeV)	Critical Points Trajectory name	Degree of confidence	zero trajectories[(+)] A B C D E F located outside (0) or inside (I).
					I I
≈ 0.600	11	1432	B	****	
					0 0 I I I 0
≈ 0.700	15	1500	C	***	
			E	**	
					0 0 0 I 0 0
≈ 0.800	19	1560	F	**	
					0 0 0 I 0 I
≈ 0.815	20	1565	C	*	
(or maybe a little lower)					0 0 I I 0 I
≈ 1.100	32	1730	E	***	
					0 0 I I I I
≈ 1.350	45	1865	B	**	
			A	****	
					I I I I I I
≈ 1.550	54	1950	A	****	
					0 I I I I I
≈ 1.835	64	2085	E	**	
					0 I I I 0 I

**** Data and polynomial approximation show that $|P| = 1$.

*** Data need to be renormalized (± 6%) to exhibit $|P| = 1$. simultaniously with polynomial approximation

** No data (in cosθ) exist where the polarization is found to be $\mp$ 1. by the polynomial approximation.

* Polarization is observed to go through an extremum (close to $\mp$ 1.) but no data exist at the energy where it is presumed to be exactly $\mp$ 1 for the considered trajectory

(+) The trajectories are named after their location in the w-plane in the Δ(1900) mass region; the zeros of $Z_-(0<\theta<\pi)$ are A(forward), C($\theta \sim 0$), and E(backward); the corresponding zeros of $Z_+(\pi<\theta<2\pi)$ being B(forward), D($\theta \sim 0$) and F(backward).

Table IV

WAVES	WHAT WE OBSERVE:	OUR CONLUSIONS	STATUS-PDG (1974) [7](+)
S31	-A counterclockwise loop between 1400 and 1700 MeV; no data with $\delta Re(T_{J\epsilon})$ and $\delta Im<.05$ -A clockwise loop ∿ 1900 MeV.	Δ(1650) exists, but M=?, Γ =? No Δ(1900) S31.	Δ(1650)**** Δ(1900)*
P31	-Clockwise loop . -Narrow signal, few data	No Δ(1910) P31 ∿ 1600 MeV.	Δ(1910)***
P33	-A counterclockwise loop on the Argand circle, very small errors -A clock-wise loop. -A narrow signal, few data.	**** No Δ(1690) P33. ∿ 1900 MeV.	Δ(1232)**** Δ(1690)* Δ(1900)* (Litchfield 1974)[8]
D33	-Large counter clockwise loop -Sharp drop at ∿ 1900 + Interf.(?)	∿ 1700 MeV. + ?	Δ(1670)***
D35	-Counter clockwise loop	M ∿ 1900 MeV. *** Γ < 100 MeV.	Δ(1960)**
F35	-From 1600 → 2000 MeV : within errors, counterclockwise loop	M ∿ 1800 MeV. *** Γ ≥ 200 MeV.	Δ(1890)***
F37	-From 1700 2000 MeV: Counterclockwise loop (Max. at ∿ 1900 MeV) -	M ∿ 1900 MeV. **** Γ = ? (No Breit-Wigner shape)	Δ(1950)****

(+) Identical to the (1976) edition.

#	PLAB (GeV/c)	√s (MeV)	Resonant waves	#Zeros CPWA	*	TRAJECTORIES** A B C D E F	CPWA: Candidate i.e. |P|≃|	CPWA: Effective Crossing	For Comparison: DATA(2) ABCDEF	For Comparison: DATA(2) Candid.& Crossing	For Comparison: ZPWA Resonant wave	Discrepancies: Zero Traject.	Discrepancies: Resonant Waves.
1	.097	1104		2	2	0 I		B	-II---				
2	.192	1160		6	2	I¯I							
3	.255	1202		6	2	I I							
4	.297	1231	- P33	6	2	I I					-P33(1232)		
5	.336	1258		6	2	I I							
6	.359	1274		6	2	I I							
7	.386	1292		6	2	I I							
8	.427	1320		6	2	I I							
9	.490	1362		6	2	I I							
10	.532	1390		6	2	I I							
11	.616	1444		6	6	0 I I I I 0	B	F	00III0	B		B	
12	.657	1470		8	6	0 I I I I I	A, D		00III0			B, F	
13	.675	1481		8	6	0 I I I I I		B, E	00III0	E		B, F	
14	.706	1500		8	6	0 0 I I 0 I	C		000I00	C		C, F	
15	.725	1512		8	6	0 0 I I 0 I			000I00			C, F	
16	.757	1531		8	6	0 0 0 I 0 I			000I00				
17	.777	1543		8	6	0 0 0 I 0 I	C		000I00	C,F			
18	.826	1572		8	6	0 0 0 I 0 I			00II0I			C,	
19	.874	1600		10	6	0 0 0 I 0 I	E	E	00II0I		-P31	C,	-P31
20	.900	1615	-S31	10	6	0 0 0 I I		C	00II0I		-S31	C, E	
21	.924	1629		10	6	0 0 I I I I		E	00II0I				
22	.975	1658		10	6	0 0 I¯I 0 I			00II0I				
23	1.001	1672		10	6	0 0 I I 0 I			00II0I				
24	1.029	1688		10	6	0 0 I I 0 I			00II0I		-D33		
25	1.081	1716	-D33	10	6	0 0 I I 0 I			00II0I				
26	1.122	1738		10	6	0 0 I I 0 I	E		00IIII	E		E	
27	1.178	1768	-P31	10	6	0 0 I I 0 I			00IIII		-F35	E	-P31
28	1.282	1822		10	6	0 0 I I 0 I	B		00IIII			E	
29	1.357	1860	-F35 -D35	10	6	0 0 I I 0 I			IIIIII	B, A		A, B, E	
30	1.437	1900	-P33 -F37	10	6	0 0 I I 0 I	B		IIIIII		-F37 -P33 -D35	A, B, E	
31	1.505	1933		12	6	0 0 I I 0 I			IIIIII			A, B, E	
32	1.578	1968		12	6	0 0 I I 0 I	A		0IIIII	A		B, E	
33	1.638	1996	-S31	12	10	0 0 I I 0 I			0IIIII			B, E	-S31
34	1.693	2022		12	10	0 0 I I 0 I			0IIIII			B. E	
35	1.737	2042		12	10	0 0 I I 0 I			0IIIII			B, E	
36	1.886	2109		12	10	0 0 I I 0 I	E		0IIIII			B, E	
37	1.979	2150	-~39	12	10	0 0 I I 0 I	A	A E F	0III0I	E		B, E	
38	2.083	2195		12	10	I 0 I I I 0			0III0I			A,B,E,F	
39	2.178	2235		12	10	I 0 I I I 0	C	C					
40	2.272	2274		12	10	I 0 0 I I 0		F					
41	2.385	2320		12	10	I 0 0 I I I		F					
42	2.548	2385	-H311	12	10	0 0 I I I 0							
43	2.773	2472		12	10	0 0 I I I 0							

* Inside ellipse of Lehman

**0(I) for Outside (Inside) the unit circle in the w-plane

Table V[9]

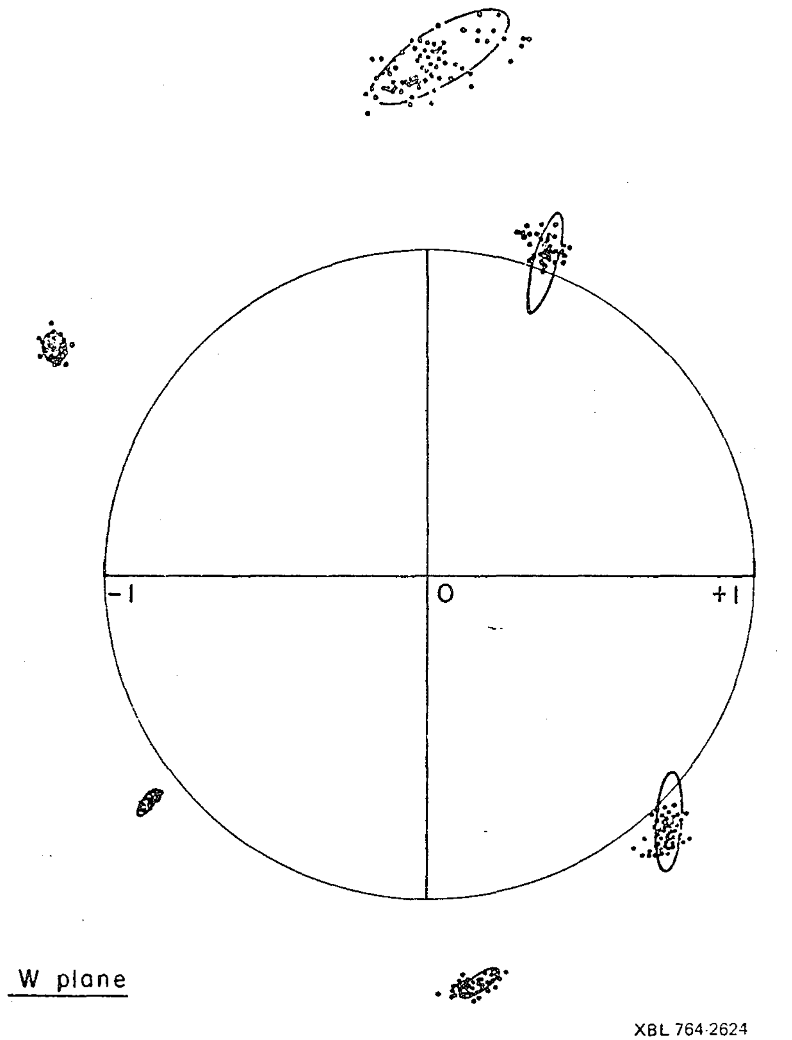


Figure 1(a).

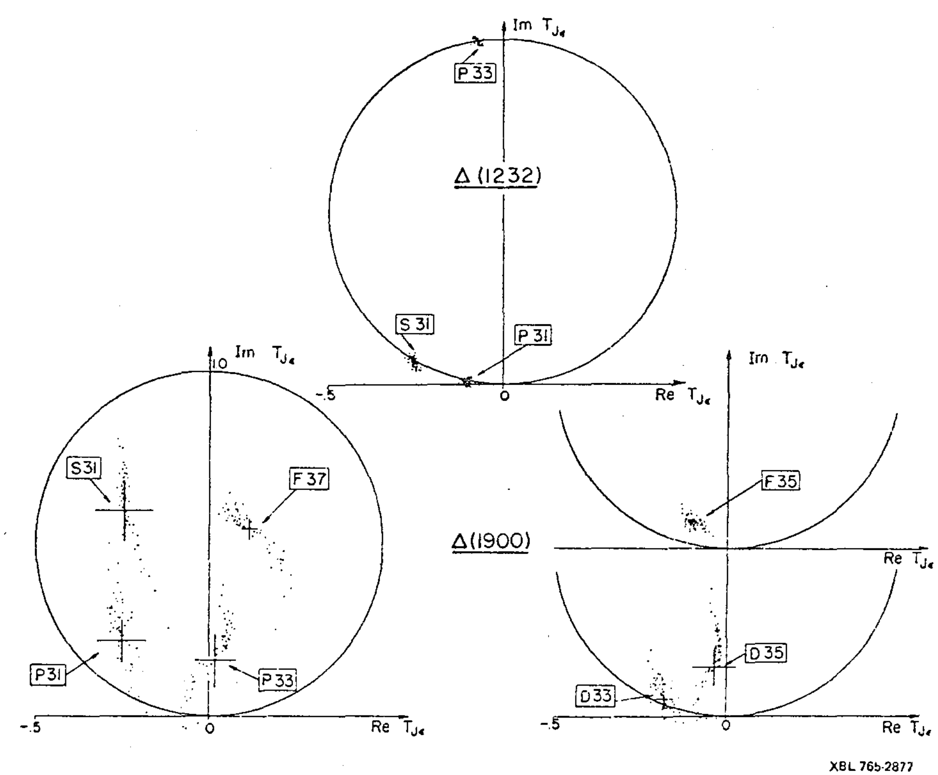


Figure 1(b)

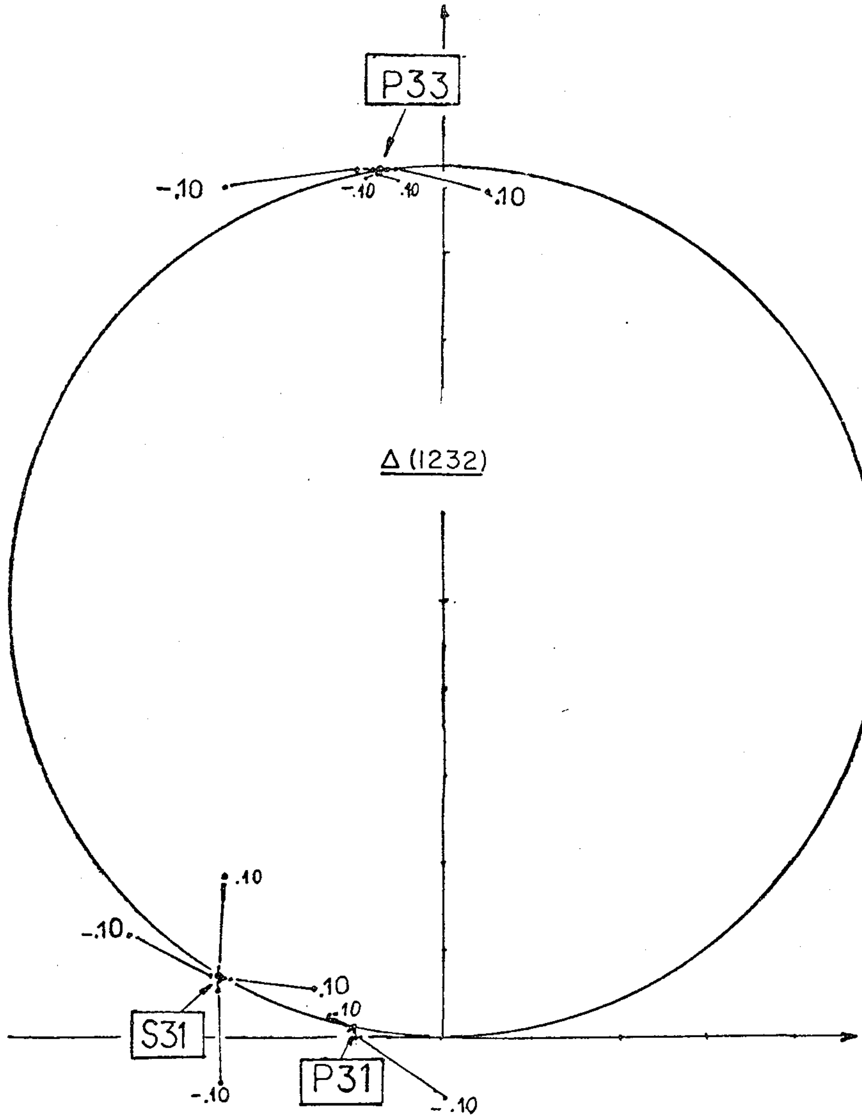


Figure 2 .

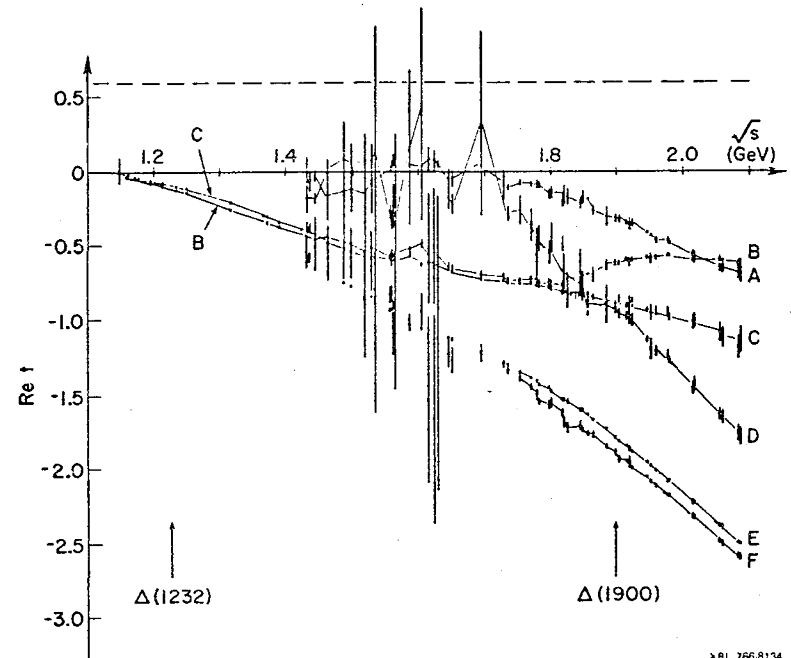


Figure 3 .

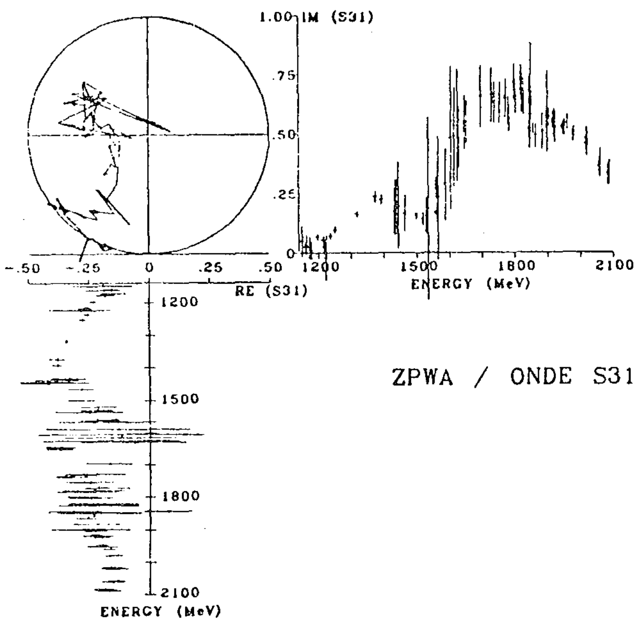


Figure 4(a) .

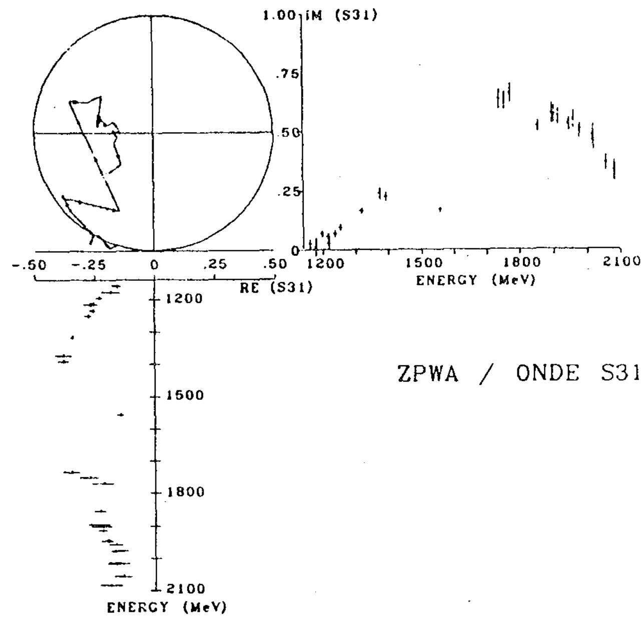


Figure 5(a).

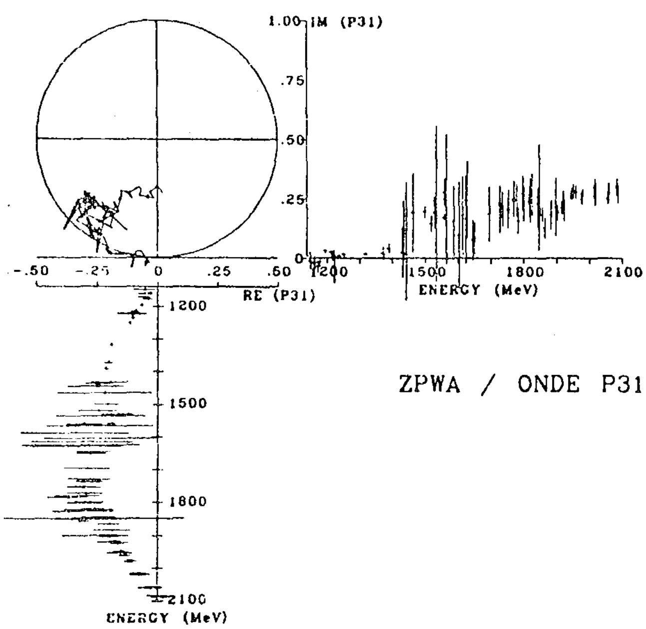


Figure 4(b)

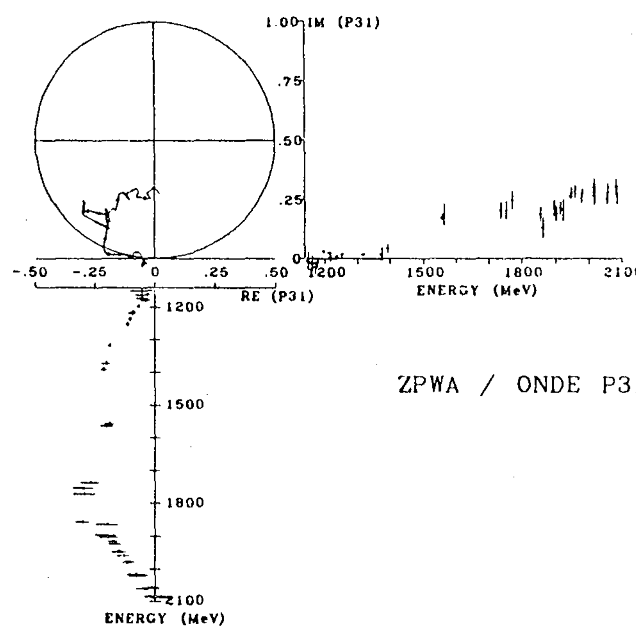


Figure 5(b)

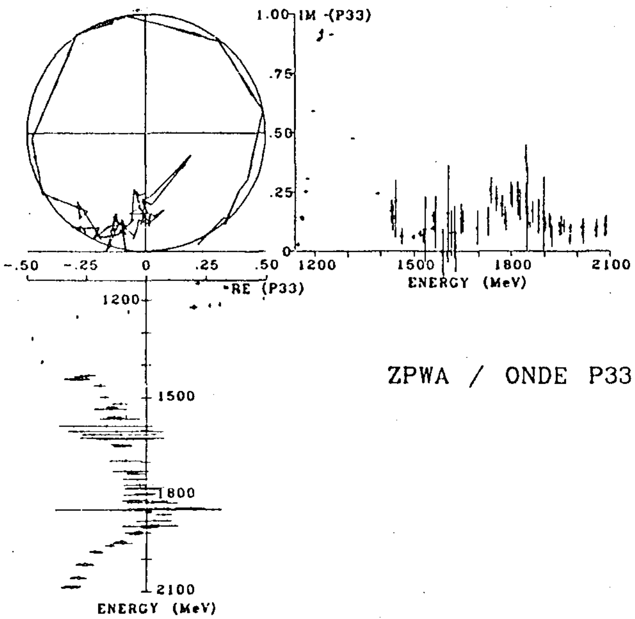


Figure 4(c).

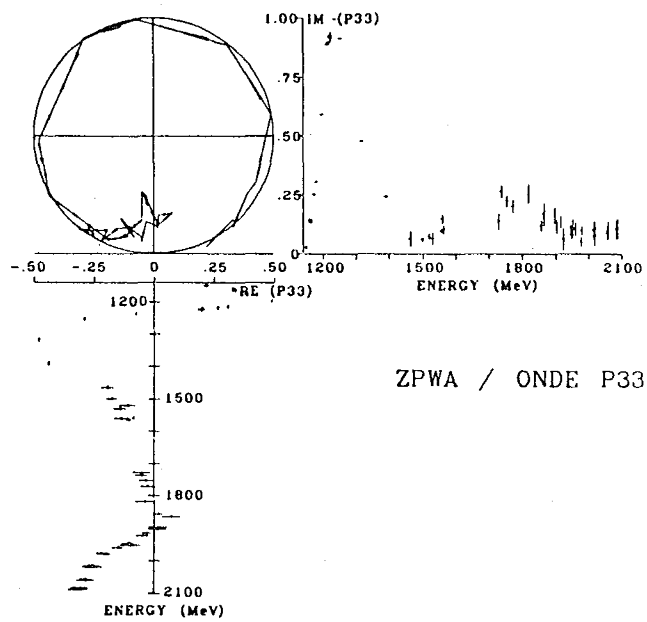


Figure 5(c).

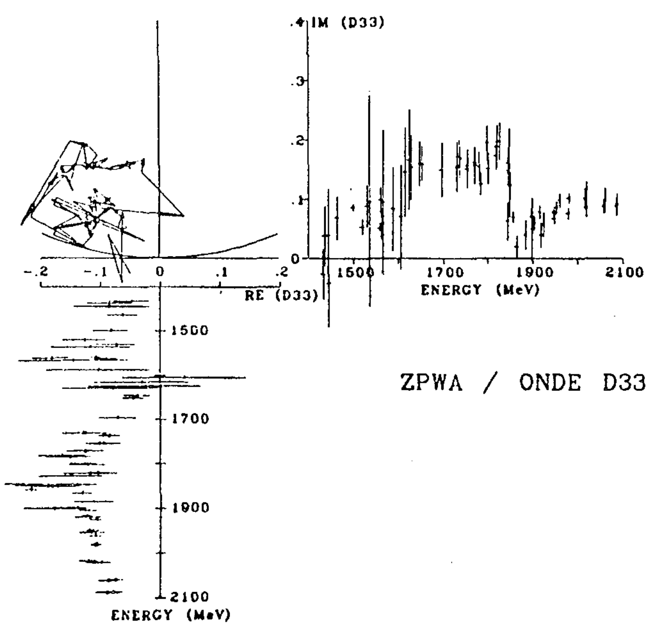


Figure 4(d).

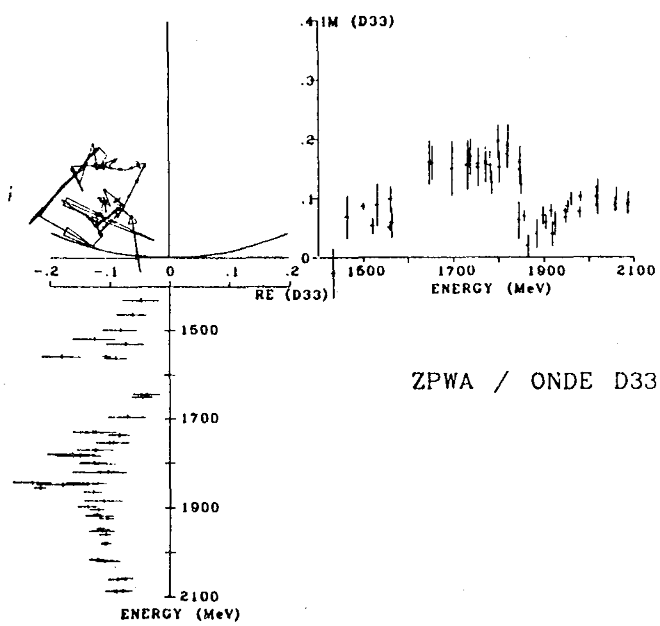


Figure 5(d).

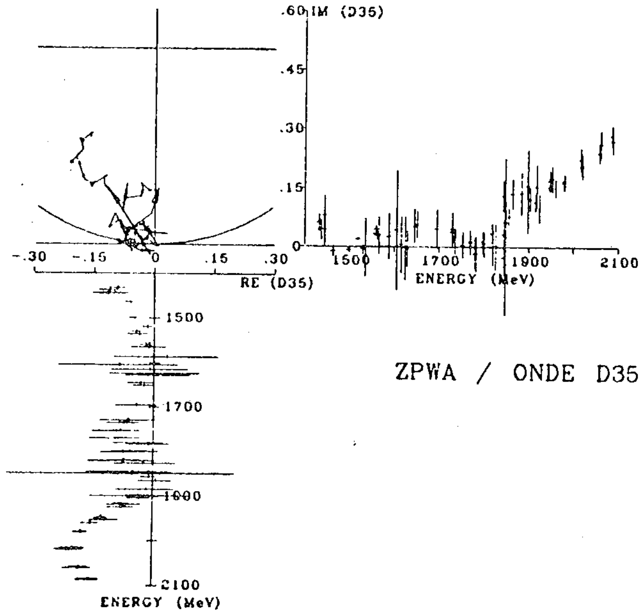


Figure 4(e).

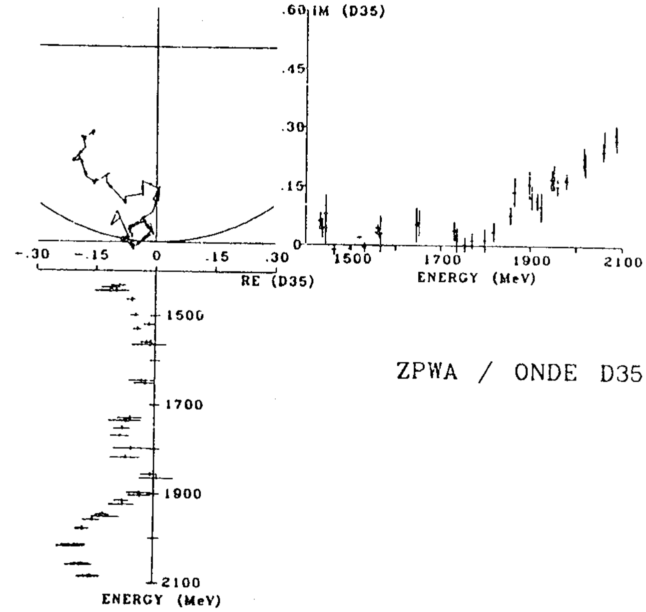


Figure 5(e).

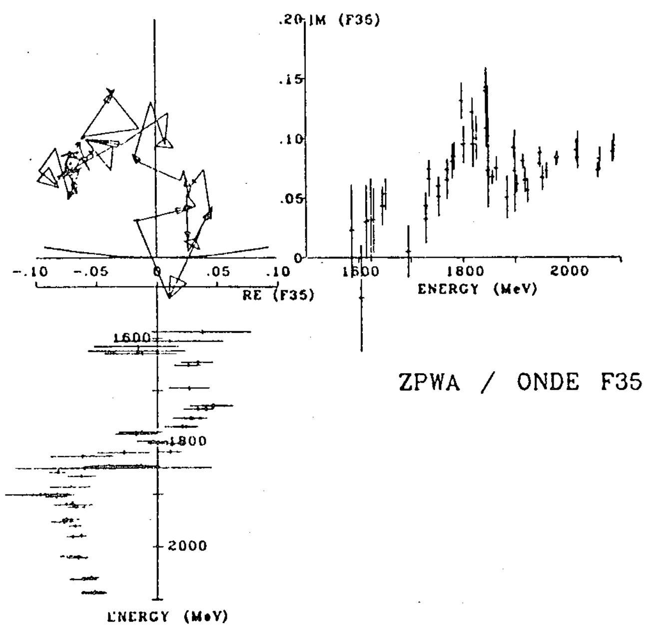


Figure 4(f).

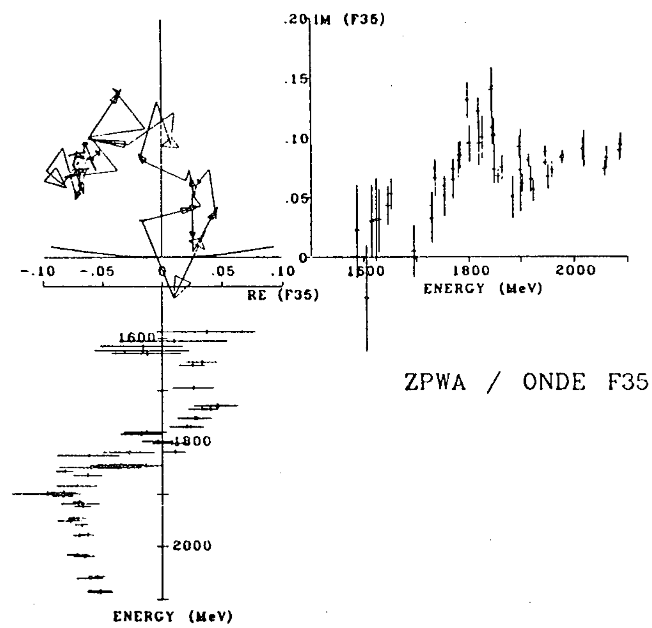


Figure 5(f).

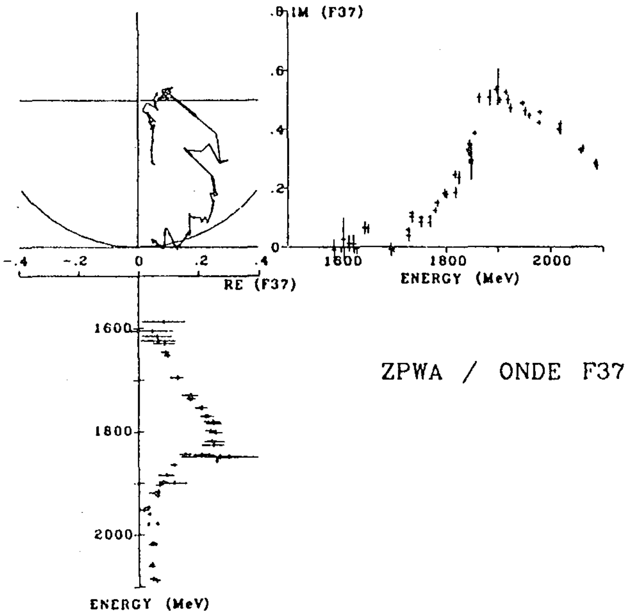


Figure 4(g).

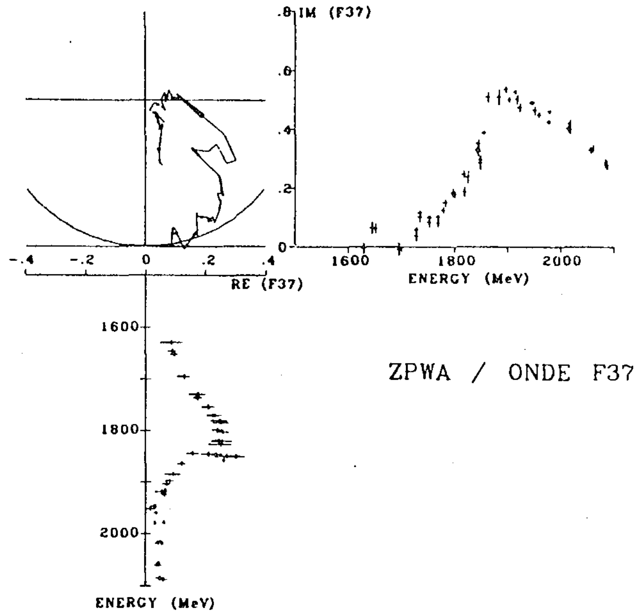


Figure 5(g).

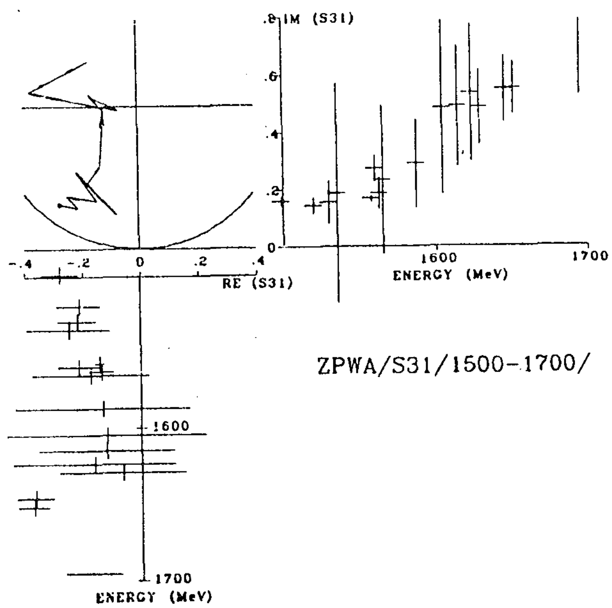


Figure 6.

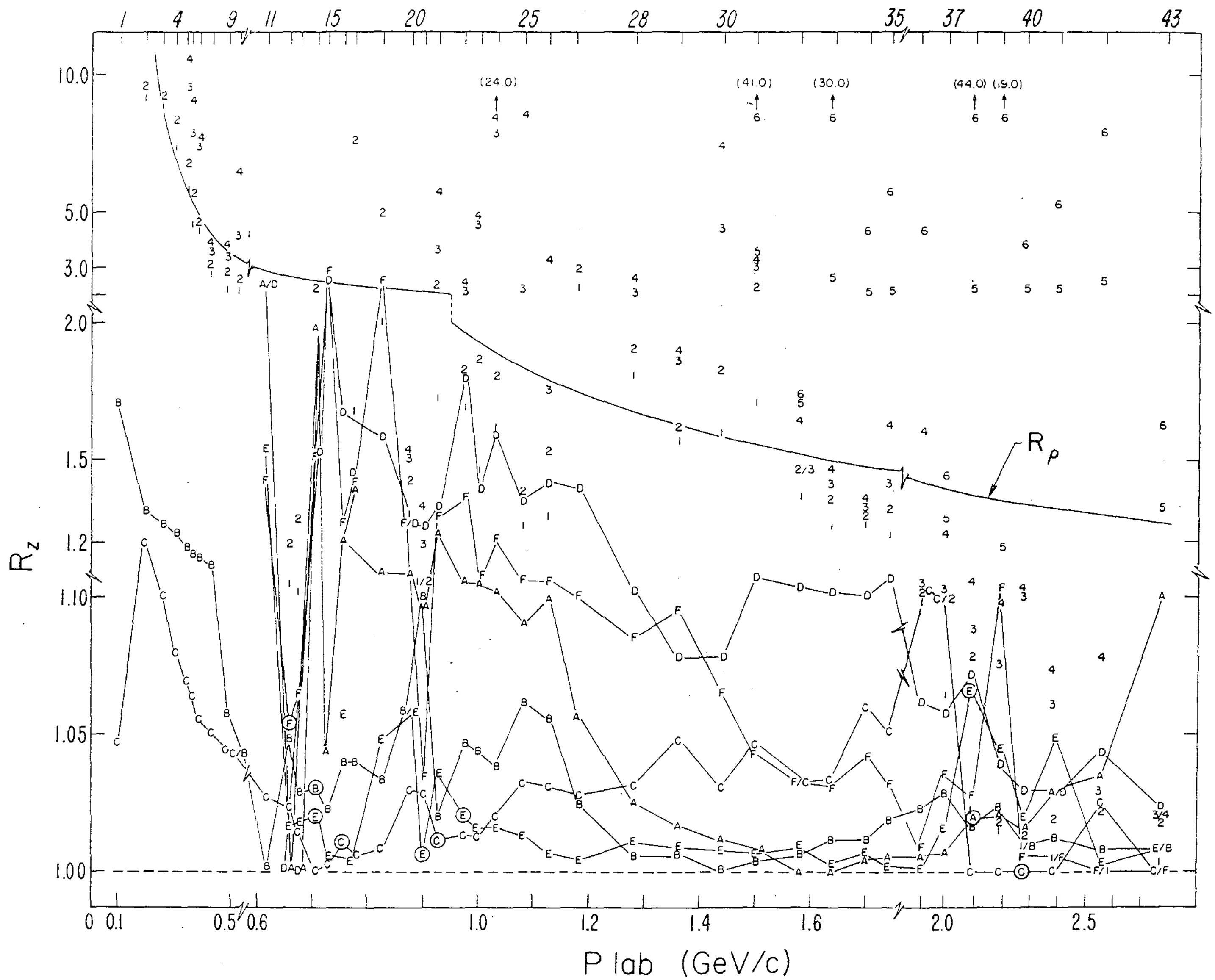

1 4 9 11 15 20 25 28 30 35 37 40 43
10.0
5.0
3.0
2.0
1.5
1.2
1.10
1.05
1.00
Rz
Rρ
(24.0)
(41.0)
(30.0)
(44.0)
(19.0)
0 0.1 0.5 0.6 1.0 1.2 1.4 1.6 2.0 2.5
P lab (GeV/c)

Figure 7.

DISCUSSION

H BURKHARDT: I must challenge your claim that this approach is model independent. The data determines the moduli, etc. of the amplitudes but there is an unknown function of angle or ω in the phase which gives the continuum ambiguity - your simple linear parametrisation hardly begins to explore it; it is, of course, completely removed by elastic unitarity so does not affect the $\Delta(1232)$ region. The particular Barrelet model you use emphasises the zeros; it assumes a simple form from this point of view and so obtains unique solutions from extra mathematical assumptions unmotivated by physics. I believe we are more likely to get the correct answers with model parametrisations which include more physics, such as the dispersion relations used by Pietarinen or Cutkosky. We have results to indicate they could be sound though the statistical error analysis has not been done, so nothing is sure.

D M CHEW: It was not at all obvious to us that in the $\Delta(1232)$ region, the amplitude constructed from the zeros (and an overall constant phase) would satisfy unitarity. We were, in fact, very much surprised by the discovery that no angle-dependent phase is needed in the elastic region. By continuity, it seemed to us that if such a phase is absent at low energy it should be absent at all energies. Our hypothesis is supported by the fact that the amplitude constructed entirely from the zeros never exceeds the unitarity bound. To the extent that we do not have a theoretical basis for the absence of an angle-dependent phase, our prescription may be described as a "model". But, to an experimenter, it is an extraordinarily simple and attractive model.

Furthermore, none of the "model parametrisations which include more physics" has so far been able to resolve the discrete ambiguity. Such models also fail to relate errors in the partial waves directly to measurement error. Thus it is unclear to me what you mean when you speak of the "correct answers".

R E CUTKOSKY: You commented that you would like to check the zeros in other analyses, those more recent than that of Saclay. I have plots showing the zero trajectories of some of the zeros calculated from our fit. These are in the forward hemisphere, corresponding to the deep dips in the transverse cross sections pointed out yesterday by Kelly. They also correspond to the two zeros in this region in which you pointed out that there is a difference between your assumption that the zeros cross the physical region, and the Saclay fit, in which the zeros do not cross the physical region.

The first figure shows the position of the zero of $T+(\theta)$ (which is related to $(1+P)\sigma$). This does not cross the physical region, and it does not approach the physical region perpendicularly. Thus we evidently agree with Saclay, as calculated by you. The second figure shows the zero of $T-(\theta)$. This also stays on the same side of the physical region in the 1700 to 2000 MeV mass region, in agreement with Saclay. There is an interesting uncertainty in our present fit above 1.8 GeV/c, associated with the second zero of T- which is shown. It is possible to simultaneously reflect both zeros in the real axis and maintain the fit. However, this is really a continuum ambiguity, if one takes into account also the errors of the data; there are large (correlated) uncertainties in the values of Re (θ).

To summarise this long comment, the objections you have raised against the Saclay analysis in the 1900 MeV mass region would also apply to us, and vice-versa. The real question is, does $|P| = 1$ at several points in this energy region for $\theta < 90^o$? We think it does not, but this is ultimately a question for the experimentalists.

D M CHEW: I thank you for providing us with the zero trajectories of your amplitude; in particular, I find very interesting the 2 zeros that you show, which are supposed to be the closest to the physical region: they give me an opportunity to show the difference of information that one can get when taking the original data carefully selected as opposed to the averaged data from an amalgamation.

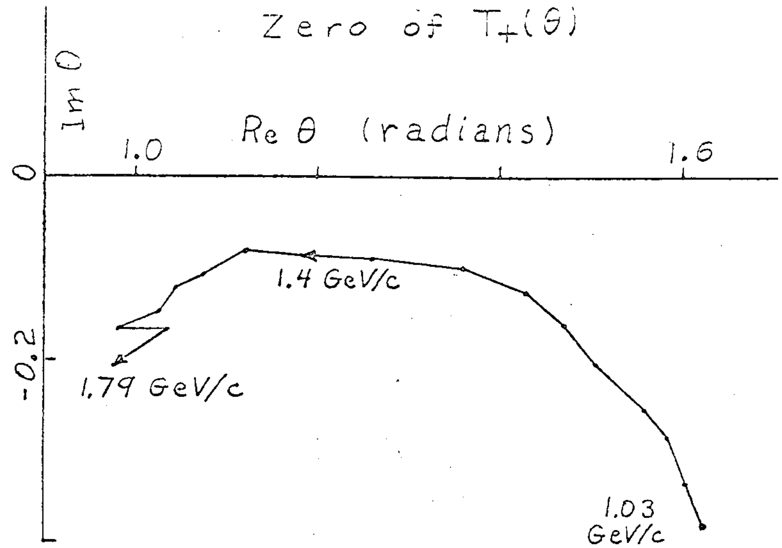
Let me first discuss the forward-hemisphere polarisation data between 1800 and 2000 MeV, which are relevant to the two zeros at issue. The question is whether these data are compatible with the polarisation reaching +1 or -1 at three different energies.

(1) When normalisation uncertainties, as well as statistical errors are considered, all experiments are compatible with the polarisation reaching -1 for $.15 < \cos\theta < .40$, at an energy near 1850 MeV (See the left-hand portion of the attached Table, together with Table II of my paper).

(2) In most experiments, the measured polarisation, for $0.6 < \cos\theta < .9$, hovers near +1 between 1850 and 1950 MeV, but the polynomial representation obtained by the method of the moments indicates that $P = +1$ only at two energies, 1850 MeV and 1950 MeV for the most accurate data (MARTIN 75) (See the right-hand portion of the attached Table).

The three attached figures (a) to (c) show our corresponding zero trajectories for three different sets of data. For each set, trajectory (B) crosses the physical region once (when $P = -1$), and trajectory (A) crosses twice (when $P = +1$). For either trajectory, in order to avoid crossing the physical region, one must introduce a sudden change of direction (cusp), in violation of analyticity for the amplitude.

The fact that you do not find any of these crossing is presumably a consequence of data amalgamation in the neighbourhood of points where the polarisation reaches ±1. Since such points are associated with extreme minima in the differential cross section, they are revealed only by high precision measurements with good resolution both in $\cos\theta$ and s. Consequently, mixing together data at different energies and with different accuracy and normalisation, tends to degrade the information present in the most accurate data.



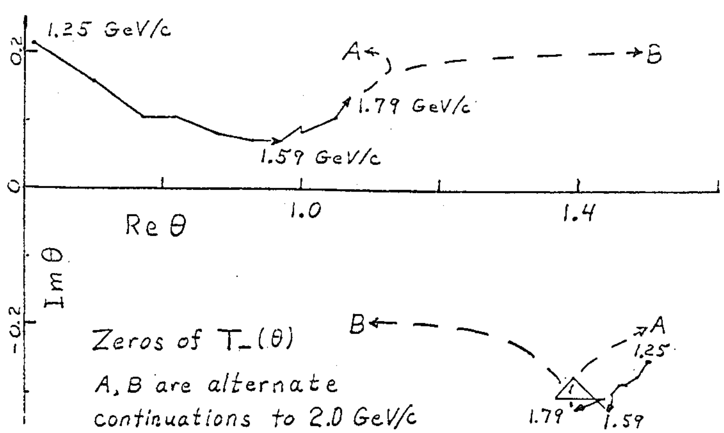
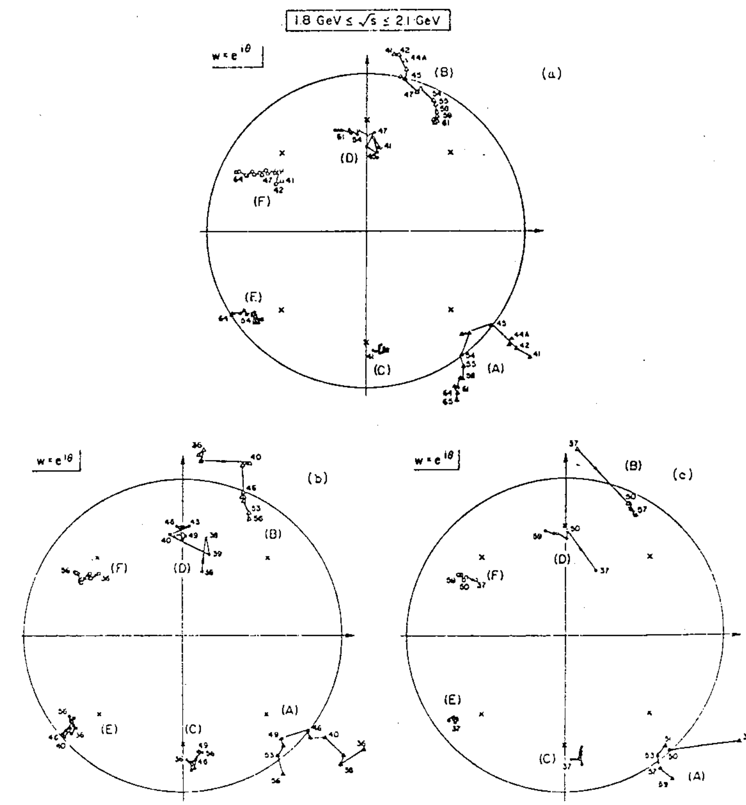


Table showing the possibility for the polarisation |P| to be equal to 1. in the Δ(1900) mass region.

P = -1				P = +1			
Plab(GeV/c)*	Reference**	cosθ	taking into account the error:	Plab(GeV/c)*	Reference**	cosθ	taking into account the error:
1.320	CERNHO 70	.26	<< 1σ Stat. only	1.274	M 75	.89	< 1σ Stat. only
1.352	JOHNSON67	.15	<< 1σ Stat. only	1.298	M 75	.86	< 1σ Stat. only
1.357	M 75	(.25)	? (NO DATA)	1.326	M 75	.86	< 1σ Stat. only
1.375	CERNHO 70	.30	≈ 1σ Stat.only	1.352	JOHNSON67	.86	≈ 1σ Stat. only
1.407	M 75	.40	< 1σ (Statistical and	1.357	M 75	(.8).87	≈ 1σ Stat. only
			Systematic)	1.375	CERNHO 70	.72	≈ 1σ Stat. only
1.439	M 75	.42	≈ 1σ Stat. only	1.439	M 75	.68	≈ 1σ Systematic
1.441	JOHNSON67	.35	< 1σ Stat. only	1.441	JOHNSON67	.72	<< 1σ Stat. only
1.460	CERNHO 70	.32	< 1σ Stat. only	1.460	CERNHO 70	.66	≈ 1σ Stat. only
1.500	CERNHO 70	.42	≈ 1σ Stat. only	1.476	M 75	.65	≈ 1σ Stat. only
				1.500	CERNHO 70	.62	<< 1σ Stat only
				1.507	M 75	.73	≈ 1σ Stat. only
				1.530	CERNHO 70	.66	< 1σ Stat. only
				1.535	M 75	(.6).58	≈ 1σ Stat. only
				1.569	M 75	.63	≈ 1σ Stat. only
				1.570	JOHNSON67	.60	< 1σ Stat. only

* Underlined value of Plab corresponds to the polynomial approximation showing |P| =1. at the underlined value of cosθ.

** 1)From the πN Two-Body Scattering Data Compilation,Particle Data Group,LBL-63 (April 1973).
-CERNHO 70 is M.G.Albrow et al.,Nucl.Phys. B25,9 (1971).
-JOHNSON67 is C.H.Johnson,Thesis,UCRL-17683 (1967).
2)M 75 is J.F.Martin et al.,Nucl.Phys. B89,253 (1975).



This report was done with support from the United States Energy Research and Development Administration. Any conclusions or opinions expressed in this report represent solely those of the author(s) and not necessarily those of The Regents of the University of California, the Lawrence Berkeley Laboratory or the United States Energy Research and Development Administration.