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The Ricci Flow on Riemannian Groupoids

by

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Abstract

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Doctor of Philosophy in Mathematics

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Professor John W Lott, Chair

We study the Ricci flow on Riemannian groupoids. We assume that these groupoids are closed and that the space of orbits is compact and connected. We derive maximum principles for groupoids and give some applications for immortal Ricci flow solutions on closed manifolds. We prove the short time existence and uniqueness of the Ricci flow on groupoids. We also define a \mathcal{F} -functional and derive the corresponding results for steady breathers on these groupoids.

To my parents.

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Chapter 1

Introduction

The Ricci flow equation, introduced by R. Hamilton in [11], is the nonlinear partial differential equation

$$\frac{\partial}{\partial t}g(t) = -2Ric(g(t)) \quad (1.1)$$

where $g(t)$ is a Riemannian metric on a fixed smooth manifold M . Hamilton proved the short time existence and uniqueness of a solution of the Ricci flow on a closed manifold. This result was extended by Shi [25] to the case of a noncompact manifold with initial Riemannian metric g_0 which is complete and of uniform bounded curvature.

The Ricci flow equation has become a well-known method to deform a Riemannian metric in order to improve it and deduce topological properties of the underlying smooth manifold. For instance, Hamilton proved that a three-dimensional simply connected closed manifold with a metric of positive Ricci curvature is diffeomorphic to the three-sphere \mathbb{S}^3 . Hamilton then outlined a program to use the Ricci flow to prove the Poincaré conjecture [10]. This program was successfully completed in 2002 and 2003 by Perelman who proved the Poincaré conjecture and, the more general conjecture, Thurston's Geometrization conjecture [22, 21, 20].

One of the basic methods used in Ricci flow and in other geometric evolution equations is the blowup or blowdown analysis. This method consists of constructing a sequence of pointed complete solutions of the Ricci flow by a parabolic rescaling of the original solution. If this sequence has uniform bounds on the curvature and if there is a uniform lower bound on the injectivity radius of the basepoints at time zero, then this sequence will converge in a smooth sense to a pointed complete Ricci flow solution ([9]). The condition on the injectivity radius ensures that the sequence does not collapse to a lower dimensional manifold. One of Perelman's key results was showing the existence of such a lower bound in the case of finite time singularities ([22]). On the other hand, if we look at Ricci flow solutions that exists for all $t \geq 0$ and try to determine the behavior as $t \rightarrow \infty$ by performing a blowdown analysis, we don't necessarily have a lower bound on the injectivity radius. Thus, the constructed sequence can collapse. However, as shown in [17], the limit can be described as a Ricci flow

on a pointed closed Riemannian groupoid of the same dimension. A Riemannian groupoid is an étale groupoid (\mathcal{G}, M) with a Riemannian metric g on the space of units M such that the local diffeomorphisms generated by the groupoid elements are local isometries. In other words, the metric is invariant under the action of \mathcal{G} on M .

An obvious problem is to determine if certain results in the theory of Ricci flow on Riemannian manifolds can be generalized to the case of closed Riemannian groupoids. For instance, this approach was used in [12] to study the long time behavior of the Ricci tensor for an immortal solution with uniformly bounded curvature and diameter.

In this dissertation, we provide some answers to this problem. We consider the simpler case where the space of orbits of the groupoid is compact and connected. The first result that we generalize is the short time existence and uniqueness of a solution of the Ricci flow.

Theorem 1. *Suppose (\mathcal{G}, M) is a closed groupoid with compact connected space of orbits $W = M/\mathcal{G}$ and suppose g_0 is a \mathcal{G} -invariant Riemannian metric on M . Then there exists a unique solution $g(t)$ of the Ricci flow defined on a maximal interval $[0, T)$ and with initial condition $g(t) = g_0$. Furthermore, if $T < \infty$, then*

$$\sup_M |Rm|(\cdot, t) \rightarrow \infty$$

as t approaches T .

As in the manifold case, we will prove the short time existence of the Ricci flow by proving the short time existence of the Ricci-DeTurck flow. However, to prove the uniqueness of the Ricci flow, we will avoid dealing with the theory of harmonic map heat flow and instead use the energy functional approach introduced by Kotschwar in [15].

The space of “arrows” \mathcal{G} of the closed Riemannian groupoid is equipped with two topologies. One that gives it the structure of an étale groupoid and another one that gives it the structure of a proper Lie groupoid. We construct a family Ξ of smooth Haar systems for this proper Lie groupoid. Recall that a Haar system is a collection of measures $\{d\rho^p\}_{p \in M}$ where, for each $p \in M$, the measure has support on $r^{-1}(p) = \mathcal{G}^p$, the preimage of p under the range map of the groupoid. These measures are \mathcal{G} -invariant in the appropriate sense and satisfy an additional property. For simplicity, we shall sometimes denote a Haar system by ρ or by $d\rho$.

The family Ξ of Haar systems that we construct can be identified with $C_{\mathcal{G}}^{\infty}(M)$, the space of smooth \mathcal{G} -invariant functions on the space of units M , as follows. For a fixed ρ_0 of Ξ , we have the bijection

$$\begin{aligned} C_{\mathcal{G}}^{\infty}(M) &\rightarrow \Xi \\ w &\rightarrow e^w d\rho_0. \end{aligned}$$

This identification is clearly not natural since it depends on the choice of the representative ρ . However, we use it to prove certain results in this paper.

Furthermore, to each element of Ξ , we will associate a \mathcal{G} -invariant closed one-form θ that comes from a Riemannian submersion generated by the closed groupoid. We prove the following

Theorem 2. *Suppose (\mathcal{G}, M) is a closed groupoid with compact connected space of orbits $W = M/\mathcal{G}$. Let $\mathcal{M}_{\mathcal{G}}$ be the space of \mathcal{G} -invariant Riemannian metrics on M and let Ξ be the family of smooth Haar systems mentioned above. We define the \mathcal{F} -functional $\mathcal{F} : \mathcal{M}_{\mathcal{G}} \times \Xi \rightarrow \mathbb{R}$ by*

$$\mathcal{F}(g, \rho) = \int_M (R + |\theta|^2) \phi^2 d\mu_g \quad (1.2)$$

where R is the scalar curvature of the metric g , $d\mu_g$ the Riemannian density, θ the one-form associated to the Haar system and ϕ any smooth nonnegative cutoff function for the Haar system. Under the flow equations

$$\begin{aligned} \frac{\partial g}{\partial t} &= -2Ric(g) \\ \frac{\partial(d\rho)}{\partial t} &= (|\theta|^2 - R - \operatorname{div}(\theta)) d\rho \end{aligned} \quad (1.3)$$

the evolution of the \mathcal{F} -functional is given by

$$\frac{d}{dt} \mathcal{F}(g(t), \rho(t)) = 2 \int_M |Ric(g) + \frac{1}{2} \mathcal{L}_{\theta^\#} g|^2 \phi^2 d\mu_g \geq 0. \quad (1.4)$$

As is shown in [7], the integrals listed above do not depend on the choice of the nonnegative cutoff function for the Haar system.

This dissertation is structured as follows. In Chapter 2, we collect and prove some results about closed Riemannian groupoids. In particular, we go over the construction of the cross-product groupoid on the orthonormal frame bundle as outlined in [7][Section 2]. This construction plays a major role in most of the proofs in this dissertation. We also define the family Ξ of Haar systems that was mentioned above.

In Chapter 3, we derive maximum principles for parabolic equations on groupoids and consider applications in the context of immortal Ricci flow solutions on closed manifolds. More precisely, let M be an n -dimensional closed manifold. An immortal Ricci flow solution $g(t)$ on M is a solution that exists for all $t \geq 0$. We set

$$\|Rm\|_{\infty}(t) = \max\{|Rm|(x, t) : x \in M\}$$

and

$$\|Ric\|_{\infty}(t) = \max\{|Ric|(x, t) : x \in M\}.$$

The diameter of the induced metric structure on M will be denoted by $diam(M; g(t))$ for each $t \geq 0$. Then we have the following results which we published in [12].

Theorem 3. *Given $n \in \mathbb{N}$ and numbers $K, D > 0$, there exists a function $F_{n,K,D} : [0, \infty) \rightarrow [0, \infty)$ with the following properties*

1. $\lim_{t \rightarrow \infty} F_{n,K,D}(t) = 0.$

2. If $g(t)$ is an immortal solution to the Ricci flow on a closed manifold M of dimension n such that $\|Rm\|_\infty(t) \leq K$ and $\text{diam}(M; g(t)) \leq D$ for all $t \geq 0$, then the Ricci tensor satisfies $\|Ric\|_\infty(t) \leq F_{n,K,D}(t)$ for all $t \geq 0$.

Theorem 4. Suppose $g(t)$ is an immortal solution to the Ricci flow on a closed 3-dimensional manifold M and suppose there exists a constant $C > 0$ such that

$$\text{diam}(M; g(t))^2 \|Rm\|_\infty(t) < C \quad (1.5)$$

for all $t \geq 0$. Then, the solution $g(t)$ is a type III solution.

Recall that an immortal solution $g(t)$ to the Ricci flow on a closed manifold M is said to have a type III singularity if $\sup_{M \times [0, \infty)} t |Rm| < \infty$.

In Chapter 3, we also prove a maximum principle which extends a result from [13].

Theorem 5. Let $(\mathcal{G}, M, g(t))$, $0 \leq t < T$ be a smooth one-parameter family of Riemannian groupoids with noncompact connected space of orbits W . Suppose that the induced length structure d_t on the space W is complete. Let $u \in C(M \times [0, T]) \cap C^{2,1}(M \times (0, T))$ be a \mathcal{G} -invariant function that satisfies the inequality

$$\frac{\partial u}{\partial t} \leq \Delta u + \langle X, \nabla u \rangle + bu$$

where X and b are \mathcal{G} -invariant and uniformly bounded. Suppose also that the function u satisfies

$$\int_0^T \int_M u_+^2(x, t) e^{-cd_t(O_{x_0}, O_x)^2} \phi^2(x) d\mu_g dt < \infty$$

where $u_+ = \max(u, 0)$ and $\phi = \phi_t$ is a smooth time dependent nonnegative cutoff function for a smooth time dependent family of Haar system ρ_t . Finally, suppose that the mean curvature form $\theta = \theta_t$, the variation of the metric and the variation of the Haar system are uniformly bounded. If $u(\cdot, 0) \leq 0$, then $u(\cdot, t) \leq 0$ for all $t \in [0, T]$.

In Chapter 4, we prove the uniqueness of the Ricci flow solution and in Chapter 5 we prove the short time existence. In Chapter 6, we briefly talk about 2-dimensional closed Riemannian groupoids. Finally, Chapter 6 is devoted to the \mathcal{F} -functional and the corresponding results for steady breathers.

Chapter 2

Riemannian groupoids

In this chapter, we list the necessary material that we will need about groupoids. For information about the basic definitions and constructions in the theory of groupoids, we refer to the books [2, 18].

In Section 2.1, we introduce some notation and review the concept of a complete effective Riemannian (étale) groupoid. In Section 2.2, we construct the “closure” of such a groupoid as defined in [8, 23]. This will give us a groupoid equipped with two topologies, one which gives it the structure of a proper Lie groupoid and a finer one which gives it the structure of a complete effective Riemannian groupoid. In Section 2.3, we briefly recall Haefliger’s local models for the structure of a closed Riemannian groupoid.

In the next three sections, we go over the same constructions done in [7][Sections 2 and 3]. We construct the cross-product groupoid on the orthonormal frame bundle and use it to generate a Haar system for the closed groupoid and the associated mean curvature form. We equip this Haar system with a nonnegative cutoff function. These quantities are used in Section 2.6 to define a measure for integrating functions on the space of units which are invariant under the groupoid action. We also derive an integration by parts formula for this measure. Then, in Section 7, we define the family of Haar systems Ξ and derive some relevant evolution equations for the Haar system and the mean curvature form. These evolution equations will be used when we will differentiate the energy functional in Chapter 3 and the \mathcal{F} -functional in Chapter 7.

The last section is devoted to \mathcal{G} -paths and \mathcal{G} -invariant vector fields. We recall how the smooth \mathcal{G} -paths can be used to define a length structure on the space of orbits and show that any \mathcal{G} -invariant vector field defines a one-parameter family of differentiable equivalences of the Riemannian groupoid in the sense of [8][Subsection 1.4].

2.1 Basic definitions

We will denote a smooth étale groupoid by a pair (\mathcal{G}, M^n) where M^n is the space of units and \mathcal{G} is the space of “arrows”. The corresponding source and range maps will be denoted

by s and r respectively. Our convention is that, for any h_1 and $h_2 \in \mathcal{G}$, the product $h_1 h_2$ is defined if and only if $s(h_1) = r(h_2)$. Given any $x \in M$, we write \mathcal{G}^x for $r^{-1}(x)$, \mathcal{G}_x for $s^{-1}(x)$, \mathcal{G}_x^x for the isotropy group $s^{-1}(x) \cup r^{-1}(x)$ of x and O_x for the orbit of x . Furthermore, given any $h \in \mathcal{G}$, we will denote by $dh : T_{s(h)}M \rightarrow T_{r(h)}M$ the linearization of h . Recall that this is simply the differential at $s(h)$ of a diffeomorphism $\phi : U \rightarrow V$ where U and V are open neighborhoods of $s(h)$ and $r(h)$ respectively and the map ϕ has the form $r \circ \alpha$ for some local section $\alpha : U \rightarrow \mathcal{G}$ of s at h . More precisely, the map α satisfies $s \circ \alpha = id$ and $\alpha(s(h)) = h$. We will consider étale groupoids up to *weak equivalence* ([18][Chapter 5.4]). A smooth étale groupoid is said to be *developable* if it is weakly equivalent to the étale groupoid associated to the action of a discrete group on a smooth manifold through diffeomorphisms.

Given a smooth manifold M , let $\Gamma(M)$ be the space of germs of diffeomorphisms $\phi : U \rightarrow V$ between open subsets of M . Then, for any smooth étale groupoid (\mathcal{G}, M) , we have the map

$$\begin{aligned} \kappa : \mathcal{G} &\rightarrow \Gamma(M) \\ h &\rightarrow germ_{s(h)}(r \circ \alpha) \end{aligned}$$

where, as before, the map α is a local section of s at h . The étale groupoid (\mathcal{G}, M) is said to be *effective* if the map κ is injective.

We will also say that the étale groupoid (\mathcal{G}, M) is *complete* if, for every pair of points x and y of M , there are neighborhoods U and V of x and y respectively so that, for any $h \in \mathcal{G}$ with $s(h) \in U$ and $r(h) \in V$, there is a corresponding section of s defined on all of U .

Finally, a smooth étale groupoid (\mathcal{G}, M) is *Riemannian* if M is equipped with a Riemannian metric g such that the elements of the pseudogroup generated by \mathcal{G} are local isometries. Equivalently, for each $h \in \mathcal{G}$, the linearization $dh : (T_{s(h)}M, g_{s(h)}) \rightarrow (T_{r(h)}M, g_{r(h)})$ is an isometry of inner product spaces. We will then say that the metric g is \mathcal{G} -invariant and we will denote the Riemannian groupoid by the triple (\mathcal{G}, M, g) . From now on, we will assume that the Riemannian groupoids are complete and effective unless otherwise stated.

2.2 The closure of a Riemannian groupoid

We will now recall how to construct the closure of a Riemannian groupoid in the sense of [8, 23]. Given a smooth manifold M^n , we let $J^1(M)$ be the space of 1-jets of local diffeomorphisms equipped with the 1-jet topology. Elements of $J^1(M)$ can be represented by triples (x, y, A) where $x, y \in M$ and $A : T_x M \rightarrow T_y M$ is a linear bijection. The pair $(J^1(M), M)$ is a smooth Lie groupoid in the sense of [18][Chapter 5] with source and range maps defined by $s(x, y, A) = x$ and $r(x, y, A) = y$ respectively. It is not étale unless $\dim(M) = 0$. A Riemannian metric g on M induces a smooth Lie subgroupoid $(J_g^1(M), M)$ of $(J^1(M), M)$ where $J_g^1(M) \subset J^1(M)$ consists of triples (x, y, A) where the map A is now a linear isometry with respect to g .

A Lie groupoid (\mathcal{G}, M) is said to be *proper* if the space \mathcal{G} is Hausdorff and the map $(s, r) : \mathcal{G} \rightarrow M \times M$ is proper. In [7][Lemma 1], it was shown that the Lie groupoid

$(J_g^1(M), M)$ generated by a Riemannian metric on M is proper. Indeed, the space $J_g^1(M)$ is Hausdorff and the map $(s, r) : J_g^1(M) \rightarrow M \times M$ is proper since it defines a fiber bundle structure with fiber diffeomorphic to the compact Lie group $O(n)$.

Given a Riemannian groupoid (\mathcal{G}, M, g) , there is a natural map of \mathcal{G} into $J^1(M)$ defined by

$$\begin{aligned} \nu : \mathcal{G} &\rightarrow J^1(M) \\ h &\rightarrow (s(h), r(h), dh). \end{aligned}$$

Since the metric g is \mathcal{G} -invariant, the image of this map is contained in $J_g^1(M)$. Furthermore, the map ν is injective. This follows from our assumption that the Riemannian groupoid is effective and the fact that the germ of a local isometry is completely determined by its 1-jet. The space \mathcal{G} can now be viewed as a subset of $J^1(M)$ and we can consider its closure $\overline{\mathcal{G}} \subset J^1(M)$ with respect to the 1-jet topology. The pair $(\overline{\mathcal{G}}, M)$ is a topological subgroupoid of $(J_g^1(M), M)$. Since $\overline{\mathcal{G}}$ is contained in $J_g^1(M)$, it follows from [24][Section 2] that $(\overline{\mathcal{G}}, M)$ is in fact a smooth Lie subgroupoid of $(J_g^1(M), M)$. The orbits of this Lie groupoid are closed submanifolds of M and correspond to the closures of the orbits of the groupoid (\mathcal{G}, M) (see [23][Section 3]). It also follows that the quotient space $W = M/\overline{\mathcal{G}}$, equipped with the quotient topology, is a Hausdorff space. The corresponding quotient map will be denoted by $\sigma : M \rightarrow W$. Finally, the fact that $\overline{\mathcal{G}}$ is a subset of $J_g^1(M)$ implies that the Lie groupoid $(\overline{\mathcal{G}}, M)$ is proper.

In addition to the subspace topology induced by the 1-jet topology on $J^1(M)$, the space $\overline{\mathcal{G}}$ can be equipped with a finer topology which gives the triple $(\overline{\mathcal{G}}, M, g)$ the structure of a complete effective Riemannian groupoid. For now on, we will assume that the Riemannian groupoid (\mathcal{G}, M, g) satisfy $\overline{\mathcal{G}} = \mathcal{G}$. We will then say that the groupoid is *closed*. Also, when we will work with the space \mathcal{G} , we will assume, unless otherwise stated, that it is equipped with the subspace topology induced by the 1-jet topology. We will call the quotient space W the *space of orbits* and we will in general assume that it is compact and connected.

2.3 Local structure of the closed groupoid

Let (\mathcal{G}, M, g) be a closed Riemannian groupoid as before and let \mathfrak{g}_M be the corresponding Lie algebroid as defined in [18][Chapter 6]. As mentioned in [7][Subsection 2.5], \mathfrak{g}_M is a \mathcal{G} -equivariant flat vector bundle over M whose fibers are copies of a fixed Lie algebra \mathfrak{g} called the **structural Lie algebra**. If $an : \mathfrak{g}_M \rightarrow TM$ is the anchor map of the Lie algebroid and if $P : U \times \mathfrak{g} \rightarrow \mathfrak{g}_M$ is a local parallelization over an open set $U \subset M$, then the map $an \circ P$ describes a Lie algebra of Killing vector fields on U isomorphic to \mathfrak{g} .

For a fixed point $p \in M$, let K denote the isotropy group of the closed groupoid at p . K is a compact Lie group and we will denote its Lie algebra by \mathfrak{k} . Since \mathfrak{g} can be realized as the Lie algebra of Killing vector fields on some neighborhood of p , there is a natural inclusion $i : \mathfrak{k} \rightarrow \mathfrak{g}$. Furthermore, there is a representation $Ad : K \rightarrow Aut(\mathfrak{g})$ such that

1. Ad leaves invariant the subalgebra $\mathfrak{k} = i(\mathfrak{k})$ of \mathfrak{g} and $Ad|_{\mathfrak{k}}$ is the adjoint representation of K on \mathfrak{k} .
2. The differential of Ad is the adjoint representation of \mathfrak{k} on \mathfrak{g} .

Finally, there is a representation $T : K \rightarrow Gl(V)$ given by the action of K on the subspace V of $T_p M$ which is the orthogonal complement of the tangent space $T_p O_p$ of the orbit space O_p . The tangent space $T_p O_p$ is isomorphic to the vector space $\mathfrak{g}/\mathfrak{k}$ and the linear representation of K on $T_p M$, which is faithful, corresponds to the direct sum representation $T \oplus Ad$ on $V \oplus \mathfrak{g}/\mathfrak{k}$.

It is shown in [8][Section 4] that the quintuple $(\mathfrak{g}, K, i, Ad, T)$ determines the weak equivalence class of the restriction of \mathcal{G} (with the étale topology) to a small invariant neighborhood of the orbit O_p . Furthermore, given such a quintuple, we can construct an explicit local model for the structure of the groupoid.

2.4 The induced groupoid over the orthonormal frame bundle.

Given a Riemannian groupoid (\mathcal{G}, M, g) , let $\pi : F_g \rightarrow M$ be the orthonormal frame bundle associated to the metric g . This is a principal $O(n)$ -bundle over M . There is a right action of \mathcal{G} on F_g defined by saying that if $h \in \mathcal{G}$ and f is an orthonormal frame at $r(h)$ then $f \cdot h$ is the frame $(dh)^{-1}(f)$ at $s(h)$. We can then construct the corresponding cross-product groupoid with space of “arrows”

$$\hat{\mathcal{G}} = F_g \rtimes \mathcal{G} = \{(f, h) \in F_g \times \mathcal{G} : \pi(f) = r(h)\}$$

and space of units F_g . The source and range maps are defined by $s(f, h) = f \cdot h$ and $r(f, h) = f$ respectively. It is clear that $(\hat{\mathcal{G}}, F_g)$ has the structure of a smooth proper Lie groupoid and the structure of a smooth étale groupoid. The latter property implies, in particular, that the linearization $d\hat{h} : T_{s(\hat{h})} F_g \rightarrow T_{r(\hat{h})} F_g$ is well-defined for any $\hat{h} \in \hat{\mathcal{G}}$. Also, the groupoid $(\hat{\mathcal{G}}, F_g)$ has trivial isotropy groups.

Let ∇ be the Levi-Civita connection of the Riemannian metric g . This connection defines an $O(n)$ -invariant Riemannian metric \hat{g} on F_g such that the map $\pi : (F_g, \hat{g}) \rightarrow (M, g)$ is a Riemannian submersion and such that the restriction of \hat{g} to a π -fiber is equal to the bi-invariant metric on $O(n)$ with unit volume. Since the metric g on M is \mathcal{G} -invariant, the metric \hat{g} on F_g is $\hat{\mathcal{G}}$ -invariant.

Let Z be the space of orbits of the groupoid $(\hat{\mathcal{G}}, F_g)$ with quotient map $\hat{\sigma} : F_g \rightarrow Z$. The connection ∇ defines a canonical parallelism of F_g which is invariant under the action of $\hat{\mathcal{G}}$. It follows from [23][Theorem 4.2] that Z is a smooth manifold and the quotient map $\hat{\sigma}$ is a smooth submersion. In fact, since the metric \hat{g} on F_g is $\hat{\mathcal{G}}$ -invariant, it induces a metric \bar{g} on Z which makes $\hat{\sigma}$ a Riemannian submersion. Furthermore, the action of $O(n)$ on F_g

naturally induces an action of $O(n)$ on Z and the metric \bar{g} is invariant under this action. We have a commutative diagram

$$\begin{array}{ccc} F_g & \xrightarrow{\hat{\sigma}} & Z \\ \pi \downarrow & & \downarrow \hat{\pi} \\ M & \xrightarrow{\sigma} & W \end{array} \quad (2.1)$$

where the map $\hat{\sigma}$ is $O(n)$ -equivariant and the maps π and $\hat{\pi}$ correspond to taking $O(n)$ -quotients. The space Z is compact since W is compact.

The space $\hat{\mathcal{G}}$ comes from an equivalence relation on F_g defined by saying that f, f' are equivalent if and only if $\hat{\sigma}(f) = \hat{\sigma}(f')$. More precisely, there is an $O(n)$ -equivariant isomorphism $\mathcal{G} \simeq F_g \times_Z F_g$.

2.5 The Haar system generated by an invariant metric and the associated mean curvature form

As we mentioned earlier, a Haar system for a proper Lie groupoid (\mathcal{G}, M) is a collection of positive measures $\{d\rho^p\}_{p \in M}$ which is \mathcal{G} -invariant in the appropriate sense and such that the support of the measure $d\rho^p$ is \mathcal{G}^p for any $p \in M$. Furthermore, given any function $\alpha \in C_c(\mathcal{G})$, the function

$$p \rightarrow \int_{\mathcal{G}^p} \alpha(h) d\rho^p(h)$$

is in $C_c(M)$ ([27][Definition 1.1]).

Let (\mathcal{G}, M, g) be a closed Riemannian groupoid as above. For $f \in F_g$, let $d\rho^f$ be the measure on $\hat{\mathcal{G}}$ which is supported on $\hat{\mathcal{G}}^f \simeq \hat{\sigma}^{-1}(\hat{\sigma}(f))$ and is given there by the fiberwise Riemannian density. The family $\{d\rho^f\}_{f \in F_g}$ is a Haar system for the proper Lie groupoid $(\hat{\mathcal{G}}, F_g)$. In particular, the family of measures $\{d\rho^f\}_{f \in F_g}$ is $\hat{\mathcal{G}}$ -invariant in an appropriate sense.

Let $p \in M$ and $f \in F_g$ be such that $\pi(f) = p$. There is a diffeomorphism $i_{p,f} : \mathcal{G}^p \rightarrow \hat{\mathcal{G}}^f$ given by $i_{p,f}(h) = (f, h)$. Let $d\rho^p$ be the measure on \mathcal{G} which is supported on \mathcal{G}^p and is given there by the pullback measure $i_{p,f}^* d\rho^f$. Since the family of measures $\{d\rho^f\}_{f \in \pi^{-1}(p)}$ is $O(n)$ -equivariant, the measure $d\rho^p$ is independent of the choice of f .

The family of measures $\{d\rho^p\}_{p \in M}$ is a Haar system for the proper Lie groupoid (\mathcal{G}, M) . As follows from [7][Lemma 4] or [26][Proposition 6.11], we can construct a smooth nonnegative cutoff function for this Haar system. That is, there exists a nonnegative smooth function $\phi \in C^\infty(M)$ such that

1. For any compact subset K of M , the set $\text{supp}(\phi) \cap s(r^{-1}(K))$ is also compact.
2. $\int_{\mathcal{G}^p} \phi^2(s(h)) d\rho^p(h) = 1$ for any $p \in M$.

Since we are assuming that the space of orbits W is compact, the first condition simply means that ϕ has compact support.

Let $\hat{\theta}$ be the mean curvature form of the induced Riemannian submersion $\rho : (F_g, \hat{g}) \rightarrow (Z, \bar{g})$. By [7][Lemma 3], the form $\hat{\theta}$ is a closed one-form which is $\hat{\mathcal{G}}$ -basic and $O(n)$ -basic. Let θ be the unique one-form on M such that $\pi^*\theta = \hat{\theta}$. The form θ is closed and \mathcal{G} -invariant.

Let $E \rightarrow M$ be a vector bundle on M equipped with a right \mathcal{G} -action and let ∇^E be a \mathcal{G} -invariant connection on E . Given $h \in \mathcal{G}$ and $e \in E_{s(h)}$, let $e \cdot h^{-1} \in E_{r(g)}$ denote the action of h^{-1} on e . For any compactly supported element $\xi \in C_c^\infty(M; E)$, we can construct the section

$$\int_{\mathcal{G}^p} \xi_{s(h)} \cdot h^{-1} d\rho^p(h) \tag{2.2}$$

which will be a \mathcal{G} -invariant element of $C^\infty(M; E)$ whose value at $p \in M$ is given by (2.2). By [7][Lemma 6], we have the following identity in $\Omega^1(M; E)$:

$$\begin{aligned} \nabla^E \int_{\mathcal{G}^p} \xi_{s(h)} \cdot h^{-1} d\rho^p(h) &= \int_{\mathcal{G}^p} (\nabla^E \xi)_{s(h)} \cdot h^{-1} d\rho^p(h) \\ &\quad + \theta_p \int_{\mathcal{G}^p} \xi_{s(h)} \cdot h^{-1} d\rho^p(h). \end{aligned}$$

It follows that, for any $\omega \in \Omega_c^*(M)$, we have

$$\begin{aligned} d \int_{\mathcal{G}^p} \omega_{s(h)} \cdot h^{-1} d\rho^p(h) &= \int_{\mathcal{G}^p} (d\omega)_{s(h)} \cdot h^{-1} d\rho^p(h) \\ &\quad + \theta_p \wedge \int_{\mathcal{G}^p} \omega_{s(h)} \cdot h^{-1} d\rho^p(h). \end{aligned} \tag{2.3}$$

In particular, if we take ω to be equal to the function ϕ^2 and apply the definition of the nonnegative cutoff function, we get

$$\theta_p = - \int_{\mathcal{G}^p} (d\phi^2)_{s(h)} \cdot h^{-1} d\rho^p(h) \tag{2.4}$$

for every $p \in M$.

2.6 Integration of \mathcal{G} -invariant functions.

We denote by $d\mu_g$ the Riemannian measure on (M, g) and by $C_{\mathcal{G}}(M)$ the space of continuous \mathcal{G} -invariant functions on M . We will integrate these functions by using the measure $\phi^2 d\mu_g$ on M . It is shown in [7][Proposition 1] that the integral

$$\int_M f \phi^2 d\mu_g$$

does not depend on the nonnegative cutoff function ϕ for any $f \in C_{\mathcal{G}}(M)$. More precisely, let $d\mu_{\bar{g}}$ be the Riemannian measure on (Z, \bar{g}) and let $d\eta(g) = \hat{\pi}_* d\mu_{\bar{g}}$ be the pushforward measure on W . A \mathcal{G} -invariant continuous function on M can be viewed as a continuous function on W and as an $O(n)$ -invariant continuous function on Z . It follows from [7][Proposition 1] that the following identity holds

$$\int_M f \phi^2 d\mu_g = \int_Z f d\mu_{\bar{g}} = \int_W f d\eta(g)$$

for any $f \in C_{\mathcal{G}}(M)$. Furthermore, the proof of [7][Proposition 1] can be adapted to obtain the following integration by parts formula.

Lemma 6. *Let α be a (r, s) -tensor on M and ω an $(r-1, s)$ -tensor on M . If α and ω are \mathcal{G} -invariant, then*

$$\int_M \langle \operatorname{div}(\alpha), \omega \rangle \phi^2 d\mu_g = - \int_M \langle \alpha, \nabla \omega \rangle \phi^2 d\mu_g + \int_M \langle \iota_{\theta^\sharp} \alpha, \omega \rangle \phi^2 d\mu_g \quad (2.5)$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product defined by the metric g and $\iota_{\theta^\sharp} \alpha$ is the interior product of the dual θ^\sharp of θ with α .

Proof. Since the nonnegative cutoff function ϕ has compact support, we can use the regular integration by parts formula.

$$\int_M \langle \operatorname{div}(\alpha), \omega \rangle \phi^2 d\mu_g = - \int_M \langle \alpha, \nabla(\phi^2 \omega) \rangle d\mu_g.$$

This gives

$$\int_M \langle \operatorname{div}(\alpha), \omega \rangle \phi^2 d\mu_g = - \int_M \langle \alpha, \nabla \omega \rangle \phi^2 d\mu_g - \int_M \langle \alpha, \nabla \phi^2 \otimes \omega \rangle d\mu_g.$$

We can rewrite the second integrand of the last equation as

$$- \int_M \langle \alpha, \nabla \phi^2 \otimes \omega \rangle d\mu_g = - \int_M \langle \alpha * \omega, \nabla \phi^2 \rangle d\mu_g$$

where $\alpha * \omega$ is the 1-form given by $\alpha * \omega(X) = \langle \iota_X \alpha, \omega \rangle$ for any vector X . Then, just as in the proof of [7][Proposition 1], we rewrite

$$\begin{aligned} - \int_M \langle \alpha * \omega, \nabla \phi^2 \rangle d\mu_g &= - \int_{F_g} \pi^*(\langle \alpha * \omega, \nabla \phi^2 \rangle) d\mu_{\hat{g}} \\ &= - \int_Z \left(\int_{F_g/Z} \pi^*(\langle \alpha * \omega, \nabla \phi^2 \rangle) d\mu_{\hat{g}} \right) d\mu_{\bar{g}}. \end{aligned} \quad (2.6)$$

The inner integral of the last equation corresponds to integrating over the “fibers”

$$\sigma^{-1}(\sigma(f)) \simeq \hat{\mathcal{G}}^f$$

where $f \in F_g$. Note that the function $\hat{\phi} = \pi^*\phi$ is a cutoff function for the Haar system $\{d\rho^f\}_{f \in F_g}$. Also, just like Equation (2.4), we have

$$\hat{\theta}_f = - \int_{\hat{\mathcal{G}}^f} (d\hat{\phi}^2)_{s(\hat{h})} \cdot \hat{h}^{-1} d\rho^f(\hat{h})$$

Hence, if χ is a $\hat{\mathcal{G}}$ -invariant 1-form on F_g , we have

$$\begin{aligned} \langle \hat{\theta}_f, \chi_f \rangle &= - \int_{\hat{\mathcal{G}}^f} \langle (\nabla \hat{\phi}^2)_{s(\hat{h})}, \chi_{s(\hat{h})} \rangle d\rho^f(\hat{h}) \\ \int_{\hat{\mathcal{G}}^f} \langle \hat{\theta}_{s(\hat{h})}, \chi_{s(\hat{h})} \rangle \hat{\phi}^2(s(\hat{h})) d\rho^f(\hat{h}) &= - \int_{\hat{\mathcal{G}}^f} \langle (\nabla \hat{\phi}^2)_{s(\hat{h})}, \chi_{s(\hat{h})} \rangle d\rho^f(\hat{h}). \end{aligned}$$

If we apply this to (2.6), we obtain

$$\begin{aligned} - \int_M \langle \alpha * \omega, \nabla \phi^2 \rangle d\mu_g &= \int_Z \left(\int_{F_g/Z} \langle \pi^*(\alpha * \omega), \hat{\theta} \rangle \hat{\phi}^2 d\mu_{\hat{g}} \right) d\mu_{\bar{g}} \\ &= \int_M \langle \alpha * \omega, \theta \rangle \phi^2 d\mu_g \\ &= \int_M \langle \iota_{\theta^\#} \alpha, \omega \rangle \phi^2 d\mu_g. \end{aligned}$$

So

$$\int_M \langle \operatorname{div}(\alpha), \omega \rangle \phi^2 d\mu_g = - \int_M \langle \alpha, \nabla \omega \rangle \phi^2 d\mu_g + \int_M \langle \iota_{\theta^\#} \alpha, \omega \rangle \phi^2 d\mu_g.$$

This completes the proof. \square

One can easily generalize this result to the case where the space of orbits W is not necessarily compact. We will have to assume then that the image, under the quotient map σ , of the support of the tensor α or the tensor ω is compact.

2.7 The family of Haar systems Ξ

Suppose (\mathcal{G}, M^n) is a smooth complete effective étale groupoid and suppose that, for some $T < \infty$, we have a smooth time-dependent family of \mathcal{G} -invariant Riemannian metrics $g(t)$ on M defined for $t \in [0, T]$. The closed groupoid $(\bar{\mathcal{G}}, M^n)$ does not depend on the choice of the \mathcal{G} -invariant metric g . As before, we will assume that $\bar{\mathcal{G}} = \mathcal{G}$ and that the space of

orbits W is compact and connected. We will determine how the Haar system and the mean curvature form constructed in the previous subsections vary with respect to time. Given two \mathcal{G} -invariant metrics g_1 and g_2 , one can see that, based on the way we constructed the Haar system, there is some \mathcal{G} -invariant smooth function w such that $d\rho_2^p = e^w d\rho_1^p$ where $\{d\rho_1^p\}_{p \in M}$ and $\{d\rho_2^p\}_{p \in M}$ are the Haar systems generated by g_1 and g_2 respectively. So given a one parameter family of invariant metrics $g(t)$, there will be a time-dependent \mathcal{G} -invariant smooth function β such that the evolution equation of the Haar system will be given by $\frac{\partial(d\rho^p)}{\partial t} = \hat{\beta}(p)d\rho^p$. We will now show this precisely. The Haar system and the associated mean curvature form were obtained by looking at the action of the groupoid on the orthonormal frame bundle. This space depends on the choice of the metric g . To bypass this problem, we will use the ‘‘Uhlenbeck trick’’.

Suppose that the evolution equation of the metric $g(t)$ is given by

$$\frac{\partial g}{\partial t} = v \quad (2.7)$$

where $v = v(t)$ is a smooth \mathcal{G} -invariant symmetric 2-tensor on M . Let E be a vector bundle over M isomorphic to the tangent bundle TM via a vector bundle isomorphism $u_0 : E \rightarrow TM$. We equip E with the bundle metric $(\cdot, \cdot) = u_0^* g_0$ where $g_0 = g(0)$. The map u_0 is now a bundle isometry.

For $t \in [0, T]$, consider the one-parameter family of bundle isomorphisms $u_t : E \rightarrow M$ that satisfy the ODE

$$\begin{aligned} \frac{\partial u}{\partial t} &= -\frac{v}{2} \circ u \\ u(0) &= u_0 \end{aligned} \quad (2.8)$$

where we use the metric $g(t)$ to view $v(t)$ as a bundle map $TM \rightarrow TM$. This ODE has a solution defined on $[0, T]$. Furthermore, by construction, the map u_t is a bundle isometry $(E, (\cdot, \cdot)) \rightarrow (T_{g(t)}M, g_t)$ for each t .

The right action of \mathcal{G} on TM induces, for each $t \in [0, T]$, a right action of \mathcal{G} on E given by

$$w \cdot h = u_t^{-1} ((u_t(w)) \cdot h) = u_t^{-1} \circ (dh)^{-1} \circ u_t(w) \quad (2.9)$$

where $h \in \mathcal{G}$ and $w \in E_{r(h)}$.

Lemma 7. *The right action of \mathcal{G} on E does not depend on t .*

Proof. If we differentiate Equation (2.9) with respect to time, we get

$$\frac{\partial(w \cdot h)}{\partial t} = \frac{\partial u^{-1}}{\partial t} \circ (dh)^{-1} \circ u(w) + u^{-1} \circ (dh)^{-1} \circ \frac{\partial u}{\partial t}(w).$$

Using (2.8), we get

$$\frac{\partial u^{-1}}{\partial t} = u^{-1} \circ \frac{v}{2}.$$

Hence, Equation (2.10) becomes

$$\frac{\partial(w \cdot h)}{\partial t} = u^{-1} \circ \frac{v}{2} \circ (dh)^{-1} \circ u(w) - u^{-1} \circ (dh)^{-1} \circ \frac{v}{2} \circ u(w).$$

Since the tensor v is \mathcal{G} -invariant, it follows that

$$\frac{v}{2} \circ (dh)^{-1} = (dh)^{-1} \circ \frac{v}{2}.$$

Therefore,

$$\frac{\partial(w \cdot h)}{\partial t} = 0.$$

This completes the proof. \square

Let $\pi : \mathbf{F} \rightarrow M$ be the orthonormal frame bundle of the metric bundle $(E, h) \rightarrow M$. For each $t \in [0, T]$, the map u_t induces a bundle isomorphism $\mathbf{F} \rightarrow F_{g(t)}$. Also, it follows from the previous lemma that the induced cross-product groupoid over \mathbf{F} with $\hat{\mathcal{G}} = \mathbf{F} \rtimes \mathcal{G}$ does not depend on t .

Just as in Section 2.4, the space of orbits $Z = \mathbf{F}/\hat{\mathcal{G}}$ is a smooth manifold and the quotient map $\hat{\sigma} : \mathbf{F} \rightarrow Z$ is a smooth submersion. Indeed, for each $t \in [0, T]$, the pullback connection $A(t) = u_t^* \nabla_{g(t)}$ induces a canonical parallelism on \mathbf{F} which is preserved by $\hat{\mathcal{G}}$. Furthermore, we get a smooth time-dependent $\hat{\mathcal{G}}$ -invariant and $O(n)$ -invariant metric $\hat{g}(t)$ on \mathbf{F} which induces a smooth time dependent $O(n)$ -invariant Riemannian metric $\bar{g}(t)$ on Z . This makes $\hat{\sigma} : (\mathbf{F}, \hat{g}(t)) \rightarrow (Z, \bar{g}(t))$ an $O(n)$ -equivariant Riemannian submersion for each t . We can then construct the time dependent Haar system $\{d\rho^f\}_{f \in \mathbf{F}}$ and mean curvature form $\hat{\theta}$ as described in Section 2.5. For each t , this will induce the Haar system $\{d\rho^p\}_{p \in M}$ and the mean curvature form θ corresponding to $g(t)$. We can then construct a smooth time-dependent nonnegative cutoff function ϕ for the Haar system.

The evolution equation for the Riemannian metric \hat{g} on \mathbf{F} has the form

$$\frac{\partial \hat{g}}{\partial t} = \hat{v} \tag{2.10}$$

where \hat{v} is a $\hat{\mathcal{G}}$ and $O(n)$ -invariant symmetric 2-tensor on \mathbf{F} . Recall that, for each $f \in \mathbf{F}$ and each $t \in [0, T]$, the measure $d\rho^f$ on $\hat{\mathcal{G}}^f \simeq \hat{\sigma}^{-1}(\sigma(f))$ is the fiberwise Riemannian density with respect to the metric \hat{g} . So the evolution equation for the measure $d\rho^f$ is given by

$$\frac{\partial(d\rho^f)}{\partial t} = \hat{\beta} d\rho^f. \tag{2.11}$$

where $2\hat{\beta}$ is equal to the trace of \hat{v} along the fiber $\hat{\sigma}^{-1}(\sigma(f))$ with respect to the metric \hat{g} . The function $\hat{\beta}$ is a $\hat{\mathcal{G}}$ and $O(n)$ -invariant smooth function on \mathbf{F} . It induces a \mathcal{G} -invariant function β on M .

Lemma 8. *The evolution equations for the Haar system $\{d\rho^p\}_{p \in M}$ and the mean curvature form θ are given by*

$$\frac{\partial(d\rho^p)}{\partial t} = \beta(p)d\rho^p. \quad (2.12)$$

for every $p \in M$ and

$$\frac{\partial\theta}{\partial t} = d\beta. \quad (2.13)$$

Proof. The first equation follows easily from (2.11) and the fact that β is \mathcal{G} -invariant. To prove the second equation, we start by writing the Haar system as

$$d\rho_t^p = e^{w(p,t)}d\rho_0^p$$

where $\{d\rho_0^p\}_{p \in M}$ is the Haar system at time $t = 0$ and where the \mathcal{G} -invariant function w is defined by $w(p, t) = \int_0^t \beta(p, s) ds$. It follows that a nonnegative cutoff function for time t is given by $\phi_t^2 = e^{-w}\phi_0^2$ where ϕ_0^2 is the nonnegative cutoff function at time $t = 0$. Then, using Equation (2.4), we get

$$\theta = dw + \theta_0$$

where θ_0 is the mean curvature form at time $t = 0$. Differentiating with respect to time gives us equation (2.13). \square

So, as we stated earlier, the smooth Haar system constructed in Section 2.5 is independent of the \mathcal{G} -invariant metric g modulo scaling by smooth positive \mathcal{G} -invariant functions. We will denote the family of these Haar systems by Ξ : an element of Ξ is a Haar system that can be written in the form $\{e^w d\rho^p\}_{p \in M}$ where w is a \mathcal{G} -invariant function and where $\{d\rho^p\}_{p \in M}$ is the Haar system generated by some \mathcal{G} -invariant Riemannian metric g . In particular, if we fix an element $d\rho_0$ of Ξ and if we denote by $C_{\mathcal{G}}^\infty(M)$ the space of \mathcal{G} -invariant smooth functions on M , we get a bijection $C_{\mathcal{G}}^\infty(M) \rightarrow \Xi$ defined by $w \rightarrow e^w d\rho$.

It also follows from the previous Lemma that the \mathcal{G} -invariant de Rham cohomology class of the mean curvature form θ is independent of the \mathcal{G} -invariant metric g . We will denote this class by Θ . Furthermore, one can see that if a representative θ_1 of Θ is written in the form $\theta_1 = \theta_2 + dw$ where θ_2 is the mean curvature form associated to a Haar system $\{d\rho_2\}_{p \in M}$ generated by a metric g , then θ_1 corresponds to the Haar system $\{d\rho_1\}_{p \in M} = \{e^w d\rho_2\}_{p \in M}$ which can be thought as coming from a \mathcal{G} -invariant and $O(n)$ -invariant rescaling of the metric \hat{g} along the $\hat{\sigma}$ -fibers of F_g . A nonnegative cutoff function for the Haar system $\{d\rho_1\}_{p \in M}$ will be said to be compatible with θ_1 and it is clear that, if ϕ_2 is a nonnegative cutoff function for $\{d\rho_2\}_{p \in M}$, then a nonnegative cutoff function for $\{d\rho_1\}_{p \in M}$ is given by $\phi_1^2 = e^{-w}\phi_2^2$. One can then easily deduce the following result.

Corollary 9. *If g is a \mathcal{G} -invariant metric and if θ is any representative of the class Θ , then the integration by parts formula of Lemma 6 is valid for any nonnegative cutoff function ϕ that is compatible with θ .*

We can also adapt the proof of Lemma 6 to show that, given a smooth one-parameter family of \mathcal{G} -invariant metrics g_t , a smooth time dependent \mathcal{G} -invariant function f , a smooth family of Haar systems $\{d\rho_t\}_{p \in M}$ in Ξ and a smooth compatible one-parameter family of nonnegative cutoff functions ϕ_t , the evolution equation of the integral $\int_M f \phi^2 d\mu_g$ is given by

$$\frac{\partial}{\partial t} \int_M f \phi^2 d\mu_g = \int_M \left(\frac{\partial f}{\partial t} + \frac{V}{2} - \beta \right) \phi^2 d\mu_g \quad (2.14)$$

where $V = \text{Tr}_g(v)$, the trace of v with respect to the metric g , and

$$\frac{\partial(d\rho^p)}{\partial t} = \beta(p)d\rho^p.$$

Example 10. Let G be a compact Lie group which acts isometrically and effectively on a connected compact Riemannian manifold (M, g) . Then, we can consider the cross-product groupoid $M \rtimes G$ over M which will be a closed groupoid. The corresponding étale structure comes from equipping G with the discrete topology.

As discussed in [7][Example 4], the Haar system $\{d\rho^p\}_{p \in M}$ generated by the metric g can be described as follows. Let $\{e^i\}$ be a basis of the Lie algebra \mathfrak{g} of G such that the normalized Haar measure is $d\mu_G = \bigwedge_i (e^i)^*$. Let $\{V^i\}$ be the corresponding Killing vector fields on M . The action of V^i on the orthonormal frame bundle F_g breaks up as $V^i \oplus \nabla V^i$ with respect to the decomposition $TF_g = \pi^*TM \otimes TO(n)$ of TF_g into horizontal and vertical subbundles. The Haar system is given by $d\rho^p = \sqrt{\det(\overline{M(p)})} d\mu_G$ where $M = (M_{ij})$ is a function of positive definite matrices defined by

$$M_{ij}(p) = \langle V_i(p), V_j(p) \rangle + \langle \nabla V_i(p), \nabla V_j(p) \rangle$$

The corresponding cutoff function is given by $\phi^2(p) = \frac{1}{\sqrt{\det(M(p))}}$ and the mean curvature form is $\theta = dw$ with $w = \ln \left(\sqrt{\det(M(p))} \right)$. Hence, the class Θ is equal to zero. Also, the family Ξ of Haar systems can be naturally identified with the space of G -invariant smooth functions via the map $C_G^\infty(M) \rightarrow \Xi$ defined by $w \rightarrow e^w d\rho_G$. Finally, Corollary 9 reduces to the integration by parts formula on a weighted manifold with G -invariant quantities.

2.8 \mathcal{G} -paths and \mathcal{G} -invariant vector fields

As before, we let (\mathcal{G}, M, g) be a closed Riemannian groupoid with space of orbits $W = M/\mathcal{G}$. A *smooth \mathcal{G} -path* defined on an interval $[a, b]$ and from a point $x \in M$ to a point $y \in M$ consists of a partition $a = t_0 \leq t_1 \leq \dots \leq t_k = b$ of the interval $[a, b]$ and a sequence $c = (h_0, c_1, h_1, \dots, c_k, h_k)$ where each c_i is a smooth path $c_i : [t_{i-1}, t_i] \rightarrow M$, $h_i \in \mathcal{G}$, $c_i(t_{i-1}) = s(h_{i-1})$, $c_i(t_i) = r(h_i)$, $r(h_0) = x$ and $s(h_k) = y$. The groupoid is said to be *\mathcal{G} -path-connected* if any two points of M can be joined by a \mathcal{G} -path.

Since M is equipped with the $\overline{\mathcal{G}}$ -invariant metric g , there is a natural notion of length of a smooth $\overline{\mathcal{G}}$ -path. This defines a pseudometric d on the space of orbits W given by saying that, for any $x, y \in M$, $d(O_x, O_y)$ is the infimum of the lengths of smooth $\overline{\mathcal{G}}$ -paths joining x to y . We will also assume for now on that the groupoid (\mathcal{G}, M) is *Hausdorff* in the sense of [2, Appendix G Definition 2.10]: the space \mathcal{G} equipped with the étale topology is Hausdorff and for every continuous map $c : (0, 1] \rightarrow \mathcal{G}$ such that $\lim_{t \rightarrow 0} s \circ c$ and $\lim_{t \rightarrow 0} r \circ c$ exist, $\lim_{t \rightarrow 0} c$ exists. Using this property, we can prove the following lemma

Lemma 11. *The pseudometric d on W is a metric. Furthermore, the topology on W induced by d coincides with the quotient topology. Hence, (W, d) is a length space.*

Proof. Suppose that x and y are elements of M such that $d(O_x, O_y) = 0$. Let $\epsilon > 0$ be small enough so that the closed ball $\overline{B}(x, \epsilon)$ is complete as a metric space. It follows from [2, Appendix G Lemma 2.14] that $\overline{B}(x, \epsilon)$ will contain an element of O_y . We can then construct a sequence $y_i \in O_y$ such that $y_i \rightarrow x$. Since O_y is a closed subset of M , we must have $x \in O_y$. Hence, $O_x = O_y$. The pseudometric d is indeed a metric.

Now, suppose that $x \in M$ with orbit $O_x \in W$. Let $B(O_x, b)$ be a ball of radius b with respect to the metric d on W . We want to show that $B(O_x, b)$ is open in W with respect to the quotient topology or, equivalently, that $\sigma^{-1}(B(O_x, b))$ is open in M . If $y \in \sigma^{-1}(B(O_x, b))$, then, by definition of the metric d , there exists a smooth \mathcal{G} -path c of length less than b joining x to y . Then, it is clear that, for $\epsilon > 0$ small enough, the ball $B(y, \epsilon)$ is a subset of $\sigma^{-1}(B(O_x, b))$ since for any $q \in B(y, \epsilon)$, we can join x to q by a smooth \mathcal{G} -path of length less than b obtained by “concatenating” the smooth \mathcal{G} -path c with a geodesic joining y to q . This proves that $\sigma^{-1}(B(O_x, b))$ is open in M and hence $B(O_x, b)$ is open in W with respect to the quotient topology.

Now suppose $U \subset W$ is open with respect to the quotient topology. Then $\sigma^{-1}(U)$ is open in M . Let x be an element of $\sigma^{-1}(U)$ and let $\epsilon > 0$ be small enough so that $\overline{B}(x, \epsilon)$ is complete and contained in $\sigma^{-1}(U)$. Then, it follows again from [2, Appendix G Lemma 2.14] that every orbit in $B(O_x, \epsilon)$ intersects the ball $B(x, \epsilon)$. In other words, $B(O_x, \epsilon) = \sigma(B(x, \epsilon)) \subset U$. Hence, U is open in W with respect to the metric topology. The metric topology coincides with the quotient topology. Finally, it follows from [2, Part I Lemma 5.20] that (W, d) is a length space. This completes the proof. \square

Corollary 12. *(\mathcal{G}, M, g) is \mathcal{G} -path-connected if and only if the quotient space W is connected.*

Let X be a \mathcal{G} -invariant vector field on the manifold M . Hence, under the right action of \mathcal{G} on TM , we have $X_{r(h)} \cdot h = X_{s(h)}$. We can assign to such a vector field a collection of smooth \mathcal{G} -paths defined up to equivalence. Recall that two smooth \mathcal{G} -paths c_1 and c_2 defined on the same interval $[a, b]$ are said to be equivalent if one can pass from one to the other by performing the following operations:

1. Suppose $c = (h_0, c_1, h_1, \dots, c_k, h_k)$ is a \mathcal{G} -path with the subdivision $a = t_0 \leq t_1 \leq \dots \leq t_k = b$. We get a new \mathcal{G} -path by adding a new subdivision point $t'_i \in [t_{i-1}, t_i]$

together with the unit element $g'_i = 1_{c_i(t'_i)}$. We replace c_i by c'_i , g'_i and c''_i , where c'_i and c''_i are the restrictions of c_i to $[t_{i-1}, t'_i]$ and $[t'_i, t_i]$ respectively.

2. We replace a \mathcal{G} -path c by a new path $c' = (h'_0, c'_1, h'_1, \dots, c'_k, h'_k)$ over the same division as follows: for each $i = 1, \dots, k$, choose a smooth path $\gamma_i : [t_{i-1}, t_i] \rightarrow \mathcal{G}$ with respect to the étale structure on (\mathcal{G}, M) such that $s(\gamma_i(t)) = c_i(t)$ and define $c'_i(t) = r(\gamma_i(t))$, $h'_i = \gamma_i(t_i)h_i\gamma_{i+1}(t_i)^{-1}$ for $i = 1, \dots, k-1$, $h'_0 = h_0h_1^{-1}(t_0)$ and $h'_k = \gamma_k(t_k)h_k$.

The smooth \mathcal{G} -paths defined by a \mathcal{G} -invariant vector field X will be of the form $c = (h_0, c_1, h_1, \dots, c_k, h_k)$ where, for each $i = 1, \dots, k$, the smooth path $c_i : [t_{i-1}, t_i] \rightarrow M$ is an integral curve of the vector field X . It is also clear that if $\alpha_i : U \rightarrow M$ is a local section of s at h_i , then, for some small enough $\epsilon > 0$, the path $c'_i : [t_{i-1}, t_i + \epsilon] \rightarrow M$ given by

$$c'_i(t) = \begin{cases} c_i(t) & \text{if } t_{i-1} \leq t \leq t_i; \\ r \circ \alpha_i \circ c_{i+1}(t), & \text{if } t_i \leq t \leq t_i + \epsilon. \end{cases}$$

is an integral curve of the vector field X . A \mathcal{G} -path c generated by X and defined on an interval $[a, b]$ will induce a continuous path $\tilde{c} : [a, b] \rightarrow W$. Any other smooth \mathcal{G} -path generated by X and defined on the same interval $[a, b]$ will induce the same path in W if and only if it is equivalent to c . With abuse of terminology, we will also call these induced paths the integral curves of the vector field X on W and the image of these curves will be called the flow lines of X on W . It is clear that there is a unique flow line passing through any element of W . The following lemma is a simple generalization of the ‘‘Escape Lemma’’ for vector fields on smooth manifolds.

Lemma 13. *Suppose γ is an integral curve of X on W . If the maximal domain of definition of γ is not all of \mathbb{R} , then the image of γ cannot lie in any compact subset of W .*

Proof. Let (a, b) denote the maximal domain of γ and assume that $b < \infty$ but $\gamma(a, b)$ lies in a compact set $K \subset W$. We will show that γ can be extended past b , thus contradicting the maximality of (a, b) . The case $a > -\infty$ is similar. Given any sequence of times $\{t_j\}$ such that $t_j \nearrow b$, the sequence $\{\gamma(t_j)\}$ lies in K . After passing to a subsequence, we may assume that $\{\gamma(t_j)\}$ converges to the orbit O_p for some $p \in M$. By [2, Appendix G Lemma 2.14], we can choose $\epsilon > 0$ small enough so that the closed ball $\overline{B}(p, \epsilon)$ is complete and such that the points $\gamma(t_j)$ will have a representative in $B(p, \frac{\epsilon}{2})$ for j large enough.

Since X is \mathcal{G} -invariant, we can view $|X| = |X|_g$ as a function on W . Since K is compact, we will have $|X| \leq C$ on $\sigma^{-1}(K)$ for some positive constant C . So, if $c = (h_0, c_1, h_1, \dots, c_k, h_k)$ is a \mathcal{G} -path representing γ on any subinterval $[d, e]$ of (a, b) , we will have $|\dot{c}_i(t)| \leq C$ for any $t \in [t_{i-1}, t_i]$. Let j_0 large enough so that the point $\gamma(t_{j_0})$ has a representative in $B(p, \frac{\epsilon}{2})$ and such that $C \cdot (b - t_{j_0}) \leq \frac{\epsilon}{2}$. We now choose a smooth \mathcal{G} -path $c = (h_0, c_1, h_1, \dots, c_k, h_k)$ representing γ on the interval $[0, t_{j_0}]$. There exists a point q in $O_{c_k(t_{j_0})}$ which lies in $B(p, \frac{\epsilon}{2})$. Let c' be the integral curve of X such that $c'(t_{j_0}) = q$. This integral curve must be defined on an interval bigger than $[t_{j_0}, b)$. Otherwise, we would have

$\text{Length}(c') \leq C \cdot (b - t_{j_0}) \leq \frac{\epsilon}{2}$ and thus $c'([t_{j_0}, b))$ would lie in $\overline{B}(p, \epsilon)$, thus contradicting the “Escape Lemma”. So, c' is defined on $[t_{j_0}, b + \epsilon')$ for some $\epsilon' > 0$. We can then extend the smooth \mathcal{G} -path c by “concatenating” it with the path c' . This proves that γ can be extended past b , contradicting the maximality of (a, b) . \square

Corollary 14. *If the space of orbits W is compact, the integral curves of X on W are defined for all time. In fact, for any $p \in M$ and $t \in \mathbb{R}$, there exists a smooth \mathcal{G} -path $c = (h_0, c_1, h_1, \dots, c_k, h_k)$ generated by X , defined on the interval $[0, t]$ if $t > 0$ (or $[t, 0]$ if $t < 0$) and satisfying $r(h_0) = p$ (or $s(h_k) = p$).*

Assume that W is compact and connected. We will show that the \mathcal{G} -invariant vector field X generates a self-equivalence for each $t \in \mathbb{R}$. First, we recall the notion. Let \mathbf{H} be the pseudogroup of an effective étale groupoid (\mathcal{G}, M) . So the elements of \mathbf{H} are the locally defined diffeomorphisms of the form $r \circ \alpha$ where α is a local section of s . A differentiable equivalence between two effective étale groupoids (\mathcal{G}_1, M_1) and (\mathcal{G}_2, M_2) is a maximal collection $\Psi = \{\psi_i : U_i \rightarrow V_i\}_{i \in \Lambda}$ of diffeomorphisms of open subsets of M_1 to open subsets of M_2 such that:

1. $\{U_i\}_{i \in \Lambda}$ and $\{V_i\}_{i \in \Lambda}$ are open coverings of M_1 and M_2 respectively.
2. If $\psi \in \Psi, \tau_1 \in \mathbf{H}_1$ and $\tau_2 \in \mathbf{H}_2$, then $\tau_2 \circ \psi \circ \tau_1 \in \Psi$.
3. If $\psi, \psi' \in \Psi, \tau_1 \in \mathbf{H}_1$ and $\tau_2 \in \mathbf{H}_2$, then $\psi' \circ \tau_1 \circ \psi^{-1} \in \mathbf{H}_2$ and $\psi^{-1} \circ \tau_2 \circ \psi' \in \mathbf{H}_1$.

It is clear that a differentiable equivalence induces a weak equivalence in the sense of [18][Chapter 5.4]. If $(\mathcal{G}_1, M_1) = (\mathcal{G}_2, M_2)$, we say that Ψ is a self-equivalence. Now, suppose that (\mathcal{G}, M, g) is a closed Riemannian groupoid with compact connected orbit space and let $\Psi = \{\psi\}_{i \in \Lambda}$ be a self-equivalence. We can define the pull back $\Psi^*\omega$ of a \mathcal{G} -invariant tensor ω as follows. For any $x \in M$, choose a $\psi_i \in \Psi$ such that x is in the domain of ψ_i . We set $(\Psi^*\omega)_x = (\psi_i^*\omega)_x$. This does not depend on the choice of ψ_i . Furthermore, the tensor $\Psi^*\omega$ is \mathcal{G} -invariant. Applying this to the \mathcal{G} -invariant Riemannian metric g , we obtain another \mathcal{G} -invariant Riemannian metric $\Psi^*g = \tilde{g}$. If we denote by d and \tilde{d} the metrics generated by g and \tilde{g} on W , we see that Ψ induces an isometry $(W, \tilde{d}) \rightarrow (W, d)$ which we will also call Ψ for simplicity. This isometry satisfies $\Psi^*d\eta(g) = d\eta(\tilde{g})$. Indeed, note that the self equivalence Ψ induces a differentiable equivalence between the groupoids $(F_{\tilde{g}} \rtimes \mathcal{G}, F_{\tilde{g}})$ and $(F_g \rtimes \mathcal{G}, F_g)$ where $F_{\tilde{g}}$ and F_g are the orthonormal frame bundles generated by \tilde{g} and g respectively. This differentiable equivalence will pull back the Riemannian metric on F_g to the Riemannian metric on $F_{\tilde{g}}$ and it will be “equivariant” with respect to the $O(n)$ -action on these spaces. Furthermore, it will induce an isometry $\tilde{Z} \rightarrow Z$ where \tilde{Z} and Z are the spaces of orbits of $F_{\tilde{g}}$ and F_g under the groupoid action. Now, if we recall the definition of the measure $d\eta(g)$, we see that $\Psi^*d\eta(g) = d\eta(\tilde{g})$.

Finally, based on the definition of the family Ξ , we see that we can define the pullback of an element of Ξ under a self-equivalence Ψ . More precisely, if we write an element ρ of Ξ as $d\rho = e^w d\rho_g$ where w is a smooth \mathcal{G} invariant function and where $d\rho_g$ is the Haar system

generated by an invariant metric g , then $\Psi^*d\rho = e^{\Psi^*w}d\rho_{\Psi^*g}$. It is clear that, if θ is the mean curvature form associated to $d\rho$, then $\Psi^*\theta$ is the mean curvature form for $\Psi^*d\rho$. We can also see that, if ϕ is any nonnegative cutoff function for a Haar system $\rho \in \Xi$ and if $\underline{\phi}$ is any nonnegative cutoff function for the pullback $\Psi^*d\rho$, then

$$\int_M f \phi^2 d\mu_g = \int_M \Psi^* f \underline{\phi}^2 d\mu_{\Psi^*g}$$

for any \mathcal{G} -invariant smooth function f and any \mathcal{G} -invariant smooth metric g .

Given a \mathcal{G} -invariant vector field X , we get a one-parameter family of self equivalences in an obvious way. First, for each $t \in \mathbb{R}$, we denote by γ_t the flow map of X on M at time t . More precisely, a point $p \in M$ is in the domain of γ_t if there is, in the regular sense, an integral curve c of X on M defined on $[0, t]$ (assuming $t \geq 0$) and starting at p . We then take $\gamma_t(p) = c(t)$. The domain of γ_t is an open subset of M . Now suppose $c = (h_0, c_1, h_1, \dots, c_k, h_k)$ is a smooth \mathcal{G} -path generated by X , defined on the interval $[0, t]$, starting at a point $p \in M$ and with the subdivision $0 = t_0 \leq t_1 \leq \dots \leq t_k = t$. For each i , we let τ_i be the diffeomorphisms of open subsets given by $r \circ \alpha_i$ where α_i is a local section of s at h_i . We then define a diffeomorphism ψ_c on an open neighborhood of p by

$$\psi_c = \tau_0^{-1} \circ \gamma_{t_1} \circ \tau_1^{-1} \circ \gamma_{(t_2-t_1)} \circ \dots \circ \tau_{k-1}^{-1} \circ \gamma_{(t-t_{k-1})} \circ \tau_k^{-1}.$$

The diffeomorphism ψ_c is just a composition of flow maps of the vector field X and of local diffeomorphisms generated by the groupoid. It is clear that $\psi_c(p) = s(h_k)$. The collection of all such locally defined diffeomorphism ψ_c defines a self equivalence Ψ_t . It is also clear that $\Psi_0 = \mathbf{H}$ and, for any $t_1, t_2 \in \mathbb{R}$, we have $\Psi_{t_1+t_2} = \Psi_{t_1} \circ \Psi_{t_2}$ where $\Psi_{t_1} \circ \Psi_{t_2}$ denotes the collection of locally defined diffeomorphisms of the form $\psi_c \circ \psi_{c'}$ where $\psi_c \in \Psi_{t_1}$ and $\psi_{c'} \in \Psi_{t_2}$. We can easily show that the formula

$$\frac{d}{dt} \Big|_{t=t_0} \Psi_t^* \omega = \Psi_{t_0}^* (\mathcal{L}_X \omega)$$

holds for any \mathcal{G} -invariant tensor ω . Finally, note that the results listed in this section can easily be extended to the case of smooth time dependent \mathcal{G} -invariant vector fields $X = X(t)$.

Chapter 3

Maximum principles

3.1 The weak maximum principle

In this chapter, we derive maximum principles for groupoids and prove Theorems 3 and 4. We first start by proving a weak maximum principle

Lemma 15 (Weak maximum principle for scalars on groupoids). *Suppose $g(t)$, $0 \leq t \leq T < \infty$, is a smooth one-parameter family of metrics on a smooth étale groupoid (\mathcal{G}, M) with compact connected orbit space $W = M/\mathcal{G}$. Let $X(t)$ be a smooth \mathcal{G} -invariant time-dependent vector field on M and let $F : \mathbb{R} \times [0, T] \rightarrow \mathbb{R}$ be a smooth function. Suppose that $u : M \times [0, T] \rightarrow \mathbb{R}$ is a smooth \mathcal{G} -invariant function which solves*

$$\frac{\partial u}{\partial t} \leq \Delta_{g(t)} u + \langle X(t), \nabla u \rangle + F(u, t). \quad (3.1)$$

Suppose further that $\phi : [0, T] \rightarrow \mathbb{R}$ solves

$$\begin{cases} \frac{d\phi}{dt} &= F(\phi(t), t) \\ \phi(0) &= \alpha \in \mathbb{R} \end{cases} \quad (3.2)$$

If $u(\cdot, 0) \leq \alpha$, then $u(\cdot, t) \leq \phi(t)$ for all $t \in [0, T]$.

Proof. The proof is basically the same as the corresponding proof for closed manifolds. For $\epsilon > 0$, we consider the ODE

$$\begin{cases} \frac{d\phi_\epsilon}{dt} &= F(\phi_\epsilon(t), t) + \epsilon \\ \phi_\epsilon(0) &= \alpha + \epsilon \in \mathbb{R}, \end{cases} \quad (3.3)$$

for a new function $\phi_\epsilon : [0, T] \rightarrow \mathbb{R}$. Using basic ODE theory, we see that the existence of ϕ asserted in the hypotheses and the fact that $T < \infty$ imply that for some $\epsilon_0 > 0$ there exists a solution ϕ_ϵ on $[0, T]$ for any $0 < \epsilon \leq \epsilon_0$. Furthermore, $\phi_\epsilon \rightarrow \phi$ uniformly as $\epsilon \rightarrow 0$. Thus, it suffices to show that $u(\cdot, t) < \phi_\epsilon(t)$ for all $t \in [0, T]$ and arbitrary $\epsilon \in (0, \epsilon_0)$.

If this were not true, then we could choose $\epsilon \in (0, \epsilon_0)$ and $t_0 \in (0, T]$ where $u(\cdot, t_0) < \phi_\epsilon(t_0)$ fails. We may assume that t_0 is the earliest such time, and pick $x \in M$ such that $u(x, t_0) = \phi_\epsilon(t_0)$. Indeed, such a pair (x, t_0) exists because the \mathcal{G} -invariant function u descends to a continuous function $\tilde{u} : W \times [0, T] \rightarrow \mathbb{R}$. Based on the hypothesis that W is compact and connected, we can find t_0 such that it is the earliest time where $\tilde{u}(\cdot, t_0) < \phi_\epsilon(t_0)$ fails and pick an orbit O_x so that $\tilde{u}(O_x, t_0) = \phi_\epsilon(t_0)$. This gives the required pair (x, t_0) . Now, we know that $u(x, s) - \phi_\epsilon(s)$ is negative for $s \in [0, t_0)$ and zero for $s = t_0$. Hence, we must have

$$\frac{\partial u}{\partial t}(x, t_0) - \phi'_\epsilon(t_0) \geq 0.$$

Moreover, since $u(\cdot, t_0)$ achieves a maximum at x , we have $\nabla u(x, t_0) = 0$ and $\Delta u(x, t_0) \leq 0$.

Combining these facts with the inequality (3.1) for u and equation (3.3) for ϕ_ϵ , we get the contradiction

$$\begin{aligned} 0 &\geq \left[\frac{\partial u}{\partial t} - \Delta_{g(t)} u - \langle X, \nabla u \rangle - F(u, \cdot) \right] (x, t_0) \\ &\Rightarrow 0 \geq \phi'_\epsilon(t_0) - F(\phi_\epsilon(t_0), t_0) = \epsilon > 0. \end{aligned}$$

This completes the proof. □

3.2 Application: Immortal Ricci flow solutions on closed manifolds

In the hypothesis of Theorem 3, we assume a uniform bound on the diameter and on the curvature of the closed Riemannian manifolds $(M, g(t))$. Before starting the proof of Theorem 3, we first consider the case where we also have a uniform lower bound on the injectivity radius. This simpler case will give us an outline of the proof of Theorem 3. Given a Ricci flow solution $g(t)$ on a closed manifold M , the injectivity radius corresponding to each metric $g(t)$ will be denoted by $\text{inj}(M; g(t))$.

Theorem 16. *Given $n \in \mathbb{N}$ and numbers $K, D, \iota > 0$, there exists a function $F_{n,K,D,\iota} : [0, \infty) \rightarrow [0, \infty)$ with the following properties*

1. $\lim_{t \rightarrow \infty} F_{n,K,D,\iota}(t) = 0$.
2. *If $g(t)$ is an immortal solution to the Ricci flow on a closed manifold M of dimension n such that $\text{inj}(M; g(t)) \geq \iota$, $\|Rm\|_\infty(t) \leq K$ and $\text{diam}(M; g(t)) \leq D$ for all $t \geq 0$, then $\|Ric\|_\infty(t) \leq F_{n,K,D,\iota}(t)$ for all $t \geq 0$.*

Proof. For every $t \geq 0$, we set $F_{n,K,D,\iota}(t)$ to be the supremum of $\|Ric\|_\infty(t)$ over all immortal solutions $(M^n, g(\cdot))$ satisfying the conditions of the given hypothesis. This function is well

defined since the bound on the curvature gives us a bound on the Ricci curvature. We just need to prove that $\lim_{t \rightarrow \infty} F_{n,K,D,\iota}(t) = 0$.

Suppose that this did not hold. Then, for some $\epsilon > 0$, there is a sequence $t_i \rightarrow \infty$ such that $F_{n,K,D,\iota}(t_i) > \epsilon$. This implies that there exists a sequence (M, g_i, x_i) of pointed immortal solutions to the Ricci flow such that $|Ric|(x_i, t_i) > \epsilon$. Consider the new sequence (M, \tilde{g}_i, x_i) where $\tilde{g}_i(t) = g_i(t + t_i)$ for $t \in [-t_i, \infty)$. Since we have a uniform bound on the curvature and a uniform lower bound on the injectivity radius, we can apply the Hamilton-Cheeger-Gromov compactness theorem [9]. After passing to a subsequence, we can assume that the sequence $(M_i, \tilde{g}_i(t), x_i)$ converges to an eternal Ricci flow solution $\hat{g}(t)$ on a pointed n -dimensional manifold (\hat{M}, \hat{x}) . The uniform bound on the diameters of (M_i, \tilde{g}_i) imply that \hat{M} is a closed manifold.

The eternal Ricci flow solution $\hat{g}(t)$ must be Ricci flat. Indeed, based on [4, Lemma 2.18], the solution is either Ricci flat or its scalar curvature is positive. Since \hat{M} is closed, the latter cannot happen for there would be a finite time singularity and thus would contradict the fact that $\hat{g}(t)$ is an eternal solution. Therefore, the Ricci flow solution \hat{g} satisfies $Ric(\hat{g}(t)) \equiv 0$ as claimed. On the other hand, the choice of the points $x_i \in M_i$ implies that $|Ric|(\hat{x}, 0) > \epsilon$. We reach a contradiction. The function $F_{n,K,D,\iota}(t)$ must satisfy $\lim_{t \rightarrow \infty} F_{n,K,D,\iota}(t) = 0$. This completes the proof. \square

The proof of the previous Theorem is a simple application of the following results.

1. Hamilton's compactness theorem for solutions to the Ricci flow.
2. An eternal solution to the Ricci flow on a closed manifold is Ricci flat. This result is a simple application of the weak and strong maximum principles.

In the proof of Theorem 3, we will use the more general version of Hamilton's compactness theorem which was introduced by J. Lott in [17] in the bounded curvature case. We will again construct a pointed sequence of Ricci flow solutions. But in this case, after passing to a subsequence if necessary, the pointed sequence will converge, in the sense of [17, Section 5], to an eternal Ricci flow solution on a pointed étale groupoid with compact connected orbit space. The result will then follow from the following lemma

Lemma 17. *Every eternal solution to the Ricci flow on a smooth étale groupoid with compact connected orbit space is Ricci flat.*

Proof. Suppose $(\mathcal{G}, M, g(t))$ is an eternal solution to the Ricci flow on a smooth étale groupoid with compact connected orbit space W . It follows from Lemma 15 that the basic evolution equations for the Ricci flow which are derived by applying the weak maximum principle on a closed manifold are still valid in the case of the Ricci flow on a smooth étale groupoid with compact connected orbit space. Hence, we can show that for any solution to the Ricci flow defined on an interval $[0, T]$ the scalar curvature satisfies

$$R \geq -\frac{n}{2t} \tag{3.4}$$

for $t \in [0, T]$. Fix $c \in \mathbb{R}$. By translating time, we see that the above inequality implies that

$$R(x, t) \geq -\frac{n}{2(t-c)}$$

for $(x, t) \in M \times (c, \infty)$. Taking the limit as $c \rightarrow -\infty$, we conclude that $R(x, t) \geq 0$ for all $(x, t) \in M \times \mathbb{R}$.

Now suppose that $R(x_0, t_0) > 0$ for some $(x_0, t_0) \in M \times \mathbb{R}$. By the strong maximum principle ([4, Corollary 6.55]), this implies that $R(x, t) > 0$ for all $t \geq t_0$ and for all x in the connected component of M which contains x_0 . In fact, this holds for all $x \in M$. Indeed, since W is connected, it follows that for any $x \in M$, one can find a sequence of points $x_0 = p_0, q_0, p_1, q_1, \dots, p_n, q_n = x$ such that p_i and q_i are in the same connected component of M , and q_{i-1} and p_i are in the same \mathcal{G} -orbit for each i . Since R is \mathcal{G} -invariant, it follows that $R(p_1, t_0) = R(q_0, t_0) > 0$. So, by the maximum principle, $R(x, t) > 0$ for every $t \geq t_0$ and for all x in the connected component of M which contains p_1 . In particular, $R(q_1, t) > 0$ for all $t \geq t_0$. After applying this argument a finite number of times, we deduce that $R(x, t) > 0$ for all $t \geq t_0$. This proves the claim.

The \mathcal{G} -invariant function R induces a continuous function \tilde{R} on W which satisfies $\tilde{R}(\cdot, t) > 0$ for $t \geq t_0$. Since W is compact, there exists some $\alpha > 0$ such that $\tilde{R}(\cdot, t_0) \geq \alpha$. Hence the scalar curvature satisfies $R(\cdot, t_0) \geq \alpha$. Just as in the closed manifold case, we can apply Lemma 15 to show that we have a finite time singularity. This contradicts the hypothesis that the Ricci flow solution is eternal. Therefore, we must have $R \equiv 0$. Based on the evolution equation of the scalar curvature

$$\frac{\partial R}{\partial t} = \Delta R + 2|Ric|^2$$

we deduce that $Ric \equiv 0$. This completes the proof. \square

Proof. Theorem 3. We define the function $F_{n,K,D}(t)$ as before: for every $t \geq 0$, we set $F_{n,K,D}(t)$ to be the supremum of $\|Ric\|_\infty(t)$ over all immortal solutions $(M^n, g(\cdot))$ satisfying the conditions of the given hypothesis. We just need to prove that $\lim_{t \rightarrow \infty} F_{n,K,D}(t) = 0$. This will again be proved by contradiction.

Suppose that this did not hold. Then, for some $\epsilon > 0$, there is a sequence $t_i \rightarrow \infty$ such that $F_{n,K,D}(t_i) > \epsilon$. This implies that there exists a sequence (M, g_i, x_i) of pointed immortal solutions to the Ricci flow such that $|Ric|(x_i, t_i) > \epsilon$. Consider the new sequence (M, \tilde{g}_i, x_i) where $\tilde{g}_i(t) = g_i(t + t_i)$ for $t \in [-t_i, \infty)$. After passing to a subsequence, we can assume that the sequence $(M_i, \tilde{g}_i(t), x_i)$ converges to an eternal Ricci flow solution $\hat{g}(t)$ on a pointed n -dimensional closed étale groupoid $(\hat{\mathcal{G}}, \hat{M}, O_{\hat{x}})$ (see [17, Theorem 5.12]). The uniform bound on the diameters of (M_i, \tilde{g}_i) imply that the quotient space W equipped with the metric induced by $g(t)$ is a locally compact complete length space with finite diameter. So, by the Hopf-Rinow theorem ([2, Part I Proposition 3.7]), W is a compact connected space. We can now apply Lemma 17 to deduce that the eternal Ricci flow solution on the closed groupoid $(\hat{\mathcal{G}}, \hat{M}, O_{\hat{x}})$ is Ricci flat. On the other hand, the choice of the points $x_i \in M_i$

implies that $|Ric| > \epsilon$ on $O_{\hat{x}}$. We reach a contradiction. The function $F_{n,K,D}(t)$ must satisfy $\lim_{t \rightarrow \infty} F_{n,K,D}(t) = 0$. This completes the proof. \square

We also give the proof of Theorem 4.

Proof. Theorem 4. The proof will again be by contradiction. Suppose the solution $g(t)$ satisfies $\sup_{M \times [0, \infty)} t|Rm| = \infty$. As outlined in [4, Chapter 8, Section 2], we can construct a sequence of pointed solutions to the Ricci flow which converges to an eternal solution to the Ricci flow on an étale groupoid. We first choose a sequence of times $T_i \rightarrow \infty$ and then we choose a sequence $(x_i, t_i) \in M \times (0, T_i)$ such that

$$t_i(T_i - t_i)|Rm|(x_i, t_i) = \sup_{M \times [0, T_i]} t(T_i - t)|Rm|(x, t)$$

For each i , we set $K_i = |Rm|(x_i, t_i)$ and we define the pointed Ricci flow solution (M, g_i, x_i) by $g_i(t) = K_i g(t_i + \frac{t}{K_i})$. If we set $\alpha_i = -t_i K_i$ and $\omega_i = (T_i - t_i) K_i$, then $\alpha_i \rightarrow -\infty$ and $\omega_i \rightarrow \infty$, and that the curvature $Rm(g_i)$ of the metric g_i satisfies

$$|Rm(g_i)|(x, t) \leq \frac{\alpha_i}{\alpha_i - t} \frac{\omega_i}{\omega_i - t}$$

for $(x, t) \in M \times [\alpha_i, \omega_i]$. Furthermore, the condition (1.5) implies that

$$\text{diam}(M; g_i(0))^2 = K_i \text{diam}(M; g(t_i))^2 < C$$

Hence, after passing to a subsequence if necessary, the pointed sequence (M, g_i, x_i) converges to an eternal Ricci flow solution $\hat{g}(t)$ on a pointed 3-dimensional étale groupoid $(\hat{\mathcal{G}}, \hat{M}, O_{\hat{x}})$ with compact connected orbit space. By construction, $|Rm(\hat{g})|(\cdot, 0) = 1$ on the orbit $O_{\hat{x}}$. But, by Lemma 17, the solution $\hat{g}(t)$ is flat. We get a contradiction. Therefore, the immortal solution $g(t)$ must be of type III. \square

Theorem 4 implies the following result.

Corollary 18. *Suppose $g(t)$ is an immortal Ricci flow solution on a closed 3-dimensional manifold M . Then the following are equivalent:*

1. $\text{diam}(M; g(t)) = O(t^{\frac{1}{2}})$ and $\|Rm\|_{\infty}(t) = O(t^{-1})$
2. $\text{diam}(M; g(t))^2 \|Rm\|_{\infty}(t) < C$ for some constant $C > 0$

Therefore, if $g(t)$ is an immortal Ricci flow solution on a closed connected orientable 3-manifold that satisfies

$$\text{diam}(M; g(t))^2 \|Rm\|_{\infty}(t) < C$$

then M is irreducible, aspherical and its geometric decomposition contains a single geometric piece ([16]/Theorem 1.2).

Proof. It is clear that (1) \Rightarrow (2). Now suppose that (2) holds. One can show that the variation of the length of a fixed smooth path γ in M is given by

$$\frac{dL_{g(t)}(\gamma)}{dt} = - \int_{\gamma} Ric \left(\frac{d\gamma}{ds}, \frac{d\gamma}{ds} \right) ds$$

where s is the arc length parameter. If we apply the condition (2), then we will have

$$\frac{dL_{g(t)}(\gamma)}{dt} \leq \frac{C}{diam(M; g(t))^2} L_{g(t)}.$$

This implies that

$$\frac{d(diam(M; g(t)))}{dt} \leq \frac{C}{diam(M; g(t))}$$

from which we deduce that $diam(M; g(t)) = O(t^{\frac{1}{2}})$. Combining this with Theorem 4, we conclude that (2) implies (1). \square

3.3 A maximum principle for Riemannian groupoids with complete noncompact space of orbits

We start by stating Theorem 5 in a more precise way.

Theorem 19 (Theorem 5). *Let $(\mathcal{G}, M, g(t))$, $0 \leq t < T$ be a smooth one-parameter family of Riemannian groupoids. Let ρ_t be a smooth family of Haar systems with corresponding mean curvature form $\theta = \theta_t$ and time variation given by $\frac{\partial \rho}{\partial t} = \beta d\rho$. Suppose that for each $t \in [0, T)$ the orbit space W equipped with the induced length structure d_t is a complete metric space.*

Let $X(\cdot, t)$ be a \mathcal{G} -invariant time-dependent vector field on M and let $b(\cdot, t)$ be a \mathcal{G} -invariant time-dependent function on M which satisfy

$$\begin{aligned} \sup_{M \times [0, T)} |X|_{g(t)} &\leq C_1 \\ \sup_{M \times [0, T)} |b| &\leq C_1 \end{aligned} \tag{3.5}$$

for some constant $C_1 > 0$. Suppose also there exists a constant $C_2 > 0$ such that the metric $g(t)$, the mean curvature form θ and the function β satisfy

$$-C_2 g \leq \frac{\partial g}{\partial t} \leq C_2 g \tag{3.6}$$

and

$$\begin{aligned} |\beta| &\leq C_2 \\ |\theta|_{g(t)} &\leq C_2 \end{aligned} \tag{3.7}$$

for every $(x, t) \in M \times [0, T)$. Let $u \in C(M \times [0, T)) \cap C^{2,1}(M \times (0, T))$ be a \mathcal{G} -invariant function that satisfies the inequality

$$\frac{\partial u}{\partial t} \leq \Delta u + \langle X, \nabla u \rangle + bu \tag{3.8}$$

under the condition

$$\int_0^T \int_M u_+^2(x, t) e^{-cd_t(O_{x_0}, O_x)^2} \phi^2(x) d\mu_g dt < \infty \tag{3.9}$$

for some constant $c > 0$ where x_0 is a fixed point in M , $u_+ = \max(u, 0)$ and ϕ is a cutoff function for the Haar system ρ . If $u(\cdot, 0) \leq 0$, then $u(\cdot, t) \leq 0$ for all $t \in [0, T)$.

Proof. Our proof is based on the arguments in [13, Section 1]. For each $t \in [0, T)$, the function $r_t(x) = d_t(O_{x_0}, O_x)$ is a Lipschitz function. It is differentiable almost everywhere on M and satisfies $|\nabla r_t|_{g_t} \leq 1$. We construct the \mathcal{G} -invariant function

$$h(x, t) = -\frac{r_0(x)^2}{4(2\alpha - t)}$$

on $M \times [0, \alpha]$ where the constant $0 < \alpha < C_2^{-1} \log(\frac{9}{8})$ will be determined later. The function h satisfies

$$\frac{\partial h}{\partial t} + |\nabla h|_{g(0)}^2 \leq 0 \tag{3.10}$$

almost everywhere in $M \times [0, \alpha]$. Now by (3.6),

$$\left| \frac{\partial}{\partial t} (d\mu_g) \right| \leq \frac{nC_2}{2} d\mu_g \tag{3.11}$$

and

$$\begin{aligned} e^{-C_2 t} g(0) &\leq g(t) \leq e^{C_2 t} g(0) \\ \Rightarrow e^{-C_2 t} g^{-1}(0) &\leq g^{-1}(t) \leq e^{C_2 t} g^{-1}(0) \end{aligned}$$

Then, the inequality (3.10) implies

$$\frac{\partial h}{\partial t} + e^{-C_2\alpha} |\nabla h|_{g(t)}^2 \leq 0 \quad (3.12)$$

almost everywhere in $M \times [0, \alpha]$. Furthermore, since the metric d_t on W is defined by the infimum of the length of \mathcal{G} -paths, the condition (3.6) also implies

$$e^{-\frac{C_2T}{2}} r_0(x) \leq r_t(x) \leq e^{\frac{C_2T}{2}} r_0(x) \quad (3.13)$$

We now choose a smooth function $\kappa : \mathbb{R} \rightarrow \mathbb{R}$ such that $0 \leq \kappa \leq 1$, $\kappa(y) = 1$ for all $y \leq 0$, $\kappa(y) = 0$ for all $y \geq 1$ and $-2 \leq \kappa'(y) \leq 0$ for any $y \in \mathbb{R}$. For any $R \leq 1$, we define the function $\kappa_R : \mathbb{R} \rightarrow \mathbb{R}$ by $\kappa_R(x) = \kappa(r_0(x) - R)$. Then $|\nabla \kappa_R|_{g(t)} \leq 2e^{\frac{C_2T}{2}}$ on $M \times [0, T]$. Furthermore, η_R is \mathcal{G} -invariant and the induced function on the space W has support in the ball $B(O_{x_0}; R+1)$ with respect to the metric d_0 on W . The space W is locally compact since it is the quotient of the locally compact space M by the \mathcal{G} -action. Thus, if we apply the Hopf-Rinow theorem for locally compact complete length spaces (see [2, Part I Proposition 3.7]), we deduce that the induced function κ_R on W has compact support. Based on Section 2.6 and 2.7, this implies that the integral

$$\int_M \kappa_R^2 e^h u_+^2 \phi^2 d\mu_g$$

exists for each $t \in [0, T]$. Furthermore, we can apply the integration by parts formula that we derived in these sections. If we differentiate the above integral with respect to time, we obtain

$$\begin{aligned} \frac{\partial}{\partial t} \left(\int_M \kappa_R^2 e^h u_+^2 \phi^2 d\mu_g \right) &= \int_M \kappa_R^2 e^h \frac{\partial h}{\partial t} u_+^2 d\rho_t + 2 \int_M \kappa_R^2 e^h u_+ \frac{\partial u}{\partial t} \phi^2 d\mu_g \\ &\quad + \int_M \eta_R^2 e^h u_+^2 (q - \beta) \phi^2 d\mu_g \end{aligned} \quad (3.14)$$

where the function q is given by $\frac{\partial}{\partial t}(d\mu_g) = qd\mu_g$. Based on (3.10), the first integral on the right hand side of (3.14) satisfies the inequality

$$\int_M \kappa_R^2 e^h \frac{\partial h}{\partial t} u_+^2 \phi^2 d\mu_g \leq -e^{-C_2\alpha} \int_M \kappa_R^2 e^h |\nabla h|^2 u_+^2 \phi^2 d\mu_g. \quad (3.15)$$

Based on (3.7) and (3.11), the third integral satisfies

$$\int_M \kappa_R^2 e^h u_+^2 (q - \beta) \phi^2 d\mu_g \leq \frac{(n+2)C_2}{2} \int_M \kappa_R^2 e^h u_+^2 \phi^2 d\mu_g \quad (3.16)$$

Finally, for the second integral on the right hand side of (3.14), we apply (3.8).

$$\begin{aligned} \int_M \kappa_R^2 e^h u_+ \frac{\partial u}{\partial t} \phi^2 d\mu_g &\leq \int_M \kappa_R^2 e^h u_+ \Delta u \phi^2 d\mu_g + \int_M \langle \nabla u_+, X \rangle \kappa_R^2 e^h u_+ \phi^2 d\mu_g \\ &\quad + \int_M \kappa_R^2 e^h b u_+^2 \phi^2 d\mu_g \end{aligned}$$

We now integrate by parts and obtain

$$\begin{aligned} \int_M \kappa_R^2 e^{hu_+} \frac{\partial u}{\partial t} \phi^2 d\mu_g &\leq -2 \int_M \langle \nabla \kappa_R, \nabla u_+ \rangle \kappa_R e^{hu_+} \phi^2 d\mu_g - \int_M |\nabla u_+|^2 \kappa_R^2 e^h \phi^2 d\mu_g \\ &\quad - \int_M \langle \nabla h, \nabla u_+ \rangle \kappa_R^2 e^{hu_+} \phi^2 d\mu_g + \int_M \langle \nabla u_+, \theta \rangle \kappa_R^2 e^{hu_+} \phi^2 d\mu_g \\ &\quad + \int_M \langle \nabla u_+, X \rangle \kappa_R^2 e^{hu_+} \phi^2 d\mu_g + \int_M \kappa_R^2 e^{hu_+} \phi^2 d\mu_g. \end{aligned}$$

Then we apply the conditons (3.5) and (3.7).

$$\begin{aligned} 2 \int_M \kappa_R^2 e^{hu_+} \frac{\partial u}{\partial t} \phi^2 d\mu_g &\leq \bar{C}_1 \int_M |\nabla u_+|^2 \kappa_R^2 e^{hu_+} \phi^2 d\mu_g + 2C_1 \int_M \kappa_R^2 e^{hu_+} \phi^2 d\mu_g \\ &\quad + 4 \int_M |\nabla \kappa_R| |\nabla u_+| \kappa_R e^{hu_+} \phi^2 d\mu_g - 2 \int_M |\nabla u_+|^2 \kappa_R^2 e^h \phi^2 d\mu_g \\ &\quad + 2 \int_M |\nabla h| |\nabla u_+| \kappa_R^2 e^{hu_+} \phi^2 d\mu_g \end{aligned}$$

where $\bar{C}_1 = 2(C_1 + C_2)$. Therefore, if we apply the above inequality with (3.15) and (3.16), we deduce that

$$\begin{aligned} \frac{\partial}{\partial t} \left(\int_M \kappa_R^2 e^{hu_+} \phi^2 d\mu_g \right) &\leq 4 \int_M |\nabla \kappa_R| |\nabla u_+| \kappa_R e^{hu_+} \phi^2 d\mu_g - 2 \int_M |\nabla u_+|^2 \kappa_R^2 e^h \phi^2 d\mu_g \\ &\quad + 2 \int_M |\nabla h| |\nabla u_+| \kappa_R^2 e^{hu_+} \phi^2 d\mu_g - e^{-C_2\alpha} \int_M |\nabla h|^2 \kappa_R^2 e^{hu_+} \phi^2 d\mu_g \\ &\quad + \bar{C}_2 \int_M \kappa_R^2 e^{hu_+} \phi^2 d\mu_g + \bar{C}_1 \int_M |\nabla u_+|^2 \kappa_R^2 e^{hu_+} \phi^2 d\mu_g. \end{aligned}$$

where $\bar{C}_2 = 2C_1 + \frac{(n+2)}{2}C_2$. We now have

$$4 \int_M |\nabla \kappa_R| |\nabla u_+| \kappa_R e^{hu_+} \phi^2 d\mu_g \leq \frac{1}{2} \int_M |\nabla u_+|^2 \kappa_R^2 e^h \phi^2 d\mu_g \quad (3.17)$$

$$+ 8 \int_M |\nabla \kappa_R|^2 e^{hu_+} \phi^2 d\mu_g \quad (3.18)$$

and

$$\begin{aligned} 2 \int_M |\nabla h| |\nabla u_+| \kappa_R^2 e^{hu_+} \phi^2 d\mu_g &\leq e^{-C_2\alpha} \int_M |\nabla h|^2 \kappa_R^2 e^{hu_+} \phi^2 d\mu_g \\ &\quad + e^{C_2\alpha} \int_M |\nabla u_+|^2 \kappa_R^2 e^h \phi^2 d\mu_g. \end{aligned}$$

Based on the condition on α , this gives us

$$\begin{aligned} 2 \int_M |\nabla h| |\nabla u_+| \kappa_R^2 e^h u_+ \phi^2 d\mu_g &\leq e^{-C_2\alpha} \int_M |\nabla h|^2 \kappa_R^2 e^h u_+^2 \phi^2 d\mu_g \\ &+ \frac{9}{8} \int_M |\nabla u_+|^2 \kappa_R^2 e^h \phi^2 d\mu_g. \end{aligned} \quad (3.19)$$

We also have

$$\begin{aligned} \bar{C}_1 \int_M |\nabla u_+| \kappa_R^2 e^h u_+ \phi^2 d\mu_g &\leq \frac{1}{4} \int_M |\nabla u_+|^2 \kappa_R^2 e^h \phi^2 d\mu_g \\ &+ \bar{C}_1^2 \int_M \kappa_R^2 e^h u_+^2 \phi^2 d\mu_g. \end{aligned} \quad (3.20)$$

By setting $C = \bar{C}_1^2 + \bar{C}_2$, we obtain

$$\begin{aligned} \frac{\partial}{\partial t} \left(\int_M \kappa_R^2 e^h u_+^2 \phi^2 d\mu_g \right) &\leq -\frac{1}{8} \int_M |\nabla u_+|^2 \kappa_R^2 e^h \phi^2 d\mu_g + C \int_M \kappa_R^2 e^h u_+^2 \phi^2 d\mu_g \\ &+ 8 \int_M |\nabla \kappa_R|^2 e^h u_+^2 \phi^2 d\mu_g. \end{aligned}$$

This implies that

$$\frac{\partial}{\partial t} \left(e^{-Ct} \int_M \kappa_R^2 e^h u_+^2 \phi^2 d\mu_g \right) + \frac{e^{-Ct}}{8} \int_M |\nabla u_+|^2 \kappa_R^2 e^h \phi^2 d\mu_g \leq 8 \int_M |\nabla \kappa_R|^2 e^h u_+^2 \phi^2 d\mu_g.$$

We know that $|\nabla \kappa_R|^2 \leq 4e^{C_2T}$. Furthermore, if we denote by B_a the ball of radius $a > 0$ centered at O_{x_0} in (W, d_0) , we see that the support of the function $|\nabla \kappa_R|^2$ is contained in $\sigma^{-1}(B_{R+1}/B_R)$. Hence

$$8 \int_M |\nabla \kappa_R|^2 e^h u_+^2 \phi^2 d\mu_g \leq 32e^{C_2T} \int_{\sigma^{-1}(B_{R+1}/B_R)} e^h u_+^2 \phi^2 d\mu_g.$$

Furthermore, since $u_+ = 0$ at time $t = 0$, we obtain

$$\begin{aligned} e^{-Ct} \int_M \kappa_R^2 e^h u_+^2 e^h u_+^2 \phi^2 d\mu_g + \frac{e^{-C\alpha}}{8} \int_0^t \int_M |\nabla u_+|^2 \kappa_R^2 e^h \phi^2 d\mu_g dt \\ \leq 32e^{C_3T} \int_0^\alpha \int_{\sigma^{-1}(B_{R+1}/B_R)} e^h u_+^2 \phi^2 d\mu_g dt \end{aligned} \quad (3.21)$$

for any $0 \leq t \leq \alpha$. By (3.9) and (3.13)

$$\int_0^\alpha \int_M u_+^2 e^{-c_1 r_0^2} \phi^2 d\mu_g < \infty \quad (3.22)$$

where $c_1 = ce^{C_2 T}$. We now choose $\alpha = \min(\frac{1}{8c_1}, C_2^{-1} \log(\frac{9}{8}))$. Then

$$h(x) \leq -c_1 r_0^2(x)$$

for any $x \in M$. Combining this with (3.22), we deduce that

$$\int_0^\alpha \int_M e^h u_+^2 \phi^2 d\mu_g dt < \infty.$$

Therefore, if we let $R \rightarrow \infty$ in (3.21), we get

$$e^{-Ct} \int_M e^h u_+^2 \phi^2 d\mu_g + \frac{e^{-C\alpha}}{8} \int_0^t \int_M |\nabla u_+|^2 e^h \phi^2 d\mu_g dt = 0$$

This implies that $u_+(x, t) = 0$ for all $(x, t) \in M \times [0, \alpha]$. If $T \leq \alpha$, we are done. Otherwise, we repeat the argument a finite number of times and the theorem follows. \square

Chapter 4

Uniqueness of solutions to the Ricci flow.

We will now consider the Ricci flow on closed Riemannian groupoids. We first start by proving a uniqueness theorem.

Theorem 20. *Suppose g_0 is a \mathcal{G} -invariant Riemannian metric on a closed groupoid (\mathcal{G}, M^n) with compact connected space of orbits $W = M/\mathcal{G}$ and suppose that $g(t)$ and $\tilde{g}(t)$ are \mathcal{G} -invariant solutions to the Ricci flow on the closed interval $[0, T]$ with initial value $g(0) = \tilde{g}(0) = g_0$. Then, $g(t) = \tilde{g}(t)$ for all $t \in [0, T]$.*

The energy approach of [15] will be used to prove this theorem. In [15], Kotschwar defines a time-dependent integral of the form

$$\mathcal{E}(t) = \int_M \left(t^{-1} |g - \tilde{g}|_{g(t)}^2 + t^{-\alpha} |\Gamma - \tilde{\Gamma}|_{g(t)}^2 + |Rm - \widetilde{Rm}|_{g(t)}^2 \right) e^{-\omega} d\mu_{g(t)}.$$

The function ω in the above equation depends on the distance function and satisfies certain properties and α is a suitable constant. Also Γ and $\tilde{\Gamma}$ denote the Christoffel symbols and Rm and \widetilde{Rm} denote the curvature tensors of g and \tilde{g} respectively. In our case, since we are assuming that the space of orbits is compact and since we will work with \mathcal{G} -invariant geometric quantities, we will use the measure $\phi^2 d\mu_g$ where ϕ is a nonnegative cutoff function for a time independent Haar system ρ in Ξ and we will not need the exponential decay term $e^{-\omega}$. Before starting the proof of Theorem 20, we collect the results of [15] which we will need.

We will use the metric $g(t)$ as a reference metric and will usually use $|\cdot| = |\cdot|_{g(t)}$ to denote the norms induced on $T_l^k(M)$ by $g(t)$. Just as in [15], we set

$$h = g - \tilde{g}, A = \nabla - \tilde{\nabla}, S = Rm - \widetilde{Rm}.$$

More precisely, $A_{ij}^k = \Gamma_{ij}^k - \tilde{\Gamma}_{ij}^k$ and $S_{ijk}^l = R_{ijk}^l - \tilde{R}_{ijk}^l$. The following geometric inequalities are listed in [15][Subsection 1.1]:

$$\left| \frac{\partial h}{\partial t} \right| \leq C|S| \quad (4.1)$$

$$\left| \frac{\partial A}{\partial t} \right| \leq C \left(|\tilde{g}^{-1}| |\tilde{\nabla} \widetilde{Rm}| |h| + |\widetilde{Rm}| |A| + |\nabla S| \right) \quad (4.2)$$

and

$$\begin{aligned} \left| \frac{\partial S}{\partial t} - \nabla S - \operatorname{div} U \right| &\leq C \left(|\tilde{g}^{-1}| |\tilde{\nabla} \widetilde{Rm}| |A| + |\tilde{g}^{-1}| |\widetilde{Rm}|^2 |h| \right) \\ &\quad + C(|Rm| + |\widetilde{Rm}|) |S| \end{aligned} \quad (4.3)$$

where, in each of these inequalities, C is a constant depending only on n . The quantity U is a (3,2)-tensor given by $U_{ijk}^{ab} = g^{ab} \nabla_b \tilde{R}_{ijk}^l - \tilde{g}^{ab} \tilde{\nabla}_b \tilde{R}_{ijk}^l$ and satisfies

$$|U| \leq C \left(|\tilde{g}^{-1}| |\tilde{\nabla} \widetilde{Rm}| |h| + |A| |\widetilde{Rm}| \right). \quad (4.4)$$

Since we will be working with \mathcal{G} -invariant smooth functions, these functions can be viewed as continuous functions on $W \times [0, T]$. The fact that W is compact implies that any such function is uniformly bounded on $M \times [0, T]$. In particular, we will have

$$|Rm(x, t)|_{g(t)} \leq K \text{ and } |\widetilde{Rm}(x, t)|_{\tilde{g}(t)} \leq K \quad (4.5)$$

for some $K > 0$. Let θ be the mean curvature form of the fixed Haar system ρ . We will assume that K is large enough so that we also have

$$|\theta| \leq K \quad (4.6)$$

on $M \times [0, T]$.

We showed in Chapter 3 that, on an étale groupoid with compact connected space of orbits, the maximum principle is valid when we consider \mathcal{G} -invariant quantities. It follows that we can apply the global curvature estimates of Bando and Shi (see [1, 25] and [5][Chapter 14]). In particular, we have

$$|Rm|_{g(t)} + |\tilde{Rm}|_{\tilde{g}(t)} + \sqrt{t} |\nabla Rm|_{g(t)} + \sqrt{t} |\tilde{\nabla} \widetilde{Rm}|_{\tilde{g}(t)} \leq N \quad (4.7)$$

for all $t \in [0, T]$ and where $N = N(n, K, T^*)$ with $T^* = \max\{T, 1\}$. On the other hand, the uniform curvature bounds on $g(t)$ and $\tilde{g}(t)$ imply that the metrics $g(t)$, $\tilde{g}(t)$ and g_0 are uniformly equivalent for $t \in [0, T]$. Hence, for some possibly larger constant N , the inequality

(4.7) remains valid when the norms are replaced by the norm $|\cdot| = |\cdot|_{g(t)}$. It follows that [15][Lemma 6] holds. Namely, we have

$$|h(p, t)| \leq Nt \text{ and } |A(p, t)| \leq N\sqrt{t} \quad (4.8)$$

on $M \times [0, T]$ for some constant $N = N(n, K, T^*)$.

As we mentioned before, we will define our energy integral as

$$\mathcal{E}(t) = \int_M (t^{-1}|h|^2 + t^{-\alpha}|A|^2 + |S|^2) \phi^2 d\mu_g \quad (4.9)$$

for some fixed $\alpha \in (0, 1)$. This integral is well-defined and differentiable on $(0, T]$ and it follows from (4.8) that $\lim_{t \rightarrow 0^+} \mathcal{E}(t) = 0$. Theorem 20 now follows from iterating the following result.

Proposition 21 ([15] Proposition 7). *There exists real numbers $N = N(n, K, T^*) > 0$ and $T_0 = T_0(n, \alpha) \in (0, T]$ such that $\mathcal{E}'(t) \leq N\mathcal{E}(t)$ for all $t \in (0, T_0]$. Hence $\mathcal{E} \equiv 0$ on $(0, T_0]$.*

Proof. Since the proof is basically the same as the proof of [15][Proposition 7], we will, for the most part, not be repeating the computations that are similar. We will mostly use the same notation and convention. Let $R = R(g(t))$ be the scalar curvature of the metric $g(t)$. We define

$$\begin{aligned} \mathcal{S}(t) &= \int_M |S|^2 \phi^2 d\mu_g, \quad \mathcal{H}(t) = t^{-1} \int_M |h|^2 \phi^2 d\mu_g, \\ \mathcal{I}(t) &= t^{-\alpha} \int_M |A|^2 \phi^2 d\mu_g, \quad \text{and } \mathcal{J}(t) = \int_M |\nabla S|^2 \phi^2 d\mu_g \end{aligned}$$

so $\mathcal{E}(t) = \mathcal{S}(t) + \mathcal{H}(t) + \mathcal{I}(t)$. We will denote by C a series of constants depending only on n and by N a series of constants depending on at most n, β, K and T^* . If we apply the condition (4.5), we obtain

$$\begin{aligned} \mathcal{S}' &\leq - \int_M |S|^2 R \phi^2 d\mu_g + 2 \int_M \left\langle \frac{\partial S}{\partial t}, S \right\rangle \phi^2 d\mu_g \\ &\leq N\mathcal{S} + 2 \int_M \left\langle \frac{\partial S}{\partial t}, S \right\rangle \phi^2 d\mu_g. \end{aligned}$$

Just like in the proof of [15][Proposition 7], we can use (4.3) and (4.7) to reduce this to

$$\mathcal{S}' \leq N\mathcal{S} + t\mathcal{H} + t^{\alpha-1}\mathcal{I} + 2 \int_M \langle \Delta S + \text{div}U, S \rangle \phi^2 d\mu_g.$$

We now use the integration by parts formula (2.5) to obtain

$$\begin{aligned} &2 \int_M \langle \Delta S + \text{div}U, S \rangle \phi^2 d\mu_g = \\ &-2 \int_M |\nabla S|^2 \phi^2 d\mu_g + 2 \int_M \langle \iota_{\theta^\sharp} \nabla S, S \rangle \phi^2 d\mu_g \\ &\quad -2 \int_M \langle U, \nabla S \rangle + 2 \int_M \langle \iota_{\theta^\sharp} U, S \rangle \phi^2 d\mu_g \end{aligned}$$

If we use (4.6), we get

$$\begin{aligned}
2\langle \iota_{\theta^\sharp} \nabla S, S \rangle &= 2\langle \nabla S, \theta \otimes S \rangle \\
&\leq 2|\nabla S| |\theta| |S| \\
&\leq \frac{|\nabla S|^2}{2} + 2|\theta|^2 |S|^2 \\
&\leq \frac{|\nabla S|^2}{2} + N|S|^2
\end{aligned}$$

$$2\langle \iota_{\theta^\sharp} U, S \rangle \leq |U| |S| |\theta| \leq |U|^2 + N|S|^2$$

and

$$-2\langle U, \nabla S \rangle \leq \frac{|\nabla S|^2}{2} + 2|U|^2.$$

Therefore,

$$2 \int_M \langle \Delta S + \operatorname{div} U, S \rangle \phi^2 d\mu_g = -\mathcal{J} + N\mathcal{S} + 3 \int_M |U|^2 \phi^2 d\mu_g$$

It follows from (4.4) and (4.7) that we have $|U|^2 \leq Nt^{-1}|h|^2 + N|A|^2$. Hence, putting things together, we get

$$\begin{aligned}
\mathcal{S}' &\leq N\mathcal{S} + (t + N)\mathcal{H} + (t^{\alpha-1} + Nt^\alpha)\mathcal{I} - \mathcal{J} \\
&\leq N\mathcal{S} + N\mathcal{K} + (t^{\alpha-1} + N)\mathcal{I} - \mathcal{J}
\end{aligned} \tag{4.10}$$

where the last line was obtained by using the inequalities $t \leq T$ and $t^\alpha \leq T^\alpha$.

For the quantities \mathcal{H} and \mathcal{I} , we apply the conditions (4.5) and (4.6) to obtain:

$$\mathcal{H}' \leq (N - t^{-1})\mathcal{H} + 2t^{-1} \int_M \left\langle \frac{\partial h}{\partial t}, h \right\rangle \phi^2 d\mu_g$$

and

$$\mathcal{I}' \leq (N - \beta t^{-1})\mathcal{I} + 2t^{-\alpha} \int_M \left\langle \frac{\partial A}{\partial t}, A \right\rangle \phi^2 d\mu_g.$$

We then use the conditions (4.1), (4.2) and (4.7) and do the same computations as in [15][Proposition 7] to obtain

$$\mathcal{H}' \leq \left(N - \frac{t^{-1}}{2} \right) \mathcal{H} + C\mathcal{S} \tag{4.11}$$

and

$$\mathcal{I}' \leq N\mathcal{H} + (N - \alpha t^{-1} + Ct^{-\alpha})\mathcal{I} + \mathcal{J}. \quad (4.12)$$

Combining (4.10),(4.11) and (4.12), we get

$$\mathcal{E}'(t) \leq N\mathcal{E}(t) - \frac{1}{2}t^{-1}\mathcal{H}(t) - t^{-1}(\alpha - t^\alpha + Ct^{1-\alpha})\mathcal{I}(t)$$

Thus for T_0 sufficiently small depending only on α and $C = C(n)$, and for some large enough $N = N(n, K, \alpha, T)$, we have $\mathcal{E}'(t) \leq N\mathcal{E}(t)$ on $(0, T_0]$. Since $\lim_{t \rightarrow 0^+} \mathcal{E}(t) = 0$, it follows from Gronwall's inequality that $\mathcal{E} \equiv 0$ on $(0, T_0]$. This completes the proof. \square

Chapter 5

Short time existence of solutions to the Ricci flow

We will use the Ricci-deTurck flow to prove short time existence of the Ricci flow on a closed groupoid.

Theorem 22. *If (\mathcal{G}, M) is a closed groupoid with compact connected space of orbits $W = M/\mathcal{G}$ and if g_0 is a \mathcal{G} -invariant Riemannian metric on M , then, for some $T > 0$, there exists a Ricci flow solution $g(t)$ defined for $t \in [0, T)$ and with initial condition $g(t) = g_0$.*

Proof. Let \tilde{g} be a fixed \mathcal{G} -invariant background metric. Recall that the Ricci-deTurck flow is defined by

$$\begin{aligned} \frac{\partial g_{ij}}{\partial t} &= -2R_{ij} + \nabla_i B_j + \nabla_j B_i \\ g(0) &= g_0 \end{aligned} \tag{5.1}$$

where $B = B(g)$ is the time-dependent one-form defined by

$$B_j = g_{jk} g^{pq} (\Gamma_{pq}^k - \tilde{\Gamma}_{pq}^k)$$

This is a strictly parabolic partial differential equation. In fact, as shown in [4][Chaper 7], we can rewrite this equation as

$$\frac{\partial g_{ij}}{\partial t} = P(g) = g^{pq} \tilde{\nabla}_p \tilde{\nabla}_q g_{ij} + K(g, \tilde{\nabla} g) \tag{5.2}$$

where

$$\begin{aligned} K(g, \tilde{\nabla} g) &= -g^{pq} g_{ik} \tilde{g}^{kl} \tilde{R}_{jplq} - g^{pq} g_{jk} \tilde{g}^{kl} \tilde{R}_{iplq} \\ &\quad + g^{pq} g^{kl} \left(\frac{1}{2} \tilde{\nabla}_i g_{kp} \tilde{\nabla}_j g_{lq} + \tilde{\nabla}_p g_{jk} \tilde{\nabla}_l g_{iq} - \tilde{\nabla}_p g_{jk} \tilde{\nabla}_q g_{il} \right) \\ &\quad - g^{pq} g^{kl} (\tilde{\nabla}_j g_{kp} \tilde{\nabla}_q g_{il} + \tilde{\nabla}_i g_{kp} \tilde{\nabla}_q g_{jl}). \end{aligned}$$

Since the manifold M is not necessarily compact, we can't directly use the standard theorems for parabolic partial differential equations. However, since the equation (5.2) consists of \mathcal{G} -invariant quantities, we can relate this equation to a partial differential equation defined for sections of a vector bundle with compact base space.

We showed in Section 2.4 that the orthonormal frame bundle $F = F_{\tilde{g}}$ over M defined by the metric \tilde{g} and its quotient $Z = F/\hat{\mathcal{G}}$ can be equipped with metrics \hat{g} and \bar{g} respectively such that the maps $\pi : (F, \hat{g}) \rightarrow (M, g)$ and $\hat{\sigma} : (F, \hat{g}) \rightarrow (Z, \bar{g})$ are Riemannian submersions. Given any \mathcal{G} -invariant vector bundle E over M equipped with a \mathcal{G} -invariant connection $\tilde{\nabla}^E$, we can consider the pull back bundle $\hat{E} = \pi^*E$ of E over F equipped with the pull back connection $\hat{\nabla} = \hat{\nabla}^E = \pi^*\tilde{\nabla}^E$ which will be \mathcal{G} -invariant. Also, the $O(n)$ action on F induces an action on \hat{E} which also preserves the connection $\hat{\nabla}^E$.

We apply this to the bundle $E = \text{Sym}(T^*M)$ of symmetric (2,0)-tensors on M where we equip it with the connection induced by the background \mathcal{G} -invariant metric \tilde{g} on M . We now define a parabolic partial differential equation for sections of the corresponding bundle \hat{E} as follows.

First, we denote by \mathcal{V} and by \mathcal{H} the vertical and horizontal distributions for the Riemannian submersion $\pi : (F, \hat{g}) \rightarrow (M, g)$. Let (x_1, \dots, x_n) be local coordinates defined on an open set U of M . For $1 \leq i \leq n$, we can lift the vector field ∂_{x_i} to a horizontal vector field w_i on F . Hence, $\{w_1, \dots, w_n\}$ is a local frame for $\mathcal{H}|_{\pi^{-1}(U)}$. Let $\{v_1, \dots, v_d\}$ be a local frame for $\mathcal{V}|_{\pi^{-1}(U)}$ where $d = \dim O(n) = \frac{n(n-1)}{2}$. The bi-invariant metric along the $O(n)$ -fiber is then given by $\hat{g}_{\mu\nu} = \hat{g}(v_\mu, v_\nu)$. If s is a section of \hat{E} , then for each $f \in F$, $s(f) = s_f$ is an element of $\text{Sym}(T_x^*M)$ where $x = \pi(f)$. If $x \in U$, then s_f is locally given by $s_{f;ij} = s_f(\partial_{x_i}, \partial_{x_j})$. We will suppress the index f and simply write s_{ij} . If s_{ij} is positive definite so that the inverse matrix s^{ij} is defined, the partial differential equation is then locally given by

$$\frac{\partial s_{ij}}{\partial t} = \hat{P}(s) = s^{pq} \hat{\nabla}_p \hat{\nabla}_q s + \hat{g}^{\mu\nu} \hat{\nabla}_\mu \hat{\nabla}_\nu s + \hat{K}(s, \hat{\nabla} s) \quad (5.3)$$

where

$$\begin{aligned} \hat{K}(s, \hat{\nabla} s) = & -s^{pq} s_{ik} \tilde{g}^{kl} \tilde{R}_{jplq} - s^{pq} s_{jk} \tilde{g}^{kl} \tilde{R}_{iplq} \\ & + s^{pq} s^{kl} \left(\frac{1}{2} \hat{\nabla}_i s_{kp} \hat{\nabla}_j s_{lq} + \hat{\nabla}_p s_{jk} \hat{\nabla}_l s_{iq} - \hat{\nabla}_p s_{jk} \hat{\nabla}_q s_{il} \right) \\ & - s^{pq} s^{kl} \left(\hat{\nabla}_j s_{kp} \hat{\nabla}_q s_{il} + \hat{\nabla}_i s_{kp} \hat{\nabla}_q s_{jl} \right). \end{aligned}$$

Here, the derivative terms with greek letters as indices such as $\hat{\nabla}_\mu$ denote differentiation along the vertical vectors v_μ , whereas the derivative terms with latin letters as indices denote differentiation along the horizontal vectors w_i . Equation (5.3) is a strictly parabolic partial differential equation and is invariant under the actions of $O(n)$ and $\hat{\mathcal{G}}$. In fact, the elliptic operator \hat{P} is, in a sense, simply the operator P plus the Laplacian along the $O(n)$ -fiber. Furthermore, if s is an $O(n)$ -invariant section, meaning if s is the pull back of a section g of E , we will have $\hat{P}(s) = \pi^*(P(g))$. Hence, finding a \mathcal{G} -invariant solution of (5.2) is the same as finding an $O(n)$ and $\hat{\mathcal{G}}$ -invariant solution of (5.3).

Note that the vector bundle \hat{E} over F induces a vector bundle \bar{E} over $Z = F/\hat{\mathcal{G}}$ which is equipped with a connection $\bar{\nabla} = \bar{\nabla}^E$ induced by the connection $\hat{\nabla}^E$. The action of $O(n)$ on \hat{E} induces an action of $O(n)$ on \bar{E} and the connection $\bar{\nabla}$ is invariant under this action. Furthermore, the parabolic partial differential equation (5.3) can be related to a parabolic partial differential equation

$$\frac{\partial u}{\partial t} = \bar{P}(u) \tag{5.4}$$

where \bar{P} is an $O(n)$ -invariant elliptic partial differential operator acting on sections of the bundle \bar{E} . So, if we set the initial condition to be the $O(n)$ -invariant section u_0 of \bar{E} corresponding to g_0 and since Z is a closed manifold, we will have an $O(n)$ -invariant solution $u(t)$ of (5.4) defined on $[0, T)$ for some $T > 0$. This will give us a solution $g(t)$ of (5.1).

Finally, if we consider again the time dependent \mathcal{G} -invariant one-form B that was used in the Ricci-DeTurck flow, the dual vector field B^\sharp generates a differentiable equivalence Ψ_t for short enough time t as described in Section 2.8. Furthermore, the formula

$$\frac{d}{dt}\Big|_{t=t_0} \Psi_t^* \omega_t = \Psi_{t_0}^* \left(\mathcal{L}_X \omega + \frac{d}{dt}\Big|_{t=t_0} \omega_t \right)$$

will hold for any time-dependent \mathcal{G} -invariant tensor ω_t . Applying this to the solution $g(t)$ of (5.1), we see that $g'(t) = \Psi_t^* g(t)$ is a solution of the Ricci flow. This completes the proof. \square

Just as in the closed manifold case, one can show that if (\mathcal{G}, M) is a closed groupoid with compact connected space of orbits $W = M/\mathcal{G}$ and if $g(t)$ is a solution to the Ricci flow on a maximal time interval $[0, T)$ with $T < \infty$, then

$$\sup_M |Rm|(\cdot, t) \rightarrow \infty$$

as $t \uparrow T$. The proof is the same. Indeed, we start by assuming that we have a uniform bound on the curvature and show that this implies that the Ricci flow solution can be extended past the time T , thus contradicting the maximality of $[0, T)$. The uniform bound on the curvature implies that the Riemannian metrics $g(t)$ are uniformly equivalent. More precisely, for some $K > 0$, we will have

$$e^{-2Kt} g(0) \leq g(t) \leq e^{2Kt} g(0)$$

for all $t \in [0, T]$. This implies that $g(t)$ can be extended continuously to the time interval $[0, T]$. As we mentioned before, the Bernstein-Bando-Shi global curvature estimates will hold in the case of a closed groupoid with compact connected space of orbits. Then, just as in the closed manifold case, we use these estimates to show that the extension of $g(t)$ to the closed interval $[0, T]$ is smooth. We can now take $g(T)$ as the initial metric in the short-time existence theorem in order to extend the flow to a Ricci flow on $[0, T + \epsilon)$ for some $\epsilon > 0$ giving us the contradiction.

Chapter 6

The 2-dimensional case

In this section, we look at closed Riemannian groupoids (\mathcal{G}, M, g) where $\dim(M) = 2$. If we consider the local models determined by the quintuples $(\mathfrak{g}, K, i, Ad, T)$ described in Subsection 2.3, we see that, if the structural Lie algebra has dimension at least two, then we are dealing with a locally homogeneous surface. The groupoid is then developable and the \mathcal{G} -invariant metric will have constant sectional curvature. The Ricci flow is pretty simple in this case. On the other hand, if the structural Lie algebra is trivial, then the groupoid is the groupoid of germs of change of charts of a 2-dimensional orbifold. Except for a few cases (the bad 2-orbifolds), they will be developable and the Ricci flow is well understood in this setting. Finally, if \mathfrak{g} is isomorphic to \mathbb{R} , then we should consider two cases:

1. The space of orbits W is a closed interval.
2. The space of orbits W is a circle.

The first case will be weakly equivalent to an $O(2)$ or $SO(2)$ action on the 2-sphere. By [16, Section 5], the second case is weakly equivalent to the groupoid generated by the action of $\mathbb{R} \rtimes \mathbb{Z}$ on $\mathbb{R} \times \mathbb{R}$ given by

$$(q, n) \cdot (x, y) = (c^n x + q, y + n) \tag{6.1}$$

where c is some fixed nonzero constant. The coordinate y induces a coordinate on the space of orbits which is indeed a circle. It is also clear that the case $c = 1$ is equivalent to looking at a torus. In general, after possibly changing the coordinate y , we can assume that the invariant Riemannian metric g has the form

$$g(x, y) = e^{2k(y)} dx^2 + e^{2u(y)} dy^2 \tag{6.2}$$

where $u(y+1) = u(y)$ and $k(y+1) = k(y) + \lambda$ for the real number λ defined by $e^{2\lambda} = c^2$.

Proposition 23. *Let (\mathcal{G}, M) be the closed groupoid generated by the action of $\mathbb{R} \rtimes \mathbb{Z}$ on $\mathbb{R} \times \mathbb{R}$ defined by (6.1) and equipped with an invariant metric given by (6.2). Then a solution $g(t)$*

to the Ricci flow exists for all $t \geq 0$. If $\lambda = 0$, then the solution $g(t)$ to the Ricci flow will converge in a smooth sense to a flat metric as $t \rightarrow \infty$. If the function k in (6.2) has no critical points at time $t = 0$, then $\lambda \neq 0$ and $\frac{g(t)}{t}$ will converge in a smooth sense to the hyperbolic cusp metric of scalar curvature -1 as $t \rightarrow \infty$.

Before starting the proof of this proposition, we first note that the term $e^{2u(y)} dy^2$ of the metric g induces a metric \bar{g} on the quotient space. The function k cannot be viewed as a function on the quotient space unless $\lambda = 0$. However the differential dk is well-defined on the quotient space. By doing a simple computation, we can show that the scalar curvature R of the metric g is given by

$$\begin{aligned} R &= -2(\Delta_{\bar{g}}k + |\nabla k|^2) \\ \Rightarrow R &= -2\left(\frac{d^2k}{ds^2} + \left(\frac{dk}{ds}\right)^2\right) \end{aligned} \tag{6.3}$$

where s denotes the arc length coordinate on the quotient circle. In terms of the coordinate s , we will have $k(s+L) = k(s) + \lambda$ where L is the length of the circle. Finding the metrics g of constant scalar curvature reduces to solving the differential equation

$$\left(\frac{d^2k}{ds^2} + \left(\frac{dk}{ds}\right)^2\right) = -\frac{b}{2}.$$

We cannot have positive constant scalar curvature. For if $b > 0$, the above equation would imply that $\frac{d^2k}{ds^2} < 0$. So $\frac{dk}{ds}$ would be strictly decreasing thus contradicting the fact that it is a periodic function of s . This periodicity condition also implies that, in the case $b = 0$, we must have $k(s) = \text{constant}$ (so $\lambda = 0$) and, in the case $b < 0$, the metric g is of the form

$$ds^2 + e^{2\lambda s} dx^2.$$

So the constant negative scalar curvature case corresponds to the hyperbolic cusp metric.

Proof of the Proposition. The Ricci flow equation applied to Riemannian metrics of the form (6.2) reduces to

$$\frac{\partial u}{\partial t} = \frac{\partial k}{\partial t} = \Delta_{\bar{g}}k + |\nabla k|^2. \tag{6.4}$$

Note that the evolution equation for k can be written in the simpler form

$$\frac{\partial e^k}{\partial t} = \Delta_{\bar{g}}e^k.$$

But this will not be very useful since, as we mentioned earlier, e^k cannot be viewed as a function on the quotient circle unless $\lambda = 0$.

By doing a simple computation, we can show that the evolution equation of $|\nabla k|^2$ is given by

$$\left(\frac{\partial}{\partial t} - \Delta_{\bar{g}}\right) |\nabla k|^2 = -2(|\nabla k|^2 + (\Delta_{\bar{g}}k)^2) + \langle \nabla k, \nabla(|\nabla k|^2) \rangle. \quad (6.5)$$

If C is the maximum of the function $|\nabla k|^2$ at time $t = 0$, then, by the maximum principle, $|\nabla k|^2(t) \leq C$ as long as the solution exists. We can also show that the function $H = (\Delta_{\bar{g}}k)^2$ satisfies

$$\left(\frac{\partial}{\partial t} - \Delta_{\bar{g}}\right) H \leq -8|\nabla k|^2 H + \langle \nabla k, \nabla H \rangle. \quad (6.6)$$

This means that $\Delta_{\bar{g}}k$ stays bounded. Since $R = -2(\Delta_{\bar{g}}k + |\nabla k|^2)$, the scalar curvature is bounded. So the Ricci flow exists for all time. This proves the first part of the proposition.

The case $\lambda = 0$ reduces to looking at a metric on a torus which is invariant under a S^1 action. The result follows from known results about the Ricci flow on closed surfaces. On the other hand, if we assume that $|\nabla k| \neq 0$ at time $t = 0$, then it is clear that $\lambda \neq 0$ and we can extract more information from the evolution equation (6.5). We can show that the evolution equation of $|\nabla k|$ is given by

$$\left(\frac{\partial}{\partial t} - \Delta_{\bar{g}}\right) |\nabla k| = -|\nabla k|^3 + \langle \nabla k, \nabla(|\nabla k|) \rangle. \quad (6.7)$$

So, if $0 < \alpha < |\nabla k|^2 < \beta$ at time $t = 0$, we can apply the maximum/ minimum principles to deduce that

$$\frac{\alpha}{2\alpha t + 1} \leq |\nabla k|^2 \leq \frac{\beta}{2\beta t + 1}. \quad (6.8)$$

We can then use the derived lower bound on $|\nabla k|^2$ and consider the equation (6.6) to derive the inequality

$$H \leq \frac{C}{(2\alpha t + 1)^4} \quad (6.9)$$

for some positive constant C . We now consider the new family of metrics $\hat{g}(t) = \frac{g(t)}{t}$. This will be of the form $\hat{g}(t) = e^{2\hat{k}(y,t)} dx^2 + e^{2\hat{u}(y,t)} dy^2$ where $\hat{k} = k - \log(\sqrt{t})$ and $\hat{u} = u - \log(\sqrt{t})$. By using (6.4), (6.8), and (6.9), we get

$$\frac{\alpha}{2\alpha t + 1} - \frac{1}{2t} - \frac{C}{t^4} \leq \frac{\partial \hat{u}}{\partial t} = \frac{\partial \hat{k}}{\partial t} \leq \frac{\beta}{2\beta t + 1} - \frac{1}{2t} + \frac{C}{t^4}.$$

This can be used to prove that, for t large enough, there exists a constant C such that

$$\frac{1}{C}\hat{g}(0) \leq \hat{g}(t) \leq C\hat{g}(0) \quad (6.10)$$

Furthermore, the conditions (6.8), and (6.9) imply that the scalar curvature of the original solution $g(t)$ satisfies $|R|(\cdot, t) < \frac{C}{t}$. We can repeat the proof of [3, Proposition 5.33] to show that for every positive integer k there exists a positive constant C_k depending only on $g(0)$ such that

$$|\nabla^k R|(\cdot, t) \leq \frac{C_k}{(1+t)^{k+2}}$$

From this we get corresponding bounds for the scalar curvature of the rescaled metric \hat{g} . Combining this with (6.10), we deduce that $\hat{g}(t)$ converges smoothly as $t \rightarrow \infty$ to a metric $g_\infty = e^{2k_\infty(y,t)} dx^2 + e^{2u_\infty(y,t)} dy^2$. Since $|\nabla \hat{k}|^2 = t|\nabla k|^2 \rightarrow \frac{1}{2}$ and $\hat{H} = t^2 H \rightarrow 0$ as $t \rightarrow \infty$, the scalar curvature of the metric g_∞ is equal to -1 . So it's the hyperbolic cusp metric. \square

Chapter 7

The \mathcal{F} -functional and Ricci solitons on groupoids

Let (\mathcal{G}, M) be a closed groupoid with compact connected orbit space $W = M/\mathcal{G}$. As we mentioned in the introduction, $\mathcal{M}_{\mathcal{G}}$ will denote the space of \mathcal{G} -invariant Riemannian metrics on M and we will simply denote by ρ an element of the family of Haar systems Ξ defined in Subsection 2.7.

We consider the \mathcal{F} -functional $\mathcal{F} : \mathcal{M}_{\mathcal{G}} \times \Xi \rightarrow \mathbb{R}$ defined by

$$\mathcal{F}(g, \rho) = \int_M (R + |\theta|^2) \phi^2 d\mu_g \quad (7.1)$$

where R is the scalar curvature of the metric g , $d\mu_g$ the Riemannian density, θ the mean curvature form for ρ and ϕ a corresponding nonnegative cutoff function.

If we fix a representative ρ_0 of Ξ and induce a bijection $C_{\mathcal{G}}^{\infty}(M) \rightarrow \Xi$ as described at the end of Subsection 2.7, this can be viewed as a functional on $\mathcal{M}_{\mathcal{G}} \times C_{\mathcal{G}}^{\infty}(M)$

$$\mathcal{F}(g, f) = \int_M (R + |\nabla f + \theta_0|^2) e^{-f} \phi_0^2 d\mu_g \quad (7.2)$$

where θ_0 is the mean curvature class for ρ_0 and ϕ_0 a compatible cutoff function. Notice that, in the case of Example 10, this reduces to the usual \mathcal{F} -functional on a closed Riemannian manifold.

Before computing the variation of \mathcal{F} , we define for convenience a modified divergence operator by

$$\underline{\operatorname{div}}\alpha = \operatorname{div}\alpha - \iota_{\theta\sharp}\alpha$$

for any tensor α on M . Then the integration by parts formula (2.5) can now be written as

$$\int_M \langle \underline{\operatorname{div}}(\alpha), \omega \rangle \phi^2 d\mu_g = - \int_M \langle \alpha, \nabla \omega \rangle \phi^2 d\mu_g \quad (7.3)$$

for any \mathcal{G} -invariant tensors α and ω of type (r, s) and $(r - 1, s)$ respectively. We also define the modified Laplacian operator as $\underline{\Delta} = \underline{\text{div}} \circ \nabla$ and we get the formula

$$\int_M \langle \underline{\Delta}(\alpha), \chi \rangle \phi^2 d\mu_g = - \int_M \langle \nabla \alpha, \nabla \chi \rangle \phi^2 d\mu_g \quad (7.4)$$

for any \mathcal{G} -invariant tensors α and χ of type (r, s) .

Now suppose that we have the variations $\delta g = v$ and $\delta(d\rho) = \beta d\rho$ where $v = v_{ij}$ is a \mathcal{G} -invariant symmetric $(2, 0)$ -tensor and where β is a \mathcal{G} -invariant function on M .

Proposition 24. *The variation of the \mathcal{F} -functional is given by*

$$\begin{aligned} \delta \mathcal{F} &= -\frac{1}{2} \int_M \langle 2\text{Ric} + 2\nabla\theta, v \rangle \phi^2 d\mu_g \\ &\quad + \int_M (2\underline{\text{div}}(\theta) + |\theta|^2 + R) \left(\frac{V}{2} - \beta \right) \phi^2 d\mu_g. \end{aligned}$$

or equivalently

$$\begin{aligned} \delta \mathcal{F} &= -\frac{1}{2} \int_M \langle 2\text{Ric} + \mathcal{L}_{(\theta)^\sharp} g, v \rangle \phi^2 d\mu_g \\ &\quad + \int_M (2\text{div}(\theta) - |\theta|^2 + R) \left(\frac{V}{2} - \beta \right) \phi^2 d\mu_g. \end{aligned}$$

Proof. Note that since θ is a closed one-form, the tensor $\nabla\theta$ is a symmetric $(2, 0)$ -tensor. So the expression above makes sense.

The proof of the proposition is a straightforward computation. If we apply the formula (2.14), we get

$$\delta \mathcal{F} = \int_M \delta (R + |\theta|^2) \phi^2 d\mu_g + \int_M (R + |\theta|^2) \left(\frac{V}{2} - \beta \right) \phi^2 d\mu_g.$$

It is known that

$$\delta R = -\langle \text{Ric}, v \rangle - \Delta V + \text{div}(\text{div}(v)).$$

We will rewrite this in terms of the modified divergence operator and the modified Laplacian operator. A simple computation gives us

$$\begin{aligned} \Delta V &= \underline{\Delta} V + \langle \theta, \nabla V \rangle \\ \text{div}(\text{div}(v)) &= \underline{\text{div}}(\underline{\text{div}}(v)) + 2\langle \theta, \underline{\text{div}}(v) \rangle + \langle \nabla \theta, v \rangle + v(\theta, \theta). \end{aligned}$$

So we get

$$\begin{aligned} \delta R &= -\langle \text{Ric}, v \rangle - \underline{\Delta} V - \langle \theta, \nabla V \rangle + \underline{\text{div}}(\underline{\text{div}}(v)) \\ &\quad + 2\langle \theta, \underline{\text{div}}(v) \rangle + \langle \nabla \theta, v \rangle + v(\theta, \theta). \end{aligned} \quad (7.5)$$

On the other hand, the variation of $|\theta|^2$ is given by

$$\begin{aligned}\delta(|\theta|^2) &= \delta(g^{ij}\theta_i\theta_j) \\ &= -v^{ij}\theta_i\theta_j + 2g^{ij}\theta_i\nabla_j\beta \\ &= -v(\theta, \theta) + 2\langle\nabla\beta, \theta\rangle.\end{aligned}$$

Combining this with (7.5), we can rewrite the variation of \mathcal{F} as

$$\begin{aligned}\int_M (-\langle Ric, v\rangle - \underline{\Delta}V - \langle\theta, \nabla V\rangle + \underline{\operatorname{div}}(\underline{\operatorname{div}}(v))) \phi^2 d\mu_g \\ + \int_M (2\langle\theta, \underline{\operatorname{div}}(v)\rangle + \langle\nabla\theta, v\rangle + 2\langle\nabla\beta, \theta\rangle) \phi^2 d\mu_g \\ + \int_M (R + |\theta|^2) \left(\frac{V}{2} - \beta\right) \phi^2 d\mu_g.\end{aligned}$$

Using the formulas (7.3) and (7.4), we get

$$\int_M \underline{\Delta}V \phi^2 d\mu_g = \int_M \underline{\operatorname{div}}(\underline{\operatorname{div}}(v)) \phi^2 d\mu_g = 0$$

and

$$\begin{aligned}- \int_M \langle\theta, \nabla V\rangle \phi^2 d\mu_g &= \int_M V \underline{\operatorname{div}}(\theta) \phi^2 d\mu_g \\ 2 \int_M \langle\theta, \underline{\operatorname{div}}(v)\rangle \phi^2 d\mu_g &= -2 \int_M \langle\nabla\theta, v\rangle \phi^2 d\mu_g \\ 2 \int_M \langle\nabla\beta, \theta\rangle \phi^2 d\mu_g &= -2 \int_M \beta \underline{\operatorname{div}}(\theta) \phi^2 d\mu_g.\end{aligned}$$

So we end up with

$$\begin{aligned}\delta\mathcal{F} &= - \int_M \langle Ric + \nabla\theta, v\rangle \phi^2 d\mu_g \\ &\quad + \int_M (2\underline{\operatorname{div}}(\theta) + |\theta|^2 + R) \left(\frac{V}{2} - \beta\right) \phi^2 d\mu_g.\end{aligned}$$

This completes the proof. \square

We now impose the condition

$$\frac{V}{2} - \beta = 0. \tag{7.6}$$

The variaton of the functional \mathcal{F} is now

$$\delta\mathcal{F} = -\frac{1}{2} \int_M \langle 2Ric + \mathcal{L}_{\theta^\sharp} g, v \rangle \phi^2 d\mu_g. \quad (7.7)$$

If we define a Riemannian metric on $\mathcal{M}_{\mathcal{G}}$ by

$$\langle v_{ij}, v_{ij} \rangle_g = \frac{1}{2} \int_M v^{ij} v_{ij} \phi^2 d\mu_g \quad (7.8)$$

the gradient flow of \mathcal{F} on $\mathcal{M}_{\mathcal{G}}$ is given by

$$\frac{\partial g}{\partial t} = -2(Ric(g) + \nabla\theta) = -2Ric - \mathcal{L}_{\theta^\sharp} g \quad (7.9)$$

and we will have

$$\frac{d}{dt}\mathcal{F} = 2 \int_M |Ric(g) + \frac{1}{2}\mathcal{L}_{\theta^\sharp} g|^2 \phi^2 d\mu_g \geq 0. \quad (7.10)$$

This is zero when

$$Ric(g) + \frac{1}{2}\mathcal{L}_{\theta^\sharp} g = 0. \quad (7.11)$$

This is the equation of a steady soliton. We will call such a soliton a *groupoid steady soliton*. If θ is exact, this reduces to a gradient steady soliton.

Let Ψ_t be the time dependent self-equivalences of (\mathcal{G}, M) generated by the time dependent \mathcal{G} -invariant vector field θ^\sharp . We set $g'(t) = \Psi_t^* g(t)$ and $\theta'(t) = \Psi_t^* \theta(t)$. We will then have

$$\begin{aligned} \frac{\partial g'}{\partial t} &= \Psi_t^* \left(\mathcal{L}_{\theta^\sharp} g + \frac{\partial g}{\partial t} \right) \\ \Rightarrow \frac{\partial g'}{\partial t} &= -2Ric(g'). \end{aligned} \quad (7.12)$$

The form θ' is the mean curvature form of the pullback Haar system $\rho' = \Psi^* \rho$. If we denote by β' the variation corresponding to ρ' , we have

$$\frac{\partial \theta'}{\partial t} = d\beta'.$$

On the other hand, we have

$$\begin{aligned} \frac{\partial \theta'}{\partial t} &= \Psi_t^* \left(\mathcal{L}_{\theta^\sharp} \theta + \frac{\partial \theta}{\partial t} \right) \\ &= \Psi_t^* (d(|\theta|^2) + d\beta) \\ &= \Psi_t^* (d(|\theta|^2 - R - \text{div}(\theta))) \\ &= d(|\theta'|^2 - R' - \text{div}(\theta')). \end{aligned}$$

Hence, we must have $\beta' = |\theta'|^2 - R' - \operatorname{div}(\theta') + c(t)$ where $c(t)$ is some time dependent constant. Now, since the condition (7.6) implies that

$$\int_M \phi^2 d\mu_g = \text{constant}$$

for any nonnegative cutoff function ϕ for the Haar system ρ , we will also have

$$\int_M (\phi')^2 d\mu_{g'} = \text{constant}$$

for any nonnegative cutoff function ϕ' for the Haar system ρ' . Differentiating the last equation with respect to time, we get

$$\begin{aligned} \int_M (-R' - \beta') (\phi')^2 d\mu_{g'} &= 0 \\ \int_M (-R' - |\theta'|^2 + R' + \operatorname{div}(\theta') - c(t)) (\phi')^2 d\mu_{g'} &= 0 \\ \int_M (\operatorname{div}(\theta') - c(t)) (\phi')^2 d\mu_{g'} &= 0 \\ -c(t) \int_M (\phi')^2 d\mu_{g'} &= 0 \end{aligned}$$

So $c(t)$ is equal to zero and we get

$$\beta' = |\theta'|^2 - R' - \operatorname{div}(\theta'). \quad (7.13)$$

We can relate this to a backward heat equation as follows. We can fix a smooth family of representative Haar systems $\rho_0(t)$ which induce time dependent bijections $C_{\mathcal{G}}^\infty(M) \rightarrow \Xi$. We can then write the Haar system ρ' as $d\rho' = e^{f'} d\rho_0$ for some smooth time dependent \mathcal{G} -invariant function f' . If we denote by β_0 the variation of the fixed time dependent Haar system $\rho_0(t)$ and by θ_0 the corresponding mean curvature form, then solving the equation (7.13) corresponds to solving the equation

$$\begin{aligned} \frac{\partial f'}{\partial t} &= |\nabla f' + \theta_0|^2 - R' - \operatorname{div}(\nabla f' + \theta_0) - \beta_0 \\ &= -\Delta f' + |\nabla f'|^2 + 2\langle \nabla f', \theta_0 \rangle - \operatorname{div}(\theta_0) + |\theta_0|^2 - R' - \beta_0 \end{aligned} \quad (7.14)$$

which is a backward heat equation. Indeed, we can rewrite it as

$$\frac{\partial e^{-f'}}{\partial t} = -\underline{\Delta} e^{-f'} + \langle \nabla e^{-f'}, \theta_0 \rangle + (R' + \operatorname{div}(\theta_0) + \beta_0) e^{-f'}. \quad (7.15)$$

This can be solved as outlined in [14]. We first solve (7.12) on some time interval $[t_1, t_2]$ and then solve (7.15) backwards in time on $[t_1, t_2]$. The latter operation can be done by

using the Uhlenbeck trick of Subsection 2.7. We construct the space \mathbf{F} with quotient space $Z = \mathbf{F}/\hat{\mathcal{G}}$. For each $t \in [t_1, t_2]$, we will have a commutative diagram

$$\begin{array}{ccc} (\mathbf{F}, \hat{g}(t)) & \xrightarrow{\hat{\sigma}} & (Z, \bar{g}(t)) \\ \pi \downarrow & & \downarrow \hat{\pi} \\ (M, g'(t)) & \xrightarrow{\sigma} & W. \end{array}$$

The maps π and $\hat{\pi}$ correspond to taking $O(n)$ -quotients. Furthermore, the map π is a \mathcal{G} -equivariant Riemannian submersion and the map $\hat{\sigma}$ is an $O(n)$ -equivariant Riemannian submersion for each $t \in [t_1, t_2]$. For simplicity, we shall assume that the fixed time dependent Haar system ρ_0 is the Haar system generated by the metric $g(t)$. We can view the functions f and $\Lambda = (R' + \underline{\text{div}}(\theta_0) + \beta_0)$ as $O(n)$ -invariant functions on Z . Recall that the form θ' was induced by the mean curvature form of the Riemannian submersion $\hat{\sigma}$ which is a \mathcal{G} -basic and $O(n)$ -basic one form. In particular, this form induces a corresponding $O(n)$ -invariant form on the space Z . For simplicity, we will also denote this form by θ_0 .

It follows from [6, Lemma 4.2.4] that, if $q : (X_1, g_1) \rightarrow (X_0, g_0)$ is a Riemannian submersion and if k is a smooth function on X_0 , then

$$\Delta_{g_1}(q^*k) = q^*(\Delta_{g_0}k) + \langle \alpha, \nabla(q^*k) \rangle$$

where α is the mean curvature form of the submersion. Applying this to the submersions π and $\hat{\sigma}$ and using the fact that the mean curvature form of the Riemannian submersion π is equal to zero for each t , we see that Equation (7.15) now corresponds to the $O(n)$ -invariant heat equation on Z

$$\frac{\partial v}{\partial t} = -\Delta_{\bar{g}}v + \langle \nabla v, \theta_0 \rangle + \Lambda v. \quad (7.16)$$

where $v = e^{-f}$. Since Z is compact, we can start with a value $v_2 = e^{-f_2}$ at time t_2 and solve this equation backward in time on $[t_1, t_2]$ to get an $O(n)$ -invariant solution $u(t)$ for (7.16). We can show that $u(t) > 0$. So $f(t) = -\ln(u(t))$ is well defined and will be a solution for (7.15). Using this method, we get the Haar system $\rho'(t)$ having the variation specified by (7.13). Finally, it is clear that we will have

$$\frac{d}{dt}\mathcal{F}(g'(t), \rho'(t)) = 2 \int_M |Ric(g') + \frac{1}{2}\mathcal{L}_{(\theta')^\sharp}g'|^2(\phi')^2 d\mu_{g'} \geq 0$$

for any time dependent cutoff function ϕ' compatible with ρ' . This proves Theorem 2.

Just as in the closed manifold case, we have the following result.

Proposition 25. *Given a \mathcal{G} -invariant metric g , there is a unique minimizer Haar system $\tilde{\rho}$ of $\mathcal{F}(g, \rho)$ under the constraint $\int_M \phi^2 d\mu_g = 1$ for any compatible cutoff function ϕ .*

Proof. We can rewrite the \mathcal{F} -functional as

$$\mathcal{F}(g, \rho) = \int_M (R + 2\operatorname{div}(\theta) - |\theta|^2) \phi^2 d\mu_g$$

Just like before, we fix a representative Haar system ρ_0 and write the functional as:

$$\mathcal{F}(g, f) = \int_M (R + 2\operatorname{div}(\theta_0 + \nabla f) - |\theta_0 + \nabla f|^2) e^{-f} \phi_0^2 d\mu_g$$

The constraint $\int_M \phi^2 d\mu_g = 1$ now becomes the condition $\int_M e^{-f} \phi_0^2 d\mu_g = 1$ for the \mathcal{G} -invariant smooth function f . For convenience again, we will take ρ_0 to be the Haar system generated by g and as before we consider the orthonormal frame bundle F_g generated by g and the corresponding space of orbits $Z = F_g/\hat{\mathcal{G}}$ with Riemannian metric \bar{g} . Just as in the proof of [14, Proposition 7.1], we set $\kappa = e^{-\frac{f}{2}}$. We can write the \mathcal{F} -functional as an integral over Z .

$$\mathcal{F} = \int_Z (-4\Delta_{\bar{g}}(\kappa) + B\kappa) d\mu_{\bar{g}}$$

where κ and $B = (2\operatorname{div}(\theta_0) - |\theta_0|^2 + R)$ are viewed as $O(n)$ -invariant functions on Z .

The constraint equation becomes $\int_Z \kappa^2 d\mu_{\bar{g}} = 1$. The minimum of \mathcal{F} is given by the smallest eigenvalue λ of $-4\Delta_{\bar{g}} + B$ and the minimizer $\tilde{\kappa} = e^{-\frac{f}{2}}$ is a corresponding normalized eigenvector. Since this operator is a Schrödinger operator, there is a unique normalized positive eigenvector. Finally, the equation

$$-4\Delta_{\bar{g}}(\kappa) + B\kappa = \lambda\kappa$$

translates to

$$R + 2\operatorname{div}(\theta) - |\theta|^2 = \lambda.$$

□

We define the λ -functional on $\mathcal{M}_{\mathcal{G}}$ by $\lambda(g) = \mathcal{F}(g, \tilde{\rho})$ where $\tilde{\rho}$ is the Haar system in the previous proposition.

Proposition 26. *If $g(t)$ is a solution to the Ricci flow then $\lambda(g(t))$ is nondecreasing in t .*

Proof. The proof is exactly the same as in the closed manifold case. Given a solution of the Ricci flow on a time interval $[t_1, t_2]$, we take $\rho(t_2)$ to be the minimizer $\tilde{\rho}(t_2)$. So, $\lambda(t_2) = \mathcal{F}(g(t_2), \tilde{\rho}(t_2))$. As we described right before Proposition 25, we can solve for the Haar system $\rho(t)$ backwards in time. It follows from Theorem 2 that $\mathcal{F}(g(t_1), \rho(t_1)) \leq \mathcal{F}(g(t_2), \rho(t_2))$. By the definition of λ , we have $\lambda(t_1) \leq \mathcal{F}(g(t_1), f(t_1))$. Hence $\lambda(t_1) \leq \lambda(t_2)$. This completes the proof. □

We will define a steady breather as a Ricci flow solution on an interval $[t_1, t_2]$ such that $g(t_2) = \Psi^*g(t_1)$ for some self-equivalence Ψ of (\mathcal{G}, M) . Since the λ -functional is invariant under self-equivalences, we can apply the previous proposition to get the following corollary.

Corollary 27. *A steady breather is a groupoid steady soliton.*

Proposition 28. *A groupoid steady soliton on a closed groupoid (\mathcal{G}, M) with compact connected space of orbits W is Ricci flat and the mean curvature class Θ is equal to zero.*

Proof. The Haar system $\rho(t)$ is the minimizer of $\mathcal{F}(g(t), \cdot)$ for all t . Based on the proof of Proposition 25, we have

$$R + 2\operatorname{div}(\theta) - |\theta|^2 = \lambda.$$

Since this is a groupoid steady soliton, we know that $Ric + \nabla\theta = 0$ and this gives us $R + \operatorname{div}(\theta) = 0$ by taking the trace. So the above equation reduces to

$$\begin{aligned} \operatorname{div}(\theta) - |\theta|^2 &= \lambda \\ \Rightarrow \underline{\operatorname{div}}(\theta) &= \lambda \end{aligned}$$

Integrating this over M with respect to the measure $\phi^2 d\mu_g$, we get that $\lambda = 0$.

The groupoid soliton will give an eternal solution to the Ricci flow. By [12, Lemma 6], an eternal Ricci flow solution on an étale groupoid with compact connected space of orbits must be Ricci flat. So $R = 0$. Using the previous equations, we deduce that $\theta = 0$. Hence, the class Θ is equal to zero. This completes the proof. \square

Remark 29. Suppose that we have a Ricci soliton of the form

$$Ric(g) + \nabla(\alpha) + \frac{\epsilon}{2}g = 0 \tag{7.17}$$

on a smooth manifold M where α is a closed one form and ϵ is some constant. The cases $\epsilon > 0$, $\epsilon < 0$ and $\epsilon = 0$ will correspond to expanding, shrinking and steady solitons respectively. Just as in the case of gradient Ricci solitons, we take the divergence of the above equation and get

$$\begin{aligned} g^{ki}\nabla_k R_{ij} + g^{ki}\nabla_k \nabla_i \alpha_j &= 0 \\ \Rightarrow \frac{\nabla_j R}{2} + g^{ki}\nabla_k \nabla_i \alpha_j &= 0. \end{aligned}$$

Now, if we use the fact that $\nabla\alpha$ is symmetric, we get

$$\begin{aligned} \nabla_k \nabla_i \alpha_j &= \nabla_k \nabla_j \alpha_i \\ &= \nabla_j \nabla_k \alpha_i - R_{kjim} \alpha^m. \end{aligned}$$

Hence

$$\begin{aligned}
 g^{ki}\nabla_k\nabla_i\alpha_j &= \nabla_j(\operatorname{div}(\alpha)) + R_{jm}\alpha^m \\
 &= \nabla_j(\operatorname{div}(\alpha)) - \nabla_j\alpha_m\alpha^m - \frac{\epsilon}{2}\alpha_j \\
 &= \nabla_j(\operatorname{div}(\alpha)) - \nabla_j(|\alpha|^2) - \frac{\epsilon}{2}\alpha_j.
 \end{aligned}$$

So we now have

$$\frac{\nabla_j R}{2} + \nabla_j(\operatorname{div}(\alpha)) - \nabla_j(|\alpha|^2) - \frac{\epsilon}{2}\alpha_j = 0. \quad (7.18)$$

Taking the trace of (7.17) gives us $R + \operatorname{div}(\alpha) - \frac{n\epsilon}{2} = 0$. We apply this to the above equation and end up with

$$\nabla(R + |\alpha|^2) + \epsilon\alpha = 0 \quad (7.19)$$

In the steady case ($\epsilon = 0$), this reduces to

$$\begin{aligned}
 \nabla(R + |\alpha|^2) &= 0 \\
 \Rightarrow R + |\alpha|^2 &= \text{constant}.
 \end{aligned}$$

Notice that, if we take α to be the mean curvature form θ , then the left-hand side of the above equation is the integrand of the \mathcal{F} -functional that we defined in the beginning of this section.

In the case where $\epsilon \neq 0$, Equation (7.19) implies that the closed form α is exact and this would reduce to the familiar gradient Ricci soliton setting. So we can think of the class Θ as an obstruction to defining a \mathcal{W} -functional on Riemannian groupoids.

Actually, to define the \mathcal{W} -functional and even the rescaled λ -invariant, it seems that we need the existence of a Haar system $\rho_0 \in \Xi$ such that $\Psi^*\rho_0 = \rho_0$ for any self-equivalence Ψ of the closed groupoid. Example 10 is a trivial example of this with $d\rho_0 = d\rho_G$. The existence of such a Haar system implies that the bijection

$$\begin{aligned}
 T : C_G^\infty(M) &\rightarrow \Xi \\
 w &\rightarrow e^w d\rho_0.
 \end{aligned}$$

is natural in the sense that $T(\Psi^*w) = \Psi^*(T(w))$ for any $w \in C_G^\infty(M)$ and any self-equivalence Ψ . It is easy to see that for such a Haar system ρ_0 the mean curvature class θ is equal to zero. So the class Θ is equal to zero. The \mathcal{W} -functional can then be defined as

$$\mathcal{W}(g, f, \tau) = \int_M [\tau(R + |\nabla f|^2) + f - n] e^{-f} \phi_0^2 d\mu_g$$

where ϕ_0 is any nonnegative cutoff function for ρ_0 . This functional will have symmetries similar to the symmetries in the manifold case.

We end this section by showing that there is a differential Harnack inequality associated to the system of equations (1.3). More precisely, we fix as before a time dependent Haar system $\rho_0(t)$ with variation determined by a \mathcal{G} -invariant function β_0 and with mean curvature form θ_0 . We write the solution $\rho(t)$ of (1.3) as $d\rho = e^f d\rho_0$ so that the evolution equation of ρ is now equivalent to

$$\frac{\partial f}{\partial t} = |\theta_0 + \nabla f|^2 - R - \operatorname{div}(\theta_0 + \nabla f) - \beta_0$$

Setting $v = e^{-f}$, we can rewrite the above equation as

$$P(v) = 0 \tag{7.20}$$

where P is the differential operator

$$P = \frac{\partial}{\partial t} - 2\langle \theta_0, \nabla(\cdot) \rangle - (\operatorname{div}(\theta_0) - |\theta_0|^2 + \beta_0). \tag{7.21}$$

We have the following result.

Proposition 30. *Suppose the \mathcal{G} -invariant function $v = e^{-f}$ satisfies (7.20) where the \mathcal{G} -invariant metric varies under the Ricci flow. Then*

$$P((R + 2\operatorname{div}(\theta_0 + \nabla f) - |\theta_0 + \nabla f|^2)v) = 2|\operatorname{Ric} + \nabla\nabla f + \nabla\theta_0|^2 v \geq 0.$$

Proof. We can show that the term

$$v^{-1}P((R + 2\operatorname{div}(\theta_0 + \nabla f) - |\theta_0 + \nabla f|^2)v)$$

is equal to

$$-2\langle \nabla H, \theta_0 + \nabla f \rangle + \left(\frac{\partial}{\partial t} + \Delta \right) H \tag{7.22}$$

where

$$H = R + 2\operatorname{div}(\theta_0 + \nabla f) - |\theta_0 + \nabla f|^2.$$

We know that

$$\left(\frac{\partial}{\partial t} + \Delta \right) R = \Delta R + 2|\operatorname{Ric}|^2. \tag{7.23}$$

Given a variation $\delta g = v$, the variation of the divergence of a one form α is given by

$$\delta(\operatorname{div}(\alpha)) = -\langle v, \nabla\alpha \rangle + \left\langle \left(\operatorname{div}(v) - \frac{\nabla V}{2} \right), \alpha \right\rangle.$$

Applying this, we get

$$\begin{aligned} \left(\frac{\partial}{\partial t} + \Delta\right) (2\operatorname{div}(\theta_0 + \nabla f)) &= 4\langle Ric, \nabla(\theta_0 + \nabla f) \rangle \\ &\quad + 2\Delta(\operatorname{div}(\theta_0 + \nabla f)) + 2\Delta\left(\frac{\partial f}{\partial t} + \beta_0\right) \end{aligned}$$

which reduces to

$$\begin{aligned} \left(\frac{\partial}{\partial t} + \Delta\right) (2\operatorname{div}(\theta_0 + \nabla f)) &= 4\langle Ric, \nabla(\theta_0 + \nabla f) \rangle - 2\Delta R \\ &\quad + 2\Delta|\theta_0 + \nabla f|^2. \end{aligned} \tag{7.24}$$

The last term of H gives us

$$\begin{aligned} \left(\frac{\partial}{\partial t} + \Delta\right) (-|\theta_0 + \nabla f|^2) &= -2Ric(\theta_0 + \nabla f, \theta_0 + \nabla f) \\ &\quad - 2\langle \theta_0 + \nabla f, \nabla\left(\frac{\partial f}{\partial t} + \beta_0\right) \rangle - \Delta|\theta_0 + \nabla f|^2 \end{aligned}$$

which becomes

$$\begin{aligned} \left(\frac{\partial}{\partial t} + \Delta\right) (-|\theta_0 + \nabla f|^2) &= -2Ric(\theta_0 + \nabla f, \theta_0 + \nabla f) - \Delta|\theta_0 + \nabla f|^2 \\ &\quad - 2\langle \theta_0 + \nabla f, \nabla(|\theta_0 + \nabla f|^2 - R) \rangle \\ &\quad + 2\langle \theta_0 + \nabla f, \operatorname{div}(\theta_0 + \nabla f) \rangle. \end{aligned}$$

Combining this with (7.23) and (7.24), we get

$$\begin{aligned} \left(\frac{\partial}{\partial t} + \Delta\right) H &= 2|Ric|^2 + 4\langle Ric, \nabla(\theta_0 + \nabla f) \rangle - 2Ric(\theta_0 + \nabla f, \theta_0 + \nabla f) \\ &\quad \Delta|\theta_0 + \nabla f|^2 - 2\langle \theta_0 + \nabla f, \nabla(|\theta_0 + \nabla f|^2 - R) \rangle \\ &\quad 2\langle \theta_0 + \nabla f, \operatorname{div}(\theta_0 + \nabla f) \rangle. \end{aligned}$$

For any closed one form α , we have

$$\Delta|\alpha|^2 = 2|\nabla\alpha|^2 + 2Ric(\alpha, \alpha) + 2\langle \nabla(\operatorname{div}(\alpha)), \alpha \rangle.$$

Therefore

$$\left(\frac{\partial}{\partial t} + \Delta\right) H = 2|Ric + \nabla\nabla f + \nabla\theta_0|^2 + 2\langle \nabla H, \theta_0 + \nabla f \rangle$$

which gives us

$$\begin{aligned} v^{-1}P(Hv) &= -2\langle \nabla H, \theta_0 + \nabla f \rangle + \left(\frac{\partial}{\partial t} + \Delta\right) H \\ &= 2|Ric + \nabla\nabla f + \nabla\theta_0|^2. \end{aligned}$$

This completes the proof. \square

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